NBER WORKING PAPER SERIES

AUGMENTING MARKETS WITH MECHANISMS

Samuel Antill
Darrell Duffie

Working Paper 24146
http://www.nber.org/papers/w24146

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
December 2017, Revised May 2018

The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research. Duffie is also a Research Associate of the National Bureau of Economic Research. We are grateful for expert research assistance from Yu Wu, for very helpful conversations with Bruno Biais, Piotr Dworczak, Romans Pancs, and Haoxiang Zhu, for useful feedback from Stanford faculty attending a preliminary presentation of this work on December 8, 2017, for discussions of this paper by Kerry Back at the NBER Asset Pricing Conference and Yunzhi Hu at the NSF/CEME Decentralization Conference, and for comments from seminar participants at ITAM, Stanford University, NYU, Harvard University, and MIT. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-114747.

Augmenting Markets with Mechanisms
Samuel Antill and Darrell Duffie
NBER Working Paper No. 24146
December 2017, Revised May 2018
JEL No. D47,D82,G14

ABSTRACT

We compute optimal mechanism designs for each of a sequence of size-discovery sessions, at which traders submit reports of their excess inventories of an asset to a session operator, which allocates transfers of cash and the asset. The mechanism design induces truthful reports of desired trades and efficiently reallocates the asset across traders. Between sessions, in a dynamic exchange double-auction market, traders strategically lower their price impacts by shading their bids, causing socially costly delays in rebalancing the asset across traders. As the expected frequency of size-discovery sessions is increased, market depth is further lowered, offsetting the efficiency gains of the size-discovery sessions. Adding size-discovery sessions to the exchange market has no social value, beyond that of a potential initializing session. If, as in practice, size-discovery sessions rely on price information from the exchange to set the terms of trade, then bidding incentives are further weakened, strictly reducing overall market efficiency. Keywords: mechanism design, price impact, size discovery, allocative efficiency, workup, dark pool, market design.

Samuel Antill
Graduate School of Business
Stanford University
655 Knight Way
Stanford, CA 94305-7298
santill@stanford.edu

Darrell Duffie
Graduate School of Business
Stanford University
Stanford, CA 94305-7298
and NBER
duffie@stanford.edu

An appendix is available at http://www.nber.org/data-appendix/w24146

1 Introduction

In financial markets, investors with large trading interests strategically avoid price-impact costs by executing large orders slowly. This reallocates the asset across traders more gradually than is socially optimal. This concern is exacerbated, under post-crisis regulations, by the higher costs of intermediary dealer banks for absorbing large customer orders onto their own balance sheets. Market participants have attempted to lower their price impacts by using size-discovery trading protocols such as workups and dark pools. We show that, at least in our model setting, allocative efficiency cannot be improved by augmenting price-discovery markets with size-discovery sessions, except perhaps for an initializing session. This conclusion applies whether or not size-discovery sessions have an optimal mechanism design.

In each size-discovery session, traders are induced by our mechanism design to truthfully report their excess inventories of an asset to a platform operator, which then allocates transfers of cash and the asset. In equilibrium, each session is ex-post individually rational and incentive compatible, budget balanced, and reallocates the asset perfectly efficiently among traders. Between size-discovery sessions, traders exchange the asset on an exchange that is modeled by a sequential-double-auction market,[1] modeled on the lines of Du and Zhu (2017).

It is already well understood from the work of Vayanos (1999), Rostek and Weretka (2015), and Du and Zhu (2017) that traders bid less aggressively in a financial market in order to strategically lower their price impacts, causing socially costly delays in rebalancing positions across traders.[2] Duffie and Zhu (2017) showed that some of the efficiency loss caused by rebalancing delays can be avoided by introducing a single, initializing, size-discovery session, before the exchange market opens. For this purpose, they analyzed workup, a form of size discovery that is heavily used in dealer-dominated markets, such as those for Treasurys and swaps. Duffie and Zhu (2017) also showed that workup is not a fully efficient form of size discovery because traders under-report the sizes of their positions (or equivalently, under-submit trade requests), relative to socially optimal order submissions, due to a winner's-curse effect.

As a mechanism design, the workup protocol places strong restrictions on the allowable forms of messages and transfers. We calculate the optimal mechanism design for size-discovery sessions. In equilibrium, under natural conditions, the optimal mechanism is a new form of size discovery, a direct-revelation scheme that perfectly reallocates the asset among traders. After each size-discovery session, traders' asset inventories are hit by new supply and demand

[1] Each auction is a demand-function submission game, in the sense of Wilson (1979) and Klemperer and Meyer (1989).

[2] Sannikov and Skrzypacz (2016) study a similar setting with heterogeneous traders. They also consider mechanism design, but solely as an analytical device to solve for the equilibrium of a conditional double-auction model.

shocks over time that cause a desire for further rebalancing, which is partially achieved on the exchange double-auction market, which runs continually until the next size-discovery session, and so on. For modeling simplicity, the size-discovery sessions are held at Poisson arrival times.

Even if the size-discovery mechanism designer has enough information to avoid reliance on exchange prices to set the terms of trade, we show that welfare cannot be improved by adding size-discovery sessions. As the expected frequency of size-discovery sessions is increased, the aggressiveness of exchange bidding is lowered, precisely offsetting the expected efficiency gains associated with future size-discovery sessions. Traders anticipate the opportunity to lay off excess positions at low cost in the next size-discovery session, and correspondingly lower the aggressiveness of their exchange bidding.

In practice, size discovery relies on prior exchange prices to set the terms of trade. We show that this causes traders to respond strategically in their preceding exchange order submissions, further reducing market depth and strictly reducing overall market efficiency relative to the exchange market with no size-discovery sessions (with the possible exception of an initializing size-discovery session).

Summarizing, in our model, augmenting an exchange market with size-discovery sessions has no social value, with the possible exception of an initializing session, because any allocative benefits of size-discovery sessions are fully offset, or even dominated, by a corresponding reduction in the depth of the exchange market. While one might imagine that this relatively discouraging result is caused by a size-discovery mechanism design that is "too efficient," we show that overall allocative efficiency is not helped by impairing the efficiency of the size-discovery protocol in order to better support exchange market depth and trade volumes.

We explore the performance of some alternative market designs. We also discuss some potential implications for the competition for order flow between exchange and size-discovery venues, and for potential harm to the exchange price-formation process when size-discovery venues draw sufficiently large volumes of trade away from "lit" exchange markets, a common point of debate among practitioners and policy makers, and also a point of contention in academic research.[3]

In January 2018, the European Union[4] added rules associated with the Markets in Financial Instruments Directive II (MiFiD II) that place a cap on the volume of trade transacted in dark pools, in order to "not unduly harm price formation." This "double cap" effectively restricts aggregate dark pool volume to 8% of total trade volume in affected instruments, and the fraction of trade on any dark pool to 4% of total volume.[5]

[3]See, for example, CFA Institute (2012) and the discussions of Zhu (2014) and Ye (2016).

[4]The exact implementation dates of each piece of MiFiD II vary, see `https://www.fca.org.uk/markets/mifid-ii`.

[5]Article 5 restricts the waivers of Article 4 such that "the percentage of trading in a financial instrument

The most common forms of size discovery used in current market practice are workups, matching sessions, and block-crossing dark pools. As of late 2017, according to Rosenblatt Securities, dark pools account for about 15% of U.S. equity trading volume.[6] In the market for Treasurys, workup is heavily used on the two dominant inter-dealer electronic trade platforms, BrokerTec and eSpeed. Fleming and Nguyen (2015) estimate that workup accounts for 43% to 56% of total trading volume on the largest U.S. Treasurys trade platform, BrokerTec. Once a trade is executed on BrokerTec's limit-order book at some price, a workup session can be opened for potential additional trading at the same "frozen" price. The original buyer and seller and other platform participants may submit additional buy and sell orders that are executed by time priority at this workup price. Trade on the central-limit-order book is meanwhile suspended.[7]

Matching sessions are a feature of some electronic platforms for trading corporate bonds[8] and credit default swaps (CDS). The markets for corporate bonds and CDS are distinguished by much lower trade frequency than those for Treasurys and equities. Matching sessions, correspondingly, are less frequent and of longer duration. A distinctive feature of matching sessions is that the fixed price is typically chosen by the platform operator.[9] Collin-Dufresne, Junge, and Trolle (2016) find that matching sessions and workups account for 71.3% of trade volume for the most popular CDS index product, known as CDX.NA.IG.5yr, a composite of 5-year CDS referencing 125 investment-grade firms, and 73.5% of trade volume for the corresponding high-yield index product.

Trade platforms for interest-rate swaps also commonly incorporate workup or matching-session mechanisms, as described by BGC (2015), GFI (2015), Tradeweb (2014), and Tradition (2015). The importance of workup for the interest-rate swap market is discussed by Wholesale Markets Brokers' Association (2012) and Giancarlo (2015).

---

carried out on a trading venue under those waivers shall be limited to 4% of the total volume of trading in that financial instrument on all trading venues across the Union over the previous 12 months," and "overall Union trading in a financial instrument carried out under those waivers shall be limited to 8% of the total volume of trading in that financial instrument on all trading venues across the Union over the previous 12 months." See `http://eur-lex.europa.eu/legal-content/EN/TXT/?uri=uriserv:OJ.L_.2014.173.01.0084.01.ENG` for the text of Regulation (EU) No 600/2014.

[6] See "Let There be Light, Rosenblatt's Monthly Dark Liquidity Tracker," September 2017, at `http://rblt.com/letThereBeLight.aspx?year=2017`.

[7] For more details on BrokerTec's workup protocol, see Fleming and Nguyen (2015), Fleming, Schaumburg, and Yang (2015), and Schaumburg and Yang (2016). Liu, Wang, and Wu (2015) provide additional evidence on workups in the GovPX dataset, which focuses on off-the-run Treasury securities.

[8] According to SIFMA (2016), matching sessions are provided by Codestreet Dealer Pool (pending release), Electronifie, GFI, Latium (operated by GFI Group), ICAP ISAM (pending release), ITG Posit FI, Liquidity Finance, and Tru Mid.

[9] GFI, for example, chooses a matching-session price that is based, according to SIFMA (2016), on "GFI's own data (input from the internal feeds), TRACE data, and input from traders." On the CDS index trade platform operated by GFI, the matching price "shall be determined by the Company [GFI] in its discretion, but shall be between the best bid and best offer for such Swap that resides on the Order Book."

Empirical evidence regarding the impact on exchange market performance of size-discovery trade is mixed, and limited to equity markets. Size discovery is used far more heavily in bond and swap markets. Degryse, De Jong, and van Kervel (2015) examine trading in Dutch equities across lit (exchange) and unlit trading venues, finding that a one-standard-deviation increase in dark trading activity for a particular stock reduces their metric of lit market depth in that stock by 5.5%. Nimalendran and Ray (2014) also find that dark trading is associated with greater price impact in lit markets. Hatheway, Kwan, and Zheng (2017) add to the evidence that dark venues harm exchange market liquidity. Using a natural experiment induced by an SEC rule change, Farley, Kelley, and Puckett (2017) find no effect of dark trading on exchange market depth. In these studies, however, dark trading includes not only size-discovery trade, but also other forms of trade that do not have pre-trade price transparency, or that involve hidden trades such as "iceberg" orders. Buti, Rindi, and Werner (2011) estimate that dark pools can actually improve exchange inside-quote depth.[10] Division of Trading and Markets (2013) provide a more detailed summary of empirical evidence regarding the impact of dark trade on exchange markets.

In prior work on mechanism design in dynamic settings, Bergemann and Välimäki (2010) show that a generalization of the Vickrey-Clarke-Groves pivot mechanism can implement efficient allocations in dynamic settings with independent private values.[11] Similarly, Athey and Segal (2013) and Pavan, Segal, and Toikka (2014) study optimal mechanism designs in dynamic settings with independent types. As opposed to this prior research, we focus on a market setting in which agents cannot be obliged[12] to participate in mechanism sessions or, more importantly, to abstain from trading on exchanges.

Dworczak (2017) precedes this paper in considering a mechanism design problem in which the designer cannot prevent agents from participating in a separate market. Beyond that likeness of perspective, the problems addressed by our respective models are quite different. Ollár, Rostek, and Yoon (2017) address a design problem associated with double-auction markets, but focus instead on information revelation within the market, rather than an augmenta-

[10]The inside-quote depth is the number of shares available at the best bid and best offer on the limit order book.

[11]In unreported results, and prompted by correspondence with Romans Pancs, we find that such a mechanism also implements an efficient allocation in the primitive stochastic setting of our model.

[12]Specifically, we always impose an ex-post participation condition that, at every mechanism session, all traders prefer participation to the outside option of not entering this mechanism and trading in a double-auction market until the next mechanism. In contrast, Pavan, Segal, and Toikka (2014) force agents to commit at time zero to participate in all future mechanisms (or post an arbitrarily large bond to be forfeited in the event of exit), and Bergemann and Välimäki (2010) force agents to forgo all future mechanism participation in order to sit out one mechanism event. Athey and Segal (2013) provide conditions under which efficient allocations can be reached without participation constraints, but only if agents are arbitrarily patient relative to the most extreme (finite) realization of uncertainty.

tion of the double-auction market with mechanism-based sessions. Du and Zhu (2017) considered the optimal frequency of double auctions, as an alternative design approach to reducing allocative inefficiencies associated with the strategic avoidance of price impact. Pancs (2014) analyzed the implications of workup for its ability to mitigate front-running.[13]

Our analysis is done in stages, building toward our ultimate dynamic model of a continually operating exchange market that is augmented with occasional size-discovery sessions. First, in Section 2, we develop the properties of our mechanism design in a static setting. Section 3 outlines the equilibrium behavior of a dynamic exchange market without size-discovery sessions, with a focus on strategic avoidance of price impact. Next, we consider a market with both price discovery (an exchange) and size discovery. The terms of trade in each size-discovery session rely for information on the sum Z_t of the current undesired positions of the traders. Section 4 considers the case in which Z_t is publicly observable. Although it is not realistic to assume that Z_t is public, our analysis of this case establishes the first of two channels for the welfare impact of adding size-discovery sessions, by which the expected gains from trade that will be achieved in the next size-discovery session are precisely reversed by the corresponding weakening of incentives to trade in the exchange market, in anticipation of the size-discovery session. More precisely, at each time t, for any trader holding any current position, the trader's equilibrium continuation value is invariant to the expected frequency of size-discovery sessions.

The second channel of welfare impact from size discovery arises in the more realistic case in which the operator of the size-discovery sessions cannot directly observe the aggregate position Z_t of traders. As in practice, the terms of trade in size discovery are in this case based on the immediately prior exchange price. Our analysis in Section 5 of this more realistic case incorporates the increased incentive of traders to shade their demands in order to reduce price impact. For example, a trader whose asset position is too high will "shade down" his or her exchange sell orders so as to reduce price impact in the current exchange market, and *also* in order to obtain better expected terms of trade in the next size-discovery session. This strategic response strictly reduces welfare, relative to a setting without size discovery. Indeed, every trader's continuation value is strictly lower than it would be if there were no size discovery. We provide a complete quantitative characterization of equilibria.

Finally, Section 6 offers a discussion of some additional market-design and policy implications. Here, we consider the competing incentives of exchange operators and size-discovery operators, as well as the coordination failure associated with the lack of incentive of size-discovery operators to consider the impact of their platforms on the depth of price-discovery

[13]The seller in Panc's model has private information about the size of his or her desired trade. The buyer is either a "front-runner" or a dealer. If the seller cannot sell the entire large position in workup, he would need to liquidate the remainder by relying on an exogenously given outside demand curve.

exchange markets. We mention the resulting scope for policy intervention. We also discuss some alternative design approaches in which the stand-alone allocative effectiveness of size-discovery sessions is purposefully reduced in order to mitigate the adverse impact on exchange market depth. In the setting that we consider, this does not help.

2 Static Mechanism Design

This section models a static mechanism-design problem in which a designer, say a trade platform operator, elicits reports from each of $n \geq 3$ traders about their asset positions, and based on those reports makes cash and asset transfers.

For trader i, the initial quantity z_0^i of assets is a finite-variance random variable[14] that is privately observable, meaning that z_0^i is measurable with respect to the information set $\mathcal{F}^i$ of trader i. The aggregate inventory $Z \equiv \sum_{i=1}^n z_0^i$ of assets is also observable to all traders and to the platform operator. For example, Z could be deterministic. We relax the observability of Z in Section 5.

A report from trader i is a random variable $\hat{z}^i$ that is measurable with respect to the information set of trader i. Given a list $\hat{z} = (\hat{z}^1, \ldots, \hat{z}^n)$ of trader reports, a reallocation is a list $y = (y^1, \ldots, y^n)$ of finite-variance random variables that is measurable with respect to[15] $\{Z, \hat{z}\}$ and satisfies $\sum_{i=1}^n y^i = 0$.

Anticipating the form of post-mechanism indirect utility for the equilibrium of our eventual model of a dynamic market, we assume that the value to trader i of a given reallocation y is $\mathbb{E}[V^i(z_0^i + y^i, Z) \mid \mathcal{F}^i]$, where

$$V^i(z^i, Z) = u^i(Z) + \left(\beta_0 + \beta_1 \overline{Z}\right)\left(z^i - \overline{Z}\right) - K\left(z^i - \overline{Z}\right)^2, \tag{1}$$

where $u^i : \mathbb{R} \to \mathbb{R}$ is a real-valued measurable function to be specified such that $u^i(Z)$ has a finite expectation, $\overline{Z} \equiv Z/n$, and β_0, β_1, and K are real numbers, with $K > 0$, that do not depend on i.

In our eventual model setting, the value $V^i(z^i, Z)$ is measured in units of wealth, allowing us to use a simple additive welfare criterion. A reallocation is thus welfare maximizing given a list $\hat{z}$ of reports if it solves

$$\sup_{y \in \mathcal{Y}(\hat{z}, Z)} \mathbb{E}\left[\sum_{i=1}^n V^i(z_0^i + y^i, Z)\right],$$

where $\mathcal{Y}(\hat{z}, Z)$ is the set of reallocations. A reallocation is said to be perfect if it is welfare

[14]Fixing a probability space $(\Omega, \mathcal{F}, P)$, trader i has information represented by a sub-σ-algebra $\mathcal{F}^i$ of $\mathcal{F}$. That is, trader i is initially informed of any random variable that is measurable with respect to $\mathcal{F}^i$.

[15]That is, z is measurable with respect to the sub-σ-algebra of $\mathcal{F}$ generated by $\{\hat{z}, Z\}$.

maximizing for the case in which the reports are perfectly revealing,[16] for example when $\hat{z}^i = z_0^i$. From the quadratic costs of asset dispersion across traders reflected in the last term of $V^i(z^i, Z)$, it is immediate that a reallocation y is perfect if and only if $z_0^i + y^i = \overline{Z}$ for all i.

We will now calculate a mechanism design that achieves a perfect reallocation. Specifically, a mechanism is a function that maps Z and a list $\hat{z}$ of reports to a reallocation denoted $Y(\hat{z}) = (Y^1(\hat{z}), \ldots, Y^n(\hat{z}))$ and a list $T(\hat{z}, Z) = (T^1(\hat{z}, Z), T^2(\hat{z}, Z), \ldots, T^n(\hat{z}, Z))$ of real-valued "cash" transfers with finite expectations. In the game induced by a mechanism (Y, T), $\hat{z}$ is an equilibrium if, for each trader i, the report $\hat{z}^i$ solves

$$\sup_{\tilde{z}} \; U^i((\tilde{z}, \hat{z}^{-i})),$$

where, for any list $\hat{z}$ of reports,

$$U^i(\hat{z}) = \mathbb{E}\left[V^i(z_0^i + Y^i(\hat{z}), Z) + T^i(\hat{z}, Z) \,|\, \mathcal{F}^i\,\right], \tag{2}$$

and where we adopt the standard notation by which for any $x \in \mathbb{R}^n$ and $w \in \mathbb{R}$,

$$(w, x^{-i}) \equiv (x^1, x^2, \ldots, x^{i-1}, w, x^{i+1}, \ldots, x^n).$$

In words, each trader i takes the strategies of the other traders as given and chooses a report $\hat{z}^i$ depending only on the information available to trader i that maximizes the conditional expected sum of the reallocated asset valuation and the cash transfer.

For any constant $\kappa_0 < 0$ and any Lipschitz-continuous functions $\kappa_1 : \mathbb{R} \to \mathbb{R}$ and $\kappa_2 : \mathbb{R} \to \mathbb{R}$ of the commonly observed aggregate inventory Z, we will consider the properties of the mechanism $\mathcal{M}^\kappa$ defined by the asset reallocation

$$Y^i(\hat{z}) = \frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i \tag{3}$$

and the cash transfer

$$T_\kappa^i(\hat{z}, Z) = \kappa_1(Z)\hat{z}^i + \kappa_0 \left(n\,\kappa_2(Z) + \sum_{j=1}^n \hat{z}^j\right)^2 + \kappa_1(Z)\kappa_2(Z) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2}. \tag{4}$$

The first term of (4) is analogous to compensation at a fixed marginal price of $\kappa_1(Z)$. This is the essential feature of size-discovery mechanisms, such as a dark pools, workups, and matching sessions, which is to freeze the price and thus eliminate the adverse effect of price impact.[17]

[16] A report $\hat{z}^i$ from trader i is perfectly revealing if z_0^i is measurable with respect to $\{Z, \hat{z}^i\}$.

[17] Not all dark pools are designed primarily for the purpose of mitigating price impacts for large orders.

When we later embed our size-discovery mechanism into a dynamic market, the "frozen price" $\kappa_1(Z)$ will, in equilibrium, be the immediately preceding exchange market price.

Departing from forms of size discovery that are used in practice, we include the non-linear second term of (4) in order to force trader i to internalize some of the quadratic cost of an uneven cross-sectional distribution of the asset. The sum of the final two terms in (4) comprise a fixed participation fee, which ensures that the platform operator does not lose money. That is, for any list $\hat{z}$ of reports, the mechanism $\mathcal{M}^\kappa$ always leaves a weakly positive profit for the platform operator because $\sum_{i=1}^n T^i_\kappa(\hat{z}, Z) \leq 0$.

The following proposition, proven in Appendix A, characterizes equilibrium of the mechanism report game. The proposition also shows that for a carefully chosen κ_0, each trader can actually ignore the reporting strategies of other traders.

Proposition 1. *Consider a mechanism of the form $\mathcal{M}^\kappa$, defined by any $\kappa_0 < 0$, and any Lipschitz-continuous $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$.*

1. *Suppose trader i anticipates that, for each $j \neq i$, trader j will submit the report $\hat{z}^j = z_0^j$. There is a unique solution to the optimal report problem for trader i induced by the mechanism $\mathcal{M}^\kappa$. This solution is $\hat{z}^i = z_0^i$ almost surely, if and only if*

$$\kappa_2(Z) = -\overline{Z} + \frac{-\kappa_1(Z) + \left(\frac{n-1}{n}\right)\left(\beta_0 + \beta_1 \overline{Z}\right)}{2\kappa_0 n}. \tag{5}$$

   That is, $\mathcal{M}^\kappa$ is a direct revelation mechanism if and only if $\kappa_2(Z)$ is given by (5).

2. *Suppose $\kappa_2(Z)$ is given by (5). If trader i anticipates the report $\hat{z}^j = z_0^j$ for each $j \neq i$, then the truthful report $z^{*i} = z_0^i$ is ex-post optimal, that is, optimal whether or not we take the special case in which trader i observes[18] z_0^{-i}.*

3. *For the list $z^* = (z^{*1}, \ldots, z^{*n})$ of such truthful reports, the reallocation $Y(z^*)$ of (3) is perfect. That is, $z_0^i + Y^i(z^*) = \overline{Z}$ for all i.*

4. *For any $\kappa_1(\,\cdot\,)$, for $\kappa_2(Z)$ given by (5), and for $\kappa_0 = -K(n-1)/n^2$, the mechanism $\mathcal{M}^\kappa$ is strategy proof. That is, the truthful report $z^{*i} = z_0^i$ is a dominant strategy, being an*

Drawing from an industry report by Rosenblatt Securities, Ye (2016) notes that "In May 2015, among the 40 active dark pools operating in the US, there are 5 dark pools in which over 50% of their Average Daily Volumes are block volume (larger than 10k per trade). Those pools can be regarded as "Institutional dark pools," and they include Liquidnet Negotiated, Barclays Directx, Citi Liquifi, Liquidnet H20, Instinet VWAP Cross, and BIDS Trading." Other objectives of dark pool users include a reduction in the leakage of private information motivating trade, and the avoidance of bid-ask spread costs. Some broker-dealers use their own dark pools to internalize order executions among their clients.

[18] To be able to observe z_0^{-i} means that z_0^{-i} is measurable with respect to $\mathcal{F}^i$.

optimal report for trader i regardless of the conjecture by trader i of the reports $\hat{z}^{-i}$ of the other traders.

By the ex-post optimality property stated in Part 2 of the proposition, it is a Nash equilibrium[19] of the complete information game (in which all traders know z_0) for traders to submit the list z^* of reports. For the special case $\kappa_0 = -K(n-1)/n^2$, this is the unique Nash equilibrium because, for any trader i, the report z^{*i} is a dominant strategy and because of the strict concavity of $U^i((\tilde{z}, \hat{z}^{-i}))$ with respect to $\tilde{z}$.

We have not yet considered whether trader i could do better by not entering the mechanism at all. From this point, we always fix $\kappa_2(\,\cdot\,)$ as specified by (5). For arbitrary κ_0 and $\kappa_1(\,\cdot\,)$, the mechanism $\mathcal{M}^\kappa$ need not be individually rational. That is, there could be realizations of (z_0^i, Z) at which trader i would strictly prefer $V^i(z_0^i, Z)$ over the expected equilibrium value to trader i. However, because the platform operator observes Z, he or she can choose $\kappa_1(Z)$ so as to ensure that all traders strictly prefer to participate in the mechanism, except in the trivial case in which the initial allocation is already perfect.

Proposition 2. *Fix $\kappa_2(\,\cdot\,)$ as in (5), let $\kappa_1(Z) = \beta_0 + \beta_1 \overline{Z}$, and let κ_0 be arbitrary. For the equilibrium reports z^* of the mechanism $\mathcal{M}^\kappa$, we have*

$$U^i(z^*) = V^i(z_0^i, Z) + K\left(z_0^i - \overline{Z}\right)^2. \tag{6}$$

With probability one, trader i weakly prefers this equilibrium value to the value $V(z_0^i, Z)$ of the initial inventory z_0^i. That is,

$$U^i(z^*) = V^i(z_0^i + Y^i(z^*), Z) + T_\kappa^i(z^*, Z) \geq V^i(z_0^i, Z).$$

The inequality is strict unless $z_0^i = \overline{Z}$. Provided that the probability distribution of z_0 has full support, this inequality holds with probability one if and only if $\kappa_1(Z) = \beta_0 + \beta_1 \overline{Z}$.

A proof is found in Appendix A. In summary, if the aggregate inventory Z is known to all traders and to the size-discovery platform operator, then the budget-balanced mechanism $\mathcal{M}^\kappa$ can implement a perfect reallocation in an ex-post individually rational equilibrium.[20] Proposition 2 also implies that the equilibrium payoffs do not depend upon the choice of κ_0.

[19]Likewise, this is also a Bayesian Nash equilibrium of the incomplete information game, after specifying beliefs about other traders' inventories.

[20]As noted to one of us by Romans Pancs, a Vickrey-Clarke-Groves (VCG) pivot mechanism can also implement a perfect reallocation in an ex-post equilibrium in this setting. However, the standard pivot mechanism cannot be both budget balanced and ex-post individually rational. The AGV mechanism of Arrow (1979), d'Aspremont and Gérard-Varet (1979) does not apply to this setting because the private information of traders is correlated.

For $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$ as specified in Proposition 2, some algebra shows that the equilibrium cash transfer to trader i is

$$\kappa_1(Z)\left(z_0^i - \overline{Z}\right) = \left(\beta_0 + \beta_1 \overline{Z}\right)\left(z_0^i - \overline{Z}\right). \tag{7}$$

The mechanism designer is thus free to choose any $\kappa_0 < 0$, because the choice of κ_0 has no impact on equilibrium transfers or allocations. Result 4 of Proposition 1 nevertheless indicates the strategy-proofness advantage of the particular choice $\kappa_0 = -K(n-1)/n^2$.

Figure 1 illustrates the cash and asset transfers that are obtainable by trader i for the mechanism of Proposition 2, when other traders follow the equilibrium report z^{*j}. The asset transfer schedule $\hat{z}^i \mapsto Y(\hat{z})$ is linear. The cash transfer schedule $\hat{z}^i \mapsto T^i_\kappa(\hat{z}, Z)$ can be close to linear, similar to the case of size-discovery mechanisms such as workups and dark pools. However, a report by trader i that is large enough in magnitude induces a significant cash penalty associated with the quadratic component of the cash transfer schedule. From a welfare viewpoint, this penalty appropriately disciplines trader i from over-exploiting the mechanism by trying to completely eliminate his or her excess inventory. A workup or dark pool handles this problem of disciplining demand and supply by rationing whichever side of the market has a greater absolute magnitude of excess inventory. Workup rations by time prioritization of orders (first come, first served). A typical dark pool rations the heavier side of the market pro rata to requested trade sizes. These rationing schemes, however, are only rules of thumb, and are strictly suboptimal. The mechanism $\mathcal{M}^\kappa$ of Proposition 2, on the other hand, achieves the first best.

As mentioned previously, a linear-quadratic utility of the form $V^i(z, Z)$ emerges in the next section as the equilibrium continuation value in the sequential-double-auction market, even if the market is augmented with future reallocation sessions. Proposition 1 therefore implies that if our mechanism is run at time 0, before the market opens, then all traders will instantly move to the socially efficient allocation. However, as traders receive subsequent inventory shocks over time, their allocation becomes inefficient, always leaving scope for improving the allocation.

3 The Welfare Cost of Price-Impact Avoidance

In this section, we model an exchange as a sequential-double-auction market. Traders strategically avoid price impact, causing a socially inefficient delay in the rebalancing of asset positions across traders. This issue is well covered by the results of Vayanos (1999), Rostek and Weretka (2015), Du and Zhu (2017), and Duffie and Zhu (2017). However, for our later purpose of exploring the augmentation of exchange trading with a sequence of size-discovery sessions, this sec-

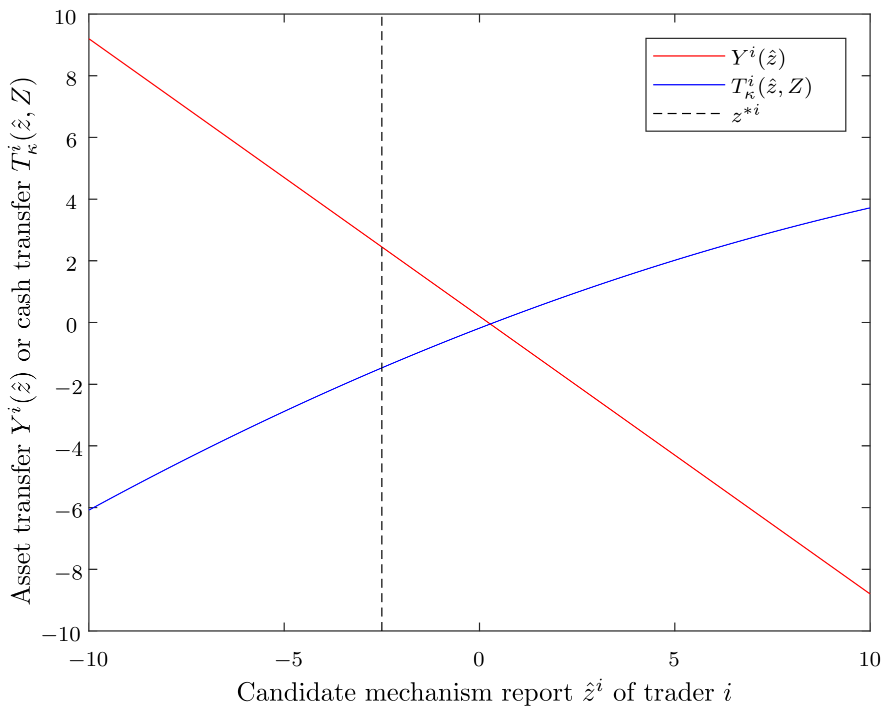

Figure 1: Mechanism Transfers and Reallocations. This figure plots the possible transfers and reallocations available in the mechanism for a trader, in an equilibrium. The parameters are $v = 0.5$, $r = 0.1$, $n = 10$, $\gamma = 0.1$, $Z = -0.5$, and $z^i_0 = -2.5$. The value function corresponds to the continuation value for the subsequent double-auction market equilibrium, so that $\beta_0 = v$, $\beta_1 = -2\gamma/r$, and $K = \gamma/[r(n-1)]$. We take $\kappa_0 = -K(n-1)/n^2$, $\kappa_1(Z) = \beta_0 + \beta_1\overline{Z}$, and $\kappa_2(Z)$ defined as in Proposition 1. The report of each of the other nine traders is fixed at the equilibrium level z^{*j}.

tion provides a suitable generalization of the dynamic-double-auction market of Duffie and Zhu (2017).

The continuous-time presentation of our results is chosen for its expositional simplicity. A discrete-time analogue of our model is found in Appendix E. While the discrete-time setting leads to messier looking results, it allows us to demonstrate a standard equilibrium robustness property, Perfect Bayes. The equilibrium behavior of the discrete-time model converges to that of the continuous-time model as the length of a time period shrinks to zero.

We fix a probability space, the time domain $[0, \infty)$, and an information filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions.[21] The market is populated by $n \geq 3$ risk-neutral traders trading a divisible asset. The payoff π of the asset is a bounded random variable with mean v. The payoff π is revealed publicly and paid to traders at a random time $\mathcal{T}$ that is exponentially distributed with parameter r. Thus $\mathbb{E}(\mathcal{T}) = 1/r$. There is no further incentive to trade once π

[21]For the "usual conditions" on a filtration see, for example, Protter (2005).

is revealed at time $\mathcal{T}$, which is therefore the ending time of the model.

Trader i has information given by a sub-filtration $\mathbb{F}^i = \{\mathcal{F}_t^i : t \geq 0\}$ of $\mathbb{F}$. The traders have symmetric information about the asset payoff. Specifically, we suppose that the conditional distribution of π given $\mathcal{F}_t$ is constant until the payoff time $\mathcal{T}$, so that no trader ever learns anything about π until the market ends. The traders may, however, have asymmetric information about their respective asset positions at each time. Price fluctuations are thus driven only by allocative concerns, and not by learning about ultimate asset payoffs. This informational setting is more relevant for markets such as those for stock index products, major currencies, and fixed income products such as swaps and government bonds. For example, there is always symmetric information about the payoff of a Treasury bill, but the price of a Treasury bill fluctuates randomly over time, partly caused by shocks to the allocation of the bills across market participants.

All asset positions, or "inventories," are measured net of traders' desired inventory levels, so that all traders would ideally wish to achieve a position of zero. The initial inventories of the asset for the n traders are specified as in Section 2 by a list $z_0 = (z_0^1, z_0^2, \ldots, z_0^n)$ of finite-variance random variables, with z_0^i measurable with respect to $\mathcal{F}_0^i$.

In a continually operating double-auction exchange market, at each time t, trader i submits an $\mathcal{F}_t^i$-measurable demand function $\mathcal{D}_t^i : \Omega \times \mathbb{R} \to \mathbb{R}$. Thus, in state ω at time t, if the outcome of the auction price is p, trader i would buy the asset at the quantity "flow" rate $\mathcal{D}_t^i(\omega, p)$. Given a double-auction price process ϕ, trader i would thus purchase the total quantity $\int_s^u \mathcal{D}_t^i(\omega, \phi_t(\omega))\, dt$ of the asset over some time interval $[s, u]$ (assuming the integral exists). We only consider equilibria in which demand functions are of the affine form

$$\mathcal{D}_t^i(\omega, p) = a + bp + cz_t^i(\omega), \tag{8}$$

for constants a, $b < 0$, and c that do not depend on i or t, and where z_t^i is the quantity of the asset held by trader i at time t. To be clear, the traders are not restricted to affine demand functions, but in equilibrium we will show that each trader optimally chooses a demand function that is affine if he or she assumes that the other traders do so.

At time t, given the demand-function coefficients (a, b, c) and the current list $z_t = (z_t^1, \ldots, z_t^n)$ of trader inventories, a price ϕ_t is chosen by a trade platform operator to clear the market. A complete equilibrium model of the demand coefficients (a, b, c) and of the evolution of the inventory processes $(z^1, \ldots, z^n)$ will be provided shortly.

Lemma 1. *Fix any demand-function coefficients (a, b, c) with $b < 0$, some time t, and some trader i. For any candidate demand $d \in \mathbb{R}$ by trader i, there is a unique price p satisfying*

$d + \sum_{j \neq i}(a + bp + cz_t^j) = 0$. *This clearing price is*

$$p = \Phi_{(a,b,c)}(d; Z_t^{-i}) \equiv \frac{-1}{b(n-1)} \left(d + (n-1)a + cZ_t^{-i}\right), \tag{9}$$

where $Z_t^{-i} \equiv \sum_{j \neq i} z_t^j$.

Thus, for any non-degenerate affine demand function used by $n-1$ of the traders, there is a unique market clearing price for each quantity chosen by the remaining trader.

The asset inventory of trader i is randomly shocked over time with additional units of the asset. The cumulative shock to the inventory of trader i by time t is H_t^i, for some finite-variance $\mathbb{F}^i$-adapted Lévy process H^i that is a martingale with respect to $\mathbb{F}$ and thus with respect to the information filtration $\mathbb{F}^i$ of trader i. A simple example is a Brownian motion with zero drift. The defining property of a Lévy process is that it has independent increments and identically distributed increments over any equally long disjoint time intervals. Without loss of generality, we take $H_0^i = 0$. The inventory shock processes $H = (H^1, \ldots, H^n)$ need not be independent across traders, but we assume that H, $\mathcal{T}$, π, and z_0 are mutually independent and that $\sum_{i=1}^n H^i$ is also a Lévy process.

Letting $\sigma_i^2 \equiv \mathrm{var}(H_1^i)$, the Lévy property[22] implies that for any time t we have $\mathrm{var}(H_t^i) = \sigma_i^2 t$. Likewise, letting $\sigma_Z^2 = \mathrm{var}\left(\sum_{i=1}^n H_1^i\right)$ and $\rho^i = \mathrm{cov}(Z_1, H_1^i)$, the Lévy property implies that $\mathrm{var}(Z_t) = \mathrm{var}(Z_0) + \sigma_Z^2 t$ and that $\mathrm{cov}(Z_t, H_t^i) = \rho^i t$.

Traders suffer costs associated with unwanted levels of inventory, whether too large or too small. One may think in terms of a market maker that is attempting to run a matched book of positions, but which may accept customer positions over time that shock its inventory. The market maker may then trade so as to lay off excess inventories with other market makers in an inter-dealer double-auction market.

The market practitioners Almgren and Chriss (2001) proposed a simple model of inventory costs for financial firms that is now popular among other practitioners and also in the related academic research literature, by which the rate of inventory cost to trader i at time t is $\gamma(z_t^i)^2$, for some coefficient $\gamma > 0$. With this model, at any time $t < \mathcal{T}$, trader i bears an expected total cost of future undesired inventory of $\mathbb{E}[\int_t^{\mathcal{T}} \gamma(z_s^i)^2 \, ds \,|\, \mathcal{F}_t^i]$.

Although financial firms do not have direct aversion to risk, broker-dealers and asset-management firms do have extra costs for holding inventory in illiquid or risky assets. These costs can be related to regulatory capital requirements, collateral requirements, financing costs, agency costs associated with a lack of transparency of the position to higher-level firm managers or clients regarding the true asset quality, as well as the expected cost of being forced to suddenly

[22]Because H^i is a finite-variance process, its characteristic exponent $\psi_i(\,\cdot\,)$ has two continuous derivatives, and $\sigma_i^2 = \psi_i''(0)$. As an example, if H^i is a Brownian motion with variance parameter φ, then $\sigma_i^2 = \varphi$.

raise liquidity by quickly disposing of remaining inventory into an illiquid market. Although it has not been given a structural foundation, the quadratic holding-cost assumption is common in dynamic market-design models, including those of Vives (2011), Rostek and Weretka (2012), Du and Zhu (2017), and Sannikov and Skrzypacz (2016).

With respect to equilibrium behavior, our model is equivalent[23] to one in which there is no shock H_t^i to the level of inventory, but there is instead a Lévy process η^i determining the net rate of benefit at time t to trader i for asset position z_t^i of $\eta_t^i z_t^i - \gamma(z_t^i)^2$.

Given the short time horizons over which inventories are typically rebalanced in practice, we have neglected the role of time preference. However, our model is behaviorally equivalent to an infinite-horizon model in which traders discount at the time preference rate r and the asset pays dividends continuously at the exogenous rate $\mu = rv$, rather than a final lump-sum dividend with mean v. This equivalence follows from an inspection of the Hamilton-Jacobi-Bellman (HJB) equation that is used in Appendix B to prove the optimality[24] of traders' candidate equilibrium trading and reporting strategies.

Lemma 1 allows trader i to reduce his or her problem to the choice of a real-valued demand process D^i, which then determines the market clearing price process $\Phi_{(a,b,c)}(D_t^i; Z_t - z_t^i)$. A demand process D^i is optimal for trader i given the demand coefficients (a, b, c) of the other traders if D^i solves the problem of maximizing total expected net profits, defined by

$$U_0^i \equiv \sup_{D \in \mathcal{A}^i} \mathbb{E}[J^i(D) \mid \mathcal{F}_0^i], \tag{10}$$

where $\mathcal{A}^i$ is the space of integrable $\mathbb{F}^i$-adapted processes such that the expectation in (10) exists,

$$J^i(D) = z_{\mathcal{T}}^D \pi - \int_0^{\mathcal{T}} \left(\gamma \left(z_t^D \right)^2 + \Phi_{(a,b,c)} \left(D_t; Z_t - z_t^D \right) D_t \right) dt, \tag{11}$$

[23]To see this equivalence, suppose that η is an exogenous Lévy process, and consider a model with no exogenous inventory shocks in which a trader with position process y, determined only by the trader's initial position and trades, benefits at time t at the rate $\eta_t y_t - \gamma y_t^2$. This preference model induces the same behavior as that associated with the benefit rate

$$\eta_t y_t - \gamma y_t^2 - \frac{\eta_t^2}{4\gamma} = -\gamma \left(y_t - \frac{\eta_t}{2\gamma} \right)^2 = -\gamma (y_t + H_t^i)^2,$$

where $H_t^i = -\eta_t/(2\gamma)$, because the extra term $\eta_t^2/(4\gamma)$ merely translates the total value by the constant $E\left(\int_0^{\mathcal{T}} \eta_t^2 \, dt\right)/(4\gamma)$. This preference model induces the same behavior as that for our basic model in which there is a cost $\gamma(z_t^i)^2$ for a position process $z_t^i = y_t + H_t^i$ that is determined by trade and by an exogenous Lévy inventory shock process H_t^i. By similar arguments, our model is also behaviorally equivalent to a model that includes both an inventory shock process and a preference shock process.

[24]Verification of optimality follows from the HJB equation and "transversality" arguments similar to those in Appendix B.

and

$$z_t^D = z_0^i + \int_0^t D_s\, ds + H_t^i. \tag{12}$$

The total expected profit (10) is finite or negative infinity for any demand process D, and is finite at any optimal demand process, given that $D = 0$ is a candidate demand process.

Demand coefficients (a, b, c) with $b < 0$ are said to constitute a symmetric affine equilibrium if, for any trader i, given (a, b, c), the demand process $D_t^i = a + b\phi_t + cz_t^i$ is optimal, where ϕ_t is the market clearing price process

$$\phi_t = \frac{a + c\overline{Z}_t}{-b},$$

where $\bar{Z}_t = Z_t/n$ and z^i solves the stochastic differential equation

$$z_t^i = z_0^i + \int_0^t (a + b\phi_s + cz_s^i)\, ds + H_t^i.$$

This definition of equilibrium implies market clearing, individual trader optimality given the assumed demand functions of other traders, and consistent conjectures about the demand functions used by other traders. This notion of equilibrium was developed by Du and Zhu (2017), who emphasized that the equilibrium demands are ex-post optimal. That is, no trader would bid differently even if he or she were able to observe the inventories of all other traders. In particular, taking the equilibrium demand coefficients (a, b, c) as given, the demand process $D_t^i = a + b\phi_t + cz_t^i$ for trader i actually solves

$$V^i(z_0^i, Z_0) \equiv \sup_{D \in \mathcal{A}} \ \mathbb{E}[J^i(D) \,|\, \mathcal{F}_0], \tag{13}$$

where $\mathcal{A}$ is the space of integrable $\mathbb{F}$-adapted processes such that the expectation in (13) exists, and z_t^D is defined by (12). Thus, the optimum initial value for trader i is $U_0^i = \mathbb{E}[V^i(z_0^i, Z_0) \,|\, \mathcal{F}_0^i]$. Likewise, the continuation value of trader i at any time $t < \mathcal{T}$ is $\mathbb{E}[V^i(z_t^i, Z_t) \,|\, \mathcal{F}_t^i]$.

Although we are working here for expositional simplicity in a continuous-time setting, the equilibria that we propose may safely be considered to be Perfect Bayesian Equilibrium. That is, in light of the ex-post optimality property, beliefs about other traders' inventories are irrelevant. This is tied down rigorously in a discrete-time analogue of our model found in Appendix E. In discrete time, the ex-post optimality property implies subgame perfection for the complete information game. Moreover, the primitive parameters of the discrete-time model and the associated discrete-time equilibrium bidding behavior converge to those for the continuous-time model as the length of a time interval shrinks to zero. This convergence was shown by Duffie and Zhu (2017) for a simpler version of this model, and applies also in the current setting.

A proof of the following proposition appears in Appendix B.

Proposition 3. *There is a unique symmetric affine equilibrium. The equilibrium market-clearing price process is*

$$\phi_t = v - \frac{2\gamma}{r}\overline{Z}_t. \tag{14}$$

In this equilibrium, for any trader i and any time t, the indirect utility of trader i defined by (13) is

$$V^i(z_t^i, Z_t) = \theta_i + v\overline{Z}_t - \frac{\gamma}{r}\overline{Z}_t^2 + \phi_t\left(z_t^i - \overline{Z}_t\right) - \frac{\gamma}{r}\frac{1}{n-1}\left(z_t^i - \overline{Z}_t\right)^2, \tag{15}$$

where

$$\theta_i = \frac{\gamma\sigma_Z^2}{r^2n^2} - \frac{\gamma}{r^2(n-1)}\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}\right) - \frac{2\gamma\rho^i}{r^2n}.$$

The equilibrium demand function of any trader i evaluated at an arbitrary price p, state ω, and time t is

$$\mathcal{D}_t^i(\omega, p) = \frac{(n-2)r^2}{4\gamma}\left(v - p - \frac{2\gamma}{r}z_t^i(\omega)\right). \tag{16}$$

That is, the equilibrium demand function is affine with coefficients

$$a = \frac{(n-2)r^2v}{4\gamma}, \qquad b = \frac{-(n-2)r^2}{4\gamma}, \qquad c = \frac{-(n-2)r}{2}. \tag{17}$$

We can now define the equilibrium welfare, given the initial list z_0 of positions, as

$$W(z_0) \equiv \sum_{i=1}^{n} V^i(z_0^i, Z_0) = \sum_{i=1}^{n}\theta_i + vZ_0 - \frac{\gamma}{r}\frac{Z_0^2}{n} - \frac{\gamma}{r(n-1)}\sum_{i=1}^{n}\left(z_0^i - \overline{Z}_0\right)^2. \tag{18}$$

An additive welfare function is appropriate for market efficiency considerations because our traders are maximizing total expected profits net of costs, measured in "dollar" values.

A social planner who is free to reallocate inventories among the n traders can obviously improve on this welfare $W(z_0)$, except in the unique trivial case in which the initial total inventory is equally split across traders (that is, $z_0^i = \overline{Z}_0$ for all i) and in which there are symmetric future inventory shocks ($H^i = H^j$ for all i, j, almost surely). By constantly reallocating inventories so as to keep $z_t^i = \overline{Z}_t$, a social planner can achieve the first-best welfare of

$$W_{fb}(Z_0) = -\frac{\gamma}{r^2}\frac{\sigma_Z^2}{n} + vZ_0 - \frac{\gamma}{r}\frac{Z_0^2}{n}. \tag{19}$$

Relative to first best, the equilibrium behavior of Proposition 3 is inefficient because each trader strategically bids so as to reduce the price impact associated with the dependence of the clearing price $\Phi_{(a,b,c)}\left(D_t; Z_t - z_t^i\right)$ on his or her demand D_t. But price impacts are mere wealth transfers, and have no direct social costs. It is not socially efficient for traders to internalize their price-impact costs. The welfare cost of strategic avoidance of price impact is well covered by the prior results of Vayanos (1999), Rostek and Weretka (2015), and Du and Zhu (2017). This paper addresses whether this welfare cost might be reduced by augmenting an existing exchange market with size-discovery sessions.

4 Augmenting Price Discovery with Size Discovery

An obvious improvement in welfare is obtained by an initializing size-discovery session. For example, Duffie and Zhu (2017) showed an improvement in welfare associated with running a workup session at time zero, before the sequential-double-auction market opens.

Workup does not optimally reallocate initial inventory. We showed in Section 2 that running an optimal mechanism at time zero achieves a perfect initial allocation, after which all traders have the same inventory $\overline{Z}_0$. If no further size-discovery reallocation sessions are run, so that after the market opens traders rely entirely on the sequential-double-auction market, then the corresponding welfare is

$$W^*(Z_0) \equiv W_{fb}(Z_0) + \frac{\gamma\sigma_Z^2}{r^2 n} + \sum_{i=1}^{n} \theta_i. \tag{20}$$

A calculation[25] then shows that

$$W^*(Z_0) \leq W_{fb}(Z_0), \tag{21}$$

with strict inequality unless $H^i = H^j$ for all i, j. The negative constant $\sum_{i=1}^n \theta_i$ reflects the aggregate costs to all traders of future random inventory shocks that are only slowly rebalanced in the subsequent sequential-double-auction market.

We are about to show that welfare is not improved by adding optimal-mechanism reallocation sessions after time zero, even though the traders' inventories are perfectly reallocated at each of these sessions. In the following section, we will show that augmenting the market with perfect reallocation sessions *strictly lowers* welfare if the size-discovery platform operator cannot directly observe the evolution of the aggregate inventory. This welfare loss is caused by incentives to strategically distort the platform operator's inference of the current inventory Z_t

[25]Rearranging terms, we have

$$\theta_i = \frac{\gamma(n-2)}{r^2(n-1)}\mathrm{var}(\bar{Z}_1 - H_1^i | Z_0) - \frac{\gamma}{r^2}\sigma_i^2.$$

We note that $\sum_{i=1}^n \mathrm{var}(\bar{Z}_1 - H_1^i|Z_0) = -n\mathrm{var}(\bar{Z}_1|Z_0) + \sum_{i=1}^n \mathrm{var}(H_1^i)$. The inequality follows from the fact that $n\sum_{i=1}^n \mathrm{var}(H_1^i) \geq \mathrm{var}(\sum_{i=1}^n H_1^i)$, with equality if and only if $H^i = H^j$ for all i, j .

from observing prior double-auction prices.

In this section, as in Section 2, the aggregate inventory Z_t is assumed to be observable by the size-discovery mechanism operator. To the model setup of the previous section, we now add a sequence of size-discovery sessions, each of which uses the perfect-reallocation mechanism developed in Section 2. These sessions occur at the event times $\tau_1, \tau_2, \ldots$ of a commonly observable Poisson process N with mean arrival rate $\lambda > 0$. The session-timing process N is independent of the other primitive processes and random variables, $\{H, \mathcal{T}, \pi, z_0\}$.

In practice, the mean frequency of size-discovery sessions varies significantly across markets. For example, workup sessions in BrokerTec's market for Treasury securities occur at an average frequency of about 600 times a day for the 2-year note, and about 1400 times a day for the 5-year note, according to statistics provided by Fleming and Nguyen (2015). These size-discovery sessions account for approximately half of all trade volume in Treasury securities on BrokerTec, which is by far the largest trade platform for U.S. Treasurys, accounting for an average of over \$30 billion in daily transactions for each of the 2-year, 5-year, and 10-year on-the-run Treasury notes. Consistent with our model, BrokerTec workup sessions are held at randomly spaced times. As opposed to our model, however, the times of BrokerTec workup sessions are not exogenous — they are chosen by market participants. In the corporate bond market, "matching sessions," another form of size discovery, occur with much lower frequency, such as once per week for some bonds. The matching sessions on Electronifie, a corporate bond trade platform, are triggered automatically by an algorithm that depends on the current-limit-order book and the unfilled portion of the last trade on the central-limit-order book. Again, this differs from our simplifying assumption that size-discovery reallocation sessions occur at independent exogenously chosen times.

In many designs for size-discovery sessions, and in the setting of the next section of our paper, the platform operator exploits prior "lit" exchange market prices as a guide to (or automatic determinant of) the "frozen price" used in the size-discovery session. This introduces additional incentive effects that we consider in the next section. In this section, because the aggregate inventory Z is observable, the size-discovery platform operator does not need to rely on prior "lit" exchange market prices to set the mechanism's cash compensation terms.

In addition to choosing an exchange market demand process D^i, as modeled in the previous section, trader i also chooses an $\mathbb{F}^i$-adapted finite-variance and jointly measurable[26] process $\hat{z}^i$ of size-discovery reports. This space of admissible $(D^i, \hat{z}^i)$ is denoted $\mathcal{C}^i$.

Our size-discovery sessions use the mechanism design (Y, T_κ) of Section 2, restricting attention to the affine functions $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$ of Z_t that exploit the properties of Propositions 1 and 2. We will calculate intercept and slope coefficients of both $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$ that are

[26]For the formal definition of adapted and jointly measurable, please refer, for example, to Protter (2005).

consistent with the resulting endogenous continuation value functions.

We will show that equilibrium exchange market demand behavior in this new setting is of the same affine form that we found in the market without size-discovery sessions, but has different demand coefficients. The traders' demands are altered by the prospect of getting a perfectly rebalanced allocation at each size-discovery session.

The demand process D^i chosen by trader i and the vector $\hat{z}$ of report processes of all traders imply that the inventory process of trader i is

$$z_t^i = z_0^i + \int_0^t D_s^i \, ds + H_t^i + \int_0^t \left(\frac{\sum_{j=1}^n \hat{z}_s^j}{n} - \hat{z}_s^i \right) dN_s. \tag{22}$$

Given the direct-revelation mechanism design (Y, T_κ) for the size-discovery sessions, an equilibrium of the associated dynamic demand and reporting game (involving symmetric affine demand functions) consists of demand coefficients (a, b, c), with the following properties:

A. If each trader i assumes that each other trader j uses these demand coefficients and truthfully reports the position $\hat{z}_t^j = z_t^j$ for the purposes of size-discovery sessions, then trader i optimally uses the same affine demand function coefficients (a, b, c) and also reports truthfully.

B. Participation in the size-discovery sessions is individually rational. Specifically, given the equilibrium strategies, at every time τ_j that a mechanism occurs, each trader i prefers, at least weakly, to participate in the session and obtain the resulting conditional expected cash and asset transfers, over the alternative of not participating.

It turns out that, in equilibrium, the continuation value of trader i at time t depends only on z_t^i and Z_t. So, it does not matter to trader i whether or not the other $n-1$ traders participate, in the off-equilibrium event that trader i opts out of the mechanism.

Our notion of equilibrium implies market clearing, rational conjectures of other traders' strategies, and individual trader optimality, including the incentive compatibility of truth-telling and individual rationality of participation in all size-discovery sessions. Appendix E analyzes the discrete-time version of this model, showing that the analogous equilibrium is Perfect Bayes.

The definition of individual trader optimality in this dynamic game is relatively obvious from the previous sections, but is now stated for completeness. Taking as given the demand coefficients (a, b, c) used by other traders and the mechanism design (Y, T_κ) for size-discovery sessions, trader i faces the problem of choosing a demand process D^i and report process $\hat{z}^i$ that solve the problem

$$U^i_{A,0} \equiv \sup_{(D,\tilde{z})\in\mathcal{C}^i} \mathbb{E}^i_0\left[J^i_A(D,\tilde{z})\right], \tag{23}$$

where $\mathbb{E}^i_0$ denotes expectation conditional on $\mathcal{F}^i_0$,

$$J^i_A(D,\tilde{z}) = z^{D,\tilde{z}}_{\mathcal{T}}\pi - \int_0^{\mathcal{T}} \gamma(z^{D,\tilde{z}}_t)^2 + \Phi_{(a,b,c)}(D_t; Z_t - z^{D,\tilde{z}}_t)D_t\,dt + \int_0^{\mathcal{T}} T^i_\kappa((\tilde{z}_t,\hat{z}^{-i}_t), Z_t)\,dN_t, \tag{24}$$

and

$$z^j_t = z^j_0 + \int_0^t \hat{D}^j_s\,ds + H^j_t + \int_0^t \left(\frac{\tilde{z}_s + \sum_{j\neq i}^n \hat{z}^j_s}{n} - \hat{z}^j_s\right) dN_s \tag{25}$$

$$z^{D,\tilde{z}}_t = z^i_0 + \int_0^t D_s\,ds + H^i_t + \int_0^t \left(\frac{\tilde{z}_s + \sum_{j\neq i}^n \hat{z}^j_s}{n} - \tilde{z}_s\right) dN_s, \tag{26}$$

taking $\hat{D}^j_t = a + b\Phi_{(a,b,c)}\left(D_t; Z_t - z^{D,\tilde{z}}_t\right) + cz^j_t$. The definition of incentive compatibility for the equilibrium is that the report process $\hat{z}^i = z^i$ must be optimal for each trader.

Here again, the equilibrium strategies are ex-post optimal in the sense described in Section 3. That is, in any equilibrium and for any trader i, we can relax the requirement that $(D,\tilde{z})$ is chosen from the space $\mathcal{C}^i$ of $\mathbb{F}^i$-adapted processes. Even if trader i could choose from the larger set $\mathcal{C}$ of $\mathbb{F}$-adapted demand and report processes, thus allowing observation of all other traders' inventory positions, trader i would have the same optimal policy. That is, we will verify that the same demand $D^i_t = a + b\phi_t + cz^i_t$ and report $\hat{z}^i_t = z^i_t$ solving (23) also solve

$$V^i_A(z^i_0, Z_0) \equiv \sup_{(D,\tilde{z})\in\mathcal{C}} \mathbb{E}_0\left[J^i_A(D,\tilde{z})\right], \tag{27}$$

where $\mathbb{E}_0$ denotes expectation conditional on $\mathcal{F}_0$.

Because the Markov stochastic control problem (27) is time-homogeneous, $V^i_A(z^i_t, Z_t)$ is the continuation value for trader i at any time $t < \mathcal{T}$. Thus, the equilibrium ex-post individual rationality condition for trader i is that, for all $t < \mathcal{T}$,

$$V^i_A(z^i_t, Z_t) \leq V^i_A\left(z^i_t + \frac{\sum_{j=1}^n \hat{z}^j_t}{n} - \hat{z}^i_t, Z_t\right) + T^i_\kappa(\hat{z}_t, Z_t). \tag{28}$$

Proposition 4. *Suppose that $\lambda < r(n-2)$. Let $\kappa_0 < 0$ be arbitrary, and fix the mechanism design (Y, T_κ) specified by (3) and (4), where*

$$\kappa_1(Z_t) = v - \frac{2\gamma\bar{Z}_t}{r}, \qquad \kappa_2(Z) = -\overline{Z}_t - \frac{\kappa_1(Z_t)}{2\kappa_0 n^2}.$$

1. *Among equilibria in the dynamic game associated with the sequential-double-auction market augmented with size-discovery sessions, there is a unique equilibrium with symmetric affine double-auction demand functions. In this equilibrium, the double-auction demand function* $\mathcal{D}_t^i$ *of trader* i *in state* ω *at time* t *is given by*

$$\mathcal{D}_t^i(\omega, p) = \frac{-r\lambda + r^2(n-2)}{4\gamma} \left(v - p - \frac{2\gamma}{r} z_t^i(\omega) \right). \tag{29}$$

That is, the coefficients (a, b, c) *of the demand function are*

$$a = \frac{[-r\lambda + r^2(n-2)]v}{4\gamma}, \qquad b = \frac{r\lambda - r^2(n-2)}{4\gamma}, \qquad c = \frac{\lambda - r(n-2)}{2}.$$

2. *The market-clearing double-auction price process* ϕ *is given by* $\phi_t = \kappa_1(Z_t)$.

3. *The mechanism design* (Y, T_κ) *achieves the perfect post-session allocation* $z^i(\tau_k) = \overline{Z}(\tau_k)$ *for each trader* i *at each session time* τ_k.

4. *For each trader* i*, the equilibrium indirect utility* $V_A^i(z_t^i, Z_t)$ *at time* t *is identical to the indirect utility* $V^i(z_t^i, Z_t)$ *given by* (15) *for the model without size-discovery sessions. Thus, welfare is invariant to this augmentation of the double-auction market with size-discovery mechanisms.*

From a comparison of the equilibrium demand schedules (16) and (29) that apply before and after augmenting the exchange market with size-discovery sessions, we see that the introduction of size-discovery sessions reduces the magnitude of the slope of the demand functions by $r\lambda/(4\gamma)$. With size-discovery sessions, traders shade their demands in the double auction to mitigate price impact even more than they would in a market without size-discovery sessions. The next size-discovery session is expected by each trader to be so effective at reducing the magnitude of that trader's excess inventory, with a low price impact, that it is individually optimal for traders to reduce the speed with which they rebalance their inventories in the double-auction market. Of course, this is not socially efficient. The welfare cost of this relaxation of order submission in the double-auction market exactly offsets the welfare improvement directly associated with the size-discovery sessions. The two market designs are not only equivalent in terms of total welfare, they are also equally desirable from the viewpoint of each individual trader. In particular, there is no incentive for any subset of traders to set up a size-discovery platform. If the size-discovery sessions are available, however, then all traders strictly prefer to participate, except in the degenerate case of identical initial inventories and identical inventory shock processes $H^1 = H^2 = \cdots = H^n$.

Figure 2 illustrates the implications of augmenting the exchange market with size-discovery sessions. This figure shows simulated sample paths for the excess inventories of two of the $n = 10$ traders, with and without size-discovery mechanisms, based on the equilibria characterized by Propositions 4 and 3, respectively. For each of the two traders whose inventories z^i are pictured, the inventory shock process H^i is an independent Brownian motion with standard deviation ("volatility") parameter $\sigma_i = 0.05$. The aggregate inventory Z_t is a Brownian motion that is independent of $\{H^1, H^2\}$, with standard deviation parameter $\sigma_Z = 0.15$. The mean frequency of size-discovery sessions is $\lambda = 0.12$. The other parameters are shown in the caption of the figure. The graphs of the asset positions are shown in heavy line weights for the market with optimal size-discovery mechanisms, and in light line weights for the market with no size-discovery sessions. In the market that is augmented with size-discovery sessions, the first such session is held at about time $t = 10$, and causes a dramatic reduction in inventory imbalances, bringing the excess inventories of all traders to the perfectly efficient level, the cross-sectional average inventory $\overline{Z}(\tau_1) = -0.05$. In the illustrated scenario, although there are no more size-discovery sessions until time 680, traders in the market that includes size discovery anticipate that they will be able to shed much of their excess inventories at the next such session, whenever it will occur, so they allow their excess inventories to wander relatively far from the efficient level $\bar{Z}_t$, avoiding price impact in the meantime by bidding relatively inaggressively in the double-auction market. Each trader i has an additional incentive to bid less aggressively, relative to the market without size-discovery sessions, because the anticipation of size-discovery sessions causes other traders to bid less aggressively, which lowers market depth and increases trader i's price impact. Indeed, as one can see, during the period that roughly spans from time 110 until time 680, the market without size discovery performs somewhat better, ex post, than the market with size discovery. However, ex ante, or looking forward from any point in time, the two market designs have the same allocative efficiency, as stated by Result 4 of Proposition 4.

5 Unobservable Aggregate Market Inventory

We now remove the unrealistic assumption that the aggregate inventory $Z_t \equiv \sum_{i=1}^{n} z_0^i + H_t^i$ is observable. If Z_t is not directly observable by the size-discovery platform operator, then the size-discovery mechanism designer cannot use the cash-transfer function T_κ, because the $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$ coefficients of T_κ depend on Z_t. As a consequence, the mechanism design and equilibrium behavior change significantly.

Even though the mechanism designer cannot directly observe Z_t, it turns out that the efficient allocation $z_t^i = \overline{Z}_t$ can be achieved at each session time because the mechanism designer

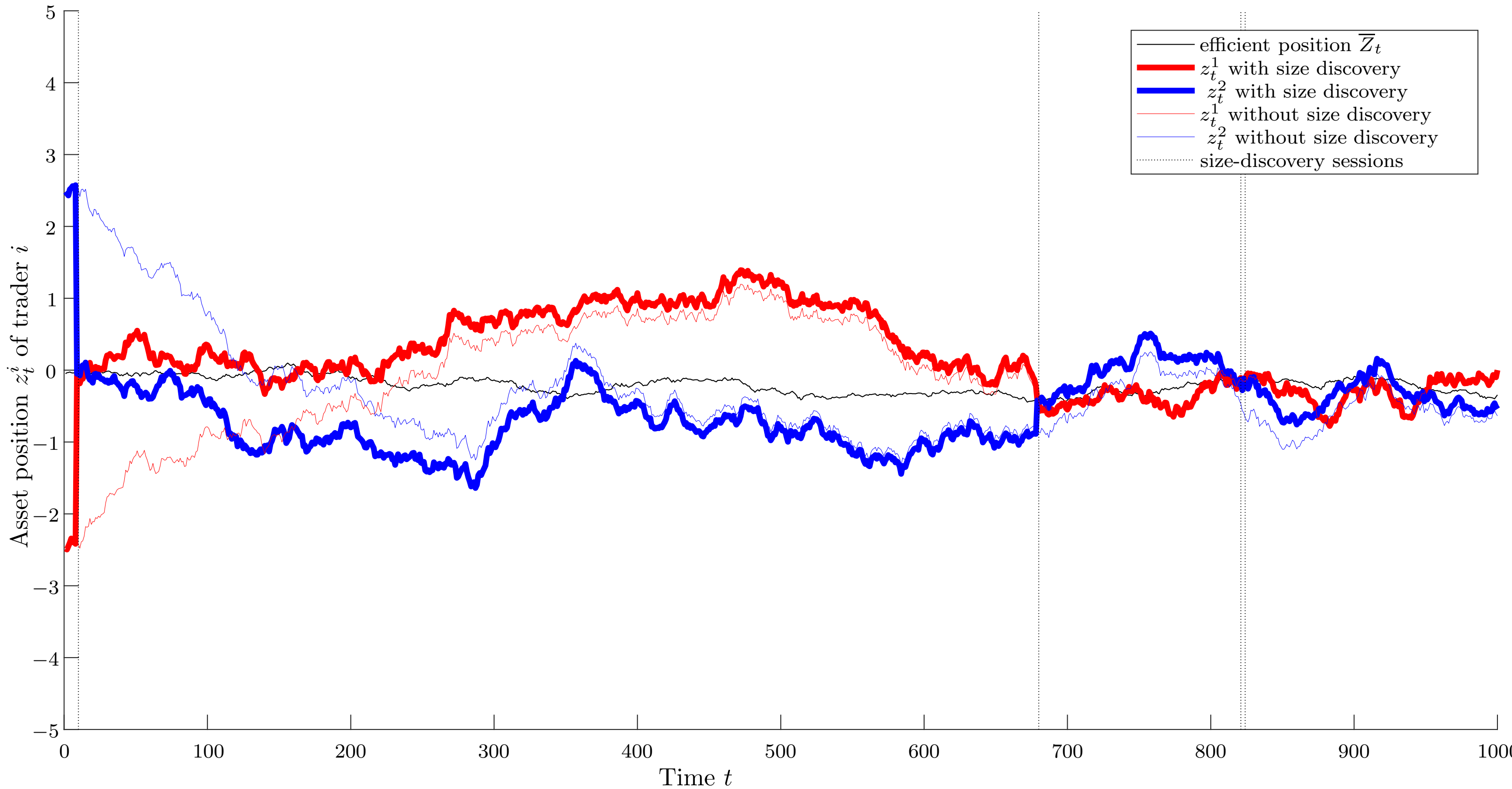

Figure 2: Inventory sample paths with and without size-discovery. This figure plots the inventory sample paths of two of the $n = 10$ traders, with and without size-discovery mechanisms, based on the equilibria characterized by Propositions 4 and 3, respectively. For each trader, the inventory shock process is an independent Brownian motion with standard-deviation parameter $\sigma_i = 0.05$. The aggregate inventory is an independent Brownian motion with standard-deviation parameter $\sigma_Z = 0.15$. The other parameters are mean asset payoff $v = 0.5$, mean rate of arrival of asset payoff $r = 0.1$, inventory cost coefficient $\gamma = 0.1$, initial aggregate market inventory $Z_0 = -0.5$, an initial asset position of trader 1 of $z_0^1 = -2.5$, an initial asset position of trader 2 of $z_0^2 = 2.5$, and a mean frequency $\lambda = 0.1167 = 0.99\bar{\lambda}$ of size-discovery sessions. The graphs of the asset positions are shown in heavy line weights for the market with optimal size-discovery mechanisms, and in light line weights for the market with no size-discovery sessions.

can infer the aggregate inventory Z_t precisely[27] from the "immediately preceding" double-auction market price $\phi_{t-} = \lim_{s\uparrow t} \phi_s$. However, traders now understand that they can strategically influence their cash compensation in the next size-discovery session by influencing the double-auction price in advance of that. For example, a buyer now has an additional incentive to lower the market clearing price, and will demand less in the double-auction market. Likewise, a seller will supply less. This further delays the rebalancing of positions across traders, strictly lowering welfare relative to a market with no size discovery.

In the double-auction market, we will limit attention to equilibria involving symmetric affine demand strategies, as in the model of the previous section, although with potentially different demand coefficients (a, b, c). We will restrict attention to a direct revelation mechanism $(Y, \hat{T})$ that exploits the perfect-reallocation scheme $Y(\,\cdot\,)$ of (3). Thus, the inventory processes are again defined by (22).

We will apply the mechanism cash transfers $\hat{T}(\hat{z}_t; \phi_{t-})$ associated with the function $\hat{T}$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined, for an arbitrary constant $\kappa_0 < 0$, by

$$\hat{T}^i(\hat{z}; p) = p\hat{z}^i + \kappa_0 \left(-n\delta(p) + \sum_{j=1}^{n} \hat{z}^j \right)^2 - p\delta(p) + \frac{p^2}{4\kappa_0 n^2}, \tag{30}$$

where

$$\delta(p) = \frac{-rv}{2\gamma} + p \left(\frac{r}{2\gamma} - \frac{1}{2n^2\kappa_0} \right). \tag{31}$$

The role of the prior price ϕ_{t-} is analogous to that applied in conventional forms of size-discovery used in practice, such as workup and dark pools. In a dark pool, as explained by Zhu (2014), the per-unit price is set by rule to the immediately preceding mid-price in a designated limit-order-book market. In BrokerTec's Treasury-market workup sessions, as explained by Fleming and Nguyen (2015), the frozen price used for workup compensation is fixed at the last trade price in the immediately preceding order-book market operated by the same platform provider. In matching sessions, the frozen price is set based on an estimate of prevailing prices in recent trades. Thus, in dark pools, workup, and other forms of size-discovery used in practice, and also in this setting for our model, there is an incentive for traders to bid strategically in the double-auction market so as to avoid worsening their cash compensation terms in the next size-discovery session, through their impact on the market price ϕ_{t-}.

As in the previous section, given the mechanism $(Y, \hat{T})$, a symmetric equilibrium for the associated dynamic game is defined by a collection (a, b, c) of demand coefficients with the same

[27]This applies except in the zero-probability event that a mechanism session happens to be held precisely at a jump time of Z. Because this event has zero probability, it can without loss of generality be ignored in our calculations.

properties described in the previous section of (A) individual optimality for each trader at all times, including optimal truth-telling, given rational conjectures of other trader's strategies, and (B) rationality of individual participation.

Here, trader i chooses a demand process D^i and a report process $\hat{z}^i$ solving

$$U_S^i \equiv \sup_{(D,\tilde{z}) \in \mathcal{C}^i} \mathbb{E}[J_S^i(D,\tilde{z}) \,|\, \mathcal{F}_0^i], \tag{32}$$

where

$$\begin{aligned} J_S^i(D,\tilde{z}) = z_{\mathcal{T}}^{D,\tilde{z}}\pi - \int_0^{\mathcal{T}} \left[\gamma\left(z_t^{D,\tilde{z}}\right)^2 + \Phi_{(a,b,c)}(D_t; Z_t - z_t^{D,\tilde{z}})D_t\right] dt \\ + \int_0^{\mathcal{T}} \hat{T}^i((\tilde{z}_t, \hat{z}_t^{-i}); \Phi_{(a,b,c)}(D_{t-}; Z_{t-} - z_{t-}^{D,\tilde{z}}))\, dN_t, \end{aligned}$$

subject to (25) and (26). Once again, in any equilibrium the strategies are ex-post optimal. That is, for each trader i, we will verify that the equilibrium strategy $(D^i, \hat{z}^i)$ also solves the complete-information version of the problem given by

$$V_S^i(z_0^i, Z_0) \equiv \sup_{(D,\tilde{z}) \in \mathcal{C}} \mathbb{E}_0[J_S^i(D,\tilde{z})], \tag{33}$$

subject to (25) and (26).

In contrast to the previous setting, for any fixed κ_0, there are exactly two symmetric equilibria. The demand function of one of these equilibria has a bigger slope than that of the other. One equilibrium therefore has lower order flow and higher price impact than the other. The following proposition characterizes these equilibria, and calculates the equilibrium associated with higher order flow, which is the more efficient of the two equilibria.

For this purpose, let $\bar{\lambda}$ be the unique positive solution of the equation

$$3\bar{\lambda} + \sqrt{8\bar{\lambda}(r+\bar{\lambda})} = (n-2)r. \tag{34}$$

Proposition 5. *Suppose $\lambda \leq \bar{\lambda}$. Fix any $\kappa_0 < 0$. Given the mechanism $(Y, \hat{T})$ defined by (3) and (30), there exist equilibria with symmetric affine double-auction demand functions for the dynamic game associated with the sequential-auction markets augmented with size-discovery sessions. Each such equilibrium has the following properties.*

1. *The market-clearing double-auction price process ϕ is given by*

$$\phi_t = v - \frac{2\gamma}{r}\overline{Z}_t. \tag{35}$$

2. *The double-auction demand of trader i at time t is $a + b\phi_t + cz_t^i$, for some coefficients (a, b, c) with $b < 0$.*

3. *The post-session allocation at each session time τ_k is the perfect allocation $z^i(\tau_k) = \overline{Z}(\tau_k)$, almost surely.*

4. *For each trader i, the equilibrium indirect utility at time t is*

$$V_S^i(z_t^i, Z_t) = \theta_i' + v\overline{Z}_t - \frac{\gamma}{r}\overline{Z}_t^2 + \phi_t\left(z_t^i - \overline{Z}_t\right) - K\left(z_t^i - \overline{Z}_t\right)^2, \tag{36}$$

where

$$K = \frac{\gamma}{r(n-1)} - \frac{\lambda}{2b(n-1)} \tag{37}$$

and

$$\theta_i' = \frac{1}{r}\left(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} - K\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}\right) - \frac{2\gamma}{r}\frac{\rho^i}{n}\right). \tag{38}$$

5. *In the more efficient equilibrium, the double-auction demand function coefficients are given by*

$$a = -vb \tag{39}$$

$$b = \frac{-r^2}{8\gamma}\left(-\frac{3\lambda}{r} + (n-2) + \sqrt{\left(\frac{\lambda}{r} - (n-2)\right)^2 - \frac{4\lambda n}{r}}\right) < 0 \tag{40}$$

$$c = \frac{2\gamma}{r}b. \tag{41}$$

6. *In the equilibrium with the demand coefficients (39)-(41), the slope b of the demand function is monotonic increasing*[28] *with respect to the mean frequency λ of size-discovery sessions.*

Part 6 of the proposition implies that the equilibrium market depth, which is the magnitude of b, is decreasing in the mean frequency λ of size-discovery sessions.

In either of the equilibria postulated by the proposition, traders are free to deviate from their affine strategies, and could consider manipulating the double-auction price so as to influence the size-discovery session operator's inference of the aggregate inventory Z_t from the market clearing price ϕ_{t-}. For example, if their inventory z_t^i is large, then trader i, absent any motive

[28] That is, for each $\lambda_0 < \bar{\lambda}$ and each associated equilibrium demand function coefficients (a_0, b_0, c_0), there is a mapping $\lambda \mapsto (a_\lambda, b_\lambda, c_\lambda)$ on a neighborhood of λ_0 to a neighborhood of (a_0, b_0, c_0), specifying the unique equilibrium demand coefficients $(a_\lambda, b_\lambda, c_\lambda)$ for each λ in the neighborhood of λ_0. The coefficient b_λ is increasing in λ.

to affect inference by the session platform operator, would naturally submit large orders to sell. By instead submitting a small buy order, the resulting (off-equilibrium-path) price ϕ_t would be higher, suggesting to the platform operator a smaller aggregate inventory. If a size-discovery session were to occur immediately afterward, the designer would then implement cash transfers based on this "distorted" price. The cash transfers would more generously compensate traders who have (and report) larger inventories, given the rebalancing objective of the platform operator. If the mechanisms are run too frequently, however, this incentive to distort the price through order submission becomes so great that the double-auction market breaks down, in that affine equilibrium demand functions cease to exist.

We now focus on the particular equilibrium defined by (39)-(41). As λ increases from zero to the solution $\bar{\lambda}$ of (34), the expected total volume of trade in the double-auction market declines. Once λ exceeds $\bar{\lambda}$, if an equilibrium were to exist, there would be so little order flow that it would be too cheap for traders to manipulate the price in order to benefit from the next size-discovery session, so markets could not clear. That is, the double-auction market would break down, and there is in fact no equilibrium with $\lambda > \bar{\lambda}$.

Given that the equilibrium double-auction demand functions have slope $b < 0$, the second term in the definition (37) of the quadratic coefficient K is positive, provided there is a non-zero mean arrival rate λ for size-discovery sessions. This implies that the inability of the platform operator to directly observe the aggregate inventory balance Z_t causes an additional reduction in allocative efficiency. In fact, in this setting, adding size-discovery sessions to the price-discovery double-auction market causes a strict reduction in welfare. The welfare at any time t in this setting is

$$\hat{W}(z_t) \equiv \sum_{i=1}^{n} V_S^i(z_t^i, Z_t) = \sum_{i=1}^{n} \theta_i' + vZ_t - \frac{n\gamma}{r}\overline{Z}_t^{\,2} - K\sum_{i=1}^{n}\left(z_t^i - \overline{Z}_t\right)^2, \tag{42}$$

which is strictly lower than the welfare for the same market without size-discovery.[29] With stochastic and unobservable total inventory, each trader shades his or her orders in the double-auction market because of the adverse expected impact of aggressive order submissions on the terms of cash compensation that will be received in the next size-discovery session.

We see from (42) that equilibrium welfare is strictly decreasing in K and strictly increasing in $\sum_{i=1}^{n}\theta_i'$. In the equilibrium of Proposition 5, K is monotonically increasing in λ,[30] while each θ_i' is monotonically decreasing in λ. That is, equilibrium welfare only gets worse as the frequency of size-discovery sessions is increased, until size-discovery sessions are so frequent that the exchange price-discovery market breaks down.

[29] The exception is of course the degenerate case of $\lambda = 0$, for which $K = -\gamma/(r(n-1))$ and the two welfare functions coincide.

[30] This follows from (37) since b is negative and increases monotonically in λ.

Moreover, each trader individually strictly prefers the market design without size discovery. That is, if size discovery exists, it is individually rational for traders to participate in each size-discovery session, but all traders would prefer to commit to a market design in which size discovery does not exist.

Figure 3 illustrates the implications of augmenting a price-discovery market with price-based size-discovery sessions. As in Figure 2, this figure shows simulated inventory sample paths of two of the $n = 10$ traders, with and without size-discovery mechanisms, now based on the equilibria characterized by Propositions 5 and 3, respectively. Figures 2 and 3 are based on the same model parameters and the same simulated scenarios for the inventory shock process $H = (H^1, \ldots, H^n)$ and size-discovery session times $\tau_1, \tau_2, \ldots$. The graphs of the asset positions shown in heavy line weights correspond to the market with optimal size-discovery mechanisms. Those paths shown in light line weights correspond to the market with no size-discovery sessions. In the market that is augmented with size discovery, the first such session is held at about time $t = 10$, and causes a dramatic reduction in inventory imbalances, bringing the excess inventories of all traders to the perfectly efficient level, the cross-sectional average inventory $\overline{Z}(\tau_1) = -0.05$. However, because traders shade their bids even more than in the equilibrium of Proposition 4, from roughly time 110 until time 680 for these inventory sample paths, the market without size discovery performed dramatically better, ex post, than the market with size discovery. This is consistent with the result that, looking forward from any point in time, the market design of Proposition 5 has strictly worse allocative efficiency than that of Proposition 3. A comparison with Figure 2 shows the degree to which the informational reliance in size-discovery sessions on prior double-auction market prices worsens the allocative efficiency of the double-auction markets.

6 Discussion and Concluding Remarks

We conclude by exploring the robustness of our results and discussing some implications for market designs involving both price discovery and size discovery.

6.1 Alternative Mechanism Designs

The central result of the paper is that augmenting a price-discovery "exchange" market (in our case a dynamic double-auction market) with optimal size-discovery mechanisms causes a reduction in allocative efficiency. Although the total welfare of market participants jumps up at each size-discovery session, the prospect of subsequent size-discovery sessions reduces the expected gains from trade in the price-discovery market between size-discovery sessions. The

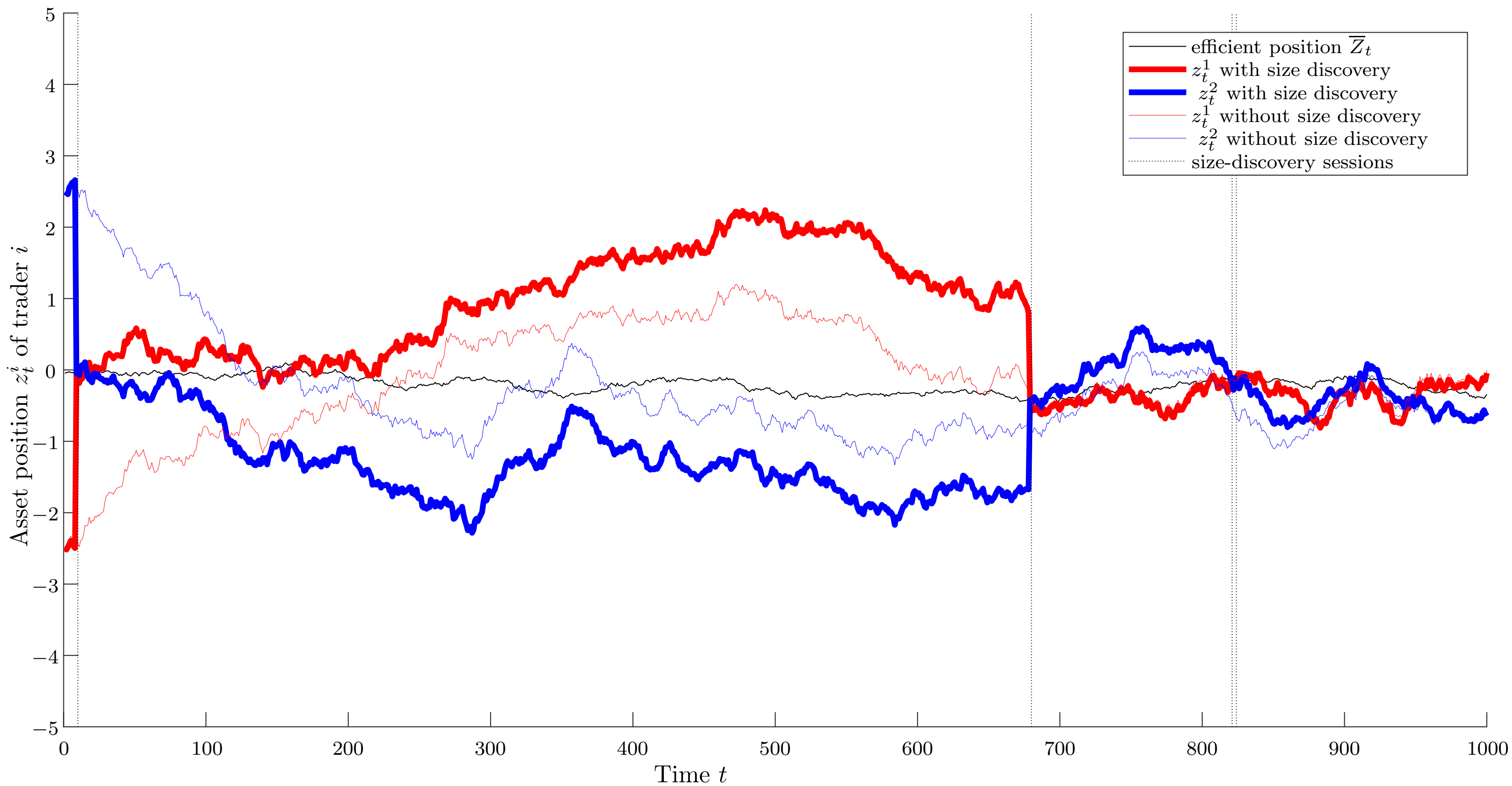

Figure 3: Inventory sample paths with and without price-based size discovery. This figure plots the inventory sample paths of two of the $n = 10$ traders, with and without size-discovery mechanisms, based on the equilibria characterized by Propositions 5 and 3, respectively. For each trader, the inventory shock process is an independent Brownian motion with standard-deviation parameter $\sigma_i = 0.05$. The aggregate inventory is an independent Brownian motion with standard-deviation parameter $\sigma_Z = 0.15$. The other parameters are mean asset payoff $v = 0.5$, mean rate of arrival of asset payoff $r = 0.1$, inventory cost coefficient $\gamma = 0.1$, initial aggregate market inventory $Z_0 = -0.5$, an initial asset position of trader 1 of $z_0^1 = -2.5$, an initial asset position of trader 2 of $z_0^2 = 2.5$, and a mean frequency $\lambda = 0.1167 = 0.99\bar{\lambda}$ of size-discovery sessions. The graphs of the asset positions are shown in heavy line weights for the market with optimal size-discovery mechanisms, and in light line weights for the market with no size-discovery sessions.

net effect is to leave welfare at least as low as that achieved without size-discovery, and strictly lower when, as in practice, the size-discovery operator relies on price information from the price-discovery market.

From a normative market-design viewpoint, this result is discouraging. We have not ruled out the possibility that some other form of size discovery could improve welfare. We have considered alternative mechanism designs, but they have lead to the same result.

For example, consider the following simple variant of a Walrasian mechanism, which allows for rationing. Suppose that the size-discovery platform operator is able to observe the aggregate inventory Z_t directly, or alternatively is able, in equilibrium, to infer Z_t exactly from observation of prior exchange prices. Either way, the platform operator will apply a mechanism that is based on some conjectured level $\tilde{Z}_t$ for the aggregate inventory, which will turn out to be Z_t in equilibrium. The platform operator announces $\tilde{Z}_t$ and some frozen price $\bar{\phi}_t$, which could in principle depend on any information available to the platform operator. The platform operator then solicits position reports $\hat{z}_t$ from the traders. If the sum of the reports, $\sum_{j=1}^n \hat{z}_t^j$, is not equal to $\tilde{Z}_t$, then the transfers of asset and cash are both zero — nothing happens. If, however, the sum of the reports is equal to $\tilde{Z}_t$, then trader i receives[31] the efficient asset transfer $Y^i(\hat{z}_t, \tilde{Z}_t)$ in exchange for cash compensation of $\bar{\phi}_t Y^i(\hat{z}_t, \tilde{Z}_t)$.

In Appendix F, we repeat the analysis of Sections 4 and 5 based on this variant of the Walrasian mechanism. We solve for equilibria of the corresponding dynamic game in which $\bar{\phi}_t$ is given by an affine function of either the aggregate inventory Z_t or the prior exchange price ϕ_{t-}. In fact, there is a unique such affine mechanism price $\bar{\phi}_t$ with the property that participation in the Walrasian rationing mechanism is individually rational, truth-telling is incentive compatible, and the mechanism achieves the efficient allocation when it is run. Somewhat surprisingly, we show in Appendix F that if the game induced by augmenting the exchange market with this version of the Walrasian mechanism admits equilibria, those equilibria must have precisely the same value functions as those we calculated in Sections 4 and 5! In particular, this version of the Walrasian mechanism cannot improve welfare relative to the market that is not augmented by mechanism sessions. Despite this formal equivalence in outcomes, the Walrasian rationing mechanism transfers are highly discontinuous in reports, and thus less robust than our base-case mechanism design in the sense that, looking outside our model, if there were to be any non-zero unanticipated noise[32] in the traders' position reports, then there would be no mechanism trade.

One might be drawn to conjecture that our mechanism design for size-discovery is "too efficient." Indeed, we have shown that the reallocative efficiency and low effective price impact

[31] To be precise, in this Walration-rationing mechanism, trader i receives the asset transfer $Y^i(\hat{z}_t, \tilde{Z}_t) = \mathbf{1}_{\{\sum_{j=1}^n \hat{z}^j = \tilde{Z}_t\}} \left(-\hat{z}^i + \sum_{j=1}^n \hat{z}^j/n\right)$ and the cash transfer $-\bar{\phi}_t Y^i(\hat{z}, \tilde{Z})$.

[32] By "noise," we are thinking in terms of the addition of an independent random variable with a density.

of our size-discovery mechanism design offer such an attractive alternative for executing trades, relative to submitting orders into the price-discovery market, that they reduce price-discovery market depth enough to offset all of the benefit of adding size discovery. We have shown that adding size discovery can actually worsen overall market efficiency.

Given this tension, one might hope to impair the efficiency of the size-discovery design just enough to raise overall market efficiency. By this line of enquiry, one would look for a loss of size-discovery efficiency that is more than offset by a gain in price-discovery allocative efficiency through an improvement of market depth.

We have discovered that this approach does not work, at least among linear-quadratic schemes for size discovery. In Appendix G, we calculate a mechanism design in which the imbalance $z^i_{t-} - \bar{Z}_t$ in the inventory of trader i is not completely eliminated in the size-discovery session. Instead, only a specified fraction ξ of this imbalance is erased by size discovery. Any parameter ξ between 0 and 1 can be supported in an equilibrium with the same properties (other than full efficiency)[33] shown in Section 2, which treats the special case $\xi = 1$. Appendix G provides a corresponding generalization of the dynamic trading model of Section 4. In this setting, overall welfare is invariant to the effectiveness ξ of size discovery. That is, welfare is the same whether one runs perfect reallocation mechanisms ($\xi = 1$), arbitrarily imperfect size-discovery mechanisms ($0 < \xi < 1$), or no size-discovery mechanisms at all.[34]

Finally, it is natural to ask whether simply getting rid of the price-discovery exchange market, and running only size-discovery sessions, could improve welfare, relative to a setting with only price discovery. Even if such a radical redesign of markets could be realistically contemplated, in all of the cases that we have studied, the size-discovery scheme must either rely on information about Z_t or violate individual rationality. In Appendix H, we consider a setting identical to that of Section 4 except that it has no exchange, only size discovery. We show that there is a unique equilibrium for the associated dynamic game, and that the first-best allocation is achieved in the limit as the frequency of size-discovery sessions approaches infinity. However, this market design requires that the aggregate inventory Z_t is observable.

For the case of unobservable Z_t, we show in Appendix I that an altered version of our size-discovery mechanism, run continuously, can achieve the first-best allocation in equilibrium. However, without information about Z_t, it is impossible to make participation in this mechanism individually rational. Other mechanisms might be able to do better. For example, in unreported results, we have found that the dynamic pivot mechanism of Bergemann and Välimäki

[33]We must, however, slightly modify our notion of budget balance. Given the equilibrium strategies, the mechanism is budget balanced with probability 1, but this might not be the case for arbitrary off-equilibrium reports.

[34]We find in unreported numerical examples that if Z_t is unobservable, and in what is otherwise the setting of Proposition 5, welfare is strictly lower with impaired mechanisms than with no mechanisms at all.

(2010) achieves the efficient allocation in a discrete-time version of our primitive model setting. However, the notion of individual rationality satisfied by that mechanism is restrictive for a practical market setting. In order for participation to be individually rational in this setting, a trader who fails to participate in any mechanism session must be absented from all future mechanism sessions. In summary, even if it were possible to eliminate exchange markets, it seems that size-discovery could replace exchange trading and achieve full efficiency only with some form of forced participation.

6.2 Cross-venue competition and stability

Our results imply that there may be a tenuous relationship between the operators of size-discovery and price-discovery platforms, respectively. Barring nearly omniscient alternative information sources, the size-discovery platform operator may need to rely heavily on the prices ϕ_t being produced in price-discovery markets. The size-discovery venue operator can draw more and more volume away from the price-discovery market by holding more and more frequent size-discovery sessions. In theory, the size-discovery venue could in some cases capture an arbitrarily large fraction of the total volume of trade across the two venues. In practice, however, the size-discovery operator would stop short, or be stopped short by others, out of practical business or regulatory concerns. CFA Institute (2012) address general concerns in this area, summarizing with the comment "The results of our analysis show that increases in dark pool activity and internalization are associated with improvements in market quality, but these improvements persist only up to a certain threshold. When a majority of trading occurs in undisplayed venues, the benefits of competition are eroded and market quality will likely deteriorate." We have already mentioned in the introduction the regulatory response to dark pools in the European Union, which in 2018 placed strict caps on volumes of trade executed in dark pools.

The conflicting volume and price discovery incentives of exchange operators and size-discovery venues could in some cases lead toward integration of the sponsors of price-discovery platforms and size-discovery platforms for trading the same asset, along the lines of BrokerTec, which operates both of these protocols for Treasurys trading on the same screen-based platform.[35] Moreover, if an exchange operating both price discovery and size-discovery platforms were to place tight restrictions on trade volumes or frequency for its size-discovery platform in order to maintain volumes and depth on its price-discovery platform, a competing platform operator could attract volume into its own size-discovery platform.

Zhu (2014) has shown that in a setting with asymmetric information about asset payoffs,

[35] Even in this case, however, Schaumburg and Yang (2016) point to some interference arising from price information arriving during size-discovery sessions from the simultaneous operation of Treasury futures trading on the Chicago Mercantile Exchange.

there tends to be a selection bias by which relatively informed investors migrate toward price-discovery markets and relatively less informed investors migrate toward dark pools. This seems to suggest support for robust trade volumes on both types of venues. On the other hand, Zhu (2014) addressed the case of dark pools that promote this selection effect with delays in dark-pool order execution caused by rationing, because rationing discourages informed investors who want to act quickly on their information. As we have pointed out, dark-pool rationing is a relatively crude mechanism design for size-discovery. Although we have not analyzed the implications in our setting of adding asymmetric information about asset payoffs, one may anticipate from our results that more efficient mechanism designs than those currently used in dark pools would be less discouraging to informed investors. This could call into question the robustness of a market design that allows size-discovery venues to free-ride on the price information coming from lit exchanges, while also having a significant ability to draw volume away from lit exchanges.

References

Almgren, R., and N. Chriss. 2001. Optimal execution of portfolio transactions. *Journal of Risk* 3:5–40.

Arrow, K. 1979. The property rights doctrine and demand revelation under incomplete information. In M. Boskin, ed., *Economics and Human Welfare*. Academic Press, Cambridge, Massachusetts.

Athey, S., and I. Segal. 2013. An efficient dynamic mechanism. *Econometrica* 81:2463–85.

Bergemann, D., and J. Välimäki. 2010. The dynamic pivot mechanism. *Econometrica* 78:771–89.

BGC. 2015. BGC Derivative Markets, L.P. Rules. https://www.cftc.gov/sites/default/files/groups/public/@otherif/documents/ifdocs/orgsefbgcexhibitm160128.pdf.

Buti, S., B. Rindi, and I. M. Werner. 2011. Diving into dark pools. Working Paper, Fisher College of Business, Ohio State University. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1630499.

CFA Institute. 2012. Dark pools, internalization, and equity market quality. White paper, CFA Institute. https://www.cfapubs.org/doi/pdf/10.2469/ccb.v2012.n5.1.

Collin-Dufresne, P., B. Junge, and A. B. Trolle. 2016. Market structure and transaction costs of index CDSs. Working Paper, EPFL. https://www.eurofidai.org/sites/default/files/pdf/parismeeting/2016/Collin_Dufresne_2016.pdf.

d'Aspremont, C., and L.-A. Gérard-Varet. 1979. Incentives and incomplete information. *Journal of Public Economics* 11:25–45.

Degryse, H., F. De Jong, and V. van Kervel. 2015. The impact of dark trading and visible fragmentation on market quality. *Review of Finance* 19:1587–622.

Division of Trading and Markets. 2013. Equity market structure literature review, Part I: market fragmentation. U.S. Securities and Exchange Commission, https://www.sec.gov/marketstructure/research/fragmentation-lit-review-100713.pdf.

Du, S., and H. Zhu. 2017. What is the optimal trading frequency in financial markets? *Review of Economic Studies* 84:1606–51.

Duffie, D., and H. Zhu. 2017. Size discovery. *The Review of Financial Studies* 30:1095–150.

Dworczak, P. 2017. Mechanism design with aftermarkets: Cutoff mechanisms. Working paper, Becker Friedman Institute, University of Chicago. http://home.uchicago.edu/~dworczak/Mechanism%20Design%20with%20Aftermarkets.pdf.

Farley, R., E. Kelley, and A. Puckett. 2017. Dark trading volume and market quality: A natural experiment. Working paper, University of Tennessee, Knoxville. http://www1.villanova.edu/content/dam/villanova/VSB/assets/marc/marc2018/SSRN-id3088715.pdf.

Fleming, M., and G. Nguyen. 2015. Order flow segmentation and the role of dark trading in the price discovery of U.S. treasury securities. Working Paper, Federal Reserve Bank of New York. https://www.newyorkfed.org/medialibrary/media/research/staff_reports/sr624.pdf.

Fleming, M., E. Schaumburg, and R. Yang. 2015. The evolution of workups in the U.S. treasury securities market. Liberty Street Economics, Federal Reserve Bank of New York. http://libertystreeteconomics.newyorkfed.org/2015/08/the-evolution-of-workups-in-the-us-treasury-securities-market.html.

GFI. 2015. GFI Swaps Exchange LLC rulebook. GFI Technical Report. https://www.cftc.gov/sites/default/files/stellent/groups/public/@otherif/documents/ifdocs/exhibitm1rulebookgfiswaps.pdf.

Giancarlo, J. C. 2015. Pro-reform reconsideration of the CFTC swaps trading rules: Return to Dodd-Frank. Commodity Futures Trading Commission Technical Report. https://www.cftc.gov/sites/default/files/idc/groups/public/@newsroom/documents/file/sefwhitepaper012915.pdf.

Hatheway, F., A. Kwan, and H. Zheng. 2017. An empirical analysis of market segmentation on U.S. equity markets. *Journal of Financial and Quantitative Analysis* 52:2399–427.

Klemperer, P. D., and M. A. Meyer. 1989. Supply function equilibria in oligopoly under uncertainty. *Econometrica* 57:1243–77.

Liu, S., J. Wang, and C. Wu. 2015. Liquidity frictions, trading and volatility: Evidence from the us treasury market. Washington State University, Working paper. http://www.fmaconferences.org/Sydney/Papers/TradingandVolatility_LiuWangWu_FMAAsia.pdf.

Nimalendran, M., and S. Ray. 2014. Informational linkages between dark and lit trading venues. *Journal of Financial Markets* 17:230–61.

Ollár, M., M. Rostek, and J. H. Yoon. 2017. Privacy in markets. Working paper, Department of Economics, University of Wisconsin, Madison. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3071374.

Pancs, R. 2014. Workup. *Review of Economic Design* 18:37–71.

Pavan, A., I. Segal, and J. Toikka. 2014. Dynamic mechanism design: A Myersonian approach. *Econometrica* 82:601–53.

Protter, P. 2005. *Stochastic integration and differential equations, second edition.* Heidelberg: Springer.

Rostek, M., and M. Weretka. 2012. Price inference in small markets. *Econometrica* 80:687–711.

———. 2015. Dynamic thin markets. *Review of Financial Studies* 28:2946–92.

Sannikov, Y., and A. Skrzypacz. 2016. Dynamic trading: Price inertia and front-running. Working paper, Graduate School of Business, Stanford University. https://economics.uchicago.edu/sites/economics.uchicago.edu/files/uploads/PDF/sannikov_dynamic_trading.pdf.

Schaumburg, E., and R. Yang. 2016. The workup, technology, and price discovery in the interdealer market for U.S. treasury securities. Liberty Street Economics, Federal Reserve Bank of New York. http://libertystreeteconomics.newyorkfed.org/2016/02/the-workup-technology-and-price-discovery-in-the-interdealer-market-for-us-treasury-se html.

SIFMA. 2016. SIFMA electronic bond trading report: US corporate & municipal securities. Technical Report, Securities Industry and Financial Markets Association. https://www.sifma.org/wp-content/uploads/2017/05/sifma-electronic-bond-trading-report-us-corporate-and-municipal-securities.pdf.

Tradeweb. 2014. Market regulation advisory notice – work-up protocol. Tradeweb Technical Report. http://www.tradeweb.com/uploadedFiles/Tradeweb/Content/About_Us/Regulation/DW%20SEF%20MRAN%20-%20Work-Up%20Protocol%20(12.29.14)v2.pdf.

Tradition. 2015. Tradition SEF platform supplement. Tradition Technical Report. http://www.traditionsef.com/assets/regulatory/rulebooks/Rulebook-2015-05.pdf.

Vayanos, D. 1999. Strategic trading and welfare in a dynamic market. *The Review of Economic Studies* 66:219–54.

Vives, X. 2011. Strategic supply function competition with private information. *Econometrica* 79:1919–66.

Wholesale Markets Brokers' Association. 2012. Comment for proposed rule 77 FR 38229. WMBA Technical Report. https://comments.cftc.gov/publiccomments/ViewComment.aspx?id=58343.

Wilson, R. 1979. Auctions of shares. *Quarterly Journal of Economics* 93:675–89.

Ye, L. 2016. Understanding the impacts of dark pools on price discovery. Working paper, The Chinese University of Hong Kong, Shenzhen. https://arxiv.org/pdf/1612.08486.pdf.

Zhu, H. 2014. Do dark pools harm price discovery? *Review of Financial Studies* 27:747–89.

Appendices

The appendices provides auxiliary results and proofs.

A Proofs of Lemma 1 and Propositions 1 and 2

This appendix contains proofs of several results.

A.1 Proof of Proposition 1

Fix a continuation value function V^i for trader i, given by

$$V^i(z^i, Z) = u^i(Z) + \left(\beta_0 + \beta_1 \bar{Z}\right)\left(z^i - \bar{Z}\right) - K\left(z^i - \bar{Z}\right)^2. \tag{43}$$

In equilibrium, trader i achieves the value

$$\sup_{\hat{z}^i} \mathbb{E}\left[V^i(z_0^i + Y^i(\hat{z}), Z) + T_\kappa^i(\hat{z}, Z) \,|\, \mathcal{F}^i\right]. \tag{44}$$

Fix reports $\hat{z}^j = z_0^j$ for $j \neq i$. Substituting (43) into (44), the quantity inside the expectation of (44) is

$$\begin{aligned} u^i(Z) + \left(\beta_0 + \beta_1 \bar{Z}\right)\left(z_0^i + Y^i(\hat{z}) - \bar{Z}\right) - K\left(z_0^i + Y^i(\hat{z}) - \bar{Z}\right)^2 \\ + \kappa_0\left(n\kappa_2(Z) + \sum_{j=1}^n \hat{z}^j\right)^2 + \kappa_1(Z)(\hat{z}^i + \kappa_2(Z)) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2}. \end{aligned} \tag{45}$$

We can write

$$Y^i(\hat{z}) = \frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i = \frac{Z - z_0^i}{n} - \frac{n-1}{n}\hat{z}^i,$$

The terms in (45) that depend on $\hat{z}^i$ sum to

$$\left(\beta_0 + \beta_1 \bar{Z}\right)\left(-\frac{n-1}{n}\hat{z}^i\right) - K\left(\frac{n-1}{n}\right)^2\left(z_0^i - \hat{z}^i\right)^2 + \kappa_0\left(n\kappa_2(Z) + Z - z_0^i + \hat{z}^i\right)^2 + \kappa_1(Z)\hat{z}^i.$$

The first derivative of this expression with respect to $\hat{z}^i$ is

$$\left(\beta_0 + \beta_1 \bar{Z}\right)\left(-\frac{n-1}{n}\right) + 2K\left(\frac{n-1}{n}\right)^2\left(z_0^i - \hat{z}^i\right) + 2\kappa_0(n\kappa_2(Z) + Z - z_0^i + \hat{z}^i) + \kappa_1(Z).$$

The second derivative of (45) with respect to $\hat{z}^i$ is negative because $K > 0$ and $\kappa_0 < 0$. It follows that the unique solution of this first order condition is the unique optimal report. Substituting

$\hat{z}^i$ with $\hat{z}^i = z_0^i$ in the first derivative and then equating the result to 0 implies that

$$0 = \left(\beta_0 + \beta_1 \bar{Z}\right)\left(-\frac{n-1}{n}\right) + 2\kappa_0(n\kappa_2(Z) + Z) + \kappa_1(Z).$$

Thus, for any fixed $\kappa_1(\,\cdot\,)$ and κ_0, we find that

$$\kappa_2(Z) = -\overline{Z} + \frac{-\kappa_1(Z) + (\frac{n-1}{n})\left(\beta_0 + \beta_1 \overline{Z}\right)}{2\kappa_0 n} \tag{46}$$

is the unique $\kappa_2(Z)$ such that trader i optimally reports $\hat{z}^i = z_0^i$. Since this report maximizes the quantity inside the expectation, it maximizes the objective state by state. This reporting strategy therefore constitutes an ex-post equilibrium of the mechanism game. At the equilibrium reports, we have

$$\frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i = -\left(z_0^i - \bar{Z}\right).$$

Thus, $z_0^i + Y^i(\hat{z}) = \bar{Z}$, as desired.

For the special case in which

$$\kappa_0 = \frac{-K(n-1)}{n^2},$$

we can define $Q \equiv \sum_{j \neq i} \hat{z}^j / n$ and calculate that

$$\begin{aligned}
\kappa_0 \left(\sum_{j=1}^n \hat{z}^j\right)^2 - K\left(z_0^i + Y^i((\hat{z}^i, \hat{z}^{-i})) - \bar{Z}\right)^2 &= \kappa_0 (nQ)^2 + \kappa_0 (\hat{z}^i)^2 + 2\kappa_0 n Q \hat{z}^i \\
&\quad - K\left(z_0^i + Q - \bar{Z}\right)^2 - K\left(\frac{n-1}{n}\right)^2 (\hat{z}^i)^2 \\
&\quad + 2K\frac{n-1}{n}\hat{z}^i\left(z_0^i + Q - \bar{Z}\right) \\
&= \kappa_0 (nQ)^2 + \kappa_0 (\hat{z}^i)^2 - K\left(z_0^i + Q - \bar{Z}\right)^2 \\
&\quad - K\left(\frac{n-1}{n}\right)^2 (\hat{z}^i)^2 + 2K\frac{n-1}{n}\hat{z}^i\left(z_0^i - \bar{Z}\right).
\end{aligned}$$

It is thus clear from equation (45) that the optimal report does not depend on Q. In this case, $\hat{z}^i = z_0^i$ is therefore a dominant strategy.

A.2 Proof of Proposition 2

Fix a continuation value as above, and let $\kappa_1(Z) = \beta_0 + \beta_1 \bar{Z}$. We see that

$$\kappa_2(Z) = -\overline{Z} - \frac{\kappa_1(Z)}{2\kappa_0 n^2}, \tag{47}$$

and thus the transfer to trader i is

$$\begin{aligned}
&\kappa_0 \left(n\kappa_2(Z) + \sum_{j=1}^{n} \hat{z}^j \right)^2 + \kappa_1(Z)(\hat{z}^i + \kappa_2(Z)) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \\
&= \kappa_0 \left(-Z - \frac{\kappa_1(Z)}{2\kappa_0 n} + Z \right)^2 + \kappa_1(Z) \left(z_0^i - \overline{Z} - \frac{\kappa_1(Z)}{2\kappa_0 n^2} \right) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \\
&= \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} + \kappa_1(Z)(z_0^i - \bar{Z}) - \frac{\kappa_1^2(Z)}{2\kappa_0 n^2} + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \\
&= \kappa_1(Z) \left(z_0^i - \bar{Z} \right) \\
&= \left(\beta_0 + \beta_1 \bar{Z} \right) \left(z_0^i - \bar{Z} \right).
\end{aligned}$$

From Proposition 1, trader i has the equilibrium post-reallocation inventory $\bar{Z}$. The equilibrium utility of trader i is then simply

$$u^i(Z) + \kappa_1(Z) \left(z_0^i - \bar{Z} \right) = u^i(Z) + \left(\beta_0 + \beta_1 \bar{Z} \right) \left(z_0^i - \bar{Z} \right).$$

Comparing this with $V^i(z_0^i, Z)$, the result follows from the fact that $K > 0$.

For the uniqueness of $\kappa_1(\cdot)$, note that for IR to hold with probability 1, by continuity, it must hold in the event that $z_0^i = \overline{Z}$ for all i. In this case, the change in utility for any trader is just the transfer received by that trader. By the definition of the transfers, straightforward algebra shows that for any vector $\hat{z}$ of reports,

$$\begin{aligned}
\sum_{i=1}^{n} T_\kappa^i(\hat{z}, Z) &= \sum_{i=1}^{n} \left(\kappa_0 \left(n\kappa_2(Z) + \sum_{j=1}^{n} \hat{z}^j \right)^2 + \kappa_1(Z)(\hat{z}^i + \kappa_2(Z)) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \right) \\
&= -n \left(\sqrt{-\kappa_0} \left(n\kappa_2(Z) + \sum_{j=1}^{n} \hat{z}^j \right) - \frac{\kappa_1(Z)}{2\sqrt{-\kappa_0} n} \right)^2.
\end{aligned}$$

Plugging in the choice of $\kappa_2(\cdot)$ suggested in Proposition 1 and using $\hat{z}^i = z_0^i$, we have

$$\sum_{i=1}^{n} T_\kappa^i(\hat{z}, Z) = -n \left(\sqrt{-\kappa_0} \, \frac{-\kappa_1(Z) + (\frac{n-1}{n}) \left(\beta_0 + \beta_1 \overline{Z} \right)}{2\kappa_0} - \frac{\kappa_1(Z)}{2\sqrt{-\kappa_0} n} \right)^2,$$

which is nonnegative if and only if $\kappa_1(Z) = \beta_0 + \beta_1 \overline{Z}$, completing the proof.

A.3 Proof of Lemma 1

Because $b \neq 0$, the following statements are equivalent:

(i) $d + \sum_{j \neq i} (a + bp + cz_t^j) = 0$.

(ii) $-b(n-1)p = d + (n-1)a + cZ_t^{-i}$.

(iii) $p = \frac{-1}{b(n-1)}\left(d + (n-1)a + cZ_t^{-i}\right)$.

B A Lemma and the Proof of Proposition 3

As noted in the text, every equilibrium in this paper is ex-post. Throughout the rest of the appendix, in every dynamic optimization we solve, we will assume all traders have access to the full filtration of information $\mathbb{F}$. In every dynamic optimization we consider, while each trader i is free to choose $\mathbb{F}$-adapted demand and report processes, their optimal demand and report processes will actually be $\mathbb{F}^i$-adapted. It is thus immediate that if trader i were forced to choose $\mathbb{F}^i$ -adapted processes, the same demand and report processes would achieve the optimal value in this more constrained problem. Mathematically, for any random variable $\mathcal{U}_\mathcal{T}$ measurable with respect to $\mathbb{F}_\mathcal{T}$, we have

$$\sup_{(D^i,\tilde{z})\in\mathcal{C}^i} \mathbb{E}\left[\mathcal{U}_\mathcal{T} \;\middle|\; \mathbb{F}_t^i\right] \leq \mathbb{E}\left[\left(\sup_{(D^i,\tilde{z})\in\mathcal{C}} \mathbb{E}\left[\mathcal{U}_\mathcal{T} \;\middle|\; \mathbb{F}_t\right]\right) \;\middle|\; \mathbb{F}_t^i\right]$$

almost surely, because $\mathcal{C}^i \subset \mathcal{C}$. If the $(D^i, \tilde{z})$ which solve the optimization on the right side of the inequality actually satisfy $(D^i, \tilde{z}) \in \mathcal{C}^i$, then $(D^i, \tilde{z})$ must also solve the optimization on the left side, and the inequality must actually hold with equality by the tower property.

In light of this, we will restrict traders to $\mathbb{F}$-adapted processes, and when those processes are $\mathbb{F}^i$-adapted, it is immediate that we have an ex-post equilibrium. For convenience, we will thus let $\mathbb{E}[\cdot]$ denote expectation with respect to $\mathbb{F}_0$ throughout the rest of the appendix.

First, we prove a technical lemma that will be useful in all subsequent proofs.

Lemma 2. *Let $c \neq 0$ be an arbitrary constant, and let $\bar{Z}_t$ and σ_Z^2 be defined as in the text. Then, for any t,*

$$\mathbb{E}\left[\int_0^t e^{-cs}\bar{Z}_s\,ds\right] = \bar{Z}_0\frac{1-e^{-ct}}{c}, \tag{48}$$

and

$$\mathbb{E}\left[\left(\int_0^t e^{-cs}\bar{Z}_s ds\right)^2\right] = \frac{(1-e^{-ct})^2}{c^2}\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}\frac{e^{-2ct}\left(2ct - 4e^{ct} + e^{2ct} + 3\right)}{2c^3}. \tag{49}$$

As $c \to 0$, these expectations converge to the expectations of the limiting integrands, and in particular

$$\mathbb{E}\left[\left(\int_0^t \bar{Z}_s\,ds\right)^2\right] = \bar{Z}_0^2 t^2 + \frac{\sigma_Z^2}{n^2}\frac{t^3}{3}. \tag{50}$$

Proof: Fixing s, because $\mathbb{E}[(\bar{Z}_s)^2] = \bar{Z}_0^2 + (\sigma_Z^2/n^2)s$ by assumption, we can apply Hölder's inequality to find that

$$\mathbb{E}\left[\left|e^{-cs}\bar{Z}_s\right|\right] \leq e^{-cs}\sqrt{\mathbb{E}\left[(\bar{Z}_s)^2\right]} = e^{-cs}\sqrt{\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}s}.$$

It follows that, for any t,

$$\int_0^t \mathbb{E}[|e^{-cs}\bar{Z}_s|]\,ds \le \int_0^t e^{-cs}\sqrt{\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}s}\;ds < \infty.$$

We may thus apply the Fubini-Tonelli theorem to write that

$$\mathbb{E}\left[\int_0^t e^{-cs}\bar{Z}_s\;ds\right] = \int_0^t \mathbb{E}\left[e^{-cs}\bar{Z}_s\right]\;ds = \bar{Z}_0\int_0^t e^{-cs}\,ds = \bar{Z}_0\frac{1-e^{-ct}}{c},$$

where we have used the fact that, from the definition of H_t, we have $\mathbb{E}[\bar{Z}_s] = \bar{Z}_0$. Henceforth, for brevity we refer to this as the "Hölder's inequality and Fubini-Tonelli theorem argument."

Now, define $W_t = \int_0^t e^{-cs}\bar{Z}_s\,ds$. By Ito's lemma,

$$W_t^2 = 2\int_0^t W_s e^{-cs}\bar{Z}_s\,ds = 2\int_0^t\int_0^s e^{-cs}\bar{Z}_s e^{-cu}\bar{Z}_u\,du\,ds.$$

By the Lévy property, $\mathbb{E}[\bar{Z}_u(\bar{Z}_s - \bar{Z}_u)] = 0$. An application of the "Hölder's inequality and Fubini-Tonelli theorem argument" implies that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t\int_0^s e^{-cs}\bar{Z}_s e^{-cu}\bar{Z}_u\,du\,ds\right] &= \int_0^t\int_0^s \mathbb{E}[e^{-cs}\bar{Z}_s e^{-cu}\bar{Z}_u]\,du\,ds \\
&= \int_0^t\int_0^s \mathbb{E}[e^{-cs}e^{-cu}(\bar{Z}_s - \bar{Z}_u + \bar{Z}_u)\bar{Z}_u]\,du\,ds \\
&= \int_0^t\int_0^s \mathbb{E}[e^{-cs}e^{-cu}\bar{Z}_u^2]\,du\,ds \\
&= \int_0^t\int_0^s e^{-cs}e^{-cu}\left(\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}u\right)\,du\,ds \\
&= \frac{(1-e^{-ct})^2}{2c^2}\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}\frac{e^{-2ct}\left(2ct - 4e^{ct} + e^{2ct} + 3\right)}{4c^3}.
\end{aligned}$$

Finally, starting at the penultimate line of the above system and plugging in $c = 0$, we arrive at

$$\mathbb{E}\left[\left(\int_0^t \bar{Z}_s\,ds\right)^2\right] = \bar{Z}_0^2 t^2 + \frac{\sigma_Z^2}{n^2}\frac{t^3}{3}. \tag{51}$$

Now, we are ready to prove Proposition 3. The proof proceeds in 4 steps. First, we use admissibility to restrict the possible set of symmetric affine equilibria. Second, we show that in any symmetric affine equilibrium, the value function must take a specific linear-quadratic form. Third, we calculate the unique value function and affine coefficients consistent with the Hamilton-Jacobi Bellman (HJB) equation. Finally, we verify that the candidate value function and coefficients indeed solve the Markov control problem. Throughout, we write simply $V(z, Z)$ in place of $V^i(z, Z)$. As in the text, we let $\sigma_i^2 \equiv \mathbb{E}[(H_1^i)^2]$.

B.1 Admissibility

In this section, we show that if there were a symmetric affine equilibrium with $c \geq r/2$, then one trader would be using an inadmissible strategy, meaning that the value achieved in the problem

$$V(z_0^i, Z_0) \equiv \sup_{D \in \bar{\mathcal{A}}} \mathbb{E}\left[z_{\mathcal{T}}^D \pi - \int_0^{\mathcal{T}} \gamma \left(z_s^D\right)^2 + \Phi_{(a,b,c)}\left(D_s; Z_s - z_s^D\right)\right) D_s \, ds \right] \tag{52}$$

would be negative infinity or undefined. In order to see this, fix candidate demand coefficients (a, b, c). Then each trader demands the asset at the flow rate $D = a + b\phi + cz$, so the market clearing price must be

$$\phi = \frac{a + c\bar{Z}}{-b}.$$

Plugging this price back into trader demands, we can write

$$D = c(z - \bar{Z}).$$

It follows that if all traders follow this strategy, the inventory of trader i at time t is

$$z_t^i = z_0^i + c \int_0^t (z_s^i - \bar{Z}_s)\, ds + H_t^i. \tag{53}$$

Applying Ito's lemma for semimartingales to $e^{-ct} z_t^i$, and multiplying both sides by e^{ct}, one can show[36] that

$$z_t^i = e^{ct} z_0^i - e^{ct} c \int_0^t e^{-cs} \bar{Z}_s \, ds + e^{ct} \int_0^t e^{-cs} \, dH_s^i. \tag{54}$$

We now show that $\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2 \, ds\right]$ is finite if and only if $2c < r$. We first must compute a few quantities. Because e^{-cs} is square integrable, the last term in the expression for z_t^i is a martingale, so by Lemma 2,

$$\mathbb{E}(z_t^i) = e^{ct} z_0^i + \bar{Z}_0 (1 - e^{ct}).$$

Next, we evaluate

$$\mathbb{E}\left[\int_0^t e^{-cs} \bar{Z}_s ds \int_0^t e^{-cs} \, dH_s^i \right].$$

Let $A_t \equiv \int_0^t e^{-cs} \bar{Z}_s ds$ and $B_t \equiv \int_0^t e^{-cs} dH_s^i$. Note that $[A, B]_t = 0$ since A is a continuous finite variation process, so by Ito's lemma for semimartingales,

$$d(A_t B_t) = A_t \, dB_t + B_t \, dA_t = A_t e^{-ct} \, dH_t^i + B_t e^{-ct} \bar{Z}_t \, dt,$$

or

$$\int_0^t e^{-cs} \bar{Z}_s \, ds \int_0^t e^{-cs} \, dH_s^i = \int_0^t e^{-cs} \int_0^s e^{-cu} \bar{Z}_u \, du \, dH_s^i + \int_0^t e^{-cs} \bar{Z}_s \int_0^s e^{-cu} \, dH_u^i \, ds.$$

[36] This is exactly the derivation of the solution of the Ornstein-Uhlenbeck process.

Since H^i_t is a martingale and $\int_0^s e^{-cu}\bar{Z}_u du$ is square integrable by Lemma 2, we have

$$\mathbb{E}\left[\int_0^t e^{-cs}\bar{Z}_s\, ds \int_0^t e^{-cs}\, dH^i_s\right] = \mathbb{E}\left[\int_0^t e^{-cs}\bar{Z}_s \int_0^s e^{-cu}\, dH^i_u\, ds\right].$$

Applying the "Hölder's inequality and Fubini-Tonelli theorem argument" this expectation is

$$\int_0^t e^{-cs}\mathbb{E}\left[\bar{Z}_s \int_0^s e^{-cu}\, dH^i_u\right] ds$$

where, by another application of Itô's formula for semimartingales and a well known result on the quadratic covariation of semimartingales,

$$\bar{Z}_t \int_0^t e^{-cu}\, dH^i_u = \int_0^t \left(\int_0^s e^{-cu}\, dH^i_u\right) d\bar{Z}_s + \int_0^t \bar{Z}_s e^{-cs}\, dH^i_s + \int_0^t e^{-cu}\, d[H^i, \bar{Z}]_u.$$

Since the integrands $\int_0^s e^{-cu}\, dH^i_u$ and $\bar{Z}_s e^{-cs}$ are square integrable, we have

$$\mathbb{E}\left[\bar{Z}_t \int_0^t e^{-cu}\, dH^i_u\right] = \mathbb{E}\left[\int_0^t e^{-cu}\, d[H^i, \bar{Z}]_u\right] = \int_0^t e^{-cu}\frac{\rho^i}{n}\, du = \frac{\rho^i}{nc}(1 - e^{-ct}).$$

Putting this together,

$$\mathbb{E}\left[\int_0^t e^{-cs}\bar{Z}_s\, ds \int_0^t e^{-cs}\, dH^i_s\right] = \int_0^t e^{-cs}\frac{\rho^i}{nc}(1 - e^{-cs})\, ds = \frac{\rho^i}{2nc^2}(1 - e^{-ct})^2.$$

Next, applying Itô isometry for martingales, and recalling that $[H^i, H^i]_t = \sigma_i^2 t$ because H^i is square-integrable, we have

$$\mathbb{E}\left[\left(\int_0^t e^{-cs}\, dH^i_s\right)^2\right] = \int_0^t e^{-2cs}\sigma_i^2\, ds = \frac{-\sigma_i^2}{2c}(e^{-2ct} - 1).$$

Combining these pieces,

$$\begin{aligned}
\mathbb{E}[(z^i_t)^2] &= e^{2ct}\left(\mathbb{E}\left[\left(z^i_0 - c\int_0^t e^{-cs}\bar{Z}_s\, ds\right)^2\right] + \mathbb{E}\left[\left(\int_0^t e^{-cs}\, dH^i_s\right)^2\right]\right) && (55)\\
&\quad + e^{2ct}\mathbb{E}\left[\left(z^i_0 - c\int_0^t e^{-cs}\bar{Z}_s\, ds\right)\left(\int_0^t e^{-cs}\, dH^i_s\right)\right] && (56)\\
&= e^{2ct}(z^i_0)^2 + 2e^{ct}z^i_0\bar{Z}_0(1 - e^{ct}) + (1 - e^{ct})^2\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}\frac{(2ct - 4e^{ct} + e^{2ct} + 3)}{2c} && (57)\\
&\quad + e^{2ct}\left(\frac{-\sigma_i^2}{2c}(e^{-2ct} - 1)\right) + e^{2ct}\left(\frac{\rho^i}{2nc^2}(1 - e^{-ct})^2\right). && (58)
\end{aligned}$$

Applying the independence of $\mathcal{T}$ and H_t^i, as well as Tonelli's theorem, we have

$$\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2\, ds\right] = \int_0^\infty re^{-rt} \int_0^t \mathbb{E}[(z_s^i)^2]\, ds\, dt \;\leq\; \int_0^\infty \int_0^t \mathbb{E}\left[re^{-rs}(z_s^i)^2\right]\, ds\, dt.$$

From (55), we see that this quantity is finite if and only if $2c < r$. In this case, it is straightforward to show that the quantity in (52) is finite, with $D = c\left(z - \bar{Z}\right)$.

B.2 Value function in a symmetric affine equilibrium

We fix demand coefficients (a, b, c) such that $c < r/2$ and $b \neq 0$. Trader i demands assets at the rate $D_t = a + b\phi_t + cz_t^i$, so the market clearing price must be

$$\phi_t = \frac{a + c\bar{Z}_t}{-b}.$$

Plugging this price back into the demand function of trader i, we can write $D_t = c(z_t^i - \bar{Z}_t)$. Because all traders follow this strategy, the inventory of trader i at time t is

$$z_t^i = z_0^i + c\int_0^t \left(z_s^i - \bar{Z}_t\right)\, ds + H_t^i. \tag{59}$$

Keeping the coefficients (a, b, c) fixed, we will now prove that in any symmetric affine equilibrium, the value function

$$V(z_0^i, Z_0) \equiv \sup_{D\in\bar{\mathcal{A}}} \mathbb{E}\left[z_{\mathcal{T}}^D \pi - \int_0^{\mathcal{T}} \gamma\left(z_s^D\right)^2 + \Phi_{(a,b,c)}\left(D_s; Z_s - z_s^D\right) D_s\, ds\right] \tag{60}$$

takes the form

$$V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z},$$

where

$$\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r - 2c} \\
\alpha_5 &= \frac{1}{r - c}\left(\frac{c^2}{b} - 2\alpha_3 c\right) \\
\alpha_4 &= \frac{1}{r}\left(\frac{c^2}{-b} - c\alpha_5\right) \\
\alpha_1 &= \frac{1}{r - c}\left(rv + \frac{ac}{b}\right) \\
\alpha_2 &= \frac{1}{r}\left(\frac{ca}{-b} - c\alpha_1\right) \\
\alpha_0^i &= \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}\right).
\end{aligned}$$

Given the α coefficients, we have

$$r\left(\alpha_0^i+\alpha_1 z+\alpha_2\bar{Z}+\alpha_3 z^2+\alpha_4\bar{Z}^2+\alpha_5 z\bar{Z}\right)=rvz-\gamma z^2+\alpha_4\frac{\sigma_Z^2}{n^2}+\alpha_3\sigma_i^2+\alpha_5\frac{\rho^i}{n}$$
$$-c(z-\bar{Z})\frac{a+c\bar{Z}}{-b}+c(z-\bar{Z})(\alpha_1+2\alpha_3 z+\alpha_5\bar{Z}).$$

Let $Y_t=1_{\{\mathcal{T}\leq t\}}$ and let $V(z,Z)$ be defined as above. Let

$$X=\begin{bmatrix} z_t^i \\ Z_t \\ Y_t \end{bmatrix}$$

and $U(X)=U(z,Z,Y)=(1-Y)V(z,Z)+Yvz$. Then, by Ito's lemma for semimartingales, for any t, we have

$$\begin{aligned}
U(X_t)-U(X_0)=&\int_{0+}^t(1-Y_{s-})V_z(z_{s-}^i,Z_{s-})+Y_{s-}v\,dz_s^i+\int_{0+}^t(1-Y_{s-})V_Z(z_{s-}^i,Z_{s-})\,dZ_s\\
&+\frac{1}{2}\int_{0+}^t(1-Y_{s-})V_{zz}(z_{s-}^i)\,d[z^i,z^i]_s^c+\frac{1}{2}\int_{0+}^t(1-Y_{s-})V_{ZZ}(z_{s-}^i)\,d[Z,Z]_s^c\\
&+\int_{0+}^t(1-Y_{s-})V_{zZ}(z_{s-}^i)\,d[z^i,Z]_s^c\\
&+\sum_{0\leq s\leq t}U(X_s)-U(X_{s-})-[(1-Y_{s-})V_z(z_{s-}^i,Z_s)+Y_{s-}v]\Delta z_s^i\\
&-\sum_{0\leq s\leq t}(1-Y_{s-})V_Z(z_{s-}^i,Z_s)\Delta Z_s,
\end{aligned}\tag{61}$$

where we have used the fact that

$$\int_{0+}^t\frac{\partial}{\partial Y}U(z_{s-}^i,Y_{s-})\,dY_s=\sum_{0\leq s\leq t}\frac{\partial}{\partial Y}U(z_{s-}^i,Y_{s-})\Delta Y_s,$$

and the fact that $[z^i,Y]^c=[Z,Y]^c=[Y,Y]^c=0$.

Now, we note that

$$\begin{aligned}
V(z_s^i,Z_s)-V(z_{s-}^i,Z_{s-})=&\alpha_1\Delta z_s^i+\alpha_2\frac{\Delta Z_s}{n}+\alpha_4\left(\frac{\Delta Z_s}{n}\right)^2+2\alpha_4\frac{Z_{s-}\Delta Z_s}{n^2}\\
&+\alpha_3(\Delta z_s^i)^2+2\alpha_3 z_{s-}^i\Delta z_s^i+\alpha_5 z_{s-}^i\frac{\Delta Z_s}{n}\\
&+\alpha_5\bar{Z}_{s-}\Delta z_s^i+\alpha_5\frac{\Delta Z_s}{n}\Delta z_s^i,
\end{aligned}$$

while

$$V_Z(z_{s-}^i,Z_{s-})\Delta Z_s=\frac{\Delta Z_s}{n}\left(\alpha_2+\alpha_5 z_{s-}^i+2\alpha_4\bar{Z}_{s-}\right)$$

$$V_z(z^i_{s-}, Z_{s-})\Delta z^i_s = \Delta z^i_s \left(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}\right).$$

Thus, the total contribution to the sum in (61) from jumps in z^i_s or Z_s is given by

$$(1 - Y_{s-}) \left(\alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + \alpha_3 (\Delta z^i_s)^2 + \alpha_5 \frac{\Delta Z_s}{n} \Delta z^i_s \right)$$

because the term $-Y_{s-} v \Delta z^i_s$ is cancelled by the same term in $U(X_s) - U(X_{s-})$.

By independence, $\Delta Y_s \Delta z^i_s = \Delta Y_s \Delta Z_s = 0$ almost surely, so $\Delta Y_s(V(z^i_s, Z_s)) = \Delta Y_s(V(z^i_{s-}, Z_{s-}))$. We note that jumps in z^i arise from jumps in H^i. We can thus write the sum as

$$\sum_{0 \le s \le t} \Delta Y_s \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-})\right) + (1 - Y_{s-}) \left(\alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + \alpha_3 (\Delta H^i_s)^2 + \alpha_5 \frac{\Delta Z_s}{n} \Delta H^i_s \right).$$

Finally, we note that

$$\begin{aligned}
\int_{0+}^t V_z(z^i_{s-}, Z_{s-})\, dz^i_s &= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})\, dz^i_s \\
&= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \left(c(z^i_s - \bar{Z}_s)\right)\, ds \\
&\quad + \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})\, dH^i_s.
\end{aligned}$$

We let

$$\chi_s = c(z^i_s - \bar{Z}_s)(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} + r(v z^i_s - V(z^i_s, Z_s)).$$

Plugging in $V_{ZZ} = 2\alpha_4/n^2$, $V_{zz} = 2\alpha_3$, $V_{zZ} = \alpha_5/n$, and evaluating (61) at $t = \mathcal{T}$, we can write

$$\begin{aligned}
U(X_{\mathcal{T}}) - U(X_0) &= \int_{0+}^{\mathcal{T}} \chi_s\, ds + \int_{0+}^{\mathcal{T}} (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})\, dH^i_s && (62)\\
&\quad + \int_{0+}^{\mathcal{T}} \frac{1}{n} \left(\alpha_2 + \alpha_5 z^i_{s-} + 2\alpha_4 \bar{Z}_{s-}\right)\, dZ_s && (63)\\
&\quad + \alpha_3 \left(-\sigma_i^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[H^i, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta H^i_s)^2 \right) && (64)\\
&\quad + \frac{\alpha_4}{n^2} \left(-\sigma_Z^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, Z]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s)^2 \right) && (65)\\
&\quad + \frac{\alpha_5}{n} \left(-\rho^i \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s \Delta H^i_s) \right) && (66)\\
&\quad + \int_0^{\mathcal{T}} \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-})\right)\, (dY_s - r\, ds), && (67)
\end{aligned}$$

where we have replaced replaced $Y_{s-}=0$ for $s \le \mathcal{T}$, by definition. Also, we have used the fact that $[z^i, z^i]^c = [H^i, H^i]^c$ and $[z^i, Z]^c = [H^i, Z]^c$, since z^i is the sum of H_t^i and a finite-variation process that is a quadratic pure-jump semimartingale (Protter (2004)).

For any deterministic $\mathcal{T}$, it is well known from the theory of Lévy processes that

$$\begin{aligned}
\mathbb{E}\left[\left(-\sigma_i^2\mathcal{T} + [H^i, H^i]^c_{\mathcal{T}} + \sum_{0\le s\le\mathcal{T}} (\Delta H_s^i)^2\right)\right] &= \mathbb{E}\left[\left(-\sigma_Z^2\mathcal{T} + [Z, Z]^c_{\mathcal{T}} + \sum_{0\le s\le\mathcal{T}} (\Delta Z_s)^2\right)\right] \\
&= \mathbb{E}\left[\left(-\rho^i\mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, H^i]^c_s + \sum_{0\le s\le\mathcal{T}} (\Delta Z_s \Delta H_s^i)\right)\right] \\
&= 0.
\end{aligned}$$

For the case of an exponentially distributed $\mathcal{T}$ that is independent of $\{Z, H^i\}$, we may apply law of iterated expectations (conditioning on $\mathcal{T}$) to show that these expectations are still zero.

Now, we let $\mathcal{G}^i_\infty$ be the σ-algebra generated by the path of $\{H_t^i, Z_t\}_{t=0}^\infty$, which is independent of $\mathcal{T}$ by assumption. Then

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\mathcal{T}} [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,(dY_s - r ds)\right] \\
&\qquad = \mathbb{E}\left[\mathbb{E}\left[\int_0^{\mathcal{T}} [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,(dY_s - r ds) \;\middle|\; \mathcal{G}^i_\infty\right]\right] \\
&\qquad = \mathbb{E}\left[\mathbb{E}\left[-r\int_0^{\mathcal{T}} [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,ds + vz^i_{\mathcal{T}} - V(z^i_{\mathcal{T}}, Z^i_{\mathcal{T}}) \;\middle|\; \mathcal{G}^i_\infty\right]\right] \\
&\qquad = \mathbb{E}\left[-r\int_0^\infty re^{-rt}\left(\int_0^t [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,ds\right) dt\right] \\
&\qquad\qquad + \mathbb{E}\left[\int_0^\infty re^{-rt}\left(vz^i_t - V(z^i_t, Z^i_t)\right)\, dt\right] \\
&\qquad = \mathbb{E}\left[-r\int_0^\infty \left(vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})\right)\int_s^\infty re^{-rt}\,dt\,ds\right] \\
&\qquad\qquad + \mathbb{E}\left[\int_0^\infty re^{-rt}\left(vz^i_t - V(z^i_t, Z^i_t)\right)\, dt\right] \\
&\qquad = \mathbb{E}\left[-\int_0^\infty \left(vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})\right) re^{-rs}\,ds\right] \\
&\qquad\qquad + \mathbb{E}\left[\int_0^\infty re^{-rt}\left(vz^i_t - V(z^i_t, Z^i_t)\right)\, dt\right] = 0,
\end{aligned}$$

where the fourth equality follows from a change of order of integration from $\int_0^\infty \int_0^t ds\,dt$ to $\int_0^\infty \int_s^\infty dt\,ds$. Finally, we have already shown that $\mathbb{E}[(z_s^i)^2], \mathbb{E}[(z_s^i)], \mathbb{E}[(\bar{Z}_s)^2]$, and $\mathbb{E}[(\bar{Z}_s)]$ are all integrable (that is, in $\mathcal{L}^1$) processes. It then follows from Hölder's inequality that $\mathbb{E}[z_s^i\bar{Z}_s]$ is also integrable. Then $(\alpha_1 + \alpha_5\bar{Z}_s + 2\alpha_3 z_s^i)$ and $\left(\alpha_2 + \alpha_5 z_s^i + 2\alpha_4\bar{Z}_s\right)$ are square-integrable processes.

So, for a deterministic $\mathcal{T}$,

$$\mathbb{E}\left[\int_{0+}^{\mathcal{T}}(\alpha_1+\alpha_5\bar{Z}_{s-}+2\alpha_3 z^i_{s-})\,dH^i_s\right]=\mathbb{E}\left[\int_{0+}^{\mathcal{T}}\frac{1}{n}\left(\alpha_2+\alpha_5 z^i_{s-}+2\alpha_4\bar{Z}_{s-}\right)\,dZ_s\right]=0,$$

since H^i and Z are martingales. Applying the law of iterated expectations after conditioning on our independent exponentially distributed $\mathcal{T}$, the same result holds. We have thus shown that taking an expectation in equation (62) reduces to

$$\mathbb{E}[U(X_{\mathcal{T}})-U(X_0)]=\mathbb{E}\left[\int_{0+}^{\mathcal{T}}\chi_s\,ds\right]. \tag{68}$$

Because α^i_0 through α_5 satisfy the system of equations specified at the beginning of this proof, we have

$$\mathbb{E}[U(X_{\mathcal{T}})-U(X_0)]=\mathbb{E}\left[\int_{0+}^{\mathcal{T}}c(z^i_s-\bar{Z}_s)\frac{a+c\bar{Z}_s}{-b}+\gamma(z^i_s)^2\,ds\right].$$

By definition, $\mathbb{E}[U(X_{\mathcal{T}})]=\mathbb{E}[vz^i_{\mathcal{T}}]=\mathbb{E}[\pi z^i_{\mathcal{T}}]$, and $\mathbb{E}[U(X_0)]=U(X_0)=V(z^i_0,Z_0)$. We can thus rearrange to find that

$$\begin{aligned}V(z^i_0,Z_0)&=\mathbb{E}\left[\pi z^i_{\mathcal{T}}+\int_{0+}^{\mathcal{T}}-c(z^i_s-\bar{Z}_s)\frac{a+c\bar{Z}_s}{-b}-\gamma(z^i_s)^2\,ds\right]\\&=\mathbb{E}\left[\pi z^i_{\mathcal{T}}+\int_{0+}^{\mathcal{T}}-c(z^i_s-\bar{Z}_s)\phi_s-\gamma(z^i_s)^2\,ds\right],\end{aligned}$$

which completes the proof.

B.3 Solving the HJB equation

For conjectured demand function coefficients a,b,c, the HJB equation is

$$\begin{aligned}rV(z,Z)=-\gamma z^2+vz+\frac{\sigma_i^2}{2}V_{zz}(z,Z)+\frac{\sigma_Z^2}{2}V_{ZZ}(z,Z)+\rho^i V_{zZ}(z,Z)\\+\sup_D\{-\Phi_{(a,b,c)}(D;Z-z)D+V_z(z,Z)D\}.\end{aligned}$$

Plugging in

$$\Phi_{(a,b,c)}(D;Z-z)=\frac{-1}{b(n-1)}[D+(n-1)a+c(Z-z)]$$

from Lemma 1, and then taking a derivative with respect to D, we have

$$\frac{1}{b(n-1)}(2D+(n-1)a+c(Z-z))+V_z(z,Z)=0,$$

or

$$D = -\frac{1}{2}[(n-1)a + c(Z-z) + b(n-1)V_z(z,Z)].$$

From the above, in any affine symmetric equilibrium, it must be that $V_z(z,Z) = \alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z$. Then

$$D = -\frac{1}{2}[(n-1)a + c(Z-z) + b(n-1)\left(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z\right)], \tag{69}$$

where the second-order condition is satisfied if and only if $b < 0$. If trader i is to find the prescribed affine strategy optimal, then D must take the form $D = a + b\phi + cz$. Further, the market clearing price must be

$$\phi = \frac{a + c\bar{Z}}{-b}.$$

Recall from the above that

$$\alpha_1 + \alpha_5\bar{Z} = \frac{1}{r-c}\left(rv - 2\alpha_3 c\bar{Z} - c\left(\frac{a + c\bar{Z}}{-b}\right)\right). \tag{70}$$

Using

$$Z = n\frac{-b\phi - a}{c},$$

we have

$$\begin{aligned}
\alpha_1 + \alpha_5\bar{Z} &= \frac{1}{r-c}\left(rv - 2\alpha_3(-b\phi - a) - c\phi\right) \\
D &= -\frac{1}{2}[(n-1)a + n(-b\phi - a) - cz + b(n-1)\left(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z\right)].
\end{aligned}$$

So, matching coefficients from D, we require that

$$\begin{aligned}
c &= \frac{1}{2}[c - 2b(n-1)\alpha_3] \\
b &= -\frac{1}{2}\left[-nb + b(n-1)\left[\frac{1}{r-c}\left(2\alpha_3 b - c\right)\right]\right] \\
a &= -\frac{1}{2}\left[(n-1)a - na + b(n-1)\frac{1}{r-c}(rv + 2\alpha_3 a)\right].
\end{aligned}$$

Cleaning up and rearranging terms,

$$c = -2b(n-1)\alpha_3 \tag{71}$$

$$(n-2)(r-c) = 2(n-1)\alpha_3 b - c(n-1). \tag{72}$$

Combining (71, 72), we see from (72) that

$$c = \frac{-(n-2)r}{2}. \tag{73}$$

Recalling from the above that

$$\alpha_3 = \frac{-\gamma}{r-2c} = \frac{-\gamma}{r(n-1)},$$

we have

$$b = \frac{-(n-2)r^2}{4\gamma}.$$

Turning to the equation for a, we use the fact that

$$\frac{1}{r-c} = \frac{2}{nr}$$

to obtain

$$-a = b(n-1)\frac{2}{nr}\left(rv - \frac{2\gamma}{r(n-1)}a\right),$$

which reduces to

$$a = \frac{(n-2)r^2v}{4\gamma}. \tag{74}$$

Plugging these in, we see that

$$\phi = \frac{a + c\bar{Z}}{-b} = v - \frac{2\gamma}{r}\bar{Z}.$$

Then returning to $\alpha_1 + \alpha_5\bar{Z}$, we see that

$$\begin{aligned}
\alpha_1 + \alpha_5\bar{Z} &= \frac{1}{r-c}\left(rv - 2\alpha_3 c\bar{Z} - c\left(\frac{a+c\bar{Z}}{-b}\right)\right) \\
&= \frac{2}{rn}\left(rv - 2\left(\frac{-\gamma}{r(n-1)}\right)\left(\frac{-(n-2)r}{2}\right)\bar{Z} - \left(\frac{-(n-2)r}{2}\right)\left(v - \frac{2\gamma}{r}\bar{Z}\right)\right) \\
&= \frac{2}{rn}\left(\frac{nrv}{2} - \frac{\gamma(n-2)}{(n-1)}\bar{Z} - (n-2)\gamma\bar{Z}\right) \\
&= v - \frac{2\gamma}{r}\bar{Z} + \frac{2\gamma}{r(n-1)}\bar{Z}.
\end{aligned}$$

This must hold for any realization of $\bar{Z}$, so $\alpha_1 = v$ and

$$\alpha_5 = -\frac{2\gamma}{r} + \frac{2\gamma}{r(n-1)}.$$

Combining this with $a/b = -v$ from above, we have

$$\alpha_2 = \frac{1}{r}\left(\frac{ca}{-b} - c\alpha_1\right) = \frac{c}{r}(v - v) = 0. \tag{75}$$

Since $c/b = 2\gamma/r$, we see that

$$\frac{c}{b} + \alpha_5 = \frac{2\gamma}{r(n-1)},$$

so

$$\alpha_4 = \frac{1}{r}\left(\frac{c^2}{-b} - c\alpha_5\right) = \frac{-c}{r}\frac{2\gamma}{r(n-1)} = \frac{\gamma(n-2)}{r(n-1)}.$$

Finally, plugging in various obtained formulas, we have

$$\begin{aligned}
\alpha_0^i &= \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}\right) \\
&= \frac{\gamma}{r^2}\left(-\frac{1}{n-1}\sigma_i^2 + \frac{n-2}{n-1}\frac{\sigma_Z^2}{n^2} + 2(\frac{1}{n-1} - 1)\frac{\rho^i}{n}\right) \\
&= \frac{\gamma\sigma_Z^2}{r^2n^2} - \frac{\gamma}{r^2(n-1)}\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}\right) - \frac{2\gamma\rho^i}{r^2 n} = \theta_i.
\end{aligned}$$

Putting this together, we see that the unique value function and demand coefficients satisfying the HJB equation are given by the constants $a, b, c, \alpha_0^i, \alpha_1, \ldots, \alpha_5$ shown above. Rearranging slightly,

$$\begin{aligned}
V(z, Z) &= \alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z} \\
&= \alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z} + v\bar{Z} - v\bar{Z} + \frac{\gamma}{r}\bar{Z}^2 - \frac{\gamma}{r}\bar{Z}^2 \\
&= \theta_i + v\bar{Z} - \frac{\gamma}{r}\bar{Z}^2 + \left(v - \frac{2\gamma}{r}\bar{Z}\right)\left(z - \bar{Z}\right) - \frac{\gamma}{r(n-1)}\left(z - \bar{Z}\right)^2.
\end{aligned}$$

B.4 Finishing the verification of optimality

We have shown that in a symmetric affine equilibrium, the value functions are quadratic and in particular must be twice continuously differentiable. The HJB equation of the previous subsection is thus a necessary condition. Moreover, there is a unique candidate symmetric affine equilibrium which satisfies this HJB equation. We have therefore shown that if each trader follows the proposed affine strategy, the traders indeed get their candidate value functions as their continuation values. It remains to verify that each trader prefers this candidate optimal strategy to any other strategy.

We adopt the notation of Section (B.2). We fix the demand-function coefficients a, b, c of the previous subsection, and the corresponding constants α_0^i, and α_1 through α_5, for some trader i. We fix an admissible demand-rate process D^i, so that the inventory of trader i at time t is

$$z_t^D = z_0^i + \int_0^t D_s^i\, ds + H_t^i, \tag{76}$$

and the trader's expected total future inventory costs is finite. Following the same steps taken in Section (B.2), we can show that

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left(\int_0^{\mathcal{T}} \zeta_s \, ds\right),$$

where

$$\zeta_s = (\alpha_1 + \alpha_5 \bar{Z}_s + 2\alpha_3 z_s^D) D_s^i + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} + r[v z_s^D - V(z_s^D, Z_s)].$$

Because the function $(z, Z) \mapsto V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z \bar{Z}$ satisfies the HJB equation, we know that

$$\zeta_s \leq \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^D) D_s^i + \gamma (z_s^D)^2.$$

Thus

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] \leq \mathbb{E}\left[\int_0^{\mathcal{T}} \Phi_s^D D_s^i + \gamma (z_s^D)^2 \, ds\right].$$

Applying the steps of Section (B.2), it follows that

$$V(z_0^i, Z_0) \geq \mathbb{E}\left[\int_0^{\mathcal{T}} -\Phi_s^D D_s^i - \gamma (z_s^D)^2 \, ds + \pi z_{\mathcal{T}}^D\right].$$

From the analysis of Section (B.2), this inequality is an equality for the proposed affine strategy $D_t^i = c(z_t^i - \bar{Z}_t)$. It follows that this affine strategy is optimal.

C Proof of Proposition 4

The proof proceeds in five steps. First, we use admissibility and the truth-telling property to restrict the possible set of equilibria. Second, we show that in any equilibrium, the value function must take a specific linear-quadratic form. Third, we use individual rationality to restrict the possible mechanism-transfer coefficients, and characterize the optimal mechanism reports in the equilibrium. Fourth, we calculate the unique value function and affine coefficients consistent with the HJB equation. Finally, we verify that the candidate value function and these coefficients indeed solve the Markov control problem. Throughout, we write $V(z, Z)$ in place of $V^i(z, Z)$.

C.1 Efficient allocations and admissibility

Fix a symmetric affine equilibrium (a, b, c). First, recall that in a symmetric affine equilibrium, the market clearing price ϕ_t satisfies $na + nb\phi_t + cZ_t = 0$, which implies that

$$\phi_t = \frac{a + c\bar{Z}_t}{-b},$$

and thus $a+b\phi_t+cz_t^i = c(z_t^i-\bar{Z}_t)$. In equilibrium each trader reports $\hat{z}^j = z^j$, so in equilibrium, the post-mechanism allocation of trader i is

$$z_t^i + \frac{\sum_{j=1}^n \hat{z}_t^j}{n} - \hat{z}_t^i = \bar{Z}_t.$$

The inventory of trader i at time t is therefore

$$z_t^i = z_0^i + c\int_0^t \left(z_s^i - \bar{Z}_s\right)\, ds + H_t^i - \int_0^t \left(z_{s-}^i - \bar{Z}_s\right) dN_s. \tag{77}$$

As in the proof of Proposition 3,

$$e^{-ct} z_t^i = z_0^i - c\int_0^t e^{-cs}\bar{Z}_s\, ds + \int_0^t e^{-cs}\, dH_s^i - \int_0^t e^{-cs}\left(z_{s-}^i - \bar{Z}_s\right)\, dN_s.$$

Letting T_1 denote the minimum of $\mathcal{T}$ and the first jump time of N, we note that

$$-\gamma\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2\, ds\right] \leq -\gamma\mathbb{E}\left[\int_0^{T_1} (z_s^i)^2\, ds\right].$$

For $t < T_1$,

$$z_t^i = e^{ct} z_0^i - ce^{ct}\int_0^t e^{-cs}\bar{Z}_s\, ds + e^{ct}\int_0^t e^{-cs}\, dH_s^i.$$

So, by Lemma 2 and the steps used in the proof of Proposition 3, we know that $\mathbb{E}\left[\int_0^{T_1}(z_s^i)^2\, ds\right]$ is finite if and only if $2c < r+\lambda$. This is true regardless of z_0^i. By a straightforward application of monotone convergence, as long as $2c < r+\lambda$, this implies that

$$\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2\, ds\right] = \mathbb{E}\left[\lim_{n\to\infty}\int_0^{T_n} (z_s^i)^2\, ds\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_0^{T_n} (z_s^i)^2\, ds\right] < \infty.$$

C.2 Linear-quadratic value function

Fix a symmetric affine equilibrium (a,b,c). As above, the market clearing price ϕ_t satisfies $na + nb\phi_t + cZ_t = 0$, which implies that

$$\phi_t = \frac{a + c\bar{Z}_t}{-b},$$

and thus $a + b\phi_t + cz_t^i = c(z_t^i - \bar{Z}_t)$.

Recall that the transfers are given by

$$\kappa_0\left(n\kappa_2(Z_t) + \sum_{j=1}^n \hat{z}_t^j\right)^2 + \kappa_1(Z_t)(\hat{z}_t^i + \kappa_2(Z_t)) + \frac{\kappa_1^2(Z_t)}{4\kappa_0 n^2}.$$

Plugging in $\hat{z}^j = z^j$, we see that for any affine $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$, this transfer takes the form

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

for constants R_0 through R_4 that depend on $\kappa_0, \kappa_1(\cdot), \kappa_2(\cdot)$.

We are now ready to show that, in any symmetric affine equilibrium, the value function

$$V(z,Z) = \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_0^{\mathcal{T}} (-\gamma(z_s^i)^2 - c(z_s^i - \bar{Z}_s)\left(\frac{a + c\bar{Z}_s}{-b}\right) ds) + \int_0^{\mathcal{T}} T_\kappa^i(\hat{z}_s, Z_s)\, dN_s\right]$$

takes the form

$$V(z,Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z},$$

where

$$\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda-2c} \\
\alpha_5 &= \frac{1}{r+\lambda-c}\left(\frac{c^2}{b} - 2\alpha_3 c + \lambda n R_3\right) \\
\alpha_4 &= \frac{1}{r}\left(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2\right) \\
\alpha_1 &= \frac{1}{r+\lambda-c}\left(rv + \frac{ac}{b} + \lambda R_4\right) \\
\alpha_2 &= \frac{1}{r}\left(\frac{ca}{-b} + (\lambda - c)\alpha_1 + \lambda n R_1\right) \\
\alpha_0^i &= \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0\right),
\end{aligned}$$

and where R_0 through R_4 are the previously defined transfer coefficients. Given the α coefficients, we have

$$\begin{aligned}
&(r+\lambda)\left(\alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z}\right) \\
&\quad = rvz - \gamma z^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_3\sigma_i^2 + \alpha_5\frac{\rho^i}{n} - c(z-\bar{Z})\frac{a+c\bar{Z}}{-b} \\
&\qquad + c(z-\bar{Z})(\alpha_1 + 2\alpha_3 z + \alpha_5\bar{Z}) + \lambda(\alpha_0^i + \alpha_1\bar{Z} + \alpha_2\bar{Z} + \alpha_3\bar{Z}^2 \\
&\qquad + \alpha_4\bar{Z}^2 + \alpha_5\bar{Z}^2 + R_0 + R_1 Z + R_2 Z^2 + R_3 Z z + R_4 z).
\end{aligned}$$

Let $Y_t = 1_{\{\mathcal{T}\le t\}}$ and $V(z,Z)$ be defined as above. Let

$$X = \begin{bmatrix} z_t^i \\ Z_t \\ Y_t \end{bmatrix}$$

and $U(X) = U(z,Z,Y) = (1-Y)V(z,Z) + Yvz$. Then, by Ito's lemma for semimartingales, for any t, we have

$$
\begin{aligned}
U(X_t) - U(X_0) = &\int_{0+}^t (1-Y_{s-})V_z(z^i_{s-}, Z_{s-}) + Y_{s-}v\, dz^i_s + \int_{0+}^t (1-Y_{s-})V_Z(z^i_{s-}, Z_{s-})\, dZ_s \\
&+ \frac{1}{2}\int_{0+}^t (1-Y_{s-})V_{zz}(z^i_{s-})\, d[z^i, z^i]^c_s + \frac{1}{2}\int_{0+}^t (1-Y_{s-})V_{ZZ}(z^i_{s-})\, d[Z,Z]^c_s \\
&+ \int_{0+}^t (1-Y_{s-})V_{zZ}(z^i_{s-})\, d[z^i, Z]^c_s \\
&+ \sum_{0\le s\le t} U(X_s) - U(X_{s-}) - [(1-Y_{s-})V_z(z^i_{s-}, Z_s) + Y_{s-}v]\Delta z^i_s \\
&- \sum_{0\le s\le t} (1-Y_{s-})V_Z(z^i_{s-}, Z_s)\Delta Z_s,
\end{aligned} \tag{78}
$$

where we have used the fact that

$$
\int_{0+}^t \frac{\partial}{\partial Y} U(z^i_{s-}, Y_{s-})\, dY_s = \sum_{0\le s\le t} \frac{\partial}{\partial Y} U(z^i_{s-}, Y_{s-})\Delta Y_s,
$$

and the fact that $[z^i, Y]^c = [Z, Y]^c = [Y, Y]^c = 0$.

Now, we note that

$$
\begin{aligned}
V(z^i_s, Z_s) - V(z^i_{s-}, Z_{s-}) = &\alpha_1 \Delta z^i_s + \alpha_2 \frac{\Delta Z_s}{n} + \alpha_4 \left(\frac{\Delta Z_s}{n}\right)^2 + 2\alpha_4 \frac{Z_{s-}\Delta Z_s}{n^2} \\
&+ \alpha_3 (\Delta z^i_s)^2 + 2\alpha_3 z^i_{s-}\Delta z^i_s + \alpha_5 z^i_{s-}\frac{\Delta Z_s}{n} \\
&+ \alpha_5 \bar{Z}_{s-}\Delta z^i_s + \alpha_5 \frac{\Delta Z_s}{n}\Delta z^i_s,
\end{aligned}
$$

while

$$
\begin{aligned}
V_Z(z^i_{s-}, Z_{s-})\Delta Z_s &= \frac{\Delta Z_s}{n}\left(\alpha_2 + \alpha_5 z^i_{s-} + 2\alpha_4 \bar{Z}_{s-}\right) \\
V_z(z^i_{s-}, Z_{s-})\Delta z^i_s &= \Delta z^i_s \left(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}\right).
\end{aligned}
$$

Thus, the total contribution to the sum in (78) from jumps in z^i_s or Z_s is given by

$$
(1-Y_{s-})\left(\alpha_4\left(\frac{\Delta Z_s}{n}\right)^2 + \alpha_3(\Delta z^i_s)^2 + \alpha_5 \frac{\Delta Z_s}{n}\Delta z^i_s\right)
$$

because the term $-Y_{s-}v\Delta z^i_s$ is cancelled by the same term in $U(X_s) - U(X_{s-})$.

We note that jumps in z^i arise from jumps in both H^i and N. By independence, $\Delta N \Delta H^i =$

$\Delta N \Delta Z = \Delta Y \Delta Z = \Delta Y \Delta z^i = 0$ with probability 1. In summary, we can write the sum as

$$\sum_{0 \le s \le t} \Delta Y_s \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-}) \right)$$
$$+ (1 - Y_{s-}) \left(\alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + \alpha_3 (\Delta H^i_s)^2 + \alpha_3 \Delta N_s (z^i_{s-} - \bar{Z}_{s-})^2 + \alpha_5 \frac{\Delta Z_s}{n} \Delta H^i_s \right).$$

It will be convenient to write

$$\begin{aligned}
\sum_{0 \le s \le t} (1 - Y_{s-}) \left(\alpha_3 \Delta N_s (z^i_{s-} - \bar{Z}_{s-})^2 \right) &= \int_0^t (1 - Y_{s-}) \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, dN_s \\
&= \int_0^t (1 - Y_{s-}) \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, (dN_s - \lambda \, ds) \\
&\quad + \int_0^t (1 - Y_{s-}) \lambda \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, ds.
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
\int_{0+}^t V_z(z^i_{s-}, Z_{s-}) \, dz^i_s &= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \, dz^i_s \\
&= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \left((c - \lambda)(z^i_s - \bar{Z}_s) \right) \, ds \\
&\quad + \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \, dH^i_s \\
&\quad + \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})(\bar{Z}_s - z^i_{s-}) \, d(N_s - \lambda \, ds).
\end{aligned}$$

We let

$$\begin{aligned}
\chi_s = {} & c(z^i_s - \bar{Z}_s)(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} \\
& - \lambda (z^i_s - \bar{Z}_s)(\alpha_1 + \alpha_5 \bar{Z}_{s-} + \alpha_3 (z^i_{s-} + \bar{Z}_{s-})) + r(v z^i_s - V(z^i_s, Z_s)).
\end{aligned}$$

Plugging in $V_{ZZ} = 2\alpha_4/n^2$, $V_{zz} = 2\alpha_3$, $V_{zZ} = \alpha_5/n$, and evaluating (78) at $t = \mathcal{T}$, we can write

$$\begin{aligned}
U(X_{\mathcal{T}}) - U(X_0) = &\int_{0+}^{\mathcal{T}} \chi_s \, ds + \int_{0+}^{\mathcal{T}} (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \, dH^i_s \\
&+ \int_{0+}^{\mathcal{T}} (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})(\bar{Z}_s - z^i_s) \, d(N_s - \lambda \, ds) \\
&+ \int_{0}^{\mathcal{T}} \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, (dN_s - \lambda \, ds) + \int_{0+}^{\mathcal{T}} \frac{1}{n} \left(\alpha_2 + \alpha_5 z^i_{s-} + 2\alpha_4 \bar{Z}_{s-}\right) \, dZ_s \\
&+ \alpha_3 \left(-\sigma_i^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[H^i, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta H^i_s)^2 \right) \\
&+ \frac{\alpha_4}{n^2} \left(-\sigma_Z^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, Z]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s)^2 \right) \\
&+ \frac{\alpha_5}{n} \left(-\rho^i \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s \Delta H^i_s) \right) \\
&+ \int_{0}^{\mathcal{T}} \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-}) \right) \, (dY_s - r \, ds),
\end{aligned}$$

where we have replaced $Y_{s-} = 0$ for $s \le \mathcal{T}$, by definition. Since H^i and Z are finite-variance processes, we can now apply arguments similar to those used in the proof of Proposition 3 to show that

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left(\int_{0+}^{\mathcal{T}} \chi_s \, ds \right).$$

Because α_0 through α_5 satisfy the system of equations specified at the beginning of this proof, we have

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left(\int_{0+}^{\mathcal{T}} \bar{\chi}_s \, ds \right),$$

where

$$\bar{\chi}_s = c(z^i_s - \bar{Z}_s) \frac{a + c\bar{Z}_s}{-b} + \gamma (z^i_s)^2 - \lambda(R_0 + R_1 Z_s + R_2 Z_s^2 + R_3 Z_s z^i_s + R_4 z^i_s).$$

Using the definitions of $U, \mathcal{T}$, and R_0 through R_4, as well as the fact that $\mathbb{E}[v z^i_{\mathcal{T}}] = \mathbb{E}[\pi z^i_{\mathcal{T}}]$, we can rearrange to find that

$$\begin{aligned}
V(z_0^i, Z_0) &= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} \bar{\chi}_s \, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} -c(z_s^i - \bar{Z}_s)\frac{a + c\bar{Z}_s}{-b} - \gamma(z_s^i)^2 + \lambda T_\kappa^i(\hat{z}_s, Z_s)\, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0}^{\mathcal{T}} -c(z_s^i - \bar{Z}_s)\frac{a + c\bar{Z}_s}{-b} - \gamma(z_s^i)^2 \, ds + \int_0^{\mathcal{T}} T_\kappa^i(\hat{z}_s, Z_s)\, dN_s\right],
\end{aligned}$$

which completes the proof.

C.3 The Mechanism

Fix a symmetric equilibrium. Recall the mechanism transfers are given by

$$\kappa_0 \left(n\kappa_2(Z_t) + \sum_{j=1}^n \hat{z}_t^j\right)^2 + \kappa_1(Z_t)(\hat{z}_t^i + \kappa_2(Z_t)) + \frac{\kappa_1^2(Z_t)}{4\kappa_0 n^2}.$$

For the purpose of this proof, we will treat $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$ as arbitrary affine functions, and then show the particular choices of $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$ stated by the proposition are the unique functions consistent with equilibrium. From the above, this transfer function with the conjectured reports leads to a linear-quadratic equilibrium value function $V(z, Z)$. Thus, maximizing $V(z + y, Z)$ with respect to y is equivalent to maximizing

$$\alpha_1(z^i + y) + \alpha_3(z^i + y)^2 + \alpha_5\bar{Z}(z^i + y),$$

which in turn is equivalent to maximizing

$$(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i)y + \alpha_3 y^2.$$

Then, when trader i chooses a report $\tilde{z}$, it must be that this choice maximizes

$$(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i)Y^i((\tilde{z}, \hat{z}^{-i})) + \alpha_3 Y^i((\tilde{z}, \hat{z}^{-i}))^2 + T_\kappa^i((\tilde{z}, \hat{z}^{-i}), Z).$$

The first order condition is,

$$-\frac{n-1}{n}(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i) - \frac{2(n-1)\alpha_3}{n}Y^i((\tilde{z}, \hat{z}^{-i})) + \kappa_1(Z) + 2\kappa_0\left(n\kappa_2(Z) + \tilde{z} + \sum_{j\neq i} \hat{z}^j\right) = 0.$$

Plugging in $\hat{z}^j = z_0^j$ and the function Y^i, we have

$$\begin{aligned}
&-\frac{n-1}{n}(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i) - \frac{2(n-1)\alpha_3}{n}\left(\frac{-(n-1)\tilde{z}}{n} + \frac{Z - z^i}{n}\right) \\
&\quad + \kappa_1(Z) + 2\kappa_0\left(n\kappa_2(Z) + \tilde{z} - z^i + Z\right) = 0
\end{aligned}$$

The second order condition is satisfied because κ_0 and α_3 are strictly negative. Since κ_2 is affine, write $\kappa_2(Z) = \hat{a} + \hat{b}Z$. The report $\tilde{z} = z^i$ satisfies this first order condition if

$$-\frac{n-1}{n}(\alpha_1 + \alpha_5 \bar{Z}) - \frac{2(n-1)\alpha_3}{n}\bar{Z} + \kappa_1(Z) + 2\kappa_0 \left(n\hat{a} + n\hat{b}Z + Z\right) = 0.$$

With this,

$$(n\hat{a} + n\hat{b}Z + Z) = \frac{\Xi}{2\kappa_0},$$

where

$$\Xi = -\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}).$$

Thus,

$$\kappa_2(Z) = \hat{a} + \hat{b}Z = -\bar{Z} + \frac{\Xi}{2\kappa_0 n},$$

implying an equilibrium change in utility of

$$\begin{aligned}\frac{\Xi^2}{4\kappa_0} + \kappa_1(Z)\left(-\bar{Z} + \frac{\Xi}{2\kappa_0 n}\right) + \frac{\kappa_1^2(Z)}{4n^2\kappa_0}& \\ + (\kappa_1(Z) - \alpha_1 - \alpha_5\bar{Z})z^i + (\alpha_1 + \alpha_5\bar{Z})\bar{Z} - \alpha_3(z^i)^2 + \alpha_3\bar{Z}^2.&\end{aligned}$$

This change in utility must be weakly positive for any z and Z. If all traders have $z = \bar{Z}$, then we need that

$$\frac{\Xi^2}{4\kappa_0} + \kappa_1(Z)\left(\frac{\Xi}{2\kappa_0 n}\right) + \frac{\kappa_1^2(Z)}{4n^2\kappa_0} = -\left(\frac{\Xi}{2\sqrt{-\kappa_0}} + \frac{\kappa_1(Z)}{2n\sqrt{-\kappa_0}}\right)^2 \geq 0,$$

which implies that $\kappa_1(Z) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}$. Plugging this in, we see that

$$\hat{a} + \hat{b}Z + z^i = z^i - \bar{Z} + \frac{\Xi}{2\kappa_0 n} = z^i - \bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2}.$$

So, we see that $n\kappa_2(Z) + \sum_{j=1}^n \hat{z}^j = -(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})/(2\kappa_0 n)$, and thus the equilibrium transfer to trader i is

$$\begin{aligned}&\frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})^2}{4n^2\kappa_0} + (\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z} - \frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})}{2\kappa_0 n^2}\right) \\ &\qquad + \frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})^2}{4n^2\kappa_0} \\ &= (\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right).\end{aligned}$$

It follows that the equilibrium change in utility for trader i from the mechanism is

$$
\begin{aligned}
(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right) &+ (\alpha_1 + \alpha_5\bar{Z})(\bar{Z} - z^i) + \alpha_3(\bar{Z})^2 - \alpha_3(z^i)^2 \\
&= 2\alpha_3\bar{Z}z^i - \alpha_3(\bar{Z})^2 - \alpha_3(z^i)^2 \\
&= -\alpha_3(z^i - \bar{Z})^2 \geq 0,
\end{aligned}
$$

where the final inequality relies on the fact that α_3 is negative in an equilibrium, from the previous section. Putting this together, as long as $\kappa_1(Z) = \alpha_1 + (\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\bar{Z}$ and $\kappa_2(Z) = \hat{a} + \hat{b}Z$ are given as above, then in equilibrium all traders find the mechanism ex-post individually rational each time it is run, and their strategy $\hat{z}^i = z^i$ is ex-post optimal. This is true only if $\kappa_1(Z)$ and $\kappa_2(Z)$ take the specified forms.

Finally, since the equilibrium transfers are $(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right)$, we see that the coefficients $\{R_m\}$ in

$$R_0 + R_1Z_t + R_2Z_t^2 + R_3Z_tz_t^i + R_4z_t^i,$$

are given by

$$
\begin{aligned}
R_0 &= 0 \\
R_1 &= -\frac{\alpha_1}{n} \\
R_2 &= -\frac{\alpha_5 + 2\alpha_3}{n^2} \\
R_3 &= \frac{\alpha_5 + 2\alpha_3}{n} \\
R_4 &= \alpha_1.
\end{aligned}
$$

Recall from the previous section that

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r + \lambda - 2c} \\
\alpha_5 &= \frac{1}{r + \lambda - c}\left(\frac{c^2}{b} - 2\alpha_3c + \lambda nR_3\right) \\
\alpha_1 &= \frac{1}{r + \lambda - c}\left(rv + \frac{ac}{b} + \lambda R_4\right).
\end{aligned}
$$

Thus, plugging in R_3, R_4, and rearranging, we have

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r + \lambda - 2c} \\
\alpha_5 &= \frac{1}{r - c}\left(\frac{c^2}{b} - 2\alpha_3c + 2\lambda\alpha_3\right) \\
\alpha_1 &= \frac{1}{r - c}\left(rv + \frac{ac}{b}\right).
\end{aligned}
$$

C.4 Solving the HJB Equation

From the above, the value function takes the form

$$V(z^i, Z) = \alpha_0^i + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3 (z^i)^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z^i \bar{Z}.$$

The associated HJB equation is

$$\begin{aligned}
0 = &-\gamma(z^i)^2 + r(vz^i - V(z^i, Z)) + \frac{\sigma_i^2}{2} V_{zz}(z, Z) + \frac{\sigma_Z^2}{n^2} V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n} V_{zZ}(z^i, Z) \\
&+ \sup_{D, \hat{z}^i} \{-\Phi_{(a,b,c)}(D; Z - z^i) D + V_z(z^i, Z) D \\
&+ \lambda \left(V(z^i + Y^i((\hat{z}^i, \hat{z}^{-i})), Z) - V(z, Z) + T_\kappa^i((\hat{z}^i, \hat{z}^{-i}), Z) \right) \}.
\end{aligned}$$

From the previous subsection, we know that fixing the equilibrium reports $\hat{z}^{-i}$ of the other traders, the report $\hat{z}^i = z^i$ achieves the supremum in the HJB equation for any D, as long as

$$\kappa_2(Z) = \hat{a} + \hat{b} Z = -\bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2}.$$

Since $V_z = \alpha_1 + 2\alpha_3 z^i + \alpha_5 \bar{Z}$, we can follow steps identical to those of the proof of Proposition 3, and see that as long as $b < 0$, the unique demand that achieves the maximum in the HJB equation is

$$D = -\frac{1}{2}[(n-1)a + n(-b\phi - a) - cz^i + b(n-1)\left(\alpha_1 + 2\alpha_3 z^i + \alpha_5 \bar{Z}\right)].$$

Plugging in $Z = n(-b\phi - a)/c$, we have

$$D = -\frac{1}{2}\left[(n-1)a + n(-b\phi - a) - cz^i + b(n-1)\left(\alpha_1 + 2\alpha_3 z^i + \alpha_5 \frac{-b\phi - a}{c}\right)\right].$$

Recall from the previous section that, after plugging in equilibrium transfers,

$$\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r + \lambda - 2c} \\
\alpha_5 &= \frac{1}{r - c}\left(\frac{c^2}{b} - 2\alpha_3 c + 2\lambda\alpha_3\right) \\
\alpha_1 &= \frac{1}{r - c}\left(rv + \frac{ac}{b}\right).
\end{aligned}$$

Then, matching coefficients in the expression for D, we have

$$
\begin{aligned}
c &= -\frac{1}{2}[-c + 2b(n-1)\alpha_3] \\
b &= -\frac{1}{2}\left[-nb + b(n-1)\left(\frac{1}{r-c}\left[2\alpha_3 b - c - \lambda 2\alpha_3 \frac{b}{c}\right]\right)\right] \\
a &= -\frac{1}{2}\left[-a + b(n-1)\frac{1}{r-c}\left(rv + 2\lambda\alpha_3(\frac{-a}{c}) + 2\alpha_3 a\right)\right].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
c &= -2b(n-1)\alpha_3 \\
(r-c)(n-2) &= \left[2\alpha_3 b(n-1) - c(n-1) - \lambda 2\alpha_3 \frac{b}{c}(n-1)\right] \\
r(n-2) &= -2c + \lambda \\
c &= \frac{\lambda - r(n-2)}{2} \\
\alpha_3 &= \frac{-\gamma}{r(n-1)} \\
b &= \frac{r\lambda - r^2(n-2)}{4\gamma}.
\end{aligned}
$$

From this, we see that b is strictly negative, satisfying the second order condition, if and only if $\lambda < r(n-2)$.

Next, we have

$$
\begin{aligned}
a &= \frac{1}{r-c}\left(-b(n-1)rv + 2\lambda\alpha_3 b(n-1)\frac{a}{c} - 2\alpha_3 ab(n-1)\right) \\
&= \frac{1}{r-c}\left(-b(n-1)rv + -\lambda a + ca\right) \\
&= \frac{2}{rn-\lambda}\left(-\frac{r\lambda - r^2(n-2)}{4\gamma}(n-1)rv + a\frac{-\lambda - r(n-2)}{2}\right).
\end{aligned}
$$

Noting that

$$\frac{\lambda + r(n-2)}{rn-\lambda} + 1 = \frac{2r(n-1)}{rn-\lambda},$$

we see that

$$a = -\frac{\left(r\lambda - r^2(n-2)\right)v}{4\gamma}.$$

From this, we see that $a = -vb$ and $c = 2\gamma b/r$, so

$$\phi_t = \frac{a + c\bar{Z}_t}{-b} = v - \frac{2\gamma}{r}\bar{Z}_t$$

and

$$\begin{aligned}\alpha_1 &= \frac{1}{r-c}\left(rv + \frac{ac}{b}\right)\\ &= \frac{1}{r-c}(rv - vc) = v.\end{aligned}$$

Likewise,

$$\begin{aligned}\alpha_5 + 2\alpha_3 &= \frac{1}{r-c}\left(\frac{c^2}{b} - 2\alpha_3 c + 2\lambda\alpha_3\right) + 2\alpha_3\\ &= \frac{1}{r-c}\left(\frac{2\gamma}{r}c + 2\alpha_3(r-c) - 2\alpha_3 c + 2\lambda\alpha_3\right)\\ &= \frac{1}{r-c}\left(\frac{2\gamma}{r}c + 2\alpha_3(r+\lambda-2c)\right)\\ &= \frac{1}{r-c}\left(\frac{2\gamma}{r}c - 2\gamma\right) = \frac{-2\gamma}{r}.\end{aligned}$$

It follows that

$$\alpha_5 = \frac{-2\gamma}{r} - 2\alpha_3 = \frac{-2\gamma}{r} + \frac{2\gamma}{r(n-1)}.$$

Plugging α_1, α_5, and α_3 into the equilibrium $\kappa_2(Z)$, we see that

$$\begin{aligned}\kappa_2(Z) &= -\bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2}\\ \kappa_2(Z) &= -\bar{Z} - \frac{v - \frac{2\gamma}{r}\bar{Z}}{2\kappa_0 n^2}.\end{aligned}$$

Likewise,

$$\kappa_1(Z) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z} = v - \frac{2\gamma}{r}\bar{Z}.$$

Recalling that $R_1 = -\alpha_1/n$ and $\alpha_1 = v$, we have

$$\begin{aligned}\alpha_2 &= \frac{1}{r}\left(\frac{ca}{-b} + (\lambda - c)\alpha_1 + \lambda n R_1\right)\\ &= \frac{1}{r}(cv + (\lambda - c)v - \lambda\alpha_1) = 0.\end{aligned}$$

Recalling that

$$R_2 = -\frac{\alpha_5 + 2\alpha_3}{n^2} = \frac{2\gamma}{rn^2},$$

we have

$$\begin{aligned}\alpha_4 &= \frac{1}{r}\left(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2\right)\\ &= \frac{1}{r}\left(\frac{-2\gamma}{r}c + (\lambda - c)\alpha_5 + \lambda\alpha_3 - \lambda(\alpha_5 + 2\alpha_3)\right)\\ &= \frac{1}{r}\left(\frac{-2\gamma}{r}c - c(\alpha_5 + 2\alpha_3) + (2c - \lambda)\alpha_3\right)\\ &= \frac{1}{r}((2c - \lambda - r)\alpha_3 + r\alpha_3)\\ &= \frac{1}{r}\left(\gamma - \frac{\gamma}{(n-1)}\right) = \frac{\gamma(n-2)}{r(n-1)}.\end{aligned}$$

Finally, since $R_0 = 0$, we have

$$\begin{aligned}\alpha_0^i &= \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0\right)\\ &= \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}\right).\end{aligned}$$

Because α_1 through α_5 are exactly the same as in Proposition 3, we know that $\alpha_0^i = \theta_i$, from the statement of Proposition 3. It follows the value function is as stated by Proposition 3.

C.5 Completing the Verification

We have shown that in a symmetric equilibrium, the traders' value functions are linear-quadratic and in particular must be twice continuously differentiable. The HJB equation of the previous subsection is thus a necessary condition, and there is a unique candidate linear-quadratic equilibrium that satisfies this equation. We have shown that if each trader follows the suggested affine strategy, they indeed get their candidate value function as a continuation value. It remains to show that each trader prefers this to any other strategy.

We take the notation of Section C.2. Fix the $a, b, c, \kappa_0, \kappa_1(Z), \kappa_2(Z)$ of the previous subsection, and the corresponding constants $\alpha_0^i, \alpha_1 - \alpha_5$ for some trader i. We fix some admissible demand process D^i, and report process $\tilde{z}$, by which the inventory of trader i at time t is

$$z_t^{(D,\tilde{z})} = z_0^i + \int_0^t D_s^i\, ds + H_t^i + \int_0^t Y^i((\tilde{z}_s, \hat{z}_s^{-i}))\, dN_s. \tag{79}$$

Following the steps of the derivation of the value function, we can show that under the laws of motion implied by D^i and $\tilde{z}$,

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left(\int_{0+}^{\mathcal{T}} \hat{\zeta}_s \, ds\right),$$

where

$$\begin{aligned}\hat{\zeta}_s = {} & D_s^i(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}}) + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} \\ & + \lambda Y^i((\tilde{z}_s, \hat{z}_s^{-i}))(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}} + \alpha_3 Y^i((\tilde{z}_s, \hat{z}_s^{-i}))) + r(v z_s^{D,\tilde{z}} - V(z_s^{D,\tilde{z}}, Z_s)).\end{aligned}$$

Since α_0 through α_5 satisfy the HJB equation, and using the fact that

$$\mathbb{E}\left[\int_0^{\mathcal{T}} \lambda T_\kappa^i((\tilde{z}_s, \hat{z}_s^{-i}), Z_s) \, ds\right] = \mathbb{E}\left[\int_0^{\mathcal{T}} T_\kappa^i((\tilde{z}_s, \hat{z}_s^{-i}), Z_s) \, dN_s\right],$$

we have

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] \leq \mathbb{E}\left[\int_{0+}^{\mathcal{T}} D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) + \gamma (z_s^{D,\tilde{z}})^2 \, ds - \int_0^{\mathcal{T}} T_\kappa^i((\hat{z}_s^i, \hat{z}_s^{-i}), Z_s) \, dN_s\right].$$

Rearranging,

$$V(z_0^i, Z_0) \geq \mathbb{E}\left[\pi z_{\mathcal{T}}^{D,\tilde{z}} + \int_{0+}^{\mathcal{T}} -D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) - \gamma (z_s^{D,\tilde{z}})^2 \, ds + \int_0^{\mathcal{T}} T_\kappa^i((\hat{z}_s^i, \hat{z}_s^{-i}), Z_s) \, dN_s\right].$$

Because this relationship holds with equality for the conjectured affine strategy, this affine strategy is optimal.

D Proof of Proposition 5

The proof proceeds in 6 steps. First, we show that transfers take a particular quadratic form in any equilibrium. Second, we show that $r + \lambda - 2c > 0$ in any equilibrium. (If not, some trader is using an inadmissible or suboptimal strategy.) Third, we show that, given the quadratic form of the transfer function, the value function in any equilibrium must take a particular linear-quadratic form. Fourth, we characterize the optimal mechanism reports and corresponding equilibrium transfers, and characterize equilibrium individual rationality (IR). Fifth, we explicitly solve for the coefficients of the value function and for the strategies that attain the maxima in the HJB equation. Finally, we show that for these candidate optimal strategies, every trader receives an inferior payoff if using any alternative strategy.

D.1 Equilibrium Transfers

We fix a symmetric equilibrium (a, b, c). First, we recall that in a symmetric affine equilibrium, the market clearing price process ϕ must satisfy

$$na + nb\phi_t + cZ_t = 0,$$

which implies that

$$\phi_t = \frac{a + c\bar{Z}_t}{-b},$$

and that $a + b\phi_t + cz_t^i = c(z_t^i - \bar{Z}_t)$.

Recall that the transfers are given by

$$\hat{T}^i(\hat{z}; p) = \kappa_0 \left(-n\delta(p) + \sum_{j=1}^{n} \hat{z}^j \right)^2 + p\left(\hat{z}^i - \delta(p)\right) + \frac{p^2}{4\kappa_0 n^2}, \tag{80}$$

where δ is an affine function. In equilibrium, ϕ_t is affine in Z_t, and everyone reports $\hat{z}^j = z^j$. It is straightforward to show then that in any symmetric equilibrium, the transfers are of the form

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i$$

for constants R_0 through R_4 that depend on δ, κ_0, and the equilibrium coefficients (a, b, c).

D.2 Admissibility

Fix a symmetric equilibrium (a, b, c). The inventory of trader i is

$$z_t^i = z_0^i + c \int_0^t (z_s^i - \bar{Z}_s)\, ds + H_t^i - \int_0^t (z_{s-}^i - \bar{Z}_{s-})\, dN_s. \tag{81}$$

Since, for fixed c, this is identical to the inventory evolution equation of Proposition 4 (Section C.1), the same proof can be used to show that

$$\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2\, ds\right]$$

is finite if and only if $2c < r + \lambda$.

D.3 The value function

We claim that in any symmetric affine equilibrium, the value function

$$V(z, Z) = \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_0^{\mathcal{T}} \left(-\gamma (z_s^i)^2 - c(z_s^i - \bar{Z}_s)\left(\frac{a + c\bar{Z}_s}{-b}\right)\right) ds + \int_0^{\mathcal{T}} \hat{T}^i(\hat{z}_s; \phi_{s-})\, dN_s\right]$$

takes the form

$$V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z},$$

where

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda-2c} \\
\alpha_5 &= \frac{1}{r+\lambda-c}\left(\frac{c^2}{b}-2\alpha_3 c+\lambda n R_3\right) \\
\alpha_4 &= \frac{1}{r}\left(\frac{c^2}{-b}+(\lambda-c)\alpha_5+\lambda\alpha_3+\lambda n^2 R_2\right) \\
\alpha_1 &= \frac{1}{r+\lambda-c}\left(rv+\frac{ac}{b}+\lambda R_4\right) \\
\alpha_2 &= \frac{1}{r}\left(\frac{ca}{-b}+(\lambda-c)\alpha_1+\lambda n R_1\right) \\
\alpha_0^i &= \frac{1}{r}\left(\alpha_3\sigma_i^2+\alpha_4\frac{\sigma_Z^2}{n^2}+\alpha_5\frac{\rho^i}{n}+\lambda R_0\right),
\end{aligned}
$$

where R_0 through R_4 are the previously defined transfer coefficients. Given the α coefficients, we have

$$
\begin{aligned}
&(r+\lambda)\left(\alpha_0^i+\alpha_1 z+\alpha_2\bar{Z}+\alpha_3 z^2+\alpha_4\bar{Z}^2+\alpha_5 z\bar{Z}\right) \\
&\quad= rvz-\gamma z^2+\alpha_4\frac{\sigma_Z^2}{n^2}+\alpha_3\sigma_i^2+\alpha_5\frac{\rho^i}{n}-c(z-\bar{Z})\frac{a+c\bar{Z}}{-b} \\
&\qquad+c(z-\bar{Z})(\alpha_1+2\alpha_3 z+\alpha_5\bar{Z})+\lambda(\alpha_0^i+\alpha_1\bar{Z}+\alpha_2\bar{Z}+\alpha_3\bar{Z}^2 \\
&\qquad+\alpha_4\bar{Z}^2+\alpha_5\bar{Z}^2+R_0+R_1 Z+R_2 Z^2+R_3 Zz+R_4 z).
\end{aligned}
$$

The rest of the proof proceeds exactly as in Section C.2, and is thus omitted.

D.4 Optimal Mechanism Reports and Equilibrium IR

In the HJB equation, trader i chooses a demand D and a report $\hat{z}^i$ to solve[37]

$$
\sup_{D,\hat{z}^i} -D\Phi_{(a,b,c)}(D;Z-z^i)+DV_z(z^i,Z)+\lambda(V(z^i+Y^i((\hat{z}^i,\hat{z}^{-i})),Z)+\hat{T}^i((\hat{z}^i,\hat{z}^{-i});\Phi_{(a,b,c)}(D;Z-z^i))).
$$

In any symmetric affine equilibrium, trader i must have a value function of the specified form. Thus, maximizing $V(z^i+y,Z)$ is equivalent to maximizing

$$
\alpha_1(z^i+y)+\alpha_3(z^i+y)^2+\alpha_5\bar{Z}(z^i+y),
$$

which is equivalent to maximizing

$$
(\alpha_1+\alpha_5\bar{Z})y+\alpha_3 y^2+2\alpha_3 z^i y.
$$

[37]For the purpose of this proof, we suppose trader i can observe Z_t. We show the corresponding optimal strategy depends only on the information in information set of trader i (which does not include Z_t). Because the resulting strategy is optimal even in the larger set of strategies, it is optimal with respect to strategies that are adapted to the information filtration of trader i.

If trader i chooses the auction demand D, thus setting the price $\phi = \Phi_{(a,b,c)}(D; Z - z^i)$ that would be used in the mechanism if one were held immediately, and given that the total of the other traders' reports is $\sum_{j\neq i} z^j = Z - z^i$, trader i gets a transfer of

$$\kappa_0 \left(-n\delta(p) + Z - z^i + \hat{z}^i\right)^2 + p\left(\hat{z}^i - \delta(p)\right) + \frac{p^2}{4\kappa_0 n^2}, \tag{82}$$

and a reallocation of

$$Y^i((\hat{z}^i, \hat{z}^{-i})) = \frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i.$$

Thus, the optimization problem faced by trader i is equivalent to maximizing the sum of (i) the quantity $-D\Phi_{(a,b,c)}(D; Z - z^i) + DV_z(z^i, Z)$ and (ii) the product of λ with

$$\begin{aligned}\mathcal{E}(\phi, Z, z^i, \hat{z}^i) \equiv (\alpha_1 + \alpha_5\bar{Z}) \left(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i\right) + \alpha_3 \left(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i\right)^2 \\ + 2\alpha_3 z^i \left(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i\right) + \kappa_0(-n\delta(\phi) + Z - z^i + \hat{z}^i)^2 + \phi(\hat{z}^i - \delta(\phi)) + \frac{\phi^2}{4\kappa_0 n^2},\end{aligned}$$

evaluated at $\phi = \Phi_{(a,b,c)}(D; Z - z^i)$.

The first order condition for optimality of $\hat{z}^i$ is

$$\begin{aligned}\frac{\partial\mathcal{E}(\phi, Z, z^i, \hat{z}^i)}{\partial\hat{z}^i} = -\frac{n-1}{n}(\alpha_1 + \alpha_5\bar{Z}) + \frac{2(n-1)^2}{n^2}\alpha_3\hat{z}^i - 2\frac{n-1}{n}\alpha_3\frac{Z - z^i}{n} \\ - \frac{n-1}{n}2\alpha_3 z^i + 2\kappa_0(-n\delta(\phi) + \hat{z}^i + Z - z^i) + \phi = 0.\end{aligned}$$

The second-order condition is satisfied if $\alpha_3 < 0$ and $\kappa_0 < 0$. For the candidate equilibrium strategy $\hat{z}^i = z^i$, we have

$$\frac{\partial\mathcal{E}(\phi, Z, z^i, \hat{z}^i)}{\partial\hat{z}^i} = -\frac{n-1}{n}(\alpha_1 + \alpha_5\bar{Z}) + \frac{2(n-1)\alpha_3}{n}(-\bar{Z}) + 2\kappa_0(-n\delta(\phi) + Z) + \phi.$$

Plugging in

$$Z = n\frac{-b\phi - a}{c},$$

which must hold in a symmetric equilibrium, and writing $\delta(\phi) = -\hat{a} - \hat{b}\phi$, we have

$$\begin{aligned}\frac{\partial\mathcal{E}(\phi, Z, z^i, \hat{z}^i)}{\partial\hat{z}^i} = -\frac{n-1}{n}\left(\alpha_1 + \alpha_5\frac{-b\phi - a}{c}\right) + \frac{2(n-1)\alpha_3}{n}\frac{b\phi + a}{c} \\ + 2\kappa_0\left(n\hat{a} + n\hat{b}\phi + n\frac{-b\phi - a}{c}\right) + \phi.\end{aligned}$$

The candidate equilibrium strategy $\hat{z}^i$ is therefore optimal provided that

$$0 = -\frac{n-1}{n}\left(\alpha_1 - \frac{\alpha_5 a}{c}\right) + \frac{2(n-1)\alpha_3 a}{nc} + 2\kappa_0 n\hat{a} - \frac{2na\kappa_0}{c}$$
$$0 = \frac{n-1}{n}\left(\frac{\alpha_5 b}{c}\right) + \frac{2(n-1)\alpha_3 b}{nc} + 2\kappa_0 n\left(\hat{b} - \frac{b}{c}\right) + 1,$$

or equivalently,

$$\hat{a} = \frac{a}{c} - \frac{1}{2n\kappa_0}\left(-\frac{n-1}{n}\left(\alpha_1 - \frac{\alpha_5 a}{c}\right) + \frac{2(n-1)\alpha_3 a}{nc}\right)$$
$$\hat{b} = \frac{b}{c} - \frac{1}{2n\kappa_0}\left(\frac{n-1}{n}\left(\frac{\alpha_5 b}{c}\right) + \frac{2(n-1)\alpha_3 b}{nc} + 1\right).$$

These equations imply that

$$\begin{aligned}\nu \equiv\ & n\hat{a} + n\hat{b}\frac{a + c\bar{Z}}{-b} + Z \\ =& -\frac{1}{2\kappa_0}\left(-\frac{n-1}{n}\left(\alpha_1 - \frac{\alpha_5 a}{c}\right) + \frac{2(n-1)\alpha_3 a}{nc}\right) \\ & - \frac{1}{2\kappa_0}\left(\frac{a + c\bar{Z}}{-b}\right)\left(\frac{n-1}{n}\left(\frac{\alpha_5 b}{c}\right) + \frac{2(n-1)\alpha_3 b}{nc} + 1\right).\end{aligned}$$

Evaluating this expression for ν at $\phi = -(a + c\bar{Z})/b$, we have

$$\nu = \frac{-1}{2\kappa_0}\left(\phi - \frac{n-1}{n}\alpha_1 + \frac{n-1}{n}\alpha_5\frac{a + b\phi}{c} + \frac{2(n-1)\alpha_3}{n}\frac{a + b\phi}{c}\right). \tag{83}$$

Consider the ex-post equilibrium IR condition that the sum of the cash transfer and the change in utility, $V(\bar{Z}, Z) - V(z^i, Z)$, must be weakly positive. This must hold even when all traders have inventory $\bar{Z}$ when entering the mechanism. In particular, the sum of the transfers must be weakly positive in this case, but it is always weakly negative by budget balance, so the transfers must sum to 0. In general, the sum of the transfers is

$$-n\left(\sqrt{-\kappa_0}\left(-n\delta(\phi) + \sum_{j=1}^{n}\hat{z}^j\right) - \frac{\phi}{2n\sqrt{-\kappa_0}}\right)^2.$$

So, if the transfers are to sum to 0, it must be that

$$\sqrt{-\kappa_0}\left(-n\delta(\phi) + \sum_{j=1}^{n}\hat{z}^j\right) - \frac{\phi}{2n\sqrt{-\kappa_0}} = 0$$

and

$$|\kappa_0|\left(-n\delta(\phi)+\sum_{j=1}^{n}\hat{z}^j\right)-\frac{\phi}{2n}=-\kappa_0\left(-n\delta(\phi)+\sum_{j=1}^{n}\hat{z}^j\right)-\frac{\phi}{2n}=0. \tag{84}$$

Recall from equation (83) that at the equilibrium strategies and the choice for $\delta(\phi)$ that is consistent with IC, we have

$$-n\delta(\phi)+\sum_{j=1}^{n}\hat{z}^j=\frac{-1}{2\kappa_0}\left(\phi-\frac{n-1}{n}\alpha_1+\frac{n-1}{n}\alpha_5\frac{a+b\phi}{c}+\frac{2(n-1)\alpha_3}{n}\frac{a+b\phi}{c}\right).$$

Thus for IR to hold, combining this with (84), it must be that

$$\begin{aligned}&\frac{1}{2}\left(\frac{n-1}{n}\phi-\frac{n-1}{n}\alpha_1+\frac{n-1}{n}\alpha_5\frac{a+b\phi}{c}+\frac{2(n-1)\alpha_3}{n}\frac{a+b\phi}{c}\right)\\&\quad=\frac{1}{2}\left((\frac{n-1}{n})\phi-\frac{n-1}{n}\alpha_1-\frac{n-1}{n}\alpha_5\bar{Z}-\frac{2(n-1)\alpha_3}{n}\bar{Z}\right)\\&\quad=0.\end{aligned}$$

Put differently, for the equilibrium strategies to satisfy IR, we need the condition

$$\phi=\alpha_1+(\alpha_5+2\alpha_3)\bar{Z}. \tag{85}$$

We conjecture and later verify that (85) holds in equilibrium. Given this, we see that, in equilibrium,

$$-n\delta(\phi)+\sum_{j=1}^{n}\hat{z}^j=\frac{-\phi}{2\kappa_0 n}.$$

Likewise, we see that

$$\begin{aligned}-\delta(\phi)+\hat{z}^i&=\hat{a}+\hat{b}\frac{a+c\bar{Z}}{-b}+z^i\\&=z^i-\bar{Z}-\frac{1}{2\kappa_0 n}\left(\phi-\frac{n-1}{n}\alpha_1+\frac{n-1}{n}\alpha_5\frac{a+b\phi}{c}+\frac{2(n-1)\alpha_3}{n}\frac{a+b\phi}{c}\right)\\&=z^i-\bar{Z}-\frac{\phi}{2\kappa_0 n^2}.\end{aligned}$$

Now, if we plug $\delta(\phi)=-\hat{a}-\hat{b}\phi$ into the definition of $\mathcal{E}(\phi,Z,z^i,\hat{z}^i)$, we arrive at

$$\begin{aligned}\mathcal{E}(\phi,Z,z^i,\hat{z}^i)=(\alpha_1+\alpha_5\bar{Z})&\left(\frac{Z-z^i}{n}-\frac{n-1}{n}\hat{z}^i\right)\\&+\alpha_3(\frac{Z-z^i}{n}-\frac{n-1}{n}\hat{z}^i)^2+2\alpha_3 z^i\left(\frac{Z-z^i}{n}-\frac{n-1}{n}\hat{z}^i\right)\\&+\kappa_0(n(\hat{a}+\hat{b}\phi)+Z-z^i+\hat{z}^i)^2+\phi(\hat{z}^i+(\hat{a}+\hat{b}\phi))+\frac{\phi^2}{4\kappa_0 n^2}.\end{aligned}$$

The partial derivative of $\mathcal{E}(\phi, Z, z^i, \hat{z}^i)$ with respect to ϕ is then

$$\mathcal{E}_\phi(\phi, Z, z^i, \hat{z}^i) = 2\kappa_0 n\hat{b}(n(\hat{a}+\hat{b}\phi) + Z - z^i + \hat{z}^i) + (\hat{z}^i + (\hat{a} + 2\hat{b}\phi)) + \frac{\phi}{2\kappa_0 n^2}.$$

Plugging in the candidate $\hat{z}^i = z^i$ and the fact from above that $\hat{a} + \hat{b}\phi = -\bar{Z} - \phi/(2\kappa_0 n^2)$, we have

$$\mathcal{E}_\phi(\phi, Z, z^i, \hat{z}^i) = 2\kappa_0 n\hat{b}\frac{-\phi}{2\kappa_0 n} + \hat{b}\phi + \left(z^i - \bar{Z} - \frac{\phi}{2\kappa_0 n^2}\right) + \frac{\phi}{2\kappa_0 n^2} = z^i - \bar{Z}.$$

Finally, using the equilibrium reports and the choice of δ consistent with IC, the equilibrium transfers are

$$\begin{aligned}
\kappa_0\left(-n\delta(\phi) + \sum_{j=1}^n \hat{z}^j\right)^2 + \phi(\hat{z}^i - \delta(\phi)) + \frac{\phi^2}{4\kappa_0 n^2} &= \frac{\phi^2}{4\kappa_0 n^2} + \phi\left(z^i - \bar{Z} - \frac{\phi}{2\kappa_0 n^2}\right) + \frac{\phi^2}{4\kappa_0 n^2} \\
&= \phi(z^i - \bar{Z}) \\
&= \frac{a + c\bar{Z}}{-b}(z^i - \bar{Z}),
\end{aligned}$$

which implies that

$$\begin{aligned}
R_0 &= 0 \\
R_1 &= \frac{a}{nb} \\
R_2 &= \frac{c}{n^2 b} \\
R_3 &= \frac{c}{-nb} \\
R_4 &= \frac{a}{-b}.
\end{aligned}$$

D.5 Solving the HJB

The optimization problem to solve is

$$\sup_{D,\hat{z}^i} -D\Phi_{(a,b,c)}(D; Z - z^i) + DV_z(z^i, Z) + \lambda\mathcal{E}(\Phi_{(a,b,c)}(D; Z - z^i), Z, z^i, \hat{z}^i).$$

Taking a total derivative with respect to $D, \hat{z}^i$, we need

$$-\Phi_{(a,b,c)}(D; Z-z^i) - D\Phi'_{(a,b,c)}(D; Z-z^i) + V_z(z^i, Z) + \lambda\Phi'_{(a,b,c)}(D; Z-z^i)\mathcal{E}_\phi(\Phi_{(a,b,c)}(D; Z-z^i), Z, z^i, \hat{z}^i) = 0$$

and

$$\mathcal{E}_{\hat{z}^i}(\Phi_{(a,b,c)}(D; Z - z^i), Z, z^i, \hat{z}^i) = 0.$$

These equalities must hold at $D = a + b\phi + cz^i$, implying that

$$\Phi_{(a,b,c)}(D; Z - z^i) = \frac{a + c\bar{Z}}{-b},$$

and at $\hat{z}^i = z^i$. We recall that

$$\Phi'_{(a,b,c)}(D; Z - z^i) = \frac{-1}{b(n-1)}.$$

From the above, the second equation is satisfied at $\phi = -(a + c\bar{Z})/b$ and at the conjectured $\hat{z}^i$ as long as

$$0 = -\frac{n-1}{n}\left(\alpha_1 - \frac{\alpha_5 a}{c}\right) + \frac{2(n-1)\alpha_3 a}{nc} + 2\kappa_0 n\hat{a} - \frac{2na\kappa_0}{c} \tag{86}$$

$$0 = \frac{n-1}{n}\left(\frac{\alpha_5 b}{c}\right) + \frac{2(n-1)\alpha_3 b}{nc} + 2\kappa_0 n\left(\hat{b} - \frac{b}{c}\right) + 1, \tag{87}$$

where we have written $\delta(\phi)$ as $\delta(\phi) = -\hat{a} - \hat{b}\phi$. For the first order conditon on D, we need

$$-\phi + \frac{1}{b(n-1)}(a + b\phi + cz^i) + (\alpha_1 + 2\alpha_3 z^i + \alpha_5\bar{Z}) - \frac{\lambda}{b(n-1)}\mathcal{E}_\phi(\phi, Z, z^i, \hat{z}^i) = 0.$$

We showed that, at equilibrium, $\mathcal{E}_\phi = z^i - \bar{Z}$. Plugging this in, and using $\bar{Z} = (-b\phi - a)/c$, we see that

$$-\phi + \frac{1}{b(n-1)}(a + b\phi + cz^i) + \left(\alpha_1 + 2\alpha_3 z^i + \alpha_5\frac{-b\phi - a}{c}\right) - \frac{\lambda}{b(n-1)}\left(z^i - \frac{-b\phi - a}{c}\right) = 0.$$

Gathering terms,

$$\begin{aligned}
0 &= -1 + \frac{1}{(n-1)} - \alpha_5\frac{b}{c} - \frac{\lambda}{c(n-1)} \\
0 &= \frac{1}{b(n-1)}c + 2\alpha_3 - \frac{\lambda}{b(n-1)} \\
0 &= \frac{1}{b(n-1)}a + \left(\alpha_1 + \alpha_5\frac{-a}{c}\right) - \frac{\lambda}{b(n-1)}\frac{a}{c}.
\end{aligned}$$

Rearranging,

$$0 = -(n-2)c - \alpha_5(n-1)b - \lambda \tag{88}$$

$$c = -2\alpha_3 b(n-1) + \lambda, \tag{89}$$

while from the derivation of the linear-quadratic value function, we have

$$\alpha_3 = \frac{-\gamma}{r+\lambda-2c}$$
$$\alpha_5 = \frac{1}{r+\lambda-c}\left(\frac{c^2}{b} - 2\alpha_3 c + n\lambda R_3\right),$$

where R_3 is the coefficient on Zz in the transfer. From the last section, in equilibrium we have $R_3 = c/(-nb)$ and thus the relevant system is

$$\alpha_3 = \frac{-\gamma}{r+\lambda-2c}$$
$$\alpha_5 = \frac{1}{r+\lambda-c}\left(\frac{c^2}{b} - 2\alpha_3 c - \frac{\lambda c}{b}\right).$$

Multiplying both sides of the α_5 equation by $b(n-1)$, we have

$$\alpha_5 b(n-1) = \frac{1}{r+\lambda-c}(c^2(n-1) - 2\alpha_3 b(n-1)c - \lambda c(n-1)),$$

and plugging in the above, we have

$$\alpha_5 b(n-1) = \frac{nc}{r+\lambda-c}(c-\lambda),$$

so

$$0 = -(n-2)c - \left(\frac{nc}{r+\lambda-c}(c-\lambda)\right) - \lambda$$
$$0 = -(n-2)c(r+\lambda-c) - nc(c-\lambda) - \lambda(r+\lambda-c)$$
$$0 = -2c^2 + c(-(n-2)(r+\lambda) + n\lambda + \lambda) - \lambda(r+\lambda)$$
$$0 = -2c^2 + c(-(n-2)r + 3\lambda) - \lambda(r+\lambda)$$
$$c = \frac{(-(n-2)r+3\lambda) \pm \sqrt{(-(n-2)r+3\lambda)^2 - 8\lambda(r+\lambda)}}{4}.$$

It is clear that either both of the roots or neither of the roots are real. By the Déscartes rule of signs, if both are real, they are either both positive, or neither are positive. In particular, assuming that $(-(n-2)r+3\lambda)^2 - 8\lambda(r+\lambda) > 0$ so that both roots exist, if we can show one is negative then they both are negative. If $-(n-2)r+3\lambda < 0$, then the smaller root must be negative and we are done. If $-(n-2)r+3\lambda \geq 0$, then the larger root is positive so both roots are positive. Thus we see we need that $-(n-2)r+3\lambda < 0$ and $(-(n-2)r+3\lambda)^2 - 8\lambda(r+\lambda) \geq 0$, which can be concisely written as

$$-(n-2)r + 3\lambda \leq -\sqrt{8\lambda(r+\lambda)}.$$

Define

$$F(c,\lambda) = -2c^2 + c(-(n-2)r + 3\lambda) - \lambda(r+\lambda).$$

For each fixed λ, an equilibrium is determined by any $c < 0$ satisfying $F(c,\lambda) = 0$. The condition that $c < 0$ is equivalent to $b < 0$, which ensures that the second order condition above holds.

We have that $F_{cc} = -4 < 0$ and $\lim_{c\to-\infty} F = \lim_{c\to\infty} F = -\infty$. Thus, as c increases from negative infinity to infinity, F_c crosses from positive to negative exactly once, at

$$c_0 = \frac{-(n-2)r + 3\lambda}{4}.$$

Since there are two roots, we see the derivative F_c must be positive at the smaller root $\underline{c}(\lambda)$ and negative at the larger root $\overline{c}(\lambda)$, so $\underline{c}(\lambda) < c_0 < \overline{c}(\lambda)$. Fix a $\lambda \in (0, \bar{\lambda})$ and consider small, disjoint neighborhoods around $(\lambda, \overline{c}(\lambda))$ and $(\lambda, \underline{c}(\lambda))$. Applying the implicit function theorem to each of these functions,

$$\frac{\partial c}{\partial \lambda} = -\frac{F_\lambda}{F_c} = -\frac{-r - 2\lambda + 3c}{F_c}.$$

Since $c < 0$ in either equilibrium, the numerator is always negative. We just showed that F_c is positive at the smaller root and thus that $\frac{\partial \underline{c}(\lambda)}{\partial \lambda} > 0$, so that c increases monotonically in λ.

Now, recall that

$$(r + \lambda - 2c)\alpha_3 = -\gamma,$$

which, combined with equation (89), implies that

$$\begin{aligned} c(r+\lambda-2c) &= -2\alpha_3 b(n-1)(r+\lambda-2c) + \lambda(r+\lambda-2c) \\ c(r+\lambda-2c) &= 2\gamma b(n-1) + \lambda(r+\lambda-2c). \end{aligned}$$

Using the above quadratic equation for c, this can be rewritten

$$\begin{aligned} c(r+\lambda) - (c(-(n-2)r+3\lambda) - \lambda(r+\lambda)) &= 2\gamma b(n-1) + \lambda(r+\lambda-2c) \\ c(r+\lambda) - (c(-(n-2)r+3\lambda)) &= 2\gamma b(n-1) - 2\lambda c \\ cr(n-1) &= 2\gamma b(n-1) \\ c &= \frac{2\gamma}{r} b, \end{aligned}$$

which implies that

$$b = \frac{r^2}{8\gamma}\left(-(n-2) + \frac{3\lambda}{r} \pm \sqrt{\left(-(n-2) + \frac{3\lambda}{r}\right)^2 - \frac{8\lambda(r+\lambda)}{r^2}}\right).$$

We note that

$$\begin{aligned}
\left[\frac{3\lambda}{r}-(n-2)\right]^2-\frac{8\lambda(r+\lambda)}{r^2} &= \frac{\lambda^2}{r^2}-\frac{6\lambda(n-2)}{r}+(n-2)^2-\frac{8\lambda}{r} \\
&= \left(\frac{\lambda}{r}-(n-2)\right)^2-\frac{4\lambda n}{r}.
\end{aligned}$$

Thus, we have shown that

$$b=\frac{-r^2}{8\gamma}\left((n-2)-\frac{3\lambda}{r}\pm\sqrt{\left(\frac{\lambda}{r}-(n-2)\right)^2-\frac{4\lambda n}{r}}\right).$$

Further, since $c<0$ and $c=2\gamma b/r$, we have $b<0$, and since c increases monotonically in λ so does b. Using the relation that $c=2\gamma b/r$ and equation (89), we have

$$\alpha_3=\frac{c-\lambda}{-2b(n-1)}=-\frac{\gamma}{r(n-1)}+\frac{\lambda}{2b(n-1)}.$$

Using (88), we now have

$$\begin{aligned}
0 &= -(n-2)c-\alpha_5(n-1)b-\lambda \\
\alpha_5 &= \frac{-(n-2)c-\lambda}{b(n-1)} \\
&= -\frac{n-2}{n-1}\frac{2\gamma}{r}-\frac{\lambda}{b(n-1)} \\
&= \frac{-2\gamma}{r}-2\alpha_3.
\end{aligned}$$

Recall that

$$\alpha_1=\frac{1}{r+\lambda-c}\left(rv+\frac{ac}{b}+\lambda R_4\right),$$

where, based on the transfers, $R_4=-a/b$, so

$$\alpha_1=\frac{1}{r+\lambda-c}\left(rv+\frac{ac}{b}-\frac{a\lambda}{b}\right).$$

From the first order condition for auction demand,

$$0=\frac{1}{b(n-1)}a+\left(\alpha_1+\alpha_5\frac{-a}{c}\right)-\frac{\lambda}{b(n-1)}\frac{a}{c}.$$

Plugging in

$$\alpha_5=\frac{-2\gamma}{r}-2\left(\frac{c-\lambda}{-2b(n-1)}\right),$$

we have

$$0 = \alpha_1 + \frac{2\gamma}{r}\frac{a}{c},$$

implying that

$$\alpha_1 = -\frac{a}{b}.$$

Now, plugging this into the above, we have

$$\alpha_1 = \frac{1}{r + \lambda - c}(rv + -c\alpha_1 + \lambda\alpha_1),$$

from which it is clear that $\alpha_1 = v$ and $a = -bv$. Returning to the coefficients $\hat{a}, \hat{b}$ defining $\delta(\phi)$, since

$$\frac{a}{c} = -v\frac{r}{2\gamma}$$

and

$$\frac{b}{c} = \frac{r}{2\gamma},$$

we have

$$\begin{aligned}
\hat{a} &= \frac{a}{c} - \frac{1}{2n\kappa_0}\left(-\frac{n-1}{n}\left(\alpha_1 - \frac{\alpha_5 a}{c}\right) + \frac{2(n-1)\alpha_3 a}{nc}\right) \\
&= \frac{-vr}{2\gamma} - \frac{1}{2n\kappa_0}\left(-\frac{n-1}{n}\left(v - v\left(\frac{2\gamma}{r}\right)\left(\frac{r}{2\gamma}\right)\right)\right) \\
&= \frac{-vr}{2\gamma}, \\
\hat{b} &= \frac{b}{c} - \frac{1}{2n\kappa_0}\left(\frac{n-1}{n}\left(\frac{\alpha_5 b}{c}\right) + \frac{2(n-1)\alpha_3 b}{nc} + 1\right) \\
&= \frac{r}{2\gamma} - \frac{1}{2n^2\kappa_0}.
\end{aligned}$$

Returning to the system of value function coefficients, it remains to calculate

$$\begin{aligned}
\alpha_4 &= \frac{1}{r}\left(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2\right) \\
\alpha_2 &= \frac{1}{r}\left(\frac{ca}{-b} + (\lambda - c)\alpha_1 + \lambda n R_1\right) \\
\alpha_0^i &= \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0\right).
\end{aligned}$$

Plugging in the equilibrium formulas for R_2, R_1, and R_0, we have

$$\alpha_4 = \frac{1}{r}\left(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \frac{c\lambda}{b}\right)$$
$$\alpha_2 = \frac{1}{r}\left(\frac{ca}{-b} + (\lambda - c)v + \frac{a\lambda}{b}\right)$$
$$\alpha_0^i = \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}\right).$$

Using the definitions of a, b, c, we thus have

$$\alpha_4 = \frac{1}{r}\left(-\frac{2\gamma}{r}c + (\lambda - c)\left(\frac{-2\gamma}{r} - 2\alpha_3\right) + \lambda\alpha_3 + \frac{c\lambda}{b}\right)$$
$$\alpha_2 = \frac{1}{r}(cv + (\lambda - c)v + -v\lambda),$$

implying that $\alpha_2 = 0$ and that

$$\alpha_4 = \frac{1}{r}\left(2c\alpha_3 + \lambda\left(\frac{-2\gamma}{r} - 2\alpha_3\right) + \lambda\alpha_3 + \frac{2\gamma\lambda}{r}\right)$$
$$= \frac{1}{r}(2c - \lambda)\alpha_3 = \frac{\gamma}{r} + \alpha_3.$$

Finally, this implies that

$$\alpha_0^i = \frac{1}{r}\left(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} + \alpha_3\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2\right) + \alpha_5\frac{\rho^i}{n}\right)$$
$$= \frac{1}{r}\left(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} + \alpha_3\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}\right) - \frac{2\gamma}{r}\frac{\rho^i}{n}\right)$$
$$= \frac{1}{r}\left(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} + \left(-\frac{\gamma}{r(n-1)} + \frac{\lambda}{2b(n-1)}\right)\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}\right) - \frac{2\gamma}{r}\frac{\rho^i}{n}\right).$$

Note that

$$\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}$$

is the variance of $Z_1/n - H_1^i$ conditional on Z_0, and is thus positive, so α_0^i declines in λ because $b < 0$ and because b increases with λ.

Finally, we must verify that in equilibrium,

$$\phi = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}. \tag{90}$$

We see from the definitions of a, b, c that

$$\phi = \frac{a + c\bar{Z}}{-b} = v - \frac{2\gamma}{r}\bar{Z}$$

while from the definition of α_5, α_3 we have

$$2\alpha_3 + \alpha_5 = \frac{-2\gamma}{r},$$

so (90) holds with probability 1.

D.6 Finishing the verification

In this section, we show that at the $V(z, Z)$ and strategies which solve the HJB equation, using any alternate admissible strategy leads to an inferior payoff for each trader. We fix some admissible demand process D^i, and report process $\tilde{z}$, by which the inventory of trader i at time t is

$$z_t^{(D,\tilde{z})} = z_0^i + \int_0^t D_s^i\, ds + H_t^i + \int_0^t Y^i((\tilde{z}_s, \hat{z}_s^{-i}))\, dN_s. \tag{91}$$

Following the steps of the derivation of the value function, we can show that under the laws of motion implied by $D^i, \tilde{z}$,

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left(\int_{0+}^{\mathcal{T}} \tilde{\zeta}_s\, ds\right),$$

where

$$\begin{aligned}\tilde{\zeta}_s = {} & D_s^i(\alpha_1 + \alpha_5\bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}}) + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_3\sigma_i^2 + \alpha_5\frac{\rho^i}{n} \\ & + \lambda Y^i((\tilde{z}_s, \hat{z}_s^{-i}))(\alpha_1 + \alpha_5\bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}} + \alpha_3 Y^i((\tilde{z}_s, \hat{z}_s^{-i}))) + r(v z_s^{D,\tilde{z}} - V(z_s^{D,\tilde{z}}, Z_s)).\end{aligned}$$

Since α_0 through α_5 satisfy the HJB equation, we have

$$\begin{aligned}\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] \le {} & \mathbb{E}\left[\int_{0+}^{\mathcal{T}} D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) + \gamma(z_s^{D,\tilde{z}})^2\, ds\right] \\ & - \mathbb{E}\left[\int_0^{\mathcal{T}} \hat{T}^i((\hat{z}_s^i, \hat{z}_s^{-i}); \Phi_{(a,b,c)}(D_{s-}^i; Z_{s-} - z_{s-}^{D,\tilde{z}}))\, dN_s\right].\end{aligned}$$

Rearranging,

$$\begin{aligned}V(z_0^i, Z_0) \ge {} & \mathbb{E}\left[\pi z_{\mathcal{T}}^{D,\tilde{z}} + \int_{0+}^{\mathcal{T}} -D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) - \gamma(z_s^{D,\tilde{z}})^2\, ds\right] \\ & + \mathbb{E}\left[\int_0^{\mathcal{T}} \hat{T}^i((\hat{z}_s^i, \hat{z}_s^{-i}); \Phi_{(a,b,c)}(D_{s-}^i; Z_{s-} - z_{s-}^{D,\tilde{z}}))\, dN_s\right].\end{aligned}$$

Since this relationship holds with equality for the conjectured affine strategy, this affine strategy is optimal.

E Discrete-Time Results

In this appendix, we analyze discrete time versions of the models of Sections 3, 4, and 5. The focus is the existence of a subgame perfect equilibrium in each complete information game, which corresponds to a Perfect Bayes equilibrium of each incomplete information game. We also show convergence results for the models of sections 3 and 4. All the results are presented informally, with focus on the calculation of the equilibrium, but these arguments can all be made fully rigorous.

The primitive setting, other than mechanisms, is identical to Duffie and Zhu (2017). Specifically, $n > 2$ traders trade in each period $k \in \{0, 1, 2, ...\}$, where trading periods are separated by clock time Δ so that the k-th auction occurs at time $k\Delta$.

In each period k, each trader i submits an auction order $x_{ik}(p_k)$ for how many units of asset they wish to purchase if the auction price is p_k. We focus on affine equilibria in which each trader chooses

$$x_{ik}(p_k) = a + bp_k + cz_{ik},$$

where z_{ik} is the inventory of trader i when entering period k, for some constants a, c and $b \neq 0$. If $n-1$ traders use such a strategy with the same constants a, b, c, then there is a unique market clearing price $\Phi_{(a,b,c)}(D, Z - z)$ for any demand D submitted by trader i, which is given by

$$\Phi_{(a,b,c)}(D, Z - z) = \frac{(n-1)a + c(Z_k - z_{ik}) + D}{-b(n-1)}.$$

Each trader also submits a contingent mechanism report $\hat{z}_{ik}(p_k)$. With probability q, a mechanism occurs, and in that event trader i receives a net reallocation

$$Y^i(\hat{z}) = \frac{\sum_{j=1}^{n} \hat{z}_{jk}}{n} - \hat{z}_{ik}$$

and a cash transfer that will be described shortly, and that might depend upon p_k. With probability $1 - q$, a double auction occurs, and each trader receives $x_{ik}(p_k)$ units of the asset at a cost $p_k x_{ik}(p_k)$. If trader i ends period k with inventory z^+_{ik}, then in between periods k and $k+1$, they receive flow expected utility

$$-\frac{\gamma}{r}(1 - e^{-r\Delta})(z^+_{ik})^2 + v(1 - e^{-r\Delta})(z^+_{ik}),$$

which can be motivated as in Duffie and Zhu (2017). Let $\mathbf{1}_{M^k}$ equal 1 if and only if a mechanism occurs in period k, and let $\mathbf{1}^c_{M^k} = 1 - \mathbf{1}_{M^k}$. Then, in any equilibrium in which mechanisms implement efficient allocations, the equilibrium inventory evolves as

$$z_{i,k+1} = w_{i,k+1} + \mathbf{1}_{M^k}\bar{Z}_k + \mathbf{1}^c_{M^k}\left((1+c)z_{i,k} - c\bar{Z}_k\right),$$

where $\{w_{i,k+1}\}$ is a sequence of $i.i.d$ zero-mean finite-variance random variables.

E.1 Observable Z_t

Suppose the aggregate inventory Z_k is observable, and that transfers are given by

$$T^i_\kappa(\hat{z}, Z) = \kappa_0 \left(n\kappa_2(Z_k) + \sum_{j=1}^{n} \hat{z}_{jk} \right)^2 + \kappa_1(Z_k)(\hat{z}_{ik} + \kappa_2(Z_k)) + \frac{\kappa_1(Z_k)^2}{4\kappa_0 n^2}.$$

Just as in the continuous-time proof, at the equilibrium reports for affine $\kappa_1(\,\cdot\,), \kappa_2(\,\cdot\,)$, this must take the form

$$R_0 + R_1 Z_k + R_2 Z_k^2 + R_3 Z_k z_{ik} + R_4 z_{ik}.$$

We solve for a subgame perfect equilibrium in which trader i submits the demand

$$x_{ik}(p_k) = a + bp_k + cz_{ik},$$

and the report

$$\hat{z}_{ik}(p_k) = z_{ik}.$$

In such an equilibrium, the continuation value $V(z, Z)$ must be linear quadratic. Specifically, the continuation value is

$$V(z, Z) = \mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} \tilde{\pi}_k \right],$$

where

$$\begin{aligned}\tilde{\pi}_k = q \Big(&R_0 + R_1 Z_k + R_2 Z_k^2 + R_3 Z_k z_{ik} + R_4 z_{ik} - \frac{\gamma}{r}(1 - e^{-r\Delta})(\bar{Z}_k)^2 + v(1 - e^{-r\Delta})(\bar{Z}_k) \Big) \\ &+ (1 - q) \Big(-x_{ik}(p_k) p_k - \frac{\gamma}{r}(1 - e^{-r\Delta})(x_{ik}(p_k) + z_{ik})^2 + v(1 - e^{-r\Delta})(x_{ik}(p_k) + z_{ik}) \Big).\end{aligned}$$

We are given that $z_{i0} = z, Z_0 = Z$, $\sum_{i=1}^n x_{ik}(p_k) = 0$, and

$$z_{i,k+1} = w_{i,k+1} + \mathbf{1}_{M^k} \left(z_{ik} + \frac{\sum_{j=1}^n \hat{z}_{jk}}{n} - \hat{z}_{ik} \right) + \mathbf{1}^c_{M^k} \left(z_{ik} + x_{ik}(p_k) \right).$$

Fix the conjectured equilibrium a, b, c with truth-telling ($\hat{z}_{ik} = z_{ik}$), so that

$$z_{i,k+1} = w_{i,k+1} + \mathbf{1}_{M^k} \bar{Z}_k + \mathbf{1}^c_{M^k} \left((1 + c) z_{i,k} - c\bar{Z}_k \right). \tag{92}$$

The expression for $V(z, Z)$ can be decomposed into a linear combination of discounted sums of moments of z_{ik}, Z_k. We calculate these now. Straightforward calculation shows that

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} Z_k\right] = \frac{Z_0}{1-e^{-r\Delta}} = S_0 Z_0$$
$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} Z_k^2\right] = \frac{Z_0^2}{1-e^{-r\Delta}} + \frac{\sigma_Z^2 e^{-r\Delta}}{1-e^{-r\Delta}} = S_0 Z_0^2 + S_1,$$

where $\sigma_Z^2 \equiv \mathrm{var}(\sum_{i=1}^n w_{i,k+1})$. Subtracting $\bar{Z}_{i,k+1}$ from both sides of equation (92), rearranging, and taking an expectation gives

$$\mathbb{E}[z_{i,k+1} - \bar{Z}_{k+1}] = (1-q)(1+c)\mathbb{E}[z_{i,k} - \bar{Z}_k].$$

Some calculation then shows that

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\right] = \frac{z_{i0} - \bar{Z}_0}{1 - e^{-r\Delta}(1+c)(1-q)} + \frac{\bar{Z}_0}{1-e^{-r\Delta}} = S_2(z_{i0} - \bar{Z}_0) + S_0 \bar{Z}_0,$$

provided that $|e^{-r\Delta}(1+c)(1-q)| < 1$. Subtracting $\bar{Z}_{i,k+1}$ from both sides of equation (92), then multiplying both sides by $\bar{Z}_{i,k+1}$, and taking an expectation gives

$$E[z_{i,k+1} Z_{k+1} - \bar{Z}_{k+1}^2] = \left(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2}\right) + (1-q)(1+c)E[z_{i,k}\bar{Z}_k - \bar{Z}_k^2],$$

where $\rho^i = \mathbb{E}[w_{i,k+1}(\sum_{i=1}^n w_{i,k+1})]$.

Then we see that

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k\right] &= \mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k}\left(z_{ik}\bar{Z}_k - \bar{Z}_k^2\right)\right] + S_0\bar{Z}_0^2 + \frac{S_1}{n^2} \\
&= z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + e^{-r\Delta}\sum_{k=1}^{\infty} e^{-r\Delta(k-1)}\mathbb{E}[z_{ik}\bar{Z}_k - \bar{Z}_k^2] + S_0\bar{Z}_0^2 + \frac{S_1}{n^2} \\
&= z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + S_0\bar{Z}_0^2 + \frac{S_1}{n^2} \\
&\quad + e^{-r\Delta}\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k}\left(\left(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2}\right) + (1-q)(1+c)E[z_{i,k}\bar{Z}_k - \bar{Z}_k^2]\right)\right] \\
&= z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + \frac{e^{-r\Delta}\left(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2}\right)}{1-e^{-r\Delta}} + (1 - e^{-r\Delta}(1-q)(1+c))\left(S_0\bar{Z}_0^2 + \frac{S_1}{n^2}\right) \\
&\quad + (1-q)(1+c)e^{-r\Delta}\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k\right].
\end{aligned}$$

Rearranging delivers

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik} \bar{Z}_k\right] = S_0 \bar{Z}_0^2 + \frac{S_1}{n^2} + \frac{z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + \frac{e^{-r\Delta}(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2})}{1-e^{-r\Delta}}}{1-(1-q)(1+c)e^{-r\Delta}}$$
$$= S_2 z_{i0}\bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3.$$

Finally, squaring both sides of equation (92) and taking an expectation shows that

$$\mathbb{E}\left[\left(z_{i,k+1} - \bar{Z}_{k+1}\right)^2\right] = \left(\frac{\sigma_Z^2}{n^2} - 2\frac{\rho^i}{n} + \sigma_i^2\right) + (1-q)(1+c)^2 \mathbb{E}\left[\left(z_{i,k} - \bar{Z}_k\right)^2\right],$$

where $\sigma_i^2 = \mathbb{E}[w_{i,k+1}^2]$.

Then,

$$\sum_{k=0}^{\infty} e^{-r\Delta} \mathbb{E}[\left(z_{i,k} - \bar{Z}_k\right)^2] = \frac{\left(z_{i,0} - \bar{Z}_0\right)^2 + \frac{(\frac{\sigma_Z^2}{n^2} - 2\frac{\rho^i}{n} + \sigma_i^2)e^{-r\Delta}}{1-e^{-r\Delta}}}{1 - e^{-r\Delta}(1-q)(1+c)^2} = S_4\left(z_{i,0} - \bar{Z}_0\right)^2 + S_5,$$

provided that $|S_4^{-1}| = |1 - e^{-r\Delta}(1-q)(1+c)^2| < 1$. It follows that

$$\sum_{k=0}^{\infty} e^{-r\Delta} \mathbb{E}[z_{i,k}^2] = S_4\left(z_{i,0} - \bar{Z}_0\right)^2 + S_5 + 2\left(S_2 z_{i0}\bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3\right) - \left(S_0\bar{Z}_0^2 + \frac{S_1}{n^2}\right).$$

In summary, letting

$$S_0 = \frac{1}{1-e^{-r\Delta}}$$
$$S_1 = \frac{\sigma_Z^2 e^{-r\Delta}}{1-e^{-r\Delta}}$$
$$S_2 = \frac{1}{1-e^{-r\Delta}(1-q)(1+c)}$$
$$S_3 = S_2\frac{e^{-r\Delta}(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2})}{1-e^{-r\Delta}}$$
$$S_4 = \frac{1}{1-e^{-r\Delta}(1-q)(1+c)^2}$$
$$S_5 = S_4\frac{(\frac{\sigma_Z^2}{n^2} - 2\frac{\rho^i}{n} + \sigma_i^2)e^{-r\Delta}}{1-e^{-r\Delta}}$$

and assuming $|S_2^{-1}|, |S_4^{-1}|$ are strictly less than 1, we have

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\right] = S_2(z_{i0} - \bar{Z}_0) + S_0\bar{Z}_0$$

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k\right] = S_2 z_{i0}\bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3$$

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} \bar{Z}_k\right] = S_0\bar{Z}_0$$

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} \bar{Z}_k^2\right] = S_0\bar{Z}_0^2 + \frac{S_1}{n^2}$$

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta} z_{i,k}^2\right] = S_4\left(z_{i,0} - \bar{Z}_0\right)^2 + S_5 + 2\left(S_2 z_{i0}\bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3\right) - \left(S_0\bar{Z}_0^2 + \frac{S_1}{n^2}\right).$$

Suppose that

$$V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z}.$$

Then the utility for having inventory z, Z immediately after an auction or mechanism is

$$\begin{aligned}
V^+(z, Z) &= -\frac{\gamma}{r}(1 - e^{-r\Delta})(z)^2 + v(1 - e^{-r\Delta})z + \mathbb{E}\left[e^{-r\Delta}V(z + w_{i,k+1}, Z + \sum_i w_{i,k+1})\right] \\
&= -\frac{\gamma}{r}(1 - e^{-r\Delta})(z)^2 + v(1 - e^{-r\Delta})z \\
&\quad + e^{-r\Delta}\left(\alpha_0^i + \alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z}\right) \\
&= u(Z) + \left(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1 - e^{-r\Delta})\right)(z - \bar{Z})^2 + (v(1 - e^{-r\Delta}) + e^{-r\Delta}\alpha_1)z \\
&\quad + \left(e^{-r\Delta}\alpha_5 + 2(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1 - e^{-r\Delta}))\right)z\bar{Z}.
\end{aligned}$$

We have thus shown that the continuation value maximized in the mechanism takes the form found in Section 2, with

$$\begin{aligned}
\beta_0 &= (v(1 - e^{-r\Delta}) + e^{-r\Delta}\alpha_1) \\
\beta_1 &= e^{-r\Delta}\alpha_5 + 2\left(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1 - e^{-r\Delta})\right).
\end{aligned}$$

To meet the IR restriction, transfers in the mechanism thus must be run with $\kappa_1(Z_k) = \beta_0 + \beta_1\bar{Z}_k$. From Proposition 1, in the equilibrium of the mechanism game that we seek (with observable Z), each trader submits $\hat{z}_{ik} = z_{ik}$ as long as

$$\kappa_2(Z_k) = -\bar{Z}_k + \frac{-(\beta_0 + \beta_1\bar{Z}_k)}{2\kappa_0 n^2},$$

so that

$$n\kappa_2(Z_k) + \sum_i \hat{z}_{ik} = \frac{-(\beta_0 + \beta_1 \bar{Z}_k)}{2\kappa_0 n}.$$

Returning to the continuation value, in equilibrium at each mechanism event, trader i receives the cash transfer $\kappa_1(Z_k)(z_{ik} - \bar{Z}) = \left(\beta_0 + \beta_1 \bar{Z}_k\right)(z_{ik} - \bar{Z})$. The equilibrium price must be $p_k = -(a + c\bar{Z})/b$ and the equilibrium double-auction demand is $x_{ik} = c(z_{ik} - \bar{Z}_k)$. Thus, plugging in, the candidate equilibrium continuation value is

$$V(z, Z) = \mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} \hat{\pi}_k\right],$$

where

$$\begin{aligned}
\hat{\pi}_k = {} & q\left((\beta_0 + \beta_1 \bar{Z}_k)(z_{ik} - \bar{Z}_k) - \frac{\gamma}{r}(1 - e^{-r\Delta})(\bar{Z}_k)^2 + v(1 - e^{-r\Delta})(\bar{Z}_k)\right) \\
& + (1 - q)\left(-c(z_{ik} - \bar{Z}_k)\frac{a + c\bar{Z}_k}{-b} - \frac{\gamma}{r}(1 - e^{-r\Delta})((1 + c)z_{ik} - c\bar{Z}_k)^2\right) \\
& + (1 - q)\left(v(1 - e^{-r\Delta})((1 + c)z_{ik} - c\bar{Z}_k)\right).
\end{aligned}$$

Collecting terms,

$$\begin{aligned}
V(z, Z) = {} & \left(q\beta_0 + (1 - q)\left[\frac{ca}{b} + v(1 - e^{-r\Delta})(1 + c)\right]\right)\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\right] \\
& + \left(q\beta_1 + (1 - q)\left[\frac{c^2}{b} + 2\frac{\gamma}{r}(1 - e^{-r\Delta})(1 + c)c\right]\right)\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k\right] \\
& - \frac{\gamma}{r}(1 - e^{-r\Delta})(1 - q)(1 + c)^2\,\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}^2\right] + \epsilon(Z).
\end{aligned}$$

Plugging in the definitions found above, it follows that

$$\begin{aligned}
\alpha_1 &= S_2\left(q\beta_0 + (1 - q)\left[\frac{ca}{b} + v(1 - e^{-r\Delta})(1 + c)\right]\right) \\
\alpha_3 &= -\frac{\gamma}{r}(1 - e^{-r\Delta})(1 - q)(1 + c)^2 S_4 \\
\alpha_5 &= S_2\left(q\beta_1 + (1 - q)\left[\frac{c^2}{b} + 2\frac{\gamma}{r}(1 - e^{-r\Delta})(1 + c)c\right]\right) - \frac{\gamma}{r}(1 - e^{-r\Delta})(1 - q)(1 + c)^2(2(S_2 - S_4)).
\end{aligned}$$

Recalling the expressions for β_0, S_2, the formula for α_1 implies that

$$\begin{aligned}
\beta_0 &= v(1-e^{-r\Delta}) + e^{-r\Delta}\alpha_1 \\
&= v(1-e^{-r\Delta}) + \frac{e^{-r\Delta}}{1-e^{-r\Delta}(1-q)(1+c)}\left(q\beta_0 + (1-q)[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)]\right).
\end{aligned}$$

So, conjecturing and later verifying that $1-e^{-r\Delta}(1-q)(1+c)-qe^{-r\Delta} \neq 0$, we have

$$\beta_0 = \left(\frac{1-e^{-r\Delta}(1-q)(1+c)}{1-e^{-r\Delta}(1-q)(1+c)-qe^{-r\Delta}}\right)\tau_c,$$

where

$$\tau_c = v(1-e^{-r\Delta}) + \frac{e^{-r\Delta}(1-q)}{1-e^{-r\Delta}(1-q)(1+c)}\left[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)\right].$$

A similar calculation shows that

$$\begin{aligned}
\beta_1 &= e^{-r\Delta}S_2q\beta_1 + e^{-r\Delta}S_2\left((1-q)\left[\frac{c^2}{b} + 2\frac{\gamma}{r}(1-e^{-r\Delta})(1+c)c\right]\right) \\
&\quad - \frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2(S_2-S_4)) + 2\left(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}\left(1-e^{-r\Delta}\right)\right).
\end{aligned}$$

and thus

$$\beta_1 = \zeta_a(\tau_d + \tau_e),$$

where

$$\begin{aligned}
\zeta_a &= \frac{1-e^{-r\Delta}(1-q)(1+c)}{1-e^{-r\Delta}(1-q)(1+c)-qe^{-r\Delta}} \\
\tau_d &= e^{-r\Delta}S_2(1-q)\left[\frac{c^2}{b} + 2\frac{\gamma}{r}(1-e^{-r\Delta})(1+c)c\right] \\
\tau_e &= -\frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2(S_2-S_4)) + 2\left(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1-e^{-r\Delta})\right).
\end{aligned}$$

Putting this all together, the continuation value for trader i in a symmetric equilibrium, immediately after an auction or mechanism is run, is

$$V^+(z,Z) = u(Z) - \frac{\gamma}{r}(1-e^{-r\Delta})[(1-q)(1+c)^2S_4e^{-r\Delta}+1](z-\bar{Z})^2 + (\beta_0+\beta_1\bar{Z})(z-\bar{Z}).$$

Plugging in the definition of S_4, this simplifies slightly to

$$V^+(z,Z) = u(Z) + \frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(z-\bar{Z})^2 + (\beta_0+\beta_1\bar{Z})(z-\bar{Z}).$$

Trader i can choose any quantity x to purchase at the price

$$\Phi(x) = \frac{1}{-b(n-1)}\left((n-1)a + c(Z-z) + x\right).$$

With observable Z, the order size x is irrelevant to the payoff and continuation value in the event of a mechanism. Thus a trader with pre-trade position z maximizes

$$-x\frac{1}{-b(n-1)}\left((n-1)a + c(Z-z) + x\right) + V^{+}(z+x, Z)$$

Differentiating this expression with respect to x leaves

$$-\Phi(x) + \frac{x}{b(n-1)} + (\beta_0 + \beta_1 \bar{Z}) - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(z+x-\bar{Z}),$$

which must be 0 with $\Phi = \phi$, $\bar{Z} = (-a - b\phi)/c$, and $x = a + b\phi + cz$. The second order condition is met if and only if $b < 0$. This also implies $x = c(z - \bar{Z})$, so

$$(z + x - \bar{Z}) = (1+c)z + (1+c)\frac{a+b\phi}{c}.$$

Plugging this in and gathering coefficients on $\phi, z, 1$, we have

$$\begin{aligned}
0 &= -1 + \frac{1}{n-1} - \frac{b\beta_1}{c} - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{b}{c} \\
0 &= \frac{c}{b(n-1)} - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c) \\
0 &= \frac{a}{b(n-1)} + \left(\beta_0 - \frac{a}{c}\beta_1\right) - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{a}{c}.
\end{aligned}$$

We seek $a, b, c, \beta_1, \beta_0$ such that these three equations and the two equations defining β_0, β_1 all hold. Let ω be the larger root of

$$e^{-r\Delta}\omega^2 + (n-1)(1-e^{-r\Delta})\omega - 1 = 0,$$

so

$$\omega = \frac{-(n-1)(1-e^{-r\Delta}) + \sqrt{(n-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}}{2e^{-r\Delta}}.$$

Then, in Duffie and Zhu (2017), where $q = 0$, we can set

$$a = \frac{rv}{2\gamma}(1-\omega), \qquad b = -\frac{r}{2\gamma}(1-\omega), \qquad c = -(1-\omega),$$

and see that

$$\frac{(1+c)(1-e^{-r\Delta})}{1-e^{-r\Delta}(1+c)^2} = \frac{\frac{1-e^{-r\Delta}\omega^2}{n-1}}{1-e^{-r\Delta}\omega^2} = \frac{1}{n-1}.$$

It follows that the above system holds with $\beta_0 = v$, $\beta_1 = -2\gamma/r$. Now, let $\hat{\omega}$ be the larger root of

$$e^{-r\Delta}(1-q)\hat{\omega}^2 + (n-1)(1-e^{-r\Delta})\hat{\omega} - 1 = 0,$$

so that

$$\hat{\omega} = \frac{-(n-1)(1-e^{-r\Delta}) + \sqrt{(n-1)^2(1-e^{-r\Delta})^2 + 4(1-q)e^{-r\Delta}}}{2(1-q)e^{-r\Delta}}.$$

This implies that, letting a, b, c be as before but replacing ω with $\hat{\omega}$,

$$\frac{(1+c)(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2} = \frac{\frac{1-e^{-r\Delta}(1-q)\hat{\omega}^2}{n-1}}{1-e^{-r\Delta}(1-q)\hat{\omega}^2} = \frac{1}{n-1}.$$

It is straightforward to show that a, b, c defined with $\hat{\omega}$, and $\beta_0 = v, \beta_1 = -2\gamma/r$ once again solve the above system. We now must verify that they satisfy the definitions of β_0, β_1. Note that under the conjectured values,

$$\begin{aligned} q\beta_0 + (1-q)\left[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)\right] &= v\left(q + (1-q)[-(1+c) + 1 + (1-e^{-r\Delta})(1+c)]\right) \\ &= v\left(1 - e^{-r\Delta}(1+c)(1-q)\right), \end{aligned}$$

from which it can be seen that $\beta_0 = v$ is consistent with the earlier system. We noted above that

$$\left(-\frac{\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2 S_4 e^{-r\Delta} - \frac{\gamma}{r}(1-e^{-r\Delta})\right) = \frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}.$$

Plugging this into the definition of β_1, we have

$$\beta_1 = \zeta_a(\tau_d + \tau_f),$$

where ζ_a and τ_d are defined above and

$$\tau_f = -\frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2(S_2 - S_4)) + 2\frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}.$$

Rearranging, we see that

$$\begin{aligned} &\frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2S_4) + 2\frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2} \\ &\qquad = 2(1-e^{-r\Delta})\frac{\gamma}{r}[e^{-r\Delta}(1-q)(1+c)^2 S_4 - S_4], \end{aligned}$$

where $e^{-r\Delta}(1-q)(1+c)^2 S_4 - S_4 = -1$.

Pulling together terms involving S_2 and noting $(1+c)c - (1+c)^2 = -(1+c)$, we have

$$\beta_1 = \zeta_a\left[e^{-r\Delta}S_2\left((1-q)\left[\frac{c^2}{b} - 2\frac{\gamma}{r}(1-e^{-r\Delta})(1+c)\right]\right) - 2(1-e^{-r\Delta})\frac{\gamma}{r}\right].$$

Multiplying and dividing the last term by S_2, we arrive at

$$\beta_1 = \zeta_a \left[e^{-r\Delta} S_2 \left((1-q)\frac{c^2}{b} - 2\frac{\gamma}{r}(1-e^{-r\Delta})e^{r\Delta} \right) \right].$$

Applying the definition of S_2,

$$\beta_1 = \frac{e^{-r\Delta}\left((1-q)\frac{c^2}{b} - 2\frac{\gamma}{r}(1-e^{-r\Delta})e^{r\Delta}\right)}{1-e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}}.$$

Finally, we can plug in the conjectured a, b, c, so that $c^2/b = (2\gamma/r)c$, and rearrange to find

$$\beta_1 = -2\frac{\gamma}{r}\frac{e^{-r\Delta}\left(-(1-q)c + (1-e^{-r\Delta})e^{r\Delta}\right)}{1-e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}} = -2\frac{\gamma}{r}.$$

Thus the conjectured equilibrium is an equilibrium (filling in the implied $\alpha_0^i, \alpha_2, \alpha_4$). Finally, note that

$$\frac{1-\hat{\omega}}{\Delta} = \frac{(n-1)(1-e^{-r\Delta}) + 2(1-q)e^{-r\Delta} - \sqrt{(n-1)^2(1-e^{-r\Delta})^2 + 4(1-q)e^{-r\Delta}}}{2(1-q)e^{-r\Delta}\Delta}.$$

Suppose that $q = \lambda\Delta$, so this becomes

$$\frac{1-\hat{\omega}}{\Delta} = \frac{(n-1)(1-e^{-r\Delta}) + 2(1-\lambda\Delta)e^{-r\Delta} - \sqrt{(n-1)^2(1-e^{-r\Delta})^2 + 4(1-\lambda\Delta)e^{-r\Delta}}}{2(1-\lambda\Delta)e^{-r\Delta}\Delta}.$$

We multiply the denominator and numerator by $e^{r\Delta}$ and obtain

$$\frac{1-\hat{\omega}}{\Delta} = \frac{(n-1)(e^{r\Delta}-1) + 2(1-\lambda\Delta) - \sqrt{(n-1)^2(1-e^{r\Delta})^2 + 4(1-\lambda\Delta)e^{r\Delta}}}{2(1-\lambda\Delta)\Delta}.$$

The derivative of this expression with respect to Δ is

$$[2(1-2\lambda\Delta)]^{-1}\left((n-1)(re^{r\Delta}) - 2\lambda\right)$$
$$-\frac{\left((n-1)^2(1-e^{r\Delta})^2 + 4(1-\lambda\Delta)e^{r\Delta}\right)^{-.5}\left(-2re^{r\Delta}(n-1)^2(1-e^{r\Delta}) + 4r(1-\lambda\Delta)e^{r\Delta} - 4\lambda e^{r\Delta}\right)}{4(1-2\lambda\Delta)}.$$

The limit of this expression as $\Delta \to 0$ is

$$\frac{1}{2}\left((n-1)r - 2\lambda\right) - .5\frac{(4)^{-.5}(4r-4\lambda)}{2} = \frac{(n-2)r-\lambda}{2}.$$

By l'Hôpital's Rule,

$$\lim_{\Delta\to 0}\frac{-(1-\hat{\omega})}{\Delta} = \frac{-(n-2)r+\lambda}{2},$$

which is the instantaneous demand in the continuous-time model. It is immediate that a, b converge to their corresponding limits, and since the strategies converge as $\Delta \to 0$, so too must the continuation values, for properly defined shocks.

E.2 Unobservable Z_t

Let the transfer $\hat{T}^i$ be defined exactly as in the continuous-time model. As in the proof for the continuous-time model, in an equilibrium with truth-telling and affine δ, cash transfers take the form

$$R_0 + R_1 Z_k + R_2 Z_k^2 + R_3 Z_k z_{ik} + R_4 z_{ik}.$$

The value function is thus linear-quadratic, so, just as in the previous section, the equilibrium value function immediately after an auction or mechanism $V^+(z, Z)$ is linear quadratic in (z, Z) and thus can be rewritten

$$V^+(z, Z) = \upsilon_0 + \upsilon_1 z + \upsilon_2 \bar{Z} + \upsilon_3 z^2 + \upsilon_4 \bar{Z}^2 + \upsilon_5 z \bar{Z},$$

for some constants $\upsilon_0, \ldots, \upsilon_5$. Then, following the steps of Section D.4, maximizing

$$V^+(z + Y^i((\hat{z}^i, \hat{z}^{-i})), Z) + \hat{T}^i((\hat{z}^i, \hat{z}^{-i}); \phi)$$

is equivalent to maximizing

$$\begin{aligned} \mathcal{E}(\phi, Z, z^i, \hat{z}^i) \equiv{} & (\upsilon_1 + \upsilon_5 \bar{Z}) \left(\frac{Z - z^i}{n} - \frac{n-1}{n} \hat{z}^i \right) + \upsilon_3 \left(\frac{Z - z^i}{n} - \frac{n-1}{n} \hat{z}^i \right)^2 \\ & + 2\upsilon_3 z^i \left(\frac{Z - z^i}{n} - \frac{n-1}{n} \hat{z}^i \right) + \kappa_0 (-n\delta(\phi) + Z - z^i + \hat{z}^i)^2 \\ & + \phi(\hat{z}^i - \delta(\phi)) + \frac{\phi^2}{4\kappa_0 n^2}, \end{aligned}$$

Following the same steps taken in the proof of Proposition 5, we can show that $\mathcal{E}_\phi = z - \bar{Z}$ when evaluated at the equilibrium ϕ and $\hat{z}^i = z^i$, for the $\delta(\phi) = -\hat{a} - \hat{b}\phi$, consistent with equilibrium. Also, the equilibrium transfers must be

$$(\upsilon_1 + (\upsilon_5 + 2\upsilon_3)\bar{Z})(z^i - \bar{Z}),$$

so it is straightforward to show the formulas for β_0, β_1 from the previous section apply here as well, for possibly different coefficients (a, b, c).

Returning now to the discrete-time first order condition, the argument to be maximized when trader i submits an order x and report $\hat{z}^i$ is in this case

$$(1 - q) \left(-x \frac{1}{-b(n-1)} \left((n-1)a + c(Z - z) + x \right) + V^+(z + x, Z) \right) + q\mathcal{E}(\phi, Z, z^i, \hat{z}^i).$$

Taking a derivative with respect to x, setting this derivative equal to 0, and using the result that $\mathcal{E}_\phi = z - \bar{Z}$ at the equilibrium $\phi, \hat{z}$, we have

$$(1-q)\tau_g - \frac{q}{b(n-1)}(z - \bar{Z}) = 0,$$

where

$$\tau_g = -\phi + \frac{x}{b(n-1)} + (\beta_0 + \beta_1 \bar{Z}) - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(z + x - \bar{Z}).$$

Plugging in $x = a + b\phi + cz$, $\bar{Z} = (-a - b\phi)/c$, and $x = a + b\phi + cz$, the second order condition is met if and only if $b < 0$. This also implies that $x = c(z - \bar{Z})$, so

$$(z + x - \bar{Z}) = (1+c)z + (1+c)\frac{a + b\phi}{c}.$$

The above can thus be rewritten

$$(1-q)\tau_h - \frac{q}{b(n-1)}\left(z + \frac{a+b\phi}{c}\right) = 0,$$

where

$$\tau_h = -\phi + \frac{a + b\phi + cz}{b(n-1)} + \beta_0 + \beta_1\frac{-a - b\phi}{c} - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}\left((1+c)z + (1+c)\frac{a+b\phi}{c}\right).$$

Gathering terms in $\phi, z, 1$, we have

$$\begin{aligned}
0 &= (1-q)\left(-1 + \frac{1}{n-1} - \frac{b\beta_1}{c} - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{b}{c}\right) - \frac{q}{c(n-1)} \\
0 &= (1-q)\left(\frac{c}{b(n-1)} - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\right) - \frac{q}{b(n-1)} \\
0 &= (1-q)\left(\frac{a}{b(n-1)} + (\beta_0 - \frac{a}{c}\beta_1) - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{a}{c}\right) - \frac{qa}{bc(n-1)}.
\end{aligned}$$

We seek $a, b, c, \beta_1, \beta_0$ such that these three equations and the two equations defining β_0, β_1 all hold. Conjecture that for some $\tilde{\omega} \in (0,1)$, there is an equilibrium with

$$a = \frac{rv}{2\gamma}(1-\tilde{\omega}), \qquad b = -\frac{r}{2\gamma}(1-\tilde{\omega}), \qquad c = -(1-\tilde{\omega}).$$

Starting with the coefficients on z, this means we need

$$0 = (1-q)\left(\frac{2\gamma}{r(n-1)} - \frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)\tilde{\omega}^2}\tilde{\omega}\right) + \frac{2\gamma q}{r(n-1)(1-\tilde{\omega})}.$$

Multiplying through by $r/(2\gamma)$, we have

$$0 = (1-q)\left(\frac{1}{(n-1)} - \frac{(1-e^{-r\Delta})\tilde{\omega}}{1-e^{-r\Delta}(1-q)\tilde{\omega}^2}\right) + \frac{q}{(n-1)(1-\tilde{\omega})}. \tag{93}$$

Suppose there exists some $\tilde{\omega} \in (0,1)$ satisfying this equality. Straightforward calculation then shows that plugging in $\beta_0 = v, \beta_1 = -2\gamma/r$, the coefficients on $\phi, 1$ above are all 0.

Following the steps in the last section, in any equilibrium, we then have

$$\beta_1 = \frac{e^{-r\Delta}\left((1-q)\frac{c^2}{b} - 2\frac{\gamma}{r}(1-e^{-r\Delta})e^{r\Delta}\right)}{1-e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}}.$$

Plugging in the conjectured a, b, c, we have

$$\beta_1 = \frac{e^{-r\Delta}\left(-\frac{2\gamma}{r}(1-q)(1-\tilde{\omega}) - 2\frac{\gamma}{r}(1-e^{-r\Delta})e^{r\Delta}\right)}{1-e^{-r\Delta}(1-q)\tilde{\omega} - qe^{-r\Delta}}.$$

For $\beta_1 = -2\gamma/r$ to be consistent, it must be that

$$1-e^{-r\Delta}(1-q)\tilde{\omega} - qe^{-r\Delta} = e^{-r\Delta}\left((1-q)(1-\tilde{\omega}) + (1-e^{-r\Delta})e^{r\Delta}\right).$$

But this conditions holds for any $\tilde{\omega}$. Likewise, conjecturing that $\beta_0 = v$, at the conjectured a, b, c, we have

$$\begin{aligned} q\beta_0 + (1-q)\left[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)\right] &= qv + (1-q)[v(1-\tilde{\omega}) + v(1-e^{-r\Delta})\tilde{\omega}] \\ &= v\left(1-(1-q)\tilde{\omega}e^{-r\Delta}\right). \end{aligned}$$

Thus $\beta_0 = v$ is consistent with

$$\beta_0 = v(1-e^{-r\Delta}) + \frac{e^{-r\Delta}\left(q\beta_0 + (1-q)[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)]\right)}{1-e^{-r\Delta}(1-q)(1+c)}.$$

We have thus shown that, as long as $\tilde{\omega}$ satisfies (93), the conjectured a, b, c satisfy the first order condition and comprise a subgame perfect equilibrium. In unreported numerical exercises, we find that for sufficiently small Δ there exists a root $\tilde{\omega}$ such that $-(1-\tilde{\omega})/\Delta$ is equal to the order-flow coefficient c from Proposition 5, up to machine precision.

F The Walrasian-Rationing Mechanism

In this appendix we show that our main results, Propositions 4 and 5, are robust to using a different efficient mechanism. The mechanism can be loosely described as a Walrasian market with rationing. Suppose that at time t a mechanism is run, and the mechanism designer believes the aggregate inventory is $\tilde{Z}_t$. In the mechanism we consider, each trader i submits a report $\hat{z}^i_t$, and if $\sum_{i=1}^n \hat{z}^i_t \neq \tilde{Z}_t$, then nothing happens. If instead the reports satisfy $\sum_{i=1}^n \hat{z}^i_t = \tilde{Z}_t$, then each trader receives an asset reallocation of

$$Y^i(\hat{z}) = \frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i$$

and a transfer of

$$\tilde{T}^i_\kappa(\hat{z}) = -\kappa Y^i(\hat{z}) = \kappa(\hat{z}^i - \frac{\sum_{j=1}^n \hat{z}^j}{n}),$$

where κ is a frozen prize that could depend on the information available to the platform operator. It is clear that this mechanism will always be exactly budget balanced regardless of which κ is chosen. Likewise, if truth-telling is incentive compatible, this mechanism leads to an efficient allocation each time it is run. In the rest of this appendix, we show that there is a unique frozen price κ that makes participation in this mechanism ex-post individually rational and incentive compatible. In either the case of observable Z or the case of unobservable Z, replacing our mechanism with this mechanism (for the choice of κ satisfying individual rationality) leads to precisely the same equilibrium value functions of Sections 4 and 5, respectively. For the case of unobserved Z, it is not clear what conditions lead to the existence of equilibria. However, if equilibria exist, the value functions exactly coincide with those given in our Proposition 5.

F.1 Observable Z_t

In this subsection, we take the setting of Section 4 of the paper, with the exception that we use the asset reallocation function

$$Y^i(\hat{z}, Z) = \mathbf{1}_{\{\sum_{j=1}^n \hat{z}^j = Z\}} \left(\frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i \right)$$

and the transfer function

$$\tilde{T}^i_\kappa(\hat{z}, Z) = -\kappa(Z) Y^i(\hat{z}, Z)$$

for an affine $\kappa(\,\cdot\,)$ that will be specified. Using the same definition of an equilibrium as in Section 4, we now provide a sketch of a proof that there can be at most one equilibrium and that, if the equilibrium exists, the value functions coincide with the value functions of Proposition 4.

Conjecture a symmetric affine equilibrium with demand constants (a, b, c) and truthful mechanism reporting. The equilibrium auction price is once again

$$p = \frac{a + c\bar{Z}}{-b},$$

which is affine in Z. Provided that $\kappa(Z)$ is affine in Z, the equilibrium mechanism transfers are

$$\kappa(Z_t)(z^i_t - \bar{Z}_t) = R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z^i_t + R_4 z^i_t,$$

for constants R_0 through R_4 that depend on $a, b, c, \kappa(\,\cdot\,)$.

Then just as in the proof of Proposition 4, the value function takes the form

$$V(z, Z) = \alpha^i_0 + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z},$$

where

$$\alpha_3 = \frac{-\gamma}{r+\lambda-2c}$$
$$\alpha_5 = \frac{1}{r+\lambda-c}\left(\frac{c^2}{b} - 2\alpha_3 c + \lambda n R_3\right)$$
$$\alpha_4 = \frac{1}{r}\left(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2\right)$$
$$\alpha_1 = \frac{1}{r+\lambda-c}\left(rv + \frac{ac}{b} + \lambda R_4\right)$$
$$\alpha_2 = \frac{1}{r}\left(\frac{ca}{-b} + (\lambda - c)\alpha_1 + \lambda n R_1\right)$$
$$\alpha_0^i = \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0\right),$$

and where R_0 through R_4 are the previously defined transfer coefficients.

We need the mechanism to be ex-post individually rational. In equilibrium, trader i conjectures all other traders are truthfully reporting, and Z_t is publicly observable. It follows that trader i has two choices: trader i can report something different from z_t^i and get $V(z_t^i, Z_t)$ (i.e., the mechanism does nothing), or report z_t^i and get

$$V(\bar{Z}_t, Z_t) + \kappa(Z_t)(z_t^i - \bar{Z}_t).$$

Our individual rationality condition requires that

$$V(\bar{Z}_t, Z_t) + \kappa(Z_t)(z_t^i - \bar{Z}_t) \geq V(z_t^i, Z_t),$$

which is equivalent to

$$0 \leq (\alpha_1 + \alpha_5\bar{Z}_t)(\bar{Z}_t - z_t^i) + \alpha_3\left(\bar{Z}_t^2 - (z_t^i)^2\right) + \kappa(Z_t)(z_t^i - \bar{Z}_t)$$
$$0 \leq (\alpha_1 + [\alpha_5 + 2\alpha_3]\bar{Z}_t)(\bar{Z}_t - z_t^i) - \alpha_3\left(\bar{Z}_t - z_t^i\right)^2 + \kappa(Z_t)(z_t^i - \bar{Z}_t)$$
$$0 \leq (\alpha_1 + [\alpha_5 + 2\alpha_3]\bar{Z}_t - \kappa(Z_t))(\bar{Z}_t - z_t^i) - \alpha_3\left(\bar{Z}_t - z_t^i\right)^2.$$

Because $\alpha_3 < 0$, we see that individual rationality is satisfied with probability 1 if and only if $\kappa(Z_t) = \alpha_1 + [\alpha_5 + 2\alpha_3]\bar{Z}_t$. It is immediate that this not only implies individual rationality, but also the incentive compatibility of truth-telling. This is because if trader i reports anything other than z_t^i, they recognize that they get the exact same outcome as if they had not participated at all. This implies that in any equilibrium, the equilibrium transfer for trader i must be

$$(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}_t)\left(z_t^i - \bar{Z}_t\right),$$

which is exactly the expression in the equilibrium of Proposition 4.

In any candidate equilibrium, the HJB equation must be satisfied, in that

$$\begin{aligned}0 = &-\gamma(z^i)^2 + r(vz^i - V(z^i, Z)) + \frac{\sigma_i^2}{2}V_{zz}(z, Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i, Z)\\ &+ \sup_{D,\hat{z}^i}\{-\Phi_{(a,b,c)}(D; Z - z^i)D + V_z(z^i, Z)D\\ &+ \lambda\mathbf{1}(\hat{z}^i = z^i)\left(V(\bar{Z}, Z) - V(z, Z) + \kappa(Z)(z^i - \bar{Z})\right)\}.\end{aligned}$$

We know that V must take the linear quadratic form from above, and then as long as $\kappa(Z_t) = \alpha_1 + [\alpha_5 + 2\alpha_3]\bar{Z}_t$, the unique optimizer is $\hat{z}_t^i = z_t^i$ with probability 1 for any D. Plugging this in, we get the exact same HJB equation for choice of D as that in the proof of Proposition 4. The rest of the solution follows accordingly, leading to the same value functions as those of Proposition 4.

F.2 Unobservable Z_t

In this subsection, we take the setting of Section 5 of the paper, with the exception that, letting p denote the exchange price immediately prior to a mechanism occuring, we replace Y^i with the asset reallocation function

$$Y_\delta^i(\hat{z}, p) = \mathbf{1}_{\{\sum_{j=1}^n \hat{z}^j = \delta(p)\}}\left(\frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i\right)$$

and we use the cash transfer function

$$\tilde{T}_\kappa^i(\hat{z}, p) = -\kappa(p)Y_\delta^i(\hat{z}, p),$$

for an affine $\kappa(\,\cdot\,)$ and a function $\delta(\,\cdot\,)$ with the property that, in equilibrium, $\delta(p_t) = Z_t$ with probability 1. Using the definition of an equilibrium given in Section 5 (adding the criterion that $\delta(p_t) = Z_t$), we now provide a sketch of a proof that if equilibria exist, the value functions must coincide with those of Proposition 5.

In any candidate equilibrium, the price is

$$p_t = \frac{a + c\bar{Z}_t}{-b},$$

so the unique candidate δ function is

$$\delta(p) = \frac{-nbp - na}{c}.$$

Just as before, on the equilibrium path, as long as $\kappa(\,\cdot\,)$ is affine the transfers are

$$\kappa(p_t)(z_t^i - \bar{Z}_t) = R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

which leads to precisely the same candidate equilibrium value functions for any a, b, c (where the α coefficients depend upon a, b, c, κ).

Once again, our individual rationality condition requires that, on the equilibrium path,

$$\kappa(p_t) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}_t = \alpha_1 + [\alpha_5 + 2\alpha_3]\frac{-bp_t - a}{c},$$

but this is no longer the same as incentive compatibility. As in the rest of the appendix, we imagine that when considering deviations, trader i observes Z_t. Then, if the price is ϕ_t, trader i knows that in order to benefit from the mechanism, he or she must report

$$z_t^i - Z_t + \delta(\phi_t) = z_t^i - Z_t + \frac{-nb\phi_t - na}{c}.$$

These reports sum to $\delta(\phi_t) = \frac{-nb\phi_t - na}{c}$, so the mechanism will "run." Trader i will get the reallocation

$$\frac{-b\phi_t - a}{c} - \left(z_t^i - Z_t + \frac{-nb\phi_t - na}{c}\right),$$

corresponding to a change in utility of

$$\begin{aligned} &V\left(\frac{-b\phi_t - a}{c} + \left(Z_t + \frac{nb\phi_t + na}{c}\right), Z\right) - V(z_t^i, Z_t) \\ &- \left(\frac{-b\phi_t - a}{c} - \left(z_t^i - Z_t + \frac{-nb\phi_t - na}{c}\right)\right)\left(\alpha_1 + (\alpha_5 + 2\alpha_3)\frac{-b\phi_t - a}{c}\right). \end{aligned}$$

Cleaning up terms, this is

$$\begin{aligned} &V\left((n-1)\frac{b\phi_t + a}{c} + Z_t, Z_t\right) - V(z_t^i, Z_t) \\ &\quad - \left((n-1)\frac{b\phi_t + a}{c} - (z_t^i - Z_t)\right)\left(\alpha_1 + (\alpha_5 + 2\alpha_3)\frac{-b\phi_t - a}{c}\right). \end{aligned}$$

Letting E take the value 1 or 0, corresponding to the choice to enter the mechanism or not, respectively, the HJB equation is

$$\begin{aligned} 0 = &-\gamma(z^i)^2 + r(vz^i - V(z^i, Z)) + \frac{\sigma_i^2}{2}V_{zz}(z, Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i, Z) \\ &+ \sup_{D,E}\{-\Phi_{(a,b,c)}(D; Z - z^i)D + V_z(z^i, Z)D + E\lambda\nu\}, \end{aligned}$$

where

$$\begin{aligned} \nu = &V\left((n-1)\frac{b\Phi_{(a,b,c)}(D; Z - z^i) + a}{c} + Z, Z\right) - V(z^i, Z) \\ &- \left((n-1)\frac{b\Phi_{(a,b,c)}(D; Z - z^i) + a}{c} - (z^i - Z)\right)\left(\alpha_1 + (\alpha_5 + 2\alpha_3)\frac{-b\Phi_{(a,b,c)}(D; Z - z^i) - a}{c}\right). \end{aligned}$$

Taking the first-order condition with respect to D and recalling that

$$\frac{\partial \Phi_{(a,b,c)}(D; Z - z^i)}{\partial D} = \frac{-1}{b(n-1)},$$

we have

$$\begin{aligned}
&-\Phi_{(a,b,c)}(D; Z - z^i) + \frac{D}{b(n-1)} + V_z(z^i, Z) + E\lambda\left(-\frac{1}{c}V_z((n-1)\frac{b\Phi_{(a,b,c)}(D; Z - z^i) + a}{c} + Z, Z)\right)\\
&\quad + \frac{E\lambda}{c}\left(\alpha_1 + (\alpha_5 + 2\alpha_3)\frac{-b\Phi_{(a,b,c)}(D; Z - z^i) - a}{c}\right)\\
&\quad - E\lambda\left((n-1)\frac{b\Phi_{(a,b,c)}(D; Z - z^i) + a}{c} - (z^i - Z)\right)\frac{\alpha_5 + 2\alpha_3}{c(n-1)} = 0,
\end{aligned}$$

which can be cleaned up as

$$\begin{aligned}
&-\Phi_{(a,b,c)}(D; Z - z^i) + \frac{D}{b(n-1)} + V_z(z^i, Z) + E\lambda\left(-\frac{1}{c}V_z((n-1)\frac{b\Phi_{(a,b,c)}(D; Z - z^i) + a}{c} + Z, Z)\right)\\
&\quad + \frac{E\lambda}{c}\left(\alpha_1 + 2[\alpha_5 + 2\alpha_3]\frac{-b\Phi_{(a,b,c)}(D; Z - z^i) - a}{c} + \frac{(z^i - Z)}{n-1}[\alpha_5 + 2\alpha_3]\right) = 0.
\end{aligned}$$

The second-order condition is satisfied as long as $b < 0$ and $\alpha_5 + 2\alpha_3 < 0$. Suppressing the arguments of Φ and plugging in $D = a + b\Phi + cz^i$ and $Z = (-nb\Phi - na)/c$, this is

$$\begin{aligned}
&-\Phi + \frac{a + b\Phi + cz^i}{b(n-1)} + [\alpha_1 + \alpha_5\frac{-b\Phi - a}{c} + 2\alpha_3 z^i] + E\lambda\left(-\frac{1}{c}[\alpha_1 + \alpha_5\frac{-b\Phi - a}{c} + 2\alpha_3\frac{-b\Phi - a}{c}]\right)\\
&\quad + \frac{E\lambda}{c}\left(\alpha_1 + 2[\alpha_5 + 2\alpha_3]\frac{-b\Phi - a}{c} + \frac{(z^i - \frac{-nb\Phi - na}{c})}{n-1}[\alpha_5 + 2\alpha_3]\right) = 0.
\end{aligned}$$

Gathering terms, we have

$$\frac{c}{b(n-1)} + 2\alpha_3 + \frac{E\lambda}{c(n-1)}[\alpha_5 + 2\alpha_3] = 0$$

and

$$-1 + \frac{1}{(n-1)} - \frac{b\alpha_5}{c} + \frac{E\lambda b}{c^2}(\alpha_5 + 2\alpha_3) + \frac{E\lambda}{c}\left(2[\alpha_5 + 2\alpha_3]\frac{-b}{c} + \frac{(-\frac{-nb}{c})}{n-1}[\alpha_5 + 2\alpha_3]\right) = 0,$$

which simplify to

$$c^2 + 2\alpha_3 cb(n-1) + E\lambda b[\alpha_5 + 2\alpha_3] = 0$$

and

$$-1+\frac{1}{(n-1)}-\frac{b\alpha_5}{c}+\frac{E\lambda b}{c^2(n-1)}\left(\alpha_5+2\alpha_3\right)=0,$$

respectively. When $E=1$, this is

$$(-n+2)c^2-bc\alpha_5(n-1)+\lambda b\left(\alpha_5+2\alpha_3\right)=0.$$

Subtracting the above from this gives

$$(-n+1)c^2-bc(n-1)(\alpha_5+2\alpha_3)=0$$

and

$$-\frac{c}{b}=(\alpha_5+2\alpha_3).$$

Plugging this back in, we can rewrite the above two equations as

$$\frac{c}{b(n-1)}+2\alpha_3-\frac{\lambda}{b(n-1)}=0$$

and

$$-1+\frac{1}{(n-1)}-\frac{b\alpha_5}{c}-\frac{\lambda}{c(n-1)}=0.$$

This is the same pair of equations characterizing equilibrium in Proposition 5. Finally, gathering the constant terms gives

$$\begin{aligned}&\frac{a}{b(n-1)}+[\alpha_1+\alpha_5\frac{-a}{c}]+E\lambda\left(-\frac{1}{c}[\alpha_1+\alpha_5\frac{-a}{c}+2\alpha_3\frac{-a}{c}]\right)\\&\quad+\frac{E\lambda}{c}\left(\alpha_1+2[\alpha_5+2\alpha_3]\frac{-a}{c}+\frac{(-\frac{-na}{c})}{n-1}[\alpha_5+2\alpha_3]\right)=0.\end{aligned}$$

Plugging in the above gives

$$\frac{a}{b(n-1)}+[\alpha_1+\alpha_5\frac{-a}{c}]+E\lambda\left(-\frac{1}{c}[\alpha_1+\frac{a}{b}]\right)+\frac{E\lambda}{c}\left(\alpha_1+2\frac{a}{b}+\frac{(-\frac{na}{b})}{n-1}\right)=0,$$

which is the same as in our proof of Proposition 5, when $E=1$. Thus, under the assumption that $E=1$ (traders always participate in the mechanism), this leads to precisely the same value functions shown in Proposition 5. However, this assumes that it is not better for the trader to strategically not participate in the mechanism (that is, to use an entirely different nonlinear demand schedule and to participate only some of the time). Of course, to be an equilibrium by our definition, it must be that $E=1$ always. So, if equilibria exist, the value functions must coincide with those shown in Proposition 5.

G The Impaired Mechanism

In this section, we consider an alternate mechanism designed to reduce a fraction ξ of the excess inventory at each implementation. Its allocations and transfers are given by

$$Y^i(\hat{z}) = \xi \left(\frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i \right) \tag{94}$$

and

$$\begin{aligned} T^i(\hat{z}, Z) = \kappa_0 \left(n\kappa_2(Z) + \xi \sum_j \hat{z}^j \right)^2 + \kappa_1(Z)(\xi \hat{z}^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} \\ + n\kappa_0 \frac{1-\xi}{\xi} \left[(\xi \hat{z}^i + \kappa_2(Z))^2 - \left((n-1)\kappa_2(Z) + \xi \sum_{j \neq i} \hat{z}^j + \frac{\xi \kappa_1(Z)}{2\kappa_0 n} \right)^2 \right], \end{aligned}$$

for a constant $\kappa_0 < 0$ and affine $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$. It is worth noting that the sum of these transfers may not be weakly negative for any reports $\hat{z}$, but we show in all the equilibria that we consider, the transfers sum to zero with probability 1.

G.1 Sketch of proof of the analogue of Proposition 4

We provide a sketch of a proof for an alternative version of Proposition 4: For any $\xi \in (0, 1]$, there exists a unique symmetric equilibrium such that, each time the mechanism is run, all traders reduce a fraction ξ of their inventory imbalance $z^i - \bar{Z}$. The auction price and value functions are identical to those of Proposition 4, and the auction demands are identical after replacing λ with $\lambda(2\xi - \xi^2)$. The mechanism reports are still truth-telling: $\hat{z}^i = z^i$.
Proof sketch: In any such equilibrium, each trader reports $\hat{z}^i = z^i$, so that

$$Y^i(\hat{z}_t) = \xi(\bar{Z}_t - z_t^i)$$

and the transfers are

$$\begin{aligned} T^i(\hat{z}, Z) &= \kappa_0(n\kappa_2(Z) + \xi Z)^2 + \kappa_1(Z)(\xi z^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} \\ &\quad + n\kappa_0 \frac{1-\xi}{\xi} \left[(\xi z^i + \kappa_2(Z))^2 - \left((n-1)\kappa_2(Z) + \xi(Z - z^i) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n} \right)^2 \right] \\ &= \kappa_0(n\kappa_2(Z) + \xi Z)^2 + \kappa_1(Z)(\xi z^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} + \tau_a\tau_b, \end{aligned}$$

where

$$\tau_a = n\kappa_0 \frac{1-\xi}{\xi}\left(\xi Z + n\kappa_2(Z) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right)$$
$$\tau_b = \left(\xi z^i + \kappa_2(Z) - \left((n-1)\kappa_2(Z) + \xi(Z - z^i) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right)\right).$$

For any affine κ_1, κ_2, the transfer can be expressed as

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

for constants $R_0, \ldots, R_4$. Receiving such transfers at independent Poisson arrival times must lead to a linear-quadratic value function, as in the proofs the previous propositions. That is, the equilibrium continuation value function V for trader i must be of the form

$$V(z^i, Z) = \alpha_0^i + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3 (z^i)^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z^i \bar{Z}. \tag{95}$$

Fixing assumed reports $\hat{z}^j = z^j$ for other traders, trader i chooses $\tilde{z}$ to maximize

$$(\alpha_1 + \alpha_5 \bar{Z}) Y^i((\tilde{z}, \hat{z}^{-i})) + \alpha_3 Y^i((\tilde{z}, \hat{z}^{-i}))^2 + 2\alpha_3 Y^i((\tilde{z}, \hat{z}^{-i})) z^i + T^i((\tilde{z}, \hat{z}^{-i}), Z),$$

where, writing $\kappa_2(Z) = \hat{a} + \hat{b} Z$ and $\hat{z}^j = z^j$,

$$T^i((\tilde{z}, \hat{z}^{-i}), Z) = \kappa_0(\xi\tilde{z} + n\hat{a} + n\hat{b}Z + \xi(Z - z^i))^2 + \kappa_1(Z)(\xi\tilde{z} + \hat{a} + \hat{b}Z) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0}$$
$$+ n\kappa_0 \frac{1-\xi}{\xi}\left[(\xi\hat{z}^i + \hat{a} + \hat{b}Z)^2 - \left((n-1)(\hat{a} + \hat{b}Z) + \xi\sum_{j\neq i}\hat{z}^j + \frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right)^2\right].$$

The first-order condition is

$$-\frac{n-1}{n}\xi(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i) - \frac{2(n-1)\alpha_3\xi}{n} Y^i((\tilde{z}, \hat{z}^{-i})) + \xi\kappa_1(Z)$$
$$+ 2\kappa_0\xi(\xi\tilde{z} + n\hat{a} + n\hat{b}Z + \xi(Z - z^i)) + 2n\kappa_0\xi\frac{1-\xi}{\xi}(\xi\hat{z}^i + \hat{a} + \hat{b}Z) = 0.$$

Plugging in $\tilde{z} = z^i$, $Y^i((\tilde{z} \text{ and } \hat{z}^{-i})) = \xi(\bar{Z} - z^i)$, and then dividing through by ξ, we have

$$-\frac{n-1}{n}(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i) - \frac{2(n-1)\alpha_3}{n}\xi(\bar{Z} - z^i) + \kappa_1(Z)$$
$$+ 2\kappa_0(n\hat{a} + n\hat{b}Z + \xi Z) + 2n\kappa_0\frac{1-\xi}{\xi}(\xi z^i + \hat{a} + \hat{b}Z) = 0.$$

It is clear that the terms involving z^i cancel if and only if $\kappa_0 = (n-1)\alpha_3/n^2$. Given this, the unique $\hat{a}, \hat{b}$ solving this must satisfy

$$\begin{aligned}
0 &= -\frac{n-1}{n}(\alpha_1 + \alpha_5 \bar{Z}) - \frac{2(n-1)\alpha_3}{n}\xi(\bar{Z}) + \kappa_1(Z) \\
&\quad + \frac{2(n-1)\alpha_3}{n^2}(n\hat{a} + n\hat{b}Z + \xi Z)) + \frac{2(n-1)\alpha_3}{n}\frac{1-\xi}{\xi}(\hat{a} + \hat{b}Z), \\
\hat{a} + \hat{b}Z &= \frac{n\xi}{2(n-1)\alpha_2}\left(-\kappa_1(Z) + (\alpha_1 + \alpha_5 \bar{Z})\frac{n-1}{n}\right) \\
&= \frac{\xi}{2n\kappa_0}\left(-\kappa_1(Z) + (\alpha_1 + \alpha_5 \bar{Z})\frac{n-1}{n}\right).
\end{aligned}$$

Manipulating the formula for transfers, we can write the equilibrium transfer to trader i, given $\hat{z}^i = z^i$ for all i, as

$$\kappa_0(n\kappa_2(Z) + \xi Z)^2 + \kappa_1(Z)(\xi z^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} + \tau_a \tau_b.$$

Defining $\kappa_1(\,\cdot\,)$ so that

$$\xi Z + n\kappa_2(Z) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n} = 0,$$

this transfer simplifies to

$$\kappa_0(n\kappa_2(Z) + \xi Z)^2 + \kappa_1(Z)(\xi z^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0}.$$

The sum of the transfers across all traders is

$$n\kappa_0\left(\frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right)^2 - \kappa_1(Z)\left(\frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n\kappa_0} = 0.$$

Some calculation shows that the above choice for $\kappa_1(\,\cdot\,)$ uniquely ensures that the transfers sum to zero with probability 1, which must be the case for IR and budget balance to hold. Plugging in the formula for $\kappa_2(\,\cdot\,)$, we see that we need the conditions

$$\begin{aligned}
0 &= \xi Z + \frac{\xi}{2\kappa_0}\left(-\kappa_1(Z) + (\alpha_1 + \alpha_5 \bar{Z})\frac{n-1}{n}\right) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n} \\
0 &= 2\kappa_0 nZ + \big(-n\kappa_1(Z) + (\alpha_1 + \alpha_5 \bar{Z})(n-1)\big) + \kappa_1(Z) \\
\kappa_1(Z) &= (\alpha_1 + \alpha_5 \bar{Z}) + \frac{2\kappa_0 n}{n-1}Z \\
&= \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}.
\end{aligned}$$

This is the unique choice for $\kappa_1(Z)$ consistent with budget balance and ex-post IR.

The HJB equation is

$$\begin{aligned}rV(z^i,Z) = &-\gamma(z^i)^2 + rvz + \frac{\sigma_i^2}{2}V_{zz}(z^i,Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i,Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i,Z)\\ &+ \sup_{D,\tilde{z}}\{-\Phi_{(a,b,c)}(D;Z-z^i)D + V_z(z^i,Z)D\\ &+ \lambda\left(V(z^i+Y^i(\tilde{z},\hat{z}^{-i}),Z) - V(z^i,Z) + T^i((\tilde{z},\hat{z}^{-i}),Z)\right)\}.\end{aligned}$$

We just showed that because V is linear-quadratic, at the unique candidate equilibrium reallocations we must have

$$V(z+Y^i(\tilde{z},\hat{z}^{-i}),Z) - V(z,Z) = (\alpha_1+\alpha_5\bar{Z})\xi(\bar{Z}-z) + \alpha_3\xi^2(\bar{Z}-z)^2 + 2\alpha_3\xi z(\bar{Z}-z).$$

By the above, the equilibrium transfer is

$$\kappa_0\left(\frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right)^2 + \kappa_1(Z)\left(\xi(z^i-\bar{Z}) - \frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right) + \frac{(2\xi-\xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} = \kappa_1(Z)\xi(z^i-\bar{Z}).$$

Plugging in $\kappa_1(Z) = \alpha_1 + (\alpha_5+2\alpha_3)\bar{Z}$ and summing the transfer and the change in continuation value gives

$$\begin{aligned}(\alpha_1+\alpha_5\bar{Z})\xi(\bar{Z}-z) + \alpha_3\xi^2(\bar{Z}-z)^2 + 2\alpha_3\xi z(\bar{Z}-z) - (\alpha_1+\alpha_5\bar{Z}+2\alpha_3\bar{Z})\xi(\bar{Z}-z)&\\ = \alpha_3\xi^2(\bar{Z}-z)^2 - 2\alpha_3\xi\left(z^2+\bar{Z}^2-2z\bar{Z}\right)&\\ = -\alpha_3(2\xi-\xi^2)(\bar{Z}-z)^2.&\end{aligned}$$

Plugging this in, the HJB equation becomes

$$\begin{aligned}rV(z^i,Z) = &-\gamma(z^i)^2 + rvz^i + \frac{\sigma_i^2}{2}V_{zz}(z^i,Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i,Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i,Z)\\ &+ \sup_{D}\{-\Phi_{(a,b,c)}(D;Z-z^i)D + V_z(z^i,Z)D - \lambda(2\xi-\xi^2)\alpha_3\left(z^i-\bar{Z}\right)^2\}.\end{aligned}$$

This is exactly the HJB equation found in the proof of Proposition 4, after replacing λ with $\lambda^* = \lambda(2\xi-\xi^2)$.

H Only Size Discovery: Observable Inventory Z_t

In the main text of the paper, we showed that augmenting a price-discovery market with future size-discovery sessions never increases welfare, and strictly reduces welfare if the size-discovery platform operator relies on the price-discovery market for information about aggregate inventory imbalances. It is then natural to ask whether simply getting rid of the price-discovery market, and running only size-discovery sessions, could improve welfare, relative to a setting with price discovery. When stand-alone size discovery is feasible and is run sufficiently frequently, and the aggregate excess inventory Z_t is observable, it strictly improves welfare, and indeed is strictly preferred by each trader individually. From a practical viewpoint, however, it could be difficult to arrange for the abandonment of price-discovery markets. Moreover, the size-discovery sessions that we analyze might be difficult to implement in practice without information coming out of the price-discovery market.

In this appendix, we consider a pure size-discovery market, for an economy with observable aggregate inventory. For example, it suffices that Z is a deterministic constant. We exploit the same perfect-reallocation size-discovery sessions developed earlier. As before, these sessions are run at the event times of an independent Poisson process N with mean arrival rate $\lambda > 0$.

Again, traders submit mechanism report processes $\hat{z} = (\hat{z}^1, \ldots, \hat{z}^n)$. The resulting excess-inventory process z^i of trader i is then determined by

$$z_t^i = z_0^i + H_t^i + \int_0^t \left(\frac{\sum_{j=1}^n \hat{z}_s^j}{n} - \hat{z}_s^i \right) dN_s. \tag{96}$$

As in Section 4, we assume that the aggregate inventory Z_t is common knowledge for all t. The size-discovery mechanism design (Y, T_κ) uses the asset reallocation determined by (3). We again apply the cash-transfer function T_κ defined by (4) for some coefficient $\kappa_0 < 0$, with

$$\kappa_1(Z_t) = v - \frac{2\gamma}{r}\overline{Z}_t \tag{97}$$

and

$$\kappa_2(Z_t) = -\overline{Z}_t - \frac{\kappa_1(Z_t)}{2\kappa_0 n^2}. \tag{98}$$

By the same reasoning used in Propositions 1 and 2, one can show these are the unique affine choices for $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ such that an equilibrium exists. Moreover, we must restrict attention to affine $\kappa_1(\cdot), \kappa_2(\cdot)$ in this dynamic setting in order to guarantee a linear-quadratic continuation-value function.

We seek a truth-telling equilibrium of the dynamic reporting game, in which each trader optimally chooses to report $\hat{z}_t^i = z_t^i$ and in which mechanism participation is always individually rational. The exact stochastic control problem solved by each trader is an obvious simplification of the control problem of Section 4, which appears in the following section. The next proposition confirms that this equilibrium exists and provides a calculation of the continuation value for each trader.

Proposition 6. *For any $\kappa_0 < 0$, consider the size-discovery session mechanism design (Y, T_κ) of (3)-(4), with (97)-(98). The truth-telling equilibrium, that with reports $\hat{z}_t^i = z_t^i$, exists and has the following properties.*

1. *At each session time τ_k, each trader i achieves the efficient post-session position $z^i(\tau_k) = \overline{Z}(\tau_k)$, almost surely.*

2. *For each trader i, the equilibrium continuation value $V_M^i(z_t^i, Z_t)$ at time t is*

$$V_M^i(z_t^i, Z_t) = \tilde{\theta}_i + v\overline{Z}_t - \frac{\gamma}{r}\overline{Z}_t^2 + \kappa_1(Z_t)\left(z_t^i - \overline{Z}_t\right) - \frac{\gamma}{r+\lambda}\left(z_t^i - \overline{Z}_t\right)^2,$$

where

$$\tilde{\theta}_i = \frac{1}{r}\left(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} - \frac{\gamma}{r+\lambda}\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}\right) - \frac{2\gamma}{r}\frac{\rho^i}{n}\right).$$

As the mean frequency λ of reallocation sessions approaches infinity, the equilibrium welfare approaches the first-best welfare $W_{fb}(Z)$. This follows from the fact that the equilibrium total expected holding costs associated with excess inventory, relative to the holding costs at first best, approaches zero[38] as $\lambda \to \infty$. This is immediate from the fact that the quadratic coefficient $\gamma/(r+\lambda)$ of the indirect utility V_M^i approaches zero as $\lambda \to \infty$. These properties hold for any choice of $\kappa_0 < 0$, but setting $\kappa_0 = -\gamma(n-1)/(n^2(r+\lambda))$ makes each trader indifferent to instantaneous deviations by other traders.[39]

H.1 Proof of Proposition 6

The proof is extremely similar to that of Proposition 4, so we leave some details to the reader. We write $V(z, Z)$ rather than $V_M^i(z, Z)$ for brevity. For any affine $\kappa_1(\,\cdot\,)$ and $\kappa_2(\,\cdot\,)$, the transfers in equilibrium take the form

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

for some constants R_0 through R_4. In any symmetric equilibrium, the value function

$$V(z, Z) = \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_0^{\mathcal{T}} -\gamma (z_s^i)^2\, ds + \int_0^{\mathcal{T}} T_\kappa^i(\hat{z}_s, Z_s)\, dN_s\right]$$

takes the form

$$V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z},$$

[38] This convergence is also intuitively obvious from the fact that $\delta_t^i \equiv (z_t^i - \overline{Z}_t)^2$ jumps to zero at each of the event times of N. The duration of time between these successive perfect reallocations has expectation $1/\lambda$, which goes to zero. Between these perfect reallocations, δ_t^i has a mean that is continuous in t and grows in expectation at a bounded rate.

[39] Formally, if we consider the static mechanism report game with the continuation value corresponding to Proposition 6, for this κ_0 truth-telling is a dominant strategy.

where

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda} \\
\alpha_5 &= \frac{1}{r+\lambda}(\lambda n R_3) \\
\alpha_4 &= \frac{1}{r}(\lambda \alpha_5 + \lambda \alpha_3 + \lambda n^2 R_2) \\
\alpha_1 &= \frac{1}{r+\lambda}(rv + \lambda R_4) \\
\alpha_2 &= \frac{1}{r}(\lambda \alpha_1 + \lambda n R_1) \\
\alpha_0^i &= \frac{1}{r}\left(\alpha_3 \sigma_i^2 + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_5 \frac{\rho^i}{n} + \lambda R_0\right),
\end{aligned}
$$

and where R_0 through R_4 are the previously defined transfer coefficients. To see this, note that given the α coefficients, we have

$$
\begin{aligned}
(r+\lambda)\left(\alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z \bar{Z}\right) &= rvz - \gamma z^2 + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} \\
&+ \lambda(\alpha_0^i + \alpha_1 \bar{Z} + \alpha_2 \bar{Z} + \alpha_3 \bar{Z}^2 + \alpha_4 \bar{Z}^2 + \alpha_5 \bar{Z}^2 + R_0 + R_1 Z + R_2 Z^2 + R_3 Z z + R_4 z).
\end{aligned}
$$

Let $Y_t = 1_{\{\mathcal{T} \le t\}}$ and $V(z, Z)$ be defined as above. Let

$$
X = \begin{bmatrix} z_t^i \\ Z_t \\ Y_t \end{bmatrix}
$$

and $U(X) = U(z, Z, Y) = (1 - Y)V(z, Z) + Yvz$. Then, following the steps of the proof of Proposition 4, if we let

$$
\chi_s = \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} - \lambda(z_s^i - \bar{Z}_s)(\alpha_1 + \alpha_5 \bar{Z}_{s-} + \alpha_3(z_{s-}^i + \bar{Z}_{s-})) + r(vz_s^i - V(z_s^i, Z_s)),
$$

we can show that

$$
\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left[\int_{0+}^{\mathcal{T}} \chi_s \, ds\right].
$$

Because α_0^i through α_5 satisfy the system of equations specified at the beginning of this proof, we have

$$
\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left[\int_{0+}^{\mathcal{T}} \bar{\chi}_s \, ds\right],
$$

where

$$
\bar{\chi}_s = \gamma(z_s^i)^2 - \lambda(R_0 + R_1 Z_s + R_2 Z_s^2 + R_3 Z_s z_s^i + R_4 z_s^i).
$$

Using the definitions of $U, \mathcal{T}$, and R_0 through R_4, as well as the fact that $\mathbb{E}[v z_{\mathcal{T}}^i] = \mathbb{E}[\pi z_{\mathcal{T}}^i]$, we can rearrange to find that

$$\begin{aligned}
V(z_0^i, Z_0) &= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} \bar{\chi}_s \, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} -\gamma (z_s^i)^2 + \lambda T_\kappa^i(\hat{z}_s, Z_s) \, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0}^{\mathcal{T}} -\gamma (z_s^i)^2 \, ds + \int_{0}^{\mathcal{T}} T_\kappa^i(\hat{z}_s, Z_s) \, dN_s\right],
\end{aligned}$$

which completes the proof that the value function $V(z, Z)$ takes the form above. The same arguments used in Appendix C.3 go through (with these different α coefficients), so it must be that

$$\kappa_1(Z) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}.$$

and the equilibrium reports are optimal as long as

$$\kappa_2(Z) = \hat{a} + \hat{b}Z = -\bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2}.$$

Once again the equilibrium transfers are $(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right)$, so the coefficients R_m in

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

are given by

$$\begin{aligned}
R_0 &= 0 \\
R_1 &= -\frac{\alpha_1}{n} \\
R_2 &= -\frac{\alpha_5 + 2\alpha_3}{n^2} \\
R_3 &= \frac{\alpha_5 + 2\alpha_3}{n} \\
R_4 &= \alpha_1.
\end{aligned}$$

From the above, we have that

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda} \\
\alpha_5 &= \frac{1}{r+\lambda}(\lambda n R_3) \\
\alpha_4 &= \frac{1}{r}(\lambda\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2) \\
\alpha_1 &= \frac{1}{r+\lambda}(rv + \lambda R_4) \\
\alpha_2 &= \frac{1}{r}(\lambda\alpha_1 + \lambda n R_1).
\end{aligned}
$$

So, plugging in R_1, R_2, R_3, R_4, and rearranging, we have

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda} \\
\alpha_5 &= \frac{1}{r}(2\lambda\alpha_3) = \frac{2\lambda}{r}\left(\frac{-\gamma}{r+\lambda}\right) \\
\alpha_4 &= \frac{1}{r}(\lambda\alpha_5 + \lambda\alpha_3 - \lambda(\alpha_5 + 2\alpha_3)) = \frac{\lambda}{r}\left(\frac{\gamma}{r+\lambda}\right) \\
\alpha_1 &= \frac{1}{r}(rv) = v \\
\alpha_2 &= \frac{1}{r}(\lambda\alpha_1 - \lambda\alpha_1) = 0.
\end{aligned}
$$

With these choices for α_1 through α_5, and with

$$
\alpha_0^i = \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}\right),
$$

we can define the value function

$$
V(z^i, Z) = \alpha_0^i + \alpha_1 z^i + \alpha_2\bar{Z} + \alpha_3(z^i)^2 + \alpha_4\bar{Z}^2 + \alpha_5 z^i\bar{Z}.
$$

This value function solves the associated HJB equation

$$
\begin{aligned}
0 = &-\gamma(z^i)^2 + r(vz^i - V(z^i, Z)) + \frac{\sigma_i^2}{2}V_{zz}(z^i, Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i, Z) \\
&+ \sup_{\hat{z}^i}\{\lambda\left(V(z^i + Y^i((\hat{z}^i, \hat{z}^{-i})), Z) - V(z^i, Z) + T_\kappa^i((\hat{z}^i, \hat{z}^{-i}), Z)\right)\}.
\end{aligned}
$$

Plugging in $\alpha_1, \alpha_3, \alpha_5$, we have

$$
\kappa_1(Z) = v - \frac{2\gamma}{r}\bar{Z}
$$

and

$$\kappa_2(Z) = -\bar{Z} - \frac{v - \frac{2\gamma}{r}\bar{Z}}{2\kappa_0 n^2}.$$

The last part of the verification, demonstrating that alternative strategies do weakly worse, is exactly the same as in the proof Proposition 4 and thus omitted. Rearranging the coefficients α_0^i through α_5 above gives the expression in Proposition 6, completing the proof.

I Only Size Discovery: Unobservable Inventory Z_t

This appendix demonstrates that a version of our mechanism can achieve the first-best allocation in our dynamic setting, even when Z_t is unobserved, if the mechanism is run continuously and there is no exchange market. However, as we will show, it is not individually rational for participants to enter this mechanism. We only provide a sketch of this proof, since the technical details are similar to the proofs in the previous appendices.

We take the primitives of Section 3. Particularly, in the absence of any mechanisms, each trader i's excess inventory z_t^i evolves as

$$z_t^i = z_0^i + H_t^i,$$

for a Lévy process H_t^i which is a martingale with respect to the filtration $\mathbb{F}^i$ of information available to trader i. If trader i has an excess inventory process z^i, in the absence of mechanisms, their utility is given by

$$\mathbb{E}\left[z_{\mathcal{T}}^i \pi - \int_0^{\mathcal{T}} \gamma(z_t^i)^2\, dt\right],$$

where as in the paper we assume $\{\mathcal{T}, H, z_0, \pi\}$ are mutually independent.

As in the paper, trader i chooses an $\mathbb{F}^i$-adapted and jointly measurable process $\hat{z}^i$ of reports, though in light of ex-post optimality we pretend they may choose any $\mathbb{F}$-adapted process. However, we now assume the mechanisms are run continuously. If the vector of traders' reports is given by $\hat{z}$, then the excess inventory of each trader i now evolves as

$$z_t^i = z_0^i + Y^i(\hat{z}_t) + H_t^i, \tag{99}$$

where

$$Y^i(\hat{z}_t) = \frac{\sum_{j=1}^n \hat{z}_t^j}{n} - \hat{z}_t^i. \tag{100}$$

We assume that trader i is continuously compensated, where the flow payment is determined by a transfer function $\tilde{T}_\kappa^i(\hat{z}_t)$. Thus, each trader i takes the reporting strategies $\hat{z}_t^{-i}$ of the other traders as given, and chooses a report process $\tilde{z}$ to solve

$$V^i(z_0^i, Z) = \sup_{\tilde{z}} \mathbb{E}\left[z_{\mathcal{T}}^{\tilde{z}}\pi + \int_0^{\mathcal{T}} \tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma(z_t^{\tilde{z}})^2\, dt\right] \tag{101}$$

subject to

$$z_t^{\tilde{z}} = z_0^i + Y^i((\tilde{z}_t, \hat{z}_t^{-i})) + H_t^i. \tag{102}$$

We now simplify the problem. By conditioning on everything except π and applying the tower property, by independence we may rewrite the objective as

$$\sup_{\tilde{z}} \mathbb{E}\left[z_{\mathcal{T}}^{\tilde{z}} v + \int_0^{\mathcal{T}} \tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma(z_t^{\tilde{z}})^2 \, dt\right].$$

By conditioning on everything except $\mathcal{T}$ and applying the tower property, by independence we may rewrite this as

$$\sup_{\tilde{z}} \mathbb{E}\left[\int_0^\infty rve^{-ru} z_u^{\tilde{z}} \, du + \int_0^\infty re^{-ru} \int_0^u \tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma(z_t^{\tilde{z}})^2 \, dt \, du\right].$$

Applying a change of order of integration this is

$$\begin{aligned}
&\sup_{\tilde{z}} \ \mathbb{E}\left[\int_0^\infty rve^{-ru} z_u^{\tilde{z}} \, du + \int_0^\infty \left(\tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma(z_t^{\tilde{z}})^2\right) \int_t^\infty re^{-ru} \, du \, dt\right] \\
&= \sup_{\tilde{z}} \ \mathbb{E}\left[\int_0^\infty rve^{-ru} z_u^{\tilde{z}} \, du + \int_0^\infty \left(\tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma(z_t^{\tilde{z}})^2\right) \left(e^{-rt}\right) \, dt\right] \\
&= \sup_{\tilde{z}} \ \mathbb{E}\left[\int_0^\infty e^{-rt} \left(rvz_t^{\tilde{z}} + \tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma(z_t^{\tilde{z}})^2\right) \, dt\right].
\end{aligned}$$

Define $\nu_t \equiv z_0^i + H_t^i$ which does not depend on $\tilde{z}_t$. Then plugging in (102) to this new objective gives

$$\sup_{\tilde{z}} \ \mathbb{E}\left[\int_0^\infty e^{-rt} \left(rv[\nu_t + Y^i((\tilde{z}_t, \hat{z}_t^{-i}))] + \tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma([\nu_t + Y^i((\tilde{z}_t, \hat{z}_t^{-i}))])^2\right) \, dt\right].$$

By additivity, if $\tilde{z}_t$ solves

$$\sup_{\tilde{z}_t} \ e^{-rt} \left(rv[\nu_t + Y^i((\tilde{z}_t, \hat{z}_t^{-i}))] + \tilde{T}_\kappa^i((\tilde{z}_t, \hat{z}_t^{-i})) - \gamma([\nu_t + Y^i((\tilde{z}_t, \hat{z}_t^{-i}))])^2\right) \tag{103}$$

for each (ω, t), then it solves the dynamic optimization. As in the main text, we define

$$Z_t \equiv \sum_{j=1}^n (z_0^j + H_t^j)$$

and $\bar{Z}_t = Z_t/n$. Then we let

$$\begin{aligned}
V^i_{static}(z^i, Z) &= u^i_{static}(Z) + (\beta_0 + \beta_1 \bar{Z})(z^i - \bar{Z}) - K(z^i - \bar{Z})^2 \\
\beta_0 &= rv \\
\beta_1 &= -2\gamma \\
K &= \gamma \\
u^i_{static}(Z) &= rv\bar{Z} - \gamma \bar{Z}^2.
\end{aligned}$$

Because multiplying by e^{rt} does not change the optimization, by definition (103) is strategically equivalent to

$$\sup_{\tilde{z}_t} \; V^i_{static}(\nu_t + Y^i((\tilde{z}_t, \hat{z}_t^{-i})), Z_t) + \tilde{T}^i_\kappa((\tilde{z}_t, \hat{z}_t^{-i})).$$

Let

$$\begin{aligned}
\kappa_0 &\equiv -K(n-1)/n^2 = -\gamma(n-1)/n^2 \\
\kappa_1(Z) &\equiv \kappa_1
\end{aligned}$$

for any constant κ_1. Then it is immediate from Proposition 1 that if we define

$$\begin{aligned}
\kappa_2(Z) &= -\bar{Z} + \frac{-\kappa_1(Z) + (\frac{n-1}{n})(\beta_0 + \beta_1 \bar{Z})}{2\kappa_0 n} \\
&= \frac{-\kappa_1 + (\frac{n-1}{n}) rv}{2\kappa_0 n} \\
&= -\frac{-n\kappa_1 + (n-1)rv}{2\gamma(n-1)} = \kappa_2
\end{aligned}$$

and

$$\tilde{T}^i_\kappa(\hat{z}) = \kappa_1 \hat{z}^i + \kappa_0 \left(n\,\kappa_2 + \sum_{j=1}^n \hat{z}^j \right)^2 + \kappa_1 \kappa_2 + \frac{\kappa_1^2}{4\kappa_0 n^2},$$

then it is a strictly dominant strategy for each trader to report $\tilde{z}_t = \nu_t = z_0^i + H_t^i$. Further, just as in the main text of the paper, the sum of the transfers in each instant is weakly negative:

$$\begin{aligned}
\sum_{i=1}^n \tilde{T}^i_\kappa(\hat{z}) &= \kappa_1 \sum_{j=1}^n \hat{z}^j + n\kappa_0 \left(n\,\kappa_2 + \sum_{j=1}^n \hat{z}^j \right)^2 + n\kappa_1\kappa_2 + \frac{\kappa_1^2}{4\kappa_0 n} \\
&= \frac{1}{4\kappa_0 n} \left(\kappa_1 + 2\kappa_0 n \left(n\kappa_2 + \sum_{j=1}^n \hat{z}^j \right) \right)^2
\end{aligned}$$

and in equilibrium, each trader has excess inventory

$$\begin{aligned}
z_t^i &= z_0^i + Y^i(\hat{z}_t) + H_t^i \\
&= z_0^i + \frac{\sum_{j=1}^n \hat{z}_t^j}{n} - \hat{z}_t^i + H_t^i \\
&= z_0^i + \frac{\sum_{j=1}^n (z_0^j + H_t^j)}{n} - (z_0^i + H_t^i) + H_t^i \\
&= \bar{Z}_t
\end{aligned}$$

almost everywhere. We have thus shown that the continuously run mechanisms achieve the first-best allocation while remaining budget balanced.

We now show that participation in this mechanism is not individually rational. Note that at the equilibrium strategy, trader i's expected payoff is

$$\mathbb{E}\left[\int_0^\infty e^{-rt}\left(rv\bar{Z}_t + \tilde{T}_\kappa^i(\hat{z}_t) - \gamma(\bar{Z}_t)^2\right) dt\right],$$

where, since $\sum_{j=1}^n \hat{z}_t^j = Z_t$, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-rt}\tilde{T}_\kappa^i(\hat{z}_t)\, dt\right] &= \mathbb{E}\left[e^{-rt}\left(\kappa_1 \hat{z}_t^i + \kappa_0\left(n\,\kappa_2 + Z_t\right)^2 + \kappa_1\kappa_2 + \frac{\kappa_1^2}{4\kappa_0 n^2}\right) dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}\kappa_1(z_0^i + H_t^i)\, dt + \int_0^\infty e^{-rt}\kappa_0\left(n\,\kappa_2 + Z_t\right)^2\, dt\right] \\
&\quad + \frac{1}{r}\left[\kappa_1\kappa_2 + \frac{\kappa_1^2}{4\kappa_0 n^2}\right].
\end{aligned}$$

Because H^i is a martingale, this equals

$$\frac{\kappa_1 z_0^i}{r} + \frac{2\kappa_0 n\kappa_2 Z_0}{r} + \mathbb{E}\left[\int_0^\infty e^{-rt}\kappa_0 Z_t^2\, dt\right] + \frac{1}{r}\left[\kappa_1\kappa_2 + \frac{\kappa_1^2}{4\kappa_0 n^2}\right].$$

Thus the expected total profit of trader i is

$$\frac{\kappa_1 z_0^i}{r} + Z_0\iota_0 + \frac{n^2\kappa_0 - \gamma}{rn^2} Z_0^2 + \iota_1,$$

for constants ι_0, ι_1. If trader i could completely exit the mechanism, the associated expected payoff would be

$$\mathbb{E}\left[z_{\mathcal{T}}^i \pi - \int_0^{\mathcal{T}} \gamma(z_t^i)^2\, dt\right] = v z_0^i - \frac{\gamma}{r}(z_0^i)^2 + \iota_2,$$

for a constant ι_2. From this, it is clear that trader i might prefer to not enter the mechanism (for example, if $z_0^i = 0$ and Z_0 is expected to be very large) rather than participate.