NBER WORKING PAPER SERIES

SOME SIMPLE BITCOIN ECONOMICS

Linda Schilling
Harald Uhlig

Working Paper 24483
http://www.nber.org/papers/w24483

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
April 2018

Uhlig has an ongoing consulting relationship with a Federal Reserve Bank, the Bundesbank and the ECB. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

NBER working papers are circulated for discussion and comment purposes. They have not been peer-reviewed or been subject to the review by the NBER Board of Directors that accompanies official NBER publications.

© 2018 by Linda Schilling and Harald Uhlig. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Some Simple Bitcoin Economics
Linda Schilling and Harald Uhlig
NBER Working Paper No. 24483
April 2018
JEL No. D50,E40,E42,E50

ABSTRACT

How do Bitcoin prices evolve? What are the consequences for monetary policy? We answer these questions in a novel, yet simple endowment economy. There are two types of money, both useful for transactions: Bitcoins and Dollars. A central bank keeps the real value of Dollars constant, while Bitcoin production is decentralized via proof-of-work. We obtain a "fundamental condition," which is a version of the exchange-rate indeterminacy result in Kareken-Wallace (1981), and a "speculative" condition. Under some conditions, we show that Bitcoin prices form convergent supermartingales or submartingales and derive implications for monetary policy.

Linda Schilling
Utrecht University Netherlands
lin.schilling@gmail.com

Harald Uhlig
Dept. of Economics
University of Chicago
1126 E 59th Street
Chicago, IL 60637
and NBER
huhlig@uchicago.edu

1 Introduction

Cryptocurrencies, in particular Bitcoin, have received a large amount of attention in the news as of late. Indeed, the price movements have been quite spectacular, see figure 1. Bitcoins were valued below 5$ in September 2011, with an intermittent peak near 1000$ in December 2013, and were still trading below 240$ in October 2015 and below 1000$ in January 2017, before reaching a peak of more than 19000$ in December 2017 and then falling drastically below 7000$ in February 2018.

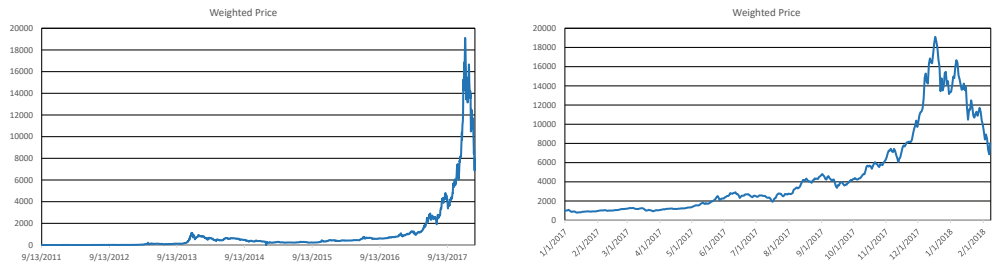

Figure 1: The Bitcoin Price since 2011-09-13 and "zooming in" only since 2017-01-01. BitStamp data per quandl.com.

These developments have given rise to a number of questions by the public and policy makers alike. How do Bitcoin prices evolve? What are the consequences for monetary policy? The purpose of this paper is to shed light on some of these questions, using a rather simple model framework, described in section 3 and analyzed in section 4. One may think of the model as a simplified version of the Bewley model, see Bewley (1977), the turnpike model of money as in Townsend (1980) or the new monetarist view of money as a medium of exchange as in Kiyotaki-Wright (1989) or Lagos-Wright (2005). With these models as well as with Samuelson (1958), we share the perspective that money is an intrinsically worthless asset, useful for executing trades between people who do not share a double-coincidence of wants. Our aim here is decidedly not to provide a new microfoundation for the use of money, but to provide a simple starting point for our analysis.

That said, our model is new, relative to the existing literature. We assume

that there are two types of infinitely-lived agents, who alternate in the periods, in which they produce and in which they wish to consume. This lack of the double-coincidence of wants then provides a role for a medium of exchange. We assume that there are two types of money: Bitcoins and Dollars. A central bank keeps the real value of Dollars constant via appropriate monetary injections, while Bitcoin production is decentralized via proof-of-work. Both monies can be used for transactions.

The key perspective for much of the analysis is the celebrated exchange-rate indeterminacy result in Kareken-Wallace (1981). We therefore obtain a version of their result, and show that the Bitcoin price in Dollar follows a martingale, adjusted for the pricing kernel, under a certain "fundamental condition".However, and in contrast to the two-period OLG-based results in Kareken-Wallace (1981), we also find that there is a "speculative condition", in which the Dollar price for the Bitcoin is expected to rise, and some agents start hoarding Bitcoin in anticipation of the price increase. We then provide conditions, so that this speculation in Bitcoins does not arise in equilibrium. Under some conditions, we show that Bitcoin prices form convergent supermartingales or convergent submartingales.

In section 5 we discuss how the presence of Bitcoins affect monetary policy. We discuss two different scenarios. In a first scenario, we assume that the Bitcoin price evolves independently of the central bank's policy. The central bank then sets transfers for price stabilization depending on the Bitcoin price. Under this 'conventional' view, we can show that under some conditions, the Bitcoin price is a bounded martingale and thus converges. Therefore, up or downward trends in the Bitcoin price are unlikely in the long run. In the second scenario we adapt the view that the central bank can set her transfers independently of the Bitcoin price and production in the economy, but still achieves price stability. In that case, the market clearing Bitcoin price is driven by the central bank's policy and we can characterize the Bitcoin price distribution as function of the Dollar quantity, the Bitcoin quantity and the distribution of production. We show that high Bitcoin price realizations become less likely as the central bank increases the Dollar quantity or as the Bitcoin quantity

grows over time. Also the Bitcoin price is higher in expectation if the economy becomes more productive.

Section 6 provides a general method for constructing equilibria, as well as specific examples, illustrating the key results. In section A we extend our analysis allowing for inflation. We finally draw our conclusions in section 7.

The most closely related contribution in the literature to our paper is Garratt-Wallace (2017). Like us, they adopt the Kareken-Wallace (1981) perspective to study the behavior of the Bitcoin-to-Dollar exchange rate. However, there are a number of differences. They utilize a two-period OLG model: the speculative condition does not arise there. They focus on fixed stocks of Bitcoins and Dollar (or "government issued monies"), while we allow for Bitcoin production and monetary policy. Production is random here and constant there. There is a carrying cost for Dollars, which we do not feature here. They focus on particular processes for the Bitcoin price. The analysis and key results are very different from ours. We discuss further related literature in section 2.

2 Some background and literature

The original Bitcoin idea and the key elements of its construction and trading system are described by the mysterious author Nakamoto (2008). They can perhaps be summarized and contrasted with traditional central-bank issued money as follows: this will be useful for our analysis. While there are many currencies, each currency is traditionally issued by a single monopolist, the central bank, which is typically a government organization. Most currencies are fiat currencies, i.e. can be produced at near-zero marginal costs. Central banks conduct their monetary operations typically with the primary objective of maintaining price stability as well as a number of perhaps secondary economic objectives. Private intermediaries can offer inside money, but in doing so are regulated and constrained by central banks or other bank regulators, and often need to obtain central bank money to do so. By contrast, Bitcoin is issued in a decentralized manner. A Bitcoin is an entry in

an electronic, publicly available ledger or blockchain. Issuing or creating (or "mining") a Bitcoin requires solving a changing mathematical problem. Anyone who solves the problem can broadcast the solution to the Bitcoin-using community. Obtaining a solution is hard and becoming increasingly harder, while checking the correctness of the solution is relatively easy. This "proof of work" for creating a Bitcoin thus limits the inflow of new Bitcoins. Bitcoins and fractions of Bitcoins can be transferred from one owner to the next, per broadcasting the transaction to the community, and adding that transaction to the ledger information or blockchain. Transaction costs may be charged by the community, which keeps track of these ledgers.

There is an increasing number of surveys or primers on the phenomenon, its technical issues or its regulatory implications, often provided by economists working for central banks or related agencies and intended to inform and educate the public as well as policy makers. Velde (2013) as well as Brito and Castillo (2013) provide early and excellent primers on Bitcoin. Weber (2013) assesses the potential of the Bitcoin system to become a useful payment system, in comparison to current practice. Badev and Chen (2014) provide an in-depth account of the technical background. Digital Currencies have received a handbook treatment by Lee (2015), collecting contributions by authors from various fields and angles. Böhme et al (2015) provide an introduction to the economics, technology and governance of Bitcoin in their Journal-of-Economics-Perspective contribution. There is now a journal called Ledger, available per ledgerjournal.org, devoted to publishing papers on cryptocurrencies and blockchain since its inaugural issue in 2016. Bech and Garratt (2017) discuss, whether central banks should introduce cryptocurrencies, finding its echo in the lecture by Carstens (2018). Borgonovo et al (2017) examine this issue from a financial and political economics approach. Chohan (2017) provides a history of Bitcoin.

The phenomenon of virtual currencies such as Bitcoin is increasingly attracting the attention of serious academic study by economists. Rather than an exhaustive literature overview, we shall only provide a sample. Some of the pieces cited above contain data analysis or modeling as well. Yermack (2015),

a chapter in the aforementioned handbook, concludes that Bitcoin appears to behave more like a speculative investment than a currency. Brandvold et al (2015) examine the price discovery on Bitcoin exchanges. Fantazzini et al (2016, 2017) provide a survey of econometric methods and studies, examining the behavior of Bitcoin prices, and list a number of the publications on the topic so far. Fernández-Villaverde and Sanches (2016) examine the scope of currency competition in an extended Lagos-Wright model, and argue that there can be equilibria with price stability as well as a continuum of equilibrium trajectories with the property that the value of private currencies monotonically converges to zero. Bolt and Oordt (2016) examine the value of virtual currencies, predicting that increased adoption will imply that the exchange rate will become less sensitive to the impact of shocks to speculators' beliefs. This accords with Athey et al (2016), who develop a model of user adoption and use of a virtual currency such as Bitcoin in order to analyze how market fundamentals determine the exchange rate of fiat currency to Bitcoin, focussing their attention on an eventual steady state expected exchange rate. They further analyze its usage empirically, exploiting the fact that all individual transactions get recorded on Bitcoin's public ledgers. They argue that a large share of transactions are related to illegal activities, thus agreeing with Foley et al (2018), who investigate this issue in additional detail. Trimborn and Härdle (2016) propose an index called CRIX for the overall cryptocurrency market. Catallini and Gans (2016) provide "some simple economics of the blockchain", a key technological component of Bitcoin, but which has much broader usages. Schnabel and Shin (2018) draw lessons from monetary history regarding the role of banks for establishing trust and current debates about cryptocurrencies. Cong and He (2018) examine the role of the blockchain technology, a key component of the Bitcoin technology, for "smart contracts". The most closely related contribution in the literature to our paper is Garratt-Wallace (2017), as we discussed at the end of the introduction.

It is hard not to think of bubbles in the context of Bitcoin. There is a large literature on bubbles, that should prove useful in that regard. In the original analysis of Samuelson (1958), money is a bubble, as it is intrinsically worthless.

We share that perspective here for both the Dollar and the Bitcoin. It might be tempting to think that prices for Bitcoin could rise forever, as agents speculate to receive even higher prices in the future. Tirole (1982) has shown that this is ruled out in an economy with infinite-lived, rational agents, a perspective which we share, whereas Burnside-Eichenbaum-Rebelo (2015) have analyzed how bubbles may arise from agents catching the "disease" of being overly optimistic. Our model shares some similarity with the bubbles perspective in Scheinkman-Xiong (2003), where a bubble component for an asset arises due to a sequence of agents, each valuing the asset for intermittent periods. Guerrieri-Uhlig (2016) provides some overview of the bubble literature. The Bitcoin price evolution can also be thought about in the context of currency speculation and carry trades, analyzed e.g. by Burnside-Eichenbaum-Rebelo (2012): the three perspectives given there may well be relevant to thinking about the Bitcoin price evolution, but we have refrained from pursuing that here.

3 The model

Time is discrete, $t = 0, 1, \ldots$. In each period, a publicly observable, aggregate random shock $\theta_t \in \Theta \subset \mathbf{R}$ is realized. All random events in period t are assumed to be functions of the history $\theta^t = (\theta_0, \ldots, \theta_t)$ of these shocks, i.e. measurable with respect to the filtration generated by the stochastic sequence $(\theta_t)_{t \in \{0,1,\ldots\}}$. Note that the length of the vector θ^t encodes the period t: thus, functions of θ^t are allowed to be functions of t itself.

There is a continuum of mass 2 of two types of agents. We shall call the first type of agents "red", and the other type "green". Red agents $j \in [0, 1)$ inelastically produce (or: are endowed with) y_t units of a consumption good in even periods t, while green agents $j \in [1, 2]$ do so in odd periods. We assume that $y_t = y(\theta^t)$ is stochastic with support $y_t \in [\underline{y}, \bar{y}]$, where $0 < \underline{y} \leq \bar{y}$. As a special case, we consider the case, where y_t is constant, $\underline{y} = \bar{y}$ and $y_t \equiv \bar{y}$ for all t. The consumption good is not storable across periods. Both types of agents j enjoy utility from consumption $c_{t,j}$ at time t per $u(c_{t,j})$, as well as

loathe providing effort $e_{t,j}$, all discounted at rate $0 < \beta < 1$ to yield life-time utility

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \left(\xi_{t,j} u(c_{t,j}) - e_{t,j}\right)\right] \tag{1}$$

where $u(\cdot)$ is strictly increasing and concave. Note that effort units are counted here in units of disutility, as it does not have some other natural scale. We assume that red agents only enjoy consuming the good in odd periods, while green agents only enjoy consuming in even periods. Formally, we impose this per $\xi_{t,j} = 1_{t \text{ is odd}}$ for $j \in [0,1)$ and $\xi_{t,j} = 1_{t \text{ is even}}$ for $j \in [1,2]$. This creates the absence of the double-coincidence of wants, and thereby reasons to trade.

Trade is carried out, using money. More precisely, we assume that there are two forms of money. The first shall be called Bitcoins and its aggregate stock at time t shall be denoted with B_t. The second shall be called Dollar, and its aggregate stock at time t shall be denoted with D_t. These labels are surely suggestive, but hopefully not overly so, given our further assumptions. In particular, we shall assume that there is a central bank, which governs the aggregate stock of Dollars D_t, while Bitcoins can be produced privately.

The sequence of events in each period is as follows. First, θ_t is drawn. Next, given the information on θ^t, the central bank issues or withdraws Dollars, per "helicopter drops" or lump-sum transfers and taxes on the agents ready to consume that particular period. The central bank can produce Dollars at zero cost. Consider a green agent entering an even period t, holding some Dollar amount $\tilde{D}_{t,j}$ from the previous period. The agent will receive a Dollar transfer $\tau_t = \tau(\theta^t)$ from the central bank, resulting in

$$D_{t,j} = \tilde{D}_{t,j} + \tau_t \tag{2}$$

We allow τ_t to be negative, while we shall insist, that $D_{t,j} \geq 0$: we therefore have to make sure in the analysis below, that the central bank chooses wisely enough so as to not withdraw more money than any particular green agent has at hand in even periods. Red agents do not receive (or pay) τ_t in even

period. Conversely, the receive transfers (or pay taxes) in odd periods, while green agents do not. The aggregate stock of Dollars changes to

$$D_t = D_{t-1} + \tau_t \tag{3}$$

The green agent then enters the consumption good market holding $B_{t,j}$ Bitcoins from the previous period and $D_{t,j}$ Dollars, after the helicopter drop. The green agent will seek to purchase the consumption good from red agents. As is conventional, let $P_t = P(\theta^t)$ be the price of the consumption good in terms of Dollars. We could likewise express the price of goods in terms of Bitcoins, but it will turn out to be more intuitive (at the price of some initial asymmetry) to use the inverse and to let $Q_t = Q(\theta^t)$ denote the price of a Bitcoin in terms of the consumption good. Let $b_{t,j}$ be the amount of the consumption good purchased with Bitcoins and $d_{t,j}$ be the amount of the consumption good purchased with Dollars. The green agent cannot spend more of each money than she owns, but may choose to not spend all of it. This implies the constraints

$$0 \leq b_{t,j} \leq Q_t B_{t,j} \tag{4}$$

$$0 \leq P_t d_{t,j} \leq D_{t,j} \tag{5}$$

The green agent then consumes

$$c_{t,j} = b_{t,j} + d_{t,j} \tag{6}$$

and leaves the even period, carrying

$$B_{t+1,j} = B_{t,j} - b_{t,j}/Q_t \geq 0 \tag{7}$$

$$D_{t+1,j} = D_{t,j} - P_t d_{t,j} \geq 0 \tag{8}$$

Bitcoins and Dollars into the next and odd period $t+1$.

At the beginning of that odd period $t+1$, the aggregate shock θ_{t+1} is drawn and added to the history θ^{t+1}.

The green agent produces y_{t+1} units of the consumption good. The agent expands effort $e_{t+1,j} \geq 0$ to produce additional Bitcoins according to the production function

$$A_{t+1,j} = f(e_{t+1,j}; B_{t+1}) \tag{9}$$

where we assume that $f(\cdot;\cdot)$ satisfies $f(0;\cdot) \equiv 0$, is strictly increasing and strictly concave in the first argument, is strictly decreasing in the second argument and twice differentiable. This specification captures the idea that individual agents can produce Bitcoins at a cost or per "proof-of-work", and that these costs are rising, as the entire stock of Bitcoins increases. As an example, one may think of

$$f(e; B) = \phi \left(\frac{e}{B}\right)^{\alpha} \tag{10}$$

for some $0 < \phi$ and $0 < \alpha < 1$. As a benchmark, we shall consider the limiting case of $\phi \to 0$, implying a fixed stock of Bitcoins, due to prohibitively high effort production costs of new Bitcoins. In odd periods, only green agents may produce Bitcoins, while only red agents get to produce Bitcoins in even periods.

The green agent sells the consumption goods to red agents. Given market prices Q_{t+1} and P_{t+1}, he decides on the fraction $x_{t+1,j} \geq 0$ sold for Bitcoins and $z_{t+1,j} \geq 0$ sold for Dollars, where

$$x_{t+1,j} + z_{t+1,j} = y_{t+1}$$

as the green agent has no other use for the good. After these transactions, the green agent holds

$$\tilde{D}_{t+2,j} = D_{t+1,j} + P_{t+1} z_{t+1,j}$$

Dollars, which then may be augmented per central bank lump-sum transfers at the beginning of the next period $t+2$ as described above. As for the Bitcoins, the green agent carries the total of

$$B_{t+2,j} = A_{t+1,j} + B_{t+1,j} + x_{t+1,j}/Q_{t+1}$$

to the next period.

The aggregate stock of Bitcoins has increased to

$$B_{t+2} = B_{t+1} + \int_{j=0}^{2} A_{t+1,j} dj$$

noting that red agents do not produce Bitcoins in even periods.

The role of red agents and their budget constraints is entirely symmetric to green agents, per simply swapping the role of even and odd periods. There is one difference, though, and it concerns the initial endowments with money. Since green agents are first in period $t = 0$ to purchase goods from red agents, we assume that green agents initially have all the Dollars and all the Bitcoins and red agents have none.

While there is a single and central consumption good market in each period, payments can be made with the two different monies. We therefore get the two market clearing conditions

$$\int_{j=0}^{2} b_{t,j} dj = \int_{j=0}^{2} x_{t,j} dj \tag{11}$$

$$\int_{j=0}^{2} d_{t,j} dj = \int_{j=0}^{2} z_{t,j} dj \tag{12}$$

where we adopt the convention that $x_{t,j} = z_{t,j} = 0$ for green agents in even periods and red agents in odd periods as well as $b_{t,j} = d_{t,j} = 0$ for red agents in even periods and green agents in odd periods.

We finally need to make an assumption regarding the monetary policy of the central bank. For that, it is important that the central bank can predict the effect of its transfers on the resulting price levels. Note that we have assumed that the random shock θ_t is known by the time the central bank picks its transfer payments τ_t. The central bank therefore can pick among the resulting equilibria, which are now functions of τ_t.

For the main part of the analysis, we assume that the central bank aims at keeping Dollar prices stable, and seeks to achieve

$$P_t \equiv 1 \tag{A1}$$

We assume further, that the central bank can find a transfer amount τ_t to do so, among the potential equilibria. We therefore instead turn around and impose (A1) as part of the equilibrium definition. Note that we are imposing that the Dollar has a nonzero value. As is well known, it is generally not hard to generate equilibria, in which the Dollar has no value. These equilibria seem understood sufficiently well in the literature, however. It therefore seems appropriate to restrict attention here to equilibria with a nonzero Dollar value.

A few remarks are in order. First, one could easily introduce another random shock within the period, but after the central bank has chosen the transfer amount, and thereby allow P_t to be a function of that random shock as well. Our assumptions above exclude such a second shock, and should thus be seen as a somewhat special case. Second, it is not a priori clear, that the restriction to equilibria with $P_t \equiv 1$ is justified. We show in section 4 that this is so per calculating equilibria with that property, and thereby demonstrating their existence. In the course of that analysis, we also calculate the monetary transfers, that achieve such equilibria. Finally, price stability is unlikely to be the welfare-maximizing objective for a central bank. Aside from potential stabilization concerns due to the random shocks, it is seems a priori likely that a Friedman-type rule and an ensuing deflation close to the time discount factor will maximize welfare. This is not a novel insight, of course, and this paper is not about optimal monetary policy per se, though. We thus take the more widely adopted policy objective in practice of price stability as our benchmark for the analysis. We return to this issue in the extension section 5.

So far, we have allowed individual green agents and individual red agents to make different choices. We shall restrict attention to symmetric equilibria, in which all agents of the same type end up making the same choice. Thus, instead of subscript j and in slight abuse of the notation, we shall use subscript g to indicate a choice by a green agent and r to indicate a choice by a red agent. With these caveats and remarks, we arrive at the following definition.

Definition 1 *An* **equilibrium** *is a stochastic sequence*

$$(A_t, B_t, B_{t,r}, B_{t,g}, D_t, D_{t,r}, D_{t,g}, \tau_t, P_t, Q_t, b_t, c_t, d_t, e_t, x_t, y_t, z_t)_{t\in\{0,1,2,\ldots\}}$$

which is measurable[1] *with respect to the filtration generated by* $(\theta_t)_{t\in\{0,1,\ldots\}}$*, such that*

1. **Green agents** *optimize: given aggregate money quantities* (B_t, D_t, τ_t)*, production* y_t*, prices* (P_t, Q_t) *and initial money holdings* $B_{0,g} = B_0$ *and* $D_{0,g} = D_0$*, a green agent* $j \in [1,2]$ *chooses consumption quantities* b_t, c_t, d_t *in even periods and* x_t, z_t*, effort* e_t *and Bitcoin production* A_t *in odd periods as well as individual money holdings* $B_{t,g}$*,* $D_{t,g}$*, all non-negative, so as to maximize*

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \left(\xi_{t,g} u(c_t) - e_t\right)\right]$$

where $\xi_{t,g} = 1$ *in even periods,* $\xi_{t,g} = 0$ *in odd periods, subject to the budget constraints*

$$0 \leq b_t \leq Q_t B_{t,g} \tag{13}$$

$$0 \leq P_t d_t \leq D_{t,g} \tag{14}$$

$$c_t = b_t + d_t \tag{15}$$

$$B_{t+1,g} = B_{t,g} - b_t/Q_t \tag{16}$$

$$D_{t+1,g} = D_{t,g} - P_t d_t \tag{17}$$

[1] More precisely, $B_t, B_{t,g}$ and $B_{t,r}$ are "predetermined", i.e. are measurable with respect to the $\sigma-$-algebra generated by θ^{t-1}

in even periods t and

$$A_t = f(e_t; B_t), \text{ with } e_t \geq 0 \tag{18}$$

$$y_t = x_t + z_t \tag{19}$$

$$B_{t+1,g} = A_t + B_{t,g} + x_t/Q_t \tag{20}$$

$$D_{t+1,g} = D_{t,g} + P_t z_t + \tau_{t+1} \tag{21}$$

in odd periods t.

2. **Red agents** *optimize: given aggregate money quantities (B_t, D_t, τ_t), production y_t, prices (P_t, Q_t) and initial money holdings $B_{0,r} = 0$ and $D_{0,r} = 0$, a red agent $j \in [0, 1)$ chooses consumption quantities b_t, c_t, d_t in odd periods and x_t, z_t, effort e_t and Bitcoin production A_t in even periods as well as individual money holdings $B_{t,r}$, $D_{t,r}$, all non-negative, so as to maximize*

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \xi_{t,r} u(c_t)\right]$$

where $\xi_{t,r} = 1$ in odd periods, $\xi_{t,r} = 0$ in even periods, subject to the budget constraints

$$D_{t,r} = D_{t-1,r} + \tau_t \tag{22}$$

$$0 \leq b_t \leq B_{t,r}/Q_t \tag{23}$$

$$0 \leq P_t d_t \leq D_{t,r} \tag{24}$$

$$c_t = b_t + d_t \tag{25}$$

$$B_{t+1,r} = B_{t,r} - b_t/Q_t \tag{26}$$

$$D_{t+1,r} = D_{t,r} - P_t d_t \tag{27}$$

in odd periods t and

$$A_t = f(e_t; B_t), \text{ with } e_t \geq 0 \tag{28}$$

$$y_t = x_t + z_t \tag{29}$$

$$B_{t+1,r} = A_t + B_{t,r} + x_t/Q_t \tag{30}$$

$$D_{t+1,r} = D_{t,r} + P_t z_t + \tau_{t+1} \tag{31}$$

in even periods t.

3. *The* **central bank** *achieves $P_t \equiv 1$, per choosing suitable transfers τ_t.*

4. **Markets clear***:*

$$\textit{Bitcoin market:} \quad B_t = B_{t,r} + B_{t,g} \tag{32}$$

$$\textit{Dollar market:} \quad D_t = D_{t,r} + D_{t,g} \tag{33}$$

$$\textit{Bitcoin denom. cons. market:} \quad b_t = x_t \tag{34}$$

$$\textit{Dollar denom. cons. market:} \quad d_t = z_t \tag{35}$$

4 Analysis

The equilibrium definition quickly generates the following accounting identities. The aggregate evolution for the stock of Bitcoins follows from the Bitcoin market clearing condition and the bitcoin production budget constraint,

$$B_{t+1} = B_t + f(e_t) \tag{36}$$

The aggregate evolution for the stock of Dollars follows from the Dollar market clearing constraint and the beginning-of-period transfer of Dollar budget constraint for the agents,

$$D_t = D_{t-1} + \tau_t \tag{37}$$

The two consumption markets as well as the production budget constraint

$$y_t = x_t + z_t$$

delivers that consumption is equal to production[2]

$$c_t = y_t \tag{38}$$

It will be convenient to bound the degree of consumption fluctuations. The following somewhat restrictive assumption will turn out to simplify the analysis of the Dollar holdings. In a more generalized setting, across-time insurance motives might arise, which however seem to have little to do with the analysis of our core issue, the analysis of Bitcoin pricing. We therefore impose this assumption throughout the rest of the analysis.

Assumption A. 1 *For all t,*

$$u'(y_t) - \beta^2 E_t[u'(y_{t+2})] > 0 \tag{39}$$

The following is a consequence of a central bank policy aimed at price stability, inducing an opportunity cost for holding money. This contrasts with a monetary policy implementing the Friedman rule. Note that assumption (39) holds.

Proposition 1 (All Dollars are spent:) *Agents will always spend all Dollars. Thus, $D_t = D_{t,g}$ and $D_{t,r} = 0$ in even periods and $D_t = D_{t,r}$ and $D_{t,g} = 0$ in odd periods.*

Proof: *Let $\underline{D}_{\infty,g} = \liminf_{t\to\infty} D_{t,g}$. It is clear, that $\underline{D}_{\infty,g} = 0$. By assumption it cannot undercut zero. Also it cannot be strictly positive, since otherwise a green agent could improve his utility per spending $\underline{D}_{\infty,g}$ on consumption goods in some even period, without adjusting anything else except*

[2]Note that the analysis here abstracts from price rigidities and unemployment equilibria, which are the hallmarks of Keynesian analysis, and which could be interesting to consider in extensions of the analysis presented here.

reducing Dollar holdings subsequently by $\underline{D}_{\infty,g}$ in all periods. Note that Dollar holdings for green agents in odd periods are never higher than the Dollar holdings in the previous even period, since at best, they can choose to not spend any Dollars in the even periods. Consider then some odd period, such that $D_{t,g} > 0$, i.e. suppose the green agent has not spent all her Dollars in the previous even period $t-1$. The agent can then increase his consumption in $t-1$ at the cost of reducing his consumption in $t+1$ by the same amount, for a marginal utility gain of

$$\beta^{t-1}\left(u'(c_{t-1}) - \beta^2 E_{t-1}[u'(c_{t+1})]\right)$$

given θ^{t-1}. Since $c_t = y_t$ in all periods, this gain is strictly positive per assumption 1, a contradiction. For red agents, this argument likewise works for all even periods $t \geq 2$. •

Proposition 2 (Dollar Injections:) *In equilibrium, the post-transfer amount of total Dollars is*

$$D_t = z_t$$

and the transfers are

$$\tau_t = z_t - z_{t-1}$$

Proof: *Assume, $t+1$ is even. The Dollar-denominated consumption market clearing condition implies*

$$D_{t+1} = D_{t+1,g} + D_{t+1,r} = D_{t+1,g} = P_t z_t + \tau_{t+1} \tag{40}$$

where we used $D_{t+1,r} = 0 = D_{t,g}$. If $t+1$ is odd, then analogously $D_{t+1,g} = 0 = D_{t,r}$. The evolution of the amount of Dollar is given as

$$D_{t+1} = D_t + \tau_{t+1} \tag{41}$$

Comparing (40) and (41) and using $P_t = 1$, we have

$$D_t = z_t \tag{42}$$

Likewise, $D_{t+1} = z_{t+1}$. Plugging $D_{t+1} = z_{t+1}$ into (40) and using that equation for t rather than $t+1$ delivers

$$\tau_t = z_t - z_{t-1} \tag{43}$$

•

Note that we can have $z_t = 0$, in principle. This is a situation, in which goods are sold at $P_t = 1$ Dollar, but where no transactions at that price happen. We shall typically consider the more conventional situation, where $z_t > 0$ and at least parts of the transactions are carried out with Dollars. We are interested in the question, whether transactions are also carried out in Bitcoin or not, and what that implies.

As for the production of Bitcoins, we obtain the following result.

Proposition 3 (Bitcoin Production Condition:) *Suppose that Dollar sales are nonzero, $z_t > 0$ in period t. Then*

$$1 \geq \beta E_t \left[u'(c_{t+1}) \frac{\partial f(e_t; B_t)}{\partial e_t} Q_{t+1} \right] \tag{44}$$

This inequality is an equality, if there is positive production $A_t > 0$ of Bitcoins and associated positive effort $e_t > 0$ at time t as well as positive spending of Bitcoins $b_{t+1} > 0$ in $t+1$.

Proof: *This is a standard comparison of costs to benefits, and can be derived formally, using the usual Kuhn-Tucker conditions and some tedious calculations. Briefly, providing effort causes disutility of -1 at time t at the margin. Likewise, at the margin, this effort generates $\frac{\partial f(e_t;B_t)}{\partial e_t}$ Bitcoins. A Bitcoin buys Q_{t+1} units of the consumption good at time $t+1$, to be evaluated at marginal*

utility $u'(c_{t+1})$. This gives rise to condition (44) with equality, when Bitcoin production is positive as well as $b_{t+1} > 0$. For that latter condition, note that the agent may value Bitcoins more highly than Q_{t+1} at $t+1$, if she chooses to keep them all, i.e. $b_{t+1} = 0$. •

Keep in mind that we are assuming in the proposition above, that Bitcoin production can happen right away, if this turns out to be profitable. In practice, capacity such as computing farms together with programmers capable of writing the appropriate code may need to build up gradually over time. One could extend the model to allow for such time-to-build of capacity, and it may interesting to do so.

The following proposition establishes properties of the Bitcoin goods price Q_t in the "fundamental" case, where Bitcoins are used in transactions. The following is a version of the celebrated result in Kareken-Wallace (1981).

Proposition 4 (Fundamental Condition:)
Suppose that sales happen both in the Bitcoin-denominated consumption market as well as the Dollar-denominated consumption market at time t as well as at time $t+1$, i.e. suppose that $x_t > 0$, $z_t > 0$, $x_{t+1} > 0$ and $z_{t+1} > 0$. Then

$$E_t\left[u'(c_{t+1})\right] = E_t\left[u'(c_{t+1})\frac{Q_{t+1}}{Q_t}\right] \tag{45}$$

In particular, if consumption and production is constant at $t+1$, $c_{t+1} = y_{t+1} \equiv \bar{y} = \underline{y}$, then

$$Q_t = E_t\left[Q_{t+1}\right] \tag{46}$$

i.e., the price of a Bitcoin in Dollar is a martingale.

Proof: *If $x_t > 0$ and $z_t > 0$, selling agents at time t must be marginally indifferent between accepting Dollars and accepting Bitcoins. If they sell one marginal unit of the consumption good for Dollars, they receive one Dollar per $P_t = 1$ and can buy one unit of the consumption good at date $t+1$ at price $P_{t+1} = 1$ and evaluating the extra consumption at the marginal utility $u'(c_{t+1})$. If they sell one marginal unit of the consumption good for Bitcoin,*

they receive $1/Q_t$ Bitcoins and can thus buy Q_{t+1}/Q_t units of the consumption good at date $t+1$, evaluating the extra consumption at the marginal utility $u'(c_{t+1})$. Indifference implies (45). •

Equation (45) can be understood from a standard asset pricing perspective. As a slight and temporary detour for illuminating that connection, consider some extension of the current model, in which the selling agent enjoys date t consumption with utility $v(c_t)$. The agent would have to give up current consumption, marginally valued at $v'(c_t)$ to obtain an asset, yielding a real return R_{t+1} at date $t+1$ for a real unit of consumption invested at date t. Consumption at date $t+1$ is evaluated at the margin with $u'(c_{t+1})$ and discounted back to t with β. The well-known Lucas asset pricing equation then implies that

$$1 = E_t\left[\beta\frac{u'(c_{t+1})}{v'(c_t)}R_{t+1}\right] \tag{47}$$

One such asset are Dollars. They yield a safe return of $R_{D,t+1} = 1$, due to constant price level $P_t \equiv 1$. The asset pricing equation (47) then yields

$$1 = E_t\left[\beta\frac{u'(c_{t+1})}{v'(c_t)}\right] \tag{48}$$

Likewise, Bitcoins provide the real return$R_{B,t+1} = \frac{Q_{t+1}}{Q_t}$, resulting in the asset pricing equation

$$1 = E_t\left[\beta\frac{u'(c_{t+1})}{v'(c_t)}\frac{Q_{t+1}}{Q_t}\right] \tag{49}$$

One can now solve (49) for $v'(c_t)$ and substitute it into (49), giving rise to equation (45). The difference to the model at hand is the absence of the marginal disutility $v'(c_t)$.

The next proposition establishes properties of the Bitcoin goods price Q_t, if potential good buyers prefer to keep some or all of their Bitcoins in possession, rather than using them in transaction, effectively speculating on lower Bitcoin goods prices or, equivalently, higher Dollar prices for a Bitcoin in the future.

Proposition 5 (Speculative Condition:)
Suppose that $B_t > 0$, $Q_t > 0$, $z_t > 0$ and that $b_t < Q_t B_t$. Then,

$$u'(c_t) \leq \beta^2 E_t \left[u'(c_{t+2}) \frac{Q_{t+2}}{Q_t} \right] \tag{50}$$

where this equation furthermore holds with equality, if $x_t > 0$ and $x_{t+2} > 0$.

Proof: *Market clearing implies that demand equals supply in the Bitcoin-denominated consumption market. Therefore, $b_t < Q_t B_t$ implies that buyers choose not to use some of their Bitcoins in purchasing consumption goods. For a (marginal) Bitcoin, they could obtain Q_t units of the consumption good, evaluated at marginal utility $u'(c_t)$ at time t. Instead, they weakly prefer to hold on to the Bitcoin. The earliest period, at which they can contemplate purchasing goods is $t + 2$, where they would then obtain Q_{t+2} units of the consumption good, evaluated at marginal utility $u'(c_{t+2})$, discounted with β^2 to time t. As this is the weakly better option, equation (50) results. If good buyers use Bitcoins both for purchases as well as for speculative reasons, both uses of the Bitcoin must generate equal utility, giving rise to equality in (50).*
•

Proposition 6 (Zero-Bitcoin-Transactions Condition:)
Suppose that $B_t > 0$, $Q_t > 0$, $z_t > 0$ and that there is absence of goods transactions against Bitcoins $x_t = b_t = 0$ at date t. Then

$$E_t \left[u'(c_{t+1}) \right] \geq E_t \left[u'(c_{t+1}) \frac{Q_{t+1}}{Q_t} \right] \tag{51}$$

If consumption and production is constant at t and $t + 1$, i.e. if $c_t = c_{t+1} \equiv \bar{y} = \underline{y}$, absence of goods transactions against Bitcoins at date t implies

$$Q_t \geq E_t \left[Q_{t+1} \right] \tag{52}$$

Proof: *If $x_t = b_t = 0$, it must be the case, that sellers do not seek to*

sell positive amounts of goods against Bitcoin at the current price in Bitcoin Q_t, and at least weakly prefer to sell using Dollars instead. This gives rise to equation (51). •

A few remarks regarding that last two propositions are in order. To understand the logical reasoning applied here, it is good to remember that we impose market clearing. Consider a (possibly off-equilibrium) case instead, where sellers do not wish to sell for Bitcoin, i.e. $x_t = 0$, because the Bitcoin price Q_t is too high, but where buyers do not wish to hold on to all their Bitcoin, and instead offering them in trades. This is a non-market clearing situation: demand for consumption goods exceeds supply in the Bitcoin-denominated market at the stated price. Thus, that price cannot be an equilibrium price. Heuristically, the pressure from buyers seeking to purchase goods with Bitcoins should drive the Bitcoin price down until either sellers are willing to sell or potential buyers are willing to hold. One can, of course, make the converse case too. Suppose that potential good buyers prefer to hold on to their Bitcoins rather than use them in goods transactions, and thus demand $b_t = 0$ at the current price. Suppose, though, that sellers wish to sell goods at that price. Again, this would be a non-market clearing situation, and the price pressure from the sellers would force the Bitcoin price upwards.

We also wish to point out the subtlety of the right hand side of equations (45) as well as (50): these are expected utilities of the next usage possibility for Bitcoins only if transactions actually happen at that date for that price. However, as equation (50) shows, Bitcoins may be more valuable than indicated by the right hand side of (45) states, if Bitcoins are then entirely kept for speculative reasons. These considerations can be turned into more general versions of (45) as well as (50), which take into account the stopping time of the first future date with positive transactions on the Bitcoin-denominated goods market. The interplay of the various scenarios and inequalities in the preceding three propositions gives rise to potentially rich dynamics, which we explore and illustrate further in the next section.

For consumption and production constant at $t, t+1$ and $t+2$, $c_t = c_{t+1} =$

$c_{t+2} \equiv \bar{y} = \underline{y}$, absence of goods transactions against Bitcoins $x_t = 0$ at t requires

$$E_t\left[Q_{t+1}\right] \leq Q_t \leq \beta^2 E_t\left[Q_{t+2}\right] \tag{53}$$

We next show that this can never be the case. Indeed, even with non-constant consumption, all Bitcoins are always spent, provided we impose a slightly sharper version of assumption 1.

Assumption A. 2 *For all t,*

$$u'(y_t) - \beta E_t[u'(y_{t+1})] > 0 \tag{54}$$

With the law of iterated expectations, it is easy to see that assumption 2 implies assumption 1. Further, assumption 1 implies that (54) cannot be violated two periods in a row.

Theorem 1 (No-Bitcoin-Speculation Theorem.) *Suppose that $B_t > 0$ and $Q_t > 0$ for all t. Impose assumption 2. Then in every period, all Bitcoins are spent.*

Proof: *Since all Dollars are spent in all periods, we have $z_t > 0$ in all periods. Observe that then either inequality (51) holds, in case no Bitcoins are spent at date t, or equation (45) holds, if some Bitcoins are spent. Since equation (45) implies inequality (51), (51) holds for all t. Calculate that*

$$\begin{aligned}
\beta^2\mathbb{E}_t[u'(c_{t+2})Q_{t+2}] &= \beta^2\mathbb{E}_t[\mathbb{E}_{t+1}[u'(c_{t+2})Q_{t+2}]] && \textit{(law of iterated expect.)}\\
&\leq \beta^2\mathbb{E}_t[\mathbb{E}_{t+1}[u'(c_{t+2})]\cdot Q_{t+1}] && \textit{(per equ. (51) at } t+1\textit{)}\\
&< \beta\mathbb{E}_t[u'(c_{t+1})Q_{t+1}] && \textit{(per ass. 2 at } t+1\textit{)}\\
&\leq \beta\mathbb{E}_t[u'(c_{t+1})]Q_t && \textit{(per equ. (51) at } t\textit{)}\\
&< u'(c_t)Q_t && \textit{(per ass. 2 at } t\textit{)}
\end{aligned}$$

Thus, the speculative condition, i.e. equation (50) cannot hold in t. Consequently, $b_t = Q_t B_t$, i.e. all Bitcoins are spent in t. Since t is arbitrary, all Bitcoins are spent in every period. •

Corollary 1 (Bitcoin price bound) *Suppose that $B_t > 0$ and $Q_t > 0$ for all t. The Bitcoin price is bounded by*

$$0 \leq Q_t \leq \bar{Q}$$

where

$$\bar{Q} = \frac{\bar{y}}{B_0} \tag{55}$$

Proof: *It is clear that $Q_t \geq 0$. Per theorem 1, all Bitcoins are spent in every periods. Therefore, the Bitcoin price satisfies*

$$Q_t = \frac{b_t}{B_t} \leq \frac{b_t}{B_0} \leq \frac{\bar{y}}{B_0} = \bar{Q}$$

•

Obviously, the current Bitcoin price is far from that upper bound. The bound may therefore not seem to matter much in practice. However, it is conceivable that Bitcoin or digital currencies start playing a substantial transaction role in the future. The purpose here is to think ahead towards these potential future times, rather than restrict itself to the rather limited role of digital currencies so far.

Heuristically, assume agents sacrifice consumption today to keep some Bitcoins as investment in order to increase consumption the day after tomorrow. Tomorrow, these agents produce goods which they will need to sell. Since all Dollars change hands in every period, sellers always weakly prefer receiving Dollars over Bitcoins as payment. The Bitcoin price tomorrow can therefore not be too low. But with a high Bitcoin price tomorrow, sellers today will weakly prefer receiving Dollars only, if the Bitcoin price today is high as well. But at such a high Bitcoin price today, it cannot be worth it for buyers today to hold back Bitcoins for speculative purposes, a contradiction.

We can rewrite equation (45) as

$$Q_t = \frac{\text{cov}_t(u'(c_{t+1}), Q_{t+1})}{E_t[u'(c_{t+1})]} + E_t[Q_{t+1}] \tag{56}$$

With that, we obtain the following corollary to theorem 1.

Corollary 2 (Bitcoin Correlation Pricing Formula:)
Suppose that $B_t > 0$ and $Q_t > 0$ for all t. Impose assumption 2. In equilibrium,

$$Q_t = \kappa_t \cdot corr_t(u'(c_{t+1}), Q_{t+1}) + E_t[Q_{t+1}] \tag{57}$$

where

$$\kappa_t = \frac{\sigma_{u'(c)|t}\, \sigma_{Q_{t+1}|t}}{E_t[u'(c_{t+1})]} > 0 \tag{58}$$

where $\sigma_{u'(c)|t}$ is the standard deviation of marginal utility of consumption, $\sigma_{Q_{t+1}|t}$ is the standard deviation of the Bitcoin price and $corr_t(u'(c_{t+1}), Q_{t+1})$ is the correlation between the Bitcoin price and marginal utility, all conditional on time t information.

Proof: *With theorem 1, the fundamental condition, i.e. proposition 4 and equation (45) always applies in equilibrium. Equation (45) implies equation (56), which in turn implies (57).* •

Corollary 3 (Martingale Properties of Equilibrium Bitcoin Prices:)
Suppose that $B_t > 0$ and $Q_t > 0$ for all t. Impose assumption 2. If and only if in equilibrium and for all t, marginal utility of consumption and Bitcoin price at $t+1$ are positively correlated, given date-t information, the Bitcoin price is a supermartingale and strictly falls in expectation.

$$Q_t > E_t[Q_{t+1}] \tag{59}$$

If and only if marginal utility and the Bitcoin price are always negatively corre-

lated, the Bitcoin price is a submartingale and strictly increases in expectation,

$$Q_t < E_t[Q_{t+1}] \tag{60}$$

The Bitcoin price is a martingale,

$$Q_t = E_t[Q_{t+1}] \tag{61}$$

if and only if the Bitcoin price and marginal utility are uncorrelated.

Proof: *This is an immediate consequence of corollary 2, since all terms, production, marginal utility and prices are positive there.* •

If the Bitcoin price is a martingale, today's price is the best forecast of tomorrow's price. However, if Bitcoin prices and marginal utility of consumption are positively correlated, then Bitcoins depreciate over time. Essentially, holding Bitcoins offers insurance against the consumption fluctuations, for which the agents are willing to pay an insurance premium in the form of Bitcoin depreciation. Conversely, for negative correlation of Bitcoin prices and marginal utility, the risk premium in form of expected Bitcoin appreciation induces the agents to hold them.

Theorem 2 (Bitcoin Price Convergence Theorem.) *Suppose that $B_t > 0$ and $Q_t > 0$ for all t. Impose assumption 2. For all t and conditional on information at date t, suppose that marginal utility $u'(c_{t+1})$ and the Bitcoin price Q_{t+1} are either always nonnegatively correlated or always non-positively correlated. Then the Bitcoin price Q_t converges almost surely pointwise as well as in L^1 norm to a (random) limit Q_∞,*

$$Q_t \to Q_\infty \text{ a.s. } \quad \text{and } E\left[\mid Q_t - Q_\infty \mid\right] \to 0 \tag{62}$$

Proof: *With corollary 1, the price process for Q_t is bounded. With corollary 3, Q_t is thus a bounded supermartingale or a bounded submartin-*

gale. The Bitcoin price is a uniformly integrable random variable since the set $\{\theta \in \Theta : Q_t(\theta) > C\}$ *has measure zero for* $C > \overline{Q}_1$*. Doob's first and second martingale convergence theorem then imply (62).* •

The following result shows that Bitcoins cannot loose their value completely, with certainty. This may provide some solace to those that have purchased Bitcoins already. It is easy to find generalizations of this result, but we skip them, in the interest of space.

Corollary 4 *Suppose* Q_t *is a martingale. Then it cannot be the case that* Q_t *converges to zero in probability,* $Q_t \xrightarrow{P} 0$*.*

Proof: *Suppose to the contrary, that* $Q_t \xrightarrow{P} 0$*. For any probability level* $0 \leq p < 1$ *and any* $\epsilon > 0$*, there is then a* T*, so that* $P_0(Q_t \leq \epsilon) \geq p$ *for all* $t \geq T$*, where* P_0 *is the probability, given information at date* $t = 0$*. Pick* p *and* ϵ *such that*

$$\frac{Q_0 - \epsilon}{1 - p} > \bar{Q}$$

For $t \geq T$*, calculate*

$$\begin{aligned} Q_0 &= E_0[Q_t] \\ &= E_0[Q_t \mid Q_t > \epsilon]\, P_0(Q_t > \epsilon) + E_0[Q_t \mid Q_t \leq \epsilon]\, P_0(Q_t \leq \epsilon) \\ &\leq E_0[Q_t \mid Q_t > \epsilon]\, P_0(Q_t > \epsilon) + \epsilon \end{aligned}$$

Thus,

$$E_0[Q_t \mid Q_t > \epsilon] \geq \frac{Q_0 - \epsilon}{1 - P_0(Q_t \leq \epsilon)} \geq \frac{Q_0 - \epsilon}{1 - p} > \bar{Q}$$

a contradiction to corollary 1. •

It is important to note that corollary 3 as well as theorem 2 make an assumption about the price sequence Q_t of the Bitcoin price, which is an

equilibrium object. There is no guarantee (so far), that these assumptions hold, given the fundamentals of the environment.

5 Bitcoins and Monetary Policy

In the following we discuss two scenarios and its consequences for monetary policy. Throughout the section we impose assumption 2.

5.1 Scenario 1 - Conventional approach

Assume that Bitcoin prices move independently of central bank policies. Then, we can explicitly characterize the impact of the Bitcoin price on policy.

Proposition 7 (Conventional Monetary Policy:)
The equilibrium Dollar quantity is given as

$$D_t = y_t - Q_t B_t \tag{63}$$

The central bank's transfers are

$$\tau_t = y_t - Q_t\, B_t - z_{t-1} \tag{64}$$

Proof: *[Proposition 7]*

$$D_t = y_t - b_t = y_t - B_t Q_t$$

since all agents spend all their Bitcoins and using proposition 2. •

The Dollar quantity is pinned down through the central bank's transfer policy to maintain price stability for given production realization, Bitcoin price realization, and Bitcoin quantity.

Proposition 8 (Dollar Stock Evolution:)
Tomorrow's expected Dollar quantity equals today's Dollar quantity corrected

for deviation from expected production, purchasing power of newly produced Bitcoin and correlation

$$\mathbb{E}_t[D_{t+1}] = D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t + \kappa_t \; B_{t+1} \cdot corr_t(u'(c_{t+1}), Q_{t+1}) \quad (65)$$

Likewise, the central bank's expected transfers satisfy

$$\mathbb{E}_t[\tau_{t+1}] = -\left(y_t - \mathbb{E}_t[y_{t+1}]\right) - A_t Q_t + \kappa_t \; B_{t+1} \cdot corr_t(u'(c_{t+1}), Q_{t+1}) \quad (66)$$

If the Bitcoin price is a martingale, then

$$\mathbb{E}_t[D_{t+1}] = D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t \quad (67)$$

$$\mathbb{E}_t[\tau_{t+1}] = -\left(y_t - \mathbb{E}_t[y_{t+1}]\right) - A_t Q_t \quad (68)$$

Proof: *Since B_{t+1} is known at time t, using Corollary 2 and $B_{t+1} = B_t + A_t$ we obtain*

$$\begin{aligned}
\mathbb{E}_t[D_{t+1}] &= \mathbb{E}_t[y_{t+1}] - B_{t+1}\mathbb{E}_t[Q_{t+1}] && (69)\\
&= \mathbb{E}_t[y_{t+1}] - B_t \; Q_t - A_t Q_t + \kappa_t \; B_{t+1} \cdot \mathrm{corr}_t(u'(c_{t+1}), Q_{t+1}) && (70)\\
&= D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t + \kappa_t \; B_{t+1} \cdot \mathrm{corr}_t(u'(c_{t+1}), Q_{t+1}) && (71)
\end{aligned}$$

where

$$\kappa_t = \frac{\sigma_{u'(c)|t} \, \sigma_{Q_{t+1}|t}}{E_t[u'(c_{t+1})]} > 0 \quad (72)$$

Similarly, with $D_t = z_t$,

$$\begin{aligned}
\mathbb{E}_t[\tau_{t+1}] &= \mathbb{E}_t[y_{t+1}] - B_{t+1}\mathbb{E}_t[Q_{t+1}] - \mathbb{E}_t[z_t]\\
&= D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t + \kappa_t \; B_{t+1} \cdot \mathrm{corr}_t(u'(c_{t+1}), Q_{t+1}) - z_t\\
&= -\left(y_t - \mathbb{E}_t[y_{t+1}]\right) - A_t Q_t + \kappa_t \; B_{t+1} \cdot \mathrm{corr}_t(u'(c_{t+1}), Q_{t+1})
\end{aligned}$$

If the Bitcoin price is a martingale, then $\mathrm{corr}_t(u'(c_{t+1}), Q_{t+1}) = 0$, and corresponding terms drop. •

5.2 Scenario 2 - Unconventional approach

We now adapt the less standard view that the central bank can maintain the price level $P_t \equiv 1$ independently of the transfers she sets. Further we assume that she sets transfers independently of production. This may surely appear to be an unconventional perspective. The key here is, however, that these assumptions are not inconsistent with our definition of an equilibrium. The analysis would then be incomplete without the consideration of such an unconvential perspective.

Note that the market clearing condition implies that the Bitcoin price satisfies

$$Q_t = \frac{y_t - D_t}{B_t} \tag{73}$$

Intuitively, the causality is in reverse compared to scenario 1: now central bank policy drives Bitcoin prices. However, the process for the Dollar stock cannot be arbitrary. To see this, suppose that $y_t \equiv \bar{y}$ is constant. We already know that Q_t must then be a martingale. Suppose B_t is constant as well. Equation (73) now implies that D_t must be a martingale too. Intuitively, the central bank has been freed from concerns regarding the price level, and is capable to steer the Bitcoin price this way or that, per equation (73). But in doing so, the equilibrium asset pricing conditions imply that the central bank is constrained in its policy choice.

Proposition 9 (Submartingale Implication:)
If the Dollar quantity is set independently of production, the Bitcoin price process is a submartingale, $\mathbb{E}_t[Q_{t+1}] \geq Q_t$.

Proof: *Conditional on time t information, and combined with corollary 2,*

we obtain

$$\begin{aligned}\mathbb{E}_t[Q_{t+1}] &= Q_t - \kappa_t \, corr_t \left(u'(c_{t+1}), Q_{t+1}\right) \\ &= Q_t - \kappa_t \, corr_t \left(u'(c_{t+1}), \frac{y_{t+1} - D_{t+1}}{B_{t+1}}\right) \\ &= Q_t - \frac{\kappa_t}{B_{t+1}} \, corr_t \left(u'(y_{t+1}), y_{t+1}\right)\end{aligned}$$

since the Dollar quantity is uncorrelated with production. Note that we have $cov_t \left(u'(y_{t+1}), y_{t+1}\right) \leq 0$ *since marginal utility is a decreasing function of consumption. Therefore,* $\mathbb{E}_t[Q_{t+1}] > Q_t$. •

Suppose that production y_t is iid. Let F denote the distribution of y_t, $y_t \sim F$. The distribution G_t of the Bitcoin price is then given by

$$G_t(s) = \mathbb{P}(Q_t \leq s) = F(B_t \, s + D_t). \tag{74}$$

As a consequence, changes in expected production or production volatility translate directly to changes in the expected Bitcoin price or price volatility.

Proposition 10 (Bitcoin Price Distribution:)
If Bitcoin quantity or Dollar quantity is higher, high Bitcoin price realizations are less likely in the sense of first order stochastic dominance.

Proof: *Note that* $F(B_t \, s + D_t)$ *is increasing in* B_t *and increasing in* D_t. •

Intuitively, by setting the Dollar quantity, the central bank can control the Bitcoin price. Further, a growing quantity of Bitcoins calms down the Bitcoin price.

Compare two economies which differ only in terms of their productivity distributions, F_1 vs F_2. We say economy 2 is more productive than economy 1, if the productivity distribution of economy 2 first order stochastically dominates the productivity distribution of economy 1. We say economy 2

has more predictable production than economy 1, if the productivity distribution of economy 2 second order stochastically dominates the productivity distribution of economy 1.

Proposition 11 (Bitcoins and Productivity)
In more productive economies or economies with higher predictability of production, the Bitcoin price is higher in expectation.

Proof: *First order stochastic dominance implies second order stochastic dominance. We therefore have to show the result only if F_2 second-order stochastically dominates F_1. Let $G_{2,t}$ and $G_{1,t}$ be the resulting distributions for Bitcoin prices. Since the Bitcoin price is positive,*

$$\begin{aligned}
\mathbb{E}[Q_{t,2}] &= \int_0^\infty x \, dG_{2,t}(x) \\
&= \int_0^\infty (1 - G_{2,t}(x)) \, dx \\
&= \int_0^\infty (1 - F_2(B_t x + D_t)) \, dx \\
&\geq \int_0^\infty (1 - F_1(B_t x + D_t)) \, dx \\
&= \mathbb{E}[Q_{t,1}]
\end{aligned}$$

•

6 Examples

6.1 General construction

One can generally construct equilibria as follows. First, pick a process for the iid θ_t: they provide the probabilistic "canvas" on which to draw everything else. Next, pick a stochastic process of marginal utilities $u'(c_t) = u'(c(\theta^t))$ in such a way, that assumption 1 or even assumption 2 is satisfied. Since we start

with these marginal utilities as primitives for this construction, denote them as $m_t = u'(c_t)$, noting that $m_t = m(\theta^t)$. Pick a sequence of random shocks[3] $\epsilon_t = \epsilon(\theta^t)$, such that $E_t[\epsilon_{t+1}] = 0$ and such that their infinite sum is bounded, $\mid \sum_t \epsilon_t \mid \leq \xi < \infty$ for some ξ. Pick an initial Bitcoin value Q_0. Q_0 should be large enough, see equation (75) below, in order to achieve $Q_t \geq 0$ for all t. That formula may sometimes not be very practical, however. Instead, one can always start from some Q_0 and increase it later towards the end of the construction, adding some fixed amount to all Q_t. Given Q_t, let

$$Q_{t+1} = Q_t + \epsilon_{t+1} - \frac{\mathrm{cov}_t(m_{t+1}, \epsilon_{t+1})}{E_t[m_{t+1}]}$$

Note that the conditional covariance of m_{t+1} and ϵ_{t+1} equals the conditional covariance of m_{t+1} and Q_{t+1}. It is now easy to see that (56) is satisfied. Q_0 needs to be large enough, so that $Q_t \geq 0$ always. To achieve that, set

$$Q_0 > \xi + \sum_{t=0}^{\infty} \frac{\mathrm{cov}_t(m_{t+1}, \epsilon_{t+1})}{E_t[m_{t+1}]} \tag{75}$$

In order to complete the construction and to examine allocations, postulate some strictly concave utility function $u(\cdot)$. The process for marginal utilities m_t then generates a sequence of outputs $y_t = (u')^{-1}(m_t)$. Start with some initial amount of Bitcoins B_0. Suppose B_t is known. With Q_t, proposition 3 and equation (44) deliver the amount of new Bitcoin production A_t and thus B_{t+1}. If assumption 2 has been imposed, theorem 1 now implies the amount of purchases $x_t = b_t = Q_t/B_t$ with Bitcoins and the Dollar-financed purchases $z_t = d_t = y_t - b_t$. One needs to be careful in picking B_0 such that $b_t \leq y_t$ for all t. One can achieve this "ex post" either by reducing B_0 or by rescaling the utility function and thereby the y_t's.

[3]Note that θ^t encodes the date t per the length of the vector θ^t. Therefore, we are formally allowed to change the distributions of the ϵ_t as a function of the date as well as the past history.

6.2 Specific examples

For a more specific example, suppose that $\theta_t \in \{L, H\}$, achieving each value with probability 1/2. Let m_t be iid, $m_t = m(\theta_t)$, with $m(L) \leq m(H)$ and $E[m_t] = (m_L + m_H)/2 = 1$. Pick $0 < \beta < 1$ such that $m(L) > \beta$, thereby implying assumption 2. At date t and for $\epsilon(\theta^t) = \epsilon_t(\theta_t)$, consider two cases

Case A: $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$

Case B: $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$

The absolute value of the sum of the ϵ_t is bounded by $\xi = 2$. Note that

$$\frac{\mathrm{cov}_t(m_{t+1}, \epsilon_{t+1})}{E_t[m_{t+1}]} = \begin{cases} 2^{-(t+2)}(m(H) - m(L)) & \text{if case A at } t+1 \\ -2^{-(t+2)}(m(H) - m(L)) & \text{if case B at } t+1 \end{cases}$$

With equation (75), pick

$$Q_0 > \xi + (m(H) - m(L))/2$$

Consider three constructions,

Always A: Always impose case A, i.e. $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$.

Always B: Always impose case B, i.e. $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$

Alternate: In even periods, impose case A, i.e. $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$. In odd periods, impose case B, i.e. $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$

For each of these, complete the calculations of the Q_t sequences as generally described above. If $m(L) = m(H) = 1$, then all three constructions result in a martingale for $Q_t = E_t[Q_{t+1}]$. Suppose that $m(L) < m(H)$. The "Always A" construction results in $Q_t > E_t[Q_{t+1}]$ and Q_t is a supermartingale. The "Always B" construction results in a submartingale $Q_t < E_t[Q_{t+1}]$. The "Alternate" construction results in a price process that is neither a supermartingale nor a submartingale, but which one still show to converge almost surely and in L_1 norm. One can vary the "Alternate" construction and impose

the case switch after possibly increasing stretches of adjacent periods: convergence still obtains. We therefore conjecture that the convergence results in theorem 2 always hold, provided assumption 2 is satisfied.

These examples were meant to illustrate the possibility, that supermartingales, submartingales as well mixed constructions can all arise, starting from the same assumptions about the fundamentals. Sample paths of these price processes are unlikely to look like the saw tooth pattern shown in figure 1, however. To get somewhat closer to that, the following construction may help. Once again, let $\theta_t \in \{L, H\}$, but assume now that $P(\theta_t = L) = p < 0.5$. Suppose that $m(L) = m(H) = 1$. Pick some $\underline{Q} > 0$ as well as some $Q^* > \underline{Q}$. Pick some $Q_0 \in [\underline{Q}, Q^*]$. If $Q_t < Q^*$, let

$$Q_{t+1} = \begin{cases} \frac{Q_t - p\underline{Q}}{1-p} & \text{if } \theta_t = H \\ \underline{Q} & \text{if } \theta_t = L \end{cases}$$

If $Q_t \geq Q^*$, let $Q_{t+1} = Q_t$. Therefore Q_t will be a martingale and satisfies (56). If Q_0 is sufficiently far above $\bar{Q}$ and if p is reasonably small, then typical sample paths will feature a reasonably quickly rising Bitcoin price Q_t, which crashes eventually to $\underline{Q}$ and stays there, unless it reaches the upper bound Q^* first. Further modifications of this example can generated generate repeated sequences of rising prices and crashes.

For the relationship between output and marginal utilities as well as the covariance of output with Bitcoin prices, consider some specific cases for utility functions.

1. Suppose that utility is linear or that consumption is constant. In that case, $u'(c_t)$ is constant. Assumption 2 is satisfied, since $0 < \beta < 1$. Consequently, all Bitcoins are spent every period and Q_t is a bounded martingale, converging to a limit Q_∞.

2. Suppose that utility is quadratic, $u(c) = \alpha c^2 + \beta c + \gamma$, with $\alpha < 0$. Then, marginal utility of consumption at date t equals $2\alpha y_t + \beta = 2\alpha y_t + \beta$, exploiting $c_t = y_t$. The correlation between marginal utility and the Bitcoin price is then equal to the negative of the correlation of production

and the Bitcoin price.

3. Suppose that utility is logarithmic, $u(c) = \log(c_t)$. The correlation between marginal utility and the Bitcoin price is then equal to the correlation of inverse of production and the Bitcoin price. With CRRA utility $u(c) = (c^{1-\eta} - 1)/(1 - \eta)$, that inverse of production has to be taken to the power of η.

7 Conclusions

This paper has analyzed the evolution of Bitcoin prices and the consequences for monetary policy in a model, in which two groups of infinitely lived agents alternate in their production and consumption possibilities. A central bank keeps the real value of Dollars constant via appropriate monetary injections, while Bitcoin production is decentralized via proof-of-work. Both monies can be used for transactions. We have shown a "fundamental" condition and a "speculative" condition for Bitcoin prices. We have provided conditions, under which no speculation in Bitcoins arises. Under some conditions, we have shown that Bitcoin prices form convergent supermartingales or convergent submartingales. We have studied the implications for monetary policy under a "conventional" as well an "unconventional" scenario. We have provided a general method for constructing equilibria, as well as specific examples, and extended our analysis allowing for inflation.

The simple model environment considered here already gives rise to a rich set of insights and results. Future research may examine the robustness of the results obtained here as well as their general applicability to the market of digital currencies.

References

[1] Athey, Susan, Ivo Prashkevov, Vishnu Sarukkai, Jing Xia (2016), "Bitcoin Pricing, Adoption, and Usage: Theory and Evidence", draft, Stanford

University.

[2] Badev, Anton and Matthew Chen (2014), "Bitcoin: Technical Background and Data Analysis," Federal Reserve Board, Washington, D.C., Staff Working Paper 2014-104.

[3] Bech, Morten and Rodney Garratt (2017), "Central Bank Cryptocurrencies," BIS Quarterly Review (September), 55-70.

[4] Truman Bewley, Truman (1977), "The permanent income hypothesis: a theoretical formulation," Journal of Economic Theory, vol. 16, no. 2, pp. 252292.

[5] Böhme, Rainer, Nicolas Christin, Benjaimin Edelman and Tyler Moore (2015), "Bitcoin: Economics, Technology and Governance," Journal of Ecoomic Perspectives (Spring), vol. 29, no. 2, pp. 213-238.

[6] Bolt, Wilko and Maarten R.C. van Oordt (2016), "On the Value of Virtual Currencies," Financial Stability Department, Bank of Canada, draft.

[7] Borgonovo, Emanuele, Stefano Caselli, Alessandra Cillo and Donato Masciandaro (2017), "Beyond Bitcoin and Cash: do we would like a central bank digital currency? A financial and political economics approach," Bocconi WP 65.

[8] Brito, Jerry and Andrea Castillo (2013), "Bitcoin. A Primer for Policymakers," Mercatus Center, George Mason University, draft.

[9] Brandvold, Morten, Peter Molnár, Kristian Vagstad, Ole Christian ANdreas Valstad, "Price discovery on Bitcoin exchanges," Journal of International Financial Markets, Instiutions & Money, vol. 36 (2015), 18-35.

[10] Burnside, Craig, Martin Eichenbaum and Sergio Rebelo (2011), "Carry Trade and Momentum in Currency Market," Annual Review of Financial Economics, vol. 3, pp. 511-535

[11] Burnside, Craig, Martin Eichenbaum and Sergio Rebelo (2015), "Understanding Booms and Busts in Housing Markets," draft, Northewestern University.

[12] Catalini, Christian and Joshua S. Gans (2016), "Some Simple Economics of the BLockchain," NBER WP 22952.

[13] Carstens, Augustin (2018), "Money in the digital age: what role for central banks?", Lecture at the House of Finance, Goethe University Frankfurt February 6th, 2018, available as draft from BIS.

[14] Cole, Harold L. and Narayana Kocherlakota (1998), "Zero Nominal Interest Rates: Why They're Good and How to Get Them," Federal Reserva Bank of Minneapolis, Quarterly Review (Spring).

[15] Cong, Lin William and Zhiguo He (2018), "Blockchain Disruption and Smart Contracts," draft, Booth School of Business, University of Chicago.

[16] Chohan, Usman W. (2017), "A History of Bitcoin," School of Business and Economics, University of New South Wales (draft).

[17] Fantazzini, Dean, Erik Nigmatullin, Vera Sukhanovskaya and Sergey Ivliev (2016), "Everything you always wanted to know about Bitcoin modelling but were afraid to ask," draft (in English), Moscow School of Economics, published (in Russian) in Applied Econometrics (2016), vol. 44, 5-24 as part I and Applied Econometrics,vol. 45, pp. 5-28 as part 2.

[18] Fernández-Villaverde, Jesús and Daniel Sanches (2016), "Can Currency Competition Work?", draft, University of Pennsylvania.

[19] Foley, Sean, Jonathan R. Karlsen, Talis J. Putnins (2018), "Sex, drugs, and Bitcoin: how much illegal activity is financed through cryptocurrencies?", draft, University of Sydney.

[20] Garratt, Rodney and Neil Wallace (2017), "Bitcoin 1, Bitcoin 2, ... : An Experiment in Privately Issued Outside Monies," draft, UC Santa Barbara and Penn State University.

[21] Guerrieri, Veronica and Harald Uhlig (2016), "Housing and Credit Markets: Booms and Busts," chapter 17 in *Handbook of Macroeconomics*, John Taylor and Harald Uhlig, eds., Elsevier, pp. 1427-1496

[22] Kareken, John and Neil Wallace (1981), "On the Indeterminacy of Equilibrium Exchange Rates," The Quarterly Journal of Economics, vol. 96, No. 2 (May), pp. 207-222.

[23] Kiyotaki, Nobuhiro and Randall Wright (1989), "On Money as a Medium of Exchange," Journal of Political Economy, Vol. 97, No. 4 (August), pp. 927-954.

[24] Lagos, Ricardo and Randall Wright (2005), "A Unified Framework for Monetary Theory and Policy Analysis," Journal of Political Economy, Vol. 113, No. 3 (June), pp. 463-484.

[25] Lee Kuo Chuen, David, ed. (2015), *Handbook of Digital Currency,* 1st edition, Elsevier.

[26] Nakamoto, Satoshi (2008), "Bitcoin: A Peer-to-Peer Electronic Cash System," www.bitcoin.org.

[27] Samuelson, Paul A. (1958), "An Exact Consumption Loan Model of Interest, With or Without the Social Contrivance of Money," Journal of Political Economy 66(5), pp. 467482.

[28] Scheinkman, José A. and Wei Xiong (2003), "Overconfidence and speculative bubbles," Journal of Political Economy, vol. 111 no. 6, pp. 1183-1220.

[29] Schnabel, Isabel and Hyun Song Shin (2018), "Money and trust: lessons from the 1620s for money in the digital age," BIS Working Paper no 698.

[30] Tirole, Jean (1982), "On the Possibility of Speculation under Rational Expectations," Econometrica, vol. 50, no. 5, pp. 1163-1181

[31] Townsend, Robert (1980), "Models of Money with Spatially Separated Agents," in *Models of Monetary Economies*, John Kareken and Neil Wallace, eds., Federal Reserve Bank of Minneapolis, pp. 265-303.

[32] Trimborn, Simon and Wolfgang Härdle, "CRIX or evaluating blockchain based currencies," SFB 649 Discussion Paper 2016-021, Humboldt University Berlin.

[33] Weber, Beat (2013), "Bitcoin and the legitimacy crisis of money,,' Oesterreichische Nationalbank.

[34] Yermack (2015), David, "Is Bitcoin a real currency?" in D. Lee (ed), The Handbook of Digital Currency, Elsevier, pp. 31-44.

FOR ONLINE PUBLICATION

A Extension: Inflation

So far, we have imposed that the Dollar price level stays constant. In this section, we modify our analysis, allowing for possibly random inflation,

$$\pi_{t+1} = \frac{P_{t+1}}{P_t} \in \mathbb{R} \tag{76}$$

Proposition 12 (Constant Inflation Targeting:)
Suppose the central bank targets a constant inflation $\pi_t \equiv \pi > 1$. Then all Dollars are spent in each period.

Proof: *Following the proof of Proposition 1 and without constant prices, agents spend all Dollars if*

$$u'(c_{t-1}) > \beta^2 \mathbb{E}_{t-1}\left[u'(c_{t+1})\frac{P_{t-1}}{P_{t+1}}\right] \tag{77}$$

With constant inflation, this becomes

$$u'(c_{t-1}) > \left(\frac{\beta}{\pi}\right)^2 \quad \mathbb{E}_{t-1}[u'(c_{t+1})] \tag{78}$$

With $\pi > 1$, this is implied by assumption 1. •

With deflation, agents may either possibly hoard Dollars or a stronger assumption than assumption 1 is required. For constant consumption and with deflation $\pi = \beta$, there no longer is a cost to holding money.

Proposition 13 (Modified Fundamental Condition:)
Suppose that inflation π_t is random. Suppose that sales happen both in the Bitcoin-denominated consumption market as well as the Dollar-denominated

consumption market at time t as well as at time $t+1$, i.e. suppose that $x_t > 0$, $z_t > 0$, $x_{t+1} > 0$ and $z_{t+1} > 0$. Then

$$\mathbb{E}_t \left[u'(c_{t+1}) \frac{Q_{t+1}}{Q_t} \right] = \mathbb{E}_t \left[u'(c_{t+1}) \frac{1}{\pi_{t+1}} \right] \tag{79}$$

In particular, if consumption and production is constant at $t+1$, $c_{t+1} = y_{t+1} \equiv \bar{y} = \underline{y}$, then

$$Q_t \, \mathbb{E}_t \left[\frac{1}{\pi_{t+1}} \right] = E_t \left[Q_{t+1} \right] \tag{80}$$

Proof: *Under a general Dollar price level, indifference between investment in Dollars or Bitcoin now requires (79).* •

Interestingly, the speculative condition of proposition 5 remains unchanged, as it only compares the real purchasing power of Bitcoins in t to their real purchasing power in $t+2$. The Zero-Bitcoin-Transactions Condition of proposition 6 requires modification, though, since it concerns the choice between accepting Dollars versus accepting Bitcoins, when selling goods.

Proposition 14 (Modified Zero-Bitcoin-Transactions Condition:) *Suppose that $B_t > 0$, $Q_t > 0$, $z_t > 0$ and that there is absence of goods transactions against Bitcoins $x_t = b_t = 0$ at date t. Then*

$$E_t \left[u'(c_{t+1}) \frac{1}{\pi_{t+1}} \right] \geq E_t \left[u'(c_{t+1}) \frac{Q_{t+1}}{Q_t} \right] \tag{81}$$

Proof: *Direct.* •

If the central bank follows a constant inflation target, theorem 1 continues to hold:

Theorem 3 (No Bitcoin Speculation) *Suppose that $B_t > 0$ and $Q_t > 0$ for all t. Impose assumption 2. With constant inflation $\pi_t = \pi > 1$, all Bitcoins are spent in every period.*

Proof: *As in the proof of theorem 1,*

$$
\begin{aligned}
\beta^2 \mathbb{E}_t[u'(c_{t+2})Q_{t+2}] &= \beta^2 \mathbb{E}_t[\mathbb{E}_{t+1}[u'(c_{t+2})Q_{t+2}]] && \textit{(law of iterated expectation)} \\
&\leq \beta^2 \mathbb{E}_t[\mathbb{E}_{t+1}[u'(c_{t+2})\frac{Q_{t+1}}{\pi}]] && \textit{(equation (81))} \\
&< \beta^2 \mathbb{E}_t[\mathbb{E}_{t+1}[u'(c_{t+2})]\ Q_{t+1}] && \textit{(inflation)} \\
&< \beta \mathbb{E}_t[u'(c_{t+1})Q_{t+1}] && \textit{(per ass. 2 in t+1)} \\
&\leq \beta \mathbb{E}_t[u'(c_{t+1})\frac{Q_t}{\pi}] && \textit{(equation (81))} \\
&\leq \beta \mathbb{E}_t[u'(c_{t+1})Q_t] && \textit{(inflation)} \\
&\leq u'(c_t)Q_t && \textit{(per ass. 2 in t)}
\end{aligned}
$$

•