INSTRUMENT-BASED VS. TARGET-BASED RULES

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ABSTRACT

We develop a simple delegation model to study rules based on instruments vs. targets. A principal faces a better informed but biased agent and relies on joint punishments as incentives. Instrument-based rules condition incentives on the agent's observable action; target-based rules condition incentives on outcomes that depend on the agent's action and private information. In each class, an optimal rule takes a threshold form and imposes the worst punishment upon violation. Target-based rules dominate instrument-based rules if and only if the agent's information is sufficiently precise. An optimal hybrid rule relaxes the instrument threshold whenever the target threshold is satisfied.

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1 Introduction

The question of whether to base incentives on agents' actions or the outcomes of these actions arises in various contexts. Perhaps most prominently, scholars and policymakers have long debated on the merits of using instruments vs. targets for monetary policy. The US House of Representatives and several notable economists support the use of a Taylor (1993) rule that guides the interest rate choice of the central bank,[1] whereas numerous central banks such as the Bank of England and the Bank of Canada rely on inflation targeting rules that are based on outcomes.[2] Fiscal policy rules also vary in this dimension, with some US states constraining instruments like tax rates and spending and others using rules contingent on targets like deficits.[3] More recently, these considerations have received attention in the design of environmental policies. Environmental regulation may focus on technology mandates—requirements on firms' production processes, such as the choice of equipment—or on performance standards—requirements on output, such as maximum emission rates.[4,5]

In this paper, we develop a stylized model to study and compare *instrument-based* and *target-based* rules. Using mechanism design, we present a simple theory that elucidates the benefits of each class of rule and shows which class will be preferred as a function of the environment. Additionally, we characterize the optimal unconstrained or *hybrid* rule and examine how combining instruments and targets can improve welfare.

Our model builds on a canonical delegation framework. A principal delegates decision-making to an agent who is biased towards higher actions. The agent's action is observable, but the agent has private information about its value, with a higher agent *type* corresponding to a higher expected marginal benefit of the action for both the principal and the agent. We extend this delegation setting by introducing an observable noisy outcome that is a function of the agent's action and his private information. For example, the agent may be a policymaker who is biased towards expansionary monetary policy relative to society, and the outcome is inflation which depends on the choice of policy and the realization of economic shocks about which the

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[1]The Financial Choice Act of 2017 passed by the US House of Representatives requires the Federal Reserve to report on a rule to Congress. For discussions for and against Taylor rules, see Svensson (2003), Appelbaum (2014), Blinder (2014), Taylor (2014), Bernanke (2015), and the Statement on Policy Rules Legislation signed by a number of economists, available at `http://web.stanford.edu/~johntayl/2016_pdfs/Statement_on_Policy_Rules_Legislation_2-29-2016.pdf`.

[2]See Bernanke and Mishkin (1997), Bernanke et al. (1999), Mishkin (1999), Svensson (2003), and Mishkin (2017) for discussions of inflation targeting regimes.

[3]See National Conference of State Legislatures (1999).

[4]See for example US Congress, Office of Technology Assessment (1995) and Goulder and Parry (2008).

[5]These issues are also relevant to organizations, where for example promotion and firing policies may be informed by both workers' decisions and their performance.

agent has ex-ante private information. Due to his bias, the agent's preferred outcome exceeds that of the principal.[6]

As is standard in delegation settings, transfers between the parties are infeasible, but the principal can make use of joint punishments as incentives. That is, the principal can engage in "money burning" by taking measures that mutually harm the principal and the agent, like imposing sanctions or firing the agent. We distinguish between different classes of rules depending on how punishments are structured: we say that the principal's rule is instrument-based if punishments depend only on the agent's action, and the rule is target-based if punishments depend only on the realized outcome. In the context of monetary policy, an instrument-based rule conditions punishments on the choice of policy, whereas a target-based rule conditions punishments on realized inflation.

We begin by showing that, within each class, an optimal rule takes a threshold form, with violation of the threshold leading to the worst punishment. In the case of an optimal instrument-based rule, the principal allows the agent to choose any action up to a threshold and maximally punishes him for exceeding it. The logic is analogous to that in other delegation models; since the agent prefers higher actions than the principal, this punishment structure is optimal to deter the agent from taking actions that are excessively high. In the case of an optimal target-based rule, the principal specifies a threshold for the outcome, maximally punishing the agent if the realized outcome is above it. This punishment structure also incentivizes the agent to not choose excessively high actions, as these actions result in higher outcomes in expectation. High-powered incentives of this form are common in moral hazard settings with hidden action.[7]

Our main result uses this characterization of the optimal rules for each class to compare their performance. We show that target-based rules dominate instrument-based rules if and only if the agent's private information is sufficiently precise. To illustrate, suppose that the agent's information is perfect. Then the principal guarantees her preferred action by providing steep incentives under a target-based rule, where punishments do not occur *on path* because the perfectly informed agent chooses the action that delivers the target outcome. This target-based rule strictly dominates any instrument-based rule, as the latter cannot incentivize the agent while giving him enough flexibility to respond to his information. At the other extreme, suppose that the agent has no private information. Then the principal guarantees her ex-ante preferred action with an instrument-based rule that ties the hands of the agent, namely that

punishes the agent if any higher action is chosen. This instrument-based rule strictly dominates any target-based rule, as the latter gives the agent unnecessary discretion and requires on-path punishments to provide incentives.

We prove that this result holds more generally as we vary the precision of the agent's private information away from the extremes of perfect and no information. Furthermore, we show that the benefit of using a target-based rule over an instrument-based rule is decreasing in the bias of the agent and increasing in the severity of punishment. Intuitively, the less biased is the agent, the less costly is incentive provision under a target-based rule, as the principal can deter the agent from choosing high actions with less frequent punishments. Similarly, the harsher is the punishment imposed on the agent for missing the target, the less often the principal needs to exercise punishment on path to implement a target outcome. These two forces therefore make target-based rules more appealing than instrument-based rules on the margin.

A natural question is how the principal can combine instruments and targets to improve upon the above rules that rely exclusively on one of these tools. We study the optimal hybrid rule, that is the optimal unconstrained rule in which punishments can depend freely on the agent's action and the realized outcome.[8] We show that this rule admits a simple implementation: the principal sets an instrument threshold which is relaxed whenever a target threshold is satisfied. The optimal hybrid rule dominates instrument-based rules by allowing the agent more flexibility to choose high actions under a target-based criterion, and it dominates target-based rules by more efficiently limiting the agent's discretion with direct punishments. An example of an optimal hybrid rule in the context of monetary policy would be a Taylor rule which, whenever violated, switches to an inflation target. Notably, some policymakers and economists advocated such an approach in the US in the aftermath of the Global Financial Crisis, when the Federal Reserve's policy deviated significantly from the Taylor rule but realized inflation remained near the target.[9]

This paper is related to several literatures. First, the paper fits into the mechanism design literature that studies the tradeoff between commitment and flexibility in policymaking, including Athey, Atkeson, and Kehoe (2005), Amador, Werning, and Angeletos (2006), and Halac and Yared (2014, 2017a,b). Second, the paper contributes to an extensive literature on delegation in principal-agent settings that builds on the insights of Holmström (1977, 1984).[10] We extend the theoretical frameworks in both of these literatures by introducing an observable outcome that partially reflects the agent's information, and by studying incentives that condi-

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[8]This rule yields the principal the highest welfare that she can achieve given the agent's private information.
[9]See for example Yellen (2015, 2017).
[10]See Alonso and Matouschek (2008), Amador and Bagwell (2013), and Ambrus and Egorov (2017), among others.

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tion on this outcome. Third, the paper relates to other theoretical literatures on optimal policy design, including in the context of monetary policy where instruments and targets have been analyzed.[11] We contribute to this literature by characterizing rules as optimal mechanisms in a private information setting and by contrasting incentive provision under each class of rule.[12]

2 Model

We consider a stylized model with a principal and an agent. The agent observes a signal $s \in \{s^L, s^H\}$, which is the agent's private information or *type*, and chooses an action $\mu \in \mathbb{R}$. Given this action choice, an outcome $\pi = \mu - \theta$ is realized, where $\theta \in \mathbb{R}$ is a shock. A possible interpretation is that the action μ is a policy instrument, such as the level of monetary policy expansion; the shock θ is a stochastic macroeconomic fundamental, such as the level of economic slack; and the outcome π is a payoff-relevant outcome, such as the level of inflation.

The agent's signal is informative about the shock. Specifically, we assume that the conditional distribution of the shock is normal with mean equal to the signal, i.e., $\theta|_{s^i} \sim \mathcal{N}(s^i, \sigma^2)$ for $i = L, H$. The precision of the agent's information is given by $\sigma^{-1} > 0$. We take $s^L = -\Delta$ and $s^H = \Delta$ for some $\Delta > 0$ and assume that each signal occurs with equal probability. The shock's unconditional distribution is thus a mixture of two normal distributions with mean and variance given by

$$\mathbb{E}(\theta) = 0 \ \text{ and } \ Var(\theta) = \sigma^2 + \Delta^2.$$

The principal observes the agent's action μ and the realized shock θ (or equivalently the realized outcome π). She cannot however deduce the agent's private information s^i from these observations, as the distribution of θ has full support over the entire real line for each s^i.

As is standard in settings of delegation, transfers between the principal and the agent are not feasible. Instead, as a function of the action μ and the shock θ, the principal can commit to a continuation value $V(\mu, \theta) \in [\underline{V}, \overline{V}]$, for some finite $\underline{V}$ and $\overline{V}$. This continuation value represents rewards and punishments such as sanctions or replacement of the agent.[13]

Denote by $\phi(z|\overline{z}, \sigma_z^2)$ the normal density of a variable z with mean $\overline{z}$ and variance σ_z^2, and

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[11]See for example Svensson (2010) and Giannoni and Woodford (2017). Models of monetary policy are concerned with additional issues such as the role of inflationary expectations, which our paper does not consider. Frankel and Kartik (2017) examine how inflationary expectations and thus equilibrium outcomes depend on not only the amount but also the kind of information that a central bank has.

[12]Although the settings and analyses differ in many aspects, the paper also relates to some models of career concerns for experts which study how the information a principal has affects reputational incentives and efficiency. In particular, Prat (2005) distinguishes between information on actions and outcomes.

[13]Our assumption that the signal s is privately observed by the agent can equivalently be interpreted as a restriction on these rewards and punishments, which cannot explicitly condition on the signal.

by $\Phi(z|\bar{z}, \sigma_z^2)$ the corresponding normal cumulative distribution function. The agent's expected welfare conditional on information s^i and action μ^i, for $i = L, H$, is

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V\left(\mu^i, \theta\right) \right] \phi(\theta|s^i, \sigma^2)d\theta, \tag{1}$$

where $\alpha > 0$. The principal's expected welfare is

$$\sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + V\left(\mu^i, \theta\right) \right] \phi(\theta|s^i, \sigma^2)d\theta. \tag{2}$$

The principal and the agent receive utility as a function of the observable outcome $\pi = \mu - \theta$ which is concave and single-peaked. For both the principal and the agent, the preferred action μ that maximizes this utility is increasing in the shock θ. However, as captured by the parameter $\alpha > 0$, the agent is biased relative to the principal. For each signal s^i, the agent's preferred action, or *flexible* action, is equal to $s^i + \alpha$ (this follows from (1) since $\mathbb{E}(\theta|s^i, \sigma^2) = s^i$). This level exceeds the principal's preferred action, or *first-best* action, which is equal to s^i. Therefore, conditional on the signal, the agent always prefers a higher action than the principal. To facilitate the exposition, we assume:

Assumption 1. $\alpha \geq 2\Delta$.

Assumption 1 is analogous to an assumption in Halac and Yared (2014), and its role is to take agent types which are relatively "close" to each other, i.e. with Δ relatively small.[14] The implication of this assumption is that inducing the first-best action given the signal is not incentive compatible for the agent. In particular, Assumption 1 yields

$$s^L < s^H \leq s^L + \alpha < s^H + \alpha, \tag{3}$$

so the agent's flexible action $s^i + \alpha$ exceeds the first-best action under each signal.

As shown in (1)-(2), the principal and the agent can be jointly rewarded with a high continuation value $V(\mu, \theta) \in \left[\underline{V}, \overline{V}\right]$ or jointly punished with a low such value. This common continuation value captures the fact that incentivizing the agent is costly for the principal. We assume that the principal has a sufficient breadth of incentives to use in her relationship with the agent; specifically, our analysis in the next sections will assume:

Assumption 2. $\overline{V} - \underline{V} \geq \dfrac{\alpha^2}{2\phi(1|0, 1)}.$

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[14]In this sense, our results will not rely on a discrete distance between the types. See Section 5 for a discussion.

We distinguish between different classes of rules according to how the principal structures incentives. We say that a rule is instrument-based if the principal commits to a continuation value $V(\mu, \theta)$ which depends only on the action μ. A rule instead is target-based if the continuation value $V(\mu, \theta)$ depends only on the realized outcome $\pi = \mu - \theta$. Finally, if the continuation value $V(\mu, \theta)$ depends freely on μ and θ—and therefore freely on μ and π—we say that the rule is hybrid.

We are interested in comparing the performance of these different classes of rules as the environment changes. Our analysis will consider varying the precision of the agent's private information while holding fixed the mean and variance of the shock θ. At one extreme, we can take $\sigma \to \sqrt{Var(\theta)}$ and $\Delta \to 0$, so the agent is uninformed with signal $s^L = s^H = 0$. At the other extreme, we can take $\sigma \to 0$ and $\Delta \to \sqrt{Var(\theta)}$, so the agent is perfectly informed with signal $s^i = \theta$.[15] Note that since Assumption 1 holds for all feasible $\sigma > 0$ and $\Delta > 0$ given $Var(\theta)$ fixed, the assumption implies $\alpha \geq 2\sqrt{Var(\theta)}$.

3   Instrument-Based and Target-Based Rules

3.1   Optimal Instrument-Based Rule

An instrument-based rule specifies an action μ^i for each agent type $i = L, H$ and a continuation value $V(\mu, \theta)$ as a function of the action μ only. Let $V^i \equiv V(\mu^i)$ for $i = L, H$. An optimal instrument-based rule solves the following program:

$$\max_{\mu^L, \mu^H, V^L, V^H} \sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + V^i \right] \phi(\theta | s^i, \sigma^2) d\theta \tag{4}$$

subject to, for $i = L, H$,

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V^i \right] \phi(\theta | s^i, \sigma^2) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{(\mu^{-i} - \theta - \alpha)^2}{2} + V^{-i} \right] \phi(\theta | s^i, \sigma^2) d\theta, \tag{5}$$

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V^i \right] \phi(\theta | s^i, \sigma^2) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{(s^i - \theta)^2}{2} + \underline{V} \right] \phi(\theta | s^i, \sigma^2) d\theta, \tag{6}$$

$$V^i \in [\underline{V}, \overline{V}]. \tag{7}$$

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[15] In our setting, welfare under instrument-based and target-based rules depends only on the mean and variance of θ. This avoids additional complications stemming from the fact that higher moments of the distribution of θ vary with σ and Δ.

The constraints in (5) are the private information constraints, guaranteeing that an agent of type i has no incentive to misrepresent his type and deviate privately to action μ^{-i}. The constraints in (6) are the enforcement constraints, guaranteeing that an agent of type i has no incentive to deviate publicly to any action $\mu \neq \mu^i$. Without loss we specify the worst punishment $\underline{V}$ for any such public (off-path) deviation, and thus the agent's most profitable deviation entails choosing his flexible action $s^i + \alpha$, as reflected in (6). Finally, the constraints in (7) guarantee that rewards and punishments are feasible.

Define a *maximally-enforced instrument threshold* μ^* as a rule that specifies the maximal reward $\overline{V}$ if the agent's action is below a threshold μ^* and the maximal punishment $\underline{V}$ if the action exceeds this threshold. We find:

Proposition 1. *The optimal instrument-based rule admits $\mu^L = \mu^H = 0$ and $V^L = V^H = \overline{V}$. This rule can be implemented with a maximally-enforced instrument threshold $\mu^* = 0$.*

The optimal instrument-based rule assigns both agent types the action that maximizes the principal's ex-ante welfare. The agent is given no discretion, and punishments occur only off path, if the agent were to publicly deviate to a different action.

To prove Proposition 1, we solve a relaxed version of (4)-(7) which ignores the private information constraint (5) for the high type and the enforcement constraints (6) for both types. We show that under Assumption 1, the solution to this relaxed problem entails no discretion, and it thus satisfies (5). Moreover, Assumption 2 guarantees that (6) is also satisfied.

Proposition 1 is in line with the results of an extensive literature on delegation, which provides conditions under which threshold delegation with no money burning is optimal. This result extends to a continuum of agent types under some additional assumptions on the distribution of types; see Amador, Werning, and Angeletos (2006) and Amador and Bagwell (2013) among others. The analysis in Halac and Yared (2017b) is related in that it considers enforcement constraints like those in (6) and shows the optimality of maximally-enforced thresholds, where on- and off-path violations lead to the worst punishment. In our setting, enforcement constraints are non-binding by Assumption 2, so punishments occur only off path. Halac and Yared (2017b) study the issues that arise when the analog of this assumption is relaxed in their context.

3.2 Optimal Target-Based Rule

A target-based rule specifies an action μ^i for each agent type $i = L, H$ and a continuation value $V(\mu, \theta)$ as a function of the outcome $\pi = \mu - \theta$ only. We denote such a continuation value by

$V(\pi)$, where note that $V(\pi)$ is defined for $\pi \in (-\infty, \infty)$ since θ is normally distributed. An optimal target-based rule solves the following program:

$$\max_{\mu^L, \mu^H, V(\pi)} \sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + V(\mu^i - \theta) \right] \phi\left(\theta | s^i, \sigma^2\right) d\theta \tag{8}$$

subject to, for $i = L, H$,

$$\mu^i \in \arg\max_{\mu} \left\{ \int_{-\infty}^{\infty} \left[-\frac{(\mu - \theta - \alpha)^2}{2} + V(\mu - \theta) \right] \phi\left(\theta | s^i, \sigma^2\right) d\theta \right\}, \tag{9}$$

$$V(\pi) \in \left[\underline{V}, \overline{V}\right] \text{ for all } \pi. \tag{10}$$

We restrict attention to rules in which $V(\pi)$ is piecewise continuously differentiable. Note that integration by substitution yields

$$\int_{-\infty}^{\infty} V(\mu - \theta) \phi\left(\theta | s^i, \sigma^2\right) d\theta = \int_{-\infty}^{\infty} V(\pi) \phi\left(\mu - s^i - \pi | 0, \sigma^2\right) d\pi, \tag{11}$$

where we have used the fact that $\phi\left(\theta | s, \sigma^2\right) = \phi\left(\theta - s | 0, \sigma^2\right)$ since $\phi(\cdot)$ is the density of a normal distribution. Using (11) to substitute in (9), action μ^i must satisfy the following first-order condition of the agent:

$$\alpha - \left(\mu^i - s^i\right) + \int_{-\infty}^{\infty} V(\pi) \phi'\left(\mu^i - s^i - \pi | 0, \sigma^2\right) d\pi = 0 \text{ for } i = L, H. \tag{12}$$

Condition (12) is necessary for the rule to be incentive compatible. Its solution is $\mu^i = s^i + \kappa$ for $i = L, H$ and some $\kappa \gtreqless 0$, where κ is independent of the agent's type i. The latter observation allows us to simplify the principal's problem as welfare then also becomes independent of i.

Define a *maximally-enforced target threshold* π^* as a rule that specifies the maximal reward $\overline{V}$ if the outcome is below a threshold π^* and the maximal punishment $\underline{V}$ if the outcome exceeds this threshold. We find:

Proposition 2. *The optimal target-based rule admits* $\mu^i = s^i + \kappa$, $V(\pi) = \overline{V}$ *if* $\pi \leq \pi^*$, *and* $V(\pi) = \underline{V}$ *if* $\pi > \pi^*$, *for* $i = L, H$, *some* $\kappa \in (0, \alpha)$, *and some* $\pi^* > \kappa$. *This rule can be implemented with a maximally-enforced target threshold* π^*.

The optimal target-based rule provides incentives with a maximally-enforced target threshold π^*. Since a higher action μ results in a higher outcome π in expectation, an agent of type

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i responds to this target by choosing an action $s^i + \kappa$ which is below his flexible action $s^i + \alpha$. In contrast to the optimal instrument-based rule, here punishment occurs along the equilibrium path whenever $\pi > \pi^*$, so as to appropriately incentivize the agent. Proposition 2 shows that since punishment is costly, the principal limits its frequency by keeping the agent's action above the first-best action s^i. Moreover, it is shown that the induced average outcome is below the target, namely $\mathbb{E}(\pi) = \kappa < \pi^*$. A rule that yields $\mathbb{E}(\pi) = \kappa = \pi^*$ would be suboptimal, as it would entail punishing the agent half of the time (the frequency with which π would exceed π^*). In the optimal rule, the realized outcome π exceeds π^* less than half of the time so that the principal punishes the agent less often.

To prove Proposition 2, we follow a first-order approach and solve a relaxed version of (8)-(10) that replaces (9) with the agent's first-order condition (12). We establish that the solution to this relaxed problem takes the threshold form described above, and we show that Assumption 1 and Assumption 2 are sufficient to guarantee the validity of this first-order approach.[16]

High-powered incentives as those described in Proposition 2 are common in moral hazard settings; see Abreu, Pearce, and Stacchetti (1990).[17] These incentives arise here because punishments cannot directly depend on the agent's action under a target-based rule. We note that the results of this section extend to a setting with a continuum of agent types: as noted above, the agent's first-order condition implies that the principal's welfare is independent of i, and hence also independent of the number of types.

3.3 Optimal Class of Rule

Our main result uses the characterizations in Proposition 1 and Proposition 2 to compare the performance of instrument-based and target-based rules. We find that which class of rule is optimal for the principal depends on the precision of the agent's private information:

Proposition 3. *Take instrument-based and target-based rules and consider changing σ while keeping $Var(\theta)$ unchanged. There exists $\sigma^* > 0$ such that a target-based rule is strictly optimal if $\sigma < \sigma^*$ and an instrument-based rule is strictly optimal if $\sigma > \sigma^*$. The cutoff σ^* is decreasing in the agent's bias α and the worst continuation value $\underline{V}$.*

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[16]We consider a doubly-relaxed problem that takes (12) as a weak inequality constraint (cf. Rogerson, 1985) in order to establish the sign of the Lagrange multiplier on (12) and characterize the solution.

[17]Incentives are *bang-bang* because punishment enters the principal and agent's welfare functions linearly and the continuous outcome satisfies a monotone likelihood ratio condition. See Abreu, Pearce, and Stacchetti (1990) for further discussion.

To see the logic, consider first how the principal's welfare under each class of rule changes as we vary the precision of the agent's information σ^{-1}, while keeping the shock variance $Var(\theta)$ unchanged. Since the optimal instrument-based rule gives no flexibility to the agent to use his private information, the principal's welfare under this rule is invariant to σ. In fact, by Proposition 1 and $Var(\theta) = \mathbb{E}(\theta^2)$ (since $\mathbb{E}(\theta) = 0$), the principal's welfare under the optimal instrument-based rule is given by

$$-\frac{Var(\theta)}{2} + \overline{V},$$

independent of σ. In contrast, using Proposition 2, one can verify that the principal's welfare under the optimal target-based rule is decreasing in σ, that is increasing in the precision of the agent's information. Intuitively, a better informed agent can more closely tailor his action to the shock, and is less likely to trigger punishment by overshooting the target specified by the principal. As a result, a higher precision reduces the outcome volatility and the principal's cost of providing high-powered incentives under a target-based rule.

These comparative statics imply that to prove the first part of Proposition 3, it suffices to show that a target-based rule is optimal for high enough precision of the agent's information whereas an instrument-based rule is optimal otherwise. Consider the extreme in which the agent is perfectly informed, that is, $\sigma \to 0$ and $\Delta \to \sqrt{Var(\theta)}$. In this case, the optimal target-based rule sets a threshold $\pi^* = 0$, providing steep incentives and inducing the first-best action. Note that this rule involves no punishments along the equilibrium path, as a perfectly informed agent of type $i = L, H$ chooses $\mu^i = s^i$ to avoid punishment. Consequently, in this limit case, the optimal target-based rule yields welfare

$$\overline{V} > -\frac{Var(\theta)}{2} + \overline{V},$$

and thus it dominates the optimal instrument-based rule.

Consider next the extreme in which the agent is uninformed, that is, $\sigma \to \sqrt{Var(\theta)}$ and $\Delta \to 0$. In this case, the optimal instrument-based rule guarantees the principal her preferred outcome given no information by tying the hands of the agent. Instead, the principal cannot implement her ex-ante optimum with a target-based rule, which gives the agent unnecessary discretion and requires punishments to provide incentives. The optimal target-based rule in this limit case sets a threshold $\pi^* > 0$, inducing an agent of type $i = L, H$ to choose $\mu^i = s^i + \kappa$ for $\kappa > 0$ and yielding welfare

$$-\frac{Var(\theta)}{2} + \overline{V} - \frac{\kappa^2}{2} - \Phi\left(\kappa - \pi^* | 0, \sigma^2\right)\left(\overline{V} - \underline{V}\right) < -\frac{Var(\theta)}{2} + \overline{V}.$$

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Thus, this rule is dominated by the optimal instrument-based rule.

The second part of Proposition 3 shows that the benefit of using a target-based rule over an instrument-based rule is decreasing in the bias of the agent and increasing in the severity of punishment. The less biased is the agent, the less costly is incentive provision under a target-based rule, as relatively infrequent punishments become sufficient to deter high actions. Similarly, the harsher is the punishment experienced by the agent for missing the target, the less often punishment needs to be used on the equilibrium path to provide incentives under a target-based rule. In contrast, the optimal instrument-based rule is independent of the agent's bias and the severity of punishment. As such, target-based rules dominate instrument-based rules for a larger range of parameters if the agent's bias is relatively low or punishment is relatively severe.

4 Hybrid Rules

A hybrid rule combines features of instrument-based and target-based rules, with a continuation value $V(\mu, \theta)$ that depends freely on μ and θ. For $i = L, H$, denote by $V^i(\theta)$ the continuation value assigned to agent type i as a function of the shock θ. An optimal hybrid rule solves the following program:

$$\max_{\mu^L, \mu^H, V^L(\theta), V^H(\theta)} \sum_{i=L,H} \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta)^2}{2} + V^i(\theta) \right] \phi(\theta|s^i, \sigma^2) d\theta \tag{13}$$

subject to, for $i = L, H$,

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V^i(\theta) \right] \phi(\theta|s^i, \sigma^2) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{(\mu^{-i} - \theta - \alpha)^2}{2} + V^{-i}(\theta) \right] \phi(\theta|s^i, \sigma^2) d\theta, \tag{14}$$

$$\int_{-\infty}^{\infty} \left[-\frac{(\mu^i - \theta - \alpha)^2}{2} + V^i(\theta) \right] \phi(\theta|s^i, \sigma^2) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{(s^i - \theta)^2}{2} + \underline{V} \right] \phi(\theta|s^i, \sigma^2) d\theta, \tag{15}$$

$$V^i(\theta) \in [\underline{V}, \overline{V}] \text{ for all } \theta. \tag{16}$$

The solution to this program gives the principal the highest welfare that she can achieve given the private information of the agent. Note that constraints (14)-(15) are analogous to (5)-(6) in the program that solves for the optimal instrument-based rule, but they now allow the continuation value to depend on the shock θ in addition to the agent's type i. Note also

that by analogous arguments as those used to solve for the optimal target-based rule, the continuation value $V^i(\theta)$ can be equivalently written as a function of the outcome, $V^i(\pi)$. We use this formulation in what follows to ease the interpretation.

Define a *maximally-enforced hybrid threshold* $\{\mu^*, \mu^{**}, \pi^*(\mu)\}$ as a rule that specifies the maximal reward $\overline{V}$ if the outcome is below a threshold $\pi^*(\mu)$ and the maximal punishment $\underline{V}$ if the outcome exceeds this threshold, where $\pi^*(\mu)$ is a function of the agent's action:

$$\pi^*(\mu) = \begin{cases} \infty & \text{if } \mu \le \mu^* \\ h(\mu) & \text{if } \mu \in (\mu^*, \mu^{**}] \\ -\infty & \text{if } \mu > \mu^{**} \end{cases} \tag{17}$$

for some continuous function $h(\mu) \in (-\infty, \infty)$ which satisfies $\lim_{\mu \downarrow \mu^*} h(\mu) = \infty$. The cutoff μ^* is a soft instrument threshold, where any action $\mu \le \mu^*$ is rewarded independently of the outcome with maximal reward $\overline{V}$. The cutoff $\mu^{**} > \mu^*$ is a hard instrument threshold, where any action $\mu > \mu^{**}$ is punished independently of the outcome with maximal punishment $\underline{V}$. Intermediate actions $\mu \in (\mu^*, \mu^{**}]$ are maximally rewarded if the outcome satisfies $\pi \le \pi^*(\mu)$ and maximally punished if the outcome satisfies $\pi > \pi^*(\mu)$. Therefore, an interior target threshold only applies to intermediate actions.

We find:

Proposition 4. *The optimal hybrid rule admits $\mu^L < \mu^H$, $V^L(\pi) = \overline{V}$ for all π, $V^H(\pi) = \overline{V}$ if $\pi \le \pi^*(\mu^H)$, and $V^H(\pi) = \underline{V}$ if $\pi > \pi^*(\mu^H)$, for some $\pi^*(\mu^H) \in (-\infty, \infty)$. This rule can be implemented with a maximally-enforced hybrid threshold $\{\mu^*, \mu^{**}, \pi^*(\mu)\}$, where $\mu^* = \mu^L$ and $\mu^{**} = \mu^H$.*

The optimal hybrid rule assigns a relatively low action and the maximal reward to the low type, while specifying a higher action and a target threshold for the high type. To prove this result, we solve a relaxed version of (13)-(16) which ignores the information constraint (14) for the high type and the enforcement constraints (16) for both types. We establish that the solution to this relaxed problem takes the form described in Proposition 4 and satisfies these constraints.

The optimal hybrid rule dominates instrument-based rules by giving the agent more flexibility to respond to his private information while preserving incentives. Specifically, under a hybrid rule, the principal can allow the agent to choose actions $\mu > \mu^*$ and still deter excessively high actions by using a target-based criterion. Analogously, the optimal hybrid rule dominates target-based rules by more efficiently limiting the agent's discretion to choose actions that are excessively high. That is, under a hybrid rule, the principal can avoid punishments under

actions $\mu \le \mu^*$ and directly punish the agent for actions $\mu > \mu^{**}$, regardless of the realized outcome. As a result, a hybrid rule allows the principal to reduce the frequency of punishment along the equilibrium path.

While combining instruments and targets could in principle yield rules with complicated forms, Proposition 4 shows that the optimal hybrid rule admits an intuitive implementation. This rule essentially consists of an instrument threshold μ^* which is relaxed to μ^{**} whenever the target threshold is satisfied. As noted in the Introduction, rules of this form have been advocated in practice in the context of monetary policy.

5   Concluding Remarks

Using mechanism design, we have characterized optimal instrument-based and target-based rules, studied the conditions under which each class is optimal, and characterized the optimal hybrid rule that combines instruments and targets. As discussed in the Introduction, our results may shed light on a number of applications. For example, in the context of monetary policy, our analysis implies that inflation targeting should be adopted if the central bank has significantly superior information relative to the public; otherwise a Taylor rule would perform better. We found that inflation targeting has a larger advantage over Taylor rules if the central bank's inflationary bias is relatively small or the sanctions that can be imposed for missing the inflation target are relatively large. Furthermore, we have shown that an optimal hybrid rule would guide the choice of the interest rate as in a Taylor rule but relax these requirements when realized inflation satisfies some specified target, similar to the measures proposed by policymakers in the aftermath of the Global Financial Crisis.

We considered a stylized model that could be extended in different directions. First, while we have limited attention to two agent types, one can show that our main insights continue to hold under a continuum of types. As discussed in Section 3, given certain distributional assumptions, the optimal instrument-based and target-based rules continue to take a threshold form, and under suitably modified versions of our Assumption 1 and Assumption 2, the analogs of Proposition 1 and Proposition 2 continue to hold.[18] Consequently, our result in Proposition 3, on when one class of rule dominates the other, applies to a continuum of types.

A second possible extension would be to relax our assumption of a common continuation value, capturing the fact that, in reality, sanctions may harm the agent more than the principal.

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[18]The equivalent of Assumption 1 would guarantee that the agent's bias is sufficiently large so that the optimal instrument-based rule—assuming full enforcement—admits bunching. The equivalent of Assumption 2 would guarantee that enforcement constraints are non-binding in the instrument-based rule.

Specifically, consider multiplying the continuation value $V(\mu^i, \theta)$ in the agent's welfare function (1) by a finite constant $c > 1$, similar to the "leaky bucket" setting studied by Amador and Bagwell (2013). We can show that for an agent bias α large enough relative to c, the optimal instrument-based rule continues to be a maximally-enforced instrument threshold with no punishments on path, and thus the principal's welfare under this rule is unchanged by c. Moreover, the optimal target-based rule continues to be a maximally-enforced target threshold, and since raising c relaxes the principal's problem, her welfare in this case increases.[19] Therefore, we find that our main results extend, with asymmetric punishments favoring target-based rules over instrument-based rules on the margin.

Finally, a third extension could consider an agent bias that is unknown to the principal and may take different signs. In the application to monetary policy, central bankers may be biased in favor or against inflation relative to society, and their preferences may not be public. Under assumptions analogous to our Assumption 1 and Assumption 2, an instrument-based rule that bunches all agent types regardless of their bias would be optimal, so our characterization in Proposition 1 would remain valid. A characterization of the optimal target-based rule, on the other hand, would have to deal with the problem that assigned actions may now depend on the agent's bias. We leave this question for future research.

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[19]The claim about the optimal target-based rule is straightforward to verify given our proof of Proposition 2. As for the optimal instrument-based rule, we can show that replacing Assumption 1 with $\alpha/c \geq 2\Delta$ is sufficient for Proposition 1 to hold. Details are available upon request.

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A   Appendix

A.1   Proof of Proposition 1

We proceed in three steps.

Step 1.   We solve a relaxed version of (4)-(7) which ignores (5) for $i = H$ and (6) for $i = L, H$. Step 2 verifies that the solution to this relaxed problem satisfies these constraints.

<u>Step 1a.</u>   We show that the solution satisfies (5) for $i = L$ as an equality. If this were not the case, then the principal would optimally set $\mu^i = s^i$ and $V^i = \overline{V}$ for $i = L, H$. However, (5) for $i = L$ would then become

$$\int_{-\infty}^{\infty} \left[-\frac{\left(\left(s^L - \theta\right) - \alpha\right)^2}{2} \right] \phi\left(\theta | s^L, \sigma^2\right) d\theta \geq \int_{-\infty}^{\infty} \left[-\frac{\left(\left(s^L - \theta\right) + \left(s^H - s^L - \alpha\right)\right)^2}{2} \right] \phi\left(\theta | s^L, \sigma^2\right) d\theta,$$

which after some algebra yields

$$\left(s^H - s^L\right)\left(s^H - s^L - 2\alpha\right) \geq 0.$$

This inequality contradicts Assumption 1. Thus, (5) for $i = L$ must bind.

<u>Step 1b.</u>   We show that the solution satisfies $\mu^H \geq \mu^L$. Suppose by contradiction that $\mu^H < \mu^L$. Consider two perturbations, one assigning μ^L and $\overline{V}$ to both types, and another assigning μ^H and $\overline{V}$ to both types. Since these perturbations are feasible and incentive compatible, the contradiction assumption requires that neither of them strictly increase welfare, which requires:

$$s^H \mu^H - \frac{(\mu^H)^2}{2} \geq s^H \mu^L - \frac{(\mu^L)^2}{2} \quad \text{and} \quad s^L \mu^L - \frac{(\mu^L)^2}{2} \geq s^L \mu^H - \frac{(\mu^H)^2}{2}.$$

This is equivalent to

$$\left(\mu^H - \mu^L\right)\left[s^H - \frac{\left(\mu^H + \mu^L\right)}{2}\right] \geq 0 \quad \text{and} \quad \left(\mu^H - \mu^L\right)\left[s^L - \frac{\left(\mu^H + \mu^L\right)}{2}\right] \leq 0.$$

For $\mu^H < \mu^L$, these inequalities require

$$s^H - \frac{\left(\mu^H + \mu^L\right)}{2} \leq 0 \leq s^L - \frac{\left(\mu^H + \mu^L\right)}{2},$$

which cannot hold given $s^H > s^L$. Therefore, $\mu^H \geq \mu^L$.

<u>Step 1c.</u> We show that the solution satisfies $V^L = V^H = \overline{V}$. Note first that if $V^L < \overline{V}$, then an increase in V^L is feasible, relaxes constraint (5) for $i = L$, and strictly increases the objective. Hence, $V^L = \overline{V}$, and therefore (5) for $i = L$ (which binds by Step 1a) can be rewritten as:

$$\left(s^L + \alpha\right)\mu^L - \frac{(\mu^L)^2}{2} + \overline{V} = \left(s^L + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} + V^H. \tag{18}$$

This equation implies that, up to an additive constant independent of the allocation, the high type's welfare satisfies

$$\left(s^H + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} + V^H = \left(s^L + \alpha\right)\mu^L - \frac{(\mu^L)^2}{2} + \overline{V} + \left(s^H - s^L\right)\mu^H. \tag{19}$$

Now suppose by contradiction that $V^H < \overline{V}$. Then it follows from (18) and Step 1b that $\mu^H > \mu^L$. Substituting (19) into the objective in (4), the principal's welfare up to an additive constant independent of the allocation is equal to

$$\left(s^L + \frac{1}{2}\alpha\right)\mu^L - \frac{(\mu^L)^2}{2} - \frac{1}{2}\left(\alpha - \left(s^H - s^L\right)\right)\mu^H + \overline{V}. \tag{20}$$

Consider a perturbation that reduces μ^H to μ^L and increases V^H to $\overline{V}$. This perturbation is feasible, satisfies (18), and strictly increases the principal's welfare given the representation in (20) and Assumption 1. It follows that $V^H < \overline{V}$ cannot hold, and thus $V^H = \overline{V}$ in the solution.

<u>Step 1d.</u> We show that the solution satisfies $\mu^L = \mu^H = 0$. By Step 1b, if $\mu^L \neq \mu^H$, then $\mu^H > \mu^L$. However, a perturbation that reduces μ^H to μ^L is then feasible, satisfies (18), and strictly increases the principal's welfare given the representation in (20) and Assumption 1, yielding a contradiction. It follows that $\mu^L = \mu^H$, and since $\mathbb{E}(\theta) = 0$, the principal's welfare in (4) conditional on $\mu^L = \mu^H$ is maximized at $\mu^L = \mu^H = 0$.

Step 2. We verify that the solution to the relaxed problem in Step 1 satisfies the constraints of the original problem. Since $\mu^L = \mu^H$ and $V^L = V^H$, constraint (5) for $i = H$ is satisfied. As for the constraints in (6), given $\mu^L = \mu^H = 0$ these require, for $i = L, H$,

$$\overline{V} \geq \left(s^i + \alpha\right)\left(s^i + \alpha\right) - \frac{(s^i + \alpha)^2}{2} + \underline{V},$$

which reduces to

$$\overline{V} - \underline{V} \geq \frac{(s^i + \alpha)^2}{2}.$$

By Assumption 1, this inequality holds for $i = L, H$ if

$$\overline{V} - \underline{V} \geq \frac{\alpha^2/4 + \alpha^2 + \alpha^2}{2},$$

which is satisfied by Assumption 2 and the fact that $2\phi(1|0,1) < 0.5$.

Step 3. We verify that a maximally-enforced instrument threshold $\mu^* = 0$ implements the solution. Given (1) and (3), conditional on choosing an action $\mu \leq \mu^*$ and receiving continuation value $\overline{V}$, the agent's optimal action choice is $\mu = \mu^*$ regardless of his type. Moreover, conditional on choosing an action $\mu > \mu^*$ and receiving continuation value $\underline{V}$, the agent's optimal choice is $s^i + \alpha$ for each $i = L, H$. The enforcement constraints in (6) guarantee that the agent has no incentive to deviate to $\mu > \mu^*$.

A.2   Proof of Proposition 2

We proceed in two steps.

Step 1. We follow a first-order approach by solving a relaxed version of (8)-(10) that replaces (9) with the agent's first-order condition (12). Step 2 verifies the validity of this approach.

As noted in the text, the solution to (12) is $\mu^i = s^i + \kappa$ for $i = L, H$ and some $\kappa \gtrless 0$. Hence, the relaxed problem can be written as:

$$\max_{\kappa, V(\pi)} \left\{ -\frac{\kappa^2}{2} + \int_{-\infty}^{\infty} V(\pi)\, \phi(\kappa - \pi|0, \sigma^2) d\pi \right\} \tag{21}$$

subject to

$$\alpha - \kappa + \int_{-\infty}^{\infty} V(\pi)\, \phi'(\kappa - \pi|0, \sigma^2) d\pi = 0, \tag{22}$$

$$V(\pi) \in \left[\underline{V}, \overline{V}\right] \text{ for all } \pi. \tag{23}$$

Step 1a.   Denote by λ the Lagrange multiplier on (22). We show that $\lambda < 0$. To do this, we consider a doubly-relaxed problem in which constraint (22) is replaced with an inequality constraint (cf. Rogerson, 1985):

$$\alpha - \kappa + \int_{-\infty}^{\infty} V(\pi)\phi'(\kappa - \pi|0, \sigma^2) d\pi \leq 0. \tag{24}$$

Since this is an inequality constraint, the multiplier satisfies $\lambda \leq 0$. We show that (24) holds as an equality in the solution to the doubly-relaxed problem, and thus this problem is equivalent to (21)-(23) with $\lambda < 0$. Suppose by contradiction that (24) holds as a strict inequality. Then to maximize (21) the principal chooses $\kappa = 0$ and $V(\pi) = \overline{V}$ for all π. However, substituting back into the left-hand side of (24), using the fact that $\phi'(\kappa - \pi|0, \sigma^2) = \frac{\pi - \kappa}{\sigma^2}\phi(\kappa - \pi|0, \sigma^2)$, yields

$$\alpha + \overline{V} \int_{-\infty}^{\infty} \frac{\pi}{\sigma^2}\phi(-\pi|0, \sigma^2)d\pi = \alpha \leq 0,$$

which is a contradiction since $\alpha > 0$. Therefore, (24) holds as an equality in the doubly-relaxed problem and $\lambda < 0$.

Step 1b. We show that the solution to (21)-(23) satisfies $V(\pi) = \overline{V}$ if $\pi \leq \pi^*$ and $V(\pi) = \underline{V}$ if $\pi > \pi^*$, for some $\pi^* \in (-\infty, \infty)$. Denote by $\overline{\psi}(\pi)$ and $\underline{\psi}(\pi)$ the Lagrange multipliers on the upper bounds and the lower bounds on $V(\pi)$. The first-order condition with respect to $V(\pi)$ is

$$\phi(\kappa - \pi|0, \sigma^2) + \lambda\phi'(\kappa - \pi|0, \sigma^2) + \underline{\psi}(\pi) - \overline{\psi}(\pi) = 0. \tag{25}$$

Suppose that $V(\pi)$ is interior with $\underline{\psi}(\pi) = \overline{\psi}(\pi) = 0$. Then (25) yields

$$-\frac{1}{\lambda} = \frac{\phi'(\kappa - \pi|0, \sigma^2)}{\phi(\kappa - \pi|0, \sigma^2)} = \frac{\pi - \kappa}{\sigma^2}. \tag{26}$$

Since the right-hand side of (26) is strictly increasing in π whereas the left-hand side is constant, it follows that (26) holds for only one value of $\pi \in (-\infty, \infty)$, which we label π^*. By (25) and (26), the solution has $V(\pi) = \overline{V}$ if $\pi \leq \pi^*$ and $V(\pi) = \underline{V}$ if $\pi > \pi^*$.

Step 1c. We show that $\pi^* > \kappa$ and $\kappa \in (0, \alpha)$. To show the first inequality, recall from Step 1a that $\lambda < 0$; hence, (26) yields $\pi^* > \kappa$. To show $\kappa < \alpha$, note that by Step 1b, (22) can be rewritten as

$$\alpha - \kappa - \phi(\kappa - \pi^*|0, \sigma^2)(\overline{V} - \underline{V}) = 0. \tag{27}$$

Since $\phi(\kappa - \pi^*|0, \sigma^2)(\overline{V} - \underline{V}) > 0$, (27) requires $\kappa < \alpha$.

We are left to show that $\kappa > 0$. By Step 1b, we can write the Lagrangian of the principal solving for the optimal level of κ and π^* as

$$-\frac{\kappa^2}{2} + (1 - \Phi(\kappa - \pi^*|0, \sigma^2))\overline{V} + \Phi(\kappa - \pi^*|0, \sigma^2)\underline{V} \tag{28}$$
$$+\lambda\left[\alpha - \kappa - \phi(\kappa - \pi^*|0, \sigma^2)(\overline{V} - \underline{V})\right].$$

The first-order condition with respect to κ is

$$- \kappa - \phi(\kappa - \pi^* | 0, \sigma^2) \left(\overline{V} - \underline{V}\right) - \lambda \left[1 + \phi'(\kappa - \pi^* | 0, \sigma^2) \left(\overline{V} - \underline{V}\right)\right] = 0, \qquad (29)$$

and the first-order condition with respect to π^* is

$$\phi(\kappa - \pi^* | 0, \sigma^2) \left(\overline{V} - \underline{V}\right) + \lambda \phi'(\kappa - \pi^* | 0, \sigma^2) \left(\overline{V} - \underline{V}\right) = 0. \qquad (30)$$

Substituting (30) into (29) yields

$$- \lambda = \kappa. \qquad (31)$$

Since $\lambda < 0$ by Step 1a, (31) implies $\kappa > 0$.

Step 2. We verify the validity of the first-order approach: we establish that the choice of κ in the relaxed problem satisfies (9) and therefore corresponds to the agent's global optimum.

Step 2a. We begin by showing that the agent has no incentive to choose some $\kappa' \neq \kappa$, $\kappa' \leq \pi^*$. Differentiating the first-order condition (27) with respect to κ yields

$$- 1 - \phi'(\kappa - \pi^* | 0, \sigma^2) \left(\overline{V} - \underline{V}\right). \qquad (32)$$

Note that (32) is strictly negative for all $\kappa \leq \pi^*$, and thus the agent's welfare is strictly concave over this range. Since by Step 1c the solution to the relaxed problem sets $\kappa < \pi^*$, we conclude that this κ is a maximum and dominates any other $\kappa' \leq \pi^*$.

Step 2b. We next show that the agent has no incentive to choose some $\kappa' \neq \kappa$, $\kappa' > \pi^*$. To prove this, we first establish that in the solution to the relaxed problem, given π^*, κ satisfies $\kappa - \pi^* \leq -\sigma$. Suppose by contradiction that $\kappa - \pi^* > -\sigma$. Note that by (26) and (31), $\kappa - \pi^* = -\frac{\sigma^2}{\kappa}$. Hence, the contradiction assumption implies $\kappa > \sigma$. Substituting $\kappa - \pi^* = -\frac{\sigma^2}{\kappa}$ into (27) yields

$$\alpha - \kappa - \phi\left(-\frac{\sigma^2}{\kappa} | 0, \sigma^2\right) \left(\overline{V} - \underline{V}\right) = 0. \qquad (33)$$

Since the left-hand side of (33) is decreasing in κ and (by the contradiction assumption) $\kappa > \sigma$, (33) requires

$$\alpha - \sigma - \phi\left(-\sigma | 0, \sigma^2\right) \left(\overline{V} - \underline{V}\right) > 0.$$

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Multiply both sides of this equation by $\sigma > 0$ to obtain:

$$\sigma(\alpha - \sigma) - \sigma\phi\left(-\sigma|0,\sigma^2\right)\left(\overline{V} - \underline{V}\right) > 0. \tag{34}$$

Note that since $0 < \sigma < \sqrt{Var(\theta)}$ and, by Assumption 1, $\sqrt{Var(\theta)} \leq \alpha/2$, we have $\sigma(\alpha - \sigma) < \alpha^2/2$. Hence, (34) yields

$$\frac{\alpha^2}{2\sigma\phi\left(-\sigma|0,\sigma^2\right)} > \overline{V} - \underline{V}. \tag{35}$$

However, this inequality violates Assumption 2 since $\sigma\phi\left(-\sigma|0,\sigma^2\right) = \phi\left(1|0,1\right)$. Thus, given π^*, κ satisfies $\kappa - \pi^* \leq -\sigma$.

We can now establish that the agent has no incentive to deviate to $\kappa' \neq \kappa$, $\kappa' > \pi^*$. Consider some $\kappa' > \pi^*$ that is a local maximum for the agent. The difference in welfare for the agent from choosing the value of κ given by the solution to the relaxed problem versus κ' is equal to

$$\alpha\kappa - \frac{\kappa^2}{2} - \left(\alpha\kappa' - \frac{(\kappa')^2}{2}\right) + \left(\Phi\left(\kappa' - \pi^*|0,\sigma^2\right) - \Phi\left(\kappa - \pi^*|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right). \tag{36}$$

Note that by the arguments in Step 1c and κ and κ' satisfying the agent's first-order condition, it follows that both κ and κ' are between 0 and α. Thus, (36) is bounded from below by

$$-\frac{\alpha^2}{2} + \left(\Phi\left(\kappa' - \pi^*|0,\sigma^2\right) - \Phi\left(\kappa - \pi^*|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right). \tag{37}$$

Since (27) is satisfied for both κ and κ' and $\kappa' > \pi^* > \kappa$, we must have $\phi\left(\kappa - \pi^*|0,\sigma^2\right) > \phi\left(\kappa' - \pi^*|0,\sigma^2\right)$. Moreover, by the symmetry of the normal distribution, $\phi\left(\kappa - \pi^*|0,\sigma^2\right) = \phi\left(-(\kappa - \pi^*)|0,\sigma^2\right)$ and thus $\Phi\left(-(\kappa - \pi^*)|0,\sigma^2\right) < \Phi\left(\kappa' - \pi^*|0,\sigma^2\right)$. Therefore, (37) is bounded from below by

$$-\frac{\alpha^2}{2} + \left(\Phi\left(-(\kappa - \pi^*)|0,\sigma^2\right) - \Phi\left(\kappa - \pi^*|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right). \tag{38}$$

Since, as shown above, $\kappa - \pi^* \leq -\sigma$, we obtain that (38) is itself bounded from below by

$$-\frac{\alpha^2}{2} + \left(\Phi\left(\sigma|0,\sigma^2\right) - \Phi\left(-\sigma|0,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right) = -\frac{\alpha^2}{2} + \left(\Phi\left(1|0,1\right) - \Phi\left(-1|0,1\right)\right)\left(\overline{V} - \underline{V}\right) > 0,$$

where the last inequality follows from Assumption 2 and the fact that $\phi\left(1|0,1\right) < \Phi\left(1|0,1\right) - \Phi\left(-1|0,1\right)$. Therefore, the agent strictly prefers κ over κ'.

A.3 Proof of Proposition 3

We begin by proving the following lemma:

Lemma 1. *Consider changing σ while keeping $Var(\theta)$ unchanged. The principal's welfare is independent of σ under the optimal instrument-based rule and it is strictly decreasing in σ under the optimal target-based rule.*

Proof. By Proposition 1, an optimal instrument-based rule sets $\mu^i = 0$ and $V^i = \overline{V}$ for $i = L, H$. Since $Var(\theta) = \mathbb{E}(\theta^2) - (\mathbb{E}(\theta))^2 = \mathbb{E}(\theta^2)$ (by $\mathbb{E}(\theta) = 0$), the principal's welfare under this rule is equal to $-\frac{Var(\theta)}{2} + \overline{V}$, which is independent of σ.

To evaluate the principal's welfare under an optimal target-based rule, consider the Lagrangian taking into account the conditional variance term (which is exogenous and thus excluded from (28)):

$$-\frac{\sigma^2}{2} - \frac{\kappa^2}{2} + \left(1 - \Phi(\kappa - \pi^*|0, \sigma^2)\right)\overline{V} + \Phi(\kappa - \pi^*|0, \sigma^2)\underline{V}$$
$$+\lambda\left[\alpha - \kappa - \phi(\kappa - \pi^*|0, \sigma^2)\left(\overline{V} - \underline{V}\right)\right].$$

The derivative with respect to σ is:

$$-\sigma + \left(\overline{V} - \underline{V}\right)\left[\int_{\kappa-\pi^*}^{\infty}\left(-\frac{\sigma^2 - z^2}{\sigma^3}\right)\phi(z|0, \sigma^2)dz + \lambda\frac{\sigma^2 - (\kappa - \pi^*)^2}{\sigma^3}\phi(\kappa - \pi^*|0, \sigma^2)\right].$$

The first term is strictly negative. Using (26) and (31) to substitute in for λ and π^*, the sign of the second term is the same as the sign of

$$-\int_{-\frac{\sigma^2}{\kappa}}^{\infty}\left(\sigma^2 - z^2\right)\phi(z|0, \sigma^2)dz - \kappa\left[\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right]\phi\left(-\frac{\sigma^2}{\kappa}|0, \sigma^2\right). \tag{39}$$

We next show that this expression is strictly negative, which proves the claim. To show this, consider the derivative of (39) with respect to κ:

$$\left[\left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right)\frac{\sigma^2}{\kappa^2} - \left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right) - 2\left(\frac{\sigma^2}{\kappa}\right)^2 - \left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right)\frac{\sigma^2}{\kappa^2}\right]\phi\left(-\frac{\sigma^2}{\kappa}|0, \sigma^2\right).$$

This derivative takes the same sign as

$$-\left(\sigma^2 - \left(\frac{\sigma^2}{\kappa}\right)^2\right) - 2\left(\frac{\sigma^2}{\kappa}\right)^2,$$

which is strictly negative. Hence, since $\kappa > 0$, it suffices to show that the sign of (39) is weakly negative for $\kappa \to 0$. By the definition of variance, the first term in (39) goes to zero as $\kappa \to 0$. The second term in (39) can be rewritten as:

$$-\sigma^2 \kappa \phi \left(-\frac{\sigma^2}{\kappa} | 0, \sigma^2 \right) + \frac{\sigma^4}{\kappa} \phi \left(-\frac{\sigma^2}{\kappa} | 0, \sigma^2 \right). \tag{40}$$

As $\kappa \to 0$, the first term in (40) goes to zero. Moreover, applying L'Hopital's Rule on

$$\frac{1/\kappa}{\phi \left(-\frac{\sigma^2}{\kappa} | 0, \sigma^2 \right)^{-1}}$$

shows that the second term also goes to zero. □

We now proceed with the proof of Proposition 3. By Lemma 1, welfare under the optimal instrument-based rule is invariant to σ, whereas welfare under the optimal target-based rule is decreasing in σ. To prove the first part of the proposition, it thus suffices to show that a target-based rule is optimal at one extreme, for $\sigma \to 0$, and an instrument-based rule is optimal at the other extreme, for $\sigma \to \sqrt{Var(\theta)}$. This is what we prove next.

Consider first the case of $\sigma \to 0$. By the arguments in Step 1c and Step 2b of the proof of Proposition 2, $0 < \kappa \le \sigma$. Hence, $\kappa \to 0$ as $\sigma \to 0$. Moreover, as $\sigma \to 0$, $\phi(z|0,\sigma^2)$ corresponds to a Dirac's delta function, with cumulative distribution function $\Phi(z|0,\sigma^2) = 0$ if $z < 0$ and $\Phi(z|0,\sigma^2) = 1$ if $z \ge 0$. Therefore, since $\kappa - \pi^* < 0$ in the optimal target-based rule, the limit of the principal's welfare under this rule, as $\sigma \to 0$, is given by

$$\lim_{\sigma \to 0} \left\{ -\frac{\kappa^2}{2} + \left(1 - \Phi \left(\kappa - \pi^* | 0, \sigma^2 \right) \right) \overline{V} + \Phi \left(\kappa - \pi^* | 0, \sigma^2 \right) \underline{V} \right\} = \overline{V}.$$

Since the principal's welfare under the optimal instrument-based rule is $-\frac{Var(\theta)}{2} + \overline{V}$, it follows that the optimal target-based rule dominates the optimal instrument-based rule.

Consider next the case of $\sigma \to \sqrt{Var(\theta)}$ and thus $\Delta \to 0$. Since κ in the optimal target-based rule satisfies equation (33), the solution in this case admits $\kappa > 0$. The principal's welfare under the optimal target-based rule is then equal to

$$-\frac{Var(\theta)}{2} + \overline{V} - \frac{\kappa^2}{2} - \Phi \left(\kappa - \pi^* | 0, \sigma^2 \right) \left(\overline{V} - \underline{V} \right).$$

Since this is strictly lower than $-\frac{Var(\theta)}{2} + \overline{V}$, it follows that the optimal instrument-based rule dominates the optimal target-based rule.

Finally, to prove the second part of the proposition, note that the principal's welfare under the optimal instrument-based rule is independent of the agent's bias α and the punishment $\underline{V}$. Thus, it suffices to show that the principal's welfare under the optimal target-based rule is decreasing in α and $\underline{V}$. The former follows from the fact that the derivative of the Lagrangian in (28) with respect to α is equal to λ, which is strictly negative by Step 1a in the proof of Proposition 2. To evaluate how welfare changes with $\underline{V}$, consider the representation of the program in (8)-(10). A reduction in $\underline{V}$ relaxes constraint (10). Since this constraint is binding in the solution (by Step 1b of the proof of Proposition 2), it follows that a reduction in $\underline{V}$ strictly increases the principal's welfare under the optimal target-based rule.

A.4   Proof of Proposition 4

We proceed in three steps.

Step 1.   We solve a relaxed version of (13)-(16) which ignores (14) for $i = H$ and (15) for $i = L, H$. Step 2 verifies that the solution to this relaxed problem satisfies these constraints.

<u>Step 1a.</u>   We show that the solution satisfies (14) for $i = L$ as an equality. The proof of this claim is analogous to that in Step 1a of the proof of Proposition 1 and thus omitted.

<u>Step 1b.</u>   We show that the solution satisfies $\mu^H \geq \mu^L$. The proof of this claim is analogous to that in Step 1b of the proof of Proposition 1 and thus omitted.

<u>Step 1c.</u>   We show that the solution satisfies $V^L(\theta) = \overline{V}$ for all θ. If $V^L(\theta) < \overline{V}$ for some θ, then an increase in $V^L(\theta)$ is feasible, relaxes constraint (14) for $i = L$, and strictly increases the objective. The claim follows.

<u>Step 1d.</u>   We show that the solution satisfies $V^H(\theta) = \underline{V}$ if $\theta < \theta^*$ and $V^H(\theta) = \overline{V}$ if $\theta \geq \theta^*$, for some $\theta^* \in (-\infty, \infty)$. Denote by λ the Lagrange multiplier on (14) and by $\overline{\psi}(\theta)$ and $\underline{\psi}(\theta)$ the Lagrange multipliers on the upper bounds and the lower bounds on $V^H(\theta)$. The first-order condition with respect to $V^H(\theta)$ yields

$$\frac{1}{2}\phi(\theta|s^H, \sigma^2) - \frac{1}{2}\lambda\phi(\theta|s^L, \sigma^2) + \underline{\psi}(\theta) - \overline{\psi}(\theta) = 0. \tag{41}$$

Suppose that $V^H(\theta)$ is interior with $\underline{\psi}(\theta) = \overline{\psi}(\theta) = 0$. Then (41) implies

$$\lambda = \frac{\phi\left(\theta - s^H|0, \sigma^2\right)}{\phi\left(\theta - s^L|0, \sigma^2\right)}. \tag{42}$$

Since the right-hand side of (42) is strictly increasing in θ whereas the left-hand side is constant, it follows that (42) holds only for one value of $\theta \in (-\infty, \infty)$, which we label θ^*. By (41) and (42), the solution has $V(\theta) = \underline{V}$ if $\theta < \theta^*$ and $V(\theta) = \overline{V}$ if $\theta \geq \theta^*$.

<u>Step 1e.</u> We show that the solution satisfies $\mu^L \in [s^L, s^H]$ and $\mu^H \in [s^L, s^H]$ with $\mu^H > \mu^L$. Since $\lambda > 0$, it follows that θ^* satisfying (42) is interior. Hence, given Step 1b, the binding constraint (14) for $i = L$ implies $\mu^H > \mu^L$. The principal's first-order condition with respect to μ^L yields

$$\mu^L = s^L + \frac{\lambda}{1 + \lambda} \alpha,$$

which implies that μ^L, and thus μ^H, exceed s^L. The first-order condition with respect to μ^H yields

$$\mu^H = s^H - \frac{\lambda}{1 - \lambda} (\alpha - 2\Delta),$$

and the second-order condition yields $\lambda < 1$. Using Assumption 1, it follows that μ^H, and thus μ^L, are below s^H.

We end this step by observing that since $\lambda < 1$ and $s^H = -s^L = \Delta$, (42) implies $\theta^* < 0$.

Step 2. We verify that the solution to the relaxed problem satisfies the constraints of the original problem. The binding constraint (14) for $i = L$ implies

$$\overline{V} - \left(1 - \Phi\left(\theta^*|s^L, \sigma^2\right)\right)\overline{V} - \Phi\left(\theta^*|s^L, \sigma^2\right)\underline{V} = \left(s^L + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^L + \alpha\right)\mu^L + \frac{(\mu^L)^2}{2}. \quad (43)$$

Since $s^H > s^L$ and $\mu^H > \mu^L$, the right-hand side of (43) is strictly smaller than

$$\left(s^H + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^H + \alpha\right)\mu^L + \frac{(\mu^L)^2}{2}. \quad (44)$$

Moreover, the left-hand side of (43) is strictly larger than

$$\overline{V} - \left(1 - \Phi\left(\theta^*|s^H, \sigma^2\right)\right)\overline{V} - \Phi\left(\theta^*|s^H, \sigma^2\right)\underline{V}. \quad (45)$$

Therefore, (44) is strictly larger than (45), implying that constraint (14) for $i = H$ is satisfied.

To verify that constraint (15) for $i = L$ is satisfied, recall from Step 1e that $\mu^L \in [s^L, s^H]$. Given this range, the low type's welfare in the optimal rule is no less than that under $\mu^L = s^L$, and thus (15) for $i = L$ is guaranteed to hold if

$$\left(s^L + \alpha\right)s^L - \frac{(s^L)^2}{2} + \overline{V} \geq \left(s^L + \alpha\right)\left(s^L + \alpha\right) - \frac{(s^L + \alpha)^2}{2} + \underline{V}.$$

This inequality simplifies to

$$\overline{V} - \underline{V} \geq \frac{\alpha^2}{2},$$

which is guaranteed to hold by Assumption 2.

Finally, we verify that constraint (15) for $i = H$ is also satisfied. Note that by constraint (14) for $i = H$ being satisfied, the high type's welfare in the optimal rule is no less than that achieved from mimicking the low type under $\mu^L = s^L$. Thus, (15) for $i = H$ is guaranteed to hold if

$$\left(s^H + \alpha\right) s^L - \frac{(s^L)^2}{2} + \overline{V} \geq \left(s^H + \alpha\right) \left(s^H + \alpha\right) - \frac{(s^H + \alpha)^2}{2} + \underline{V}.$$

This inequality simplifies to

$$\overline{V} - \underline{V} \geq \frac{(2\Delta + \alpha)^2}{2},$$

which is guaranteed to hold by Assumption 1 and Assumption 2.

Step 3. We verify that a maximally-enforced hybrid threshold $\{\mu^*, \mu^{**}, \pi^*(\mu)\}$ implements the solution. Let $\mu^* = \mu^L$ and $\mu^{**} = \mu^H$. Construct $\pi^*(\mu)$ as described in (17), with $h(\mu)$ solving

$$\overline{V} - \left(1 - \Phi\left(\mu - h\left(\mu\right)|s^L, \sigma^2\right)\right)\overline{V} - \Phi\left(\mu - h\left(\mu\right)|s^L, \sigma^2\right)\underline{V} \tag{46}$$
$$= \left(s^L + \alpha\right)\mu - \frac{\mu^2}{2} - \left(s^L + \alpha\right)\mu^L + \frac{(\mu^L)^2}{2}.$$

The left-hand side of (46) is increasing in $\mu - h(\mu)$ and the right-hand side is increasing in μ. Note that $\lim_{\mu\downarrow\mu^*} h(\mu) = \infty$ and, by (43), $h(\mu^{**}) = \mu^{**} - \theta^*$. If follows that a solution for $h(\mu)$ exists and $\mu - h(\mu)$ is increasing in μ.

We verify that both agent types choose their prescribed actions, $\mu^L = \mu^*$ and $\mu^H = \mu^{**}$, under this maximally-enforced hybrid threshold. By Step 2, neither type has incentives to deviate to $\mu > \mu^{**}$, as the best such deviation entails choosing $\mu = s^i + \alpha$ for $i = L, H$ which is suboptimal by (15). The low type has no incentive to deviate to $\mu = \mu^{**}$ by (14) for $i = L$, and this type has no incentive to deviate to $\mu < \mu^*$ either as he is better off by instead choosing $\mu^* < s^L + \alpha$ and receiving the same continuation value. The high type has no incentive to deviate to $\mu \leq \mu^*$, as the best such deviation entails choosing $\mu^* < s^H + \alpha$ which is suboptimal by (14) for $i = H$. Therefore, it only remains to be shown that neither type has incentives to deviate to $\mu \in (\mu^*, \mu^{**})$. This follows immediately from (46) for the low type, as this equation ensures that the low type is indifferent between choosing μ^* and choosing any $\mu \in (\mu^*, \mu^{**})$.

To show that the high type has no incentive to deviate, combine (43) and (46) to obtain:

$$\left(\Phi\left(\theta^*|s^L,\sigma^2\right) - \Phi\left(\mu - h\left(\mu\right)|s^L,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right) = \left(s^L + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^L + \alpha\right)\mu + \frac{\mu^2}{2}.$$
(47)

Since $s^H > s^L$ and $\mu^H > \mu$, the right-hand side of (47) is strictly smaller than

$$\left(s^H + \alpha\right)\mu^H - \frac{(\mu^H)^2}{2} - \left(s^H + \alpha\right)\mu + \frac{\mu^2}{2}.$$
(48)

Moreover, note that $\mu - h\left(\mu\right) < \theta^*$ for $\mu \in (\mu^*,\mu^{**})$, and $\theta^* < 0$ by Step 1e. Hence, the left-hand side of (47) is strictly larger than

$$\left(\Phi\left(\theta^*|s^H,\sigma^2\right) - \Phi\left(\mu - h\left(\mu\right)|s^H,\sigma^2\right)\right)\left(\overline{V} - \underline{V}\right).$$
(49)

Therefore, (48) is strictly larger than (49), implying that the high type has no incentive to deviate to $\mu \in (\mu^*,\mu^{**})$.