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MACROECONOMICS WITH HETEROGENEOUS AGENTS AND INPUT-OUTPUT NETWORKS

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Macroeconomics with Heterogeneous Agents and Input-Output Networks

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Abstract

The goal of this paper is to simultaneously unbundle two interacting reduced-form building blocks of traditional macroeconomic models: the representative agent and the aggregate production function. We introduce a broad class of disaggregated general equilibrium models with Heterogeneous Agents and Input-Output networks (HA-IO). We characterize their properties through two sets of results describing the propagation and the aggregation of shocks. Our results shed light on many seemingly disparate applied questions, such as: sectoral comovement in business cycles; factor-biased technical change in task-based models; structural transformation; the effects of corporate taxation; and the dependence of fiscal multipliers on the composition of government spending.

Introduction

This paper takes inspiration from François Quesnay's approach to economics as put forth in his classic *Tableau Economique*. While largely forgotten today, the Tableau Economique, published in 1758, was one of the first great works of economic theory. Quesnay's Tableau conceived of economies as systems of interacting parts: merchants, farmers, and artisans, trading in laissez-faire marketplaces, produced and consumed intermediate and final goods and services. Quesnay's conception of the economy, which marked the beginning of general equilibrium theory and influenced luminaries from Adam Smith to Wassily Leontief, emphasized heterogeneity in production and in consumption.

We revive this tradition by introducing and elucidating the properties of a broad class of general equilibrium models with Heterogeneous Agents and Input-Output networks (HA-IO).

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Our goal is to simultaneously unbundle two interacting reduced-form building blocks of traditional macroeconomic models: the representative agent and the aggregate production function. Our hope is to contribute to the general foundations of a disaggregated approach to the study of macroeconomic phenomena.

Our framework is general and does not rely on specific functional forms. It features an arbitrary number of households with heterogeneous preferences supplying different factors of production to an arbitrary network of producers who combine factors and intermediate inputs using arbitrary neoclassical technologies. It also allows for an arbitrary pattern of distortions captured as explicit or implicit wedges. The model can be applied intra-temporally or inter-temporally using the Arrow-Debreu construct of indexing commodities by dates and states. The latter interpretation allows us to build a theoretical bridge between input-output models of production networks and heterogeneous-agent models with idiosyncratic risk, incomplete markets, and borrowing constraints.[1]

We investigate the patterns of shock propagation and aggregation generated by the model, and by pinning them down, clarify the way shocks, mediated by elasticities of substitution and general equilibrium forces, transmit through production networks and to consumers.

We show that in models with representative agents and balanced-growth preferences, a class that includes many present-day models of production networks, the patterns of propagation are incredibly constrained.[2] Specifically, in such models, the response of the sales of a producer (or collection of producers) i to a productivity shock to j is the same as j's response to a shock to i. Symmetric propagation in these models is a consequence of the first welfare theorem, and so can be established without taking a stance on the parametric structure of the model. This improbable property suggests that such efficient representative-agent production-networks models, despite their considerable influence in empirical and quantitative analyses of comovement, are also restrictive in important and unexpected dimensions. The practical relevance of this observation obviously depends on the applied question under consideration, but it underscores the surprising importance of apparently innocuous modeling choices in disaggregated approaches.

We use symmetry and symmetry-breaking as organizing devices for the paper. We start by providing general non-parametric comparative statics for how shocks propagate in efficient representative agent models with homothetic preferences over consumption goods and inelastic factor supplies where symmetric propagation holds. Then, we introduce various ingredients which break symmetry: heterogeneous consumers; non-homothetic preferences over consumption goods; elastic factor supplies with non-balanced growth preferences; and distor-

[1]Under this interpretation, consumer- and commodity-specific wedges capture the different endogenous shadow rates of returns on different assets by different consumers implicit in decentralizations of these models.

[2]By balanced-growth preferences, we mean preferences that are separable between consumption and leisure, homothetic over consumption, and balanced-growth over consumption and leisure. See King, Plosser, and Rebelo (1988) for more information on these preferences.

tions. These features all materially change the way the same production network transmits shocks from one producer to another.

These results show how to interpret regression-based empirical studies of the effects of shocks on prices and quantities. Many such analyses use instruments to trace out the effects of exogenous microeconomic shocks, and interpret their results using a partial equilibrium framework where the exclusion restriction is that non-shocked good and factor prices are fixed. Of course, only general equilibrium responses can be observed in practice. The presumption is that general equilibrium effects are small if the shock only hits a small part of the economy. HA-IO linkages weaken these interpretations and the underlying exclusion restrictions. They blur the line between partial and general equilibrium by introducing "local" general equilibrium effects. In a nutshell, even if the parts of the economy of the economy that are directly hit by the shock are small, other parts of the economy can be hit indirectly through the HA-IO network. Our analysis shows how to map regression coefficients to structural primitives while properly taking into account these general equilibrium forces.

Our results on propagation are also helpful in terms of thinking about aggregation in disaggregated economies. We propose new definitions of "industry-level" productivity and markups, for any collection of producers, and characterize the behavior of these aggregates. Our notion of industry-level productivity growth has a structurally interpretable decomposition into pure changes in technology and changes in allocative efficiency. When the economy is efficient, there are no changes in allocative efficiency, and our definition then coincides with the usual Solow residual.

In inefficient economies, both the measure of industry-level productivity and industry-level markup have non-trivial aggregation properties. Changes in these industry-level aggregates are *endogenous* in the sense that their evolution is not a simple average of the changes in microeconomic productivities and markups in that industry. They depend on shocks outside of the industry. They can also be the subject of fallacies of composition whereby the behavior of aggregates is divorced from the behavior of the individual components. For instance, it is possible for industry-level productivity to fall even as all firms in that industry become more productive, or for industry markups to fall even as all firms in the industry increase their markups.

To streamline the exposition we restrict our attention to nested-CES microeconomic production and utility functions for most of the paper, with an arbitrary number of nests, input-output shares, and elasticities. This choice is partly driven by the popularity of these functional forms in the literature, and partly due to the fact that they are relatively parsimonious. However, we show that with a simple and relatively minor modification, all of our results can be readily extended to arbitrary neoclassical production functions. Perhaps more importantly, this also shows that the essential intuition built from the CES benchmark survive to the more general case. Our non-CES results are particularly useful for empirical applications, such as structural estimation, where more flexibility in the microeconomic functional forms is especially desirable.

By characterizing the way shocks diffuse throughout the economy using a very general framework, our formulas nest most models in the literature, and can be used to help extend the insights in simple models to quantitatively more realistic environments. Taken together, our propagation and aggregation results can be used to trace out the effects of changes in technology or distortions at various levels of aggregation. This allows us to speak to many seemingly disparate questions in a variety of contexts ranging from sectoral comovement in business cycles, factor-biased technical change in task-based models, Baumol's cost disease and structural transformation, the effects of corporate taxation on output, and the dependence of fiscal multipliers on the composition of government spending. We sketch some example applications along these lines in the paper.

The outline of the paper is the following: in Section 1, we set up the general model, the necessary notation, and the notion of equilibrium. In Section 2, we establish general conditions for symmetric propagation. To make the paper more digestible and more modular, we begin by analyzing a special case, and then introduce successive generalizations, to highlight what each additional ingredient brings to the table. Section 3 analyzes the baseline case: an efficient representative agent model with homothetic preferences and inelastic factor supplies (non-homothetic preferences are treated in Appendix G). We enrich the baseline model by allowing for: heterogeneous consumers in Section 4; elastic factor supplies, in Section 5; and distortions, in Section 6. Having fully characterized propagation, in Section 7, we introduce our industry-level aggregates of productivity and markups, whose evolution can be immediately established with the aid of the earlier results on propagation. Although we restrict our focus on nested-CES economies, in Section 8, we show how all of our results can be generalized to arbitrary neoclassical production functions with one weird trick.

Related Literature

This paper is closely related to Baqaee and Farhi (2017a) and Baqaee and Farhi (2017b) and uses some of the tools developed in those papers. However, whereas those papers focused on the effects of shocks on GDP and aggregate TFP, this paper focuses on the propagation of shocks from one producer to another. Rather than aggregating value-added for the whole economy and characterizing its properties, we define and characterize the behavior of sub-aggregates of producers. Furthermore, the analysis in this paper is strictly more general since we allow for heterogeneous agents, whereas those papers worked with a representative agent. Allowing for heterogeneous agents is especially important given our focus, since the representative agent assumption has important implications for the comovement patterns.

More broadly, this paper relates to the literature on multi-sector models and models with production networks. Much attention in this literature has focused on the implications of heterogeneity for the behavior of GDP with less emphasis on comovement, for example, Gabaix (2011), Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), Jones (2011), Jones (2013), Bigio and La'O (2016), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017), as well as the aforementioned Baqaee and Farhi (2017a), and Baqaee and Farhi (2017b).

Theoretical interest in comovement in production networks started with Long and Plosser (1983) and Shea (2002) who use efficient representative agent models with homothetic preferences.[3] More recently, the question of how shocks propagate in production networks has become a topic of a vibrant empirical literature; examples include Foerster, Sarte, and Watson (2011), Di Giovanni, Levchenko, and Méjean (2014), Stella (2015), Barrot and Sauvagnat (2016), Acemoglu, Akcigit, and Kerr (2015), Atalay (2017), and Carvalho, Nirei, Saito, and Tahbaz-Salehi (2016). These analyses, although primarily empirical, use efficient representative agent models with balanced-growth preferences and production networks to interpret their empirical findings.

Our model nests the models in these papers as special cases, and shows that they all feature symmetric propagation. This is due to different reasons in different papers: the representative agent assumption, efficiency of the equilibrium, Cobb-Douglas functional forms, or having a single factor. By relaxing these common assumptions, our paper sheds light on the generic structure of propagation in general equilibrium.

Finally, our paper also relates to the literature on heterogeneous agent models. Broadly speaking, this literature can be divided in two parts: dynamic models, which focus on heterogeneity in marginal propensities to consume across different periods of time, and static models, which focus on heterogeneity in marginal propensities to consume across different types of goods. The former types of models are increasingly popular in macroeconomics, whereas the latter are common in international and regional economics. By adopting an Arrow-Debreu view where goods can be indexed by states of nature and dates, our result speak to both sets of models.

The literature focusing on heterogeneous agents in a dynamic context dates back to Bewley (1977), Huggett (1993), Aiyagari (1994), and Krusell and Smith (1998). Indexing commodities by dates and states, our framework can in principle capture these economies via consumer- and commodity-specific wedges that capture the different endogenous shadow rates of returns on different assets faced by different consumers implicit in decentralizations of these economies with heterogeneous agents, idiosyncratic risk, incomplete markets, and borrowing constraints. More recently, a large literature has extended this framework to environments with nominal rigidities. To illustrate how our formulas can be applied to study such economies, we set up a simple example inspired by Baqaee (2015), which extends the environments in Bilbiie (2008),

[3]Some recent papers that study propagation in inefficient environments with representative agents and specific functional forms include Baqaee (2018), Altinoglu (2016), and Grassi (2017).

Eggertsson and Krugman (2012), and Farhi and Werning (2016). The model features input-output linkages, sticky-wages, and two classes of agents (borrowers and savers) with different marginal propensities to consume because one class is up against a borrowing constraint while the other is not. We use it to study the dependence of fiscal multipliers on the composition of government spending.

The literature focusing on heterogeneous agents in a static context, once one includes models of international and inter-regional trade, is too voluminous to list. However, a more recent and smaller literature, for example Jaravel (2016), Argente and Lee (2017), Clayton, Jaravel, and Schaab (2018), and Cravino, Lan, and Levchenko (2018), shows that static marginal propensities to consume can be heterogeneous not just across regions or countries, but also across sectors or product categories. We present a simple illustrative example with three sectors (agriculture, manufacturing, and servinces) and two classes of workers (skilled and unskilled) with different sources of income and spending patterns, to characterize the reaction of the the skill premium to productivity shocks to the different sectors.

1 Setup

In this section, we setup the model, define the equilibrium, and lay down some input-output definitions.

1.1 Model

The model has a set of consumers C, a set of producers N, and a set of factors F with supply functions L_f. What distinguishes goods from factors is the fact that goods are produced by combining factors and goods, whereas factors are produced ex nihilo. The output of each producer is produced using intermediate inputs and factors, and is sold as an intermediate good to other producers and as a final good. What distinguishes consumers from producers is their ownership of the factors.

Households

Each agent c has preferences

$$U_c\left(\mathcal{D}_c(c_{c1},\ldots,c_{cN}), L_{c1},\ldots,L_{cf}\right),$$

where $\mathcal{D}_c$ is homothetic, c_{cj} is consumer c's consumption of good j, and L_{cj} is consumer c's supply of factor j. Each consumer faces the budget constraint

$$\sum_{i=1}^{N}(1+\tau_{ck})p_i c_{ci} = \sum_{f=1}^{F} w_f L_{cf} + \pi_c + \tau_c,$$

where p_i and c_{ci} are the price and quantity of good i used by consumer c, w_f and L_{cf} are the price and quantity of the factor f owned by consumer c, π_c is profits and τ_c is net revenues earned by taxes and subsidies rebated back to household c. The numbers τ_{ci} denote consumption taxes or subsidies on consumer c.

Without loss of generality, we can assume $\mathcal{D}_c$ is constant-returns-to-scale, allowing us to define a composite consumption good $Y_c = \mathcal{D}_c(\boldsymbol{c})$ for each consumer. To model factor supply, we often use an alternative formulation where factor supplies are not derived from utility maximization and are instead given by factor supply functions $L_{cf} = G_{cf}(w_f, Y_c)$.[4] When we do so, we maintain the assumption that given factor supplies, final demand for consumer c is given by the maximization of $\mathcal{D}_c(c_{c1}, \ldots, c_{cN})$ subject to the budget constraint and denote by Y_c is the corresponding real consumption index.[5]

Producers

Each good k is produced with some constant or decreasing returns to scale production function. Without loss of generality, we can assume that all production functions are constant-returns-to-scale simply by adding producer-specific fixed factors to the economy. Crucially, this means that our results will apply to economies with arbitrary non-homothetic production functions, as long as they do not have increasing-returns-to-scale.[6] Hence, we can write the cost function of each producer as

$$\frac{y_k}{A_k}\mathbf{C}_k\left((1+\tau_{k1})p_1, \ldots, (1+\tau_{kN})p_N, (1+\tau^f_{k1})w_1, \ldots, (1+\tau^f_{kF})w_F\right),$$

where y_k is the total output of k, and $\mathbf{C}_k/A_k$ is the marginal cost of producing good k. The number A_k is a Hicks-neutral productivity shifter, and τ_{ki} and τ_{kf} are producer-input-specific taxes or subsidies.

[4]We use this specific formulation, which assumes zero cross-factor-price elasticites, for simplicity and ease of exposition. Our results can easily be generalized away from this case.

[5]The distinction between these two formulations is obviously irrelevant when factor supplies are inelastic, but it will be useful for some of our applications, for example for comparative statics of the steady-state of a multi-sector Ramsey model with capital and labor where capital supply is infinitely elastic.

[6]See Baqaee (2018) for analysis of a production network economy with increasing returns to scale.

Notation

Going forward, and to make the exposition more intuitive, we slightly abuse notation in the following way. For each factor f, we interchangeably use the notation w_f or p_{N+f} to denote its wage, the notation L_{if} or $x_{i(N+f)}$ to denote its use by producer i, and the notation L_f or y_f or to denote total factor supply. We interchangeably use the notation c_{ci} or x_{ci} to denote the consumption of good i in final demand.

Furthermore, we represent all wedges in the economy as markups by adding additional producers. For example, the wedge τ_{ij} can be modeled in a modified setup as a markup charged by a new producer that buys input j and sells it to producer i. Going forward, we take advantage of this equivalence and assume that all wedges take the form of markups. Each producer i charges a price p_i equal to a markup μ_i over its marginal cost.

Equilibrium

Given productivities A_i and markups μ_i, a general equilibrium is a set of prices p_i, intermediate input choices x_{ij}, factor input choices l_{if}, outputs y_i, and final demands c_{ci}, such that:

i. each producer chooses inputs to minimize its costs taking prices as given;

ii. each household maximizes utility subject to its budget constraint taking prices as given;

iii. the markets for all goods and factors clear.

Although markups are primitives in our model, our results can also be applied to models with endogenous markups along the lines mentioned in Baqaee and Farhi (2017b).

1.2 Input-Output Definitions

We introduce some input-output notation and definitions. We define input-output objects such as input-output matrices, Leontief inverse matrices, and Domar weights. In the presence of markups/wedges, a distinction must be made between cost-based and revenue based input-output concepts. We denote the cost-based concepts with a tilde. Of course, when there are no wedges, the cost-based and revenue-based definitions will coincide.

Final Expenditure Shares

Let $b_{(c)}$ be the $N \times 1$ vector whose ith element is equal to the share of good i in consumer c's expenditures

$$b_{ci} = \frac{p_i c_{ci}}{\sum_{i \in N} p_i c_{ci}}.$$

Let χ_c be consumer c's share of total expenditure

$$\chi_c = \frac{\sum_{i \in N} p_i c_{ci}}{\sum_{j \in N} \sum_{d \in C} p_j c_{dj}}.$$

HA-IO Matrix

We define the revenue-based HA-IO matrix to be the $(C+N+F) \times (C+N+F)$ matrix Ω whose ijth element is equal to i's expenditures on inputs from j as a share of its total revenues

$$\Omega_{ij} \equiv \frac{p_j x_{ij}}{p_i y_i}.$$

Note that the HA-IO matrix Ω includes expenditures on the factors of production and of the consumers. We also define the factor-distribution matrix Φ to be the $C \times (F+1)$ matrix whose cfth element is

$$\Phi_{cf} = \frac{w_f L_{cf}}{w_f L_f}.$$

In words, Φ_{cf} is the share of factor f's income accruing to consumer c.[7] We let $F+1$ index net income due to taxes and profits.

By analogy, define the cost-based HA-IO matrix $\tilde{\Omega}$ as

$$\tilde{\Omega}_{ij} \equiv \frac{p_j x_{ij}}{\sum_l p_l x_{il}}.$$

Its ijth element $\tilde{\Omega}_{ij}$ records the expenditure of producer i on good j as a fraction of the total cost of producer i. By Shephard's lemma, $\tilde{\Omega}_{ij}$ is also the elasticity of the cost of i to the price of j, holding the prices of all other producers constant.

HA-IO Leontief Inverse Matrix

We define the HA-IO Leontief inverse matrix as

$$\Psi \equiv (I - \Omega)^{-1} = I + \Omega + \Omega^2 + \dots,$$

and the cost-based HA-IO Leontief-inverse matrix as

$$\tilde{\Psi} \equiv (I - \tilde{\Omega})^{-1} = I + \tilde{\Omega} + \tilde{\Omega}^2 + \dots.$$

[7] In the body of the paper, we analyze the case with heterogeneous consumers and the case with distortions separately. Since we do not simultaneously consider heterogeneous consumers and pure profits, we do not need to track how profits are disbursed across different consumers. In the Appendix H, where we show how heterogeneous consumers and wedges can be analyzed at the same time, we will augment the Φ matrix so that, in addition to factor payments, it also records how profits are being disbursed to different households.

While the input-output matrix Ω records the *direct* exposures of one producer to another, the Leontief inverse matrix Ψ records instead the *direct and indirect* exposures through the production network. This can be seen most clearly by noting that $(\Omega^n)_{ij}$ measures the weighted sums of all paths of length n from producer i to producer j.

By Shephard's lemma, $\tilde{\Psi}_{ij}$ is also the elasticity of the cost of i to the price of j holding fixed the price of factors but taking into account how the price of all other goods in the economy will change. Note that this is still a *partial-equilibrium* elasticity, which does not take into account changes in factor prices that occur in general equilibrium (when the requirement that factor markets clear is imposed). These general equilibrium effects are complex and will be fully characterized below.

GDP and Domar Weights

GDP or nominal output is the total sum of all expenditures on final consumption by all consumers

$$GDP = \sum_{i \in N} \sum_{c \in C} p_i c_{ci}.$$

We define the Domar weight λ_i of producer i to be its sales share as a fraction of GDP

$$\lambda_i \equiv \frac{p_i y_i}{GDP}.$$

Note that $\sum_{i=1}^{N} \lambda_i > 1$ in general since some sales are not final sales but intermediate sales.

For expositional convenience, for a factor f, we sometimes use Λ_f instead of λ_f. Note that the revenue-based Domar weight Λ_f of factor f is simply its total income share. Then, we can write consumer c's share in aggregate income as

$$\chi_c = \frac{\sum_i p_i c_{ci}}{GDP} = \sum_{f \in F} \Phi_{cf} \Lambda_f + \Phi_{cF+1} \Lambda_{F+1},$$

where $F+1$ indexes net income due to taxes and profits.

We can also define the vector b to be final demand expenditures as a share of GDP

$$b_i = \frac{\sum_{c \in C} p_i c_{ci}}{GDP} = \sum_{c \in C} \chi_c \Omega_{ci}.$$

The accounting identity

$$p_i y_i = \sum_{c \in C} p_i c_{ci} + \sum_j p_i x_{ji} = \sum_{c \in C} \Omega_{ci} \chi_c GDP + \sum_j \Omega_{ji} \lambda_j GDP$$

links the revenue-based Domar weights to the Leontief inverse via

$$\lambda' = b'\Psi = b'I + b'\Omega + b'\Omega^2 + \dots .$$

Another way to see this is that the i-th element of $b'\Omega^n$ measures the weighted sum of all paths of length n from producer i to final demand.

We can decompose λ_i into the sum of all paths from producer i to consumer c weighted by that consumer's size. Let λ_i^c be

$$\lambda_i^c = \sum_{j \in N} \Omega_{cj} \Psi_{ji}.$$

Using language from Baqaee (2015), we can think of λ_i^c as the *network-adjusted* consumption share of good i for agent c. Then

$$\lambda_i = \sum_{c \in C} \chi_c \lambda_i^c,$$

so that λ^c provides a decomposition of λ_i by consumers. For a factor f, we sometimes use Λ_f^c instead of λ_f^c.

We also define the share of the sales of good j as input to producer i as a fraction of aggregate output

$$\lambda_{ij} \equiv \frac{p_j x_{ij}}{GDP} = \Omega_{ij} \lambda_i,$$

and for a factor f, we sometimes use Λ_{if} instead of λ_{if}.

By analogy, the cost-based Domar weights are

$$\tilde{\lambda} \equiv b'\tilde{\Psi} = b' + b'\tilde{\Omega} + b'\tilde{\Omega}^2 + \dots .$$

As above, for a factor f, we sometimes use $\tilde{\Lambda}_f$ instead of $\tilde{\lambda}_f$.

Real Output and GDP Deflator

Since our economy has heterogeneous households the level of real GDP or output is ambiguous to define. Hence, we do not offer a definition for the *level* of real GDP, defining instead only the *changes* in real GDP using the Divisia index. We define the change in real GDP as

$$\mathrm{d} \log Y = \sum_i b_i \, \mathrm{d} \log c_i.$$

Similarly, we can define changes in the GDP deflator as

$$\mathrm{d} \log P = \sum_c b_i \, \mathrm{d} \log p_i.$$

If there exists a representative consumer with homothetic preferences in this economy, then $\mathrm{d}\log Y$ and is equivalent, to a first order, to the change in real GDP defined via the representative consumer's ideal price index. Similarly, $\mathrm{d}\log P$ is equivalent to the change in the representative agent's ideal price index.[8] Through out the rest of the paper, we measure changes in prices in real terms using the GDP deflator.

1.3 Interpretation

Before stating our results, we briefly discuss two important points of interpretation.

Intratemporal vs. Intertemporal

There are several ways to interpret this model and we will explicitly make use of all of them in our examples: (1) we could view it as a purely static model; (2) we could interpret final demand as a per-period part of a larger dynamic problem, where the inelastically supplied factors are pre-determined state variables; (3) we could interpret final demand as an intertemporal consumption function where goods are also indexed by time and states à la Arrow-Debreu.

Interpretation (1) is the most straightforward. In interpretation (2), final demand encompasses consumption demand and investment demand, and the formulation with factor supply functions must be used. In interpretation (3), the process of factor (capital or human capital for example) accumulation is captured via intertemporal production functions that transform goods in one period into goods in other periods.[9] Our formulas would apply to these economies without change, but of course, in such a world, input-output definitions are expressed in net-present value terms.

In principle, interpretation (3) also allows us to capture borrowing constraints and incomplete markets for consumers facing idiosyncratic risk, with the consumer- and commodity-specific wedges capturing the different endogenous shadow rates of returns on different assets by different consumers implicit in decentralizations of these models.

Biased Technical Change and Demand Shocks

Although the model is written in terms of Hicks-neutral productivity shocks, this is done without loss of generality. We can always capture non-neutral productivity shocks, say factor-augmenting shocks, by relabelling the relevant factor of a given producer to be a separate producer. Then, Hicks-neutral productivity shocks to that industry would be identical to factor-biased productivity shocks in the original model.

[8]See Hulten (1973) for more details on the relationship between Divisia indices and cardinal measures of welfare and output.

[9]This modeling choice would also be well-suited to handle technological frictions to the reallocation of factors such as adjustment costs and variable capacity utilization.

Demand shocks can also be modeled in this way. A demand shock for a certain input used by i can be modelled via a positive consumer-specific productivity shock for that input along with negative consumer-specific productivity shocks to all other inputs, leaving the overall productivity of i unchanged.

1.4 Standard Form for CES Economies

Any nested-CES economy, with an arbitrary numbers of agents, producers, factors, CES nests, elasticities, and intermediate input use, can be re-written in what we call *standard form*, which is more convenient to study. Throughout the paper, variables with over-lines are normalizing constants equal to the values at some initial allocation.

A CES economy in standard form is defined by a tuple (ω, θ, F). The $(N+F+C) \times (N+F+C)$ matrix ω is a matrix of input-output parameters. The $(N+C) \times 1$ vector θ is a vector of microeconomic elasticities of substitution. Finally, for economies with distortions, we supplement the definition with the specification of a $N \times 1$ vector μ is a vector of markups/wedges for the N goods. Each good k in N or in C is produced with the production function

$$\frac{y_k}{\overline{y}_k} = \frac{A_k}{\bar{A}_k} \left(\sum_l \omega_{kl} \left(\frac{x_{kl}}{\overline{x}_{kl}} \right)^{\frac{\theta_k - 1}{\theta_k}} \right)^{\frac{\theta_k}{\theta_k - 1}},$$

where x_{lk} are intermediate inputs from l used by k. Without loss-of-generality, we represent the final good Y_c consumed by each consumer c as being purchased by the household from a producer producing the final good. When there is only one consumer, we can define aggregate output in levels using the consumer's consumption aggregator.

Through a relabelling, this structure can represent any CES economy with an arbitrary pattern of nests and wedges and elasticities. Intuitively, by relabelling each CES aggregator to be a new producer, we can have as many nests as desired.

For the rest of paper, except in Section 2, we work with nested-CES economies. However, in Section 8, we show that all of our nested-CES results can be easily extended to non-CES economies with a simple modification using the concept of a substitution operator.

1.5 Outline of Analysis

Our results are comparative statics describing how, starting from an initial equilibrium, the equilibrium levels of various quantities change in responses to shocks to productivities or wedges. To help build intuition, we focus on nested-CES economies, which it turns out, capture much of the important intuition of more general models.

In Section 2, we establish general conditions for symmetric propagation. In Section 3, we study the basic model with a representative consumer, inelastically supplied factors, and no distortions. We characterize propagation and see a concrete demonstration of symmetric propagation. In Sections 4-6, we enrich the basic model by introducing ingredients which change the patterns of propagation and break symmetry. In Section 4, we allow for heterogeneous consumers, in Section 5, we allow for elastic factor supplies that are not derived from balanced-growth perferences, and in Section 6 we allow for wedges. We study each of these generalizations in isolation to keep the exposition clear. In the appendix, we provide results for the general model, which simultaneously allows for heterogeneous agents, elastic factors, and distortions. Once we end our analysis of propagation, we close out the analysis in Section 7 by defining and characterizing the properties of industry aggregates of productivity and wedges in this class of models. As previously indicated, all of our results easily generalize beyond CES functional forms as explained in Section 8.

2 Symmetric Propagation

In this section, we establish a surprising (and most likely counterfactual) symmetry result for this class of models when there is a representative-agent with balanced-growth preferences. This result helps organize our analysis in the rest of the paper, since we can show how adding more ingredients can break this symmetry. We use the most general version of the model, which does not impose a nested CES structure.

Proposition 1 (Symmetric Propagation)**.** *Consider the efficient model without markups/wedges. For two producers i and j, symmetric propagation*

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log A_j}$$

holds in equilibrium if either of the following conditions is satisfied:

(i) There is a representative agent with balanced-growth preferences

$$U(\mathcal{D}(c_1, \ldots, c_N), L_1, \cdots, L_F) = U(\mathcal{D}(\boldsymbol{c})v(\boldsymbol{L})),$$

where $\mathcal{D}$ is homogenous of degree one, and factor supply is derived from these preferences; or

(ii) There is a single primary factor, indexed by L, and preferences are

$$U(\mathcal{D}(c_1, \ldots, c_N), L),$$

where $\mathcal{D}$ is homothetic.

Note that in the definition of balanced-growth preferences in (i), the disutility of factor supply function $v(\boldsymbol{L})$ need not be homothetic.

Many papers in the multisector literature and almost all papers in the production network literature work with either a single factor of production or balanced-growth preferences, meaning that most of these papers feature symmetric propagation of shocks.[10] We discuss these sufficient conditions in turn.

We start with condition (i): symmetric propagation is consequence of chaining together three facts: (1) efficiency of the equilibrium, (2) the existence of a representative agent, (3) homotheticity of preferences $\mathcal{D}$ over consumption goods and balanced-growth preferences over consumption goods and factors.

To understand the intuition for this result, it is easier to start with the case where factors supplies are inelastic where condition (i) boils down to the requirement that preferences over goods are homothetic. Efficiency (1) implies that proportional impact of real output of a shock to the productivity of producer i is given by its sales share λ_i. With a representative agent (2) with homothetic preferences over consumption goods (3), we can define a price index to deflate nominal GDP to obtain real output in levels. Hence λ_i is the derivative of log real output with respect to the log productivity of producer i, i.e. $\lambda_i = \mathrm{d}\log Y / \mathrm{d}\log A_i$. The symmetry of partial derivatives $\mathrm{d}^2 \log Y / (\mathrm{d}\log A_j \, \mathrm{d}\log A_i) = \mathrm{d}^2 \log Y / (\mathrm{d}\log A_i \, \mathrm{d}\log A_j)$ then immediately implies symmetric propagation in sales shares $\mathrm{d}\,\lambda_i / \mathrm{d}\log A_j = \mathrm{d}\,\lambda_j / \mathrm{d}\log A_i$.

The generalization to the case of elastic factor supplies and balanced-growth preferences involves several modifications. We can define an extended price index for welfare which loads not just on goods but also on factor prices and starts not with nominal GDP but with nominal GDP net of factor payments. An envelope theorem then implies that the sales share of producer i as a fraction of nominal GDP net of factor payments is equal to the elasticity of welfare to the productivity of this producer. Because of balanced growth preferences, the expenditures on consumption goods are proportional to nominal GDP net of factor payments and welfare is proportional to real output. As a consequence, the elasticity of real output to productivities are again given by sales shares as a fraction of nominal GDP. The result follows.

As usual, Proposition 1 can be applied both to static or to dynamic environments. In a dynamic setting, if per-period preferences are log-balanced-growth, then the environment satisfies condition (i) (as in Foerster, Sarte, and Watson (2011) for example), and Proposition 1 applies.

Symmetric propagation follows from (ii) for the following reasons. First, changes in factor prices do not lead to any redistribution across consumers and so final demand is de facto homothetic as if there were a representative agent. Second, when there is a single factor, an in-

[10] This symmetry property involves general equilibrium responses and is therefore different from the "reciprocity relations" discussed by Samuelson (1953) which involve the equilibrium change in quantities (or shares) of goods to changes in factor supplies in partial equilibrium holding goods prices constant.

crease in the supply of that factor affects the marginal cost of all producers in exactly the same way: one-for-one. Hence, changes in the supply of the factor do not change relative prices, and therefore, propagation in sales-shares in the one factor model does not depend on the elasticity of factor supply. The result then follows from (i).

When the conditions in Proposition 1 are satisfied, symmetric propagation holds not only for sales shares but also for sales, as long as GDP is used as the numeraire. In fact, symmetric propagation holds for sales even in situations where it does not for sales shares. To state this result, temporarily relax our maintained assumption of homotheticity over consumption and separability between consumption and factors.

Proposition 2. *Consider a modification of the efficient model without markups/wedges with a representative agent whose preferences are given by*

$$U(c_1, \ldots, c_N, L_1, \ldots, L_F),$$

where preferences over consumption goods are not necessarily homothetic. Then there exists some price index P_u such that

$$\frac{\mathrm{d}\, p_i y_i}{\mathrm{d} \log A_j} = \frac{\mathrm{d}\, p_j y_j}{\mathrm{d} \log A_i},$$

when P_u is used as the numeraire. If U is homothetic, then P_u is the ideal price index associated with U. If U is homothetic and factors are inelastically supplied, then P_u is the GDP deflator.

Interestingly, the logic for symmetry in sales is even stronger than for sales shares. In particular, if we measure prices using the household's ideal price index (one which accounts appropriately for leisure), then symmetric propagation in sales holds for *any* representative agent model where the first welfare theorem holds. However, symmetric propagation in sales is not readily observable since it applies to *real* sales measured using the household's unobservable ideal price index. Since we do not directly observe these real sales, this makes symmetric propagation in sales less interesting from an applied perspective (except when utility is homothetic and factors supplies are inelastic).

In the rest of the paper, we start in Section 3 by analyzing a setup with is a representative agent and inelastic factor supplies where Proposition 1 holds. We then proceed to show how Proposition 1 can be broken. In Section 4, we break homotheticity of final demand by introducing hetereogenous consumers.[11] In Section 5, we break symmetry by allowing for multiple

[11]In Appendix G, we also show an example where symmetry is broken with a representative consumer with non-homothetic preferences. The example in Appendix G uses the non-homothetic demand system used by Comin, Lashkari, and Mestieri (2015) and proposed by Sato (1977). Non-homothetic preferences are a key ingredient in the sizeable literature on structural change, and Appendix G connects our results to this literature. We show how to recover the two key facts emphasized by Comin, Lashkari, and Mestieri (2015): the hump-shaped consumption share of manufacturing over time, and the decline in the quantity manufacturing goods relative to services despite faster productivity growth in manufacturing.

factors which cannot be derived from balanced-growth preferences. Finally, we break symmetric propagation in Section 6 by allowing for distortions, which severs the link between sales and derivatives of the welfare function established by the first-welfare theorem.

3 Basic Model

We begin by stating our comparative static results for nested-CES economies with a *representative consumer*, *inelastic factors*, and *no distortions*. We then work through some examples. From now on, unless stated otherwise, we work with nested-CES economies. Section 8 shows how to generalize the results to non-CES economies.

3.1 Comparative Statics

In this section, we characterize the elasticities to the different productivities of aggregate output, shares, sales, prices, and quantities.

Aggregate Output and Shares

We start by characterizing the elasticities to the different productivities of the sales shares or Domar weights. For this result, it is useful to use a notation which explicitly differentiate factors from other producers. The following proposition is taken from Baqaee and Farhi (2017a).[12]

Proposition 3. *(Aggregate Output and Shares) The elasticities of aggregate output to the different productivities are given by*

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k} = \lambda_k. \tag{1}$$

The elasticities of the sales shares or Domar weights of i is given by

$$\frac{\mathrm{d}\log \lambda_i}{\mathrm{d}\log A_k} = \sum_j (\theta_j - 1)\frac{\lambda_j}{\lambda_i} Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(i)}) - \sum_g \sum_j (\theta_j - 1)\frac{\lambda_j}{\lambda_i} Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(i)})\frac{\mathrm{d}\log \Lambda_g}{\mathrm{d}\log A_k}. \tag{2}$$

[12] We first established Proposition 3 in Baqaee and Farhi (2017a) where our focus was the characterization to the second order of the macroeconomic impacts of microeconomic shocks in order to account for nonlinearities. Since the linear or first-order macroeconomic impact of microeconomic shocks is given by Hulten's theorem as $\mathrm{d}\log Y/\mathrm{d}\log A_i = \lambda_i$, it follows that their second-order or nonlinear impact is given by $\mathrm{d}^2\log Y/(\mathrm{d}\log A_i\,\mathrm{d}\log A_k) = \mathrm{d}\,\lambda_i/\log A_k = \lambda_i\,\mathrm{d}\log\lambda_i/\mathrm{d}\log A_k$. Our focus here is different: we aim to characterize to the first order the propagation of microeconomic shocks in production networks, i.e. the impact of a shock to producer k on producer i. Proposition 3 turns out to be useful for these two different objectives.

The elasticities of the factor shares solve the following system of linear equations

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_k} = \sum_j (\theta_j - 1)\frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(f)}) - \sum_g \sum_j (\theta_j - 1)\frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(f)})\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_k}. \tag{3}$$

In these equations, we make use of the input-output covariance operator introduced by Baqaee and Farhi (2017a):

$$Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}) = \sum_i \Omega_{ji}\Psi_{ik}\Psi_{il} - \left(\sum_i \Omega_{ji}\Psi_{ik}\right)\left(\sum_i \Omega_{ji}\Psi_{il}\right), \tag{4}$$

where $\Omega^{(j)}$ corresponds to the jth row of Ω, $\Psi_{(k)}$ to kth column of Ψ, and $\Psi_{(l)}$ to the lth column of Ψ. In words, this is the covariance between the kth column of Ψ and the lth column of Ψ using the jth row of Ω as the distribution. Since the rows of Ω always sum to one for a reproducible (non-factor) good j, we can formally think of this as a covariance, and for a non-reproducible good, the operator just returns 0. The input-ouput covariance operator turns out to be a key statistic for nested-CES economies.

Equation (1) is an output *aggregation* equation, the content of which is simply Hulten's theorem: the elasticity of aggregate output to productivity of a producer is given by its Domar weight. We call equations (2) and (3) the *share propagation equations*. Of course, equation (3) is obtained simply by letting $i = f$ in (2).

Our analysis will show that this basic structure, where all equilibrium relationships can be deduced by combining an aggregation equation with share propagation equations, holds in general, and is not an artefact of the simplifications made in this section. Even as we open the door to distortions, elastic factor supplies, and non-CES functional forms, an aggregation equation along with share propagation equations will pin down the equilibrium. Intuitively, the share propagation equations determine how each quantity's share of the pie changes, and the aggregation equations determines how the size of the pie changes.

Note that we can rewrite the system of linear factor share propagation equations (3) as

$$\frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k} = \Gamma\frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k} + \delta_{(k)}, \tag{5}$$

with

$$\Gamma_{fg} = -\sum_j (\theta_j - 1)\frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}}\left(\Psi_{(g)}, \Psi_{(f)}\right),$$

and

$$\delta_{fk} = \sum_j (\theta_j - 1)\frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}}\left(\Psi_{(k)}, \Psi_{(f)}\right).$$

We call δ the *factor share impulse matrix*. Its kth column encodes the direct or first-round effects of a shock to the productivity of producer k on factor income shares, taking relative factor prices as given. We call Γ the *factor share propagation matrix*. It encodes the effects of changes in relative factor prices on factor income shares, and it is independent of the source of the shock k.

Imagine a positive shock $\mathrm{d}\log A_k > 0$ to producer k. For fixed relative factor prices, every producer i will substitute across its inputs in response to this shock. Suppose that $\theta_j > 1$, so that producer j substitutes (in shares) *towards* those inputs i that are more reliant on producer k, captured by Ψ_{ik}, the more so, the higher is $\theta_j - 1$. Now, if those inputs are also more reliant on factor f, captured by a high $Cov_{\Omega^{(j)}}\left(\Psi_{(k)}, \Psi_{(f)}\right)$, then substitution by j will increase demand for factor f and hence the income share of factor f. These substitutions, which happen at the level of each producer j, must be summed across producers leading to the term δ_{fk}.

This first round of changes in the demand for factors triggers changes in relative factor prices which then sets off additional rounds of substitution in the economy that we must account for, and this is the role Γ plays. For a given set of factor prices, the shock to k affects the demand for each factor, hence factor income shares and in turn factor prices, as measured by the $F \times 1$ vector $\delta_{(k)}$. These changes in factor prices then cause further substitution through the network, leading to additional changes in factor demands and prices. The impact of the change in the relative price of factor g on the share of factor f is measured by the fgth element of the $F \times F$ matrix Γ. The movements in factor shares are the fixed point of this process, i.e. the solution of equation (3).

The intuition for equation (2) is similar. The first term on the right-hand side accounts for the effect of the shock for given relative factor prices, and the second term on the right-hand side accounts for the effects of the changes in relative factor prices.

In the case where there is only one factor, which we then denote by L, then we have $\mathrm{d}\log \Lambda_L / \mathrm{d}\log A_k = 0$ since Λ_L is always equal to 1. Equation (3) becomes trivial, and the second covariance terms on the right-hand side of equations 2 drop outs.

Prices

We now characterize the elasticities of prices to the different productivities.

Proposition 4. *(Prices) The elasticities of the prices of the different producers to the different productivities are given by*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \tag{6}$$

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -\Psi_{ik} + \sum_g \Psi_{ig} \frac{\mathrm{d}\log w_g}{\mathrm{d}\log A_k}, \tag{7}$$

where $\mathrm{d}\log Y / \mathrm{d}\log A_k$ *and* $\mathrm{d}\log \Lambda_f / \mathrm{d}\log A_k$ *are given in Proposition 3.*

Equation (6) characterizes the general-equilibrium elasticities of wages to the different productivities. These are necessary to complete the partial-equilibrium elasticities encoded in the Leontief inverse matrix in order to get the general-equilibrium elasticities of prices to productivities in equation (7).

Note that with one factor, denoted by L, equations (6) and (7) become $\mathrm{d}\log w_f / \mathrm{d}\log A_k = \lambda_k$ and $\mathrm{d}\log p_i / \mathrm{d}\log A_k = -\Psi_{ik} + \lambda_k$. Since λ_k and Ψ_{ik} only depend on the downstream input-output linkages of producer k, shocks propagate downstream in prices. This property no longer holds with multiple factors, because productivity shocks then propagate downstream and upstream. Indeed, they lead to upstream changes in the relative prices of factors which depend on all the input-output linkages in the economy. These upstream changes in relative factor prices in turn propagate downstream, and since all producers are downstream from factors, affect all prices. [13]

Sales, and Quantities

Armed with Propositions 3 and 4, we can characterize the elasticities of sales and output quantities of the different producers to the different productivities.

Corollary 1. *(Sales and Quantities) The elasticities of the sales and output quantities of the different producers to the different productivities are given by*

$$\frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log \lambda_i}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \tag{8}$$

$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} - \frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k}, \tag{9}$$

where $\mathrm{d}\log \lambda_i / \mathrm{d}\log A_k$, $\mathrm{d}\log Y / \mathrm{d}\log A_k$, and $\mathrm{d}\log p_i / \mathrm{d}\log A_k$ are given in Propositions 3 and 4. These formulas can be applied to factors by treating them as producers of non-reproducible goods using $i = f$ and replacing p_i by w_f, y_i by L_f, and λ_i by Λ_f.

The formula for sales is straightforward. Indeed, equation (8) can be obtained by combining Hulten's theorem as stated in equation (1) and the formula for sales shares as stated in equation (2) since $p_i y_i = Y\lambda_i$.

The formula for quantities in (9) can then be obtained by combining the formula for sales in equation (8) and the formula for prices in equation (7) since $y_i = p_i y_i / p_i$.

[13] An interesting implication of Hulten's theorem (see Proposition 3) in efficient economies with inelastic factor supplies is that changes to the composition of final demand have no effect on aggregate output. When factor supplies are elastic as in Section 5, this property remains true as long as there is only one factor, but fails in general when there are multiple factors and the different producers are differentially exposed to the different factors (see Proposition 6), a point previously made in Baqaee (2015). It also fails in general when there are distortions as in Section 6, since then changes in the composition of demand can affect aggregate TFP by directing resources towards or away from more distorted parts of the economy and impacting the degree of allocative efficiency (see Propositions 8 and 14).

Input Shares, Input Expenditures, and Input Quantities

Using Propositions 3 and 4 as well as Corollary 1, it is easy to derive the elasticities of input shares, expenditures and quantities of the different producers to the different productivities. These results can actually be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l. They are collected in Corollary 2 in Appendix B.

Symmetric Propagation

Since the economy in this section satisfies the conditions of Proposition 1, it features symmetric propagation. $\lambda_k = \mathrm{d}\log Y / \mathrm{d}\log A_k$, we have[14]

$$\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log A_j} = \frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log A_i} = \frac{\mathrm{d}^2\log Y}{\mathrm{d}\log A_i\,\mathrm{d}\log A_j}.$$

Of course, one can also prove symmetry directly by relying on the shares propgation equations (2) and (3).

3.2 Regressions in General Equilibrium

Proposition 3 shows the propagation of shocks from one micro producer to another are profoundly affected by general equilibrium assumptions. Modelling decisions about whether or not there is a representative consumer or whether factor markets for some goods are shared non-trivially affect the way that shocks propagate, for a fixed input-output matrix and vector underlying micro-elasticities of substitution. Specifically, we can write expressions of the form

$$\mathrm{d}\log x_i = \epsilon^x_{ij}\,\mathrm{d}\log A_j,$$

for the effect of an exogenous shock to j on the sales share ($x_i = \lambda_i$), sales, ($x_i = p_i y_i$), output ($x_i = y_i$), or price ($x_i = p_i$) of i, where ϵ^x_{ij} depends on all the primitives of the model, including the HA-IO matrix, the micro-elasticities of substitution, and the structure of factor markets. Once we disaggregate the economy, “local” factor markets and “local” input-output connections introduce “local” general equilibrium responses. To see why this matters, we work through two simple examples.

[14]Symmetric propagation also applies to factors, for example, replacing i by f, λ_i by Λ_f, and A_i by L_f.

Elasticities of Substitution

Consider regressions of the form

$$\Delta \log(\lambda_i/\lambda_j) = a\Delta \log A_j + controls + \varepsilon,$$

where $\Delta \log A_j$ is an exogenous shock to the production of j, a is the regression coefficient and ε is the error term. To interpret the coefficient a, local general equilibrium forces must be taken into account.

For example, this regression could be the second stage of an IV regression designed to estimate elasticities of substitution. Anticipating an upcoming example, suppose that GDP is given by

$$Y = \left(\sum_i b_i \left(\frac{y_i}{\overline{y}} \right)^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}.$$

Imagine each producer has a production function $y_i = A_i L_i$ where L_i is the labor used by producer i. In such a world, we know that

$$\mathrm{d}\log(\lambda_i/\lambda_j) = -(1-\theta)\,\mathrm{d}\log p_j.$$

Under the exclusion restriction that $\Delta \log A_j$ moves p_j and no other prices or wages, we can use the regression equation to estimate the elasticity of substitution θ. However, in this model, suppose that we say $L_i = M_i^{\beta} F_i^{1-\beta}$, where M_i corresponds to mobile labor, which can be reallocated across producers, and F_i corresponds to fixed labor, which cannot be reallocated.[15] In this case, if we apply Proposition 3, we find

$$\mathrm{d}\log(\lambda_j/\lambda_i) = \frac{1-\theta}{1+(\theta-1)(1-\beta)}\,\mathrm{d}\log A_i.$$

Hence, the estimated coefficient a is not a simple function of the elasticity of substitution, but also depends on how mobile the labor force is, and this holds *regardless* of the size of i and j. In other words, simply knowing that i and j are small is not enough to logically rule out the importance of general equilibrium mechanisms. In essence, the aforementioned exclusion restriction is violated due to the existence of local labor markets.

Output Elasticity

Consider regressions of the form

$$\Delta \log p_i = a\Delta \log A_j + controls + \varepsilon,$$

[15] See Appendix B.1 for more details on this example.

where $\Delta \log A_j$ is an exogenous shock to the production of j, a is the regression coefficient and ε is the error term. For example, this regression could be the second stage of an IV regression designed to estimate elasticities of the cost function. In general,

$$\mathrm{d}\log p_i = \Omega_{ij}\,\mathrm{d}\log p_j,$$

where Ω_{ij} is the elasticity of i's marginal cost with respect to j's price. Under the exclusion restrict that $\Delta \log A_j$ moves only p_j and no other prices or wages, we can use the regression equation to estimate output elasticities. However, applying Proposition 4, we know that

$$\mathrm{d}\log p_i = \Psi_{ij}\,\mathrm{d}\log A_j + \sum_g \Psi_{ig}\,\mathrm{d}\log w_g.$$

Even if i and j are infinitesimal and the prices of factors are assumed to be constant, if there exists some producer k who sells to k and buys from j, so that $\Omega_{ik}\Psi_{kj} \neq 0$ then our regressions are contaminated by local GE effects. In this case, the existence of local supply chains violate the exclusion restriction.

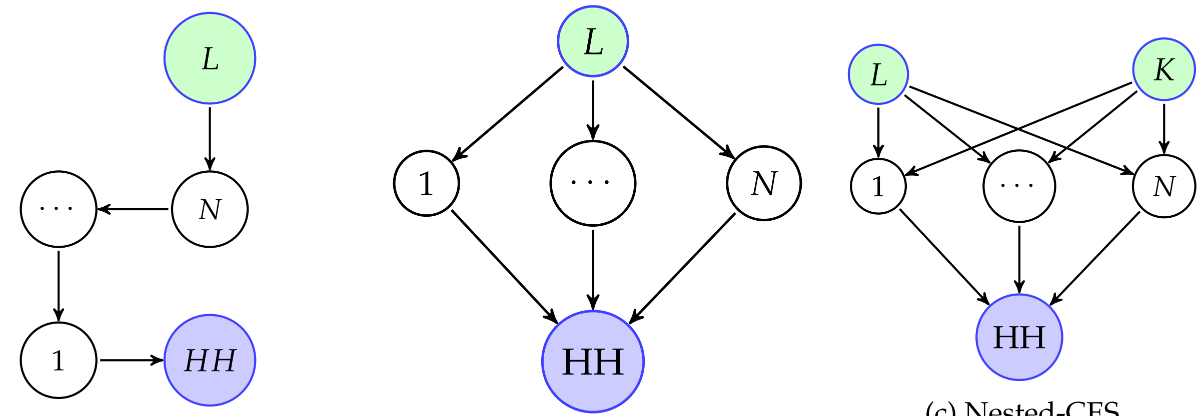
Propositions 3 and 4, and their extensions and generalizations in subsequent sections, are useful for two reasons. First, they suggest caution in inferring structural objects from coefficients of regressions due to the presence of these general equilibrium forces. Second, they offer a way to map the results of regressions to structural primitives once one takes a stance on the nature of the general equilibrium forces.

3.3 Simple Illustrative Examples

In this section, we illustrate the results derived in Section 3.1 with three simple economies: the vertical economy, the horizontal economy, and a double-nested CES economy. In each of these economies, we apply our results to characterize some selected propagation results. We end this section by applying our results, using the intertemporal interpretation, to the classic model of Long and Plosser (1983).

In the first two examples, there is a single factor called labor. The vertical economy is a chain of producers: producer N produces linearly using labor and downstream producers transform linearly the output of the producer immediately upstream from them. The household purchases the output of the most downstream producer. The horizontal economy features downstream producers who produce linearly from labor. The household purchases the output of the downstream producers according to a CES aggregator. The last economy has two factors, L and K, used by CES producers who then sell to the household.

We will be particularly interested in the possibility of generating positive comovement in a given producer-level variable, by which we mean that these variables all move in the same di-



(a) Vertical Economy

(b) Horizontal Economy

(c) Nested-CES

Figure 1: The solid arrows represent the flow of goods. The flow of profits and wages from firms to households has been suppressed in the diagram.

rection across producers i in response to the productivity shock of a given producer k. We will identify two distinct channels for positive comovement in output: an intermediate-input channel (vertical economy); and a labor reallocation channel in the presence of complementarities (horizontal economy).

In the final example, we show how we can capture the insights of models of capital-biased technical change and automation, like Acemoglu and Restrepo (2018), in our framework, and to extend those models to more complex and quantitatively realistic production structures.

Example: Vertical Economy

In the vertical economy, we have

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k} = 1,$$

$$\frac{\mathrm{d}\log \lambda_i}{\mathrm{d}\log A_k} = 0, \quad \frac{\mathrm{d}\log \Lambda_L}{\mathrm{d}\log A_k} = 0,$$

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = 1_{\{i>k\}}, \quad \frac{\mathrm{d}\log w_L}{\mathrm{d}\log A_k} = 1,$$

$$\frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} = 1, \quad \frac{\mathrm{d}\log w_L L}{\mathrm{d}\log A_k} = 1,$$

$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A_k} = 1_{\{i<k\}}, \quad \frac{\mathrm{d}\log L_i}{\mathrm{d}\log A_k} = 0.$$

In this economy, shocks propagate downstream for quantities and upstream for prices, and in both directions for sales. A shock $\mathrm{d}\log A_k > 0$ to the productivity of producer k generates positive comovement in sales, positive comovement in quantities for all for all downstream producers $i \leq k$. Positive comovement in output is entirely due to propagation via intermediate inputs.

Example: Horizontal Economy

In the horizontal economy, we have

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k} = \lambda_k,$$

$$\frac{\mathrm{d}\log \lambda_i}{\mathrm{d}\log A_k} = (\theta-1)(\delta_{ik}-\lambda_k), \quad \frac{\mathrm{d}\log \Lambda_L}{\mathrm{d}\log A_k} = 0,$$
$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -(\delta_{ik}-\lambda_k), \quad \frac{\mathrm{d}\log w_L}{\mathrm{d}\log A_k} = \lambda_k,$$
$$\frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} = \lambda_k + (\theta-1)(\delta_{ik}-\lambda_k), \quad \frac{\mathrm{d}\log w_L L}{\mathrm{d}\log A_k} = \lambda_k,$$
$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A_k} = \delta_{ik} + (\theta-1)(\delta_{ik}-\lambda_k), \quad \frac{\mathrm{d}\log L_i}{\mathrm{d}\log A_k} = (\theta-1)(\delta_{ik}-\lambda_k),$$

where δ_{ik} is a Kronecker delta. Consider a positive shock $\mathrm{d}\log A_k > 0$ to the productivity of producer k. Aggregate output increases by $\lambda_k \,\mathrm{d}\log A_k$. The price of producer k decreases by $(1-\lambda_k)\,\mathrm{d}\log A_k$ and the prices of the other producers increase by $\lambda_k\,\mathrm{d}\log A_k$.

Suppose that producers are substitutes so that $\theta > 1$. Then the share of producer k increases and those of the other producers decrease. Labor is reallocated towards producer k and away from the other producers. The output of producer k increases by more than $\mathrm{d}\log A_k$ and the output of the other producers decreases. The sales of producer k increase by more than $\lambda_k\,\mathrm{d}\log A_k$ and those of other producers increase by less than $\lambda_k\,\mathrm{d}\log A_k$ (they actually decrease if $\theta > 2$). These propagation patterns are reversed when producers are complements with $\theta < 1$, which provides a simple illustration of the notion of cost disease emphasized by Baumol (1967): a positive shock to the productivity of producer k causes the output of this producer to expand more than the rest of the economy and at the same time its relative share to decrease because of a strong adverse relative price effect while labor is reallocated to other producers.

In the substitutes case $\theta > 1$, it is possible to get positive comovement in sales but not in output. In the complements case $\theta < 1$ by contrast, we can get positive comovement in both sales and output. Positive comovement in output then comes about through labor reallocation in the presence of complementarities, a channel entirely different from the intermediate input channel at work in the vertical economy. We can never get positive comovement in labor since total labor is fixed.

This example emphasizes the importance of the reallocation of inputs for comovement. Indeed, if labor could not be reallocated, then the equations would coincide with the formulas for the Cobb-Douglas case with full labor reallocation, no matter what the true elasticity of substitution θ between producers is. In Appendix B.1, we consider cases with intermediate amounts of reallocation.

This example confirms a noticeable implication of Hulten's theorem: that impediments to factor reallocation are irrelevant for the effects on aggregate output. However we see here that such impediments matter for the propagation of shocks and comovement: they mitigate the channel of positive comovement through complementarities that we identified above.

Example: Capital-biased Technical Change in a Task-Based Model

Our results are well-suited to studying questions of structural change. For instance, a large and growing literature studies the causes of the recent decline in labor's share of income. This literature emphasizes the importance of the substitution patterns in the production technologies available to society.

One of the themes of this literature is that a single-good aggregate production function $Y = F(K, L)$ is not complex enough to capture the data. Karabarbounis and Neiman (2013), Oberfield and Raval (2014), Rognlie (2016) and Acemoglu and Restrepo (2018) use more complex production structures to draw out the implications of capital-augmenting shocks on labor's share of income.Proposition 3, and its generalizations in the subsequent sections, can transport the key intuitions of this literature into environments with realistic input-output linkages, frictions, disaggregated factor markets, and non-parametric production functions.

As an example, in this section, we consider an example along the lines of Acemoglu and Restrepo (2018), and show how their insights can be recovered in our framework. They argue that a possible consequence of capital-biased technical change and automation has been a simultaneous decline in both labor's share of income and the real wage. However, they point out that capital-biased shocks cannot generate a decline in the real wage with an aggregate production function, since the positive shock to capital will always increase labor's marginal product.

To capture their intuition, suppose that each producer, associated to a "task", produces from capital and labor according to

$$\frac{y_i}{\bar{y}_i} = \left(\omega_{iL} \left(\frac{\tilde{L}_i}{\bar{\tilde{L}}_i} \right)^{\frac{\theta_{KL-1}}{\theta_{KL}}} + \omega_{iK} \left(\frac{\tilde{K}_i}{\bar{\tilde{K}}_i} \right)^{\frac{\theta_{KL-1}}{\theta_{KL}}} \right)^{\frac{\theta_{KL}}{\theta_{KL}-1}}$$

with

$$\tilde{K}_i = \frac{A_{iK}}{\bar{A}_{iK}} K_i \qquad \text{and} \qquad \tilde{L}_i = \frac{A_{iL}}{\bar{A}_{iL}} L_i.$$

The consumer values the output of these producers according to a CES aggregator with elasticity of substitution θ.

We characterize the elasticity of wages and of the labor share in response to shocks, using Proposition 3.[16] We follow Acemoglu and Restrepo (2018) and assume that $\theta = 1$, so that

$$\frac{\mathrm{d}\log \Lambda_L}{\mathrm{d}\log A_{kK}} = \frac{-(\theta_{KL}-1)\lambda_k \omega_{kK}\frac{\omega_{kL}}{\Lambda_L}}{1+(\theta_{KL}-1)\sum_i \lambda_i \frac{\omega_{iL}}{\Lambda_L}\frac{\omega_{iK}}{\Lambda_K}},$$

and

$$\frac{d\log w_L}{d\log A_{kK}} = \lambda_k \omega_{kK}\frac{1+(\theta_{KL}-1)\sum_i \lambda_i \left(\frac{\omega_{iL}}{\Lambda_L}-\frac{\omega_{kL}}{\Lambda_L}\right)\frac{\omega_{iK}}{\Lambda_K}}{1+(\theta_{KL}-1)\sum_i \lambda_i \frac{\omega_{iL}}{\Lambda_L}\frac{\omega_{iK}}{\Lambda_K}}.$$

So, a capital-augmenting shock to task k decreases labor's share of income as long as labor and capital are substitutes $\theta_{KL} > 1$. However, the effect of such a shock on the real wage is ambiguous. If task k is more labor intensive than the average task, and capital and labor are highly substitutable, then the real wage falls. This is because as task k substitutes from labor to capital, labor is reallocated to other tasks who use labor less productively. This reallocation of labor reduces labor's marginal product, and hence the real wage.

It is easy to see that a single-good model could not generate these patterns: simply let k's sales share equal $\lambda_k = 1$. In that case, $\omega_{kL} = \Lambda_L$ and $\omega_{kK} = \Lambda_k$, so that $\mathrm{d}\log w_L/\,\mathrm{d}\log A_{kK} = \Lambda_k/\theta_{KL} > 0$ for all θ_{KL}.

Our analysis also shows that the assumption that the consumer have Cobb-Douglas preferences is important: to get a decline in the wage, we require consumption goods to be less substitutable than inputs into production. This is because as task k substitutes from labor to capital, it does not expand relative to other producers in the economy, and hence the labor moves from task k to other tasks. If consumption goods were highly substitutable, then this channel would break down. For instance, if instead consumption goods were perfect substitutes, then using (10), we would have

$$\lim_{\theta\to\infty}\frac{\mathrm{d}\log w_L}{\mathrm{d}\log A_{kK}} = \lambda_k\omega_{kK}\left(1+\frac{\left(\frac{\omega_{kL}}{\Lambda_L}-1\right)}{\frac{1}{\Lambda_K}Var_\lambda(\omega_{(L)})}\right) > 0$$

since $\omega_{kL} > \sum_i \lambda_i \omega_{iL} = \Lambda_L$.

[16] In general, when $\theta \neq 1$, Proposition (3) gives

$$\frac{\mathrm{d}\log \Lambda_L}{\mathrm{d}\log A_{kK}} = \frac{(\theta-1)\lambda_k\omega_{kK}(\frac{\omega_{kL}}{\Lambda_L}-1)-(\theta_{KL}-1)\lambda_k\omega_{kK}\frac{\omega_{kL}}{\Lambda_L}}{1+(\theta-1)\frac{1}{\Lambda_L\Lambda_K}Var_\lambda(\omega_{(L)})+(\theta_{KL}-1)\sum_i\lambda_i\frac{\omega_{iL}}{\Lambda_L}\frac{\omega_{iK}}{\Lambda_K}}. \tag{10}$$

Combine this with $\mathrm{d}\log Y/\,\mathrm{d}\log A_{kK} = \lambda_k\omega_{kK}$ to get an expression for $\mathrm{d}\log w_L = \mathrm{d}\log \Lambda_L + \mathrm{d}\log Y$.

Example: The Long-Plosser Model

Finally, we give the example of the Long and Plosser (1983) model. There is a representative consumer. Per-period preferences are logarithmic in a Cobb Douglas consumption aggregator c_t across the N sectors with $c_t = \Pi_{i=1}^{N} c_{it}^{b_i}$. Labor supply in all periods L_t is inelastic.[17] The discount factor is $\beta < 1$. Good i in period $t+1$ is produced using labor and the different goods from period t according to a constant-returns Cobb-Douglas aggregator $y_{i(t+1)} = A_{i(t+1)} L_{it}^{\omega_{iL}} \Pi_{j=1}^{N} X_{ijt}^{\omega_{ij}}$.

We imagine that the economy is at a steady state with $A_{it} = 1$ for all t. At $t = 0$, a one-time unanticipated shock hits the economy in the form of a new path for the productivities. The solution is available in closed-form and can be found in Long and Plosser (1983). Our purpose here is to show how this economy and its propagation mechanisms can be captured by our formalism in its inter-temporal interpretation.

The goods it are the different goods i in the different periods $t \geq 1$. The factors are labor L_t in the different periods $t \geq 0$, and the goods $i0$ in the initial period. The input-output matrix is given by $\Omega_{(it)L_{t-1}} = \omega_{iL}$, $\Omega_{(it)(j(t-1))} = \omega_{ij}$, and all the other entries are 0. In addition, we have $\varrho = 1$ and $\lambda_t = (1-\beta) b' (I - \beta\omega)^{-1}$ where λ_t is the vector $[\lambda_{1t}, \cdots, \lambda_{Nt}]$. Define ω to be the $(N+1) \times (N+1)$ matrix given by ω_{ij}, with labor treated as the $(N+1)$th element. We get

$$\frac{d \log Y}{d \log A_{kt}} = \lambda_{kt}$$

$$\frac{d \log \lambda_{is}}{d \log A_{kt}} = 0, \quad \frac{d \log \Lambda_{L_s}}{d \log A_{kt}} = 0,$$

$$\frac{d \log p_{is}}{d \log A_{kt}} = -1_{\{t \leq s\}} \left(\omega^{s-t}\right)_{ik} + \lambda_{kt}, \quad \frac{d \log w_{L_s}}{d \log A_{kt}} = \lambda_{kt},$$

$$\frac{d \log (p_{is} y_{is})}{d \log A_t} = \lambda_{kt}, \quad \frac{d \log (w_{L_s} L_s)}{d \log A_t} = \lambda_{kt},$$

$$\frac{d \log y_{is}}{d \log A_{kt}} = 1_{\{t \leq s\}} \left(\omega^{s-t}\right)_{ik}, \quad \frac{d \log L_s}{d \log A_{kt}} = 0.$$

These equations demonstrate how positive comovement through intermediate inputs arises in the Long-Plosser model. The model is particularly tractable because all the elasticities of substitution are unitary, so that all shares are invariant to all the shocks.

When the economy has only one sector, then the Long-Plosser model can be reinterpreted as the Brock-Mirman specification of the neoclassical growth model with log balanced-growth preferences, Cobb-Douglas production from labor and capital, and full depreciation of capi-

[17] In the original Long-Plosser model, labor supply is elastic with log balanced-growth per-period preferences $\log c_t - \nu(1 + \zeta_L^{-1}) L_t^{(1+\zeta_L^{-1})}$. Equilibrium labor supply is constant and so the model is isomorphic to the one with exogenous labor supply that we study here.

tal.[18] There positive comovement arises through capital accumulation because capital is an intermediate input. The Brock-Mirman specification is also extremely tractable, lending itself to a closed-form solution, and for the same reason.

4 Heterogeneous Agents

In this section, we characterize how shocks propagate in economies with heterogeneous consumers. For simplicity, we focus on the special case with inelastically supplied factors and no distortions. The more general case is treated in the appendix.

4.1 Comparative Statics

We start by characterizing the elasticities to the different productivities of the sales shares or Domar weights.

Proposition 5. *(Aggregate Output and Shares) The elasticities of aggregate output to the different productivities are given by*

$$\frac{d\log Y}{d\log A_k} = \lambda_k. \tag{11}$$

The elasticities of the sales shares or Domar weights of i is then given by

$$\frac{d\log \lambda_i}{d\log A_k} = \sum_j \frac{\lambda_j}{\lambda_i}(\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(i)}) - \sum_g \sum_j \frac{\lambda_j}{\lambda_i}(\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(i)})\frac{d\log \Lambda_g}{d\log A_k}$$
$$+ \frac{1}{\lambda_i}\sum_g \sum_c (\lambda_i^c - \lambda_i)\Phi_{cg}\Lambda_g \, d\log \Lambda_g. \tag{12}$$

The elasticities of the factor shares solve the following system of linear equations

$$\frac{d\log \Lambda_f}{d\log A_k} = \sum_j \frac{\lambda_j}{\Lambda_f}(\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(f)}) - \sum_g \sum_j \frac{\lambda_j}{\Lambda_f}(\theta_j - 1)Cov_{\Omega^{(j)}} v(\Psi_{(g)}, \Psi_{(f)})\frac{d\log \Lambda_g}{d\log A_k}$$
$$+ \frac{1}{\Lambda_f}\sum_g \sum_c (\Lambda_f^c - \Lambda_f)\Phi_{cg}\Lambda_g \frac{d\log \Lambda_g}{d\log A_k}. \tag{13}$$

The overall effect on real GDP, as measured by the Divisia index, are exactly the same as before, following Hulten's theorem. Relative to (5), with heterogeneous consumers, the factor

[18]There is a small difference: inputs are imposed to be purchased one period in advance in the Long-Plosser version of the model while they can in principle be freely adjusted in the Brock-Mirman model. However, because labor is fixed and capital must be accumulated, these distinctions are irrelevant for the equilibrium allocation.

share equations solve

$$\frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k}=\Gamma\frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k}+\Theta\frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k}+\delta_{(k)},$$

where Γ and $\delta_{(k)}$ are exactly as they were in the representative consumer economy:

$$\Gamma_{fg}=-\sum_j(\theta_j-1)\frac{\lambda_j}{\Lambda_f}Cov_{\Omega^{(j)}}\left(\Psi_{(g)},\Psi_{(f)}\right),$$

and

$$\delta_{fk}=\sum_j(\theta_j-1)\frac{\lambda_j}{\Lambda_f}Cov_{\Omega^{(j)}}\left(\Psi_{(k)},\Psi_{(f)}\right).$$

The new term Θ is an $F\times F$ matrix whose fgth element is

$$\Theta_{fg}=\frac{1}{\Lambda_f}\sum_{c\in C}(\Lambda_f^c-\Lambda_f)\Phi_{cg}\Lambda_g.$$

This captures how changes in the price of the factor g change the distribution of income across consumers, and how this change in the distribution of income, in turn, affects demand for the factor f (since different households are differently exposed, directly and indirectly, to the different factors). If for some good l all consumers are symmetrically exposed $\lambda_l^c=\lambda_l$, then the changes in the distribution of income will have no effect and the final term will disappear.

Intuitively, in the representative agent model, factor prices change in response to substitution across inputs, captured by $\delta_{(k)}$ and Γ. In the hetereogenous agent model, relative factor prices can also change in response to changes in the distribution of income Θ. Furthermore, whereas due to the symmetry of Γ and $\delta_{(k)}$, the propagation of shocks was symmetric for the efficient, representative agent model, the addition of the non-symmetric income effects Θ breaks symmetry. It is easy to see that if there exists a representative consumer with homothetic preferences, the income distribution effect disappears.

Prices and Quantities

Given Proposition 5 characterizing the output and share propagation equations, we can easily characterize comparative statics for how various prices, quantities, and input choices change in response to shocks along the exact same lines as in Section 3. In fact, the corresponding equations are exactly the same, and the only difference is that different output and share propagation equations must be plugged in. We do not spell all of this out for the sake of brevity.

Asymmetric Propagation

As alluded to earlier, models with heterogeneous consumers have asymmetric propagation in sales shares and in sales. This comes about from the non-homotheticities in final demand generated via the income distribution channel. To see how this can happen, consider two goods i and j. Suppose that $\lambda_i^c = \lambda_i$ for every c, but that $\lambda_j^c \neq \lambda_j$ for some c such that c's preferences are different from average preferences and c's factor ownership is different from average factor ownership. Then, the income distribution term disappears for the propagation of a shock to j on i in equation (12), but is present for the effect of a shock to i on j, thereby generating an asymmetry in propagation. We fully work out a concrete example to demonstrate this logic in Appendix C.

Equation (12) also makes clear that symmetry will return if each consumer's share of aggregate income is fixed. In this case, there exists a representative consumer with stable preferences: namely, a geometric social welfare function where the Pareto-weights are equal to each consumer's fixed share of aggregate income would implement the decentralized equilibrium. Since this social welfare function is homothetic, that immediately implies that propagation must be symmetric, since Hulten's theorem would then tie the derivatives of the welfare function to the sales shares.

4.2 Simple Illustrative Example

To illustrate the implications of abandoning the representative agent assumption, we take the nested-CES economy in Figure 1c considered in Section 3 and add multiple consumers. This economy is depicted in Figure 2 and has two factors of production, two consumers, and three goods which use the factors in different ways.

Skill-biased Technology and the Skill Premium

To make the exercise more concrete, we can think of the first factor as high-skill labor H and the second factor as low-skill labor L. Agent 1 is high-skilled and agent 2 is low-skilled. The three producers are services S, manufacturing M, and agriculture A. Services are relatively more intensive in H, agriculture is relatively more intensive in L, and manufacturing uses H and L evenly. Finally, high-skilled agents consume relatively more services and low-skilled agents consumer relatively less services.[19] Finally, for simplicity, consumption of both households is Cobb-Douglas, but the elasticity of substitution between H and L is $\sigma > 1$ for all producers.

Consider a positive productivity shock to high-skill labor $d\log A_H > 0$. Then, by Proposition 5, high-skill labor's share of income solves the following fixed-point equation:

[19] These patterns are broadly consistent with the data, see for example Jaravel (2016).

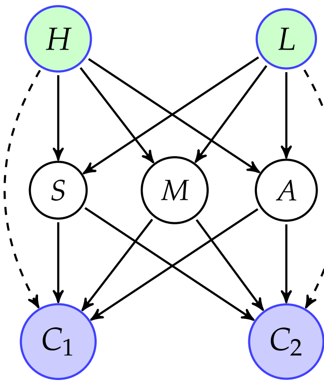


Figure 2: The solid arrows represent the flow of goods. The dashed arrows represent the flow of wages to households.

$$\mathrm{d}\,\Lambda_H = (\sigma-1)\sum_i \lambda_i \omega_{iH}\omega_{iL}\,\mathrm{d}\log A_H - \frac{(\sigma-1)}{\Lambda_L\Lambda_H}\sum_i \lambda_i \omega_{iH}\omega_{iL}\,\mathrm{d}\,\Lambda_H + \sum_i \omega_{iH}(b_{1i}-b_{2i})\,\mathrm{d}\,\Lambda_H, \tag{14}$$

where i indexes A, M, and S.

The overall change in equilibrium is the fixed point of the system depicted graphically in Figure 3, which it is tempting to call a "Quesnaysian" cross and to associate with a "Quesnaysian" multiplier given by the denominator of the solution

$$\mathrm{d}\,\Lambda_H = \frac{(\sigma-1)\sum_i \lambda_i \omega_{iH}\omega_{iL}\,\mathrm{d}\log A_H}{1+\frac{(\sigma-1)}{\Lambda_L\Lambda_H}\sum_i \lambda_i \omega_{iH}\omega_{iL} + \sum_i \omega_{iH}(b_{1i}-b_{2i})}.$$

The first term on the right-hand side of (14), the positive intercept, is the effect of substitution towards H and away from L holding fixed the factor prices. This is the partial equilibrium effect. In equilibrium, as producers substitute towards H, the price of H rises relative to L, and general equilibrium forces attenuate the amount of substitution. This effect is the second term, which is negative. However, the increase in high-skilled labor's share of income redistributes income towards high-skilled agents, who's consumption is more H intensive. This final "income distribution" effect, which is also a general equilibrium force, is positive and acts in the opposite direction because skilled labor exposure and bias in consumption are positively associated. The first two terms are present both in representative agent and heterogeneous agent models, but the third term appears only in the latter.

To see the effects of the shock to high-skill labor on the skill premium, we use $\mathrm{d}\log\Lambda_L = -(\Lambda_H)/(\Lambda_L)\,\mathrm{d}\log\Lambda_H$ to get

$$\mathrm{d}\log\left(\frac{w_H}{w_L}\right) = \mathrm{d}\log\Lambda_H - \mathrm{d}\log\Lambda_L = \frac{1}{\Lambda_L}\,\mathrm{d}\log\Lambda_H.$$

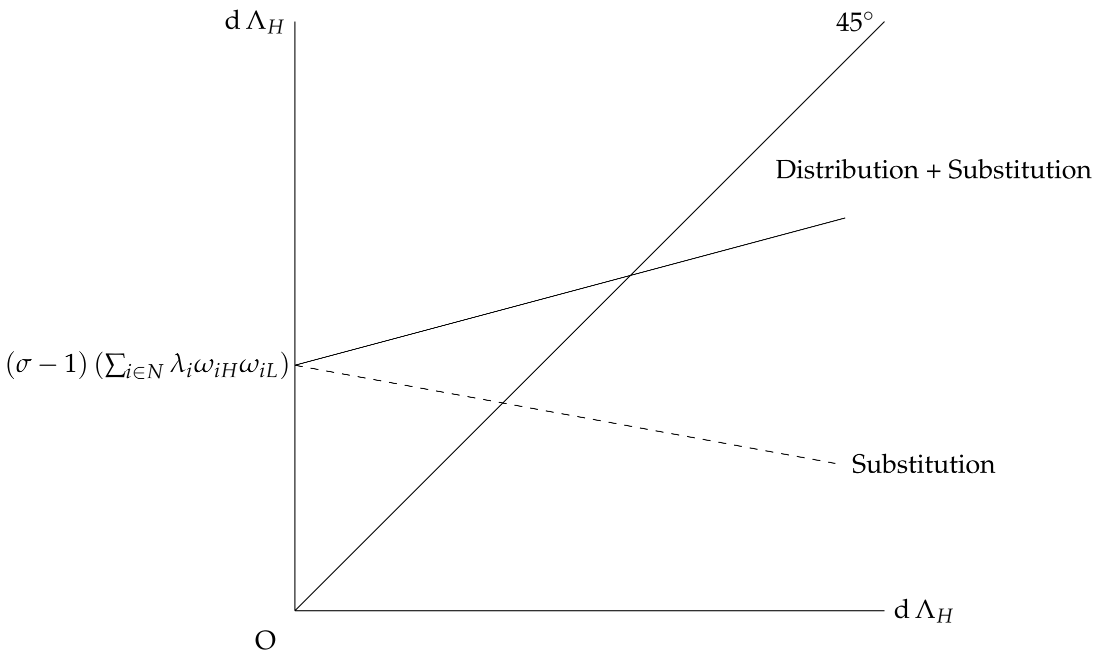


Figure 3: The "Quesnaysian Cross": the substitution line has slope $\frac{(\sigma-1)}{\Lambda_L \Lambda_H} \sum_i \lambda_i \omega_{iH} \omega_{iL}$; and the distribution + substitution line has slope $\frac{(\sigma-1)}{\Lambda_L \Lambda_H} \sum_i \lambda_i \omega_{iH} \omega_{iL} + \sum_i \omega_{iH}(b_{1i} - b_{2i})$.

Whether general equilibrium forces amplify or attenuate the partial equilibrium effect depends on whether the line in Figure 3 is upward sloping or not. In the representative agent model, general equilibrium always attenuates the partial equilibrium effect. However, with more consumers, if the income-distribution effect is strong enough, then general equilibrium can amplify the impact of the shock on the skill premium.

In Section 6.2, we connect the Quesnaysian cross to the Keynesian cross, and use it to analyze the dependence of fiscal multipliers on the composition of government spending, in a dynamic setting with a production network and heterogeneous agents.

5 Elastic Factor Supplies

In this section, we characterize how shocks propagate in economies with elastic factor supplies. For simplicity, we focus on the special case with a representative agent and no distortions. The more general case is treated in the appendix.

To model elastic factor supplies, we focus for simplicity on the case where cross-factor-price elasticities of factor supplies are zero. The general case can be derived along the same lines. We can therefore write the supply of factor f as $L_f = G_f(w_f, Y)$, where w_f is the price of the factor and Y is aggregate output.

Let $\zeta_f = \partial \log G_f / \partial \log w_f$ be the elasticity of the supply of factor f to its real wage, and $\gamma_f = -\partial \log G_f / \partial \log Y$ be its income elasticity. The setup of Section 3 can be obtained as the special case where $\gamma_f = \zeta_f = 0$ for all f.

Our derivations apply whenever factor supplies can be modeled with such factor supply functions, whether or not they arise from the maximization of utility functions. This generalization will prove useful when we apply the model to a setup with labor and capital to derive comparative statics on the steady-state of a Ramsey model.

5.1 Comparative Statics

In this section, we characterize the elasticities to the different productivities of aggregate output, shares, sales, prices, and quantities.

Aggregate Output and Shares.

Hulten's theorem fails with elastic factor supplies and this complicates the analysis. The following proposition, taken from Baqaee and Farhi (2017b), provides a joint characterization of the elasticities of aggregate output and factor shares to the different productivity shocks.

Proposition 6. *(Aggregate Output and Shares with Elastic Factors) In economies with elastic factor supplies, the elasticities of aggregate output and and factor shares to the different productivities are given by the solution of the following system of equations*

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k} = \varrho\left[\lambda_k - \sum_g \frac{1}{1+\zeta_g}\Lambda_g \frac{\mathrm{d}\log \Lambda_g}{\mathrm{d}\log A_k}\right], \tag{15}$$

$$\begin{aligned}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_k} &= \sum_j (\theta_j - 1)\frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(f)}) \\ &- \sum_j (\theta_j - 1)\frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}}(\sum_g \Psi_{(g)} \frac{1}{1+\zeta_g}\frac{\mathrm{d}\log \Lambda_g}{\mathrm{d}\log A_k} + \sum_g \Psi_{(g)}\frac{\gamma_g - \zeta_g}{1+\zeta_g}\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \Psi_{(f)}),\end{aligned} \tag{16}$$

where $\varrho \equiv 1/(\sum_f \Lambda_f \frac{1+\gamma_f}{1+\zeta_f})$. The elasticities of the shares of the other producers to the different productivities are given by

$$\begin{aligned}\frac{\mathrm{d}\log \lambda_i}{\mathrm{d}\log A_k} &= \sum_j (\theta_j - 1)\frac{\lambda_j}{\lambda_i} Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)}, \Psi_{(i)}) \\ &- \sum_j (\theta_j - 1)\frac{\lambda_j}{\lambda_i} Cov_{\Omega^{(j)}}(\sum_g \Psi_{(g)} \frac{1}{1+\zeta_g}\frac{\mathrm{d}\log \Lambda_g}{\mathrm{d}\log A_k} + \sum_g \Psi_{(g)}\frac{\gamma_g - \zeta_g}{1+\zeta_g}\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \Psi_{(i)}).\end{aligned} \tag{17}$$

In this case, the share propagation equations need to be solved jointly with the aggregation equation for aggregate output. The share propagation matrix and the share impulse matrix must be adjusted accordingly.

With only one factor, which we denote by L, the results take a simpler form. We then get $\mathrm{d}\log Y/\,\mathrm{d}\log A_k = \varrho\lambda_k$ with $\varrho = (1+\zeta_L)/(1+\gamma_L)$. The parameter ϱ encodes the relative strength of the income and substitution effects on factor supply $\mathrm{d}\log L = \zeta_L \,\mathrm{d}\log w - \gamma_L \,\mathrm{d}\log Y$ associated with a productivity shock $\mathrm{d}\log A_k > 0$: on the one hand, the shock increases output and so reduces factor supply via an income effect, the strength of which depends on γ_L; on the other hand, it increases the real wage of this factor and so increases factor supply via a substitution effect, the strength of which depends on ζ_L. Only in the special case where the balanced-growth condition is verified so that $\zeta_L = \gamma_L$ and $\varrho = 1$ do we recover Hulten's theorem $\mathrm{d}\log Y/\,\mathrm{d}\log A_k = \lambda_k$.

When there are several factors, then we need not only to average these effects over factors but also to keep track of changes in relative factor prices, and the changes in relative factor prices in turn depend on each other and on changes in output. We will show below that the quantities $(1+\zeta_g)^{-1}\,\mathrm{d}\log\Lambda_g/\,\mathrm{d}\log A_k + (\gamma_g - \zeta_g)(1+\zeta_g)^{-1}\,\mathrm{d}\log Y/\,\mathrm{d}\log A_k$ are simply equal to $\mathrm{d}\log w_g/\,\mathrm{d}\log A_k - \mathrm{d}\log Y/\,\mathrm{d}\log A_k$ and therefore encode relative changes in prices across factors, exactly as in the case with inelastic factor supply examined in Proposition 3. This is why we need to jointly solve for aggregate output elasticities and factor share elasticities to the different productivity shocks in equations (15) and (16). These elasticities are also needed to compute the elasticities to the different productivities of the shares of the producers in equation (17). We only recover Hulten's theorem when $\zeta_g = \gamma_g$ is independent of g.

Prices

We now characterize the elasticities of prices to the different productivities.

Proposition 7. *(Prices) The elasticities of the prices of the different producers to the different productivities are given by*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} = \frac{1}{1+\zeta_f}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_k} + \frac{1+\gamma_f}{1+\zeta_f}\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \tag{18}$$

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -\Psi_{ik} + \sum_g \Psi_{ig}\frac{\mathrm{d}\log w_g}{\mathrm{d}\log A_k}, \tag{19}$$

where $\mathrm{d}\log Y/\,\mathrm{d}\log A_k$ *and* $\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_k$ *are given in Proposition 3.*

The only difference with the case with inelastic factors covered in Proposition 4 is in equation (18) which describes the general equilibrium elasticity of the price of factor f to the productivity of producer k. The mapping of the elasticities of the factor share $\mathrm{d}\log\Lambda_f/\,\mathrm{d}\log A_k$ and of aggregate output $\mathrm{d}\log Y/\,\mathrm{d}\log A_k$ to these elasticities now involves the substitution and income elasticities ζ_f and γ_f, which are necessary to decompose the elasticity of the factor share into the elasticity of the factor price $d \log w_f / d \log A_k$ and the elasticity of the factor quantity $d \log L_f / d \log A_k$. The latter will be characterized below.

Sales, and Quantities

Armed with Propositions 6 and 7, we can characterize the elasticities of the sales and output quantities of the different producers to the different productivities, along the same lines as in Corollary 1. In fact, equations (8) and (9) in Corollary 1 still apply. The only difference is that now $d \log \lambda_i / d \log A_k$, $d \log Y / d \log A_k$, and $d \log p_i / d \log A_k$ must be taken from Propositions 6 and 7 instead of Propositions 3 and 4.

Input Shares, Input Expenditures, and Input Quantities

Using Propositions 6 and 7, it is easy to derive the elasticities of input shares, expenditures and quantities of the different producers to the different productivities. As in the case of inelastic factor supplies, these results can actually easily be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l. The results can be found in the appendix.

Asymmetric Propagation

With elastic factors, propagation is still symmetric in sales as long factor supplies satisfy balanced growth with $\gamma_f = \zeta_f$ for all f. We actually saw earlier that this result applies more generally to balanced-growth preferences, even if factor supplies are not separable.

Below, we provide a simple example of asymmetric propagation with elastic factor supply in the context of our example on structural transformation and capital deepening with fully elastic capital and inelastic labor, a configuration of factor supplies which does not satisfy balanced growth. Asymmetric propagation between producers i and j obtains if the two producers do not have the same capital intensity.

5.2 Simple Illustrative Examples

In this section, we provide some simple examples to illustrate our formulas with elastic factors.

Example: Vertical Economy with Elastic Labor

We start with the case of a horizontal economy with a single factor, which we call labor. Let $L = G_L(w_L, Y)$ be labor supply, where w_L is the wage and Y is aggregate output. Let $\zeta_L = \partial \log G_L / \partial \log w_L$ be the elasticity of labor supply to its real wage, and $\gamma_L = -\partial \log G_L / \partial \log Y$ be its income elasticity, and define $\varrho = (1 + \zeta_L)/(1 + \gamma_L)$.

In the vertical economy with elastic labor, we have

$$\frac{d\log Y}{d\log A_k} = \varrho,$$

$$\frac{d\log \lambda_i}{d\log A_k} = 0, \quad \frac{d\log \Lambda_L}{d\log A_k} = 0,$$

$$\frac{d\log p_i}{d\log A_k} = 1_{\{i>k\}}, \quad \frac{d\log w_L}{d\log A_k} = 1,$$

$$\frac{d\log p_i y_i}{d\log A_k} = \varrho, \quad \frac{d\log w_L L}{d\log A_k} = \varrho,$$

$$\frac{d\log y_i}{d\log A_k} = 1_{\{i<k\}} + (\varrho - 1), \quad \frac{d\log L}{d\log A_k} = \varrho - 1.$$

When labor supply is inelastic so that $\zeta_L = \gamma_L = 0$, we get $\varrho = 1$ and we recover the formulas in Section 3.3. These formulas also apply more generally when preferences satisfy the balanced-growth condition $\zeta_L = \gamma_L = 1$, which also implies $\varrho = 1$, since then labor supply is invariant to productivity shocks.

When $\varrho > 1$, labor supply $d\log L = (\varrho - 1)\, d\log A_k$ increases in response to a productivity shock $d\log A_k > 0$ since the substitution effect outweighs the income effect. Compared to the case with inelastic labor supply, this increase in labor supply contributes to increasing labor employment, output, sales, and wage payments in all sectors proportionally. Elastic factor supply therefore represents a new force for positive comovement, which is different from the other two forces that we have already identified, namely intermediate inputs and complementarities. Unlike the other two forces, it can generate positive comovement not only in output and sales, but also in labor.

Example: Horizontal Economy with Elastic Labor

We continue with the case of a horizontal economy with a single factor called labor. We define G_L, ζ_L, γ_L and ϱ exactly as in the vertical economy example above.

In the horizontal economy with elastic labor, we have

$$\frac{d\log Y}{d\log A_k} = \varrho\lambda_k,$$

$$\frac{d\log \lambda_i}{d\log A_k} = (\theta - 1)(\delta_{ik} - \lambda_k), \quad \frac{d\log \Lambda_L}{d\log A_k} = 0,$$

$$\frac{d\log p_i}{d\log A_k} = -(\delta_{ik} - \lambda_k), \quad \frac{d\log w_L}{d\log A_k} = \lambda_k,$$

$$\frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} = \varrho\lambda_k + (\theta-1)(\delta_{ik}-\lambda_k), \quad \frac{\mathrm{d}\log w_L L}{\mathrm{d}\log A_k} = \varrho\lambda_k,$$

$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A_k} = \delta_{ik} + (\theta-1)(\delta_{ik}-\lambda_k) + (\varrho-1)\lambda_k,$$

$$\frac{\mathrm{d}\log L}{\mathrm{d}\log A_k} = (\varrho-1)\lambda_k, \quad \frac{\mathrm{d}\log L_i}{\mathrm{d}\log A_k} = (\theta-1)(\delta_{ik}-\lambda_k) + (\varrho-1)\lambda_k,$$

The comments that we made about the vertical economy also apply here. When labor supply is inelastic so that $\zeta_L = \gamma_L = 0$, we get $\varrho = 1$ and we recover the formulas in Section 3.3. These formulas also apply more generally when preferences satisfy the balanced-growth condition $\zeta_L = \gamma_L = 1$, which also implies $\varrho = 1$, since then labor supply is invariant to productivity shocks. When $\varrho > 1$, labor supply $\mathrm{d}\log L = (\varrho-1)\lambda_k \,\mathrm{d}\log A_k$ increases in response to a productivity shock $\mathrm{d}\log A_k > 0$ since the substitution effect outweighs the income effect, which increases labor, employment, output, sales, and wage payments in all sectors proportionally. Elastic factor supply contributes to positive comovement, not only in output and sales, but also in labor.

Example: Structural Transformation and Capital Deepening

We can also use our formulae to think about the long-run steady-state of a Ramsey model with two factors: capital and labor. Suppose capital is perfectly elastically supplied at some price $w_K = r_K$ reflecting its user cost. The other factor is labor and is fully inelastic at some quantity L. We can use this model to capture some of the insights in the literature on structural transformation.

Each producer produces from capital and labor according to a CES production function

$$\frac{y_i}{\bar{y}_i} = \frac{A_i}{\bar{A}_i}\left(\omega_{iL}\left(\frac{L_i}{\bar{L}_i}\right)^{\frac{\theta_{KL}-1}{\theta_{KL}}} + \omega_{iK}\left(\frac{K_i}{\bar{K}_i}\right)^{\frac{\theta_{KL}-1}{\theta_{KL}}}\right)^{\frac{\theta_{KL}}{\theta_{KL}-1}},$$

with $\omega_{iK} + \omega_{iL} = 1$. At the initial allocation, the labor and capital shares of producer i are given by ω_{iL} and ω_{iK}, the aggregate labor and capital shares are given by $\Lambda_L = \sum_i \lambda_i \omega_{iL}$ and $\Lambda_K = \sum_i \lambda_i \omega_{iK}$, and in addition we have $\varrho = 1/\Lambda_L$.

In Appendix D, we fully characterize the patterns of comovement in this model by applying Proposition 6. Here we only use the general version of the model to emphasize how this example can generate asymmetric propagation. We have

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}\log A_j} = -(\theta-1)\lambda_i\lambda_j + (\theta-1)\frac{\lambda_i\lambda_j}{\Lambda_L}(\omega_{iK}-\Lambda_K),$$

Assuming $\theta \neq 1$, this implies that $\mathrm{d}\lambda_i/\mathrm{d}\log A_j \neq \mathrm{d}\lambda_j/\mathrm{d}\log A_i$ as long as $\omega_{iK} \neq \omega_{jK}$ so that, we get asymmetric propagation if and only if i and j do not have the same capital intensity.

We now make the analysis more concrete. We take inspiration from Acemoglu and Guerrieri (2008), and specialize the model to Cobb-Douglas production functions within sectors with $\theta_{KL} = 1$, and complementarities in consumption with $\theta < 1$. We consider an aggregate productivity Hicks-neutral productivity shock with $\mathrm{d}\log A_k = \mathrm{d}\log A$ for all k. We find

$$\frac{\mathrm{d}\log \Lambda_L}{\mathrm{d}\log A} = -(\theta - 1)\frac{Var_\lambda(\omega_{(L)})}{\Lambda_L^2},$$

$$\frac{\mathrm{d}\log \lambda_i}{\mathrm{d}\log A} = (\theta - 1)\frac{1}{\Lambda_L}(\omega_{iK} - \Lambda_K),$$

and

$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A} = 1 + (\theta - 1)\frac{1}{\Lambda_L}(\omega_{iK} - \Lambda_K) + \omega_{iK}\frac{1}{\Lambda_L} - \frac{\mathrm{d}\log \Lambda_L}{\mathrm{d}\log A}.$$

In response to an aggregate productivity shock, sectors that are more capital intensive expand relatively more than the rest of the economy, but their value expands relatively less because of complementarities in consumption, so that their relative share decreases and factors are reallocated to other producers—a form of cost disease à la Baumol (1967) originating in differences in capital intensities rather than in productivities. As a result, the labor share increases and the capital share decreases. The plausibility of these empirical patterns is documented in Acemoglu and Guerrieri (2008).

6 Distortions

In this section, we characterize how shocks propagate in economies with distortions. For simplicity, we focus on the special case with a representative agent and inelastically supplied factors. The more general case is treated in the appendix.

We allow for an arbitrary set of wedges modeled (without loss of generality) as markups. These wedges do not dissipate resources but result instead in the misallocation of resources. They could arise from distortions in factor or intermediate input markets, credit constraints, as well as markups. We start with the special case where factors are inelastically supplied, and then generalize to the case with elastic factors. We end the section by looking at some examples. As before, we stick to a representative agent model for clarity.

6.1 Comparative Statics

In this section, we derive the counterparts of the results in Sections 3 for inefficient economies.

Aggregate Output and Shares

In Section 5, we showed that Hulten's theorem fails when factors supplies are not inelastic. Hulten's theorem also fails when factor supplies are inelastic but when the economy is inefficient. The following proposition, taken from Baqaee and Farhi (2017b), provides a joint characterization of the elasticities of aggregate output and factor shares to the different productivity shocks.

Proposition 8. *(Aggregate Output and Shares in Inefficient Economies) In inefficient economies with markups/wedges, the elasticities of aggregate output to the different productivities are given by*

$$\frac{d\log Y}{d\log A_k} = \tilde{\lambda}_k - \sum_g \tilde{\Lambda}_g \frac{d\log \Lambda_g}{d\log A_k}, \tag{20}$$

where the elasticities of the factor shares to the different productivities are given by the solution of the following system of equations

$$\frac{d\log \Lambda_f}{d\log A_k} = \sum_j (\theta_j - 1)\frac{\mu_j^{-1}\lambda_j}{\Lambda_f} Cov_{\tilde{\Omega}^{(j)}}\left(\tilde{\Psi}_{(k)} - \sum_g \tilde{\Psi}_{(g)} \frac{d\log \Lambda_g}{d\log A_k}, \Psi_{(f)}\right). \tag{21}$$

The elasticities of the shares of the other producers to the different productivities are given by

$$\frac{d\log \lambda_i}{d\log A_k} = \sum_j (\theta_j - 1)\frac{\mu_j^{-1}\lambda_j}{\lambda_i} Cov_{\tilde{\Omega}^{(j)}}\left(\tilde{\Psi}_{(k)} - \sum_g \tilde{\Psi}_{(g)} \frac{d\log \Lambda_g}{d\log A_k}, \Psi_{(i)}\right). \tag{22}$$

The intuition for the difference between these results and those that arise in efficient economies derived in Proposition 3 is as follows. In Baqaee and Farhi (2017b), we show that the effects of productivity shocks on output can be decomposed into pure changes in technology and changes in allocative efficiency, and that this decomposition corresponds to the two terms on the right-hand side of the output aggregation equation (20).

In efficient economies, $\tilde{\lambda}_k = \lambda_k$, $\tilde{\Lambda}_g = \Lambda_g$, and so $\sum_g \tilde{\Lambda}_g \, d\log \Lambda_g / \, d\log A_k = 0$: changes in technology are captured by revenue-based Domar weights and changes in allocative efficiency are zero so that we recover Hulten's theorem as stated in equation (1). In inefficient economies with markups/wedges, allocative efficiency improves when the weighted average of changes in factor shares decreases. It can only happen if resources are re-allocated to parts of the economies that are more distorted downwards because of higher downstream markups/wedges. This improves allocative efficiency because these parts of the economy were too small to begin with.

We now turn to the share propagation equations (21) and (22). In inefficient economies, changes in the share of i or f to a productivity shock to producer k, must now combine exposures to k in costs as captured by $\tilde{\Psi}_k$ and exposure to i and f in revenues as captured by Ψ_i and Ψ_f. In efficient economies, $\tilde{\Psi}_k = \Psi_k$ and $\mu_j = 1$ and we recover equations (2) and (3). The share propagation matrix and the share impulse matrix must be adjusted accordingly.

Prices

We now characterize the elasticities of prices to the different productivities.

Proposition 9. *(Prices) In inefficient economies with markups/wedges, the elasticities of the sales, prices, and output quantities of the different producers to the different productivities are given by*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \tag{23}$$

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -\tilde{\Psi}_{ik} + \sum_g \tilde{\Psi}_{ig} \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \tag{24}$$

where $\mathrm{d}\log \Lambda_f / \mathrm{d}\log A_k$ *and* $\mathrm{d}\log Y / \mathrm{d}\log A_k$ *are given in Proposition 8.*

The only difference with the case of efficient economies with no markups/wedges covered in Proposition 4 is in equation (24) which describes the general equilibrium elasticities of the prices of the different producers to the different productivities, and which now involves the cost-based Leontief-inverse matrix instead of the revenue-based input-output matrix. This is because in the presence of markups/wedges, it is the cost-based Leontief inverse matrix which encodes the partial-equilibrium elasticities of the prices of the different producers to the prices of the other producers.

Sales, and Quantities

Armed with Propositions 8 and 9, we can characterize the elasticities of the sales and output quantities of the different producers to the different productivities, along the same lines as in Corollary 1. In fact, equations (8) and (9) in Corollary 1 still apply. The only difference is that now $\mathrm{d}\log \lambda_i / \mathrm{d}\log A_k$, $\mathrm{d}\log Y / \mathrm{d}\log A_k$, and $\mathrm{d}\log p_i / \mathrm{d}\log A_k$ must be taken from Propositions 8 and 9 instead of Propositions 3 and 4.

Input Shares, Input Expenditures, and Input Quantities

Using Propositions 8 and 9, it is easy to derive the elasticities of input shares, expenditures and quantities of the different producers to the different productivities. As above, these results can actually easily be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l. They can be found in Appendix E.

Shocks to Markups/Wedges

In Appendix D, we give a detailed characterization of the propagation of shocks to markups and wedges with elastic and inelastic factors. In the same appendix, we apply these formulas to study the long-run effects of corporate-taxation in the presence of markup heterogeneity.

Asymmetric Propagation

Even with a representative agent, the propagation of shocks is asymmetric for sales shares and for sales in inefficient economies. The reason is that wedges sever connection between the elasticities of the output function and the sales shares. Symptoms of asymmetry can be seen directly in Proposition 8, since the input-output covariance operator, and the factor share propagation matrix Γ, is no longer symmetric due to the involvement of both Ψ and $\tilde{\Psi}$.

6.2 Simple Illustrative Examples

In this section, we illustrate these results with some examples. We work through the vertical and horizontal economies with inelastic factor supply, where we assume that the distortions take the form of markups. Then we work through an example with heterogeneous agents, elastic labor supply, nominal rigidities, and borrowing constraints to analyze the dependence of fiscal multipliers on the composition of government spending. In the appendix, we also consider an application to the incidence of corporate income taxes in a model with markups, elastic capital, and inelastic labor

Vertical Economy with Inelastic Labor

In the vertical economy, the formulas derived in Section 3.3 are unchanged. The difference is that in the calculations that lead to these formula, we must take into account the difference between cost-based and revenue-based input-output objects. For example we have $\tilde{\lambda}_k = 1$ and $\tilde{\Lambda}_L = 1$ but $\lambda_k = \Pi_{i=1}^{k-1}\mu_i^{-1}$ and $\Lambda_L = \Pi_{i=1}^{N}\mu_i^{-1}$.[20]

Horizontal Economy with Inelastic Labor

In the horizontal economy, we have

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k} = \lambda_k + (\theta-1)\lambda_k(1 - \frac{\mu_k^{-1}}{\sum_j \lambda_j \mu_j^{-1}}),$$

[20]Similarly, we have $\tilde{\Omega}_{ij} = \delta_{j(i+1)}$ and $\tilde{\Omega}_{iL} = \delta_{Ni}$ but $\tilde{\Omega}_{ij} = \mu_i^{-1}\delta_{j(i+1)}$ and $\tilde{\Omega}_{iL} = \mu_i^{-1}\delta_{Ni}$. And likewise, we have $\tilde{\Psi}_{ij} = \delta_{ij} + 1_{\{j<i\}}$ and $\tilde{\Psi}_{iL} = 1$ but $\Psi_{ij} = \delta_{ij} + 1_{\{j>i\}}\Pi_{l=i}^{j-1}\mu_i^{-1}$ and $\Psi_{iL} = 1_{\{j>i\}}\Pi_{l=i}^{N}\mu_i^{-1}$.

$$\frac{\mathrm{d}\log\lambda_i}{\mathrm{d}\log A_k} = (\theta-1)(\delta_{ik}-\lambda_k), \quad \frac{\mathrm{d}\log\Lambda_L}{\mathrm{d}\log A_k} = -(\theta-1)\lambda_k(1-\frac{\mu_k^{-1}}{\sum_j \lambda_j\mu_j^{-1}}),$$

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -(\delta_{ik}-\lambda_k), \quad \frac{\mathrm{d}\log w_L}{\mathrm{d}\log A_k} = \lambda_k-(\theta-1)\lambda_k(1-\frac{\mu_k^{-1}}{\sum_j \lambda_j\mu_j^{-1}}),$$

$$\frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} = \lambda_k+(\theta-1)(\delta_{ik}-\lambda_k)+(\theta-1)\lambda_k(1-\frac{\mu_k^{-1}}{\sum_j \lambda_j\mu_j^{-1}}),$$
$$\frac{\mathrm{d}\log w_L L}{\mathrm{d}\log A_k} = \lambda_k-(\theta-1)\lambda_k(1-\frac{\mu_k^{-1}}{\sum_j \lambda_j\mu_j^{-1}}),$$

$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A_k} = \delta_{ik}+(\theta-1)(\delta_{ik}-\lambda_k)+(\theta-1)\lambda_k(1-\frac{\mu_k^{-1}}{\sum_j \lambda_j\mu_j^{-1}}),$$
$$\frac{\mathrm{d}\log L_i}{\mathrm{d}\log A_k} = (\theta-1)(\delta_{ik}-\frac{\lambda_k\mu_k^{-1}}{\sum_j \lambda_j\mu_j^{-1}}).$$

To derive these results, we must distinguish cost-based and revenue-based input-output objects. For example, we have $\lambda_i = \tilde{\lambda}_i$ but $\tilde{\Lambda}_L = 1 \neq \Lambda_L = (\sum_j \lambda_j \mu_j^{-1})^{-1}$.

Overall, the difference with the case of an efficient economy studied in Section 3.3 comes from the terms $(\theta-1)\lambda_k[1-\mu_k^{-1}/(\sum_j \lambda_j\mu_j^{-1})]$ which appear in various of these expressions. To understand the intuition behind this term, consider its effect on the output aggregation equation, in the case where producer k charges a higher markup than the (harmonic) average of markups across producers. Suppose that producers are substitutes so that $\theta > 1$. Then labor is reallocated towards producer k and away from the other producers. This improves allocative efficiency by $(\theta-1)\lambda_k[1-\mu_k^{-1}/(\sum_j \lambda_j\mu_j^{-1})]\,\mathrm{d}\log A_k$ because this producer was too small to begin with. Aggregate output increases by more than $\lambda_k\,\mathrm{d}\log A_k$. This increase in allocative efficiency also boosts the output and the sales of all producers. However, it reduces the labor share and the real wage because it increases the average markup. The effects are reversed when producers are complements so that $\theta < 1$, or when producers are substitutes with $\theta > 1$ but producer k charges a lower markup than average.

Overall, the implications for comovement are subtle. Depending on whether producers are complements or substitutes, and whether the producer experiencing the shock has a relatively high or low markup, the effects identified in the efficient case are magnified or mitigated. Of course, as before, it is impossible to get positive comovement in labor.

To see asymmetric propagation in action, consider the effects of a productivity shock to a good $\mathrm{d}\log A_i$ or a productivity shock to the factor $\mathrm{d}\log A_L$. Since all goods are produced from labor, $\mathrm{d}\,\lambda_i / \mathrm{d}\log A_L = 0$. However, $\mathrm{d}\,\Lambda_L / \mathrm{d}\log A_i = -(\theta - 1)\lambda_k(1 - \mu_i^{-1}/\Lambda_L)$, and so $\mathrm{d}\,\lambda_i / \mathrm{d}\log A_L \neq \mathrm{d}\,\Lambda_L / \mathrm{d}\log A_i$.

Fiscal Multipliers – The Quesnaysian-Keynesian Cross

The baseline model in Section 3 is a representative agent efficient model with inelastic factors. In Sections 4-6, we provide, one by one, extensions of the baseline model to allow for heterogeneous consumers, elastic factors, and distortions. In the Appendix, we describe the formulas when all these these ingredients are present together. The example in this section is an applications of these general results, bringing together all the elements that have been emphasized separately so far. Specifically, we investigate the "bang-for-buck" from different types of government spending in terms of stimulating employment. This example features heterogeneous agents (the government, borrowers, and savers), frictions (borrowing constraints, sticky wages and the zero lower bound), and elastic labor supply.

We use our formulas to investigate the "bang-for-buck" from different types of government spending in terms of stimulating employment. We show that heterogeneous agents and input-output linkages can shape the Keynesian cross, and how the composition of government spending can affect the fiscal multiplier through both the slope and the intercept of the Keynesian cross. This example builds heavily on the analysis of fiscal multipliers in production networks by Baqaee (2015). It extends the environments in Bilbiie (2008) and Eggertsson and Krugman (2012) as well as the analysis in Farhi and Werning (2016) to incorporate input-output linkages.

Suppose there are two types of households indexed by $o \in \{b, s\}$. These households have differing discount factors ρ_o: the more patient household is the saver (s) and the less patient one the borrower (b), with $\rho_s > \rho_b$. Each household maximizes

$$\sum_t \rho_o^t \log(c_{o,t}),$$

where $c_{o,t} = \prod_k (c_{o,t,k})^{b_k}$ is total private consumption. The household has budget constraint

$$\sum_k p_{t,k} c_{o,t,k} + D_{o,t} = \sum_i (w_{it} l_{it})\,\Phi_{oi} + r_t K_t \Phi_{oK} + (1 + i_{t-1}) D_{o,t-1} - \tau_t,$$

where $p_{t,k}$ is the price of good k in time t, $D_{o,t}$ is assets, and i_t is the nominal interest rate. The household receives a fraction Φ_{oi} of the labor employed in each industry $w_{it} l_{it}$ and capital income $r_t K_t$ in proportion Φ_{oK} to its share of factor ownership. Finally, the household faces lump sum taxes τ_t. In addition, there is a time-varying borrowing constraint $D_{o,t} \geq D_t$, which we assume binds always for borrowers and never for savers. This binding borrowing constraint can be captured in our framework via intertemporal wedges for the borrower.

The firms are competitive and rent capital and labor on spot markets from the household and re-optimize every period. Therefore, their problems are static.

$$\max_{y_{it},l_{it},x_{ijt}} \quad p_{it}y_{it} - \sum_j p_{jt}x_{ijt} - w_t l_{it} - r_t k_{it},$$

such that

$$y_{it} = (l_{it})^{\alpha_i} k_{it}^{\eta_i} \prod x_{ijt}^{\omega_{ij}}.$$

The government faces the budget constraint

$$B_t = (1 + i_{t-1})B_{t-1} + \sum_k p_{t,k}g_{t,k} - \tau_t,$$

where τ_t is income from lump sum taxation. Denote the total size of government outlays in period t by $G_t = \sum_k p_{t,k}g_{t,k}$. We let δ_i be the share of government expenditures $p_{t,i}g_{t,i}/G_t$ on good i. We assume that there is no government spending, and constant taxes from period 2 onwards.

Prices are flexible and the market for the goods and services clears:

$$p_{t,k}y_{t,k} = p_{t,k}(g_{t,k} + c^b_{t,k} + c^s_{t,k}) + \sum_j p_{t,k}x_{t,j,k}.$$

The rental rate of capital is also flexible and so capital, which is perfectly inelastically supplied, is always fully employed. In period 1, wages are perfectly sticky and households supply labor elastically at this nominal wage. From period 2 onwards, the labor market clears, and the labor supply function is perfectly inelastic at a level given by the endowment of labor.

We consider situation where the borrowing constraint is tighter in period 1 than in future periods, so much so that the natural rate of the economy is negative in period 1 but positive from period 2 onwards. In addition, we assume that the central bank does not tolerate wage inflation. From period 2 onwards, this intolerance for inflation does not prevent the economy from operating at full potential: the central bank simply sets the nominal interest rate at the natural interest rate and achieves full employment and no inflation. However, it generates a recession in period 1: the central bank sets the nominal interest rate at the zero lower bound, there is no inflation, hence the real interest is higher than the natural interest rate and there is a recession.

The zero lower bound, sticky wages and the insistence on no wage inflation, can all be captured in our framework via a set of endogenous markups/wedges. These markups/wedges are endogenous to any shock or policy hitting the economy and we can solve for this depen-

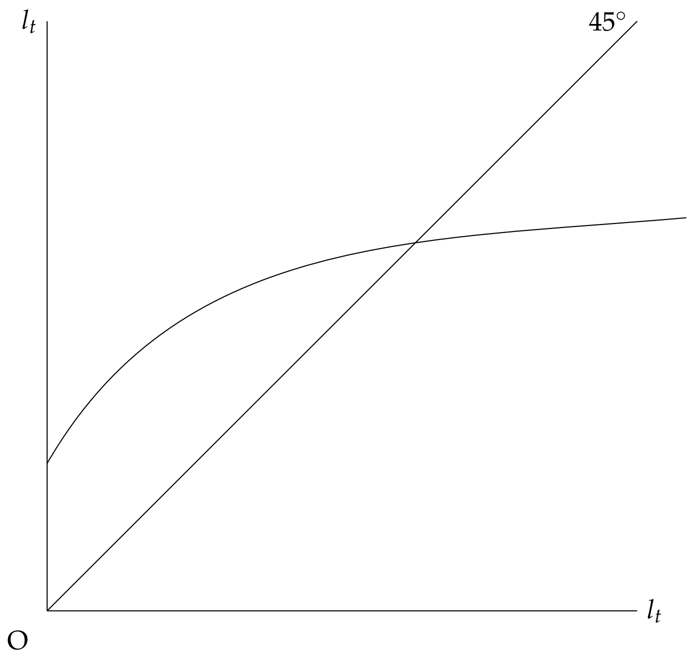


Figure 4: The Quesnaysian-Keynesian Cross. Both the slope and the intercept of the Keynesian cross depend on the composition of government spending across different goods.

dence explicitly. [21] The same goes for the borrowing constraint.[22]

Equilibrium labor l_1 during the recession in period 1 is given by

$$w_1 l_1 = \left[\sum_i (\lambda_i^c(1-\mathcal{B}) + \lambda_i^g \mathcal{B})\alpha_i \Phi_{bi} + (1-\Lambda_L^c)\Phi_{bK}\right] w_1 l_1 \\ + \left(\sum_i (\lambda_i^c(1-\mathcal{B}) + \lambda_i^g \mathcal{B})\alpha_i (1-\Phi_{bi}) + (1-\Phi_{bK})\right)(\Lambda_L^g - \Lambda_L^c) G_t + const \qquad (25)$$

where Φ_{bK} is the share of capital and Φ_{bi} is the share of each labor type owned by the borrower, λ^g is the network-adjusted consumption share of each good from the perspective of the government given by $\delta'\Psi$, where δ is the vector of government expenditure shares on different goods. The term $\mathcal{B}$ is the size of the government spending relative to GDP given by

$$\mathcal{B} = \frac{G_1}{w_1 l_1 + r_1 K_1} = \frac{G_1}{w_1 l_1}\frac{w_1 l_1}{w_1 l_1 + r_1 k_1} = \frac{G_1}{w_1 l_1}\frac{\Lambda_L^c}{1 + (\Lambda_L^c - \Lambda_L^g) G_1 / w_1 l_1}.$$

In equation (25), *const* is a term which is independent of the composition δ of government spending and employment $w_1 l_1$, but does depend on the level G_1 of government spending. To streamline the exposition, and since our main focus is on the dependence of fiscal multipliers

[21] In the particular context of this simple example, it suffices to use a consumption tax and a markup on labor in period 1. The point, however, is more general.

[22] In the model, it can be captured via a set of increasing consumption taxes specific to the borrower.

on the composition of government spending for a given level of government spending, we refrain from giving its full expression as a function of primitive parameters.[23]

Equation (25) is a Keynesian cross, which can be written in closed form in levels thanks to our Cobb-Douglas assumptions. Similar to Baqaee (2015), we can use equation (25) to understand the effects of changes to the composition of government spending, given by the vector δ, holding fixed the amount of government spending G_1. Unlike Baqaee (2015), this Keynesian cross is nonlinear. We refer to the first term on the right side of (25) as the "slope" effect and the second term as the intercept. Whereas traditionally, government spending only affects the intercept term, in this example, both the intercept term, and the slope terms are affected by the composition of government spending.

The intercept is increasing in $(\Lambda_L^g - \Lambda_L^c) = (\delta - b)'\Psi_{(L)}$, which is the same force as emphasized by Baqaee (2015): $\Psi_{(L)}$ is the network-adjusted labor intensity of each good. If the government spends relatively more on labor intensive goods than the private sector, then the initial boost in demand for labor is higher.

The slope of the Keynesian cross depends on $\lambda_i^g \alpha_i (1 - \Phi_{bi})$. Hence, if the government spends in such a way that relatively more income is generated for borrowers, then this makes the multiplier effect on employment and output larger, because borrowers have a higher marginal propensity to consume out of current income than savers.

The slope effect, which depends on the fraction of income going to borrowers versus savers, depends on the size of government spending relative to GDP $\mathcal{B}$. This is because differences between the pattern of spending of the government and the private sector causes changes in the fraction of income going to borrowers versus savers. Hence, as output expands, holding fixed government spending G_1, causes the share of government expenditures in output $\mathcal{B}$ to decline. If the government spending stimulates borrower income relatively more than private spending, then the slope effect declines in output. Since output increases as employment increases, this gives rise to concavity in the Keynesian cross. This nonlinearity is stronger when the government spending is relatively more labor intensive than private spending $\Lambda_L^g - \Lambda_L^c = (\delta - b)'\Psi_{(L)} > 0$.

The aforementioned Quesnaysian cross, in this example, would be the linearized version of equation (25), showing the change in employment in response to a disturbance, say to the composition of government spending. This linearized system has the same form as a Keynesian cross, except that now, both the slope and intercept terms depend on how the government chooses to spend its budget across different goods.

[23] The details are in the Appendix.

7 Industry-level Aggregation

In this section, we introduce and characterize the movements of industry-level aggregates of productivity and markups/wedges. We introduce a new measure of industry-level TFP for economies with distortions, and we characterize its properties, along with the properties of the industry-level markup/wedge. We conclude with a discussion of aggregating industry-level information back up to the level of the whole economy. Our analysis is motivated by the proliferation of high-quality industry-level data, which whilst having a lot of detail, often lack universal coverage and fall short of the requirements for general equilibrium aggregation à la Baqaee and Farhi (2017b) or Baqaee and Farhi (2018).

For our industry-level analysis, we group producers into an arbitrary partition of "industries." We make sure that producers of non-factor goods are never in the same industry as producers of factor goods, however, an industry might be using some of its own goods as inputs. We start by defining and analyzing industry-level productivity, and then move on to industry-level markups/wedges.

7.1 Industry-level Productivity

Productivity measures how much output can be produced per unit of (appropriately weighted) inputs. For efficient economies, output can only change either because inputs have changed or because the production technology has changed. We can therefore measure changes in technology by constructing a Solow residual and subtracting input growth from output growth.

This intuition breaks down in inefficient models since, for a given set of production technologies and inputs, the amount of output produced can vary due to changes in the efficiency of the allocation. Alternative definitions of TFP have been proposed for inefficient economies, e.g. Hall (1990), Basu and Fernald (2002), and Petrin and Levinsohn (2012). In Baqaee and Farhi (2017b), we show that these measures all suffer from shortcomings, and we introduce a notion of *aggregate* TFP growth which can be decomposed into productivity growth due to improvements in technology and improvements in allocative efficiency. When applied to efficient models, the definition collapses to the usual definition offered by Solow (1957). A downside of our approach in Baqaee and Farhi (2017b) is that we only define and decompose aggregate TFP for the whole economy. In practice, we may want to measure the productive efficiency of a set of producers even when that set does not constitute the whole of the economy. In this section, we extend our approach to cover subsets of the economy.

We show that we can compute an interpretable notion of industry TFP growth, and decompose it into pure changes in technology and changes in allocative efficiency. When the economy is efficient, we recover the standard industry-level Solow residuals and there are no changes in allocative efficiency. When the economy is not efficient, the measure is different from the

Solow residual, there are changes in allocative efficiency driven by shocks *inside and outside* the industry, and the pure changes in technology are weighted according to some cost-based modifications of sales shares.

Consider some collection of producers indexed by the set $\mathcal{I}$, which we call an industry. Industry $\mathcal{I}$ receives some factors and intermediate inputs from outside of the industry, these resources are distributed across producers inside the industry, potentially combined with some goods the industry produces itself, and then resources are sent out to the rest of the economy.

The net output of the industry is its output minus intermediate consumption of its own inputs. Denote industry $\mathcal{I}$'s total use of good l by

$$x_{\mathcal{I}l} \equiv \sum_{i \in \mathcal{I}} x_{il}.$$

Define the industry *net* output to be

$$c_i^{\mathcal{I}} \equiv y_i - x_{\mathcal{I}i}.$$

Letting

$$\lambda_i^{\mathcal{I},c} \equiv \frac{p_i c_i^{\mathcal{I}}}{\sum_{j \in \mathcal{I}} p_j c_j^{\mathcal{I}}},$$

denote the net output shares of the different producers in the industry, we can define industry $\mathcal{I}$'s net-output growth using the Divisia index

$$\mathrm{d}\log c_{\mathcal{I}} \equiv \sum_{i \in \mathcal{I}} \lambda_i^{\mathcal{I},c} \, \mathrm{d}\log c_i^{\mathcal{I}}.$$

Note that when $\mathcal{I}$ is the whole economy, then $\mathrm{d}\log c_{\mathcal{I}}$ corresponds to real GDP growth.

Let e_j be the jth standard basis vector. For $j \notin \mathcal{I}$, define the revenue-based exposure of net industry output to producer j to be

$$\Lambda_j^{\mathcal{I}} \equiv \left(\lambda^{\mathcal{I},c}\right)' \left(I - \Omega^{\mathcal{I},\mathcal{I}}\right)^{-1} \Omega^{\mathcal{I},\mathcal{NI}} e_j = \frac{p_j x_{\mathcal{I}j}}{\sum_{i \in \mathcal{I}} p_i c_i^{\mathcal{I}}},$$

the cost-based exposure of net industry output to producer j to be

$$\tilde{\Lambda}_j^{\mathcal{I}} \equiv \left(\lambda^{\mathcal{I},c}\right)' \left(I - \tilde{\Omega}^{\mathcal{I},\mathcal{I}}\right)^{-1} \tilde{\Omega}^{\mathcal{I},\mathcal{NI}} e_j,$$

and for $i \in \mathcal{I}$, the cost-based exposure of net industry output to producer i to be

$$\tilde{\lambda}_i^{\mathcal{I}} = \left(\lambda^{\mathcal{I},c}\right)' \left(I - \tilde{\Omega}^{\mathcal{I},\mathcal{I}}\right)^{-1} e_i,$$

with obvious vector, matrix, and block sub-matrix notations.

Proposition 10 (Industry-level Productivity Growth)**.** *Net industry output growth can be written as*

$$\mathrm{d}\log c_{\mathcal{I}} = \underbrace{\sum_{j\notin\mathcal{I}} \tilde{\Lambda}_j^{\mathcal{I}}\, \mathrm{d}\log x_{\mathcal{I}j}}_{\Delta\ input} + \underbrace{\sum_{i\in\mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}}\, \mathrm{d}\log A_i}_{\Delta\ technology} \underbrace{- \sum_{i\in\mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}}\, \mathrm{d}\log \mu_i - \sum_{j\notin\mathcal{I}} \tilde{\Lambda}_j^{\mathcal{I}}\, \mathrm{d}\log \Lambda_j^{\mathcal{I}}}_{\Delta\ allocative\ efficiency},$$

where $\mathrm{d}\log \Lambda_j^{\mathcal{I}}$ *and* $\mathrm{d}\log x_{\mathcal{I}j}$ *are characterized by the formulas in the preceding sections.*

The left-hand side is the industry net-output growth. The right-hand side is the contribution of input growth, technology growth, and changes in allocative efficiency, respectively. We now explain each of these terms and their interpretation.

Intuitively, the change in the industry's net-output depends on the inputs to the industry $x_{\mathcal{I}j}(j \notin \mathcal{I})$, the productivity level of producers in the industry $A_i(i \in \mathcal{I})$, and the distribution of resources across producers in industry $\mathcal{I}$, which we index by the proportion $\mathcal{X}_{\mathcal{I}ij} = x_{ij}/x_{\mathcal{I}j}$ of the total use of input j by industry $\mathcal{I}$ coming from producer i:

$$\mathrm{d}\log c_{\mathcal{I}} = \frac{\partial \log c_{\mathcal{I}}}{\partial \log x_{\mathcal{I}}}\, \mathrm{d}\log x_{\mathcal{I}} + \frac{\partial \log c_{\mathcal{I}}}{\partial \log A}\, \mathrm{d}\log A + \frac{\partial \log c_{\mathcal{I}}}{\partial \mathcal{X}_{\mathcal{I}}}\, \mathrm{d}\,\mathcal{X}_{\mathcal{I}}.$$

Proposition 10 gives us a way to decompose these various effects, since

$$\frac{\partial \log c_{\mathcal{I}}}{\partial \log x_{\mathcal{I}j}} = \tilde{\Lambda}_j^{\mathcal{I}}, \quad \frac{\partial \log c_{\mathcal{I}}}{\partial \log A_i} = \tilde{\lambda}_i^{\mathcal{I}}, \quad \frac{\partial \log c_{\mathcal{I}}}{\partial \mathcal{X}_{\mathcal{I}}}\, \mathrm{d}\,\mathcal{X}_{\mathcal{I}} = -\sum_{i\in\mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}}\, \mathrm{d}\log \mu_i - \sum_{j\notin\mathcal{I}} \tilde{\Lambda}_j^{\mathcal{I}}\, \mathrm{d}\log \Lambda_j^{\mathcal{I}}.$$

Intuitively, following a shock, the net-output of industry $\mathcal{I}$ changes for two reasons: (1) holding fixed the distribution of resources $\mathcal{X}_{\mathcal{I}}$ across all producers in industry $\mathcal{I}$, the industry's net-output changes due (1a) to a change in the inputs and (1b) productivity; and (2) the distribution of resources across producers in the industry may change.

We think of (1a) as the input growth effect, of (1b) as the *pure* change in technology effect, and of (2) as the change in *allocative efficiency.* If the economy is efficient, it follows from the envelope theorem that the change in allocative efficiency effect is zero. By contrast, when the economy is inefficient, the envelope theorem fails and changes in allocative efficiency are nonzero.

By subtracting input growth from output growth, we arrive at a measure of industry-level productivity growth its decomposition into changes in technology and changes in allocative efficiency:[24]

[24]When $\mathcal{I}$ is the whole economy, then these definitions and decompositions are identical to the definition and decomposition of aggregate TFP in Baqaee and Farhi (2017b). When $\mathcal{I}$ is a single producer that does not use its own output as an input, then there is no scope for changes in allocative efficiency, since there is only one way to allocate a fixed amount of resources to a production function.

$$\mathrm{d}\log c_{\mathcal{I}} - \sum_{j\notin\mathcal{I}} \tilde{\Lambda}_j^{\mathcal{I}}\, \mathrm{d}\log x_{\mathcal{I}j} = \underbrace{\sum_{i\in\mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}}\, \mathrm{d}\log A_i}_{\Delta\text{technology}} \underbrace{- \sum_{i\in\mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}}\, \mathrm{d}\log \mu_i - \sum_{j\notin\mathcal{I}} \tilde{\Lambda}_j^{\mathcal{I}}\, \mathrm{d}\log \Lambda_j^{\mathcal{I}}}_{\Delta\text{allocative efficiency}}.$$

When there are markups/wedges, productivity and markups/wedges shocks inside and outside the industry influence allocative efficiency by changing the relative demands for the outputs of the different producers in the industry. It follows that industry TFP growth depends on productivity and markup/wedges shocks inside and outside the industry.

When there are no markups/wedges, these changes in relative demands originating from shocks outside the industry have no impact on the TFP growth of the industry. In this case, there are no changes in allocative efficiency, and industry TFP growth depends only on productivity shocks inside the industry. Moreover, our definition collapses to a traditional industry-level Solow residual.

Example: Fallacy of Composition

Consider an industry where each producer produces linearly from a common input $c_i = A_i x_i$ and charges a markup μ_i over marginal cost. Let $x = \sum_i x_i$ is the total input use of the industry and assume for simplicity that total input supply x to the industry fixed. In addition, assume that the consumer and all the producers in the economy are exposed to producers in this industry via a CES aggregator

$$c_{\mathcal{I}} = \left(\sum_{i\in\mathcal{I}} \omega_{\mathcal{I}i} \left(\frac{c_i}{\bar{c}_i} \right)^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

with $\theta > 1$.

Now, suppose that a series of positive productivity shocks hits producers in industry $\mathcal{I}$ so that *every* producer in industry $\mathcal{I}$ becomes *more* productive. In the presence of heterogeneous markups, it is possible for the industry as a whole to become *less* productive.

To see how, let $\mathrm{d}\log A_i = (\mu_{\mathcal{I}}/\mu_i)/(\theta - 1) > 0$. In other words, firms with higher markups receive smaller productivity shocks. Now, using Proposition 10, we can show that as long as θ is sufficiently large or markups are sufficiently dispersed, we have

$$\mathrm{d}\log c_{\mathcal{I}} = \frac{1}{\theta - 1} - \frac{Var_{\lambda_i/\lambda_{\mathcal{I}}}(\mu_i^{-1} | i \in \mathcal{I})}{E_{\lambda_i/\lambda_{\mathcal{I}}}(\mu_i^{-1} | i \in \mathcal{I})} < 0.$$

Of course, when the model is efficient, dispersion in markups is zero, and the industry-level productivity *must* increase.

The first term in the sum corresponds to the pure technology effect of the productivity shock, while the second term corresponds to the change in allocative efficiency. Intuitively, industry $\mathcal{I}$ becomes less productive, despite the increase in microeconomic productivity, because allocative efficiency can fall more quickly than technology improves.

7.2 Industry-level Markups/Wedges

We define an industry markup/wedge $\mu_{\mathcal{I}}$ by requiring that it matches the share in aggregate output of the profits of industry $\mathcal{I}$ so that $\lambda_{\mathcal{I}}(\mu_{\mathcal{I}}-1)/\mu_{\mathcal{I}} = \sum_{i\in\mathcal{I}} \lambda_i(\mu_i-1)/\mu_i$. This means that the industry markup is the sales-weighted harmonic average of the markups of the producers in the industry

$$\mu_{\mathcal{I}} \equiv \left(\sum_{i\in\mathcal{I}} \frac{\lambda_i}{\lambda_{\mathcal{I}}}\mu_i^{-1}\right)^{-1},$$

where

$$\lambda_{\mathcal{I}} \equiv \sum_{i\in\mathcal{I}} \lambda_i.$$

Industry markups are endogenous objects, in a way that we can characterize.

Proposition 11. *(Industry Markup/Wedge) The elasticities of the industry markup/wedge to the different productivities and markups/wedges are given by*

$$\mathrm{d}\log\mu_{\mathcal{I}} = \sum_{i\in\mathcal{I}} \frac{\lambda_i}{\lambda_{\mathcal{I}}}\frac{\mu_i^{-1}}{\mu_{\mathcal{I}}^{-1}}\,\mathrm{d}\log\mu_i - \sum_{i\in\mathcal{I}} \frac{\lambda_i}{\lambda_{\mathcal{I}}}\frac{\mu_i^{-1}}{\mu_{\mathcal{I}}^{-1}}(\mathrm{d}\log\lambda_i - \mathrm{d}\log\lambda_{\mathcal{I}}). \tag{26}$$

where $\mathrm{d}\log\lambda_i$ *is given by the formulas in are given by the formulas in the preceding sections.*

The first term on the right-hand side of equation (26) is the weighted average of the change of the markups/wedges of the producers in the industry, which captures a partial-equilibrium effect taking the relative sizes of the different producers as given. The second term captures the effects of reallocations effects across producers in the industry with different markups/wedges, which captures the endogenous equilibrium changes in the industry markup delivered by the model. These arise from changes in productivity shocks and markups/wedges from producers in the industry, but also from producers outside the industry. By implication, even if markups/wedges and productivities are independent across producers, industry markups are not independent across industries.

Once again, inference about industry-level markups is subject to fallacies of composition, just like industry-level productivity. It is easy to construct examples where the markups of all producers in an industry increase and yet the industry markup decreases.

An important message that emerges from this analysis is that aggregate markups and aggregate productivity are highly endogenous objects. As a result, many patterns are possible. See

Burstein, Carvalho, and Grassi (2018) for an investigation of the cyclical properties of aggregate markups in a model with an industry structure and with endogenous markups.

Since we have characterized the comovement patterns the model generates, our results can be used to study changes in industry-level productivity and markups while taking the endogenous compositional effects into account.

7.3 Aggregating Industry-level Outcomes

We have shown how to define and characterize industry aggregates for a given partition of producers into industries. A different question is whether theses industry aggregates are sufficient statistics for the model to the first order, a property which we call first-order economic aggregation.

We say that first-order economic aggregation holds for a variable of interest if: given the *initial* industry-aggregates (input-output information at the industry level, industry markup/wedges), *changes* in industry productivities and in industry markups/wedges are enough to compute *changes* in output. In general, first-order economic aggregation does not hold, and industry-level aggregates are not enough to deduce the movements in output, unless we make some very restrictive assumptions. We discuss these issues in more detail in the appendix, and show a special case where first-order economic aggregation does obtain: a situation where all producers in each industry use the same production function, and all producers in each industry enter into the production functions of other industries via a homothetic aggregator.

8 Beyond CES

The input-output covariance operator defined in equation (4) is useful in characterizing the substitution patterns in economies where all production and utility functions are nested CES functions. In this section, we generalize this input-output covariance operator in such a way that allows us to work with arbitrary production functions.

For a producer j with cost function $\mathbf{C}_j$, let $\theta_j(x,y) = \mathbf{C}_j \mathbf{C}_{j,xy} / \left(\mathbf{C}_{j,x}\mathbf{C}_{j,y}\right)$ denote the Allen-Uzawa elasticity of substitution between inputs $x \neq y$, noting that $\theta_j(x,y) = \theta_j(y,x)$ due to symmetry of partial derivatives.

Then, we define the *input-output substitution operator* for j as

$$\Phi_j(\Psi_{(k)}, \Psi_{(l)}) = \sum_{\substack{x,y \\ x\neq y}} \Omega_{jx}\Omega_{jy}(\theta_j(x,y) - 1)\Psi_{xl}\Psi_{yk}, \tag{27}$$

$$= \frac{1}{2} E_{\Omega^{(j)}} \left((\theta_j(x,y) - 1)(\Psi_k(x) - \Psi_k(y))(\Psi_l(x) - \Psi_l(y)) \right), \tag{28}$$

where $\Psi_k(x) = \Psi_{xk}$.

When all the Allen-Uzawa elasticities $\theta_j(x,y)$ are identical $\theta_j(x,y) = \theta_j$, as happens for example when the production function of producer j is CES with elasticity θ_j, then we recover recover the product of the input-output covariance operator $Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)})$ and of the deviation from one of the common elasticity of substitution $\theta_j - 1$:

$$\Phi_j(\Psi_{(k)}, \Psi_{(l)}) = (\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}).$$

Even when the Allen-Uzawa elasticities $\theta_j(x,y)$ are not identical across couples (x,y), the input-output substitution operator shares many properties with the CES case.[25] It is immediate to verify, for example, that: $\Phi_j(\Psi_{(k)}, \Psi_{(l)})$ is bilinear in $\Psi_{(k)}$ and $\Psi_{(l)}$; $\Phi_j(\Psi_{(k)}, \Psi_{(l)})$ is symmetric in $\Psi_{(k)}$ and $\Psi_{(l)}$; and $\Phi_j(\Psi_{(k)}, \Psi_{(l)}) = 0$ whenever $\Psi_{(l)}$ or $\Psi_{(k)}$ is a constant.

Intuitively, $\Phi_j(\Psi_{(k)}, \Psi_{(l)})$ captures the way in which j redirects demand (in cost shares) towards l in response to one percent change in the price of j. To see this, we make use of the following well-known result (see for example Russell, 2017): the derivative of the expenditure share in cost of input x with respect to the price of input y is given by $\Omega_{jx}\Omega_{jy}(\theta_j(x,y) - 1)$. The first line in equation (27) says that this requires considering, for each pair of inputs x and y, how much the percentage change Ψ_{yk} in the price of y induced by a one percent change in the price of k causes j to substitute towards x (in cost shares) as measured by $\Omega_{jx}\Omega_{jy}(\theta_j(x,y) - 1)\Psi_{yk}$ and on on the exposure of x to l as measured by Ψ_{xl}.

Equation (28) exploits the symmetry of Allen-Uzawa elasticities of substitution to say that this amounts to considering, for each pair of inputs x and y, whether or not increased exposure to k as measured by $\Psi_k(x) - \Psi_k(y)$, corresponds to increased exposure to l as measured by $\Psi_l(x) - \Psi_l(y)$, and whether x and y are complements or substitutes as measured by $(\theta_j(x,y) - 1)$. So, for example, if x and y are substitutes, so that $\theta_j(x,y) > 1$, and if x is more exposed to the shock to the price of k than y, so $\Psi_k(x) - \Psi_k(y) > 0$, then substitution by j between x and y will result in increased demand for l if, and only if, x is also more exposed to l, given by $\Psi_l(x) - \Psi_l(y) > 0$.

Luckily, it turns out that all of the results stated so far can be generalized to non-CES economies simply by replacing terms of the form $(\theta_j - 1)Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)})$ by $\Phi_j(\Psi_{(k)}, \Psi_{(l)})$.

[25] The function Φ is related to the *distance covariance* function defined by Székely, Rizzo, and Bakirov (2007). A probabilistic interpretation of Φ is given by

$$\begin{aligned}\Phi_j(\Psi_{(k)}, \Psi_{(l)}) &= \frac{1}{2} E_{\Omega^{(j)}}\left((\Psi_l(x) - \Psi_l(y))(\Psi_k(x) - \Psi_k(y))(\theta_j(x,y) - 1)\right), \\ &= \frac{1}{2}\left(-2Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}) + \overline{\theta}_j E^j\left((\Psi_l(x) - \Psi_l(y))(\Psi_k(x) - \Psi_k(y))\right)\right),\end{aligned}$$

where $\overline{\theta}_j$ is the grand mean of $\theta_j(x,y)$ and E^j is the expectation taken with respect to the bivariate distribution defined by the marginal distributions $\Omega_j(x)$, $\Omega_j(x)$ and the copula $\theta_j(x,y)/\overline{\theta}_j$.

So, for example, Proposition 3 becomes

$$\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log A_k} = \sum_{j=0} \Phi_j(\Psi_{(k)}, \Psi_{(i)}) - \sum_f \sum_j \Phi_j(\Psi_{(i)}, \Psi_{(f)}) \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_k}.$$

By replacing the input-output covariance operator with the input-output substitution operator, we fully characterize the comovement patterns of the general economy described in Section 1, with arbitrary, and potentially, non-homothetic production functions, arbitrary distortions, an arbitrary number of factors, and arbitrary patterns of input-output linkages. The substitution operator can be used in a similar way to extend the results in all the other sections.

Conclusion

In this paper, we characterize the effects of microeconomic shocks on all prices and quantities in a general equilibrium environment. Our results allow for any neoclassical production structure and patterns of distorting wedges.

We show that without non-homotheticities in final demand, non-balanced growth preferences, or wedges, propagation patterns in these models is constrained to have strong symmetries. Neglecting these features can therefore jeopardize the realism of the model in important and surprising ways. The importance of this observation obviously depends on the question at hand, but it underscores the importance of carefully-deliberated modeling choices in disaggregated approaches.

With the aid of these propagation results, we define and characterize the properties of industry-level aggregates. Aggregation is non-trivial, and can easily be susceptible to fallacies of composition whereby propositions that are true about the productivity or markup of every single producer in an industry is false for the aggregate.

The generality of the results also show how seemingly disparate questions like "how does automation affect the skill premium" are, theoretically speaking, similar to questions like "how do sectors co-move in business cycles ", and "how do corporate taxes affect productivity,": the same set of tools and relationships determine their answer.

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