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ABSTRACT

Ambiguity aversion alone does not explain the market nonparticipation puzzle. We show that in a rational expectations equilibrium model with a fund offering the risk-adjusted market portfolio (RAMP), ambiguity averse investors hold the fund and an information-based portfolio, and thus participate in all asset markets, directly or indirectly. This result follows from a new separation theorem which states that an investor's equilibrium portfolio can be decomposed into components, each matching the optimal portfolio based on only one information source (price versus private signal). Asset risk premia satisfy the CAPM with the fund as the pricing portfolio.

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1 Introduction

Nonparticipation in domestic public equity markets and home bias in international investing are two of the main stylized facts of household finance. With respect to nonparticipation, a large fraction of investors hold no equity.[1] As for home bias, investors very often fail to participate in foreign stock markets despite the benefits to diversification and international risk sharing (French and Poterba 1991; Tesar and Werner 1995; Lewis 1999; Cooper, Sercu, and Vanpée 2013).

These puzzles of nonparticipation raise a challenge to frictionless optimal portfolio theories (Campbell 2006). Furthermore, frictions such as information asymmetry and transaction costs do not seem to fully explain the puzzle.[2]

To address this puzzle, a major strand of research proposes that nonparticipation derives from *ambiguity aversion*. In this explanation, investors do not know perfectly certain parameters of the distribution of assets payoffs—they face *model uncertainty*. In making investment decisions, ambiguity averse investors place heavy weight upon worst-case scenarios for these parameters. For example, several papers model financial markets in which some investors are subject to model uncertainty and have multiple-priors utility functions, and provide conditions under which investors do not hold certain assets or asset classes.[3]

This literature focuses on how ambiguity aversion affects direct security holdings, and therefore does not address whether introducing a fund might ameliorate the problem. In particular, a fund run by a money manager who knows all the parameters of the economy could commit to offering a portfolio that is a function of those parameters to try to induce investors to participate in all assets markets indirectly. We view such a fund as *passive* if it does not possess any private information about asset returns. Whether such a passive fund can actually induce participation is not obvious, because

[1] Only a minority of relatively well-off individuals—those with $100, 000 in liquid assets—participate in the equity market (Mankiw and Zeldes 1991), and poorer investors tend to participate even less. Similarly, almost 20% of households at the 80^{th} percentile of the wealth distribution own no public equity (Campbell 2006). According to more recent evidence in the 2013 Survey of Consumer Finances, less than 15% of U.S. households report owning stocks directly, and only about 50% of households own stocks either directly or indirectly through mutual funds or retirement accounts (Bricker et al. (2014)).

[2] Information asymmetry can explain why investors underweight certain assets, but not zero positions (Van Nieuwerburgh and Veldkamp 2009). Moderate transaction costs of participation also do not seem to explain why even wealthy households would fail to participate; as discussed by Gouskova, Juster, and Stafford (2004), they are not "the major consideration."

[3] See Bossaerts et al. (2010), Cao, Wang, and Zhang (2005), Easley and O'Hara (2009), Easley and O'Hara (2010), Epstein and Schneider (2010), and Cao, Han, Hirshleifer, and Zhang (2011).

an ambiguity averse investor may view such a fund as extremely risky. Furthermore, how risky such a fund is perceived to be depends on the investing strategies of other investors. Hence, only an analysis of market equilibrium can determine whether such a fund can bring about full participation.

It has recently been argued that a key role for investment advisors and managers is to gain investor trust, reducing their anxiety about participation in the stock market. Such trust in "money doctors" can benefit investors by encouraging them to participate in the stock market (Gennaioli, Shleifer, and Vishny 2015). However, previous analysis of this issue takes as exogenously given that a persuasive advisor can induce trust and participation. Allowing for investor ambiguity aversion and asymmetric information provides a formal way of endogenizing and analyzing explicitly the sources of investor anxiety about the stock market, and whether the presence of reputable investment funds run by sophisticated managers will actually allay investor anxiety enough to induce participation.

Since investors can directly trade all assets, the fund might seem like a redundant asset to investors. However, the investor does not have the information needed to replicates the fund's portfolio ex ante. So the question of how investors who do not know the ambiguous parameters form optimal portfolios that include the fund and other assets is economically meaningful and determinate. Further interesting questions are how the investor's degree of knowledge about the ambiguous parameter of the economy, or the investor's private information, affects the optimal holdings in the fund; and how the existence of the fund affects asset risk premia.

We examine these questions in a rational expectations equilibrium setting modified to allow for model uncertainty and ambiguity aversion. Investors differ in the supports of their beliefs about the exogenous parameters of the market, and hence hold different beliefs about the fund's portfolio composition. Nevertheless, we identify a portfolio strategy (a contingent plan to choose holdings as a function of the model parameters), such that if the passive fund commits to following this strategy, all investors hold identical positions in the fund as a common component of their portfolios. This portfolio differs from the value-weighted market portfolio, but includes positive positions in all traded assets. Hence, all investors participate in all assets' markets directly or via the fund. In consequence, in our setting—a fairly standard rational expectations setting supplemented with ambiguity aversion and with the ability of agents to offer funds—ambiguity aversion does not explain the nonparticipation puzzle. In developing our analysis, we also provide a new separation theorem for optimal security holdings under asymmetric information, and a version of the CAPM that holds under ambiguity aversion and asymmetric information.[4]

Specifically, in our setting there is a continuum of investors with strictly positive endowments of all risky assets. Investors hold a common uninformative prior about risky asset payoffs, so based upon the prior, holding non-zero positions of the assets is infinitely risky. Security endowments are subject to random supply shocks. Prices are set to clear the markets for all assets. For each asset, investors are divided into two groups. Members of one group receive conditionally independent private signals about the asset payoff and know the precision of the supply shock. Members of the other group neither receive any private signals about the asset payoff, nor know the precision of the supply shock.[5] In particular, the uninformed investors' subjective belief about the precision of the supply shock includes the possibility of precisions that are arbitrarily close to zero. As a result, since the uninformed investors cannot extract information about the asset payoff from its price when the precision of the supply shock is arbitrarily close to zero, they may perceive the assets to be extremely risky.

Ambiguity averse investors choose optimal portfolio to maximize expected utility

[4]This separation theorem is the reason that delegation to a fund induces participation in our model. In general an individual will be willing to delegate to an agent who credibly commits to a contingent strategy (based on the agent's knowledge) of doing whatever the individual himself would have done based on that knowledge. But it does not follow that the nonparticipation problem is cured by delegating to a fund whose manager knows the model parameters. The problem is that different potential investors in the fund have different information sets (as well as different degrees of ignorance about model parameters). So in general (without imposing the equilibrium condition, and without our portfolio information separation theorem) there is no guarantee that a single fund (or fund of funds) can persuade different investors that the fund is offering part or all of what each investor would have chosen herself (knowing what the fund knows). (In the model, in equilibrium the fund does not offer each investor the overall portfolio that she would want to hold, only a key component of such a portfolio.) It is only in equilibrium, and by virtue of our new separation theorem, that we can conclude that there is a single fund (which we characterize) which offers a key common component of the portfolios that the diverse investors themselves would have chosen if they knew what the fund knows.

It might further be suggested that even without imposing the equilibrium condition, the problem of nonparticipation could be solved by offering a separate tailored fund to each investor consisting of the portfolio that this investor would have chosen if the investor knew the ambiguous model parameters. However, even if this were a relevant option in practice, this approach does not solve the problem because it is not clear exactly what portfolios such funds should offer.

[5]Most of the literature considers ambiguity aversion about asset payoffs, with the exception of Watanabe (2016), who assumes that investors are ambiguous about the mean of the asset's random supply shock. For tractability, we similarly assume that investors are ambiguous about precisions of assets' supply shocks. However, our main result that all investors in equilibrium hold the passive fund holds even when investors are ambiguous about other asset market parameters.

under worst-case assumptions for the values of the supply volatility parameters that they are uncertain about. So for any portfolio contemplated by an investor, expected utility is calculated contingent on the unknown parameters having values that minimize traditional CARA expected utility (following the max-min expected utility proposed by Gilboa and Schmeidler (1989)).

There is a passive fund whose manager knows the supply shock precisions of all assets. The assumption that fund managers observe parameters that individual investors do not is based on the idea that fund managers are professionals who understand financial markets.[6] The fund offers all investors a single portfolio which is a deterministic function of the exogenous parameters, including the supply volatilities of all the assets. Though investors who face model uncertainty do not know the exact weights of the portfolio, the function used for constructing the portfolio is common knowledge. We refer to the portfolio offered as 'the fund.'

The key intuition derives from a new separation theorem which applies in the setting with no model uncertainty. In this setting, there is a rational expectations equilibrium in which any investor's equilibrium risky asset holding can be decomposed into two components. The first is a common deterministic component that plays a role in our model somewhat similar to the market portfolio in the CAPM, but is distinct from the endowed market portfolio. We call this component the Risk-Adjusted Market Portfolio (*RAMP*). The second is the investor's *information-based portfolio*, which includes a non-zero position in an asset if and only if she receives a private signal about the asset.

This new separation theorem differs from the separation theorem derived in the literature in that any individual investor's optimal portfolio is separated by the different parts of her *information set*. We therefore call it the *Information Separation Theorem*. Specifically, *RAMP* is constructed based only on the information extracted from asset prices, and is independent of the investor's private signal. So another name for it is the learning-from-price-based portfolio. Obversely, the information-based portfolio depends only on the information derived from the private signal; it is independent of the information extracted from asset prices.

This separation derives from the model assumptions of CARA utilities and normal random variables. As is standard in such settings, any investor's optimal position in

[6]In particular, we assume that for each asset, there is a positive measure of investors who know the precision of its random supply shock. Hence, the knowledge needed to construct an appropriate fund is available in the economy. In Section 6, we discuss how the passive fund can be implemented by a fund of funds.

each asset is proportional to the product of her information precision about the asset payoff and the difference between the conditional expectation of the asset payoff (given her information) and the asset price. The precision of the investor's overall information about an asset is the sum of the precisions of the price signal and her private signal. Because of normality, her conditional expectation of the asset payoff is the average of the conditional expectations based on each of her two signals, weighted by the signal precisions. Hence, the investor's optimal position is the sum of two components. Each is the product of the precision of one signal and the difference between the conditional expectation of the asset payoff, based on only this signal, and the asset price. *RAPM* is the investor's optimal portfolio based only on the price signal, and her information-based portfolio is her optimal portfolio based only on her private signal.

The Information Separation Theorem provides new insight into how ambiguity averse investors will participate in asset markets when there is model uncertainty. Consider again a setting where investors are subject to model uncertainty and a passive fund offers *RAMP*. One share of the passive fund represents one unit of *RAMP*. Consider the following proposed strategy profile: each investor holds exactly one share of the fund, and additionally holds her investor-specific information-based portfolio (which could be a nullity). Given that all other investors behave as prescribed, no investor has an incentive to deviate.

The key insight is that the fund uses its knowledge to do precisely what each investor would choose in her non-information-based portfolio if she knew what the fund knows. Consider, for example, an investor named Lucy and a vector of precisions of assets' supply shocks that is possible according to her subjective belief. Given the value of this vector, she would be in a possible world without model uncertainty. In such a possible world, since all other investors are holding exactly one share of the passive fund and their own information-based portfolios, they are holding the same portfolios as they would in the rational expectations equilibrium in this world. Hence, the market clearing condition implies that the pricing function is the same as the one in the rational expectations equilibrium. Therefore, if Lucy knew the parameter values that characterize this possible world, her optimal portfolio choice would consist of *RAMP* and her own information-based portfolio.

By holding the passive fund and her own information-based portfolio, Lucy implements exactly such an investment strategy in every possible world. In each possible world, *RAMP*, and therefore the composition of the passive fund, differ. But Lucy's information-based portfolio does not. So even when Lucy is ambiguous about some or all assets, her optimal portfolio choice is to hold exactly one share of the passive fund together with her own information-based portfolio.

In the above argument, given other investors' strategies, we first fix a possible world, and then calculate Lucy's optimal investment strategy. Hence, we are implicitly assuming that Lucy has a min-max utility. However, since Lucy's optimal investment strategy, holding one unit of the fund and her own information-based portfolio, is constant across all possible worlds in her belief support, her min-max utility is the same as her max-min utility. Put differently, a strong min-max property holds in the equilibrium, and thus holding one unit of the fund and her own information-based portfolio is also the optimal investment strategy with her max-min utility.

This argument for the optimality of investing in the fund (if all other investors behave according to the proposed equilibrium) is a powerful one, as it requires only that an ambiguity averse investor's investment strategy be time-consistent. Imagine that Lucy could pay a fee to learn the precise values of all parameters that she does not know. This would eliminate her ambiguity aversion, so she would subsequently trade as an uninformed investor who is not subject to model uncertainty. The fund is doing just what Lucy herself would do if she were to pay the fee. So by time consistency, given equilibrium behavior of all other investors, Lucy would strictly prefer holding the fund over paying the fee, however small, to learn the parameters. In other words, delegation can replace information acquisition as a means of addressing ambiguity aversion.[7]

The presence of the fund affects asset risk premia. The fact that an investor with no private information optimally holds the passive fund implies that it is mean-variance efficient, so that the CAPM security market line holds with *RAMP* as the pricing portfolio. Because the pricing portfolio does not depend on the realization of the random supply shock, and the weight of each asset in the pricing portfolio are conditional on asset prices, which are publicly observable, the portfolio is potentially observed by an econometrician. This makes the model empirically testable.

Lucy's willingness to hold the passive fund is an equilibrium outcome; the reasoning relies on her conjecture about the willingness of all other investors to hold the fund and their own information-based portfolios. Otherwise, to consider an off-equilibrium

[7] Mele and Sangiorgi (2015) analyze a model in which ambiguity averse investors can acquire costly information about model parameters. They show that, without a fund, investors have strong incentives to do so.

possibility, if other investors' trading were to lead to asset prices that are almost uninformative, Lucy would not hold the passive fund, because *RAMP* would be extremely risky to her in this case. We consider such a scenario in detail in Subsection 4.1.

This setting endogenizes investor trust in fund managers (Gennaioli, Shleifer, and Vishny 2015). One subtle point that this approach reveals is that inducing investors to participate requires more than investor trust in the honesty and superior knowledge of fund managers about the financial market. It is crucial that investors foresee an equilibrium in which other investors also trust the fund managers and trade accordingly: as we discuss in Subsection 4.1, off the equilibrium, it is possible that an investor is not willing to hold the fund, even if she trusts the fund manager.

The equilibrium argument also highlights the fact that investors hold *RAMP* for risk-sharing rather than mere individual-level diversification reasons, which would apply even without analysis of market clearing. As further indication that diversification incentives are not at the heart of the result, we show that an equilibrium with 'fund' delegation exists even in the special case in which only one risky asset is traded in the market.[8]

It may seem surprising that all investors take the same position in the passive fund, even though their beliefs about the precisions of supply shocks and thus the fund composition have different supports. Investors with different priors have different worst-case scenarios, and therefore differ in how risky they view the fund. However, owing to the Information Separation Theorem, in equilibrium all investors agree that it is good to delegate their non-information-based investments to a fund that has access to the true values of the supply volatilities. *RAMP* is based upon those actual values.

Overall, these findings suggest that when an appropriate passive fund is available, investors' ambiguity aversion alone does not explain nonparticipation. Since in fact there is limited participation, this means that our model is not an accurate description of reality. We view the primary contributions of this model as four-fold. First is providing conceptual clarification that ambiguity aversion does not, in a fairly standard setting, explain the nonparticipation puzzle unless there is also a failure of the market to offer an appropriate passive mutual fund. Second, when the fund is available, there exists an equilibrium with several strong and interesting properties. For example, a version of CAPM holds with the fund being the pricing portfolio. Third is the Information Separation Theorem. Fourth is offering normative implications, about the possible ben-

[8]We consider a multi-asset setting in order to derive cross-sectional asset pricing implications.

efits of investor education and of funds offering *RAMP*. In particular, the index fund industry has been growing rapidly in recent decades, yet the nonparticipation puzzle remains. So the model raises the possibility that innovation in the fund industry, and in particular the introduction of funds that offer *RAMP*, can benefit investors.

2 A Model with Investor Ambiguity Aversion

There are two dates, date 0 and date 1. The economy is populated by a continuum of investors with measure one, who are indexed by i and uniformly distributed over $[0, 1]$. All investors trade at date 0 and consume at date 1. Any investor i invests in a riskfree asset and $N \geq 2$ independent risky assets by herself. (In this section, we assume that the number N is common knowledge.) The riskfree asset pays r units, and risky asset n pays f_n units of the single consumption good. Taking the riskfree asset to be the numeraire, let P be the price vector of the risky assets and D_i be the vector of shares of the risky assets held by investor i. Investor i can hold a passive fund that commits to offering a portfolio X, which is an N-dimension column vector with the n^{th} element being the shares of the n^{th} risky asset in X. Then, by holding d_i (a scalar) shares of the fund, investor i effectively holds the portfolio $d_i X$. Therefore, an investor i's risky assets holdings are $d_i X + D_i$.

Let $W_i = (w_{i1}, w_{i2}, \ldots, w_{iN})'$ be the endowed shareholdings of investor i, and let $W = \int_0^1 W_i \mathrm{d}i >> 0$ be the aggregate endowments of shares in the capital market. So any investor i's final wealth at date 1 is

$$\Pi_i = r \left[W_i' - \left(d_i X' + D_i' \right) \right] P + \left(d_i X' + D_i' \right) F, \tag{1}$$

where $F = (f_1, f_2, \ldots, f_N)'$. The first term in (1) is the return of investor i's investment in the riskfree asset, and the second term is the total return from her investments in risky assets.

We assume that all investors share a common uniform improper prior of F, and so no investor has prior information about any risky asset's payoff. Hence, any investor i's information consists of the equilibrium price vector and the realization of a private information signal S_i only. In particular, $S_i = F + \epsilon_i$, where F and ϵ_i are independent; and ϵ_i and ϵ_j are also independent. Each ϵ_i is normally distributed, with mean zero and precision matrix $\mathbf{\Omega}_i$. We assume that $\mathbf{\Omega}_i$ is diagonal for all $i \in [0, 1]$, and so investor i's private signal about asset n's payoff is uninformative about asset k's payoff.

We call investor i an informed investor of asset n if and only if the n^{th} diagonal entry of $\mathbf{\Omega}_i$ is strictly positive. Let the n-dimension column vector λ summarize the measures of informed investors of each asset, with the n^{th} element being the measure of the informed investors of asset n; we assume $\lambda_n \in (0,1)$. Let $\text{Diag}(\lambda)$ be the diagonal matrix with the n^{th} diagonal entry being the n^{th} element of λ. We assume that any investor i is uninformed about at least one asset. We call an investor i with precision matrix $\mathbf{\Omega}_i = 0$ an uninformed investor. We assume that $\gamma \in (0,1)$ fraction of investors are uninformed, so $\gamma \leq \min_j \left(1 - \lambda_j\right)$.

For simplicity, we assume that the private signals of all informed investors of asset n have the same precision $\kappa_n > 0$. Let $\mathbf{\Omega}$ be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being κ_n. Denote by $\mathbf{\Sigma}$ the matrix of the average precision of private signals, we have

$$\mathbf{\Sigma} = \int_0^1 \mathbf{\Omega}_i \mathrm{d}i = \mathbf{\Omega}\text{Diag}(\lambda) \tag{2}$$

As is standard, the independence of the errors implies that in the economy as a whole signal errors average out, so that the equilibrium pricing function does not depend on the error realizations (though it does depend on their distributions).

Let Z denote the random vector of supplies of all risky assets. We assume that Z is independent of F and of ϵ_i (for all $i \in [0,1]$). We further assume that Z is normally distributed with mean 0 and the precision matrix $\boldsymbol{U}$. By independence of assets, $\boldsymbol{U}$ is diagonal and positive definite, with the n^{th} diagonal entry being τ_n.

Investors commonly know all parameters except $\boldsymbol{U}$. Specifically, we assume that only informed investors of asset n know τ_n; any uninformed investor i of asset n will have her own subjective prior belief about τ_n with the support $\left(\underline{\tau}_n^i, \bar{\tau}_n^i\right)$, where $\bar{\tau}_n^i > \underline{\tau}_n^i \geq 0$. Denote by $\mathcal{U}_i$ the set of investor i's belief about $\boldsymbol{U}$, and by $\boldsymbol{U}_i$ a typical element in $\mathcal{U}_i$. We allow different uninformed investors of a particular asset n to have different supports of their beliefs about τ_n. To establish the benchmark, we assume that for any uninformed investor i, $\underline{\tau}_n^i = 0$ for all n; that is, any uninformed investor believes that any asset's supply shock could be extremely volatile.

A passive fund knows all the model parameters but does not observe any signals about assets' payoffs. The fund offers the portfolio X below to investors:

$$X = \left[I + \frac{1}{\rho^2}\left(\mathbf{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W, \tag{3}$$

where ρ is investors' common risk tolerance coefficient. Importantly, the portfolio X

does not include any signals of assets' payoffs. In addition, one share of the fund represents the effective asset holding X, and so for any given price vector P, one share of the fund is sold at the price $X'P$. While investors know that conditional on a $\boldsymbol{U}$, the portfolio offered by the passive fund is X, they do not know the true composition of X, unless they are informed about all assets.

All investors are risk averse, so when all model parameters are common knowledge, at date 0 their expected utility is CARA,

$$\mathbb{E}_i u(\Pi_i) = \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{4}$$

However, investor i may be subject to model uncertainty about the precisions of some assets' random supplies, and will choose an investment strategy (d_i, D_i) to maximize the infimum of her CARA utility. Formally, each investor i's decision problem is[9]

$$\max_{d_i, D_i} \inf_{\boldsymbol{U}_i \in \mathcal{U}_i} \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{5}$$

There are other utility representations of investors' ambiguity aversion preferences, such as the smooth ambiguity-aversion preference proposed by Klibanoff, Marinacci, and Mukerji (2005). Our main results also hold if we adopt the smooth ambiguity-aversion preference.

We are interested in a rational expectations equilibrium defined as follows.

Definition 1 *A pricing vector P^* and a profile of all investors' risky assets holdings $\{d_i^*, D_i^*\}_{i \in [0,1]}$ constitute a rational expectations equilibrium, if*

1. *Given P^*, (d_i^*, D_i^*) solves investor i's maximization problem in equation (5), for all $i \in [0,1]$; and*

2. *P^* clears the market, that is,*

$$\int_0^1 (d_i^* X + D_i^*)\, \mathrm{d}i = W + Z, \quad \text{for any realizations of } F \text{ and } Z. \tag{6}$$

[9] An investor's utility in this paper differs slightly from that defined in Gilboa and Schmeidler (1989). Because any investor's subjective prior about the precision of the random supply shock has a non-compact support, the investor maximizes the infimum, rather than the minimum, of her CARA utility among all possible precisions.

As in the literature on rational expectations models, equilibrium prices play two roles: clearing the market and aggregating private information. Hence, a rational expectations equilibrium differs from a Walrasian equilibrium mainly in that the asset prices convey information about the asset payoffs to investors. This is especially important in our setting with ambiguity averse investors. Since they hold neither informative priors nor private signals about the asset payoffs, observation of asset prices is what allows them to update their beliefs to have finite conditional variances, making them willing to participate in the markets for risky assets. When the precision of the supply shock of an asset is arbitrarily close to zero, the asset price becomes uninformative, and so ambiguity averse investors will hold a zero position of the asset.

3 Benchmark: No Passive Fund

We next establish a benchmark for comparison by studying a model without the passive fund. In such a setting, any investor i's investment strategies are constrained by $d_i = 0$. Since all asset realizations are independent, we can first focus on investor i's decision whether to hold an asset n that she is uninformed about.

Investor i is risk averse, so she will not hold any non-zero position of asset n, unless the distribution of asset n's payoff has a finite variance, conditional on her information. Investor i, however, has neither prior information nor private information about asset n's payoff. Hence, she estimates the payoff based only on the price, which partially aggregates informed investors' private information. The informativeness of price increases in the precision of the supply shock. When the supply shock has a zero precision, price becomes completely uninformative.

Investor i does not know the precision of asset n's supply shock. By assumption, investor i's subjective prior belief about τ_n has the support $\left(0, \bar{\tau}_n^i\right)$. Investor i may extract some information about τ_n from the price of asset n. However, because all random variables in our model are normally distributed, the support of investor i's belief about τ_n does not change. So as she considers the worst-case scenario in making the investment decision, investor i focuses on the possibility that the true τ_n is very close to 0, since in such a case, asset n's price is almost uninformative.

Suppose that investor i holds a non-zero position of asset n. As the price becomes almost uninformative, the payoff variance conditional upon price diverges to infinity. So holding a non-zero position is extremely risky in the worst-case scenario. To avoid this risk, investor i optimally chooses a zero position. Proposition 1 below summarizes the argument above.

Proposition 1 *If an investor i is uninformed about asset n, and $\underline{\tau}_n^i = 0$, then investor i will hold a zero position of asset n.*

Since all asset realizations are independent, investors can evaluate assets' conditional (on prices) expected return and variance one by one. Then, because an uninformed investor i has $\underline{\tau}_n^i = 0$ for all n, her belief about any asset's payoff has potentially extremely large conditional variance. Therefore, given any $D_i \neq 0$, the infimum of investor i's utility will be $-\infty$; so, $D_i \neq 0$ is strictly dominated by $D_i = 0$. That is, any uninformed investor i refrains from participating in any asset market. Since there are γ measure of uninformed investors, Corollary 1 below shows that limited participation presents in this benchmark model without a passive fund, consistent with the prediction in the literature.

Corollary 1 *In the model without a passive fund, there are γ measure of investors who do not participate in risky assets' markets.*

In addition, Proposition 1 also implies that the value-weighted market portfolio currently offered by the index funds does not resolve the limited participation problem in this setting. Indeed, the value-weighted market portfolio in our model is equivalent to the portfolio W. Since the only way in which the ambiguity averse investors participate in the assets markets is to hold the fund, their tradings change the total supplies of the assets proportionally. Such an effect will not change the price informativeness, which is still determined by the precisions of supply shocks. Then, when the precisions of supply shocks are arbitrarily close to zero, it follows from Proposition 1 that ambiguity averse investors will not hold the index fund.

Corollary 2 *The existing index funds that offer the value-weighted market portfolio W cannot encourage investors to participate in the markets of the assets they are ambiguous about.*

4 Introducing a Passive Fund Results in Full Participation

In this section, we show that in equilibrium an appropriate passive fund induces all investors to participate in all asset markets. We also show how investors will allocate their initial wealth among the fund, their direct holdings of risky assets, and the riskfree asset, even when investors do not know the exact composition of the passive fund. We then argue that the full participation with a passive fund follows from the Information Separation Theorem that applies in financial markets without the passive fund and without ambiguity aversion.

4.1 An Equilibrium with Full Participation

In the model with a passive fund that offers the portfolio X, any investor i's investment strategy (d_i, D_i) leads to effective asset holdings $d_i X + D_i$. If investor i is uninformed, she does not know $\boldsymbol{U}$ and hence does not know the exact composition of X. However, all investors commonly know X as a function of $\boldsymbol{U}$ that is specified in equation (3).

The main result of our paper is presented in Proposition 2 below, which shows that in an equilibrium, all investors hold exactly one share of the passive fund and thus participate in all asset markets. Hence, with an appropriately constructed passive fund, even with ambiguity aversion, there is full (though intermediated) participation.

Proposition 2 *In the model with a passive fund that commits to offering the portfolio X specified in equation* (3)*, there is an equilibrium in which*

1. *All investors will buy one share of the passive fund, and so $d_i^* = 1$ for all $i \in [0, 1]$;*
2. *Any investor i will hold an extra portfolio $\rho \boldsymbol{\Omega}_i (S_i - rP)$; and*
3. *For any given F and Z, the equilibrium price is*

$$P = \frac{1}{r}\left[F - \frac{1}{\rho}\left(\boldsymbol{\Sigma} + \rho^2 \boldsymbol{\Sigma U \Sigma}\right)^{-1} W - \frac{1}{\rho}\boldsymbol{\Sigma}^{-1} Z\right]. \tag{7}$$

The intuition of Proposition 2 arises from a new separation theorem that applies in the setting without ambiguity aversion and the passive fund. Since such an intuition is not straightforward, we discuss it in detail in Section 4.2. In the rest of this subsection, we discuss some properties of the equilibrium characterized in Proposition 2.

First, in equilibrium, uninformed investors are indifferent between holding the passive fund and not participating in asset markets. When contemplating a position in the passive fund, uninformed investors believe that the fund's holdings of all assets are very close to zero, when the precisions of all assets' supply shocks are almost zero. Hence, by holding the fund, the infima of the uninformed investors' utilities are the same as the utility from not participating.

Nevertheless, the only reasonable conclusion is that an uninformed investor who is ambiguity averse and otherwise-rational holds the passive fund (when other investors follow equilibrium behavior). In particular, since any uninformed investor i's subjective prior about the precision of any asset n's supply shock is $(0, \bar{\tau}_n^i)$, she knows for sure that $\tau_n > 0$. For any given τ_n, holding the passive fund is strictly better than not participating. So while investor i has the same (infimum) utility ex ante, once τ_n is realized, she knows she will be *strictly* better off holding the passive fund. So only holding the passive fund is time-consistent.

Given this, it is not surprising that there are ways to express preferences that capture formally the fact that an investor is not indifferent, even ex ante, as to whether to invest in the fund. This can be done by considering perturbations of the model. Consider a sequence of perturbed models in which all uninformed investors' priors about $\boldsymbol{U}$ have strictly positive lower bounds. When the perturbed lower bounds converge to zero, the perturbed models converge to our original model. In any of these perturbed models, strictly positive lower bounds of investors' priors about $\boldsymbol{U}$ imply that holding the passive fund is investor i's unique best response to other investors' strategies in an equilibrium, as shown in the proof of Proposition 2. Hence, when investor i's prior knowledge about model parameters switches a little bit, investor i strictly prefers to hold the passive fund; then, by the revealed preference, investors would like to choose the passive fund in the original model, given all other investors' strategies. Therefore, the equilibrium characterized in Proposition 2 is *near strict*, an equilibrium refinement concept defined by Fudenberg, Kreps, and Levine (1988).[10]

Second, while investors have heterogeneous priors about $\boldsymbol{U}$ and thus different beliefs about the fund's composition, they all hold exactly one share of the fund. Take two investors, Lucy and Martin, for an example. Lucy is uninformed and believes that τ_n (for any n) could be arbitrarily close to 0; Martin does not receive private information about asset payoffs either, but he knows the true precisions of all supply shocks. According to Proposition 2, both Lucy and Martin will hold one share of the passive fund, but neither Lucy nor Martin holds any extra positions because they don't have any pri-

[10]Formally, a strategy profile σ is *near strict* in a game Γ if there exists a sequence of games $\{\Gamma^n\}$ and a sequence of strategy profiles $\{\sigma^n\}$, such that (i) $\lim_n \Gamma^n = \Gamma$; (ii) for each n, σ^n is a strict equilibrium of Γ^n; and (iii) $\lim_n \sigma^n = \sigma$. Here, a *strict equilibrium* is an equilibrium in which any investor's strategy is her unique best response to all other investors' strategies in the equilibrium.

vate information about assets' payoffs. Hence, Lucy and Martin are effectively holding the same portfolio. So differences in investors' holdings arise only from differences in their information signals, not from differences in their model uncertainty or ambiguity aversion.

Third, Proposition 2 shows the importance of risk sharing among investors in their optimal portfolio choices. Specifically, consider an investor who faces model uncertainty about a subset of traded assets, and views the return distributions as exogenous. Even if she can indirectly trade those assets through a passive fund, it may not be optimal for her to do so, because she cannot calculate the fund's expected return and risk. Therefore, arguments based on the incentive of individuals to diversify do not, under radical ignorance, justify holding of the fund. In contrast, in our equilibrium setting, an investor optimally holds the fund, given her belief that other investors will also do so (together with their direct portfolios). Hence, she is willing to hold the fund too, which achieves the benefit of optimally sharing risk with other investors.

The fact that equilibrium rather than just diversification considerations are crucial for the full participation result can be seen more concretely by considering the off-equilibrium possibility that other investors trade in a fashion that causes asset prices to be almost uninformative. In such a scenario, an ambiguity averse investor (Lucy) would not hold the passive fund, because *RAMP* would be perceived as extremely risky. Specifically, suppose that the off-equilibrium trading strategy profile of other investors leads asset price informativeness to converge to zero as the random supply shock precisions go to zero. This convergence could be even faster than the convergence of the asset positions in *RAMP* to zero. Hence, taking any non-zero position of the passive fund will give Lucy infinite risks in the worst-case scenario, since she believes that random supply shock precisions could be extremely close to zero. Therefore, Lucy will not hold the fund. In contrast, in such a case, an uninformed investor who knows the supply shock precisions may still hold asset positions that are bounded away from zero, because the investor can extract asset payoff information from asset prices, resulting in finite risk.

Proposition 2 more broadly suggests that the reason why actual investors often fail to diversify goes beyond investor ambiguity aversion. In particular, for an investor to hold the fund, all other investors need to behave according to the prescribed equilibrium strategy profile. If imperfectly rational investors reason about possible portfolios based solely on partial equilibrium risk and return arguments, portfolios containing assets that investors are ambiguous about might seem extremely risky (or in the limiting case, infinitely risky). Proposition 2 shows that, owing to equilibrium considerations, even ambiguity averse investors, if otherwise rational, will hold such assets. But actual investors may not understand the equilibrium reasoning which underlies this result.

Instructors in finance know that it is hard for students (or even experts), to keep in mind equilibrium considerations. This is reflected in the portfolio advice given to investors in Cochrane (1999), which repeatedly emphasizes that even when investors are heterogeneous, the average investor must hold the market portfolio. This implies that when investors are rational, an investor should not deviate from that norm unless there is a specific circumstance that makes such a choice especially appropriate for that investor and not others, who in aggregate must take the opposite position. For example, Cochrane points out that, counter to naive intuition, in a rational setting, the low expected returns of growth stocks do not make growth a bad deal, and the fact that market returns are predictable does not make market timing a good deal.

Why is there such a need to emphasize these points, even for the rather sophisticated audience that Cochrane's article was addressed to? Because equilibrium considerations are not immediately intuitive; careful thought, training, and vigilance is required to avoid errors.[11]

4.2 The Information Separation Theorem

Proposition 2 is a surprising result. It is true that investors are willing to hold the fund because the fund knows the precisions of all assets' supply shocks. However, the result is not driven by any overall informational superiority of the fund over investors. Informed investors of an asset receive private signals about it that are not observed by the fund.

Nor is the fund offering investors great safety. For the strategy profile described in Proposition 2 to be an equilibrium, the fund has to offer the portfolio X, specified in equation (3). We verify that if the fund offers another portfolio

$$X' = \left[I + \frac{1}{\rho^2} \left(\mathbf{\Sigma} \boldsymbol{U}^{\frac{1}{4}} \right)^{-1} \right]^{-1} W,$$

[11] In our setting, owing to asymmetric information, neither the uninformed nor the informed hold the market portfolio (though of course the 'average' investor must hold the market inclusive of supply shocks). A further point analogous to Cochrane's also applies with respect to *RAMP*. When a passive fund is available, an ambiguity averse investor always holds *RAMP* as a portfolio component despite its severe apparent riskiness, because investors should only deviate from this holding if they have a special reason to do so (i.e., if they have private information).

which is also a function of $\boldsymbol{U}$ and will converge to 0 as $\boldsymbol{U}$ converges to zero, uninformed investors will not hold the fund and thus will refrain from participating in the financial markets. This is because when $\boldsymbol{U}$ converges to 0, X' converges to 0 much slower than $\boldsymbol{U}$. But the conditional variance of holding any non-zero positions diverges to infinity at the same speed as $\boldsymbol{U}$ converges to 0. Hence, the risk of holding X' diverges to infinity as $\boldsymbol{U}$ converges to 0, implying that holding the passive fund is extremely risky for uninformed investors in the worst-case scenarios.

Hence, the intuition of Proposition 2 must go beyond the passive fund's superior knowledge about the financial markets. We now provide greater insight into this result based upon a new separation theorem for financial markets with asymmetric information, but without model uncertainty or funds.

We now modify the model described in Section 2 by assuming that $\boldsymbol{U}$ is common knowledge among all investors and that there is no passive fund.[12] Then the model is a traditional rational expectations equilibrium model with multiple risky assets, analyzed by Admati (1985). Proposition 3 characterizes a linear rational expectations equilibrium and shows investors' optimal risky assets holding when all parameters are common knowledge.

Proposition 3 *In the model whose parameters are all common knowledge among investors, there exists an equilibrium with the pricing function*

$$P = \boldsymbol{B}^{-1}\left[F - A - \boldsymbol{C}Z\right], \tag{8}$$

where

$$A = \frac{1}{\rho}\left[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}\right]^{-1} W \tag{9}$$

$$\boldsymbol{B} = r\boldsymbol{I} \tag{10}$$

$$\boldsymbol{C} = \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}. \tag{11}$$

[12] The theorem we are about to state does not require the assumption of an uninformative prior. Also, the assumption of independent assets in our model is without loss of generality. We could allow for correlated assets with full rank prior variance-covariance matrix; in such a case, asset payoffs can be decomposed into orthogonal risk factors, and investors' private signals are about these risk factors. Van Nieuwerburgh and Veldkamp (2009), Van Nieuwerburgh and Veldkamp (2010) and Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) show that such a model has the same solution (but with different interpretations) as in the setting with independent assets. Hence, we prove a general version of Proposition 3 in the appendix for the case of independent assets with normal priors. Since both the equilibrium pricing function and investors' equilibrium holdings are continuous in the prior precisions of assets' payoffs, substituting zero prior precisions will lead to exact Proposition 3.

Any investor i's risky asset holding is

$$D_i = \left[I + \frac{1}{\rho^2} \left(\mathbf{\Sigma} \boldsymbol{U} \right)^{-1} \right]^{-1} W + \rho \mathbf{\Omega}_i \left(S_i - rP \right). \tag{12}$$

Owing to supply shocks, asset prices are not fully revealing, so information asymmetry persists in equilibrium and different investors have different asset holdings. An investor's asset holding is the sum of two components. The first term in equation (12),

$$\left[I + \frac{1}{\rho^2} \left(\mathbf{\Sigma} \boldsymbol{U} \right)^{-1} \right]^{-1} W$$

is the *risk-adjusted market portfolio* (*RAMP*), which is deterministic. *RAMP* differs from the ex-ante endowed market portfolio W, because it is also influenced by the informativeness of the equilibrium price, as reflected in the variance of supply shocks and signal noise. Investors take the informativeness of asset prices into account when trading to share risks. When the supply shock to an asset becomes more volatile, or on average investors' private information of such an asset is less precise, the equilibrium price contains less precise information about this asset. This increases risk, which, other things equal, reduces investor holdings of this asset.

The second component of any investor's risky asset holding, the second term in (12), is what we call information-based portfolio. This position, $\rho \mathbf{\Omega}_i \left(S_i - rP \right)$, consists of extra holdings in the securities about which the investor has information. Investor i holds such an extra position of an asset n if and only if the n^{th} diagonal entry of $\mathbf{\Omega}_i$ is $\kappa_n > 0$. This suggests that any investor i holds direct positions of a risky asset because possessing an informative signal about such an asset reduces its conditional volatility (independent of the signal realization). Investor i's direct positions of a risky asset also come from her speculation, which is taken to exploit superior information. Different investors, even if they are informed about asset n, hold different speculative portfolios, because they receive heterogeneous private signals.

A critical feature of any investor's equilibrium asset holdings in equation (12) is that its two components are influenced differently by investors' information sets. The first component, *RAMP*, is formed based only on the information that the investor gleans from asset prices; it is independent of the investor's private information. In contrast, the second component, the information-based portfolio, can be formed based only on the investor's own private information; it is independent of the information content of the market price. Since the supply shock precisions do not affect the distributions of the private signals, it follows that the information-based portfolio is *independent of the supply shock precisions*. The reason for this independence is that private signals and the random supply shocks are normally distributed, so that the conditional expectation of the asset payoffs is linear in private signals and the price signal.[13]

This independence implies a new separation theorem under asymmetric information.

Theorem 1 (The Information Separation Theorem) *When the characteristics of all assets are common knowledge, equilibrium portfolios have three components: a deterministic risk-adjusted market portfolio (RAMP); an information-based portfolio based upon private information and equilibrium prices but no extraction of information from prices; and the riskfree asset.*

This separation turns out to be important for understanding market participation and asset prices when ambiguity averse investor face model uncertainty, and can hold the risky assets through a passive fund, as analyzed in Subsection 4.1. In particular, information separation implies in that setting that the unknown model parameter, the noise supply shock precision, does not affect the information-based portfolio, so that investor holdings outside their fund holdings can be analyzed simply.

Theorem 1 indicates that investors can form an optimal portfolios in separate steps: (1) buy one share of *RAMP*; (2) buy the information-based portfolio using only private information, not the information extracted from price; and (3) put any left-over funds into the riskfree asset. This separation theorem derives from market equilibrium as well as optimization considerations. This differs from those (non-informational) separation theorems in the literature that are based solely on individual optimization arguments.[14]

RAMP is exactly the same as the portfolio X specified in equation (3). The passive fund can provide such a portfolio because the passive fund knows all the model parameters, and X does not include any investor's private information. Meanwhile, the information-based portfolio is exactly the same as the direct holdings of the risky assets

[13]Vives (2008) derives investors' equilibrium asset holdings in a single-asset environment with a normal prior and zero aggregate endowment. Therefore, his result cannot be directly used in our analysis when investors are ambiguity averse about some assets.

[14]It may seem puzzling that none of the three portfolio components depend on the information that an investor extracts from price. How then does this information enter into the investor's portfolio decision? The answer is that *RAMP* is optimal precisely because of the ability of investors to extract information from price. As mentioned before, *RAMP* is deterministic; it does not depend on the private signals. But the fact that *RAMP* is an optimal choice is true only because investors update their beliefs based on price. So the optimal portfolio choice is indeed influenced by such information extraction.

in Proposition 2. To form the information-based portfolio, an investor does not need to extract information from the equilibrium price: she can treat the equilibrium prices as given parameters, and solve for the information-based portfolio from her CARA utility maximization problem as in a partial equilibrium model.

The Information Separation Theorem provides the intuition of investors' equilibrium investment strategies in the setting with model uncertainties. Consider the model in which investors are uncertain about the precisions of some assets' supply shocks. For each possible world $\boldsymbol{U}_i \in \mathcal{U}_i$, investor i can solve her optimal risky assets holdings, assuming that the equilibrium pricing function is the one in equation (7) with $\boldsymbol{U}$ being $\boldsymbol{U}_i$. Importantly, because all other investors are holding one share of the fund and their own direct information-based portfolio, they are effectively holding the risky assets as in the world with $\boldsymbol{U}_i$ being common knowledge. Therefore, in the possible world $\boldsymbol{U}_i$, the market clearing condition implies that the pricing function is the one specified in equation (7) with $\boldsymbol{U}$ being $\boldsymbol{U}_i$. That is, investor i's belief about the pricing function is correct. So, she would like to hold the risky assets as in the world $\boldsymbol{U}_i$. Such risky assets holdings can be implemented by holding one share of the passive fund and her information-based portfolio, so investor i would like to use the investment strategy in Proposition 2. Furthermore, investor i is still uncertain about $\boldsymbol{U}$, so holding the risk-adjusted market portfolio through holding one share of the fund is strictly preferred.

In the above, investor i chooses the investment strategy to maximize her expected CARA utility for any fixed possible world (given that all other investors trade according to the prescribed strategy profile). Here, we indeed implicitly assume that investor i has a min-max utility. However, because investor i's optimal investment strategy, holding one share of the fund and her own information-based portfolio, is a constant across all possible worlds, her max-min utility is the same as her min-max utility. That is, a strong max-min property holds in the equilibrium, and hence, in our model with investor i having max-min utilities, holding one share of the fund and the information-based portfolio is also investor i's optimal investment strategy.

The argument above shows how the Information Separation Theorem helps understand the full participation of ambiguity averse investors in an equilibrium. Indeed, the same argument can also be applied when investors are unaware of some assets or when investors have heterogeneous risk tolerances. In the online appendix, we extend our model to allow for investor unawareness (defined as a diffuse prior over the parameter values that characterize the capital market) or for heterogeneous risk tolerances. We find that in each of these extensions, there exists an equilibrium with full participation.

5 CAPM Pricing with a Passive Fund

Propositions 2 and 3 indicate that the model with ambiguity aversion and the passive fund has an equilibrium in which investors' effective risky assets holdings are exactly the same as in the rational expectations equilibrium in the setting without model uncertainties. Therefore, the passive fund induces full participation even with ambiguity aversion. It can also reduce asset risk premia, because in the equilibrium, uninformed investors are sharing risks with informed ones.

Since the portfolio offered by the passive fund is effectively *RAMP* in the setting without model uncertainty, to analyze the effect of the passive fund on risk premia, we return to the setting without model uncertainty. In such a model, the supply shocks make the asset prices in equilibrium imperfectly revealing, and so in the equilibrium, there are information asymmetries among investors and different risky asset holdings. Hence, the setting is very different from the classic CAPM setting, which assumes identical beliefs and has the implication that all investors hold the same risky asset portfolio.

Since holding the market is equivalent to the CAPM pricing relation, it might seems that in our setting there would not be a way to identify a portfolio that prices all assets and is identifiable ex ante based upon publicly available information. Nevertheless, even with information asymmetry, we identify an efficient portfolio in the model and therefore an implementable version of the CAPM pricing relationship.

From Proposition 3, we know that in the setting without model uncertainties, investors hold the risk-adjusted market portfolio as a common component of their holdings. Therefore, it is natural to consider the risk-adjusted market portfolio, which is just X specified in equation (3), as a candidate for CAPM pricing. From equation (8), the equilibrium pricing function is

$$P = \frac{1}{r}\left[F - A - \frac{1}{\rho}\mathbf{\Sigma}^{-1}Z\right], \tag{13}$$

where $A = \frac{1}{\rho}[\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}]^{-1}W$.

Given any realized equilibrium price P, the volatility of asset payoffs derives from the supply shock only. Let $\text{diag}(P)$ be an $N \times N$ diagonal matrix, whose off-diagonal elements are all zero and whose n^{th} diagonal element is just the n^{th} element of the vector

P. Generically, as no asset has a zero price, $\mathrm{diag}(P)$ is invertible. Then, by the definition of $\mathrm{diag}(P)$,

$$\mathrm{diag}(P)^{-1}P = \mathbb{1}, \tag{14}$$

where $\mathbb{1} = (1, 1, \ldots, 1)'$. From the equilibrium pricing (equation (13)), we have

$$\mathrm{diag}(P)^{-1}\mathbb{E}(F) - r\mathbb{1} = \mathrm{diag}(P)^{-1}A. \tag{15}$$

Here, $\mathbb{E}(F)$ is the expected payoff conditional on the equilibrium price. The LHS of equation (15) is just the vector of the risky assets' equilibrium risk premia.

Given a realized equilibrium price, the risk-adjusted market portfolio X has value $P'X$. Then the vector of the weights of risky assets in the risk-adjusted market portfolio is

$$\omega = \frac{1}{P'X}\mathrm{diag}(P)X.$$

Hence, conditional on the price P, the difference between the expected return of *RAMP* and the riskfree rate is

$$\begin{aligned}
\mathbb{E}(R_X) - r &= \omega'\mathrm{diag}(P)^{-1}\mathbb{E}(F) - r \\
&= \frac{1}{P'X}X'\mathrm{diag}(P)\mathrm{diag}(P)^{-1}(A + rP) - r \\
&= \frac{1}{P'X}X'A, \qquad (16)
\end{aligned}$$

where the expectations are all conditional on the equilibrium price.

The variance of *RAMP* is

$$\mathbb{V}(R_X) = \mathbb{E}\left[\left(\omega'\mathrm{diag}(P)^{-1}\boldsymbol{C}Z\right)\left(\omega'\mathrm{diag}(P)^{-1}\boldsymbol{C}Z\right)'\right] = \left(\frac{1}{P'X}\right)^2 X'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X, \tag{17}$$

and the covariance between all risky assets and *RAMP* is

$$Cov(R, R_X) = \frac{1}{P'X}\mathrm{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X. \tag{18}$$

Let α be the CAPM alpha. From equations (15)-(18), and since $X = \rho(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C})^{-1}A$, we have the following proposition.

Proposition 4 (Risk Premia with Supply Shocks) *In the model with all parameters being common knowledge, asset risk premia satisfy the CAPM where the relevant market portfolio for pricing is the risk-adjusted market portfolio.*

This result may seem surprising, since investors have heterogeneous asset holdings, and since the portfolios held by informed investors are not mean-variance efficient with respect to the public information set. Nevertheless, in equilibrium, there are no extra risk premia incremental to those predicted by the CAPM using *RAMP*.

The CAPM pricing relation using *RAMP* is equivalent to the assertion that *RAMP* is mean-variance efficient conditional only on asset prices. This efficiency can be seen from the utility maximization problem of an investor who is uninformed about all assets. Such an investor balances the expected returns and the risks of her holdings, and her information consists of the equilibrium price only. In equilibrium, such an investor holds *RAMP*, implying that *RAMP* is mean-variance efficient conditional only on equilibrium prices.

Privately informed investors also hold *RAMP* as a component of their portfolios; this is the piece that does not depend upon their private signals (except to the extent that their signals are incorporated into the publicly observable market price). In addition, they have other asset holdings taking advantage of the greater safety of assets they have more information about, and for speculative reasons based upon their private information. *RAMP* is not mean-variance efficient with respect to their private information sets, but it is efficient with respect to the information set that contains only publicly available information.

A very different version of the CAPM has been derived in somewhat similar model setups (see, for example, Easley and O'Hara (2004), Biais, Bossaerts, and Spatt (2010), and the online appendix of Van Nieuwerburgh and Veldkamp (2010)). In these models, the market portfolio for CAPM pricing is the ex-post total supply of the risky assets, the sum of the endowed risky assets and the random supply of risky assets ($W + Z$ in our model). This market portfolio is mean-variance efficient *conditional on the average investor's information set*, and so the CAPM return-covariance relation holds from the perspective of the average investor. The version of the CAPM presented in Proposition 4 differs in that the pricing portfolio is determined ex ante (prior to the realization of the random supply shocks) and that risk premia are conditional only upon the *public* information set (market prices). This makes the market portfolio more directly observable to an econometrician.

In the model whose parameters are all common knowledge among investors, *RAMP* is a natural candidate for the CAPM pricing portfolio, because it is the common component in all investors' risky asset holdings. We show that *RAMP* is mean-variance efficient unconditional on any investor's private information. Therefore, the CAPM security market line relation holds without conditioning on private information, with respect to *RAMP*. One of the further contributions here is to establish that increasing information asymmetry, and its effect on investor participation, does not clearly predict whether there will be an increase versus decrease in risk premium.

We are now in a position to see how a passive fund affects asset risk premia in the setting with model uncertainty. Proposition 2 shows that in the model where investors are uncertain about the precisions of asset supply shocks, they all hold one share of the passive fund. The portfolio provided by the passive fund is just the risk-adjusted market portfolio in the setting with all parameters commonly known. Therefore, the passive fund makes assets' risk premia satisfy the CAPM, even if investors have heterogeneous information and are uncertain about different model parameters. Corollary 3 presents this even more surprising result.

Corollary 3 *In the model where investors are uncertain about the precisions of some assets' supply shocks, and a passive fund offering portfolio X specified in equation* (3)*, asset risk premia satisfy the CAPM using X as the pricing portfolio.*

6 Implementation of the Passive Fund

We have identified a portfolio, *RAMP*, such that if a passive fund commits to offering it, there exists an equilibrium in which all investors hold exactly one share of the fund. Since *RAMP* includes positive positions of all traded assets, investors participate in all assets markets via the fund in the equilibrium. However, the fund needs to have full knowledge about the parameter values of the capital market to construct *RAMP*. Hence, if there is no single individual in the economy who knows all the parameter values, then this knowledge is dispersed. As we have assumed, for each asset, there is a positive measure of investors who know the parameter value for that asset. In this section, we show that the passive fund can still be implemented via a fund of funds.

Suppose that the set of all traded assets can be partitioned into M subsets. In the partition j, there are $m_j \geq 1$ assets. We assume that there is a positive measure of investors, who know all parameters about assets in partition j but do not have any private signals about the payoffs of such assets. We call these investors "Group j uninformed investors." (In the extreme case, $M = N$, and so, in each partition, there is only one asset. Then, we are in the setting described in Section 2.)

We consider the following equilibrium. Each of the Group j uninformed investors commits to offering a portfolio Y_j. Here, Y_j can be seen as a "local" fund, which includes only assets in partition j. For each asset n included in asset partition j, Y_j includes exactly the same position as in the portfolio X.

First, the fund fee will be zero in an equilibrium. Since there are infinitely many funds who are committing to offering Y_j, the j^{th} local fund industry is perfectly competitive. Hence, the fund fee should be the same as the marginal cost of offering Y_j, which is zero.

Second, and more importantly, as required in the existing index fund industry, all local funds are required to disclose their asset holdings at the end of the period. Then, if a local fund of Group j that deviates from Y_j, its portfolio holding will differ from other Group j local funds portfolio holdings. Hence, such a deviation is observable ex post and verifiable. Ex post, once a fund's deviation is detected, We assume that the fund will be heavily punished or incur a large reputation cost. It follows that no local fund is willing to deviate from its commitment to invest in Y_j.

Finally, any investor will first buy one share of the Group j local fund, for each j. By doing so, any investor will form an asset holding $(Y_1', Y_2', \dots, Y_M')' = X'$, which is exactly the passive fund specified in equation (3). Then, investors will hold their own information-based portfolios. Obviously, investors are effectively holding one share of the passive fund and their own information-based portfolios, which are their optimal investment strategies in the equilibrium described in Proposition 2.

7 Concluding remarks

A leading explanation for nonparticipation puzzles is investor ambiguity aversion. This literature focuses on direct trading of assets by investors in the face of model uncertainty. We study here whether ambiguity aversion can still solve the puzzle when an appropriately designed passive fund is available run by a manager who observes the model parameters that investors are uncertain about (though the manager does not observe any private information signals about fundamentals). We show that when there is a passive fund that offers the *risk-adjusted market portfolio* (*RAMP*), all investors prefer to hold the fund and thus participate in all asset markets, even if they do not know the passive fund's composition. This conclusion arises from applying a new portfolio information separation theorem which holds in a setting without model uncertainty and implies that assets' equilibrium risk premia conditional only on public information satisfy the CAPM, with the passive fund (i.e., *RAMP*) as the pricing portfolio.

Since ambiguity aversion does not, by itself, explain the limited market participation puzzle, what does? Dimmock et al. (2016) find that ambiguity aversion is associated with lower stock market participation. Their tests do not distinguish participation via funds versus direct investment in individual stocks, so their finding does not speak specifically to how ambiguity aversion affects the choice between these alternatives. However, our findings suggest that to understand the Dimmock et al. evidence, it is important to investigate what additional frictions or irrationalities might contribute to nonparticipation and prevent the passive fund solution to ambiguity aversion from working perfectly.

With regard to frictions, there could be heavy trading costs, though as discussed in the introduction, it seems unlikely that this is the full explanation for nonparticipation puzzles.

A second possibility is that existing funds may not offer *RAMP*, so that our solution to nonparticipation is unavailable to investors. This may be because, prior to this paper, it was not understood that *RAMP* solves the problem. (Most existing index funds offer proxies for the value weighted market rather than *RAMP*.) Each active fund provides a different portfolio strategy, so at most only one of these (and probably none) closely matches *RAMP*. This possibility suggests a policy implication of our approach: that fund firms should introduce *RAMP* portfolios as a service to ambiguity averse investors who can benefit from diversification and risk sharing.

A third, and related, possibility derives from agency problems. If investors cannot be sure that a fund manager who claims to hold *RAMP* really does, then our conclusion of full participation does not follow. Whether disclosure policies could address agencies problems is an open question.

Finally, it could be that imperfect rationality explains the puzzle. This possibility has a bearing on the argument that providing investors with better information might encourage market participation. Unfortunately, when there is psychological bias, providing more information could make the problem worse. More information does *not* always debias decision makers, since extraneous information can be distracting or overwhelming. For example, providing extensive information about numerous assets could make investors feel less competent about evaluating their investments. This could exacerbate ambiguity aversion. Similarly, such information might push investors toward the use of simple judgment heuristics such as narrow framing, which is another leading possible explanation for nonparticipation.

Other forms of irrationality provide a further possible reason for nonparticipation. Our finding that the availability of *RAMP* induces full participation is based on the premise that ambiguity aversion is their only mistake. However, investors may make psychological errors other than those that come just from ambiguity aversion. If, for example, some investors do not perfectly understand the concept of a market equilibrium, they may regard participation as too risky even if *RAMP* is available. Furthermore, if it is common knowledge that some investors will fail to hold the fund, the equilibrium will be different. We can no longer conclude that other ambiguity averse investors (even those who do understand the concept of equilibrium) will be willing to participate via a fund.

This suggests a further normative implication, that it is valuable to educate investors more deeply about the concept of market equilibrium. Specifically, as our model makes clear, in equilibrium participation can be much safer than it might otherwise seem. It would be possible to explain to investors at a nontechnical level why holding even assets that one knows little about can sometimes improve reward/risk ratios. In particular, it is intuitive that in equilibrium prices need to be set so that even a very risky asset becomes attractive enough for some investors to want to hold it. Finally, even if there is some aspect of the world that an investor feels she knows almost nothing about, it is intuitive that a trustworthy agent (the caveat 'trustworthy' being crucial) could choose on the investor's behalf the weights on the affected assets that the investor would herself have chosen if she knew what the agent knows.

A Omitted Proofs

Proof of Proposition 1:

Because investor i is uniformed about asset n, by assumption, $\kappa_i = 0$. Hence, investor i's only information about the distribution of asset n's payoff is its price, which may partially aggregate informed investors' private signals. Suppose the uninformed investors' aggregate demand for asset n is $(1-\lambda_n)D(p_n)$. Since uninformed investors do not observe τ_n, $D(p_n)$ is not a function of τ_n.

Given any P and any $\tau_n \in \left(0, \bar{\tau}_n^i\right)$, we derive investor i's expected utility conditional on P_n as follows. Suppose asset n's pricing function in a linear equilibrium is

$$f_n = a + bp_n + cz_n,$$

where a, b, and c are undetermined parameters. Since informed investors know τ_n, they can extract information from the price without any ambiguity. Therefore, any informed investor j's demand is

$$D_j = \rho\left[\kappa_n s_j + \frac{\tau_n}{c^2}a + \frac{\tau_n}{c^2}(b-r)p_n - r\kappa_n p_n\right].$$

Then, the informed investors' aggregate demand will be

$$\lambda_n\rho\left[\kappa_n f_n + \frac{\tau_n}{c^2}a + \frac{\tau_n}{c^2}(b-r)p_n - r\kappa_n p_n\right].$$

Then, the market clearing condition implies that

$$\lambda_n\rho\left[\kappa_n f_n + \frac{\tau_n}{c^2}a + \frac{\tau_n}{c^2}(b-r)p_n - r\kappa_n p_n\right] + (1-\lambda_n)D(P_n) = w_n + z_n.$$

Matching the coefficient of the market clearing condition and the pricing function, we have

$$\begin{aligned} a &= \frac{w_n}{\lambda_n\kappa_n\rho} - \frac{\tau_n}{c^2\kappa_n}a \\ bp_n &= -\frac{(1-\lambda_n)D(p_n)}{\lambda_n\kappa_n\rho} - \frac{\tau_n}{c^2\kappa_n}(b-r)p_n + rp_n \\ c &= \frac{1}{\lambda_n\kappa_n\rho} \end{aligned}$$

Therefore, for any given $\tau_n \in \left(0, \bar{\tau}_n^i\right)$, conditional on the price P_n, $|\mathbb{E}(f_n - rp_n|p_n)| < +\infty$. On the other hand, the variance of asset n's payoff conditional on p_n is

$$\mathbb{V}\left(f_n|p_n\right) = c^2\tau_n^{-1},$$

which diverges to $+\infty$ as τ_n goes to 0. Hence, any non-zero position D_i of asset n brings investor i a utility

$$-\exp\left(-\frac{1}{\rho}w_i r p_n\right)\exp\left[-\frac{1}{\rho}D_i\mathbb{E}\left(f_n - rp_n|p_n\right)+\frac{D_i^2}{2\rho^2}\mathbb{V}\left(f_n|p_n\right)\right], \tag{19}$$

which goes to $-\infty$ as τ_n goes to 0. Therefore, if investor i is uninformed about asset n, and $\underline{\tau}_n^i = 0$, investor i refrains from participating in the market of asset n.

Q.E.D.

Proof of Proposition 2:

We first verify that the market clearing condition holds. Each investor i's effective risky assets holding is

$$d_i^* X + D_i^* = \left[I + \frac{1}{\rho^2}\left(\mathbf{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\mathbf{\Omega}_i\left(S_i - rP\right).$$

Then, using the pricing function (equation (7)), the aggregate demand can be calculated as

$$\begin{aligned}
& \int_0^1 \left(d_i^* X + D_i^*\right)\mathrm{d}i \\
= & \left[I + \frac{1}{\rho^2}\left(\mathbf{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\mathbf{\Sigma}\left(F - rP\right) \\
= & \left[I + \frac{1}{\rho^2}\left(\mathbf{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\mathbf{\Sigma}\left(\frac{1}{\rho}\left(\mathbf{\Sigma}+\rho^2\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}\right)^{-1} W + \frac{1}{\rho}\mathbf{\Sigma}^{-1}Z\right) \\
= & \left[I + \frac{1}{\rho^2}\left(\mathbf{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \left[\boldsymbol{I}+\rho^2\mathbf{\Sigma}\boldsymbol{U}\right]^{-1} W + Z \\
= & \rho^2\mathbf{\Sigma}\boldsymbol{U}\left[\boldsymbol{I}+\rho^2\mathbf{\Sigma}\boldsymbol{U}\right]^{-1} W + \left[\boldsymbol{I}+\rho^2\mathbf{\Sigma}\boldsymbol{U}\right]^{-1} W + Z \\
= & W + Z.
\end{aligned}$$

Therefore, the market clears.

Now, for any investor i, we consider a general investment strategy $d_i X + D_i$. Denote by D_{in} investor i's direct holding of asset n. Suppose that investor i is informed about asset n. Then, the pricing function (7) implies that investor i's optimal holding of asset n is

$$\left[1 + \frac{1}{\rho^2}\left(\lambda_n\kappa_n\tau_n\right)^{-1}\right]^{-1} w_n + \rho\kappa_n\left(s_{in} - rp_n\right) = x_n + \rho\kappa_n\left(s_{in} - rp_n\right).$$

Therefore, any combination of d_i and D_{in} such that

$$d_i x_n + D_{in} = x_n + \rho\kappa_n \left(s_{in} - rp_n\right)$$

can lead to the optimal holding of asset n for investor i.

Now, consider an asset n that investor i is uninformed about. For any given d_i and D_{in}, investor i is effectively holding a position $d_i x_n + D_{in}$ of asset n. Then, for any given τ_n Such a holding will bring invest i a utility

$$-\exp\left(-\frac{1}{\rho}w_{in}rp_n\right)\exp\left[-\frac{1}{\rho}\left(d_i x_n + D_{in}\right)\mathbb{E}\left(f_n - rp_n|p_n\right) + \frac{\left(d_i x_n + D_{in}\right)^2}{2\rho^2}\mathbb{V}\left(f_n|p_n\right)\right]. \tag{20}$$

There are two cases. In the first case where $\underline{\tau}_n^i = 0$, similarly to Proposition 1, if $D_{in} \neq 0$, the infimum of such a utility is $-\infty$, since $\mathbb{V}(f_n|P_n) \to +\infty$ as $\tau_n \to 0$. Therefore, $D_{in}^* = 0$. Next, substituting X_n into equation (20), the investor's utility given τ is

$$-\exp\left(-\frac{1}{\rho}w_{in}rp_n\right)\exp\left[-\left(d_i - \frac{1}{2}d_i^2\right)\frac{\rho\tau_n\lambda_n^2\kappa_n^2 w_n^2}{\left[\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2\right]^2}\right]. \tag{21}$$

It follows from equation (21) that for any d_i, the infimum of the investor's utility is at most $-\exp\left(-\frac{1}{\rho}w_{in}rP_n\right)$. Since the investor can get the utility at least $-\exp\left(-\frac{1}{\rho}w_{in}rP_n\right)$ by employing the investment strategy $d_i^* = 1$, there is no profitable deviation.

In the second case, $\underline{\tau}_n^i > 0$. We first assume that any investor i has min-max utility, and then finally show that her max-min utility is the same as her min-max utility, which implies a strong min-max property. Then, investor i's optimal investment strategy with a max-min utility is the same as the optimal investment strategy with a min-max utility. Since investor i does not know τ_n, d_i and D_{in} are not functions of τ_n. For any given τ_n, we can solve d_i^* and D_{in}^* by the first order condition of the following maximization problem:

$$\max_{d_i, D_{in}} \left(d_i x_n + D_{in}\right)\frac{w_n}{\rho\left[\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2\right]} - \frac{\left(d_i x_n + D_{in}\right)^2}{2\rho}\frac{1}{\rho^2\lambda_n^2\kappa_n^2\tau_n}. \tag{22}$$

The second order condition of such a maximization problem holds, because the utility function in equation (22) is strictly concave.

Differentiating the utility function in equation (22) with respect to d_i, we get one of the first-order conditions:

$$x_n\frac{w_n}{\rho\left[\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2\right]} - \frac{\left(d_i x_n + D_{in}\right)x_n}{\rho}\frac{1}{\rho^2\lambda_n^2\kappa_n^2\tau_n} = 0.$$

So,

$$d_i x_n + D_{in} = d_i \frac{\rho^2 \tau_n \lambda_n^2 \kappa_n^2}{\lambda_n \kappa_n + \rho^2 \tau_n \lambda_n^2 \kappa_n^2} w_n + D_{in} = \frac{\rho^2 \tau_n \lambda_n^2 \kappa_n^2}{\lambda_n \kappa_n + \rho^2 \tau_n \lambda_n^2 \kappa_n^2} w_n + D_{in}.$$

Then, $d_i^* = 1$ and $D_i^* = 0$, because they are not functions of τ_n. Therefore, with a min-max utility, if an investor i is uninformed about asset n, she will hold exactly one share of the passive fund and a zero position of asset n.

Because investor i's optimal investment strategy $(d_i^*, D_{in}^*) = (1, 0)$ is constant across all possible τ_n, we have

$$\min_{\tau_n} \max_{d_i, D_{in}} u((d_i, D_{in}), \tau_n) = \min_{\tau_n} u((1,0), \tau_n) \leq \max_{d_i, D_{in}} \min_{\tau_n} u((d_i, D_{in}), \tau_n).$$

Generally, by the min-max utility, we have

$$\min_{\tau_n} \max_{d_i, D_{in}} u((d_i, D_{in}), \tau_n) \geq \max_{d_i, D_{in}} \min_{\tau_n} u((d_i, D_{in}), \tau_n).$$

Then, we have

$$\min_{\tau_n} \max_{d_i, D_{in}} u((d_i, D_{in}), \tau_n) = \max_{d_i, D_{in}} \min_{\tau_n} u((d_i, D_{in}), \tau_n).$$

This implies a strong min-max property, and hence, $(d_i^*, D_{in}^*) = (1, 0)$ is also the optimal investment strategy of investor i, when she has a max-min utility.

In sum, given the pricing function specified in equation (7), it is optimal for any investor i to choose the investment strategy $d_i^* = 1$ and

$$D_{in}^* = \begin{cases} 0, & \text{if she is uninformed about asset } n; \\ \rho \kappa_n \left(S_{in} - r P_n \right), & \text{if she is informed about asset } n. \end{cases}$$

Q.E.D.

Proof of Theorem 3:

Let's first prove a more general version of Proposition 3, when investors hold a common prior belief about F, $F \sim \mathcal{N}\left(\overline{F}, \boldsymbol{V}\right)$. As is standard in the literature of rational expectations equilibrium, we consider the linear pricing function

$$F = A + \boldsymbol{B} P + \boldsymbol{C} Z, \quad \text{with } C \text{ nonsingular.} \tag{23}$$

If and only if B is nonsingular, equation (23) can be rearranged to

$$P = -\boldsymbol{B}^{-1}A + \boldsymbol{B}^{-1}F - \boldsymbol{B}^{-1}\boldsymbol{C}Z, \tag{24}$$

which solves for prices. Recall that $S_i = F + \epsilon_i$, so conditional on F, P and S_i are independent. Therefore, we can write down assets' payoffs' posterior means and posterior variances conditional on all information that are available to investor i as follows.

First consider investor i's belief about F conditional on P. Conditional on P, F is normally distributed with mean $A + \boldsymbol{B}P$ and precision $[\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}']^{-1}$. On the other hand, conditional on S_i, investor i's belief about F is also normally distributed, with mean S_i and precision $\boldsymbol{\Omega}_i$. Therefore, investor i's belief about F conditional on what the investor observes, P and S_i, is also normally distributed. The mean of the conditional distribution of F is the weighted average of the expectation conditional on the price P, the expectation conditional on investor i's private signal S_i, and the prior mean $\overline{F}$. Therefore, the conditional mean of F is

$$\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right]^{-1}\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right]. \tag{25}$$

The precision of the conditional distribution of F is

$$(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}. \tag{26}$$

Then, from any investor i's first order condition, investor i's demand is

$$\begin{aligned}
D_i &= \rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right] \\
&\quad \left\{\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right]^{-1}\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right] - rP\right\} \\
&= \rho\left\{\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right] - \left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right] rP\right\} \\
&= \rho\left\{(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\boldsymbol{\Omega}_i - r\boldsymbol{V}^{-1}\right\} P \\
&\qquad + \rho\boldsymbol{\Omega}_i S_i + \rho[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}].
\end{aligned} \tag{27}$$

Integrating across all investors' demands gives the aggregated demand as

$$\begin{aligned}
\int_0^1 D_i \mathrm{d}i =& \rho\left\{(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\left(\int_0^1 \boldsymbol{\Omega}_i \mathrm{d}i\right) - r\boldsymbol{V}^{-1}\right\} P \\
&+ \rho\left(\int_0^1 \boldsymbol{\Omega}_i S_i \mathrm{d}i\right) + \rho[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}].
\end{aligned} \tag{28}$$

By equation (2), we have $\int_0^1 \mathbf{\Omega}_i \mathrm{d}i = \mathbf{\Sigma}$. Also, note that

$$\int_0^1 \mathbf{\Omega}_i S_i \mathrm{d}i = \mathbf{\Sigma} F.$$

Therefore, from the market clearing condition, we have

$$\int_0^1 D_i \mathrm{d}i = Z + W. \tag{29}$$

In an equilibrium, both equation (23) and equation (29) hold simultaneously for any realized F and Z, therefore, by matching coefficients in these two equations, we have

$$\rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}\right] - W = -\boldsymbol{C}^{-1}A \tag{30}$$

$$\rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\mathbf{\Sigma} - r\boldsymbol{V}^{-1}\right] = -\boldsymbol{C}^{-1}\boldsymbol{B} \tag{31}$$

$$\rho\mathbf{\Sigma} = \boldsymbol{C}^{-1} \tag{32}$$

Therefore, from equation (32), we have

$$\boldsymbol{C} = \frac{1}{\rho}\mathbf{\Sigma}^{-1}$$

Obviously, $\boldsymbol{C}$ is positive definite and symmetric. Then from equation (30), we have

$$[\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}]A = \frac{1}{\rho}W - \boldsymbol{V}^{-1}\overline{F}.$$

Because both $(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma})$ and $\mathbf{\Sigma}$ are both positive definite, we have

$$A = [\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}]^{-1}\left(\frac{1}{\rho}W - \boldsymbol{V}^{-1}\overline{F}\right).$$

From equation (31), we have

$$[\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}](\boldsymbol{B} - r\boldsymbol{I}) = r\boldsymbol{V}^{-1}.$$

Again, because $[\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}]$ is positive definite, we have

$$\boldsymbol{B} = r\boldsymbol{I} + r[\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}]^{-1}\boldsymbol{V}^{-1}.$$

Obviously, $\boldsymbol{B}$ is invertible. By substituting A, $\boldsymbol{B}$, and $\boldsymbol{C}$ into equation (24), we solve the equilibrium pricing function.

Now, let's look at any investor i's holding. Substituting the coefficients into investor i's holding function (27), we have

$$D_i = \left(\boldsymbol{I} + \frac{1}{\rho^2} (\boldsymbol{\Sigma U})^{-1} \right)^{-1} W + \rho \left[I + \rho^2 \boldsymbol{\Sigma U} \right]^{-1} \boldsymbol{V}^{-1} (\overline{F} - rP) + \rho \boldsymbol{\Omega}_i (S_i - rP).$$

Finally, because the pricing function P and any investor i's demand function D_i are continuous in $\boldsymbol{V}^{-1}$, we can substitute $\boldsymbol{V}^{-1} = 0$ to get Proposition 3.

Q.E.D.

Proof of Proposition 4:

By equations (16), (17), and (18), we have

$$\begin{aligned} & \frac{\frac{1}{P'X} \text{diag}(P)^{-1} \boldsymbol{C U}^{-1} \boldsymbol{C} X}{\left(\frac{1}{P'X}\right)^2 X' \boldsymbol{C U}^{-1} \boldsymbol{C} X} \frac{X'A}{P'X} \\ = \quad & \frac{\text{diag}(P)^{-1} \boldsymbol{C U}^{-1} \boldsymbol{C} X}{X' \boldsymbol{C U}^{-1} \boldsymbol{C} X} X'A. \end{aligned}$$

This is the RHS of the Security Market Line relation. We want to show that this equals the difference between the risky assets' rates of return and the riskfree asset's rate of return, which is shown to be $\text{diag}(P)^{-1} A$ from equation (15).

Then, we have

$$\begin{aligned} & \frac{\text{diag}(P)^{-1} \boldsymbol{C U}^{-1} \boldsymbol{C} X}{X' \boldsymbol{C U}^{-1} \boldsymbol{C} X} X'A = \text{diag}(P)^{-1} A \\ \Leftrightarrow \quad & \text{diag}(P)^{-1} \boldsymbol{C U}^{-1} \boldsymbol{C} X X' A = \text{diag}(P)^{-1} A X' \boldsymbol{C U}^{-1} \boldsymbol{C} X \\ \Leftrightarrow \quad & \boldsymbol{C U}^{-1} \boldsymbol{C} X X' A = A X' \boldsymbol{C U}^{-1} \boldsymbol{C} X. \end{aligned}$$

The last equation holds because $X = \rho (\boldsymbol{C U}^{-1} \boldsymbol{C})^{-1} A$ and $(\boldsymbol{C U}^{-1} \boldsymbol{C})^{-1}$ is a symmetric matrix.

Q.E.D.

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B Online Appendix

To evaluate the robustness of our conclusions about investor participation, we now consider two possible model generalizations. First, investors may be unaware of certain traded assets, making it unattractive or infeasible for them to hold such assets. Previous literature considers this another important possible reason for limited participation. Second, investors may have heterogeneous risk tolerances. These cases also suggest further empirical implications.

B.1 Uncertainty about Other Parameters and Unawareness

Section 2 assumed that investors were uncertain about the precisions of assets' supply shocks, and maximized their CARA utilities based upon worst-case scenarios. In addition, we assumed that the number of the risky assets is common knowledge. Hence, all investors know the existence of all assets, and can observe their prices.

Investors unawareness is an important alternative possible explanation for nonparticipation. Specifically, investors may not know certain traded risky assets, and so they do not observe such assets' prices. It is infeasible for investors to directly hold assets they are unaware of (Merton 1987; Easley and O'Hara 2004). For example, if an investor has never heard of FLIR Systems (an S&P 500 firm), it seems natural for the investor not to participate in this market.

Such a definition of investor unawareness is rather restricted. We relax the definition of investor unawareness to allow for extreme ignorance about some of the asset's characteristics, even when the investor can observe its price. This makes it physically possible (though not necessarily attractive) for an investor to hold an asset the investor is unaware of.

Formally, we say that investor i is unaware of asset n, if she holds a diffuse uniform prior about the precision of asset n's supply shock;[15] that is, $\tau_n \sim U(0, +\infty)$. Here, we allow investor i to know all characteristics of asset n (other than the precision of the supply shock) and observe asset n's price.

Since our focus is now on unawareness, not ambiguity aversion, instead of maximizing her CARA utility in the worst case scenario, investor i maximizes her 'average' CARA utility over the set of all possible precisions of asset n's supply shock. Proposition 5 below shows that in this setting, the investor will not hold asset n directly.

Proposition 5 *When there is no passive fund, investors will not participate in the markets of assets they are unaware of.*

Proof of Proposition 5:

[15] If we assume that the prior about τ_n is $\tau_n \sim U(0, \bar{\tau}_n)$ for some $\bar{\tau}_n \in \mathbb{R}_{++}$, Proposition 5 below still holds.

Because there is no passive fund, $d_i = 0$. Consider $D_i \neq 0$ (any non-zero direct holding of asset n). For any given $\tau_n \in (0, +\infty)$, conditional on asset n's price P_n, investor i's utility is

$$- \exp\left(-\frac{1}{\rho} w_i r P_n\right) \exp\left[-\frac{1}{\rho} D_i \mathbb{E}\left(f_n - rP_n | P_n\right) + \frac{D_i^2}{2\rho^2} \mathbb{V}\left(f_n | P_n\right)\right].$$

Similarly to the proof of Proposition 1, $|\mathbb{E}(f_n - rP_n|P_n)|$ is bounded, and the variance of asset n's payoff conditional on P_n is

$$\mathbb{V}\left(f_n | P_n\right) = c^2 \tau_n^{-1},$$

where $c = 1/(\lambda_n \kappa_n \rho)$ is independent of τ_n. Then, by Jensen's inequality, we have

$$\begin{aligned} & \lim_{h\to+\infty} \int_0^h -\frac{1}{h} \exp\left(-\frac{1}{\rho} w_i r P_n\right) \exp\left[-\frac{1}{\rho} D_i \mathbb{E}\left(f_n - rP_n | P_n\right) + \frac{D_i^2}{2\rho^2} \mathbb{V}\left(f_n | P_n\right)\right] \mathrm{d}\tau_n \\ \leq \quad & -\exp\left(-\frac{1}{\rho} w_i r P_n\right) \exp\left[\lim_{h\to+\infty} \int_0^h \frac{1}{h}\left(-\frac{1}{\rho} D_i \mathbb{E}\left(f_n - rP_n | P_n\right) + \frac{D_i^2}{2\rho^2} \mathbb{V}\left(f_n | P_n\right)\right) \mathrm{d}\tau_n\right]. \end{aligned}$$

The right-hand side of this inequality diverges to $-\infty$, implying that the left-hand side, which is the investor's average CARA utility, is also $-\infty$. Therefore, $D_i \neq 0$ is dominated by $D_i = 0$; hence, investors will not directly hold the assets they are unaware of.

Q.E.D.

We now analyze whether a passive fund that offers *RAMP* can lead to full participation in the setting with investors' unawareness. This is not a trivial question, since investors still need to assess the expected return and the risk of holding the passive fund, when they allocate their initial wealth among the passive fund, the risky assets they are aware of, and the riskfree asset.

We assume that any investor i believes that the number of all traded assets is equally likely to be any integer that is greater than or equal to the number of assets she is aware of. For an asset investor i is unaware of, investor i holds diffuse uniform priors about all its parameters she does not know.[16] These priors consist of a uniform uninformative prior about its endowment over the support $(0, +\infty)$, a uniform uninformative prior about the precision of the average private information about its payoff over the support $(0, +\infty)$, and a uniform uninformative prior about the precision of the supply shock in its market over the support $(0, +\infty)$. Investors know that all random variables about assets' characteristics are independent.

We further assume that all investors are aware of the riskfree asset, and that there is a fund that invests in the risky assets. Investors know the existence and name of the

[16] The assumption of diffuse uniform priors is not necessary for Proposition 6 below. Indeed, from its proof, we can see that Proposition 6 holds for any subjective priors investors may have.

fund and are aware of its price, but are unaware of (have diffuse priors about) its return characteristics. So an investor who is unaware of some assets can have extremely poor information about the distribution of returns on this fund.

The fund manager observes the characteristics and prices of all assets and therefore is able to construct and offer to investors the portfolio X as specified in equation (3). It is common knowledge that this is the portfolio offered by the fund.

In such a setting, an investor i's investment strategy is a function mapping from her information set to positions in the passive fund and the other assets. As we argue above, if investor i is unaware of asset n, and decides not to hold the passive fund, then investor i will have a zero holding of asset n, because either it is infeasible for investor i to hold asset n, or holding asset n is infinitely risky to the investor.

We define an *admissible world* of an investor as the union of the set of assets she is informed about and a possible set of assets she is uninformed about or unaware of and hypothesized possible characteristics for these assets. Specifically, consider any investor i. We divide all traded assets into two groups, Γ_{i1} and Γ_{i2}. Suppose that investor i is informed about Γ_{i1} assets only, and so she knows all characteristics of Γ_{i1} assets. However, she is uninformed about or unaware of Γ_{i2} assets. In particular, she knows the existence of Γ'_{i2} assets and she can contemplate a possible set of assets Γ''_{i2}. Denote by $\widetilde{\Gamma}_{i2}$ the union of Γ'_{i2} and Γ''_{i2}. The combined asset set $\Gamma_{i1} \cup \widetilde{\Gamma}_{i2}$, together with an hypothesized vector of characteristics for each asset in $\widetilde{\Gamma}_{i2}$, constitutes an admissible world. The set $\widetilde{\Gamma}_{i2}$ is associated with a number $\widetilde{N} \geq \# \left(\Gamma'_{i2}\right)$ of assets in $\widetilde{\Gamma}_{i2}$. For each asset $n \in \widetilde{\Gamma}_{i2}$, the possible world specifies the specific parameters values characterizing asset n: the endowment $\widetilde{W_n} > 0$, the average precision of private information $\widetilde{\lambda_n \kappa_n}$, and the precision of the supply shock $\widetilde{\tau}_n$.

We assume that conditional upon an admissible world, the investor has the CARA utility function. However, investors maximize their average CARA utilities over all admissible worlds, when making investment decisions.

Proposition 6 below shows that in such a general case, investors will hold exactly one share of the passive fund, and thus participate in all assets' markets.

Proposition 6 *In the general model where investors are uncertain about several characteristics of the traded assets, including the number of assets:*

1. *There exists an equilibrium in which all investors hold one share of the passive fund and their own information-based portfolios.*

2. *Asset prices and investors' effective risky assets holdings are identical to those in the model without any model uncertainty.*

3. *Generically all investors take non-zero positions in all traded assets.*

Proof of Proposition 6:

Consider any investor i, who is aware of assets in Γ_{i1} but is uninformed about or unaware of assets in Γ_{i2}. Then, any of investor i's admissible world $\widetilde{\Gamma}$ consists of Γ_{i1} assets and possible $\widetilde{\Gamma}_{i2}$ assets; that is, $\widetilde{\Gamma} = \Gamma_{i1} \cup \widetilde{\Gamma}_{i2}$.

The strategy profile under consideration prescribes that all investors buy one share of the fund and hold their own information-based portfolios. Hence, in $\widetilde{\Gamma}$, by The Information Separation Theorem, all other investors' portfolio choices are effectively the same as in equation (12), because the fund offers the risk-adjusted market portfolio in $\widetilde{\Gamma}$. Therefore, in $\widetilde{\Gamma}$, the pricing function will be the same as in equation (8). Then, for any given price vector, investor i's optimal portfolio choice will be the same as in equation (12) too. Such a portfolio choice can be implemented by holding the passive fund and investor i's own information-based portfolio based only on her knowledge about Γ_{i1} assets. Therefore, in $\widetilde{\Gamma}$, it is optimal for investor i to hold the fund and her information-based portfolio, when all other investors do the same thing.

Since the admissible world $\widetilde{\Gamma}$ is constructed arbitrarily, the arguments above show that it is optimal for investor i to hold one share of the fund and her own information-based portfolio, when all other investors hold the fund and their own information-based portfolio. By similar arguments, when all other investors hold the fund and their own information-based portfolios, any investor will optimally hold the fund and her own information-based portfolio. Therefore, the strategy under consideration is an equilibrium.

Then, for any realized world, since all investors' effective holdings are exactly same as in equation (8), the market clearing condition implies that the equilibrium price function is same as in the case where all parameters are common knowledge. In addition, since all investors hold the fund who offers the risk-adjusted market portfolio, all investors will have strictly positive positions of all assets.

Q.E.D.

From information separation, the portfolio constructed by the fund described in Proposition 6 is implementable using only public information. So if a passive fund wants to provide investors with *RAMP*, it does not need to know the private signal of any investor. For investors, buying a fund share is the same as holding *RAMP*—the first component described by the information separation theorem. Therefore, intuitively, all investors are satisfied to buy fund shares, despite their extreme ignorance about the return distribution of the fund and its assets.

Since we have assumed a uniform uninformative prior for an investor on the number of assets of which he is not aware, one might suspect that this would interfere severely with the investor's attempt to speculate even on the assets the individual is aware of. However, investors do not need to know the number of assets traded in the market when forming their information-based portfolios. Consider for example any investor i. Denote by N_i the number of assets that she is informed about. For any given $\hat{N} \geq N_i$, except the $N_i \times N_i$ block Ω_i, all other blocks in the $\hat{N} \times \hat{N}$ matrix Ω_i are 0. So lack of knowledge about the number N does not affect investors' information-based trading.

B.2 Heterogeneous Risk Tolerances

In the model described in Section 2, investors share a same risk aversion coefficient ρ. Such an assumption leads to investors' homogeneous holdings of the passive fund. Indeed, in the equilibrium characterized in Proposition 2, all investors hold one share of the passive fund. However, it is conceivably that differences in risk tolerances, and investor unawareness of other investors' risk tolerances, could resurrect investors' heterogeneous holdings of the passive fund. We extend the model in Section 2 by assuming that any investor i ($i \in [0,1]$) has the risk aversion coefficient ρ_i. Here, ρ_i is a continuous function of i. Let

$$\overline{\rho} = \int_0^1 \rho_i \mathrm{d}i \quad \text{and} \quad \overline{\Sigma} = \int_0^1 \rho_i \Omega_i \mathrm{d}i.$$

Here, $\overline{\rho}$ is the average risk tolerance, and $\overline{\Sigma}$ is the average precision of investors' private information that is weighted by their risk tolerances. We assume that any investor i knows ρ_i, but she does not know the distribution of ρ_j and thus the average risk tolerance $\overline{\rho}$. The passive fund cannot evaluate each individual investor's risk tolerance, but it has accurate information about the distribution of investors' risk tolerances; hence, it knows $\overline{\rho}$ and $\overline{\Sigma}$. Then, the passive fund offers the portfolio

$$\overline{X} = \left[\overline{\rho} + \left(\overline{\Sigma} U\right)^{-1}\right]^{-1} W. \tag{33}$$

to all investors. Proposition 7 shows that investors with different risk tolerances hold different numbers of shares of the passive fund.

Proposition 7 *In the model with investors' heterogeneous risk tolerances, there exists an equilibrium in which any investor i with the risk tolerance ρ_i holds ρ_i shares of the passive fund and her own information-based portfolio $\rho_i \Omega_i (S_i - rP)$.*

Proof of Proposition 7:

We first analyze the model in which investors have heterogeneous risk tolerances and all parameters are common knowledge. We again consider the linear pricing function as in equation (23),

$$F = A + BP + CZ, \quad \text{with } C \text{ nonsingular}.$$

Therefore, conditional on the price, assets' payoffs have the conditional distribution is

$$F|P \sim \mathcal{N}\left(A + BP, CU^{-1}C'\right).$$

An investor i gleans such information from the price. Therefore, an investor i's demand is

$$D_i = \rho_i \left[(CU^{-1}C')^{-1}(B - rI) - r\Omega_i\right] P + \rho_i \Omega_i S_i + \rho_i (CU^{-1}C')A. \tag{34}$$

Then, by integrating all investors' demands and equalizing the aggregate demand and the total supply (the aggregate endowments and the supply shocks), we can derive the pricing function

$$P = B^{-1}\left[F - A - CZ\right], \tag{35}$$

where

$$A = \left[\overline{\Sigma} + \overline{\rho}\left(\overline{\Sigma}U\overline{\Sigma}\right)^{-1}\right]^{-1} W \tag{36}$$

$$B = rI \tag{37}$$

$$C = \overline{\Sigma}^{-1}. \tag{38}$$

Any investor i's risky asset holding is

$$D_i = \rho_i\left[\overline{\rho} + \left(\overline{\Sigma}U\right)^{-1}\right]^{-1} W + \rho_i\Omega_i\left(S_i - rP\right). \tag{39}$$

Because the passive fund provides the portfolio $\overline{X}$ specified in equation (33), Equation (39) can be rewritten as

$$D_i = \rho_i\overline{X} + \rho_i\Omega_i\left(S_i - rP\right). \tag{40}$$

Then, when investors are uncertain about some parameters and thus are subject to ambiguity aversions, they still want to hold the passive fund. In particular, investor i first buys ρ_i shares of a passive fund and then use her own private information to form the information-based portfolio $\rho_i\Omega_i^{-1}\left(S_i - rP\right)$. Finally, investor i invests the rest of her endowments in the riskfree asset.

Q.E.D.