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ABSTRACT

We develop a New Monetarist model with expenditure and unemployment risks that generates equilibria with non-degenerate distribution of money holdings. Distributional effects can overturn key insights of the model with degenerate distributions such that, e.g., the value of money depends on the income distribution, a one-time money injection raises aggregate real balances in the short run – price adjustments look sluggish; anticipated inflation can raise output and welfare; there can be a long-run trade-off between inflation and unemployment. Our model features an aggregate demand channel through which transfers to workers can raise employment and a new amplification mechanism of productivity shocks.

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1 Introduction

This paper studies equilibria of a New Monetarist model with both expenditure and income risks and non-degenerate distributions of money holdings.[1] While the literature following Lagos and Wright (2005) focuses almost exclusively on equilibria with degenerate distributions, we explore the region of the parameter space where equilibria feature endogenous ex-post heterogeneity. We show that a large class of those equilibria remain analytically tractable and exhibit distributional effects that can overturn key insights of the model regarding short-run money neutrality, the effects of anticipated inflation on output and welfare, and the relationship between inflation and unemployment. Moreover, the income distribution, the distribution of real balances, and labor market outcomes are jointly determined, which has novel implications for policy and the amplification of productivity shocks.

Our environment is composed of workers who receive opportunities to consume early, in a decentralized market where money is essential (because credit is not incentive-feasible), or late, in a centralized market. In the late period, they receive an income, w, that they can use to accumulate liquid assets in the form of real balances. If w is not too large, it takes $N \geq 2$ periods, where N is endogenous, for a worker with no money to reach his targeted real balances ($N = 1$ in the Lagos-Wright model). As a result, the distribution of money holdings is non-degenerate and value functions are strictly concave in money holdings. In contrast to the Lagos-Wright model, the value of money at a steady-state equilibrium increases with workers' income, which creates a channel through which the income distribution affects firms' profits and entry, and hence unemployment.

We illustrate how the distributional effects matter for the most basic monetary policy experiments. We start with a classical experiment in monetary theory, since at least Hume (1752) and Cantillon (1755), namely, a one-time, unanticipated increase in the money supply. We characterize the transitional dynamics for allocations and prices following a "helicopter drop" of money to workers in the centralized market. If workers can reach their targeted real balances in a single period, $N = 1$, as in the Lagos-Wright model, such a money injection has

[1] Recent surveys of this literature include, e.g., Rocheteau and Nosal (2017), and Lagos, Rocheteau, and Wright (2017).

no real effect, i.e., the price level adjusts proportionally to the money supply instantaneously. In contrast, if $N \geq 2$ then our model features non-trivial transitional dynamics. For instance, if $N = 2$ then a one-time increase in the money supply raises aggregate real balances, i.e., the price level does not increase as much as the money supply. The distribution of real balances becomes less dispersed in the following decentralized goods market, which raises social welfare. We provide conditions under which the injection of money triggers a deflation in the short run, in accordance with the "price puzzle" of Eichenbaum (1992).

We incorporate income risk by assuming that w follows a two-state Markov chain where the low state is interpreted as unemployment. In the Lagos-Wright model, income risk is irrelevant since it does not affect workers' choice of real balances (e.g., Berentsen, Menzio, and Wright, 2011). In contrast, when $N \geq 2$ income risk matters and the distribution of real balances becomes a function of the income distribution across workers. It follows that the effectiveness of monetary policy depends on the state of the labor market. For instance, a one-time injection of money is more likely to have real effects when unemployment is high and the income of the unemployed is low. Anticipated inflation can raise welfare when unemployment is high.

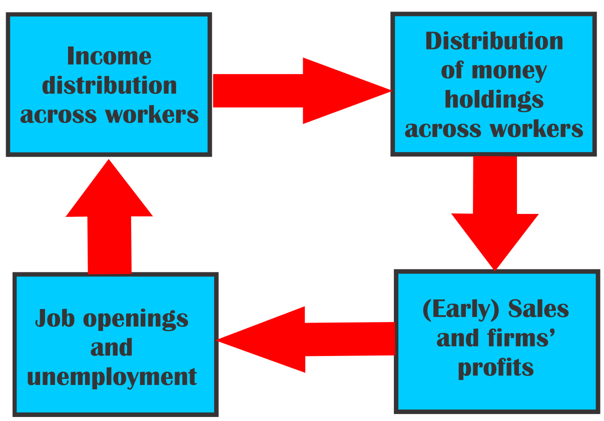

Figure 1: The aggregate demand channel

Finally, we endogenize the income risk by formalizing a labor market with matching frictions, along the lines of Mortensen and Pissarides (1994). The model generates an aggregate demand channel according to which the distribution of incomes affects the distribution of liquidity across workers, which affects firms' sales, profits, and job openings. This new channel, which is illustrated in Figure 1, leads to new policy predictions. For instance, an increase in unemployment benefits generates an increase in the distribution of real balances (in a first-order stochastic sense), which can raise employment. The aggregate demand channel also provides an amplification mechanism for productivity shocks. Indeed, as productivity goes up, the fraction of employed workers increases. Employed workers accumulate real balances faster than unemployed ones, which generates an increase in aggregate real balances, allowing firms to raise their sales to early consumers. In terms of monetary policy, we show that the model can generate a long-run Phillips curve if money growth is implemented by transfers to workers. The trade-off between inflation and unemployment is exploitable and can raise society's welfare.

1.1 Literature review

Our paper is part of the literature on search equilibria with distributions of money holdings, starting with Diamond and Yellin (1985) in a model with price posting. This literature includes Green and Zhou (1998) and Zhou (1999) under price posting, Camera and Corbae (1999), Molico (2006), Zhu (2005) under bargaining, and Menzio, Shi, and Sun (2013) in a model of posting with directed search. A key assumption of our model is that the economy is composed of two goods traded in alternating markets structures as in Lagos and Wright (2005) and Rocheteau and Wright (2005). In that vein, Chiu and Molico (2010, 2011) relax the assumption of quasi-linear preferences but they have to solve the model numerically. Zhu (2008) achieves tractability by assuming overlapping generations of finitely-lived agents while Berentsen, Camera, and Waller (2005) assume two rounds of decentralized trade before agents can readjust their money holdings. In contrast, we work with a similar environment as in Lagos-Wright and Berentsen, Menzio, and Wright (2011) but study equilibria with binding constraints (on consumption or labor supply) that have not been investigated before. Those equilibria are tractable and can be solved in closed-form. Moreover, we study out-of-steady-state dynamics, income risk, and unemployment. Chiu and Molico (2014) and Jin and Zhu (2014) also study transitional dynamics following "helicopter" drops in the context of the

Shi-Trejos-Wright model with general distributions of money holdings and show, through numerical examples, that a money injection can have a persistent effect on output and price adjustments are sluggish. Their findings are broadly consistent with our analytical results.

Our model is related to the continuous-time version of Rocheteau, Weill, and Wong (2018) that describes a competitive economy populated with ex-ante identical agents, along the lines of Bewley (1980). In contrast, we study a discrete-time economy with random matching and non-competitive pricing. The use of discrete time allows us to harness the ex-post heterogeneity, thereby facilitating the study of transitional dynamics, the introduction of income risk and a frictional labor market. The combination of both random matching risk and employment risk is related to Algan, Challe, and Ragot (2011) who study temporary and permanent changes in money growth in a Bewley economy. Related to Lippi, Ragni, and Trachter (2016) the effects of lump-sum monetary injections depend on the state of the economy. Our expenditure shocks are similar to the uncertain lumpy expenditures in the Baumol-Tobin model of Alvarez and Lippi (2013), but we do not take the consumption path as exogenous and we endogenize the income risk.

Our extension with a frictional labor market is related to the Mortensen-Pissarides model of Krusell, Mukoyama, and Şahin (2010) with risk-averse agents who self-insure by accumulating capital. In our model agents self-insure with money holdings against both expenditure and income shocks, which allows us to study monetary policy. Equity shares play no insurance role as they are held by risk-neutral entrepreneurs.[2] However, our model incorporates an aggregate demand mechanism that operates through the composition of early and late consumption and the distribution of money holdings. A key difference relative to Berentsen, Menzio, and Wright (2011) is that the income risk arising from the frictional labor market matters since wealth effects are present in equilibria with nondegenerate distributions. The induced relationship between the income distribution and the distribution of liquidity can overturn some policy predictions and generates an amplification mechanism.

Our paper complements a recent literature on heterogeneous-agent, new-Keynesian (HANK)

[2]For a version of the Mortensen-Pissarides model where claims on firms' profits are part of the liquidity, with money and government bonds, see Rocheteau and Rodriguez (2014).

models, e.g., Oh and Reis (2012), McKay and Reis (2016), and Kaplan, Moll and Violante (2017). Those models describe cashless economies with monopolistic competition, sticky prices, and uninsurable idiosyncratic risk where monetary policy takes the form of a Taylor rule. In Oh and Reis (2012), there are two groups of households: one group is more patient and has access to perfect insurance markets and the second group is relatively impatient and do not own shares in the firms. Lump sum transfers boost aggregate demand by redistributing wealth from households with a low marginal propensity to consume (MPC) to those with a high MPC. In Kaplan, Moll and Violante (2017), agents self insure against idiosyncratic income risk by accumulating different assets with different degrees of liquidity, but they are subject to transaction costs to reallocate portfolios. Their environment distinguishes between hand-to-mouth households who consume their entire current income and unconstrained households. The latter are affected by monetary policy through a change in the rate of return on liquid assets while hand-to-mouth households are affected through their labor income and fiscal effects coming from the budget constraint of the government. In our model, monetary policy is conducted through changes in the money growth rate and we do not assume nominal rigidities.[3] A key prediction of our model is that agents have a target for their real balances that they approach slowly through time. It allows us to distinguish between agents who have reached the target and agents who have not. Relative to Oh and Reis (2012), lump-sum transfers of money raise aggregate demand by redistributing the liquid wealth from the former to the latter. Relative to Kaplan, Moll, and Violante (2017), monetary policy affects the latter through both its fiscal effects and the rate of return of money while the former are only affected by the rate of return of money. The distribution of liquidity across workers also matters for the composition of firms' sales between early sales financed with liquid assets and priced at a markup and late sales, which affects firms' incentives to open jobs and allows us to endogenize the employment risk.

[3] Even though we discuss monetary policy in terms of money growth, there is a one-to-one mapping between the inflation rate and the nominal interest rate on an illiquid bond (that can only be held by entrepreneurs) through the Fisher equation. Our companion paper (Rocheteau, Weill, and Wong, 2018) considers an economy with money and bonds. Other papers in the New Monetarist literature discuss policy in terms of open-market operations. For an overview see, e.g., Rocheteau, Wright, and Xiao (2018).

2 Environment

Time, $t \in \mathbb{N}_0$, is discrete and the horizon is infinite. Each period has three stages. In the first stage, a subset of agents are subject to income/employment shocks. In the second stage, agents trade in a decentralized retail market (DM). In the third stage, they trade in a centralized market (CM). The DM and CM consumption goods are perishable and the CM good is taken as the numéraire.

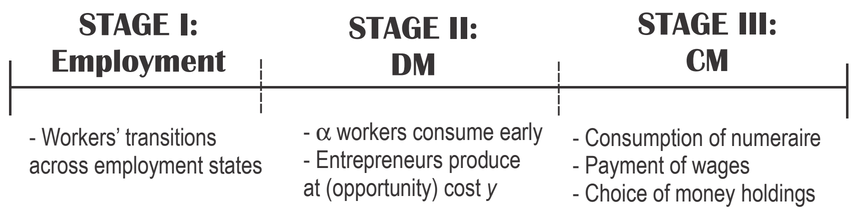

Figure 2: Timing.

The economy is populated by two types of agents: a unit measure of *workers* and a positive measure of risk-neutral *entrepreneurs*. Entrepreneurs consume in the CM stage only. Workers consume in both DM and CM stages but only work in the CM stage. The period utility function of a worker is $\varepsilon v(y) + c$ where $\varepsilon \in \{0, 1\}$ is a preference shock, $y \in \mathbb{R}_+$ is (early) DM consumption and $c \in [0, \bar{c}]$ is (late) CM consumption.[4] We assume that v is continuously differentiable, bounded, and strictly concave with $v(0) = 0$, $v'(0) = +\infty$, and $v'(+\infty) = 0$. Preference shocks, $\{\varepsilon_t\}$ are i.i.d. across agents and time with $\Pr(\varepsilon_t = 1) = \alpha$ and $\Pr(\varepsilon_t = 0) = 1 - \alpha$. The period utility of an entrepreneur is $c \in \mathbb{R}_+$. The discount factor across periods, $\beta \equiv (1 + r)^{-1} \in (0, 1)$, is common to all agents.

Workers are in one of two states, $e \in \{0, 1\}$, interpreted as employment states, i.e., $e = 1$

[4] We introduce $\bar{c}$ for technical reasons, i.e., to keep value functions bounded in order to apply standard dynamic programming results. We can choose $\bar{c}$ to be arbitrarily high so that in equilibrium $c \leq \bar{c}$ never binds. We could also introduce a minimum subsistance level, $\underline{c}$. In our analysis $\underline{c}$ has been normalized to 0.

if the worker is employed and $e = 0$ if the worker is unemployed. This state evolves according to a Markov process with transitions occurring in the first stage. A transition from $e = 1$ to $e = 0$ occurs with job separation probability δ. Likewise, a transition from $e = 0$ to $e = 1$ occurs with job finding probability λ. We endogenize λ in Section 7 with explicit search frictions in the labor market. A worker in state e receives a real income, w_e, at the beginning of the CM stage, where $w_1 \geq w_0$.

Each entrepreneur is endowed with $\bar{q}$ units of output at the beginning of each period. Output is perfectly storable across stages and can be divided into y units of DM output and $\bar{q} - y$ units of numéraire. Hence, the opportunity cost of selling y in the DM is exactly y. Output is perishable across periods.

Market structures differ in the DM and CM. In the DM, the demand from workers with a preference for early consumption is divided evenly and randomly among all entrepreneurs. We denote σ the measure of early-consumers per entrepreneur. We assume a simple trading mechanism according to which entrepreneurs charge a constant gross markup $\mu \geq 1$ over their linear cost, i.e., each unit of DM consumption costs μ units of numéraire. This trading mechanism satisfies feasibility (under a condition specified later) and individual rationality and it is similar to other mechanisms studied in the literature, e.g., bargaining, competitive pricing with strictly convex costs (e.g., Rocheteau and Wright, 2005), and monopolistic competition (e.g., Silva, 2017). The case $\mu = 1$ coincides with the commonly-used mechanism whereby the buyer makes a take-it-or-leave-it offer to the producer. Given this markup, we denote $y^* > 0$ such that $v'(y^*) = \mu$. In the CM, all agents are price-takers and markets clear.

In the absence of enforcement and monitoring technologies, debt contracts, either across stages or across periods, are not incentive feasible. There is an intrinsically useless, perfectly divisible and storable asset called money that agents can (but don't have to) use as a medium of exchange to overcome these frictions. We use M_t to denote the money supply at the start of period t. The CM price of money in terms of the numéraire is ϕ_t. The gross rate of return of money is denoted $R_t \equiv \phi_t/\phi_{t-1}$.

3 Equilibrium

We characterize an equilibrium in three steps. First, we study the decision problem of a worker who takes as given the sequence of rates of return on money, $\{R_t\}_{t=1}^{+\infty}$. Second, given the worker's optimal consumption/saving decisions, we write the law of motion for the distribution of real balances. Third, we clear the money market in every CM in order to obtain the value of money, $\{\phi_t\}_{t=0}^{+\infty}$, and hence its rate of return, $R_{t+1} = \phi_{t+1}/\phi_t$.

Value functions Consider first the problem of a worker at the beginning of the CM of period t with employment state $e \in \{0,1\}$ holding $z \geq 0$ real balances (money balances expressed in terms of numéraire). In order to characterize this problem, we make two assumptions on $\{R_{t+1}\}_{t=0}^{+\infty}$. First, there exists some $\underline{R} > 0$ such that, for all $t \geq 0$, $R_{t+1} > \underline{R}$. This first assumption rules out hyper-inflationary dynamics where the gross rate of money approaches 0. Second, we assume that

$$\sum_{i=1}^{\infty} \beta^i (1-\alpha)^{i-1} \alpha \prod_{j=1}^{i} R_j < +\infty. \tag{1}$$

This condition allows us to prove the differentiability of the value function. Both assumptions will be verified for the steady states and transitional dynamics we analyze in the paper.

The value function of a worker at the beginning of the CM solves:

$$W_t(z,e) = \max_{c,z' \geq 0} \{c + \beta E_e [V_{t+1}(z',e')]\} \quad \text{s.t.} \quad z' = R_{t+1}(z + w_e - c) \geq 0, \ \ c \leq \bar{c}, \tag{2}$$

where V_{t+1} is the value function of the worker at the beginning of $t+1$ following the realization of the employment state, e', but before entering the DM. According to (2), the worker chooses his consumption, c, and next-period real balances, z', in order to maximize his expected discounted continuation value in $t+1$. The budget identity specifies that the next-period real balances are equal to current real balances and income net of consumption multiplied by the gross rate of return of money. The value functions are indexed by t as the gross rate of return of money, R_t, might vary over time.

The lifetime expected discounted utility of a worker at the beginning of the DM is:

$$V_t(z,e) = \alpha \max_{\mu y \leq z} [v(y) + W_t(z - \mu y, e)] + (1-\alpha) W_t(z,e). \tag{3}$$

With probability α, the worker receives a preference shock for early consumption, in which case he consumes y in exchange for μy real balances. With probability $1-\alpha$, the worker does not wish to consume early and enters the next CM with z real balances.

From (2) and (3), we define W_t recursively as follows:

$$W_t(z,e) = \max\left\{w_e + z - \frac{z'}{R_{t+1}} + \beta E_e\left[\alpha\left[\upsilon(y_{e'}) + W_{t+1}\left(z' - \mu y_{e'}, e'\right)\right] + (1-\alpha)W_{t+1}(z',e')\right]\right\}$$

$$\text{s.t.} \quad z' \in \left[R_{t+1}\left(z + w_e - \bar{c}\right), R_{t+1}\left(z + w_e\right)\right] \quad \text{and } y_{e'} \leq \frac{z'}{\mu}, \tag{4}$$

where the maximization is with respect to the DM consumption plan, $\{y_{e'}\}_{e' \in \{0,1\}}$, and next-period real balances, $z' \geq 0$.

Proposition 1 *Given $\{R_{t+1}\}_{t=0}^{+\infty}$ satisfying (1), the Bellman equations (3)-(4) have unique bounded solutions, $V_t(z,e)$ and $W_t(z,e)$, that are continuous, concave, strictly increasing, and satisfy*

$$\|W\| \leq \frac{\bar{c} + \beta\alpha\|\upsilon\|}{1-\beta} \text{ and } \|V\| \leq \frac{\bar{c} + \alpha\|\upsilon\|}{1-\beta}.$$

Moreover, W_t and V_t are continuously differentiable with $W_t'(0^+, e) < \infty$ and $V_t'(0^+, e) = \infty$ for all $e \in \{0,1\}$.

In order to prove Proposition 1 (see Appendix) we use (4) to define a contraction mapping from the set of bounded functions into itself. In order to establish differentiability, we apply the envelope theorem of Rincón-Zapatero and Santos (2009).[5]

Choice of real balances Let ξ_t denote the Lagrange multiplier associated with $c \geq 0$.[6] When $c = 0$, the worker finds it optimal to forego all consumption in the current CM in order to accumulate real balances he can spend in the following DM. Substituting $c = z + w_e - z'/R_{t+1}$ into the objective, we rewrite the worker's problem as:

$$W_t(z,e) = z + w_e + R_{t+1}^{-1} \max_{z' \geq 0}\left\{-z' + \beta R_{t+1} E_e\left[V_{t+1}(z',e')\right] + \xi_t\left[R_{t+1}(z + w_e) - z'\right]\right\}. \tag{5}$$

[5] Applying the envelope theorem of Rincón-Zapatero and Santos (2009) is not immediate because two of their maintained assumptions are violated. First, our environment is non-stationary, since R_{t+1} is not constant over time. Second, some optimal choices may not lie in the interior of the state space, e.g., $y \leq z/\mu$.

[6] Throughout, we assume $\bar{c}$ is large enough so that $c \leq \bar{c}$ never binds in the decision problem of an agent with real balances z, for all z in the support of the money distribution.

If $\xi_t = 0$, then the choice of next-period real balances is independent of current wealth and W_t is linear. However, if $c \geq 0$ binds, $\xi_t > 0$, then the choice of real balances is no longer independent of current wealth and W_t is no longer linear—the two key ingredients for the tractability of the Lagos-Wright model. The first-order condition for the choice of real balances is

$$-\omega_t(z, e) + R_{t+1}\beta E_e V'_{t+1}(z', e') \leq 0, \quad \text{“} = \text{”} \text{ if } z' > 0, \tag{6}$$

where $\omega_t(z, e) \equiv W'_t(z, e) = 1 + \xi_t$ measures the cost of accumulating real balances, and $R_{t+1}\beta E_e V'_{t+1}(z', e')$ is the marginal expected benefit. We denote $z^{\star}_{e,t+1}$ the solution to (6) when $\xi_t = 0$:

$$R_{t+1}\beta E_e V'_{t+1}(z^{\star}_{e,t+1}, e') = 1. \tag{7}$$

It equalizes the marginal utility of consumption, one, with the discounted marginal value of real balances in the next DM. The constraint $c \geq 0$ does not bind if $z + w_e \geq z^{\star}_{e,t+1}/R_{t+1}$.

Choice of early consumption. The solution to the maximization problem on the right side of (3) is

$$y_t(z, e) = \upsilon'^{-1}\left[\mu W'_t(z - \mu y, e)\right] \text{ if } \upsilon'(z/\mu) < \mu W'_t(0, e); \tag{8}$$

$$= z/\mu \text{ otherwise.} \tag{9}$$

According to (8)-(9), whenever possible the worker equalizes his marginal utility of early consumption, $\upsilon'(y)$, with its opportunity cost measured by $\mu W'_t$. Using that $W'_t(z - \mu y, e)$ is non-increasing in z, it follows that $y_t(z, e)$ is non-decreasing in z. If $c \geq 0$ does not bind in the following CM, then $W'_t = 1$ and $y_t(z, e) = \upsilon'^{-1}(\mu) = y^*$. If $z \leq \bar{z}_{e,t}$, where $\bar{z}_{e,t}$ is the solution to

$$\upsilon'\left(\frac{\bar{z}_{e,t}}{\mu}\right) = \mu W'_t(0, e), \tag{10}$$

then the worker spends all his real balances. From Proposition 1, $W'_t(0, e) < +\infty$ and hence $\bar{z}_{e,t} > 0$. We summarize the optimal solution to (3) in the following proposition.

Proposition 2 ***(Early consumption)*** *For given $\{R_{t+1}\}_{t=0}^{+\infty}$ satisfying (1), the worker's problem in the DM, (3), has a unique solution, $y_t(z, e)$. This solution is continuous, increasing, satisfies* $\lim_{z\to 0} y_t(z, e) = 0$ *and* $\lim_{z\to\infty} y_t(z, e) = \infty$.

1. *For all* $z \le \bar{z}_{e,t}$, $y_t(z,e) = z/\mu$.

2. *If* $W_t'(0,e) > 1$, *i.e.*, $w_e < z^{\star}_{e,t+1}/R_{t+1}$, *then* $\bar{z}_{e,t} < \mu y^*$ *and* $y_t(z,e) < z/\mu$ *for all* $z \in \left(\bar{z}_{e,t}, \mu y^* + z^{\star}_{e,t+1}/R_{t+1} - w_e\right)$.

3. *For all* $z \in \left[\mu y^* + z^{\star}_{e,t+1}/R_{t+1} - w_e, \mu y^* + z^{\star}_{e,t+1}/R_{t+1} - w_e + \bar{c}\right]$, $y_t(z,e) = y^* = \upsilon'^{-1}(\mu)$.

4. *For all* $z > \mu y^* + z^{\star}_{e,t+1}/R_{t+1} - w_e + \bar{c}$ *then* $y_t(z,e) > y^*$.

Distribution of real balances. We denote $G_{e,t}(z)$ the measure of workers in state $e \in \{0,1\}$ holding no more than z real balances at the start of the CM stage (before late consumption) in period t. It solves:

$$G_{e,t}(z) = \int \left[\alpha \mathbb{I}_{\{x-\mu y_t(x,e) \le z\}} + (1-\alpha)\mathbb{I}_{\{x \le z\}}\right] dF_{e,t}(x) \tag{11}$$

$$G_t(z) = G_{0,t}(z) + G_{1,t}(z), \tag{12}$$

where $F_{e,t}(x)$ is the measure of workers in employment state e with less than x real balances at the start of the DM market of period t. The first term on the right side of (11) captures the measure of workers who receive a spending opportunity in the DM and enter the CM with $x - \mu y_t(x,e)$ real balances. The second term corresponds to workers who do not receive a spending opportunity. The measure of workers at the start of the DM market is given by:

$$F_{e',t+1}(z) = \sum_{e \in \{0,1\}} p_{e,e'} \int \mathbb{I}_{\{z'_{t+1}(x,e) \le z\}} dG_{e,t}(x), \tag{13}$$

where $p_{e,e'}$ is the transition probability from e to e', e.g., $p_{0,1} = \lambda$ and $p_{1,0} = \delta$, and $z'_{t+1}(x,e)$ is the CM choice of real balances conditional on holding x real balances in employment state e. Hence, the distribution of real balances across all workers is given by $F_t(z) = F_{0,t}(z) + F_{1,t}(z)$. Finally, the steady-state employment rate of workers is

$$n = \frac{\lambda}{\delta + \lambda}. \tag{14}$$

Value of money The value of money is determined by the following money market clearing condition:

$$\phi_t M = \int x dF_t(x). \tag{15}$$

It depends on the distribution, which itself depends on policy rules. The rate of return of money is

$$R_{t+1} = \frac{\phi_{t+1}}{\phi_t} = \frac{\int x dF_{t+1}(x)}{\int x dF_t(x)}. \tag{16}$$

Finally, we impose the following feasibility condition in the DM goods market:

$$\sigma \sum_{e \in \{0,1\}} \int y_t(z,e) dF_{e,t}(z) \leq \bar{q}. \tag{17}$$

This condition requires that the workers' demand for CM consumption per entrepreneur is no greater than the output of each entrepreneur. It is satisfied provided that $\bar{q}$ is sufficiently high.[7]

Definition 1 *Given some initial distribution of nominal money balances, H_0, a perfect-foresight monetary equilibrium is composed of:*

1. *A sequence of value functions, $\{V_t, W_t\}_{t=0}^{+\infty}$, that solve the Bellman equations (3) and (4).*

2. *A sequence of prices, $\{\phi_t, R_{t+1}\}_{t=0}^{+\infty}$ that solve the market-clearing condition, (15), and the definition (16).*

3. *A sequence of distributions of real money balances across workers, $\{F_{e,t}, G_{e,t}, F_t\}_{t=0}^{+\infty}$, that solves the law of motions (11)-(12), (13), $F_t(z) = F_{0,t}(z) + F_{1,t}(z)$, and $F_0(z) = H_0(z/\phi_0)$.*

4. *Distributions and policy functions satisfy the feasibility condition in the DM goods market, (17).*

The value functions cannot in general be determined independently from the sequence of prices and distributions. An exception is given by steady-state equilibria where $\{F_t, \phi_t\}$ is constant over time and the gross rate of return of money is $R_t = \phi_{t+1}/\phi_t = 1$.

[7] Notice that this feasibility constraint is less restrictive than the one under pairwise meetings. If we imposed matches to be bilateral, then feasibility would require $y_t(z,e) \leq \bar{q}$ for all (z,e) in the support of $F_{e,t}(z)$.

4 Money in the long run

In this section, we shut down employment risk, $\delta = 0$, so that all workers receive the same income w, and we specialize our analysis to steady-state equilibria where workers in the DM spend all their real balances, $z^\star < \bar{z}$. This class of equilibria encompasses equilibria with degenerate distributions studied in Lagos and Wright (2005). It also contains equilibria with non-degenerate distributions where value functions are strictly concave and wealth effects exist.

Targeted real balances The marginal value of real balances at the beginning of the DM is

$$V'(z) = \frac{\alpha}{\mu}\upsilon'\left(\frac{z}{\mu}\right) + (1-\alpha)\omega(z), \quad \forall z < \bar{z}, \tag{18}$$

where the first term on the right side is the expected marginal utility of real balances in the DM. From (7) with $\omega = 1$ and $R = 1$, because we focus on steady-state equilibria with constant money supply, $z^\star$ solves:

$$\upsilon'\left(\frac{z^\star}{\mu}\right) = \mu\left(1 + \frac{r}{\alpha}\right). \tag{19}$$

At the targeted real balances, the marginal utility of DM consumption is equal to the product of two wedges: the markup and the average holding cost of real balances due to discounting.

Distribution of real balances Workers increase their real balances by the wage, w, until they reach their target or until they receive an opportunity to consume in the DM and deplete their money holdings. Hence, the support of the distribution of real balances across workers at the beginning of a period is $\{w, 2w, ..., (N-1)w,\, z^\star\}$ where $N \in \mathbb{N}$ solves

$$(N-1)w < \mu\upsilon'^{-1}\left[\mu\left(1 + \frac{r}{\alpha}\right)\right] \leq Nw. \tag{20}$$

The distribution F is composed of N mass points, $\{f_n\}_{n=1}^{N}$, where f_n is the measure of workers holding nw for all $n \in \{1, ..., N-1\}$ and f_N is the measure of workers holding their target, $z^\star$. We have:

$$
\begin{aligned}
f_1 &= \alpha && (21)\\
f_n &= (1-\alpha) f_{n-1} \text{ for all } n \in \{2, N-1\} && (22)\\
\alpha f_N &= (1-\alpha) f_{N-1}. && (23)
\end{aligned}
$$

According to (21), a measure α of workers receive a preference shock for early consumption, in which case they spend all their real balances. Those workers start the following period with $z_1 = w$ real balances. According to (22)-(23), workers with $z_{n-1} = (n-1)w$ real balances in a given period hold $z_n = nw$ in the following period if they do not have a preference for early consumption, with probability $1-\alpha$. From (21)-(23), the distribution of real balances is a truncated geometric distribution:

$$
\begin{aligned}
f_n &= \alpha(1-\alpha)^{n-1} \text{ for all } n = 1, ..., N-1 && (24)\\
f_N &= (1-\alpha)^{N-1}. && (25)
\end{aligned}
$$

Value of money and prices. Aggregate real balances are $\phi M = \sum_{n=1}^{N} f_n z_n$. From (24)-(25), and after some calculation, this gives

$$
\phi M = w \frac{\left\{1-(1-\alpha)^{N-1}\left[(N-1)\alpha+1\right]\right\}}{\alpha} + (1-\alpha)^{N-1} z^\star. \tag{26}
$$

Aggregate real balances do not depend on the nominal money supply and hence money is neutral in the long run.

Proposition 3 ***(Distribution of real balances and workers' income)*** *Consider a steady-state equilibrium with full depletion of real balances ($z^\star < \bar{z}$) featuring a N-point distribution of real balances. If $N=1$ then $\phi M = z^\star$, which is independent of w. If $N \geq 2$, then $\partial(\phi M)/\partial w > 0$.*

If the distribution of money is degenerate, $N = 1$, then ϕM reduces to $z^\star$. In that case, workers' income does not affect aggregate real balances. In contrast, for a given $N \geq 2$ the value of money increases with the wage, w, and it decreases with the rate of time preference, r. For instance, in the case $N = 2$ that we study extensively later, aggregate real balances are a weighted average of income and target, $\phi M = \alpha w + (1-\alpha) z^\star$. The fact that income matters for the mean of the distribution will generate a new aggregate demand channel with implications for (un)employment once we endogenize the measure of firms in Section 7.

Marginal value of real balances From (6), $\omega(z) = \beta V'(z')$ where $z' = \min\{z + w, z^\star\}$. Substituting $V'(z')$ by its expression given by (18) the marginal value of money solves

$$\omega(z) = \beta \left[\frac{\alpha}{\mu} \upsilon' \left(\frac{z+w}{\mu} \right) + (1-\alpha)\omega(z+w) \right], \ \forall z < z^\star - w,$$

and $\omega(z) = 1$ for all $z \in [z^\star - w, z^\star]$. The closed-form solution is

$$\omega(z) = 1 + \frac{\alpha}{\mu} \sum_{j=1}^{+\infty} \beta^j (1-\alpha)^{j-1} \left[\upsilon' \left(\frac{z + jw}{\mu} \right) - \upsilon' \left(\frac{z^\star}{\mu} \right) \right]^+, \tag{27}$$

where $[x]^+ = \max\{x, 0\}$. The marginal value of money is equal to one, the marginal utility of late consumption, plus the discounted sum of the differences between the marginal utility of DM consumption at a point in time and the marginal utility of consumption at the targeted real balances.

From (10), the condition for full depletion of real balances, $z^\star \leq \bar{z}$, can be expressed as $\upsilon'(\ z^\star/\mu) \geq \mu\omega(0)$ or, from (27), as:

$$\upsilon' \left(\frac{z^\star}{\mu} \right) - \mu = \frac{\mu r}{\alpha} \geq \alpha \sum_{j=1}^{+\infty} \beta^j (1-\alpha)^{j-1} \left[\upsilon' \left(\frac{jw}{\mu} \right) - \upsilon' \left(\frac{z^\star}{\mu} \right) \right]^+. \tag{28}$$

We represent the condition (28) by a grey area in Figure 3 where $\upsilon'(y^\star) = \mu$. The dotted lines represent the conditions in (20), $\mu(1 + r/\alpha) = \upsilon'(Nw/\mu)$. The case studied in LRW, $N = 1$, requires the wage, w, to be large so that the buyer can readjust his money balances in a single period. If the endowment is such that $\upsilon'(w/\mu) > \mu(1 + r/\alpha)$ then it will take more than one period for the buyer to reach his targeted real balances.[8]

A steady-state, monetary equilibrium with full depletion of real balances is a list, (N, $z^\star, \phi, \{\mu_n\}_{n=1}^N$), that solves (19), (20), (24)-(25), (26), and (28).

[8]In our working paper we show that the conditions for the steady state equilibrium to display a non-degenerate distribution hold for standard calibrations of the model provided that we reduce the model frequency from annually to monthly or weekly. Moreover, this change has significant welfare implications as it reduces the welfare cost of inflation by about 20%.

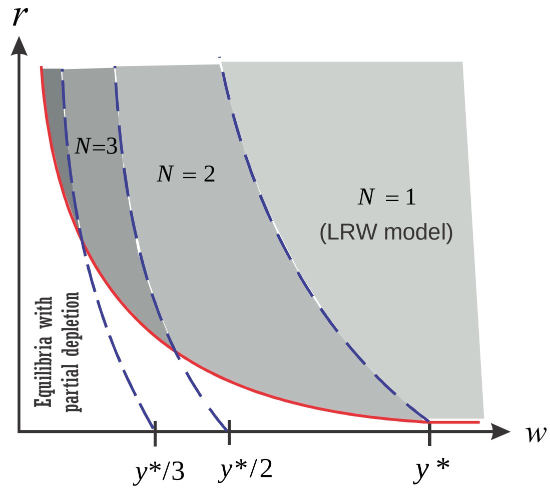

Figure 3: Typology of equilibria

Proposition 4 ***(Existence of steady-state monetary equilibria with full depletion)*** *If (28) holds, then there exists a steady-state monetary equilibrium with full depletion. If $\upsilon'(w/\mu) \leq \mu(1 + r/\alpha)$ then the equilibrium features $N = 1$, i.e., there is a degenerate distribution of workers' real balances. If $\upsilon'(w/\mu) > \mu(1 + r/\alpha)$ then the equilibrium features $N \geq 2$, i.e., the distribution of workers' real balances is non-degenerate.*

5 Money in the short run

There is a long tradition to study the effects of unanticipated changes of the money supply on the dynamics of prices and the real economy, going back to Cantillon (1755) and Hume (1752), and more recently Friedman (1969), Lucas (1972) and Wallace (1997). For instance, Hume considers the following thought experiment: "For suppose that, by miracle, every man in Great Britain should have five pounds slipped into his pocket in one night; this would much more than double the whole money that is at present in the kingdom." We consider a related experiment where the monetary authority transfers πM, with $\pi > 0$, in a lump-sum fashion to all workers at the time they enter the CM of $t = 0$.[9] We contrast the effects of

[9] Similarly, Friedman (1969) introduces the famous "helicopter drop" parable: "Let us suppose now that one day a helicopter flies over this community and drops an additional $1,000 in bills from the sky, which is,

such an "helicopter drop" experiment on equilibria with a degenerate distribution ($N = 1$) and equilibria with a nondegenerate distribution ($N = 2$).

As a benchmark, consider first equilibria with $N = 1$, which requires $\upsilon'(w/\mu) < \mu(1 + r/\alpha)$. From (7) and (18), $z_{t+1}^{\star}$ is determined by the following Euler equation:

$$\beta R_{t+1}\left\{\frac{\alpha}{\mu}\upsilon'\left(\frac{z_{t+1}^{\star}}{\mu}\right)+1-\alpha\right\}=1 \text{ for all } t \geq 1. \tag{29}$$

By substituting $R_1 = z_1^{\star}/[\phi_0(1+\pi)M]$ and $R_{t+1} = z_{t+1}^{\star}/z_t^{\star}$ into (29), we reduce a monetary equilibrium to a list, $(\phi_0, \{z_t^{\star}\}_{t=1}^{\infty})$ with $\phi_0 > 0$, that solves:

$$\phi_0(1+\pi)M = \beta z_1^{\star}\left[\frac{\alpha}{\mu}\upsilon'\left(\frac{z_1^{\star}}{\mu}\right)+1-\alpha\right] \tag{30}$$

$$z_t^{\star} = \beta z_{t+1}^{\star}\left[\frac{\alpha}{\mu}\upsilon'\left(\frac{z_{t+1}^{\star}}{\mu}\right)+1-\alpha\right] \quad \text{for all } t \geq 1. \tag{31}$$

Proposition 5 ***(Money injection with degenerate distributions)*** *There exists a solution to (30)-(31) such that $z_{t+1}^{\star} = z^{\star}$, $R_{t+1} = 1$, and $\phi_t = z^{\star}/[M(1+\pi)]$ for all $t \geq 0$.*

The value of money adjusts instantly to its new steady state so that aggregate real balances are constant. The money injection has no real effects.

In the rest of this section, we focus on equilibria with $N = 2$, which requires $w < \mu\upsilon'^{-1}[\mu(1 + r/\alpha)] \leq 2w$. The initial distribution of money, H_0, corresponds to the steady-state distribution where a measure α of workers hold $m_\ell = wM/[\alpha w + (1-\alpha)z^{\star}]$ and a measure $1-\alpha$ hold $m_h = z^{\star}M/[\alpha w + (1-\alpha)z^{\star}]$. At the beginning of the CM of $t = 0$, after a round of DM trades, the distribution of money holdings has three mass points. There is a measure α of workers holding no money (the workers who were matched in the previous DM), a measure $\alpha(1-\alpha)$ holding m_ℓ, and a measure $(1-\alpha)^2$ holding m_h.

Assuming π is close to 0, and by continuity with respect to the steady state we conjecture that the distribution of real balances at the start of the DM, F_t, $t \geq 1$, has two mass points, z_t^1 and $z_t^{\star}$. The first mass point, z_t^1, corresponds to the real balances held by the α workers who depleted their money holdings in the previous DM. At $t = 1$, $z_1^1 = R_1(w + \pi\phi_0 M)$ while

of course, hastily collected by members of the community. Let us suppose further that everyone is convinced that this is a unique event which will never be repeated."

for all $t \geq 2$, $z_t^1 = R_t w$. The second mass point, $z_t^\star$, corresponds to the targeted real balances of the remaining $1-\alpha$ workers who were unmatched in the DM of $t-1$. It solves

$$\upsilon'\left(\frac{z_t^\star}{\mu}\right) = \mu\left(1+\frac{1+r-R_t}{\alpha R_t}\right) \quad \text{for all } t \geq 1. \tag{32}$$

Aggregate real balances are equal to the population-weighted average of z_t^1 and $z_t^\star$:

$$\phi_t(1+\pi)M = \alpha z_t^1 + (1-\alpha)z_t^\star \text{ for all } t \geq 1. \tag{33}$$

From (33) written at two consecutive dates, ϕ_0 and $\{R_t\}_{t=1}^{+\infty}$ solve:

$$\phi_0(1+\pi)M = \frac{\alpha z_1^1 + (1-\alpha)z_1^\star}{R_1} \tag{34}$$

$$\alpha z_t^1 + (1-\alpha)z_t^\star = \frac{\alpha z_{t+1}^1 + (1-\alpha)z_{t+1}^\star}{R_{t+1}} \text{ for all } t \geq 1. \tag{35}$$

This system is the analog of (30)-(31) for equilibria featuring $N = 2$.

Proposition 6 ***(Money injection with nondegenerate distributions)*** *For all π close to 0, there exists a unique solution to (34)-(35) that becomes stationary starting at $t = 2$ with $R_t = 1$ and $\phi_t(1+\pi)M = \alpha w + (1-\alpha)z^\star$ for all $t \geq 2$.*

1. *R_1 is the unique solution to*

$$\frac{\alpha R_1 w + (1-\alpha)z_1^\star}{1-\alpha\pi/(1+\pi)} = \alpha w + (1-\alpha)z^\star, \tag{36}$$

where $z_1^\star = \mu\upsilon'^{-1}\left[\mu + \mu\left(1+r-R_1\right)/\alpha R_1\right]$. It is such that $R_1 < 1$ and $\phi_0 > \phi_1$, i.e. there is inflation lagging behind.

2. *Initial aggregate real balances are*

$$\phi_0(1+\pi)M = \frac{\alpha w + (1-\alpha)z^\star}{R_1} > \phi_t(1+\pi)M,\ t \geq 2. \tag{37}$$

3. *If*

$$-\frac{\upsilon''(z^\star/\mu)(z^\star/\mu)}{\upsilon'(z^\star/\mu)} > \frac{z^\star}{\left(z^\star - w\right)\beta\alpha\left(\alpha + r\right)}, \tag{38}$$

then $R_0 > 1$, i.e., there is deflation in the short run.

4. There is a mean-preserving reduction in the distribution of real balances in the DM of $t = 1$ and an increase in society's welfare.

The initial value of money, ϕ_0, does not fall in the same proportion as the increase in M so that aggregate real balances rise above their steady-state value. The economy returns to its steady state in the CM of $t = 1$.[10] Hence, $\phi_0 > \phi_1$ and $R_1 < 1$. To understand why $\phi_0 > \phi_1$, assume instead that the price adjusts instantly to its steady state value, $\phi_0 = \phi_1$. Then, since $R_1 \equiv \phi_1/\phi_0 = 1$, the real balances of unconstrained workers remain equal to their steady-state value, $z^\star$. At the same time, constrained workers find it optimal to save the lump sum transfers instead of spending it, implying that their real balances are larger than their steady-state value. On aggregate, real balances are thus larger than their steady state value, which is inconsistent with our premise. Therefore, in equilibrium, ϕ_0 has to rise above its steady state level for R_1 to fall, so that unconstrained worker find it optimal to hold less real balances.[11]

The money injection generates a redistribution of real balances and consumption. The $1 - \alpha$ workers who are unconstrained by their income when choosing z, respond to a lower R_1 by reducing z. As a result they consume less relative to the steady state, $z_1^\star < z^\star$. In contrast, the α workers who are constrained by their income in the CM can raise z by saving the lump-sum transfer, and hence they consume more in the following DM. Total real balances spent in the DM are the same as in the steady state, but welfare is higher due to the workers' concave preferences.[12]

The real value of money in the initial steady state is $(1 + \pi)\phi_1$: it exceeds the real value of money in the new steady state by a factor equal to the growth rate of the money supply, $1 + \pi$. Under condition (38), according to which the target for real balances is sufficiently

[10] We show in Rocheteau, Weill, and Wong (2015) that transitional dynamics following an unanticipated increase in the money supply are long lasting for equilibria featuring $N \geq 3$.

[11] Proposition 6 has been derived for π close to 0. If the money injection is sufficiently large, then workers who enter the CM of $t = 0$ with depleted money holdings can reach their target $z_1^\star$ solution to (32), i.e., the distribution of real balances is degenerate, and $R_1 < 1$ solves $z_1^\star = \alpha w + (1-\alpha) z^\star$. Moreover, $R_1 > (1+\pi)^{-1}$, so that prices increase relative to their initial steady-state value.

[12] Note that if the DM production cost of entrepreneurs is convex, then DM output increases. See Rocheteau, Weill, and Wong (2015).

inelastic in R, the money injection causes the real value of money to rise on impact, at $t = 0$, before falling at time $t = 1$. Hence, we obtain the paradoxical result that a small money injection generates deflation followed by future inflation.[13]

While there is more to be done on this topic, we hope our results illustrate the importance of incorporating distributional considerations to understand even the most basic monetary policy experiments.

6 Income risk

We now reintroduce the income risk, $\delta > 0$. We focus on equilibria where employed workers are similar to the agents in the LRW model in that they can reach $z^\star$ in a single period, $w_1 > z^\star$. In contrast, unemployed workers are similar to the agents in Section 5 and can accumulate $z^\star$ in $N = 2$ periods, i.e., $w_0 < z^\star \leq 2w_0$.[14] We study the effects of both one-time money injections and constant money growth.

One-time money injection We revisit first the experiment in Section 5 that consists in a small money injection through a lump-sum transfer to all workers. The rate of return of money in the first period, R_1, solves a generalized version of (36),

$$\frac{\alpha u R_1 w_0 + (1-\alpha u)\, z_1^\star}{1 - \alpha u \pi / (1+\pi)} = \alpha u w_0 + (1-\alpha u)\, z^\star, \tag{39}$$

where $z_1^\star = \mu v'^{-1}\left[\mu + \mu\,(1 + r - R_1)/\alpha R_1\right]$ and where $z^\star = \mu v'^{-1}\,(\mu + \mu r/\alpha)$. The novelty is that the size of the effect on R_1 now depends on the unemployment rate, $u = 1 - n$, where n solves (14).

Proposition 7 ***(Money injection with income risk)*** *Assuming u is small, the effect of a one-time, unanticipated money injection on the rate of return of money is approximately*

$$\frac{\partial R_1}{\partial \pi} \approx -\alpha u \left[\frac{\partial \ln(z_1^\star)}{\partial R_1}\right]^{-1} > 0.$$

[13] This finding is consistent with the "price puzzle" from Eichenbaum (1992) according to which a contractionary shock to monetary policy raises the price level in the short run.

[14] The condition for full depletion of real balances, $\omega(0,0) \leq \mu^{-1} v'\left(z^\star/\mu\right) = 1 + r/\alpha$, can be reexpressed as $r/\alpha > \alpha\beta\left[v'\left(w_0/\mu\right) - v'\left(z^\star/\mu\right)\right]/\mu$.

The effect of unanticipated inflation on the rate of return of money is the product of three terms: the frequency of expenditure shocks, α, the measure of unemployed, u, and the inverse of the interest-rate semi-elasticity of money demand. The key insight is that the state of the labor market matters for the effectiveness of monetary policy. If u is close to 0, then R_1 is close to 1 and money is almost neutral. In contrast, if u is positive, then a money injection reduces R_1 below one and raises aggregate real balances.

Anticipated inflation We now turn to the case of constant money growth, $M_{t+1} - M_t = \pi M_t$, implemented via lump-sum transfers to workers. In steady state $R = (1+\pi)^{-1}$ and $z^\star$ solves

$$\upsilon'\left(\frac{z^\star}{\mu}\right) = \mu\left(1 + \frac{i}{\alpha}\right), \tag{40}$$

where $i \equiv (1+\pi)(1+r) - 1$. The quantity i can be interpreted as the nominal interest rate on an illiquid bond that can be held by entrepreneurs only.[15] By the same reasoning as above, aggregate real balances are:

$$Z = \frac{\alpha u w_0 + (1-\alpha u)(1+\pi)z^\star}{1+\pi(1-\alpha u)}. \tag{41}$$

Aggregate real balances depend on the income distribution through u and w_0: they increase with w_0 but decrease with u. In turn, the output sold by each entrepreneur expressed in terms of the numéraire depends on aggregate real balances as follows:

$$q = \sigma\frac{\mu - 1}{\mu}Z + \bar{q}, \tag{42}$$

where σ is the measure of early-consumers per entrepreneur so that the first term corresponds to early consumption sold at a markup μ. From (41), q increases with Z because it determines how much early consumers can spend on goods sold at a markup. Given the dependence of Z on the income distribution, it follows that q increases with w_0 and decreases with u. These relationships generate an aggregate demand channel through which unemployment and income affect entrepreneurs' revenue.

[15] Notice that, in contrast to the Lagos-Wright model, a bond that cannot be traded in the DM but can be traded in the CM, would still exhibit an indirect liquidity premium by allowing workers to better manage their holdings of liquidity in the CM. See, e.g., Rocheteau, Weill, and Wong (2018).

In terms of the transmission of monetary policy, inflation reduces the demand for real balances of unconstrained workers, $z^\star$, by lowering the rate of return of money, but it increases the real balances of the αu constrained workers through lump-sum transfers.

Proposition 8 ***(The inflation-output trade-off)*** *A small inflation raises Z and q if*

$$-\frac{\upsilon''(z^\star/\mu)(z^\star/\mu)}{\upsilon'(z^\star/\mu)} > \frac{z^\star}{(r+\alpha)\,\beta\mu\alpha u\,(z^\star - w_0)}, \tag{43}$$

where $z^\star = \mu\upsilon'^{-1}\left[\mu\left(1+r/\alpha\right)\right]$.

If $z^\star$ is relatively inelastic to a change in π, as implied by (43), then an increase in π above 0 affects mostly the real balances of the poorest workers. As a result, a small increase in π raises Z and, from (42), entrepreneur's output, q. The condition for a positive output effect of inflation, (43), is more likely to hold when u is large, which is another example where the state of the labor market matters for the effects of monetary policy. Finally, inflation also raises social welfare by reducing the dispersion of F.

We conclude this section by describing succinctly equilibria where both employed and unemployed workers need two periods to reach their targeted real balances. A worker in state e in the CM with depleted real balances starts the following period with $\left(w_e + Z\pi\right)/(1+\pi)$ real balances. Hence, aggregate real balances are

$$Z = \frac{\alpha\left(uw_0 + nw_1\right) + (1-\alpha)(1+\pi)z^\star}{1+\pi(1-\alpha)}.$$

Aggregate real balances increase with the mean income, $uw_0 + nw_1$. An increase in the employment rate raises Z if $w_1 > w_0$. So we now have a channel through which wages can affect real balances, which in turn affect firms' sales. We explore this channel further in the next section in a model with firm entry.

7 Unemployment and the distribution of money

We now introduce a frictional labor market in the first stage (see Figure 2), as in Mortensen and Pissarides (1994), in order to endogenize the employment risk, λ. This extension generates equilibria of the Berentsen, Menzio, and Wright (2011) model as a special case but also new equilibria with a non-degenerate distribution of money holdings across workers. In order to start production, entrepreneurs must now hire workers in a first-stage frictional labor market. The cost to open a vacant job in period t is $k > 0$ incurred in the CM of $t-1$. The worker's job finding probability, $\lambda(\theta)$, is an increasing and concave function of labor market tightness, θ, defined as the number of vacancies per unemployed, with $\lambda(0) = 0$ and $\lambda'(0) = 1$. The vacancy filling probability is $\lambda(\theta)/\theta$.

The expected discounted profits of a filled job are $\Pi = (1+r)(q - w_1)/(r+\delta)$, where w_1 is now the wage paid by the entrepreneur to his employees and q is total sales per job expressed in terms of the numéraire as given by (42).[16] The optimal number of vacancies posted by entrepreneurs is such that

$$-k + \frac{\lambda(\theta)}{\theta}\left(\frac{q - w_1}{r+\delta}\right) \leq 0, \;\text{"} = \text{"} \text{ if } \theta > 0. \tag{44}$$

The first term on the left side is the cost to open a vacant job while the second term is the probability to fill the job times the expected discounted profits of the job. Unless specified otherwise, the revenue from money creation is distributed in a lump-sum fashion to risk-neutral entrepreneurs, and hence it does not affect job creations.

Substituting q by its expression given by (42) into (44), where $\sigma = \alpha/n$ represents now the number of early consumers per filled job, and assuming an interior solution, labor market tightness solves

$$JC(\theta) \equiv \frac{(r+\delta)\,\theta k - \lambda(\theta)(\bar{q} - w_1)}{\delta + \lambda(\theta)} = MK \equiv \alpha\left(\frac{\mu - 1}{\mu}\right)\phi M, \tag{45}$$

where $JC(\theta)$ is a measure of entry costs net of profits from sales in the CM and MK represents profit margins on early sales due to the positive markup. The following lemma gives important properties of the LHS.

Lemma 1 *Assuming $\bar{q} - w_1 > 0$, there is $\underline{\theta} \geq 0$ such that: $JC(0) = JC(\underline{\theta}) = 0$, $JC(\theta) < 0$ for all $\theta \in (0, \underline{\theta})$, and $JC(\theta) > 0$, $JC'(\theta) > 0$ for all $\theta > \underline{\theta}$.*

[16] Entrepreneurs cannot commit, and hence they cannot issue claims on the profits of the jobs they created. It follows that profits are discounted according to the entrepreneurs' rate of time preference. For related models of unemployment where claims on firms' profits are liquid and the real interest rate is endogenous, see Krusell, Mukoyama, and Şahin (2010) and Rocheteau and Rodriguez (2014).

In the textbook Mortensen-Pissarides model, $\alpha = 0$ and $MK = 0$, the intersection of JC and the horizontal axis determines $\theta = \underline{\theta}$, marked "$MP$" in the left panel of Figure 4.

We consider first equilibria studied in BMW where both employed and unemployed workers can accumulate $z^\star$ in a single period, which requires $Rw_0 > z^\star$. The profit margins on early sales are $MK = \alpha(\mu - 1) z^\star/\mu$ and the equilibrium reduces to a triple $(\theta, z^\star, n)$ solution to (14), (40) and (45). Graphically, the equilibrium is determined at the intersection of JC and the horizontal MK curve, marked "BMW".[17] Aggregate real balances, $z^\star$, are unaffected by labor market outcomes, such as w_e, $\lambda(\theta)$, and n. An increase in the inflation rate reduces R and $z^\star$, thereby shifting the MK curve downward and reducing θ.

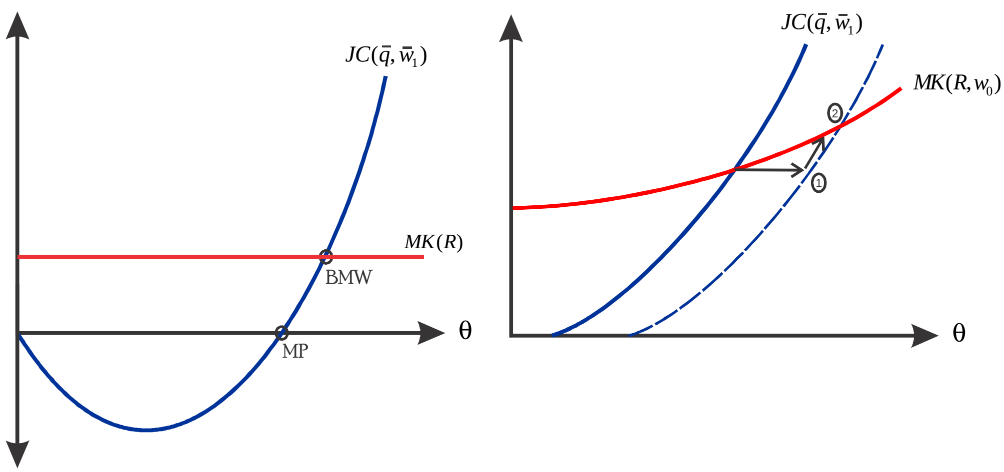

Figure 4: Equilibrium labor market tightness. Left panel: degenerate distribution of money holdings. Right panel: non-degenerate distribution of money holdings.

7.1 Ex-post heterogeneity across unemployed workers

We now consider the same type of equilibria as those studied in Section 6 where employed workers reach $z^\star$ in a single period, $Rw_1 > z^\star$, whereas unemployed workers reach $z^\star$ after two periods, $Rw_0 \in (z^\star/(1+R), z^\star)$. In a steady state αu workers hold Rw_0 real balances

[17] In Berentsen, Menzio and Wright (2011) the steady-state equilibrium might not be unique because the arrival rate of spending opportunities, α, is an increasing function of n, which creates strategic complementarities between firms' entry decision and households' choice of real balances.

and $1-\alpha u$ hold $z^{\star}$ so that

$$MK = \alpha(1-\mu^{-1})\left[(1-\alpha u)\, z^{\star} + \alpha u R w_0\right]. \tag{46}$$

In the right panel of Figure 4, the MK-curve is now upward sloping. Indeed, as θ increases, the share of workers who are employed rises. Since employed workers have a higher income than unemployed ones, they can accumulate more liquidity, which raises the aggregate demand for early consumption.

The fact that MK increases with θ generates a novel amplification mechanism to a productivity shock.[18] If $\bar{q}$ increases, more jobs are opened, which reduces the measure αu of workers with low real balances and raises the demand for early consumption. This generates an endogenous increase of q and more entry. In Figure 4, an increase in $\bar{q}$ shifts the JC-curve upward. If we keep u constant in the expression for MK, then θ rises to point 1. However, because u is endogenous and decreases with $\bar{q}$, market tightness rises further from point 1 to point 2.

Equation (46) shows that the income distribution across workers affects the distribution of liquidity and, ultimately, firms' profits. Consider an increase in unemployment benefits, w_0, financed with a lump-sum tax on entrepreneurs. The effect on the profits from early sales is

$$\frac{\partial MK}{\partial w_0} = \alpha^2 u(1-\mu^{-1})R > 0,$$

for all $\mu > 1$ and $u > 0$. The size of this effect increases with the unemployment rate, the frequency of expenditure shocks, the markup, and the rate of return of money. From (45) we have the following implications for unemployment.

Proposition 9 ***(Unemployment insurance and liquidity)*** *Suppose w_0 is financed with a lump-sum tax on entrepreneurs and w_1 is exogenous.*

1. *If the equilibrium features $N = 1$ then an increase in w_0 has no effect on aggregate real balances or (un)employment, $\partial Z/\partial w_0 = \partial u/\partial w_0 = 0$.*

[18] This amplification mechanism is distinct from the one in BMW where the frequency of early-consumption opportunities, α, is an increasing function of n. In contrast, we assume that α is constant. Our amplification mechanism operates through the distribution of real balances across workers that depends on the state of the labor market.

2. *If the equilibrium features $N = 2$ then an increase in w_0 increases the distribution of real balances in a first-order stochastic sense and it reduces unemployment, $\partial Z/\partial w_0 > 0$ and $\partial u/\partial w_0 < 0$.*

Graphically, if $N = 1$ then the MK curve is independent of w_0. If $N = 2$, then the MK curve shifts upward as w_0 increases. So more generous unemployment insurance raises the demand for early consumption, entrepreneurs' profits, and employment.

We now turn to the effects of monetary policy described as a constant money growth rate. In order to separate the monetary from the fiscal implications, we maintain the assumption that the money supply grows through lump-sum transfers to entrepreneurs. Profit margin from early sales depend on R as follows:

$$\frac{\partial MK}{\partial R} = \alpha(1-\mu^{-1})\left[(1-\alpha u)\frac{\partial z^\star}{\partial R} + \alpha u w_0\right] > 0.$$

If the target for real balances is inelastic, $\partial z^\star/\partial R \approx 0$, then the strength of the transmission mechanism is determined by uw_0. Monetary policy is more effective in economies with high unemployment and low income for the unemployed.

If the revenue from money creation is rebated in a lump-sum fashion to all workers then, from (41),

$$MK \equiv \alpha\left(\frac{\mu-1}{\mu}\right)Z = \alpha\left(\frac{\mu-1}{\mu}\right)\left[\frac{\alpha u w_0 + (1-\alpha u)(1+\pi)z^\star}{1+\pi(1-\alpha u)}\right].$$

Inflation reduces $z^\star$ but the revenue from money creation finances a transfer to workers, a fraction αu of whom are constrained by their income when choosing z. From Proposition 8, inflation raises Z if (43) holds. As a result, entrepreneurs' profits and employment increase. So the model can generate a long-run Phillips curve with an exploitable trade-off between inflation and unemployment.[19] We summarize these results with the following proposition.

Proposition 10 ***(Long-run Phillips curves)*** *Consider an equilibrium with $N = 2$ and π in the neighborhood of* 0.

[19] Rocheteau, Rupert, and Wright (2008) also obtain a long-run trade-off between inflation and unemployment in a New Monetarist model with indivisible labor by assuming substituability between CM and DM goods. However, exploiting this trade-off is not welfare improving.

1. *Money growth through lump-sum transfers to entrepreneurs raises unemployment,* $\partial u/\partial \pi > 0$*, i.e., the long-run Phillips curve is upward-sloping.*

2. *Money growth through lump-sum transfers to workers reduces unemployment,* $\partial u/\partial \pi < 0$*, if (43) holds, i.e., the long-run Phillips curve is downward-sloping.*

7.2 Ex-post heterogeneity across employed workers

Suppose now that both employed and unemployed workers need two periods to reach their targeted real balances, $Rw_1 < z^\star$ and $w_0 R(1+R) > z^\star$.[20] The margins on early sales are now:

$$MK = \alpha(1-\mu^{-1})\left\{\alpha u R w_0 + \alpha n R w_1 + (1-\alpha) z^\star\right\}, \tag{47}$$

where the right side of (47) takes into account that $F(z)$ has three mass points, i.e., αu workers hold Rw_0 real balances, αn hold Rw_1, and the remaining $1-\alpha$ hold $z^\star$. The MK-curve in the right panel of Figure 4 is now a function of w_1. As w_1 increases, the average real balances of employed workers increase and so does their early consumption. The size of this effect, is

$$\frac{\partial MK}{\partial w_1} = R\alpha^2 n(1-\mu^{-1}) > 0.$$

It increases with α and μ. In the following, we determine how an increase in w_1 affects aggregate real balances and employment. We start from an equilibrium where all workers have the same income, $w_0 = w_1$.

Proposition 11 ***(Effects of wages on liquidity and employment)*** *Suppose* $R = 1$*,* $w_0 = w_1$*, and consider an equilibrium with* $N = 2$*. A small increase in* w_1*, keeping* w_0 *constant, raises* Z *but decreases* n*.*

Even though an increase in w_1 raises aggregate real balances and sales to early consumers, the positive effect on aggregate demand is outweighed by the negative effect on the labor cost. Hence, aggregate employment decreases.

[20] The condition for full depletion of real balances is $R\beta\left[\frac{\alpha}{\mu}v'\left(Rw_0/\mu\right)/\mu + 1 - \alpha\right] \leq 1 + i/\alpha$.

The total margins on early sales increase with n. Formally,

$$\frac{\partial MK}{\partial n} = \alpha^2 R(1-\mu^{-1})(w_1 - w_0).$$

The strength of this effect increases with the income difference between the employed and the unemployed. If the increase in $\bar{q}$ also raises w_1, e.g., through wage negotiation, then the effect becomes even stronger since $\partial^2 MK/\partial w_1 \partial n > 0$. We now turn to this possibility by endogenizing wages.

Suppose $R = 1$ (to ease the exposition) and $w_1 = \gamma q + (1-\gamma)w_0$, where γ is interpreted as workers' bargaining power. Then, the revenue of a job is

$$q = \frac{\frac{\alpha}{n}(1-\mu^{-1})\left\{\alpha\left(1-n\gamma\right)w_0 + (1-\alpha)z^{\star}\right\} + \bar{q}}{1-\alpha^2(1-\mu^{-1})\gamma}. \tag{48}$$

A change in $\bar{q}$ has a multiplier effect:

$$\frac{\partial q}{\partial \bar{q}} = \frac{1}{1-\alpha^2(1-\mu^{-1})\gamma} > 1. \tag{49}$$

If $\bar{q}$ increases, then w_1 increases by γ. Because a fraction α of employed workers are constrained when choosing their real balances, the increase in w_1 raises aggregate real balances and the demand for early consumption, which raises q. This generates a further increase of w_1. And so on.

The equilibrium condition for market tightness, (45), is rewritten as

$$\frac{(r+\delta)\,\theta k - \lambda(\theta)(1-\gamma)\left(\bar{q}-w_0\right)}{\delta+\lambda(\theta)} = (1-\gamma)MK, \tag{50}$$

where the aggregate margins on early sales are

$$MK = \frac{\alpha(1-\mu^{-1})\left\{\alpha w_0 + \alpha\gamma n\left(\bar{q}-w_0\right) + (1-\alpha)z^{\star}\right\}}{1-\alpha^2(1-\mu^{-1})\gamma}. \tag{51}$$

The key novelty is that MK is now directly influenced by productivity, $\bar{q}$, since $\bar{q}$ affects w_1, which affects the distribution of real balances. Equilibrium is represented in Figure 5. We can decompose graphically the effects of an increase in $\bar{q}$ into three components. First, there is a shift to the right of the JC curve, which corresponds to the direct effect of $\bar{q}$ on θ in the MP model. Graphically, θ increases to the point marked "1" in Figure 5. Second, there is a movement along the JC curve toward the upward-sloping MK curve, from "1" to "2" because the endogenous increase in θ raises n, which in turn generates an increase in the distribution of liquidity across workers (in a first-order stochastic dominance sense). Third, there is a shift upward of the MK curve because the increase in $\bar{q}$ raises w_1 through the wage negotiation, which also improves the distribution of liquidity across workers. Market tightness increases from "2" to "3".

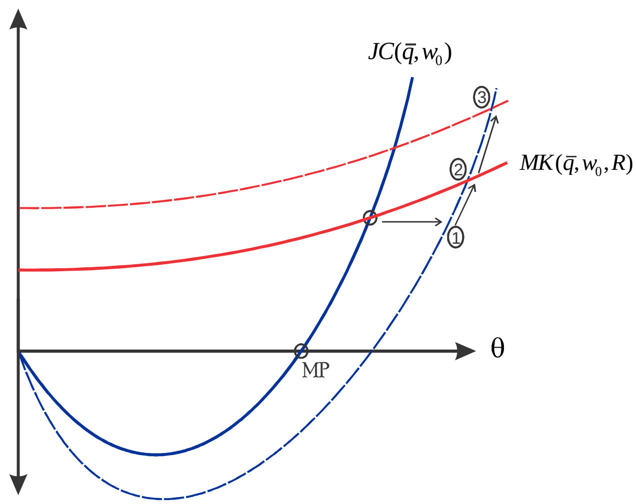

Figure 5: Amplification of productivity shocks with endogenous wages and distribution of liquidity

We have described the amplification mechanism associated with the joint determination of the income distribution, the distribution of real balances, and labor market outcomes qualitatively. More work needs to be done to quantify it in a more general setting where the real interest rate that is relevant for job creations is endogenous. We leave this part to future work.

8 Conclusion

We constructed a tractable two-sector model of monetary exchange with both expenditure and income risks featuring a non-degenerate distribution of real balances. Our model, that can be used to study the short-run and long-run effects of money growth, admits the equilibria with degenerate distribution of money holdings of the Lagos-Wright model as a special case. We showed that several insights of the Lagos-Wright model are overturn once distributional considerations are taken into account. For instance, the value of money now depends on the income distribution, which gives new policy implications. A one-time injection of money in a centralized market with flexible prices leads to higher aggregate real balances in the short run — money is not neutral — and changes in welfare. The effects are non-monotone with the size of the money injection, e.g., a small expansion of the money supply can generate deflation in the short run. In the presence of employment risk a positive money growth rate raises welfare if the unemployment rate is large and agents are sufficiently risk averse.

We extended our model to endogenize the employment risk by introducing a frictional labor market. Our model features an aggregate demand channel operating through the distribution of real balances that generates different comparative statics for policies and an amplification mechanism for productivity shocks. An increase in unemployment benefits allows workers to accumulate real balances faster, thereby providing better self-insurance against expenditure shocks. As a result, firms' sales and profits increase, which gives incentives to open more jobs. Similarly, if money is implemented through lump-sum transfers to workers, our model can generate a permanent trade-off between inflation and unemployment. Finally, an exogenous increase in productivity leads to higher employment, which raises the distribution of liquidity in a first-order stochastic sense, thereby increasing aggregate sales and employment further. This mechanism is strengthened if wages are endogenous.

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APPENDIX

Proofs of Propositions 1 and 2

Elementary Properties of Value Functions

Consider the pair of Bellman equations, for all $t \in \{0, 1, 2, \ldots\}$ and $e \in \{0, 1\}$:

$$V_t(z,e) = \sup_{y} \{\alpha [\upsilon(y) + W_t(z - \mu y, e)] + (1-\alpha) W_t(z,e)\} \tag{52}$$

$$W_t(z,e) = \sup_{z',c} \{\min\{c, \bar{c}\} + \beta E_e [V_{t+1}(z', e')]\} \tag{53}$$

s.t. $z' = R_{t+1}(z + w_e - c) \geq 0$ and where the expectation on the right side of (53) is with respect to the future employment state, $e' \in \{0, 1\}$, conditional on the current employment state, e. First, we substitute $V_t(z,e)$ from (52) into (53) to obtain the following Lemma:

Lemma 2 *The functions $W_t(z,e)$ and $V_t(z,e)$ solve* (52)-(53) *if and only if*

$$W_t(z,e) = \max \{\min\{c, \bar{c}\} + \beta E_e [\alpha [\upsilon(y_{e'}) + W_{t+1}(z' - \mu y_{e'}, e')] + (1-\alpha) W_{t+1}(z', e')]\}.$$

with respect to $c \geq 0$, $z' = R_{t+1}(z + w_e - c)$, and $0 \leq \mu y_{e'} \leq z'$ for $e' \in \{0, 1\}$.

Next, we apply standard contraction-mapping arguments to this Bellman equation. We obtain:

Lemma 3 *The Bellman equations* (52)-(53) *have unique bounded solutions, $V_t(z,e)$ and $W_t(z,e)$, that are continuous, concave, strictly increasing, and satisfy*

$$\|W\| \leq \frac{\bar{c} + \beta\alpha \|\upsilon\|}{1-\beta} \text{ and } \|V\| \leq \frac{\bar{c} + \alpha \|\upsilon\|}{1-\beta}.$$

Proof. Consider the space $C(\mathbb{N} \times \mathbb{R}_+ \times \{0, 1\})$ of bounded and continuous functions from $\mathbb{N} \times \mathbb{R}_+ \times \{0, 1\}$ to $\mathbb{R}$, equipped with the sup norm. By Theorem 3.1 in Stokey, Lucas, and Edward Prescott (1989, henceforth SLP), this is a complete metric space. Now, for any $f \in C(\mathbb{N} \times \mathbb{R}_+ \times \{0, 1\})$, consider the Bellman operator:

$$T[f]_t(z,e) = \max \{\min\{c, \bar{c}\} + \beta E_e [\alpha [\upsilon(y_{e'}) + f_{t+1}(z' - \mu y_{e'}, e')] + (1-\alpha) f_{t+1}(z', e')]\}$$

with respect to $c \geq 0$, $z' = R_{t+1}(z + w_e - c)$, and $0 \leq \mu y_{e'} \leq z'$. It is straightforward to verify that T satisfies the Blackwell sufficient condition for a contraction (Theorem 3.3 in

SLP). Moreover, the constraint set is non-empty, compact valued, and continuous. Hence by the Theorem of the Maximum (Theorem 3.6 in SLP), we obtain that $T[f]$ is continuous. It is clearly bounded since all the functions on the right-hand side of the Bellman equation are bounded. Note as well that if f is concave, then $T[f]$ is also concave since the objective is concave and the constraint correspondence has a convex graph. An application of the Contraction Mapping Theorem (Theorem 3.2 in SLP) implies that the fixed point problem $f = T[f]$ has a unique bounded solution, $W_t(z, e)$, and that this solution is continuous and concave.

Also, consider a state (z^1, e) and some feasible choice $(c^1, z^{1\prime}, y_0^1, y_1^1)$. Then, for $z^2 \geq z^1$, the following choice is feasible: $c^2 = c^1 + z^2 - z^1$, $z^{2\prime} = z^{1\prime}$ and $y_{e'}^2 = y_{e'}^1$ for $e' \in \{0, 1\}$. That is, a worker starting with z^2 can always consume $z^2 - z^1$ and otherwise behave as if he started with z^1. Since this yield (weakly) higher utility this implies that $T[W]_t(z, e) = W_t(z, e)$ is increasing. Hence, the bounded solution of the Bellman equation is increasing. One also sees that it must be strictly increasing. Indeed, a worker starting at $z^2 > z^1$ can wait to use $z^2 - z^1$ in order to pay for more consumption in the DM if an opportunity occurs with strictly positive probability, $\alpha > 0$. Since $v(y)$ is strictly increasing, this implies that $T[W]_t(z, e) = W_t(z, e)$ is strictly increasing.

Given a fixed point $W_t(z, e)$ of the Bellman operator T, we can define $V_t(z, e)$ as in (52). By identical arguments as above, one sees that $V_t(z, e)$ is bounded, continuous, concave, and strictly increasing.

Finally, we can derive upper bounds for $W_t(z, e)$ and $V_t(z, e)$. From (53) we have:

$$\|W\| \leq \bar{c} + \beta\alpha \left[\|v\| + \|W\|\right] + \beta(1-\alpha)\|W\| \Rightarrow \|W\| \leq \frac{\bar{c} + \beta\alpha\|v\|}{1-\beta}.$$

We obtain the bound on $\|V\|$ following identical arguments but for $V_t(z, e)$. ■

For the rest of this section, we slightly simplify the Bellman equation by reducing the number of optimizing variables. To do so, we first note that, since $W_t(z, e)$ is strictly increasing, it is strictly suboptimal for a worker to choose $c > \bar{c}$: the worker can instead reduce consumption, with no loss of utility, and choose higher savings z'. Hence, in the objective of the Bellman equation, we can replace $\min\{c, \bar{c}\}$ by c. Substituting $c = z + w_e - z'/R_{t+1}$ in the objective, and keeping in mind that $0 \leq c \leq \bar{c}$, we can rewrite the Bellman equation as:

$$W_t(z, e) = \max \left\{ z + w_e - \frac{z'}{R_{t+1}} + \beta E_e \left[\alpha \left[v(y_{e'}) + W_{t+1}(z' - \mu y_{e'}, e')\right] + (1-\alpha) W_{t+1}(z', e')\right] \right\}. \tag{54}$$

with respect to z' and $\{y_{e'} : e' \in \{0, 1\}\}$ and subject to $R_{t+1}(z + w_e - \bar{c}) \leq z' \leq R_{t+1}(z + w_e)$ and $0 \leq \mu y_{e'} \leq z'$.

It will be sometimes convenient to rewrite (54) more compactly as:

$$W_t(z, e) = \max_{z' \geq 0} \left\{ z + w_e - \frac{z'}{R_{t+1}} + \beta E_e \left[\alpha \Omega_{t+1}(z', e') + (1-\alpha) W_{t+1}(z', e')\right] \right\}, \tag{55}$$

subject to $R_{t+1}(z+w_e-\bar{c}) \le z' \le R_{t+1}(z+w_e)$, and where $\Omega_t(z,e)$ is the indirect utility for real balances in the DM, i.e.,

$$\Omega_t(z,e) = \max_{0\le\mu y\le z} \{\upsilon(y) + W_t(z-\mu y, e)\}. \tag{56}$$

Elementary properties of decision rules

We first consider the problem of a worker in the DM, (56).

Lemma 4 *The worker's DM problem, (56), has a unique solution, $y_t(z,e)$. This solution is continuous, increasing, satisfies $\lim_{z\to0} y_t(z,e) = 0$ and $\lim_{z\to\infty} y_t(z,e) = \infty$. Its value, $\Omega_t(z,e)$, is continuous, strictly increasing, concave, and satisfies $\Omega_t'(z^+,e) \ge \upsilon'[y_t(z,e)]/\mu$.*

Proof. The problem (56) is strictly concave since $\upsilon(y)$ is strictly concave. Hence, it has a unique solution, denoted by $y_t(z,e)$. By the Theorem of the Maximum (SLP Theorem 3.6), uniqueness implies that $y_t(z,e)$ is continuous. To show that $y_t(z,e)$ is increasing, consider any two $z_1 < z_2$. If $y_t(z_2,e) \ge z_1/\mu$, then by feasibility it immediately follows that $y_t(z_1,e) \le z_1/\mu \le y_t(z_2,e)$. Otherwise, suppose $y_t(z_2,e) < z_1/\mu$ and $y_t(z_2,e) < y_t(z_1,e)$, then first-order conditions for y are

$$\upsilon'[y_t(z_2,e)] \le \mu W_t'\left[(z_2-\mu y_t(z_2,e))^-, e\right], \tag{57}$$

$$\upsilon'[y_t(z_1,e)] \ge \mu W_t'\left[(z_1-\mu y_t(z_1,e))^+, e\right], \tag{58}$$

where we recall that $W_t(z,e)$ is concave and so it has left- and right-derivatives for all $z>0$. By concavity $W_t'(z,e)$ is decreasing in z, the fact that $y_t(z_2,e) < y_t(z_1,e)$ and $z_2 > z_1$ implies

$$\mu W_t'\left[(z_2-\mu y_t(z_2,e))^+, e\right] \le \mu W_t'\left[(z_1-\mu y_t(z_1,e))^-, e\right],$$

which implies $\upsilon'[y_t(z_2,e)] \le \upsilon'[y_t(z_1,e)]$ by using (57) and (58). Since υ is strictly concave, it contradicts the premise that $y_t(z_2,e) < y_t(z_1,e)$.

From the feasibility constraint, $0 \le \mu y \le z$, it follows that $\lim_{z\to0} y_t(z,e) = 0$. Suppose that $y_t(z,e)$ is bounded away from infinity. Indeed, since $W_t(z,e)$ is bounded, increasing, and concave, we have $0 \le W_t'(z^-,e)z \le W_t(z,e)-W_t(0,e) \le \|W\|$, so that $\lim_{z\to\infty} W_t'(z^-,e) = 0$. Then the first-order condition $\upsilon'(y) \le \mu W_t'\left[(z-\mu y)^-, e\right]$ cannot hold for z large enough because $W_t(z,e)$ must satisfy Inada condition at infinity.

The value $\Omega_t(z,e)$ is continuous by the Theorem of the Maximum. It is strictly increasing because $W_t(z,e)$ is strictly increasing. To establish the lower bound on the right derivative, we note that, $y_t(z,e)$ is feasible for any $z' \ge z$. This implies that, for all $z' \ge z$:

$$\Omega_t(z',e) \ge \upsilon\left[y_t(z,e) + \frac{z'-z}{\mu}\right] + W_t\left[z-\mu y_t(z,e), e\right],$$

with equality if $z = z'$. The result follows by subtracting the equality for $z = z'$ to the above inequality, dividing by $z' - z$, and letting $z' \to z^+$. ■

To solve for an optimal money holdings decision, we proceed as follows. We define the set of optimal real balances if w is sufficiently large so that $c \geq 0$ does not bind as:

$$Z^\star_{e,t+1} = \arg\max_{z' \geq 0} \left\{ -\frac{z'}{R_{t+1}} + \beta E_e \left[\alpha \Omega_{t+1}(z', e') + (1-\alpha) W_{t+1}(z', e') \right] \right\}.$$

Lemma 5 *The set $Z^\star_{e,t+1}$ is convex, bounded above, and bounded away from zero. Given any $z^\star_{e,t+1} \in Z^\star_{e,t+1}$, a worker's optimal choice of real balances at time t is:*

$$z' = \max \left\{ R_{t+1}(z + w_e - \bar{c}), \min \left\{ R_{t+1}(z + w_e), z^\star_{e,t+1} \right\} \right\} \tag{59}$$

Proof. The set $Z^\star_{e,t+1}$ is bounded above because both $\upsilon(y)$ and $W_{t+1}(z, e)$ are concave and bounded, implying that they satisfy Inada condition at infinity. To see that it is bounded away from zero, recall that $\Omega'_{t+1}(z, e) \geq \upsilon' \left[y(z, e) \right] / \mu$ and $\lim_{z \to 0} y(z, e) = 0$. Since $\upsilon'(0) = +\infty$, it follows that $\lim_{z \to 0} \Omega_{t+1}(z^+, e) = \infty$. This implies that, near zero,

$$-\frac{1}{R_{t+1}} + \beta E_e \left[\alpha \Omega'_{t+1}(0^+, e') + (1-\alpha) W'_{t+1}(0^+, e') \right] > 0$$

Hence $0 < \min Z^\star_{e,t+1}$. The rest of the proposition follows because the optimization program defining $Z^\star_{e,t+1}$ is concave.

The optimal rule for next period real balances is to approach $z^\star_{e,t+1}$ as closely as possible, keeping consumption below the satiation point, $\bar{c}$. Hence, for low values of z, the worker approaches $z^\star_{e,t+1}$ by lowering c to 0, in which case $z' = R_{t+1}(z + w_e)$. For large enough values of z, the worker approaches $z^\star_{e,t+1}$ by consuming up to the satiation point $\bar{c}$, in which case $z' = R_{t+1}(z + w_e - \bar{c})$. For values of z in some middle range, near $z^\star_{e,t+1}$, the worker can reach $z^\star_{e,t+1}$ in one period by consuming less than $\bar{c}$. ■

Lemma 6 *The derivative of the value function is bounded, $W'(0^+, e) < \infty$.*

Proof. Choose any $z^\star_{e,t+1} \in Z^\star_{e,t+1}$ and consider the following two cases.

If $z^\star_{e,t+1} \leq R_{t+1} w_e$ then, for all $z > 0$ close enough to zero, an optimal choice of real balances is $z' = z^\star_{e,t+1}$. Substituting $z' = z^\star_{e,t+1}$ into (55), we obtain:

$$W_t(z, e) = z + w_e - \frac{z^\star_{e,t+1}}{R_{t+1}} + \beta E_e \left[\alpha \Omega_{t+1}(z^\star_{e,t+1}, e') + (1-\alpha) W_{t+1}(z^\star_{e,t+1}, e') \right],$$

which implies $W'_t(0^+, e) = 1$.

If $z^{\star}_{e,t+1} > R_{t+1}w_e$ then for all $z > 0$ close enough to zero, an optimal choice of real balances is $z' = R_{t+1}(z + w_e)$. Substituting this expression into (55) we obtain:

$$W_t(z, e) = \beta E_e \left[\alpha \Omega_{t+1}\left[R_{t+1}(z + w_e), e'\right] + (1 - \alpha)W_{t+1}\left[R_{t+1}(z + w_e), e'\right]\right].$$

Since $R_{t+1} > 0$, $z' = R_{t+1}(z + w_e)$ lies in the interior of the domain of $\Omega_{t+1}(z', e')$ and $W_{t+1}(z', e')$. These concave functions have right-derivatives at these interior points. Hence, $W_t(z, e)$ has a right-derivative at zero, i.e., $W_t'(0^+, e) < \infty$. ■

With this result we establish:

Lemma 7 *For all $z > 0$, the optimal DM consumption is strictly positive: $y_t(z, e) > 0$. Moreover, for given $e \in \{0, 1\}$, $\Omega_t(z, e)$ is continuously differentiable over $(0, \infty)$ with $\Omega_t'(z, e) = \upsilon'\left[y_t(z, e)\right]/\mu$.*

Proof. The first result, $y_t(z, e) > 0$ for all $z > 0$, follows from Lemma 6 according to which $W_{t+1}(0^+, e) < \infty$ and the assumption $\upsilon'(0) = \infty$. For the second result consider some $z > 0$. Since $y_t(z, e) > 0$, $\mu y_t(z, e) - (z - z')$ is feasible for $z' < z$ and close enough to z. Therefore, for such z', we have

$$\Omega_t(z', e) \geq \upsilon\left[y_t(z, e) - \frac{z - z'}{\mu}\right] + W_t\left[z - \mu y_t(z, e), e\right].$$

Subtracting this inequality from the following equality,

$$\Omega_t(z, e) = \upsilon\left[y_t(z, e)\right] + W_t\left[z - \mu y_t(z, e), e\right],$$

and dividing both sides by $z - z'$, we obtain:

$$\frac{\Omega_t(z, e) - \Omega_t(z', e)}{z - z'} \leq \frac{\upsilon\left[y_t(z, e)\right] - \upsilon\left[y_t(z, e) - (z - z')/\mu\right]}{z - z'}.$$

Letting $z' \to z$, we obtain $\Omega_t'(z^-) \leq \upsilon'\left[y_t(z, e)\right]/\mu$. Since we have already shown in Lemma 4 that $\Omega_t(z^+, e) \geq \upsilon'\left[y_t(z, e)\right]/\mu$, and since $\Omega_t(z, e)$ is concave, we obtain that, for all $z > 0$, $\Omega_t(z, e)$ is differentiable with $\Omega_t'(z, e) = \upsilon'\left[y_t(z, e)\right]/\mu$. ■

Differentiability of the value function

We establish the differentiability of $W_t(z, e)$ and provide an explicit formula for its derivative. The main difficulty is that the Envelope Theorem of Benveniste and Scheinkman does not apply to our environment, because it requires optimal choices to lie in the interior of the constraint set – in contrast, in our setting, the constraint $c \geq 0$ may bind. To address this difficulty, Rincón-Zapatero and Santos (2009) have established an Envelope Theorem for a broad class of stationary dynamic optimization problems in which optimal choices may not lie in the interior of the constraint set, but must lie in the interior of the state space. We apply their results to our environment to establish differentiability. The application is not immediate however, because two of their maintained assumptions are violated. First, we consider non-stationary equilibria where R_{t+1} is not constant over time. Second, some optimal choices may not lie in the interior of the state space: namely, when a worker depletes his money holdings in full in the DM, he enters the following CM with zero money balances.

Maintained assumptions about returns. We make two assumptions on the time-path of gross rates of return of fiat money:

- (A1) There exists some $\underline{R} > 0$ such that, for all $t \geq 0$, $R_{t+1} > \underline{R}$.
- (A2) $\sum_{i=1}^{\infty} \beta^i (1-\alpha)^{i-1} \alpha \left[\Pi_{j=1}^{i} R_i\right] < \infty$.

The first assumption rules out hyper-inflationary dynamics and the second assumption helps with the proof that the expected present value of marginal utilities from real balance is finite – as required to apply the argument of Rincón-Zapatero and Santos (2009). Note that both assumptions are satisfied for the transitional dynamics we analyze in the paper, whereby $R_t \to 1$ as $t \to \infty$.

Bounds on decision variables Next, we establish bounds on decision variables.

Lemma 8 *The DM consumption satisfies $y_t(z,e) \geq \hat{y}(z)$ for some continuous, strictly increasing, and time-invariant function, $\hat{y}(z)$, that satisfies $\hat{y}(0) = 0$ and $0 < \hat{y}(z) \leq z$ for all $z > 0$.*

Proof. Suppose that $y_t(z,e) < z/\mu$. Since there is partial depletion, the first-order condition for the DM problem, (56), is

$$v'\left[y_t(z,e)\right] \leq \mu W_t'\left[(z-\mu y_t(z,e))^-, e\right] \leq \frac{\mu\|W\|}{z-\mu y_t(z,e)} \Rightarrow \left[z-\mu y_t(z,e)\right] v'\left[y_t(z,e)\right] \leq \mu\|W\|. \tag{60}$$

Consider now the equation $(z-\mu y)v'(y) = \mu\|W\|$ for $z > 0$. The left hand side is continuous and strictly decreasing in y, goes to infinity as $y \to 0$ and to zero as $y \to z/\mu$. Hence, the equation has a unique solution $\hat{y}(z)$, which satisfies $0 < \hat{y}(z) < z/\mu$. Since the equation has a unique solution and is continuous in (z,y), the function $\hat{y}(z)$ is continuous as well. Since $0 < \hat{y}(z) < z/\mu$, we can extend the function by continuity at $z = 0$ by setting $\hat{y}(0) = 0$.

Clearly, when $y_t(z,e) < z/\mu$, the inequality (60) implies that $y_t(z,e) \geq \hat{y}(z)$ for all t and e. When $y_t(z,e) = z/\mu$, this inequality is also satisfied since $\hat{y}(z) < z/\mu$. ■

The second preliminary result is:

Lemma 9 *For all $t \geq 0$, optimal real balances are bounded below by*

$$\underline{z}_e = \min \left\{ \underline{R} w_e, (\upsilon')^{-1} \left(\frac{1}{\beta \alpha \underline{R}} \right) \right\}.$$

Proof. A first-order condition for an optimal choice of target is:

$$-\frac{1}{R_{t+1}} + \beta E_e \left[\alpha \upsilon' \left[y_{t+1}(z, e') \right] + (1-\alpha) W'_{t+1} \left(z^+, e' \right) \right] \leq 0.$$

Since the value function is increasing, this implies that $\beta \alpha E_e \upsilon' \left[y_{t+1}(z, e') \right] \leq 1/R_{t+1}$. Since $z \geq \mu y_{t+1}(z, e)$, we obtain that $\beta \alpha \upsilon'(z) \leq 1/R_{t+1}$. Since $\upsilon'(z)$ is decreasing, this implies:

$$z \geq (\upsilon')^{-1} \left(\frac{1}{\beta \alpha R_{t+1}} \right) \geq (\upsilon')^{-1} \left(\frac{1}{\beta \alpha \underline{R}} \right) \text{ for all } z \in Z^{\star}_{e,t+1}.$$

The result follows from the policy function for real balances in (59). ■

The main proposition. We now can state our differentiability result:

Proposition 1 The value function, $W_t(z, e)$, is continuously differentiable in z, with:

$$W'_t(z, e) = E_e \left[\sum_{i=1}^{\infty} \beta^i (1-\alpha)^{i-1} \alpha \left(\Pi_{j=1}^{i} R_{t+j} \right) \frac{\upsilon' \left[y_{t+i}(z_{t+i}, e_{t+i}) \right]}{\mu} \right],$$

where $\{z_{t+i}\}$ is a stochastic process for optimal real balances starting from $z_t = z$ given histories of shocks and $\{e_{t+i}\}$ is the sequence of employment shocks from $e_t = e$.

Proof. We first use the Envelope Theorem for optimization problems with parameterized constraints of Milgrom and Segal (2002, Corollary 5). To see that all the conditions are satisfied, we first note that, given $R_{t+1} > 0$, for all $z \geq 0$ there exists $z' > 0$ such that $z' < R_{t+1}(z + w_e)$ and $z' > R_{t+1}(z + w_e - \bar{c})$. Note as well that the objective function and the function defining the constraint are continuous and concave, and have partial derivatives with respect to z which are continuous in (z, z'). The Lagrangian associated with the optimization problem (55) is:

$$\begin{aligned} L(z, z', \lambda_e) &= z - \frac{z'}{R_{t+1}} + \beta E_e \left[\alpha \Omega_{t+1}(z', e') + (1-\alpha) W_{t+1}(z', e') \right] \\ &\quad + \bar{\lambda}_e \left[R_{t+1}(z + w_e) - z' \right] + \underline{\lambda}_e \left[z' - R_{t+1}(z + w_e - \bar{c}) \right]. \end{aligned}$$

where $\lambda_e \equiv (\bar{\lambda}_e, \underline{\lambda}_e)$. Let $\Lambda^{\star}_e$ denote the set of Kuhn-Tucker multipliers and $\Xi^{\star}_e$ denote the set of optima associated with this optimization problem. These sets are non empty and compact under the stated conditions. Then by the above mentioned Envelope Theorem, we have:

$$W'_t(z^+, e) = \min_{\lambda \in \Lambda^{\star}_e} \max_{z' \in \Xi^{\star}_e} \frac{\partial L}{\partial z}(z, z', \lambda_e) = \min_{\lambda \in \Lambda^{\star}_e} 1 + R_{t+1} \left(\bar{\lambda}_e - \underline{\lambda}_e \right), \tag{61}$$

for all $z \geq 0$,

$$W_t'(z^-, e) = \max_{\lambda \in \Lambda_e^\star} \min_{z' \in \Xi_e^\star} \frac{\partial L}{\partial z}(z, z', \lambda_e) = \max_{\lambda \in \Lambda_e^\star} 1 + R_{t+1}\left(\bar{\lambda}_e - \underline{\lambda}_e\right), \tag{62}$$

for all $z > 0$. By taking the derivative of $L(z, z', \lambda_e)$ with respect to z', we obtain natural bounds for $\bar{\lambda}_e$ and $\underline{\lambda}_e$. Namely, fix some optimal real balances, $z_{t+1} \in \Xi_e^\star$. Then, by Theorem 28.3 in Rockafellar (1970), any $\lambda_e \in \Lambda_e^\star$ must satisfy:

$$\frac{\partial L}{\partial z'}(z, z_{t+1}^+, \lambda_e) \leq 0 \leq \frac{\partial L}{\partial z'}(z, z_{t+1}^-, \lambda_e).$$

Taking derivatives explicitly and rearranging the resulting first-order conditions, we obtain that for any $\lambda_e \in \Lambda_e^\star$:

$$\begin{aligned}
\bar{\lambda}_e - \underline{\lambda}_e &\geq -\frac{1}{R_{t+1}} + \beta E_e\left[\frac{\alpha}{\mu}\upsilon'\left[y_{t+1}(z_{t+1}, e')\right] + (1-\alpha)W_{t+1}'(z_{t+1}^+, e')\right] \\
\bar{\lambda}_e - \underline{\lambda}_e &\leq -\frac{1}{R_{t+1}} + \beta E_e\left[\frac{\alpha}{\mu}\upsilon'\left[y_{t+1}(z_{t+1}, e')\right] + (1-\alpha)W_{t+1}'(z_{t+1}^-, e')\right].
\end{aligned}$$

Plugging these inequalities back into (61) and (62), we obtain:

$$\begin{aligned}
\forall z \geq 0 : W_t'(z^+, e) &\geq \beta R_{t+1} E_e\left[\frac{\alpha}{\mu}\upsilon'\left[y_{t+1}(z_{t+1}, e')\right] + (1-\alpha)W_{t+1}'(z_{t+1}^+, e')\right] \\
\forall z > 0 : W_t'(z^-, e) &\leq \beta R_{t+1} E_e\left[\frac{\alpha}{\mu}\upsilon'\left[y_{t+1}(z_{t+1}, e')\right] + (1-\alpha)W_{t+1}'(z_{t+1}^-, e')\right].
\end{aligned}$$

Iterating forward, we obtain:

$$\begin{aligned}
W_t'(z^+, e) \geq\ & E_e\left[\sum_{i=1}^{I} \beta^i (1-\alpha)^{i-1}\alpha\left(\Pi_{j=1}^{i} R_{t+i}\right)\frac{\upsilon'\left[y_{t+i}(z_{t+i}, e_{t+i})\right]}{\mu}\right. \\
& \left. +\beta^I(1-\alpha)^I\left(\Pi_{i=1}^{I} R_{t+i}\right) W_{t+I}'(z_{t+I}^+, e_{t+I})\right],
\end{aligned}$$

for all $z \geq 0$, and

$$\begin{aligned}
W_t'(z^-, e) \leq\ & E_e\left[\sum_{i=1}^{I} \beta^i (1-\alpha)^{i-1}\alpha\left(\Pi_{j=1}^{i} R_{t+i}\right)\frac{\upsilon'\left[y_{t+i}(z_{t+i}, e_{t+i})\right]}{\mu}\right. \\
& \left. +\beta^I(1-\alpha)^I\left(\Pi_{i=1}^{I} R_{t+i}\right) W_{t+I}'(z_{t+I}^-, e_{t+I})\right],
\end{aligned}$$

for all $z > 0$, where z_{t+i} denote some sequence of optimal real balances decisions generated by ((59)) starting starting from $z_t = z$ and given the history of income shocks, $\{e_{t+j}\}_{j=0}^{i-1}$. From Lemma 9, we know that optimal real balances are bounded below by $\underline{z}_e$. By Lemma 8, this implies that optimal consumption in the DM is bounded below by $\hat{y}(\underline{z}_e)$. This implies the upper bounds $\upsilon'[y(z_{t+i},e_{t+1})] \leq \|\upsilon\|/\hat{y}(\underline{z}_e)$ and $W'_{t+I}(z^{\pm}_{t+I},e_{t+I}) \leq \|W\|/\min\{\underline{z}_0,\underline{z}_1\}$. Together with assumption (A2) stated at the beginning of the section, these upper bounds allow us to take limits as $I \to \infty$ in the above expressions, and we obtain:

$$W'_t(z^+,e) \geq E_e\left[\sum_{i=1}^{\infty}\beta^i(1-\alpha)^{i-1}\alpha\left(\Pi_{j=1}^{i}R_{t+i}\right)\frac{\upsilon'[y_{t+i}(z_{t+i},e_{t+i})]}{\mu}\right],$$

for all $z \geq 0$,

$$W'_t(z^-,e) \leq E_e\left[\sum_{i=1}^{\infty}\beta^i(1-\alpha)^{i-1}\alpha\left(\Pi_{j=1}^{i}R_{t+i}\right)\frac{\upsilon'[y_{t+i}(z_{t+i},e_{t+i})]}{\mu}\right],$$

for all $z > 0$. Given that $W'_t(z^+,e) \leq W'_t(z^-,e)$, this implies that $W_t(z)$ is differentiable at z, and that the derivative is as stated in the Proposition. The derivative is continuous by Theorem 24.1 in Rockafeller (1970). ■

Proof of Proposition 4. Provided that the condition for full depletion, (28), holds we construct a steady-state equilibrium as follows. From (19),

$$z^{\star} = \mu\upsilon'^{-1}\left[\mu\left(1+\frac{r}{\alpha}\right)\right].$$

We use (20) to compute the number of periods it takes to reach the target:

$$N-1 < \frac{\mu\upsilon'^{-1}\left[\mu\left(1+\frac{r}{\alpha}\right)\right]}{w} \leq N. \tag{63}$$

From (63) $N=1$ if $w \geq \mu\upsilon'^{-1}[\mu(1+r/\alpha)]$. Otherwise, $N \geq 2$. Given N and $z^{\star}$ the steady-state distribution of real balances is obtained from (24)-(25). Finally, the value of money is obtained from (26). ■

Proof of Proposition 5. The difference equation (31) has a unique positive fixed point,

$$z^{\star} = \mu\upsilon'^{-1}\left[\frac{\mu[1-\beta(1-\alpha)]}{\alpha\beta}\right].$$

Hence, for all $t \geq 1$ there is an equilibrium where $z^{\star}_t = z^{\star}$ and $R_{t+1} = z^{\star}_{t+1}/z^{\star}_t = 1$. By market clearing, and using that the distribution of real balances is degenerate, $\phi_t(1+\pi)M = z^{\star}$ for all $t \geq 1$. From (30), and using the definition of the fixed point $z^{\star}$, $\phi_0(1+\pi)M = z^{\star}$. Hence, $\phi_0 = \phi_1$ and $R_1 = 1$. ■

Proof of Proposition 6. The difference equation (35) for $t \geq 2$ can be rewritten as:

$$\alpha(R_t w) + (1-\alpha)z^{\star}_t = \frac{\alpha(R_{t+1}w) + (1-\alpha)z^{\star}_{t+1}}{R_{t+1}},$$

where $z_t^\star = \mu v'^{-1}\left[\mu\left(1+\frac{1+r-R_t}{\alpha R_t}\right)\right]$. It has a positive and constant solution, $R_t = 1$. Hence, there exists an equilibrium that becomes stationary starting at $t = 2$, with $z_t^\star = z^\star$, $z_t^1 = w$, and $R_t = 1$ for all $t \geq 2$. From (35) evaluated at $t = 1$ and (34), (ϕ_0, R_1) solves:

$$\phi_0(1+\pi)M = \frac{\alpha z_1^1 + (1-\alpha)z_1^\star}{R_1} \tag{64}$$

$$\alpha z_1^1 + (1-\alpha)z_1^\star = \alpha w + (1-\alpha)z^\star. \tag{65}$$

Substituting $z_1^1 = R_1\left(\bar{h} + \pi\phi_0 M\right)$ into (64) and solving for aggregate real balances at $t = 1$:

$$\phi_1(1+\pi)M = \frac{\alpha R_1 w + (1-\alpha)z_1^\star}{1 - \pi\alpha/(1+\pi)}. \tag{66}$$

From (64) the left side of (66) is equal to the right side of (65). Hence, R_1 solves (36). The left side of (36) is increasing in R_1, it is equal to 0 when $R_1 = 0$ and the numerator is equal to the right side when $R_1 = 1$. Given that the denominator is less than 1, it follows that there is a unique R_1 solution to (36) and it is such that $R_1 < 1$. Using the expression for $\phi_1(1+\pi)M$ given by (65), $\phi_0 = \phi_1/R_1$ solves (37).

There is short-run deflation if $\phi_0 > (1+\pi)\phi_1$, i.e., $R_1 < (1+\pi)^{-1}$. This condition holds in the neighborhood of $\pi = 0$ and $R_1 = 1$ if

$$\left.\frac{dR_1}{d(1+\pi)}\right|_{\pi=0} < -1 \Leftrightarrow \frac{-(z^\star/\mu)v''(z^\star/\mu)}{v'(z^\star/\mu)} > \frac{z^\star}{(z^\star - w)\,\beta\alpha\,(\alpha + r)},$$

where the inequality on the right is obtained by differentiating R_1 defined in (36) with respect to $1+\pi$. It corresponds to (38).

Individual real balances in $t = 1$ are $z_1^1 > w$ and $z_1^\star < z^\star$. Hence, there is a mean-preserving reduction in the distribution of real balances. From the concavity of the value functions, social welfare increases. ■

Proof of Proposition 7. Total differentiate (39) and evaluate at $\pi = 0^+$ to obtain:

$$\frac{\partial R_1}{\partial \pi} = -\alpha u\left[\frac{\alpha u w_0 + (1-\alpha u)\,z^\star}{\alpha u w_0 + (1-\alpha u)\,\partial z_1^\star/\partial R_1}\right] > 0.$$

Assuming u is close to 0, the numerator of the term between squared brackets is approximately equal to $z^\star$ while the denominator is approximately $\partial z_1^\star/\partial R_1$. Hence,

$$\frac{\partial R_1}{\partial \pi} \approx -\alpha u\left[\frac{1}{z^\star}\frac{\partial z_1^\star}{\partial R_1}\right]^{-1}.$$

The term between squared brackets is equal to $\partial \ln z_1^\star/\partial R_1$ evaluated at $\pi = 0^+$. ■

Proof of Proposition 8. Total differentiate (41) in the neighborhood of $\pi = 0$ to obtain:

$$\frac{\partial Z}{\partial \pi} = (1-\alpha u)\frac{\partial z^{\star}}{\partial \pi} + (1-\alpha u)(z^{\star} - Z).$$

From (40)

$$\frac{\partial z^{\star}}{\partial \pi} = \frac{\mu}{\upsilon''(z^{\star}/\mu)\beta\alpha},$$

where the derivative is evaluated at $\pi = 0^{+}$. Substitute $\partial z^{\star}/\partial \pi$ by its expression above into the expression for $\partial Z/\partial \pi$ to obtain:

$$\frac{\partial Z}{\partial \pi} = (1-\alpha u)\left[\frac{\mu}{\upsilon''(z^{\star}/\mu)\beta\alpha} + z^{\star} - Z\right].$$

Hence, $\partial Z/\partial \pi > 0$ iff

$$\frac{\mu}{\upsilon''(z^{\star}/\mu)\beta\alpha} + z^{\star} - Z > 0.$$

This inequality can be rearranged as:

$$-\upsilon''\left(\frac{z^{\star}}{\mu}\right) > \frac{\mu}{(z^{\star} - Z)\beta\alpha}.$$

Divide both sides by $\upsilon'(z^{\star}/\mu) = \mu(r+\alpha)/\alpha$, from (40), and multiply by $z^{\star}/\mu$ to obtain:

$$\frac{-\upsilon''\left(\frac{z^{\star}}{\mu}\right)\frac{z^{\star}}{\mu}}{\upsilon'(z^{\star}/\mu)} > \frac{z^{\star}}{(r+\alpha)\beta\mu(z^{\star} - Z)}.$$

Substitute Z by its expression given by (41) at $\pi = 0$:

$$\frac{-\upsilon''\left(\frac{z^{\star}}{\mu}\right)\frac{z^{\star}}{\mu}}{\upsilon'(z^{\star}/\mu)} > \frac{z^{\star}}{(r+\alpha)\beta\mu\alpha u(z^{\star} - w_0)}.$$

From (42), if $\partial Z/\partial \pi > 0$ then $\partial q/\partial \pi > 0$. ∎

Proof of Lemma 1. It is easy to check that $JC(0) = 0$, $JC(+\infty) = +\infty$ (because $\lim_{\theta\to\infty}\lambda(\theta)/\theta = 0$), and

$$JC'(\theta) = \frac{N(\theta)}{[\delta + \lambda(\theta)]^2},$$

where

$$N(\theta) \equiv (r+\delta)k[\delta + \lambda(\theta)] - \lambda'(\theta)[(r+\delta)k\theta + \delta(\bar{q} - w_1)]$$

Using that $\lambda(\theta)$ is concave, $\lambda'(\theta)$ is decreasing in θ. Assuming $\bar{q} - w_1 > 0$, $N(\theta)$ is increasing in θ. Moreover,

$$N(0) = \delta[(r+\delta)k - (\bar{q} - w_1)],$$

where we used that $\lambda(0) = 0$ and $\lambda'(0) = 1$. So if $(r+\delta)\,k \geq \bar{q} - w_1$ then $JC(\theta)$ is increasing for all θ. If $(r+\delta)\,k < \bar{q} - w_1$ then $JC(\theta)$ is first decreasing (and hence negative) and then increasing (since $N(+\infty) = +\infty$). It follows that there is $\underline{\theta} > 0$ such that: $JC(\theta) < 0$ for all $\theta < \underline{\theta}$ and $JC(\theta) > 0$ for all $\theta > \underline{\theta}$. ■

Proof of Proposition 9. Consider first equilibria where $N = 1$. From (45) market tightness solves

$$\frac{(r+\delta)\,\theta k - \lambda(\theta)(\bar{q} - w_1)}{\delta + \lambda(\theta)} = \alpha\left(\frac{\mu - 1}{\mu}\right) z^\star$$

where, from (40), $z^\star = \mu v'^{-1}\left[\mu\left(1 + i/\alpha\right)\right]$. Clearly, $\partial\theta/\partial w_0 = 0$ and $\partial z^\star/\partial w_0 = 0$, which implies $\partial Z/\partial w_0 = \partial u/\partial w_0 = 0$.

Consider next equilibria featuring $N = 2$. From (45) and (46) market tightness solves

$$\frac{(r+\delta)\,\theta k - \lambda(\theta)(\bar{q} - w_1)}{\delta + \lambda(\theta)} = \alpha\left(\frac{\mu - 1}{\mu}\right)\left[z^\star - \alpha\frac{\delta}{\delta + \lambda(\theta)}\left(z^\star - Rw_0\right)\right],$$

where we used that $u = \delta/\left[\delta + \lambda(\theta)\right]$. After some calculation the equation can be rearranged as:

$$(r+\delta)\,\theta k - \lambda(\theta)\left[\alpha\left(1 - \mu^{-1}\right) z^\star + (\bar{q} - w_1)\right] = \alpha\left(1 - \mu^{-1}\right)\delta\left[\alpha R w_0 + z^\star(1-\alpha)\right].$$

The left side is strictly convex in θ and it is increasing when it is positive. The right side is increasing in w_0. Hence, $\partial\theta/\partial w_0 > 0$ and $\partial u/\partial w_0 < 0$. Using that $Z = \left[(1-\alpha u)\,z^\star + \alpha u R w_0\right]$ it follows that

$$\frac{\partial Z}{\partial w_0} = \alpha u R - \alpha\frac{\partial u}{\partial w_0}\left(z^\star - Rw_0\right) > 0,$$

where we used the fact that $z^\star > Rw_0$ in any equilibrium with $N = 2$. ■

Proof of Proposition 10. Part 1: money growth implemented with lump-sum transfers to entrepreneurs. We established in the proof of Proposition 9 that equilibrium market tightness is the unique solution to:

$$(r+\delta)\,\theta k - \lambda(\theta)\left[\alpha\left(1 - \mu^{-1}\right) z^\star + (\bar{q} - w_1)\right] = \alpha\left(1 - \mu^{-1}\right)\delta\left[\alpha R w_0 + z^\star(1-\alpha)\right],$$

where, from (40), $z^\star$ solves

$$v'\left(\frac{z^\star}{\mu}\right) = \mu\left(1 + \frac{(1+r)(1+\pi) - 1}{\alpha}\right).$$

An increase in π lowers the right side (which is independent of θ) and raises the left side (which is increasing in θ when the left side is positive), which leads to a decrease in θ and an increase in u.

Part 2: money growth implemented with lump-sum transfers to workers. From (45) market tightness is the solution to

$$\frac{(r+\delta)\,\theta k - \lambda(\theta)(\bar{q}-w_1)}{\delta+\lambda(\theta)} = MK,$$

where, from (41),

$$MK \equiv \alpha\left(\frac{\mu-1}{\mu}\right) Z = \alpha\left(\frac{\mu-1}{\mu}\right)\left[\frac{\alpha u w_0 + (1-\alpha u)(1+\pi)z^\star}{1+\pi(1-\alpha u)}\right].$$

We established above that there is a unique θ solution to this equation when $\pi = 0$. Hence, at the equilibrium value for θ, the left side intersects the right side by below. See right panel of Figure 4. From Proposition 8, a small increase of π from $\pi = 0$ raises Z if (43) holds. Hence, $\partial MK/\partial\pi > 0$ for π close to 0, which increases the right side of the equation above. Graphically, the MK curve shifts upward. As a result, an increase in π raises market tightness and reduces unemployment, $\partial u/\partial \pi < 0$ when (43) holds. ■

Proof of Proposition 11. From (47) $\partial MK/\partial w_1 > 0$ and $\partial MK/\partial n = 0$ since $w_0 = w_1$ (where we used that $u = 1-n$). Hence, a small increase in w_1 raises Z and MK since there is no first-order effect of n on Z when all workers receive the same income. From (45), market tightness solves:

$$\frac{(r+\delta)\,\theta k - \lambda(\theta)(\bar{q}-w_1)}{\delta+\lambda(\theta)} = \alpha(1-\mu^{-1})\left\{\alpha u w_0 + \alpha n w_1 + (1-\alpha)z^\star\right\}.$$

It can be reexpressed as:

$$(r+\delta)\,\theta k - \lambda(\theta)\left\{\bar{q} - \left[1-\alpha^2(1-\mu^{-1})\right] w_1 + (1-\alpha)z^\star\right\} = \alpha(1-\mu^{-1})\delta\left[\alpha w_0 + (1-\alpha)z^\star\right].$$

The left side is a convex function of θ which is equal to zero when $\theta = 0$. The right side is positive and independent of θ. Hence, there is a unique θ solution to this equation. As w_1 increases, the left side increases, which implies that θ decreases (since the left side must be increasing in θ when it intersects the right side). Hence, n decreases as w_1 increases. ■