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ABSTRACT

I generalize a benchmark model of directed technical change in order to reconcile it with the historical experience of energy transitions. I show that the economy becomes increasingly locked-in to the dominant energy source when machines and energy resources are substitutes, but a transition away from the dominant energy source is possible when machines and energy resources are complements. Consistent with history, a transition in research activity leads any transition in resource supply. A calibrated numerical implementation shows that innovation is critical to climate change policy. A policymaker uses a temporary research subsidy to permanently redirect innovation towards low-emission resources, avoiding much more warming than would a policymaker restricted to an emission tax instrument.

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1 Introduction

Energy historians have emphasized the multiple dramatic transformations in energy use that accompanied industrialization, as societies have shifted from reliance on biomass to coal and then to oil and gas. According to Smil (2010, 2), the preindustrial era saw "only very slow changes" in energy use, "but the last two centuries have seen a series of remarkable energy transitions." Rosenberg (1994, 169) notes that "the diversity of energy inputs and the changing usage of those inputs over time is a central feature of the historical record." And in their seminal analysis, Marchetti and Nakicenovic (1979, 15) observe that the transitions have been so regular that "it is as though the system had a schedule, a will, and a clock." It is important to understand the economic drivers of these transitions. First, energy use is closely linked to the First Industrial Revolution (via coal), to the Second Industrial Revolution (via electricity and oil), and to the distribution of output across countries. Yet growth theory has largely abstracted from energy. Second, policymakers around the world are currently attempting to induce a new transition to low-carbon resources in order to avoid dangerous climate change. Understanding the drivers of past transitions should improve policies that aim to stimulate and sustain a new transition.

Resource economists have long focused on how depletion or exhaustion can induce transitions between resources (e.g., Nordhaus, 1973; Chakravorty and Krulce, 1994; Chakravorty et al., 1997). For example, the Herfindahl (1967) rule holds that resources should be exploited in order of increasing cost. In contrast, energy and economic historians have argued that technological change, not depletion, has been critical to past transitions between different types of resources (e.g., Marchetti, 1977; Marchetti and Nakicenovic, 1979; Rosenberg, 1983; Grübler, 2004; Fouquet, 2010; Wilson and Grubler, 2011).[1] On this view, the British transition from biomass to coal was driven by technologies such as the steam engine, not by changes in the relative abundance of timber and coal. The later transition from coal to oil was driven by the technology of the internal combustion engine, not by a lack of coal.

[1] I give five examples. Marchetti and Nakicenovic (1979, 7–8) argue, "The causal importance of resource availability is weakened by the fact that oil successfully penetrated the energy market when coal still had an enormous potential, just as coal had previously penetrated the market when wood still had an enormous potential." Fouquet (2010, 6591) observes, "In all cases, cheaper or better services were the key to the switch [between sources of energy]. In a majority of cases, the driver was better or different services." Rosenberg (1994, 169) observes that "technological innovations are often not neutral with respect to their energy requirements." Flinn (1959) emphasizes that the surmounting of "technological barriers," not the scarcity of timber, drove the British to shift towards coal. Finally, Grübler (2004, 170) writes, "It is important to recognize that these two major historical shifts [from biomass to coal, and then from coal to oil and natural gas] were not driven by resource scarcity or by direct economic signals such as prices, even if these exerted an influence at various times. Put simply, it was not the scarcity of coal that led to the introduction of more expensive oil. Instead, these major historical shifts were, first of all, technology shifts, particularly at the level of energy end use. Thus, the diffusion of steam engines, gasoline engines, and electric motors and appliances can be considered the ultimate driver, triggering important innovation responses in the energy sector and leading to profound structural change."

Resource economists' emphasis on depletion may explain the development of a given type of resource, but standard models cannot capture historians' understanding of transitions between types of resources.[2]

I develop a model of directed technical change in which innovation-led transitions occur endogenously, even when resources are not depletable. A final good is produced from multiple types of energy, which are gross substitutes. Each type of energy is produced by combining an energy resource with specialized machines. For instance, coal is combined with steam engines to produce mechanical motion or electricity. A fixed measure of scientists works to improve these machines. Each scientist targets whichever type of machine provides a more valuable patent. Scientists' efforts change the quality of machines from period to period, which in turn changes equilibrium use of each energy resource from period to period.

I show that the elasticity of substitution between resources and machines determines whether a transition in energy supply can occur in the absence of policy and of depletion. Imagine that there are two types of energy and that one type of energy initially attracts the majority of scientists and uses more raw resources. I show that three forces determine how each sector's share of research and resource extraction changes in the following period. First, as the dominant sector becomes more advanced, a *market size effect* increases that sector's share of research and of extraction. The improvement in the quality of the dominant sector's machines expands the market for its energy resource, and the resulting increase in resource extraction raises the value of a patent by expanding the market for machines. This positive feedback between extraction and research works to lock in whichever sector is already dominant. Second, a *patent quality effect* drives scientists to the sector where their patent will cover a higher quality machine. This effect draws additional scientists to the sector that dominated research in the previous period, which again works to lock in whichever sector is already dominant. Third, a *supply expansion effect* reduces the value of a patent as the average quality of a sector's machines increases. An improvement in the quality of a sector's machines shifts out the supply of machine services, which reduces the price of machine services and thus reduces the value of a patent. This force pushes scientists away from the sector that dominated research effort in the previous period. It is the only force that works against lock-in and in favor of a transition away from the dominant sector.

The elasticity of substitution between resources and machines determines the relative strengths of the patent quality and supply expansion effects.[3] When that elasticity is strictly greater than 1 (machines are "resource-saving"), demand for machine services is elastic and the price of machine services does not fall by much as technology improves. The patent

[2] The overemphasis on depletion at the expense of innovation dates back to Jevons (1865), who underestimated the scope for innovation in his famous analysis of the advancing depletion of British coal reserves (Madureira, 2012).

[3] Much literature has estimated the elasticity of substitution between energy and other inputs, but there is not much literature on the elasticity of substitution between resources and other inputs in the production of energy.

quality effect dominates the supply expansion effect. Whichever sector dominates research and extraction in some period then does so to an increasing degree in all later periods.[4] However, when that elasticity is strictly less than 1 (machines are "resource-using"), demand for machine services is inelastic and the price of machine services falls by a lot as technology improves. The supply expansion effect dominates the patent quality effect. In that case, as the dominant sector becomes more advanced, scientists can begin switching to the other sector. Eventually, their research efforts raise the quality of technology in the dominated sector, which begins increasing that sector's share of extraction via market size effects. The shift in scientists away from the dominant sector can thereby generate a transition in energy supply.

To explore the implications for climate change policy, I extend the setting to allow resources to generate carbon dioxide (CO_2) emissions that eventually raise global temperature. Such warming reduces the quantity of final good produced. I explore a case with three resources, calibrated to data for coal, natural gas, and renewables (wind and solar) and to economic growth. The elasticity of substitution between resources and machines is pinned down by estimates of the marginal cost of reducing CO_2 emissions. This calibrated elasticity is less than 1, so that machines are resource-using. The laissez-faire economy evolves towards a corner solution with all research and extraction activity focused on natural gas. With enough time, the economy would leave this corner solution and begin using renewables, but that time is too distant to limit warming over the next several centuries.

I find that innovation is critical to optimal policy. A policymaker who can use a subsidy to redirect research towards the zero-emission renewable resource does so immediately and completely when renewable energy is a decent enough substitute for other energy. These scientists improve the quality of machines in the renewable sector to such a degree that the policymaker can subsequently withdraw the subsidy without deterring scientists from working on renewables. The economy then evolves towards using only renewables by the end of the present century and thereby dramatically reduces total climate change.

In contrast, an emission tax is far less effective at limiting climate change. A policycmaker who can use an emission tax but not a research subsidy designs a U-shaped emission tax: a moderate early tax redirects scientists away from the coal resource, and a high tax in the second century begins redirecting resource use towards the renewable sector. An extremely large initial emission tax could fully redirect scientists towards the renewable sector, but such a high tax creates too many other distortions to be optimal. As a result, the optimal emission tax does not redirect enough scientists to generate the self-sustaining dynamics observed in the case of the research subsidy and does not limit warming to any appreciable degree over

[4] The forces generating lock-in are similar to those explored in a related literature on path dependency in technology adoption (e.g., David, 1985; Arthur, 1989; Cowan, 1990). That literature focuses on "dynamic increasing returns" as the source of path dependency, where the likelihood of using a technology increases in the number of times it was used in the past (perhaps through learning-by-doing or network effects). In the present setting, market size and patent quality effects both act like dynamic increasing returns.

the next two centuries. The welfare gains from the emission tax are tiny in comparison to the gains from a research subsidy. A policymaker who can use both instruments chooses to use the research subsidy in the near-term and delays the emission tax until the next century, but welfare is not appreciably greater than when the policymaker does not have access to the emission tax.

Finally, I also explore the implications of directed technical change for a different type of policy instrument: a mandate to use a minimum share of renewables. Such mandates are common. For instance, around 30 U.S. states mandate a minimum share of renewable electricity. I show that accounting for innovation is critical to the evaluation of a mandate. Even seemingly modest mandates can redirect research effort to renewables through market size effects. As a result, mandates can kickstart a transition that makes themselves non-binding over time.[5] Such mandates appear far more costly if we ignore their implications for innovation, as is common in cost-benefit analyses, and they can even be welfare-improving if different types of energy are sufficiently substitutable for each other.

My theoretical setting generalizes Acemoglu et al. (2012). Their economy demonstrates a high degree of lock-in or path dependency: whichever sector initially dominates resource extraction and research effort will increase its dominance as time passes.[6] This result is not consistent with the history of energy transitions. I show that their high degree of lock-in results from their use of a Cobb-Douglas aggregator to combine resources and machines, which fixes the elasticity of substitution between resources and machines at unity. I show that a unit elasticity is the knife-edge case in which the patent quality and supply expansion effects exactly offset each other. The evolution of research and resource use in Acemoglu et al. (2012) is therefore determined entirely by market size effects (demonstrated in Section 3 below), which generate positive feedbacks between research and extraction that lock in the dominant sector. The assumption of Cobb-Douglas production has qualitatively important implications for their economy's dynamics.[7]

[5]Some have informally argued that these mandates might allow the energy sector to escape lock-in (e.g., Lehmann and Gawel, 2013). Formal analyses of this channel have focused on learning-by-doing as the mechanism for technological change (Kalkuhl et al., 2012), which means that technology matures jointly with energy production. In contrast, here renewable technology matures *before* renewables begin "escaping" lock-in. Clancy and Moschini (2018) show that mandates can induce innovation but do not analyze dynamics. Johnstone et al. (2010) report econometric evidence that mandates to produce renewable energy have increased patenting activity.

[6]An exception is when they model resources as exhaustible or depletable. Thus, when transitions arise in their setting, these transitions are driven by the same forces explored in the resource economics literature.

[7]Most analyses that combine directed technical change and energy have divided technologies between those that augment resources and those that augment other factors such as labor (Smulders and de Nooij, 2003; Di Maria and Valente, 2008; Grimaud and Rouge, 2008; Pittel and Bretschger, 2010; André and Smulders, 2012). These studies have focused on the potential for technical change to enable long-run growth even when an exhaustible resource is essential to production. In contrast, the present paper and Acemoglu et al. (2012) both allow research effort to be directed between multiple types of resources in order to study questions about energy transitions. Acemoglu et al. (2016) develop a related setting in which two types

Quantitatively, I undertake a more realistic calibration than do Acemoglu et al. (2012). In particular, I calibrate a setting with multiple types of energy to match recent data on prices, output, and research, and I estimate the optimal emission tax within a benchmark climate system. My finding that a temporary research subsidy can be critical to limiting climate change echoes one of their more novel results, despite my calibration generating an elasticity of substitution between resources and machines that does not imply the extreme lock-in seen in their setting. However, my results do not support their finding that a temporary emission tax may be similarly effective. One could in principle mimic the effects of the temporary research subsidy by adopting a sufficiently high emission tax in the first period, but the policymaker chooses not to adopt such a high tax because it would distort resource supply to a severe degree.[8]

Further, a major lesson of Acemoglu et al. (2012) was that the elasticity of substitution between types of energy determines whether a temporary policy can effectively limit long-run warming. I find that the same elasticity of substitution is critical to optimal policy, but the reason is quite different. In Acemoglu et al. (2012), a catastrophe results unless the clean resource crowds out the dirty resource completely, which happens without permanent policy intervention only if the clean resource is sufficiently substitutable for the dirty resource. In contrast, I work with a conventional climate model that has less extreme implications. The elasticity of substitution between types of energy matters here because the policymaker finds it easier to redirect research effort when that elasticity is larger. As a result, optimal policy strongly redirects research only if that elasticity is sufficiently large. The importance of the substitutability of different types of energy suggests that improved batteries could strongly increase the attractiveness of kickstarting a transition to renewable energy.

Formally, I analyze directed technical change (Acemoglu, 2002) when final good production has a nested constant elasticity of substitution structure that allows innovation and other inputs to be complements. A prominent strand of literature argues that complementarities have been a critical—and often overlooked—element of economic growth (Rosenberg, 1976; Matsuyama, 1995, 1999; Evans et al., 1998). Milgrom et al. (1991) show how complementarities between techniques and inputs can generate persistent patterns of technical change without needing to assume increasing returns. In the present setting, increasing returns to innovation can work to lock in the dominant technology, yet complementarities can nonetheless produce changes in energy technologies and supply. This theory of innovation-led transitions will apply to other settings with complementarities between machines and other

of energy technologies compete in each of many product lines. Each product line's production function is Cobb-Douglas. As a result, their setting again generates strong path dependency or lock-in. Hart (2015) analyzes the implications of alternate assumptions about knowledge spillovers.

[8] Acemoglu et al. (2016) also quantitatively evaluate the dynamics of research subsidies and emission taxes. In line with the present paper, they find that optimal policy uses a near-term research subsidy and does not use a significant emission tax for many decades. Fried (2018) finds that endogenous innovation can substantially increase the emission reductions from a given emission tax. I here consider the degree to which a policymaker would choose to reduce emissions when aware of this dynamic.

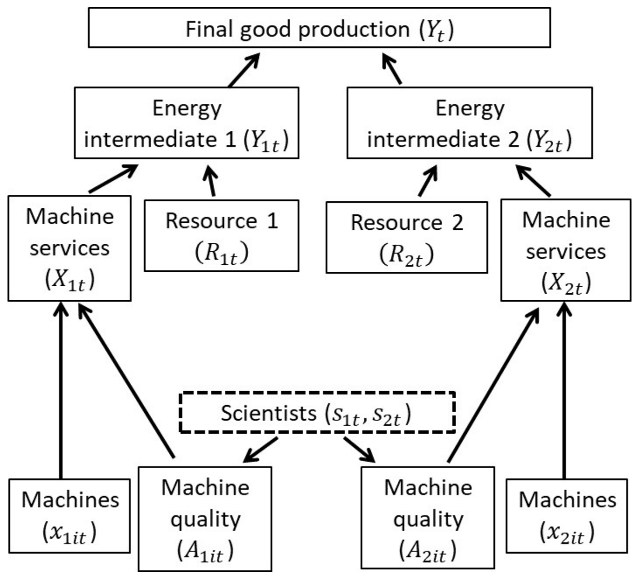


Figure 1: Overview of the theoretical setting, for $N = 2$.

factors of production. Such complementarities may be common. For instance, Grossman et al. (2017) summarize evidence that capital and labor are complements.

The next section describes the theoretical setting. Section 3 analyzes the relative incentive to research technologies in each sector. Section 4 describes the economy's laissez-faire dynamics. Section 5 numerically explores the implications for climate change policies that aim to induce a transition to renewable energy. The final section concludes. The appendix contains additional formal analysis, additional numerical results, and proofs.

2 Setting

I study a discrete-time economy in which final-good production uses multiple types of energy intermediates. Each energy intermediate is generated by combining energy resources with machines. Resources are supplied competitively. A fixed measure of households works as scientists, trying to improve the quality of machines used in producing the energy intermediates. Scientists decide which type of machine to work on. The equilibrium allocation of resources and scientists changes over time as technologies improve. Figure 1 illustrates the model setup, which I now formalize.

Begin with final-good production. The time t final good Y_t is produced competitively from N energy intermediates Y_{jt}, with $j \in \{1, 2, ..., N\}$. The final good is the numeraire in each period. The representative firm's production function takes the familiar constant elasticity of substitution (CES) form:

$$Y_t = A_Y \left(\sum_{j=1}^{N} \nu_j Y_{jt}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}.$$

The parameters $\nu_j \in (0,1)$ are the distribution (or share) parameters, with $\sum_{j=1}^{N} \nu_j = 1$. $A_Y > 0$ is a productivity parameter. I say that resource j is *higher quality* than resource k if and only if $\nu_j > \nu_k$. The parameter ϵ is the elasticity of substitution. The energy intermediates are gross substitutes ($\epsilon > 1$), consistent with evidence in Papageorgiou et al. (2017).

The energy intermediates Y_{jt} are the energy services produced by combining resource inputs R_{jt} with machine inputs X_{jt}. Production of energy intermediates has the following CES form:

$$Y_{jt} = \left(\kappa R_{jt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa) X_{jt}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

The parameter $\kappa \in (0,1)$ is the distribution (or share) parameter. The elasticity of substitution between the resource and machine inputs is σ. I call machines *resource-using* when resources and machines are gross complements ($\sigma < 1$), and I call machines *resource-saving* when resources and machines are gross substitutes ($\sigma > 1$). Resources and machines are less substitutable than are different types of energy intermediates ($\sigma < \epsilon$).

Machine services X_{jt} are produced in a Dixit-Stiglitz environment of monopolistic competition from machines of varying qualities:

$$X_{jt} = \int_0^1 A_{jit}^{1-\alpha} x_{jit}^{\alpha} \, \mathrm{d}i,$$

where $\alpha \in (0,1)$. The machines x_{jit} that work with resource j at time t are divided into a continuum of types, indexed by i. The quality (or efficiency) of machine x_{jit} is then given by A_{jit}. Machines of type i are produced by monopolists who each take the price (p_{jXt}) of machine services as given (each is small) but recognize their ability to influence the price p_{jxit} of machines of type i. The cost of producing a machine is $a > 0$ units of the final good, normalized to $a = \alpha^2$.

Scientists choose which resource they want to study and are then randomly allocated to a machine type i. Each scientist succeeds in innovating with probability $\eta \in (0,1]$. If they fail, scientists earn nothing and the quality of that type of machine is unchanged. Following Acemoglu et al. (2012), successful scientists receive a one-period patent to produce their type of machine, and using resource j as an example, they improve the quality of their machine type to

$$A_{jit} = A_{ji(t-1)} + \gamma A_{ji(t-1)}, \tag{1}$$

where $\gamma > 0$.[9] Scientists are of fixed measure, normalized to 1:

$$1 = \sum_{j=1}^{N} s_{jt}.$$

Firms that enter into production of resource j find a deposit containing one unit of the resource. Firms must pay a fixed cost (in units of the final good) to develop the nth deposit. In equilibrium, all deposits with fixed costs less than p_{jRt} get developed. Order the continuum of deposits by fixed cost. The fixed cost of the nth deposit is then $F_j(n)$, which we set to $F_j(n) = (n/\Psi_j)^{1/\psi}$ for $\psi, \Psi_j > 0$. In equilibrium, $F_j(R_{jt}) = p_{jRt}$. As a result,

$$R_{jt} = \Psi_j p_{jRt}^{\psi}. \tag{2}$$

Resources are therefore supplied isoelastically. I say that resource j is *more accessible* than resource k if and only if $\Psi_j > \Psi_k$. I impose $\psi \geq \alpha/(1-\alpha)$, which ensures that the own-price elasticity of resource supply is greater than the elasticity of machine services with respect to the resource price.

The economy's time t resource constraint is

$$Y_t \geq c_t + a \sum_{j=1}^{N} \int_0^1 x_{jit} \, \mathrm{d}i + \sum_{j=1}^{N} \int_0^{R_{jt}} F_j(n) \, \mathrm{d}n,$$

where $c_t \geq 0$ is the composite consumption good. Households have strictly increasing utility for the consumption good. Scientists therefore each choose their resource type so as to maximize expected earnings.

I study equilibrium outcomes.

Definition 1. *An* equilibrium *is given by sequences of prices for energy intermediates* ($\{p^*_{jt}\}_{j=1}^N$)*, prices for machine services* ($\{p^*_{jXt}\}_{j=1}^N$)*, prices for machines* ($\{p^*_{jxit}\}_{j=1}^N$)*, prices for resources* ($\{p^*_{jRt}\}_{j=1}^N$)*, demands for inputs* ($\{Y^*_{jt}\}_{j=1}^N, \{R^*_{jt}\}_{j=1}^N, \{X^*_{jt}\}_{j=1}^N, \{x^*_{jit}\}_{j=1}^N$)*, and factor allocations* ($\{s^*_{jt}\}_{j=1}^N$) *such that, in each period t: (i)* $\{Y^*_{jt}\}_{j=1}^N$ *maximizes profits of final good producers, (ii)* ($\{R^*_{jt}\}_{j=1}^N, \{X^*_{jt}\}_{j=1}^N$) *maximizes profits of energy intermediate producers, (iii)* ($\{p^*_{jxit}\}_{j=1}^N, \{x^*_{jit}\}_{j=1}^N$) *maximize profits of the producers of each machine i in each sector j, (iv) resource producers enter until they earn zero profits, (v)* $\{s^*_{jt}\}_{j=1}^N$ *maximizes expected earnings of scientists, (vi) prices clear the factor and input markets, and (vii) technologies evolve as in equation* (1)*.*

[9] I here follow the literature in using an increasing returns representation of innovation. I will show that an innovation-led transition is possible even though increasing returns push scientists toward the more advanced sector. See Hart (2015) for an analysis of decreasing returns to innovation in a setting closely related to Acemoglu et al. (2012).

The equilibrium prices clear all factor markets and all firms maximize profits. If scientists are employed in any two sectors, they receive the same expected reward from both, and if they are not employed in some sector, they receive a greater expected reward in some other sector that has nonzero scientists. The first appendix establishes that the equilibrium is stable in a tâtonnement sense when $N = 2$. Throughout, I drop the asterisks when clear.

3 The Equilibrium Direction of Research

The first-order condition for a producer of machine services yields the following demand curve for machines of type i in sector j:

$$x_{jit} = \left(\frac{p_{jXt}}{p_{jxit}}\alpha\right)^{\frac{1}{1-\alpha}} A_{jit}. \tag{3}$$

The monopolist producer of x_{jit} therefore faces an isoelastic demand curve and accordingly marks up its price by a constant fraction over marginal cost: $p_{jxit} = a/\alpha = \alpha$. In equilibrium, the producer of machine type i for use with resource j earns profits of:

$$\pi_{jxit} = (p_{jxit} - a)x_{jit} = \alpha(1-\alpha)p_{jXt}^{\frac{1}{1-\alpha}} A_{jit}.$$

If a scientist succeeds in innovating at time t, she exercises her patent to obtain the monopoly profit π_{jxit}. Her expected reward to choosing to research machines that work with resource type j is therefore

$$\Pi_{jt} = \eta\,\alpha\,(1-\alpha)p_{jXt}^{\frac{1}{1-\alpha}}(1+\gamma)A_{j(t-1)}, \tag{4}$$

where $A_{j(t-1)}$ is the average quality of machines in sector j. This average quality evolves as

$$A_{jt} = \int_0^1 \left[\eta s_{jt}(1+\gamma)A_{ji(t-1)} + (1-\eta s_{jt})A_{ji(t-1)}\right] \mathrm{d}i = (1+\eta\gamma s_{jt})A_{j(t-1)}, \tag{5}$$

where s_{jt} is the measure of scientists working on resource j.

Now consider the relative incentive to research technologies that work with resource j rather than technologies that work with resource k. From equation (4), we have

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)} + \gamma A_{j(t-1)}}{A_{k(t-1)} + \gamma A_{k(t-1)}} \left[\frac{p_{jXt}}{p_{kXt}}\right]^{\frac{1}{1-\alpha}}. \tag{6}$$

The intermediate-good producer's first-order conditions for profit-maximization yield

$$p_{jXt} = (1-\kappa)p_{jt}\left[\frac{X_{jt}}{Y_{jt}}\right]^{-1/\sigma} \quad \text{and} \quad p_{jRt} = \kappa\, p_{jt}\left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma}.$$

The relative incentive to research technologies for use in sector j increases in the relative price of the intermediates and decreases in the machine-intensity of sector j's output. Combining the first-order conditions, we have

$$p_{jXt} = \frac{1-\kappa}{\kappa}\left[\frac{R_{jt}}{X_{jt}}\right]^{1/\sigma} p_{jRt}. \tag{7}$$

From equation (3) and the monopolist's markup, we have

$$x_{jit} = p_{jXt}^{\frac{1}{1-\alpha}} A_{jit}.$$

Substituting into the definition of X_{jt} and using the definition of A_{jt}, we have

$$X_{jt} = p_{jXt}^{\frac{\alpha}{1-\alpha}} A_{jt}. \tag{8}$$

Substitute into equation (7) and solve for equilibrium machine prices:

$$p_{jXt} = \left[p_{jRt}\frac{1-\kappa}{\kappa}\right]^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}}. \tag{9}$$

Substituting into equation (6) and then using (2), we have:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \underbrace{\frac{(1+\gamma)A_{j(t-1)}}{(1+\gamma)A_{k(t-1)}}}_{\text{patent quality effect}} \underbrace{\left(\frac{(1+\eta\gamma s_{jt})A_{j(t-1)}}{(1+\eta\gamma s_{kt})A_{k(t-1)}}\right)^{\frac{-1}{\sigma+\alpha(1-\sigma)}}}_{\text{supply expansion effect}} \underbrace{\left(\frac{R_{jt}}{R_{kt}}\right)^{\frac{1}{\sigma+\alpha(1-\sigma)}\frac{\psi+\sigma}{\psi}}}_{\text{market size effect}} \left(\frac{\Psi_j}{\Psi_k}\right)^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}}. \tag{10}$$

Four terms determine scientists' relative incentive to research machines. The first term is a *patent quality effect* that directs research effort to the sector in which scientists will end up with the patent to better technology.[10] The other channels derive from the relative price of machine services: $(p_{jXt}/p_{kXt})^{1/(1-\alpha)}$ in equation (6). The *supply expansion effect* pushes scientists away from the more advanced sector. From equation (8), the supply of X_{jt} shifts out when its machines' average quality A_{jt} increases, and it shifts out to an especially large degree when α is small. When σ is small (machines are resource-using), the demand curve is steep because the marginal product of additional machines is constrained by the

[10] The patent quality effect depends on the realized technology, not solely on the increment to technology produced by a scientist's efforts, which introduces a type of business-stealing distortion. If γ differed by sector and were very small in the more advanced sector, scientists could have a stronger incentive to research machines in the more advanced sector even though their efforts would not improve these machines. This business-stealing distortion vanishes under the assumption of identical γ: by attracting scientists to the more advanced sector, the patent quality effect here also attracts them to the sector where they make the greatest advance.

supply of R_{jt}. By shifting out supply, the increase in A_{jt} induces a relatively large decline in the equilibrium price p_{jXt}. However, when σ is large, machines are resource-saving and the demand curve is relatively flat. The increase in A_{jt} then induces a relatively small decline in the equilibrium price p_{jXt}. Improving technology therefore pushes scientists away to a greater degree when the demand curve is steep (σ is small) or the shift in supply is large (α is small) because it then reduces p_{jXt} more strongly.

Pause to consider the net effect of a relative improvement in sector j's average technology. We have seen that this relative improvement attracts scientists through the patent quality effect and repels scientists through the supply expansion effect. From equations (5) and (10), the supply expansion effect dominates the patent quality effect if and only if $\sigma < 1$. As $\sigma \to 0$, demand for machines becomes perfectly inelastic and the supply expansion effect becomes large. As $\sigma \to \infty$, demand for machines becomes perfectly elastic and the supply expansion effect vanishes. As $\sigma \to 1$, the two effects exactly cancel, so that the incentives to research machines in one sector or the other do not directly depend on the relative quality of technology in each sector.

The remaining machine price channels in equation (10) connect research incentives to resource supply. In particular, we see research directed towards the sector with greater resource use. This is a *market size effect*. It arises for two reasons. First, from equation (7), an increase in R_{jt} shifts out demand for X_{jt}, and does so to an especially large degree when machines and resources are stronger complements (i.e., as σ becomes small). Second, also from equation (7), an increase in p_{jRt} (for given R_{jt}) also shifts out demand for X_{jt} as firms substitute machines for resources. This channel is especially strong when the elasticity of substitution between resources and machines is large, and it vanishes as that elasticity goes to zero. Each of these outward shifts in demand for X_{jt} increases scientists' incentives to work on improving machines in sector j. Therefore, the market size effect draws scientists towards whichever sector is increasing its share of resource supply over time.

Now consider how sector j's share of extraction changes from time t to $t+1$. Combining the intermediate-good producers' first-order condition for resources with the final-good producers' first-order conditions, we find demand for each resource:

$$p_{jRt} = \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma} \quad \text{and} \quad p_{kRt} = \kappa\, \nu_k A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_{kt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{kt}}{Y_{kt}}\right]^{-1/\sigma}. \tag{11}$$

Market-clearing for each resource then implies

$$\left[\frac{R_{jt}}{\Psi_j}\right]^{1/\psi} = \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma}, \tag{12}$$

$$\left[\frac{R_{kt}}{\Psi_k}\right]^{1/\psi} = \kappa\, \nu_k A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_{kt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{kt}}{Y_{kt}}\right]^{-1/\sigma}. \tag{13}$$

Demand for sector j's resources (for example) shifts inward as the share of those resources in the production of intermediate good j increases and also shifts inward as the share of intermediate good j in production of the final good increases. Rearranging equations (12) and (13) and then dividing, we have:

$$\left[\frac{R_{jt}}{R_{kt}}\right]^{\frac{1}{\sigma}+\frac{1}{\psi}} = \frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{1/\psi}\left[\frac{Y_{jt}}{Y_{kt}}\right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}}. \tag{14}$$

The change in sector j's share of resource extraction from time t to time $t+1$ therefore has the same sign as the change in sector j's share of intermediate good production. Observe that increasing the average quality of technology A_{jt} increases production of the intermediate good Y_{jt}. Thus, sector j's share of resource extraction tends to increase when the average quality of its technology is advancing relative to sector k. The sector that is advancing more rapidly tends to attract even more scientists in later periods through market size effects, which works to lock in that sector's technological advantage.

4 The Equilibrium Evolution of Resource Use and Technology

I now study the evolution of the economy in a special case with $N = 2$. Label the two sectors as j and k. I first analyze when transitions can occur and then describe long-run outcomes. I show that both the possibility of a transition and the nature of long-run outcomes are sensitive to whether machines are resource-using or resource-saving. I then study three special cases that highlight the relevant dynamics and illustrate the main ideas with a numerical example.

The following assumption will be useful for studying transitions, because it establishes a time t_0 in which sector j dominates research activity with technology that is more advanced than (or not too much less advanced than) sector k's technology:

Assumption 1. *$A_{j(t_0-1)}/A_{k(t_0-1)} > [\Psi_j/\Psi_k]^{\theta}$ and $s^*_{jt_0} > 0.5$ for some time t_0, where $\theta \triangleq 1/[(1-\alpha)(1+\psi)] \in (0,1]$.*

The next lemma establishes one set of structural conditions under which Assumption 1 holds:

Lemma 1. *If $\nu_j = \nu_k$ and $\Psi_j = \Psi_k$, then Assumption 1 holds if (i) $A_{j(t_0-1)} > A_{k(t_0-1)}$ and (ii) either $\sigma > 1$ or σ is not too much smaller than 1.*

Proof. See appendix. □

Define a *transition in research* as occurring at the first time $t \geq t_0$ at which s_{jt} begins declining, a *transition in extraction* as occurring at the first time $t \geq t_0$ at which R_{jt}/R_{kt} begins declining, and a *transition in technology* as occurring at the first time $t \geq t_0$ at which A_{jt}/A_{kt} begins declining. Finally, define resource j as being *locked-in* from time t_0 when no type of transition occurs after t_0. We have the following result:

Proposition 2. *Let Assumption 1 hold. If $\sigma > 1$, then resource j is locked-in from time t_0. If $\sigma < 1$, then a transition in extraction occurs only after a transition in research and a transition in technology occurs only after a transition in extraction. If resource j is relatively accessible ($\Psi_j \geq \Psi_k$), then a transition in technology occurs while sector j still provides the larger share of resource supply.*

Proof. See appendix. □

We see two cases. First, if machines are resource-saving ($\sigma > 1$), then a transition cannot happen. The economy is locked-in to the dominant sector. The proof shows that sector j increases its share of resource supply whenever it dominates research effort. And when sector j is both increasing its share of resource supply and dominating research effort, the market size and patent quality channels in equation (10) both pull more scientists towards sector j. Sector j therefore increases its dominance of research effort over time and continually increases its technological advantage over sector k. Sector j's increasing share of resource supply and its increasing share of research activity form a positive feedback loop that prevents sector k from ever catching up: sector j's increasingly improved technology and increasing share of resource extraction both work to attract ever more scientists to sector j, and the improving relative quality of technology in sector j works to increase its share of extraction over time.

The dynamics are qualitatively different if machines are resource-using ($\sigma < 1$). Now sector j's dominant share of research activity works to push scientists away from sector j through the supply expansion effect in equation (10), but sector j's improving relative technology works to increase its share of resource supply and thus strengthens the market size effect that pulls scientists towards sector j. The change in the market size effect is especially significant when sector j's technology is still immature, so that sector j can increasingly dominate research effort over time. However, the market size effect becomes less and less sensitive to the quality of sector j's technology as that technology becomes more advanced. The supply expansion effect eventually dominates the market size effect, which pushes scientists back towards sector k. At this point a transition in research occurs. As sector j's share of research continues to fall, a transition in extraction can occur. The transition in extraction is *innovation-led*: it can occur only after the transition in research. Even though research transitions before extraction, sector k does not begin to dominate research effort (triggering a transition in technology) until sometime after the transition in extraction, when both the market size effect and the supply expansion effect are working to push scientists towards sector k. Finally, if resource j is relatively accessible, then a transition in technology must happen while sector j still dominates resource supply. Just as the transition in extraction must follow a transition in research, so too a change in the sector that dominates resource supply must follow a change in the sector that dominates research.

Now consider long-run equilibria for any economy. The following proposition shows that a corner research allocation can persist indefinitely if and only if $\sigma > 1$.

Proposition 3. *Assume that $\sigma > 1$ and let t_0 be an arbitrary time such that, in equilibrium, either $s^*_{jt_0} = 0$ or $s^*_{jt_0} = 1$. Then $s^*_{jt} = s^*_{jt_0}$ for all $t \geq t_0$. However, if $\sigma < 1$, then for any time t_0 there exists $t > t_0$ such that $s^*_{jt} \in (0, 1)$.*

Proof. See appendix. □

Consider an allocation with all scientists working in sector j at time t_0. Improving sector j's technology increases its share of extraction over time. The corner allocation can persist only if $\Pi_{jt}/\Pi_{kt} \geq 1$ for all later times t when evaluated at $s_{jt} = 1$. When $\sigma > 1$, the market size effect works to increase Π_{jt}/Π_{kt} over time and so too does the combination of the patent quality and supply expansion effects. The corner allocation persists indefinitely. However, when $\sigma < 1$, the market size effect conflicts with the combined patent quality and supply expansion effects. The proof shows that as the average quality of technology in sector j improves, the market size effect becomes negligibly small: resources are not constrained by the availability of machines when machines become very advanced, so further improvements in their average quality does not affect resource use very much. Eventually the supply expansion effect dominates not just the patent quality effect but also the market size effect. Π_{jt}/Π_{kt} then begins to decline. If the allocation of scientists is held fixed at the corner, then eventually Π_{jt}/Π_{kt} falls below unity, at which point the corner allocation can no longer be an equilibrium. Corner allocations are self-sustaining equilibria for $\sigma > 1$ but must eventually disappear when $\sigma < 1$.

The steady state for this economy holds the research allocation fixed at some value s, so that each type of technology improves at a constant rate. The next proposition shows that a steady state with $s \in (0, 1)$ is not stable if $\sigma > 1$:

Proposition 4. *Assume that $\sigma > 1$. If $s^*_{jt} \in (0.5, 1)$, then $s^*_{j(t+1)} > s^*_{jt}$. If $s^*_{jt} \in (0, 0.5)$, then $s^*_{j(t+1)} < s^*_{jt}$.*

Proof. See appendix. □

When machines are resource-saving, any equilibrium research allocation in which the majority of scientists work in one sector at time t will have an even larger majority of scientists in that sector at time $t + 1$. In contrast, the next proposition shows that the economy approaches an interior steady state when machines are resource-using ($\sigma < 1$).

Proposition 5. *Assume $\sigma < 1$. Then the only steady-state research allocation has $s_{jt} = 0.5$ and the following are true as $t \to \infty$:*

1. $s^*_{jt} \to 0.5$.
2. *If $\nu_j = \nu_k$ and $\Psi_j = \Psi_k$, then $R^*_{jt} = R^*_{kt}$ and $A_j = A_{kt}$.*
3. *If $\nu_j \geq \nu_k$ and $\Psi_j \geq \Psi_k$ with strict inequality for at least one, then $R^*_{jt} > R^*_{kt}$ and $A_{jt} > A_{kt}$.*

4. R^*_{jt} and R^*_{kt} become constant, and R^*_{jt}/R^*_{kt} approaches $\left[\left(\frac{\nu_j}{\nu_k}\right)^{\psi}\frac{\Psi_j}{\Psi_k}\right]^{\frac{\epsilon}{\epsilon+\psi}}$.

Proof. See appendix. □

The proposition gives four results. First, the economy approaches a steady-state research allocation in which the average quality of each technology improves at the same rate. The steady state is both unique and stable. Second, if the two resources are of the same quality and accessibility, then the steady state has identical technology and extraction in each. Third, if one sector's resource is of higher quality and more accessible, then that sector dominates resource use and has better technology. Fourth, extraction eventually approaches a constant value in each sector. As discussed previously, resource supply becomes less sensitive to further advances in machine quality as machines becomes more advanced, so that resource use cannot grow at a nonzero constant rate for all time. Observe that the long-run share of each resource is not sensitive to the magnitude of σ as long as $\sigma < 1$. These shares are instead completely determined by the characteristics of each resource (specifically, Ψ_j, Ψ_k, ν_j, and ψ) and by the elasticity of substitution between the two types of energy (ϵ).

We have previously seen when a transition can happen. We now see a case in which a transition must happen.

Corollary 6. *Assume that $\sigma < 1$, $\nu_j = \nu_k$, and $\Psi_j = \Psi_k$. Then when Assumption 1 holds, both a transition in research and a transition in extraction happen before reaching the steady-state research allocation.*

Proof. By Proposition 5, $A_{jt} = A_{kt}$ in the steady-state research allocation. But Assumption 1 ensures that $A_{jt_0} > A_{kt_0}$. Thus there exists $t_1 > t_0$ such that $s_{jt_1} < 0.5$. By Proposition 2, a transition in research, a transition in extraction, and a transition in technology must happen between t_0 and t_1. □

The sector with more advanced technology can attract the majority of researchers when neither technology is very advanced. However, the relatively backward sector must eventually dominate the research allocation because the steady state has both sectors being equally advanced. As the technologies improve, scientists must eventually start switching towards the relatively backward sector, and by Proposition 2, extraction must also start switching towards the relatively backward sector sometime before the relatively backward sector begins to dominate the research allocation. Transitions in research, extraction, and technology must happen before reaching the steady-state research allocation.

I next explore three special cases that highlight the competing effects that drive the evolution of the economy, structurally grounding Assumption 1 in each case. I then provide a numerical example with three types of energy before turning to the full, calibrated model.

4.1 Special Case With Only Market Size Effects

Begin by considering the Cobb-Douglas case studied in previous literature, which arises as $\sigma \to 1$. Let $Y_{jt} = R_{jt}^{\kappa} X_{jt}^{1-\kappa}$ and $Y_{kt} = R_{kt}^{\kappa} X_{kt}^{1-\kappa}$. Equation (10) becomes:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \left(\frac{1+\eta\gamma s_{jt}}{1+\eta\gamma s_{kt}}\right)^{-1} \left(\frac{R_{jt}}{R_{kt}}\right)^{\frac{\psi+1}{\psi}} \left[\frac{\Psi_j}{\Psi_k}\right]^{-1/\psi}. \tag{15}$$

As previously discussed, the patent quality and supply expansion effects exactly cancel, so that the market size effect completely determines the evolution of the research allocation.

How does the market size effect evolve over time? Substituting the Cobb-Douglas forms, equation (14) becomes

$$\left[\frac{R_{jt}}{R_{kt}}\right]^{\frac{\psi+1}{\psi}-\kappa\frac{\epsilon-1}{\epsilon}} = \frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{1/\psi}\left[\frac{X_{jt}}{X_{kt}}\right]^{(1-\kappa)\frac{\epsilon-1}{\epsilon}}.$$

Substituting equation (7) into equation (8) and then using equation (2), we have:

$$X_{jt} = \left[\frac{1-\kappa}{\kappa} R_{jt}^{\frac{\psi+1}{\psi}} \Psi_j^{-1/\psi}\right]^{\alpha} A_{jt}^{1-\alpha}.$$

We then have:

$$\left[\frac{R_{jt}}{R_{kt}}\right]^{\Gamma} = \frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1}{\psi}\left[1-\alpha(1-\kappa)\frac{\epsilon-1}{\epsilon}\right]}\left[\frac{A_{jt}}{A_{kt}}\right]^{(1-\alpha)(1-\kappa)\frac{\epsilon-1}{\epsilon}}, \tag{16}$$

where $\Gamma \triangleq \frac{\psi+1}{\psi} - \frac{\epsilon-1}{\epsilon}\left(\kappa + \alpha(1-\kappa)\frac{\psi+1}{\psi}\right) > 0$. Sector j's share of resource extraction increases in the relative quality of sector j's technology. The more that sector j advances relative to sector k, the more that R_{jt}/R_{kt} grows, and the more that R_{jt}/R_{kt} grows, the more that Π_{jt}/Π_{kt} shifts up for any given s_{jt}. The equilibrium s_{jt} must therefore increase as $A_{j(t-1)}/A_{k(t-1)}$ increases.[11]

Using equations (15) and (16) and the result from the appendix that the total derivative of Π_{jt}/Π_{kt} with respect to s_{jt} is negative, we have $s_{jt} > 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} > \left[\frac{\nu_j}{\nu_k}\right]^{-\frac{1}{(1-\alpha)(1-\kappa)\frac{\epsilon-1}{\epsilon}}}\left[\frac{\Psi_j}{\Psi_k}\right]^{-\frac{\kappa}{(1-\alpha)(1-\kappa)(\psi+1)}}.$$

[11]This result explains why relative technology does not directly affect research incentives in Acemoglu et al. (2012): technology matters in their equation (17) via the same patent quality effect seen here (which they call a "direct productivity effect") and also through their "price effect", but substituting in for relative output prices from their equation (A.3) shows that these two effects exactly cancel. Relative technology ends up playing a role in their setting's equilibrium (see their equation (18)) because relative market size is proportional to the relative quality of technology (see their equation (A.5)). Thus, their Cobb-Douglas assumption generates the same dynamics as in the Cobb-Douglas case analyzed here.

If $\Psi_j < \Psi_k$ and $\nu_j < \nu_k$, then Assumption 1 holds when this inequality holds. We now see how lock-in arises: $s_{jt} > 0.5$ implies that $A_{jt}/A_{kt} > A_{j(t-1)}/A_{k(t-1)}$, which ensures that $s_{j(t+1)} > s_{jt}$, which implies that $A_{j(t+1)}/A_{k(t+1)} > A_{jt}/A_{kt}$, and so on. There is a knife-edge case in which $s_{jt} = 0.5$ for all time, but if equilibrium s_{jt} ever takes on any other value, then the economy progresses to a corner allocation in research.

4.2 Special Case Without Market Size Effects

Now consider a case with $\sigma = \epsilon > 1$. From equation (14), we have

$$\frac{R_{jt}}{R_{kt}} = \left(\frac{\nu_j}{\nu_k} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right)^{\frac{\sigma\psi}{\sigma+\psi}} .$$

The shares of extraction are fixed over time, independently of the quality of technology in either sector. Because R_{jt}/R_{kt} is fixed over time, market size effects cease to steer the evolution of research activity. Substituting for R_{jt}/R_{kt} in equation (10), we have:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \left(\frac{A_{j(t-1)}}{A_{k(t-1)}} \right)^{\frac{(1-\alpha)(\epsilon-1)}{(1-\alpha)\epsilon+\alpha}} \left(\frac{1+\eta\gamma s_{jt}}{1+\eta\gamma s_{kt}} \right)^{\frac{-1}{(1-\alpha)\epsilon+\alpha}} \left(\frac{\nu_j}{\nu_k} \right)^{\frac{\epsilon}{(1-\alpha)\epsilon+\alpha}} .$$

As the average quality of technology in sector j improves, the patent quality effect shifts Π_{jt}/Π_{kt} upward and so increases the share of scientists working in sector j. If

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} > \left(\frac{\nu_j}{\nu_k} \right)^{-\frac{\epsilon}{(1-\alpha)(\epsilon-1)}} ,$$

then $s_{jt} > 0.5$. If, in addition, $\nu_j < \nu_k$, then Assumption 1 holds. In that case, sector j is locked-in insofar as its share of research increases towards a corner allocation in which sector j attracts all scientists, but this increasing dominance of research activity does not affect sector j's share of extraction. There is a knife-edge case in which $s_{jt} = 0.5$ for all time, but as with the Cobb-Douglas case analyzed above, if equilibrium s_{jt} ever takes on any other value, then the economy progresses to a corner allocation in research.

4.3 Special Case With Dominant Supply Expansion Effect

Finally, consider the special case of a Leontief production function for each intermediate good, which arises as $\sigma \to 0$. In order to aid exposition, fix $\psi = \alpha/(1-\alpha)$. Let $Y_{jt} = \min\{R_{jt}, X_{jt}\}$ and $Y_{kt} = \min\{R_{kt}, X_{kt}\}$. In equilibrium, $R_{jt} = X_{jt}$ and $R_{kt} = X_{kt}$. From equation (8), we have:

$$p_{jXt} = \left(\frac{R_{jt}}{A_{jt}} \right)^{\frac{1-\alpha}{\alpha}} .$$

From equation (6), we then have:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)}}{A_{k(t-1)}} \left(\frac{A_{jt}}{A_{kt}}\right)^{-1/\alpha} \left(\frac{R_{jt}}{R_{kt}}\right)^{1/\alpha}. \tag{17}$$

From equation (8),

$$p_{jXt} X_{jt} = X_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}.$$

And from equation (2),

$$p_{jRt} R_{jt} = \Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}}.$$

Intermediate good producers' zero-profit condition is

$$p_{jt} Y_{jt} = \Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}} + X_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}.$$

Substituting for p_{jt} from the final good producers' first-order condition and then setting $X_{jt} = R_{jt}$ and $Y_{jt} = R_{jt}$, we have:

$$\nu_j Y_t^{1/\epsilon} = R_{jt}^{\frac{1-\epsilon}{\epsilon}} \left[\Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}} + R_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}\right].$$

Using $\psi = \alpha/(1-\alpha)$, we have:

$$\nu_j Y_t^{1/\epsilon} = R_{jt}^{\frac{1-\epsilon}{\epsilon} + \frac{1}{\alpha}} \left[\Psi_j^{-\frac{1-\alpha}{\alpha}} + A_{jt}^{-\frac{1-\alpha}{\alpha}}\right].$$

Obtaining an analogous result for sector k and dividing the two, we obtain:

$$\frac{R_{jt}}{R_{kt}} = \left(\frac{\nu_j}{\nu_k} \frac{\left[\Psi_k^{-\frac{1-\alpha}{\alpha}} + A_{kt}^{-\frac{1-\alpha}{\alpha}}\right]}{\left[\Psi_j^{-\frac{1-\alpha}{\alpha}} + A_{jt}^{-\frac{1-\alpha}{\alpha}}\right]}\right)^{\frac{\epsilon\alpha}{\alpha+(1-\alpha)\epsilon}}. \tag{18}$$

If $\nu_j = \nu_k$, then we have $R_{jt} \geq R_{kt}$ if and only if A_{jt} is large enough. Substituting into equation (17), we have:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)}}{A_{k(t-1)}} \left(\frac{A_{jt}}{A_{kt}}\right)^{-1/\alpha} \left(\frac{\nu_j}{\nu_k} \frac{\left[\Psi_k^{-\frac{1-\alpha}{\alpha}} + A_{kt}^{-\frac{1-\alpha}{\alpha}}\right]}{\left[\Psi_j^{-\frac{1-\alpha}{\alpha}} + A_{jt}^{-\frac{1-\alpha}{\alpha}}\right]}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}. \tag{19}$$

Now consider how Π_{jt}/Π_{kt} evolves when $s_{jt} = 1$. If Π_{jt}/Π_{kt} decreases over time, then the corner allocation cannot persist. Take logs and differentiate with respect to $A_{j(t-1)}$, holding

s_{jt} fixed:

$$\frac{\partial \ln(\Pi_{jt}/\Pi_{kt})}{\partial A_{j(t-1)}} = \left(1-\frac{1}{\alpha}\right)\frac{1}{A_{j(t-1)}} - \frac{\epsilon}{\alpha+(1-\alpha)\epsilon}\frac{1}{\left[\Psi_j^{-\frac{1-\alpha}{\alpha}} + A_{jt}^{-\frac{1-\alpha}{\alpha}}\right]}\frac{-(1-\alpha)}{\alpha}(1+\eta\gamma s_{jt})^{\frac{-(1-\alpha)}{\alpha}} A_{j(t-1)}^{\frac{-(1-\alpha)}{\alpha}-1}$$

$$= \frac{1-\alpha}{\alpha}\frac{1}{A_{j(t-1)}}\left\{ -1 + \underbrace{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}\frac{A_{jt}^{-\frac{1-\alpha}{\alpha}}}{\left[\Psi_j^{-\frac{1-\alpha}{\alpha}} + A_{jt}^{-\frac{1-\alpha}{\alpha}}\right]}}_{\to 0 \text{ as } A_{jt}\to\infty} \right\}.$$

The right-hand term in braces decreases in A_{jt}, going to 0 as $A_{jt} \to \infty$. Therefore the derivative becomes negative as A_{jt} becomes large. As established by Proposition 3, a corner allocation in research cannot persist indefinitely. A corner allocation can persist for some finite interval when A_{jt} is not too large, but over time the weakening market size effect leads Π_{jt}/Π_{kt} to decrease as A_{jt} continues to grow.

Now consider a steady-state research allocation, with $s_{jt} = s$ for all $t \geq t_0$. Because a corner allocation cannot persist, s must be strictly greater than 0 and strictly less than 1. As t increases, $A_{j(t-1)}$ and $A_{k(t-1)}$ become arbitrarily large. From equation (19), we have:

$$\frac{\Pi_{jt}}{\Pi_{kt}} \to \frac{A_{j(t-1)}}{A_{k(t-1)}}\left(\frac{A_{jt}}{A_{kt}}\right)^{-\frac{1}{\alpha}}\left(\frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}$$

$$= \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-\frac{1-\alpha}{\alpha}}\left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{-\frac{1}{\alpha}}\left(\frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}. \tag{20}$$

At an equilibrium with $s \in (0, 1)$, $\Pi_{jt} = \Pi_{kt}$. Then, for t sufficiently large, we must have:

$$\left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{\frac{1}{\alpha}} = \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-\frac{1-\alpha}{\alpha}}\left(\frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}.$$

At a steady state, $A_{j(t-1)} = (1 + \eta\gamma s)^{\Delta} A_{j(t-1-\Delta)}$ and $A_{k(t-1)} = (1 + \eta\gamma(1 - s))^{\Delta} A_{k(t-1-\Delta)}$. Therefore the following must hold for all $\Delta \geq 0$:

$$\left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{\frac{1}{\alpha}} = \left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{-\Delta\frac{1-\alpha}{\alpha}}\left(\frac{A_{j(t-1-\Delta)}}{A_{k(t-1-\Delta)}}\right)^{-\frac{1-\alpha}{\alpha}}\left(\frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}.$$

This can hold for all $t \geq 0$ if and only if

$$\left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{-\Delta\frac{1-\alpha}{\alpha}} = 1,$$

which in turn holds if and only if $s = 0.5$. Thus, the steady state research allocation must have $s = 0.5$. From equation (20), $\nu_j \geq \nu_k$ and $\Psi_j \geq \Psi_k$ then imply $A_{j(t-1)} \geq A_{k(t-1)}$ in the steady-state research allocation, with $A_{j(t-1)} > A_{k(t-1)}$ if in addition either $\nu_j > 0.5$ or $\Psi_j > \Psi_k$. Further, from equation (18), R_{jt}/R_{kt} approaches a constant value as t becomes large and $\nu_j \geq \nu_k$ with $\Psi_j \geq \Psi_k$ imply $R_{jt} \geq R_{kt}$, with $R_{jt} > R_{kt}$ if either $\nu_j > \nu_k$ or $\Psi_j > \Psi_k$.

Finally, consider an early time t_0 at which $A_{j(t_0-1)}$ and $A_{k(t_0-1)}$ are much smaller than Ψ_j and Ψ_k, respectively, and the economy is not yet at a steady-state research allocation. Equation (19) becomes:

$$\begin{aligned}\frac{\Pi_{jt_0}}{\Pi_{kt_0}} &\approx \left(\frac{A_{j(t_0-1)}}{A_{k(t_0-1)}}\right)^{-\frac{1-\alpha}{\alpha}} \left(\frac{1+\eta\gamma s_{jt_0}}{1+\eta\gamma(1-s_{jt_0})}\right)^{-\frac{1}{\alpha}} \left(\frac{\nu_j}{\nu_k}\left[\frac{A_{jt_0}}{A_{kt_0}}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}} \\ &= \left[\left(\frac{A_{j(t_0-1)}}{A_{k(t_0-1)}}\right)^{(1-\alpha)(\epsilon-1)} \left(\frac{1+\eta\gamma s_{jt_0}}{1+\eta\gamma(1-s_{jt_0})}\right)^{-1} \left(\frac{\nu_j}{\nu_k}\right)^{\epsilon}\right]^{\frac{1}{\alpha+(1-\alpha)\epsilon}}. \end{aligned} \tag{21}$$

The right-hand side increases in $A_{j(t_0-1)}/A_{k(t_0-1)}$ and decreases in s_{jt_0}. We have that $s_{jt_0} > 0.5$ if and only if[12]

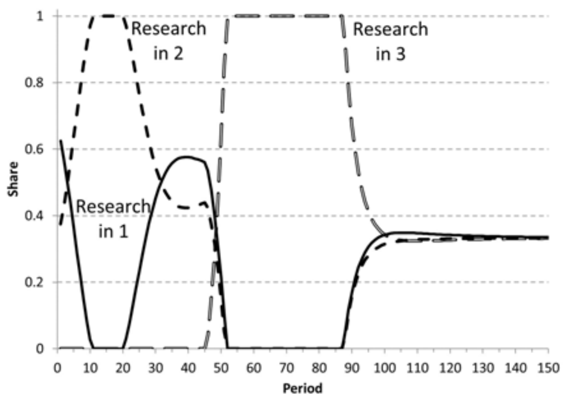
$$\frac{A_{j(t_0-1)}}{A_{k(t_0-1)}} > \left(\frac{\nu_j}{\nu_k}\right)^{\frac{-\epsilon}{(1-\alpha)(\epsilon-1)}}. \tag{22}$$

If $s_{jt_0} > 0.5$, then $A_{j(t_0-1)}/A_{k(t_0-1)}$ increases over time and the right-hand side of equation (21) shifts up over time. As a result, $s_{j(t_0+1)} > s_{jt_0}$. Therefore, sector j can increase its share of research effort over an interval of time with not-too-advanced technology. The reason is that the market size effect increasingly favors researching in sector j, as can be seen from equation (18). However, eventually sector j's technology becomes sufficiently advanced that the market size effect weakens and the supply expansion effect pushes scientists back towards sector k. A transition in research thus arises because the sensitivity of R_{jt}/R_{kt} to technological quality diminishes as technology advances, eventually making the supply expansion effect the primary determinant of research activity.

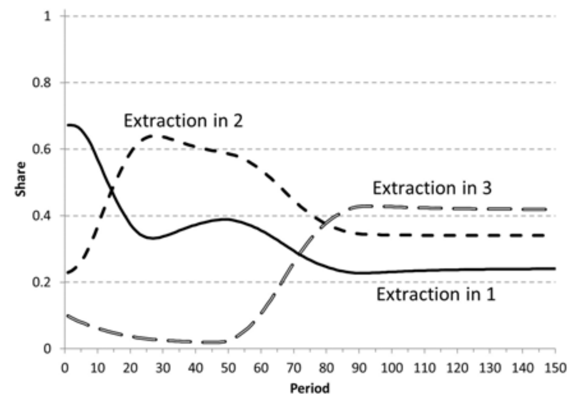
4.4 Numerical Example

I now consider a numerical example in order to make these ideas more concrete. Let there be three types of energy ($N = 3$), which differ only in their quality ν and in their initial technology. Let the first type of energy represent coal, the second represent oil, and the third represent gas. Looking back two hundred years, technologies for using coal were far more advanced than technologies for using oil, which in turn were more developed than

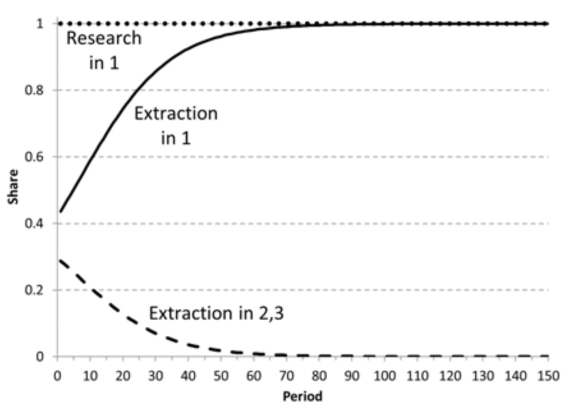
[12] If $\nu_j \leq \nu_k$ and $\Psi_j \leq \Psi_k$, then inequality (22) implies that Assumption 1 holds at t_0.



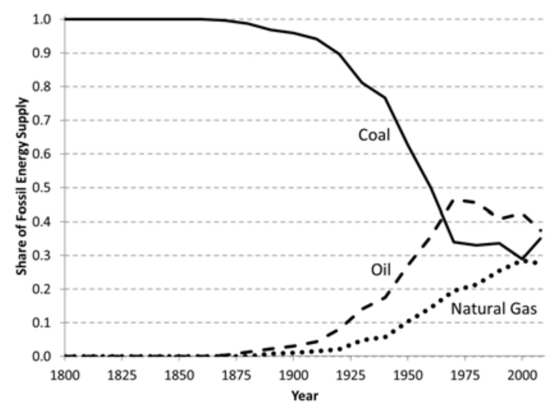
(a) Research Shares with $\sigma = 0.5$



(b) Extraction Shares with $\sigma = 0.5$



(c) Research and Extraction Shares with $\sigma = 1.5$



(d) Historical Extraction Shares

Figure 2: Top: An example of an innovation-led transition, with $\sigma = 0.5$. Bottom left: An example of lock-in, with $\sigma = 1.5$. Resources 2 and 3 have nearly identical extraction shares. Bottom right: Shares of global fossil energy supply, from Smil (2010).

technologies for using gas. I therefore fix the initial average quality of technology at 0.05 for coal, at 1% of this value for oil, and at 0.1% of this value for gas. We can think of the quality of fossil fuel resources as largely determined by the ratio of carbon to hydrogen bonds.[13] Energy derives from breaking hydrogen bonds. Fuels with a lot of carbon and little hydrogen are considered to be of lower quality because they are bulkier and more polluting. Coal is mostly carbon, oil has more hydrogen bonds per unit carbon, and natural gas has the most hydrogen bonds per unit carbon. I therefore set $\nu_1 = 0.27$ (for coal), $\nu_2 = 0.34$ (for oil), and $\nu_3 = 0.40$ (for gas).[14]

The top panels of Figure 2 plot a case with $\sigma = 0.5$, and the lower left panel plots a case with $\sigma = 1.5$. The "coal" sector 1 begins with the majority of extraction and research activity. In the case of resource-saving technologies (bottom left), research activity and extraction are locked-in to the "coal" sector 1, which attracts all research effort in all periods and increases its share of resource extraction over time. In the case of resource-using technologies, we see innovation-led transitions. Research begins transitioning immediately towards the "oil" sector 2 (top left panel), and extraction eventually follows (top right panel). The "gas" sector 3 does not attract any research effort for a while and maintains a very small share of extraction even as oil displaces coal. However, after 20 periods, research effort shifts strongly towards the gas sector, and extraction shifts towards the gas sector after 60 periods. In the long run, all sectors attract identical shares of research effort and maintain stable shares of extraction, with their ordering determined by the quality ν of each resource.

The endogenous dynamics of our setting with resource-using machines are qualitatively similar to historical patterns. The bottom right panel of Figure 2 plots resource shares since 1800. The historical patterns in these shares are similar to the patterns that emerge from our numerical simulations with resource-using machines: resource shares change rapidly as a transition occurs, and transitions do not drive formerly dominant resources out of the market. In fact, resource shares have been fairly stable since 1970. The historical patterns are nothing like the patterns that emerge from our simulations with resource-saving machines.

5 Climate Change Policy

Now consider the implications of the present model for policies to address climate change. I focus on competition among three resources in the electricity sector, with resources 1, 2, and 3 representing coal, natural gas, and renewables, respectively.

[13]Smil (2017, 245) describes how oil is of higher quality than coal because it has higher energy density, is cleaner, and is more transportable and storable. On page 270, he writes: "There has been a clear secular shift toward higher-quality fuels, that is, from coals to crude oil and natural gas, a process that has resulted in relative decarbonization (a rising H:C ratio) of global fossil fuel extraction…"

[14]The remaining parameters are $A_Y = 1$, $\epsilon = 3$, $\alpha = 0.5$, $\kappa = 0.5$, $\psi = 3$, $\Psi_1 = \Psi_2 = \Psi_3 = 1$, $\eta = 1$, and $\gamma = 0.5$. The qualitative results are not sensitive to the choice of these parameters.

I make the following extensions to the theoretical setting analyzed above. First, I allow ψ to differ by sector. Second, I allow climate change to reduce economic output. Let sector j's greenhouse gas emission intensity be e_j, and let non-electricity emissions be given by the constant $\bar{e}$. Time t emissions are

$$E_t = \bar{e} + \sum_{j=1}^{3} e_j R_{jt}.$$

Time t emissions augment the atmospheric stock of carbon dioxide (CO_2), which slowly equilibrates with land and ocean reservoirs and eventually causes warming. The three-box model of the carbon cycle and two-box model of the climate system follow the benchmark DICE climate-economy model (Nordhaus, 2008). Also following DICE, warming of T_t degrees Celsius reduces time t output to $Y_t/(1 + D(T_t))$, where $D(T_t), D'(T_t) > 0$. In the absence of policy, climate change affects equilibrium resource use and consumption only by reducing total output.

The third extension provides for a policymaker who can use policy instruments to affect the equilibrium. This policymaker seeks to maximize intertemporal welfare, which takes the standard discounted power utility form in per capita consumption, with population growing exogenously as in DICE. Consistent with Nordhaus (2008), the policymaker's utility discount rate is 1.5% per year and the elasticity of intertemporal substitution is 0.5.[15] Depending on the scenario, the policymaker can tax greenhouse gas emissions and/or can subsidize R&D into the renewable resource, financed through a lump sum tax on the representative household. In contrast to standard climate-economy models, the cost of reducing emissions at time t is endogenous: this cost depends on the supply of each energy resource, on the time t quality of the machines for using each type of resource, and on the substitutability of each type of energy for the other.

Finally, in contrast to the theoretical setting, I allow the two fossil resources to be depletable. I model depletion by replacing R_{jt} in equation (2) with $\sum_{k=0}^{t} R_{jk}$ and adjusting the economy's aggregate resource constraint appropriately. I assume that fossil resource owners have a one-period property right to extract a unit of the resource.[16]

I solve for equilibrium by (i) solving for the resource extraction allocation that equates resource supply and demand given any allocation of research effort and (ii) solving for the research allocation that maximizes scientists' expected profits given the resource allocation implied by each research allocation. I use a 10-year timestep and a policy horizon of 400 years.

I next describe the calibration before presenting laissez-faire trajectories and analyzing optimal policy.

[15] The appendix shows that the primary results are virtually unchanged for an annual utility discount rate of 0.01%, with the only difference being that the magnitude of the gains from policy increases.

[16] The results are virtually identical in the absence of depletion.

5.1 Calibration

Begin by considering the supply of each type of resource. I estimate $\psi_1 = 2.07$ and $\psi_2 = 1.61$ from supply curves for coal and gas developed for the MESSAGE energy model (McCollum et al., 2014). Drawing in part on the work of others, Johnson et al. (2017) describe the supply of power from solar photovoltaics, concentrating solar power, onshore wind, and offshore wind available by region of the world and by resource quality.[17] Aggregating across resource types and regions, I estimate $\psi_3 = 3.00$. I assume that the renewable resource does not generate emissions ($e_3 = 0$) and calculate the emission intensities of coal and gas by dividing emissions for each resource from 2010–2014 (from the Carbon Dioxide Information Analysis Center) by resource consumption over the initial timestep (described below). Other emissions $\bar{e}$ come from summing emissions from all other reported categories, which includes emissions from oil. I calibrate $\kappa = 0.04$ to the distribution parameter for energy in Golosov et al. (2014), and I fix α at 0.5 and η at 0.02.[18]

For any given σ and ϵ, there are ten remaining parameters: each A_{j0}, each Ψ_j, A_Y, ν_1, ν_2, and γ (where $\nu_3 = 1 - \nu_1 - \nu_2$). I calibrate these so that the first period's equilibrium matches conditions on each R_{j0}, each s_{j0}, each p_{j0}, and Y_0, as well as a condition on the growth rate of final-good production. Initial resource consumption comes from summing consumption from 2011–2015, as reported in the BP Statistical Review of World Energy.[19] The International Energy Agency's World Energy Investment 2017 gives R&D spending on clean energy, on thermal generation, on coal production, and on oil and gas production. I divide thermal expenditures equally between coal and gas and attribute all oil and gas spending to gas. The first period must therefore have 12% of scientists working on coal, 65% of them working on gas, and 23% of them working on renewables. I calibrate each p_{j0} to be consistent with levelized costs from the Energy Information Administration Annual Energy Outlook, using combined cycle plants to represent natural gas and solar photovoltaics to represent renewables. World Bank data for global output from 2011–2015 implies $Y_0 = 765$ trillion year 2014 dollars over the first ten-year timestep. Finally, I require Y_t to grow at an annual rate of 2% from the first to the second timestep, which is consistent with growth rates assumed in the benchmark DICE-2007 climate-economy model (Nordhaus, 2008).[20]

[17] Costs are reported in dollars per unit power and resource potential is reported in units of energy. I convert costs to dollars per unit electrical energy by using the capacity factor reported for each resource quality bin in each region. This capacity factor adjusts for the fact that the power producible from renewable resources is not available throughout the day or throughout the year. And I convert dollars per unit of electrical energy to dollars per units of energy in the resource by using the efficiency of each type of generator. From the Energy Information Administration's Annual Energy Review 2011, the efficiencies are 12% for solar photovoltaics, 21% for solar thermal, and 26% for wind.

[18] Fixing η is not interesting because I calibrate γ to market data, as described below.

[19] To obtain the energetic content of renewables from the reported tonnes of oil equivalent, use BP's assumed thermal efficiency of 38% to obtain the equivalent electrical energy and then use a 20% generator efficiency to convert electrical energy to energy in the renewable resource (see footnote 17).

[20] In order to calibrate the ten free parameters, I search for the Ψ_1, Ψ_2, Ψ_3, $A_{1,0}$, and γ that match

It remains to determine ϵ and σ. I explore several different values of ϵ, ranging from 3 to 15. I calibrate σ in order to match recent estimates of the emission reductions induced by different emission taxes.[21] The top left panel of Figure 3 plots the marginal abatement cost curve estimated in Morris et al. (2012) for the United States (adjusted to be in year 2014 dollars). A calibration with $\sigma = 0.85$ matches these estimates reasonably well across a wide range of ϵ, as measured by mean squared error. The dashed lines in the top left panel of Figure 3 plot results for the two most extreme values of ϵ studied here.[22]

Now consider the laissez-faire evolution of the calibrated economy. The top right panel of Figure 3 depicts the research allocation for $\epsilon = 15$ (the cases with alternate ϵ are similar). The gas resource quickly comes to dominate the resource allocation and continues to do so throughout the 400-year horizon. The lower two panels depict each resource's share of supply for $\epsilon = 3$ (lower left) and $\epsilon = 15$ (lower right). In both cases, the gas resource's share increases monotonically over the 400-year horizon. However, the shift from coal to gas proceeds faster in the case with $\epsilon = 15$. In both cases, the renewable resource's share increases at first before declining. In 100 (200) years, global temperature is nearly 3°C (9°C) higher than it was in 1900, due to emissions from both gas and coal. Policy will be required to avoid an environmental disaster. I now turn to an analysis of that policy.[23]

5.2 Policy

Now consider how a policymaker would steer the economy to maximize intertemporal welfare.[24] Begin by considering the case with $\epsilon = 15$, plotted in Figure 4. The top panels show that the policymaker would not use an emission tax to redirect the economy towards renewable resources in the near future. A moderate first-period tax does shift some scientists towards the renewable and gas sectors and does reduce first-period emissions, but the

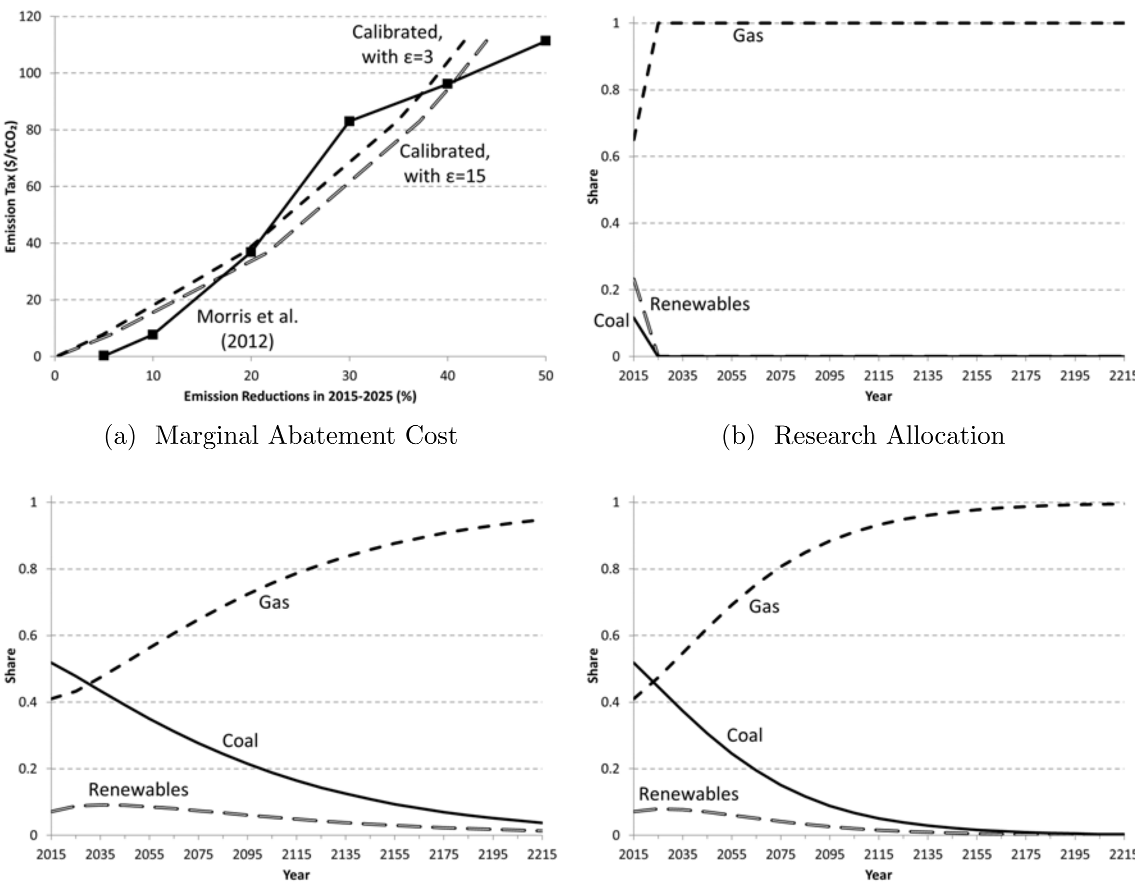
the conditions on each R_{j0}, on $p_{1,0}$, and on the growth rate of final good production. Given a vector of guesses for these parameters, I solve for $A_{2,0}$ and $A_{3,0}$ by plugging $s_{2,0}$ and $s_{3,0}$ into equation (10). I then find the A_Y that matches final good production at R_{j0}, s_{j0}, and A_{j0} to Y_0, where $\nu_{2,0}$ and $\nu_{3,0}$ are solved explicitly from substituting $p_{2,0}$ and $p_{3,0}$ into the final good producer's first-order conditions (and of course $\nu_{1,0} = 1 - \nu_{2,0} - \nu_{3,0}$). I recalibrate the model when varying ϵ and σ.

[21] In contrast to standard climate-economy models, the cost of reducing emissions will then evolve endogenously, responding to policy and to market forces.

[22] The marginal abatement cost curve is sensitive to σ. For instance, calibrations with $\sigma > 1$ shift out the marginal abatement cost curve and increase mean squared error dramatically. It is not possible to simultaneously have $\sigma > 1$, have a marginal abatement cost curve similar to Morris et al. (2012), and meet the ten conditions given in the previous paragraph.

[23] The laissez-faire evolution of the calibrated economy is largely robust to the choice of σ over the 400-year horizon.

[24] I refer to optimal policy, but I technically analyze constrained-optimal policy. The fully optimal policy would correct all of the distortions in the economy. These distortions include market power in machine production, externalities in innovation, externalities from emissions, intertemporal market failures in supply of the depletable resource, and limited patent protections. Acemoglu et al. (2012) discuss the first four, and Greaker et al. (2018) emphasize the last one.



(a) Marginal Abatement Cost

(b) Research Allocation

(c) Each Resource's Share of Supply ($\epsilon = 3$)

(d) Each Resource's Share of Supply ($\epsilon = 15$)

Figure 3: The economy's marginal abatement cost curve (top left), and the evolution of the laissez-faire economy over the first 200 years. The plotted research allocation uses $\epsilon = 15$, but results are virtually identical for alternate ϵ.

research allocation quickly reverts to its laissez-faire trajectory (top left) and so too does the allocation of resource use (top right).

However, a policymaker who can subsidize research into renewables instead of taxing emissions will choose to redirect the economy towards low-emission renewable resources. The subsidy immediately and permanently shifts all scientists into the renewable sector (top left) and achieves nearly complete decarbonization by the end of the century (top right). Without the research subsidy, the policymaker would have required an emission tax of \$570/tCO_2 to redirect all research effort to the renewable sector, which would have led renewables to supply 32% of resources in the first period.[25] The policymaker does not find it optimal to distort supply to that degree. As a result, a policymaker who can use only an emission tax does little to bend the temperature trajectory, but a policymaker who can use a research subsidy achieves a major reduction in total warming (bottom left).

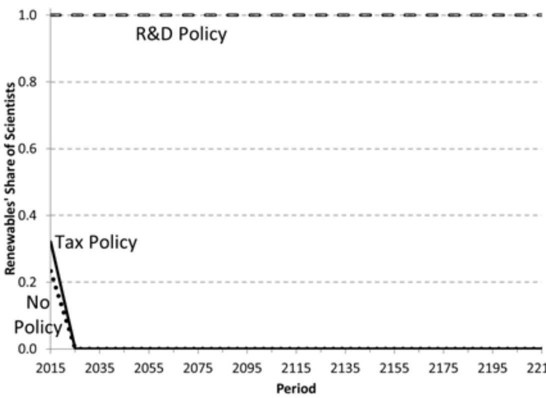
The bottom right panel of Figure 4 compares the optimal emission tax when the policymaker uses the tax on its own (crosses) and when the policymaker can employ both an emission tax and a research subsidy (squares). The availability of the research subsidy dampens the initial tax because the policymaker instead uses the research subsidy to shift scientists towards the renewable sector and thus away from the coal sector. The tax trajectories are otherwise quite similar. Whether or not the tax instrument is available, the research subsidy is nearly 14% in the first period and drops to 0 in subsequent periods. The improvement in renewables' technology from the first period of research is sufficient to keep all scientists working in that sector in subsequent periods. The policymaker can permanently redirect the economy with only a short-term research subsidy.

Table 1 describes results for several different values of ϵ. The top panel shows that scientists are more responsive to the emission tax as ϵ increases and that long-run laissez-faire use of renewables is not sensitive to ϵ. The second panel studies the case in which the policymaker only has access to an emission tax. In all cases, the policymaker's emission tax shifts scientists towards renewables in the first period. Nonetheless, renewables' share of supply in 2100 is actually smaller than in the absence of policy because the initial emission tax also shifts research towards gas and the policymaker does not subsequently use a large emission tax that would shift resource use away from gas. The balanced growth equivalent gain from optimal tax policy is small (less than 0.5%), reflecting the small role for tax policy.

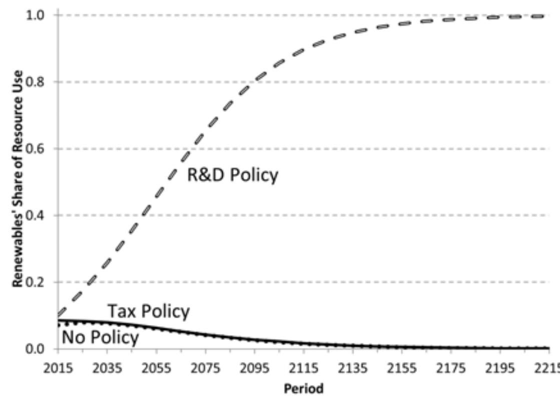
The third panel allows the policymaker to subsidize research into renewables but not to use an emission tax.[26] The optimal research subsidy is qualitatively sensitive to ϵ: for the smallest value of ϵ, the optimal research subsidy is negative for one period before dropping

[25] The distortion becomes more severe with $\epsilon = 3$: an emission tax of \$5000/tCO_2 would have only 74% of scientists working in the renewable sector even as it led renewables to supply 69% of resources in the first period.

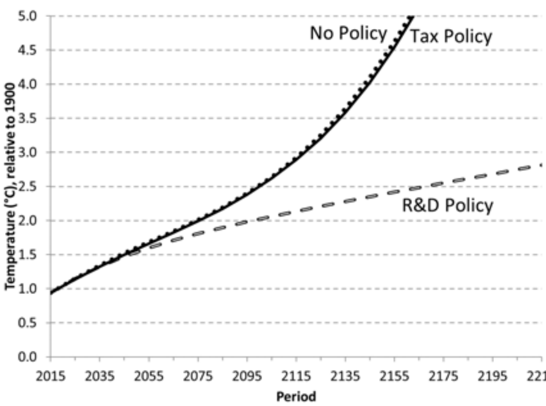
[26] The solutions in cases with only a research subsidy are sensitive to the initial guess for the allocation of scientists: guessing a high (low) allocation leads to a corner solution with all (no) scientists working in the renewable sector. I report the results for the corner solution that yields greater welfare.



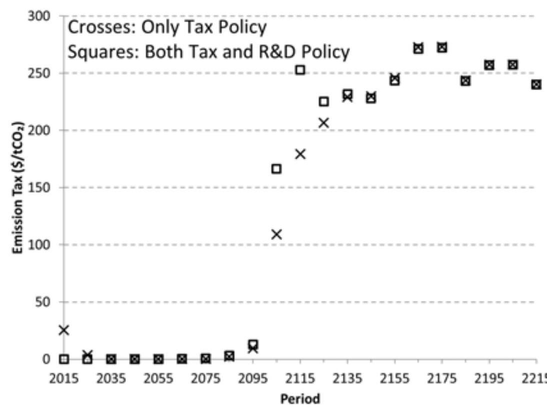
(a) Renewable Resource's Share of Research



(b) Renewable Resource's Share of Supply



(c) Temperature



(d) Emission Tax

Figure 4: The evolution of the economy when the policymaker has access to an emission tax (solid) or to a subsidy for research into renewables (dashed). Also, the emission tax chosen when that is the only instrument (crosses) and when it is combined with a subsidy for research into renewables (squares). All plots use $\epsilon = 15$.

to 0, whereas the other cases have an optimal research subsidy that is sufficiently large to immediately shift all scientists to the renewable sector. Scientists become less sensitive to policy as ϵ falls, so the policymaker must use a larger subsidy to redirect scientists towards renewables when ϵ is small. When $\epsilon = 3$, the policymaker prefers to speed the transition towards gas by redirecting scientists away from renewables. In the cases with larger ϵ, the additional research into renewables leads them to dominate resource supply within the coming century (in sharp contrast to the no-policy case), and the balanced growth equivalent gain of 3–7% reflects the large benefits of keeping temperature much closer to its pre-industrial value.

The bottom panel allows the policymaker to use both an emission tax and a subsidy for research into renewables. The policymaker opts not to use an emission tax in the first few decades, and the level of the initial research subsidy is virtually the same as in the case without an emission tax. Even though the policymaker will use a nontrivial tax in the second century (see the lower right panel of Figure 4), the balanced growth equivalent gain from policy is virtually the same whether or not the policymaker also has access to the tax: renewables dominate the resource mix by 2100 in either case.

Finally, consider a third type of policy instrument: a mandate that the renewable resource comprise a minimum share of resource use. This mandate directly amplifies the market size effect that draws scientists to the renewable sector. Figure 5 plots the evolution of the renewable resource's share of production for several different mandates and for the two extreme values of ϵ.[27] Mandates of at least 10% bind only for some number of initial periods before eventually making themselves nonbinding. The calibrated no-policy path has only 23% of scientists working in the renewable sector at first, but the 10% mandate increases this share to 76% (83%) for $\epsilon = 3$ ($\epsilon = 15$) and larger mandates increase this share to 100%.[28] This redirection of research effort improves the renewable sector's technology, which eventually increases equilibrium extraction above the share enforced by the mandate. In contrast, mandates that are too small to substantially shift initial research bind for a very long time. For instance, the plotted 8% mandate increases the share of scientists initially working in the renewable sector to around 40% for both values of ϵ, but no scientists remain in the renewable sector once a few periods have passed, just as in the laissez-faire economy. A mandate can make itself nonbinding only if it is large enough to shift research effort by a

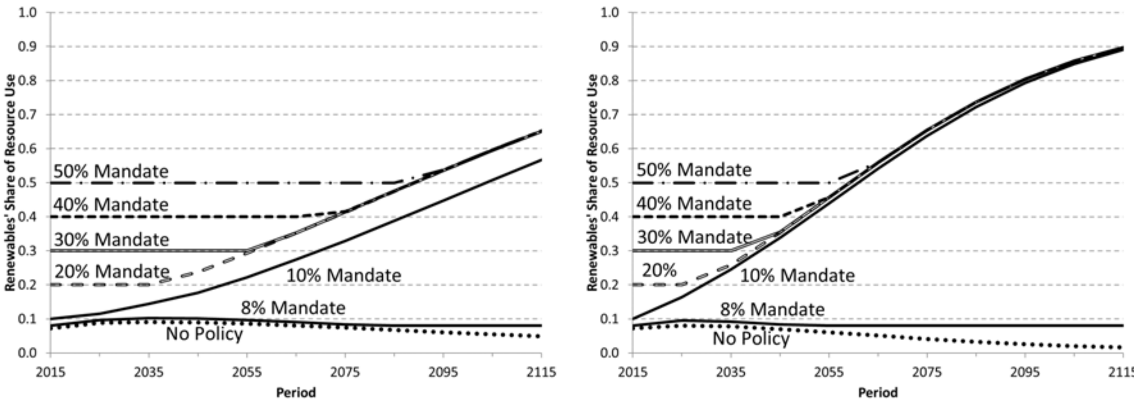
[27] The intermediate-good producer's first-order condition for resources will not hold under a binding mandate. I solve for equilibrium under a binding mandate by analyzing each intermediate-good producer's first-order condition for X_{jt}: I substitute for p_{jt} by using the final-good producer's first-order condition, I express each Y_{jt} and Y_t in terms of each X_{jt}, I substitute for each X_{jt} from equation (8), and I search for the p_{jXt} that solve these N first-order conditions for X_{jt}, given some allocation of resource extraction and research effort. The equilibrium equates supply and demand for the coal and gas resources and has scientists maximizing expected profit.

[28] For a 10% mandate, the renewable resource's share of supply follows a notably lower trajectory when $\epsilon = 3$ because scientists move towards the renewable sector only over a period of 50 years (rather than fully shifting within 10 years as in the case with $\epsilon = 15$).

Table 1: The sensitivity of business-as-usual outcomes and optimal policy to the elasticity of substitution between types of energy (ϵ).

	ϵ				
	3	6	9	12	15
No-Policy					
Tax-elasticity of clean scientists around \$20/tCO2	0.05	0.11	0.15	0.19	0.23
Tax-elasticity of clean scientists around \$100/tCO2	0.17	0.17	0.23	0.28	0.32
Renewables' share of supply in 2100 (%)	7.2	7.2	7.2	7.2	7.2
Emission Tax Only					
Initial tax (\$/tCO_2)	31	52	42	31	26
Initial share of scientists in renewables (%)	25.2	30.5	32.3	32.2	32.2
Renewables' share of supply in 2100 (%)	5.5	3.2	2.6	2.3	2.2
Balanced growth equivalent gain (%)	0.041	0.22	0.33	0.37	0.39
Research Subsidy Only					
Initial subsidy (%)	-11	29	20	16	14
Initial share of scientists in renewables (%)	0	100	100	100	100
Renewables' share of supply in 2100 (%)	4.8	77	82	84	86
Balanced growth equivalent gain (%)	1.6	3.2	4.9	5.7	6.2
Emission Tax and Research Subsidy					
Initial tax (\$/tCO_2)	101	2	0	0	0
Initial subsidy (%)	-14	29	20	16	14
Initial share of scientists in renewables (%)	0	100	100	100	100
Renewables' share of supply in 2100 (%)	4.8	77	82	89	90
Balanced growth equivalent gain (%)	1.9	3.2	4.9	5.7	6.2

In the no-policy case, the calibration ensures, for each ϵ, that 23% of scientists initially work in renewables. The balanced growth equivalent gain (Mirrlees and Stern, 1972) is the constant relative difference in consumption between two counterfactual consumption trajectories that grow at the same constant rate and also yield the exact same welfare as in the given model of policy and in the no-policy model.



(a) $\epsilon = 3$ (b) $\epsilon = 15$

Figure 5: The evolution of the economy when subject to a mandate that the renewable resource have at least a given share of extraction. The dotted line plots the evolution of the laissez-faire economy.

sufficiently large amount.

Table 2 reports the balanced growth equivalent gain for several mandates. We see that mandates of 10–30% can improve welfare relative to a no-policy case when $\epsilon = 15$, though the optimal research subsidies studied in Table 1 improved welfare by more. Smaller mandates are costly because they distort supply without kickstarting a transition to renewables, larger mandates are costly because they distort near-term resource use to a large degree, and all mandates are costly when $\epsilon = 3$ because the research allocation shifts towards renewables only slowly. Table 2 also reports the balanced growth equivalent gain if the research allocation were constrained to follow the laissez-faire path. These numbers show that endogenous innovation is critical to the evaluation of mandates. A mandate of 8% redirects the research allocation only temporarily. Because that mandate is too small to kickstart a full transition to renewables, it would actually be more valuable (even welfare-improving) if the research allocation did not respond to it. In contrast, larger mandates eventually redirect all research effort to the renewable sector. Accounting for this effect on research activity can make these mandates welfare-improving if $\epsilon = 15$ and can substantially reduce their costs if $\epsilon = 3$.

6 Conclusion

We have seen that complementarities between inputs are critical to the possibility of innovation-led transitions in factor use. In particular, complementarities between resources and machines are critical to the types of transitions seen in the history of energy supply. These complementarities eventually push scientists away from the more advanced sector, and the

Table 2: The balanced growth equivalent gain (%) from imposing a minimum share of renewable resource use, relative to the no-policy case. The "fixed research" simulations force the research allocation to follow the no-policy path.

	Mandate for Renewable Resource Use					
	8%	10%	20%	30%	40%	50%
$\epsilon = 3$						
Endogenous research	-1.7	-5.5	-1.4	-5.4	-15.3	-38.5
Fixed research	0.05	0.2	-2.4	-10.1	-24.6	-52.8
$\epsilon = 15$						
Endogenous research	-2.5	4.3	5.9	2.8	-5.3	-24.7
Fixed research	0.07	0.2	-2.8	-10.6	-24.7	-51.4

redirection of scientific effort eventually redirects factor use away from the dominant sector. A calibrated numerical simulation finds that, in the absence of policy, a transition from coal and, especially, gas resource use to renewable resource use would occur far too slowly to limit climate change to levels consistent with recent international agreements. A welfare-maximizing policymaker would use a temporary research subsidy to permanently redirect the economy away from fossil fuels. However, a policymaker who cannot use a research subsidy would accept a high degree of climate change because kickstarting a transition to renewables would require a very high emission tax. Further, the elasticity of substitution between different types of energy is critical to the policymaker's choices because it determines the responsiveness of the research allocation to policy. If improved batteries were to make renewable energy a better substitute for energy from gas and coal, then the policymaker would find it increasingly attractive to kickstart a transition to renewable energy.

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Appendices

The first appendix considers the stability of each period's equilibrium. The second appendix reports numerical results for a lower utility discount rate. The third appendix contains proofs and derivations.

First Appendix: Tâtonnement Stability

One may be concerned that interior equilibria are not "natural" equilibria in the presence of positive feedbacks from resource extraction to innovation and of potential complementarities. Indeed, Acemoglu (2002) and Hart (2012) have emphasized the role of knowledge spillovers in allowing interior research allocations to be stable in the long run. This appendix shows that interior equilibria are in fact "natural" equilibria in the present setting.

Assume $N = 2$ and label the two sectors j and k. Rearranging equation (10) and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation.[29] Substituting into equations (12) and (13) then gives us two equations in two unknowns. This system defines the equilibrium R_{jt} and R_{kt} that clear the markets for each resource.

Define the tâtonnement adjustment process and stability as follows:

Definition A-1. *A* tâtonnement adjustment process *increases R_{jt} if equation* (12) *is not satisfied and its right-hand side is greater, decreases R_{jt} if equation* (12) *is not satisfied and its left-hand side is greater, and obeys analogous rules for R_{kt} using equation* (13)*. I say that an equilibrium (R^*_{jt}, R^*_{kt}) is* tâtonnement-stable *if and only if the tâtonnement adjustment process leads to (R^*_{jt}, R^*_{kt}) from (R_{jt}, R_{kt}) sufficiently close to (R^*_{jt}, R^*_{kt}).*

The tâtonnement process changes R_{jt} and R_{kt} so as to eliminate excess supply or demand, and tâtonnement stability requires that this adjustment process converge to an equilibrium point from values close to the equilibrium. The following proposition shows that our equilibrium is tâtonnement-stable:

Proposition A-1. *The equilibrium is tâtonnement-stable.*

Proof. See third appendix. □

A Walrasian auctioneer would find our equilibrium at any time t.

[29] Technically, this function should be written to allow for corner solutions in the research allocation. The proof of stability will account for corner solutions.

Now use equations (12) and (13) to define R_{jt} and R_{kt} as functions of s_{jt},[30] and then restate equation (10) as a function only of s_{jt}:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)}}{A_{k(t-1)}} \left(\frac{A_{j(t-1)} + \eta\gamma s_{jt} A_{j(t-1)}}{A_{k(t-1)} + \eta\gamma(1-s_{jt}) A_{k(t-1)}} \right)^{\frac{-1}{\sigma+\alpha(1-\sigma)}} \left(\frac{R_{jt}(s_{jt})}{R_{kt}(s_{jt})} \right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k} \right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} . \tag{A-1}$$

The following corollary gives us the total derivative of Π_{jt}/Π_{kt} with respect to s_{jt}:

Corollary A-2. *The right-hand side of equation* (A-1) *strictly decreases in* s_{jt}*.*

Proof. See third appendix. □

The supply expansion effect makes the relative incentive to research in sector j decline in the number of scientists working in sector j. However, when sector j's share of resource extraction increases in the relative quality of its technology, a positive feedback between research and extraction maintains sector j's research incentives even as more scientists move to sector j. The proof shows, as is intuitive, that whether the relative incentive to research in sector j declines in the number of scientists working in sector j is identical to whether the equilibrium is tâtonnement-stable: tâtonnement-stability is not consistent with positive feedbacks that are strong enough to overwhelm the supply expansion effect. And we have already seen that interior equilibria are in fact tâtonnement-stable.

Second Appendix: Additional Numerical Results

Table A-1 presents simulations with a lower utility discount rate of 0.01%. The qualitative results are unchanged. The main difference is that optimal policy now generates larger welfare gains because welfare is more sensitive to longer-run climate impacts when agents are more patient.

Third Appendix: Proofs and Derivations

This appendix derives useful intermediate results before providing proofs and derivations omitted from the main text.

[30]Rearrange equations (12) and (13) to put all terms on the right-hand side. For given s_{jt}, the Jacobian of this system in R_{jt} and R_{kt} is negative definite.

Table A-1: The sensitivity of business-as-usual outcomes and optimal policy to the elasticity of substitution between types of energy (ϵ), using a lower utility discount rate of 0.01%.

	ϵ				
	3	6	9	12	15
Emission Tax Only					
Initial tax ($/t$CO_2$)	52	52	42	31	26
Initial share of scientists in renewables (%)	26.4	30.5	32.3	32.2	32.2
Renewables' share of supply in 2100 (%)	5.5	3.2	2.6	2.3	2.1
Balanced growth equivalent gain (%)	0.09	0.36	0.46	0.51	0.52
Research Subsidy Only					
Initial subsidy (%)	-11	29	20	16	14
Initial share of scientists in renewables (%)	0	100	100	100	100
Renewables' share of supply in 2100 (%)	4.8	77	82	84	86
Balanced growth equivalent gain (%)	2.4	5.2	7.2	8.2	8.8
Emission Tax and Research Subsidy					
Initial tax ($/t$CO_2$)	215	14	11	10	9
Initial subsidy (%)	-16	28	20	16	14
Initial share of scientists in renewables (%)	0	100	100	100	100
Renewables' share of supply in 2100 (%)	4.8	77	82	89	90
Balanced growth equivalent gain (%)	3.1	5.2	7.2	8.2	8.8

In the no-policy case, the calibration ensures, for each ϵ, that 23% of scientists initially work in renewables. The balanced growth equivalent gain (Mirrlees and Stern, 1972) is the constant relative difference in consumption between two counterfactual consumption trajectories that grow at the same constant rate and also yield the exact same welfare as in the given model of policy and in the no-policy model.

Useful Lemmas

First, note that equations (8) and (9) imply

$$X_{jt} = \left[\frac{1-\kappa}{\kappa} p_{jRt}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}. \tag{A-2}$$

Rearranging equation (10) and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation:

$$s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right) = \frac{(1+\eta\gamma)\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma} - 1}{\eta\gamma + \eta\gamma\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}}. \tag{A-3}$$

Let $\Sigma_{x,y}$ represent the elasticity of x with respect to y, and let $\Sigma_{x,y|z}$ represent the elasticity of x with respect to y holding z constant. The following lemma establishes signs and bounds for elasticities that will prove useful:

Lemma A-3. *The following hold, with analogous results for sector k:*

1. $\Sigma_{Y_t,Y_{jt}}, \Sigma_{Y_t,Y_{kt}} \in [0,1]$ *and* $\Sigma_{Y_t,Y_{jt}} + \Sigma_{Y_t,Y_{kt}} = 1$.
2. $\Sigma_{Y_{jt},R_{jt}|X_{jt}}, \Sigma_{Y_{jt},X_{jt}} \in [0,1]$ *and* $\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} = 1$.
3. *If* $\sigma < 1$, *then* $\Sigma_{Y_{jt},X_{jt}} \to 0$ *as* $A_{j(t-1)} \to \infty$ *and* $\Sigma_{Y_{kt},X_{kt}} \to 0$ *as* $A_{k(t-1)} \to \infty$.
4. $\Sigma_{X_{jt},A_{jt}} = \frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha} \in (0,1)$
5. $\Sigma_{X_{jt},R_{jt}} = \frac{\alpha\sigma/\psi+\alpha}{\sigma(1-\alpha)+\alpha} \in (0,1]$
6. $\Sigma_{A_{jt},s_{jt}} = \frac{\eta\gamma s_{jt}}{1+\eta\gamma s_{jt}} \in [0,1)$
7. $\Sigma_{s_{jt},R_{jt}} = \frac{\psi+\sigma}{\psi}\frac{2+\eta\gamma}{\eta\gamma s_{jt}} Z_t > 0$, *where* $Z_t \in \left[\frac{1+\eta\gamma}{(2+\eta\gamma)^2}, \frac{1}{4}\right]$. $\Sigma_{s_{jt},R_{kt}} = -\Sigma_{s_{jt},R_{jt}}$.
8. $\Sigma_{s_{jt},A_{j(t-1)}} = -\frac{(1-\sigma)(1-\alpha)}{A_{j(t-1)}}\frac{(2+\eta\gamma)}{\eta\gamma} Z_t$, *which is* < 0 *if and only if* $\sigma < 1$. Z_t *is as above.* $\Sigma_{s_{jt},A_{k(t-1)}} = -\Sigma_{s_{jt},A_{j(t-1)}}$.
9. $\Sigma_{s_{jt},s_{kt}} = -s_{kt}/s_{jt} \leq 0$

Proof. Most of the results follow by differentiation and the definition of an elasticity. #1 follows from differentiating the final-good production function $Y_t(Y_{jt}, Y_{kt})$; #2 follows from differentiating the intermediate-good production function $Y_{jt}(R_{jt}, X_{jt})$; #4 follows from differentiating equation (A-2); #5 follows from differentiating equation (A-2) after using equation (2) to substitute for p_{jRt} and using $\psi \geq \alpha/(1-\alpha)$; #6 follows from differentiating equation (5); #7 and #8 follow from differentiating equation (A-3); and #9 follows from the research constraint.

To derive #3, note that

$$\Sigma_{Y_{jt},X_{jt}} = \frac{(1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}}{\kappa R_{jt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}}.$$

From (7), (8), and (2), we have:

$$\begin{aligned} X_{jt} =& A_{jt}\left(\frac{1-\kappa}{\kappa}\left[\frac{R_{jt}}{X_{jt}}\right]^{1/\sigma} \Psi_j^{-1/\psi} R_{jt}^{1/\psi}\right)^{\frac{\alpha}{1-\alpha}} \\ =& A_{jt}\left(\frac{1-\kappa}{\kappa}\Psi_j^{-1/\psi} R_{jt}^{\frac{1}{\psi}+\frac{1}{\sigma}}\right)^{\frac{\sigma\alpha}{\sigma(1-\alpha)+\alpha}}. \end{aligned}$$

$X_{jt} \to \infty$ as $A_{j(t-1)} \to \infty$, which implies with $\sigma < 1$ that $\Sigma_{Y_{jt},X_{jt}} \to 0$ as $A_{j(t-1)} \to \infty$. Analogous results hold for sector k.

To derive #7 and #8, define

$$Z_t \triangleq \frac{\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}}{\left[1+\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}\right]^2}$$

and recognize that $s_{jt} \in (0,1)$ implies

$$\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma} \in \left(\frac{1}{1+\eta\gamma},\ 1+\eta\gamma\right)$$

from equation (10).

□

Note that $\Sigma_{X,A}$ and $\Sigma_{X,R}$ are the same in each sector. I therefore often omit the sector subscripts on these terms.

Using $s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right)$, the equilibrium is defined by equations (12) and (13), which are functions only of R_{jt} and R_{kt}. Rewrite these equations as (suppressing the predetermined technology arguments in s_{jt}):

$$1 = \kappa\,\nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{Y_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{Y_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \triangleq G_j(R_{jt}, R_{kt}),$$

$$1 = \kappa\,(1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{Y_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{Y_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \triangleq G_k(R_{jt}, R_{kt}).$$

We have:

Lemma A-4. *$\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$ and $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt} < 0$.*

Proof. Differentiating yields:

$$\begin{aligned}
\frac{\partial G_j(R_{jt}, R_{kt})}{\partial R_{jt}} =& G_j\Bigg\{ -\left(\frac{1}{\psi}+\frac{1}{\sigma}\right)\frac{1}{R_{jt}} + \left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\frac{1}{Y_{jt}}\left[\frac{\partial Y_{jt}}{\partial R_{jt}} + \frac{\partial Y_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right] \\
&+ \frac{1}{\epsilon}\frac{1}{Y_t}\left[\frac{\partial Y_t}{\partial Y_{jt}}\frac{\partial Y_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial Y_{jt}}\frac{\partial Y_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial Y_{kt}}\frac{\partial Y_{kt}}{\partial s_{kt}}\frac{\partial s_{kt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right]\Bigg\} \\
=& \frac{G_j}{R_{jt}}\Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\left[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
&- \frac{1}{\epsilon}\Bigg[\left(1-\Sigma_{Y_t,Y_{jt}}\right)\left(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
&- \Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg]\Bigg\}.
\end{aligned}$$

If the economy is at a corner in s_{jt}, then $\Sigma_{s_{jt},R_{jt}} = 0$ and, using Lemma A-3, the above expression is clearly negative. So consider a case with interior s_{jt}. The final two lines are negative. So the overall expression is negative if the third-to-last line is negative, which is the case if and only if

$$\begin{aligned}
0 \geq& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} \right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\left(\frac{\sigma+\psi}{\psi}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha} \right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[-1 + \frac{\sigma+\psi}{\psi}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha} \right],
\end{aligned} \tag{A-4}$$

where I use results from Lemma A-3. Note that $\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t \leq 3/4$, which implies that $\Sigma_{Y_{jt},X_{jt}}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha} < 1$. Using this, inequality (A-4) holds if and only if

$$\frac{\sigma}{\psi} \geq \Sigma_{Y_{jt},X_{jt}} \frac{-1 + \frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\alpha+\sigma(1-\alpha)}}{1 - \Sigma_{Y_{jt},X_{jt}}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\alpha+\sigma(1-\alpha)}}. \tag{A-5}$$

$\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t \leq 3/4$ implies that $\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\alpha+\sigma(1-\alpha)} < 1$, which implies that the right-hand side of inequality (A-5) is negative. Thus, inequality (A-5) always holds and $\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$.

The analysis of $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt}$ is virtually identical.

□

Now define the matrix G:

$$G \triangleq \begin{bmatrix} \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}} \\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}} \end{bmatrix}.$$

We have:

Lemma A-5. *The determinant of G is positive.*

Proof. Analyze $det(G)$:

$$
\begin{aligned}
det(G) \propto & \Big\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Big[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Big(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Big) \Big] \Big\} \\
& \Big\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Big[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Big(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Big) \Big] \Big\} \\
& + \Big\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Big[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Big(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Big) \\
& \qquad - \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Big] \Big\} \\
& \Big\{ \frac{1}{\epsilon} \Big[\Sigma_{Y_t,Y_{kt}} \Big(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},R_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Big) \\
& \qquad + \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Big] \Big\} \\
& + \Big\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Big[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Big(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Big) \\
& \qquad - \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Big] \Big\} \\
& \Big\{ \frac{1}{\epsilon} \Big[\Sigma_{Y_t,Y_{jt}} \Big(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Big) \\
& \qquad + \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Big] \Big\} \\
& - \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)^2 \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}},
\end{aligned}
$$

where I factored $G_j G_k / R_{jt} R_{kt}$. Use $\Sigma_{Y_t,Yjt} + \Sigma_{Y_t,Y_{kt}} = 1$ from Lemma A-3 and cancel terms with $1/\epsilon^2$ to obtain:

$$\begin{aligned}
det(G) \propto & \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}}\Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg)\Bigg]\Bigg\} \\
& \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{kt},R_{kt}|X_{kt}} - \Sigma_{Y_{kt},X_{kt}}\Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg)\Bigg]\Bigg\} \\
& - \frac{1}{\sigma}\left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}} \\
& \quad \Bigg[- \Bigg(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg) \\
& \qquad + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg] \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}} \\
& \quad \Bigg[- \Bigg(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg) \\
& \qquad + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma}\Bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg)\Bigg] \\
& \quad \Bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg)\Bigg]. \qquad \text{(A-6)}
\end{aligned}$$

All lines after the first three are positive by results from Lemma A-3. Expanding the products in those first three lines and rearranging, those first three lines become:

$$\begin{aligned}
& \frac{1}{\psi^2} \\
& + \frac{1}{\sigma^2}\Bigg[1 - \Sigma_{X,R}\Bigg]\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\Bigg(1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg) \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\Bigg[1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg] \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\Bigg[1 - \Sigma_{X,R} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg] \\
& + \frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right), \qquad \text{(A-7)}
\end{aligned}$$

where I write $\Sigma_{X,R}$ because this elasticity is the same in each sector. At corner allocations of research, $\Sigma_{s_{jt},R_{jt}} = \Sigma_{s_{jt},R_{kt}} = 0$. In this case, (A-7) is clearly positive. Now assume an interior allocation of research, so that $\Pi_{jt} = \Pi_{kt}$. Note that

$$\begin{aligned}
&1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} \\
=&\frac{1}{\psi}\frac{\sigma}{\sigma(1-\alpha)+\alpha}\left\{\psi[1-\alpha] - \alpha - (1-\alpha)[\sigma+\psi]\frac{(2+\eta\gamma)^2}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})}Z_t\right\}.
\end{aligned} \tag{A-8}$$

Substituting for Z_t and using equation (10) at $\Pi_{jt}/\Pi_{kt} = 1$, we have

$$\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}.$$

Equation (A-8) then becomes

$$1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} = -\frac{\sigma}{\psi}.$$

Substituting into (A-7), the first three lines of (A-6) are equal to

$$\begin{aligned}
&\frac{1}{\psi^2} \\
&-\frac{1}{\psi}\frac{1}{\sigma}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}} \\
&+\frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\left[1-\Sigma_{X,R}-\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right] \\
&+\frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[1-\Sigma_{X,R}-\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right] \\
&+\frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right).
\end{aligned} \tag{A-9}$$

The final line is positive. Factoring $1/\psi$, the first four lines are jointly positive if and only if:

$$\begin{aligned}
0 \leq&\frac{1}{\psi} + \frac{1}{\sigma}\Bigg[(1-\Sigma_{X,R})\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\
&\qquad - \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} - \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg] \\
=&\frac{1}{\psi} + \frac{1}{\sigma}\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\
&-\frac{1}{\sigma}\frac{\sigma+\psi}{\psi}\frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\
&\qquad + \sigma(1-\alpha)\Big(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Big)\frac{1}{2+\eta\gamma}\Bigg],
\end{aligned} \tag{A-10}$$

where we use $\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}$. Note that $\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}$ increases in $\Sigma_{Y_{jt},X_{jt}}$ and thus reaches a maximum at $\Sigma_{Y_{jt},X_{jt}} = 1$. Therefore,

$$\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}} \leq 1 + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{kt},X_{kt}} = 1.$$

Also note that $\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})$ increases in each elasticity, and each elasticity is ≤ 1. Thus,

$$\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt}) \leq (1+\eta\gamma s_{kt}) + (1+\eta\gamma s_{jt}) = 2+\eta\gamma,$$

which implies

$$\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma} \leq 1.$$

These results together imply that

$$\begin{aligned}&\alpha + \sigma(1-\alpha)\\ \geq&\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) + \sigma(1-\alpha)\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}.\end{aligned} \tag{A-11}$$

Using this, we have that inequality (A-10) holds if and only if

$$\begin{aligned}\frac{\sigma}{\psi} \geq&\Bigg\{-\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) + \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\qquad + \sigma(1-\alpha)\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\}\\ &\Bigg\{1 - \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\qquad + \sigma(1-\alpha)\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\}^{-1}.\end{aligned} \tag{A-12}$$

The denominator on the right-hand side is positive via inequality (A-11). The numerator on the right-hand side is equal to:

$$\begin{aligned}&\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\left\{-1 + \frac{1}{\sigma(1-\alpha)+\alpha}\left[\alpha + \sigma(1-\alpha)\frac{\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)}{(2+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)}\right]\right\}.\end{aligned} \tag{A-13}$$

Consider the fraction in brackets. If that fraction is ≤ 1, then the whole expression is negative and we are done. I will now prove that the fraction cannot be > 1. Assume that the fraction is > 1. Then:

$$\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right) > (2+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)$$
$$\Leftrightarrow \eta\gamma s_{kt}\Sigma_{Y_{jt},X_{jt}} + \eta\gamma s_{jt}\Sigma_{Y_{kt},X_{kt}} \geq (1+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}}\right) - (2+\eta\gamma)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}.$$

Assume without loss of generality that $\Sigma_{Y_{jt},X_{jt}} > \Sigma_{Y_{kt},X_{kt}}$. Then the left-hand side of the last line attains its largest possible value when $s_{kt} = 1$. The inequality on the last line is then satisfied only if

$$0 > \Sigma_{Y_{jt},X_{jt}} + (1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}. \tag{A-14}$$

The right-hand side is monotonic in $\Sigma_{Y_{jt},X_{jt}}$. At $\Sigma_{Y_{jt},X_{jt}} = 1$, the right-hand side is

$$1 + (1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} = 1 - \Sigma_{Y_{kt},X_{kt}} \geq 0.$$

But this contradicts inequality (A-14). Now consider the other extremum: $\Sigma_{Y_{jt},X_{jt}} = 0$. The right-hand side of inequality (A-14) becomes:

$$(1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} \geq 0,$$

which again contradicts inequality (A-14). Because the right-hand side of inequality (A-14) was monotonic in $\Sigma_{Y_{jt},X_{jt}}$ and was not satisfied for either the greatest or smallest possible values for $\Sigma_{Y_{jt},X_{jt}}$, the inequality is not satisfied for any values of $\Sigma_{Y_{jt},X_{jt}}$. Thus, the fraction in brackets in (A-13) is ≤ 1, which means that the right-hand side of inequality (A-12) is ≤ 0 and inequality (A-12) is satisfied. As a result, the first three lines of (A-6) are positive, which means that $det(G) > 0$.

□

The next two lemmas establish how relative extraction and relative profit change with the average quality of technology in sector j:

Lemma A-6. *Define* $\mathbf{R}(A_{jt}, A_{kt}) \triangleq [R_{jt}(A_{jt}, A_{kt})/R_{kt}(A_{jt}, A_{kt})]$. *Then (i)* $\partial\mathbf{R}/\partial A_{jt} > 0$ *and (ii)* $\partial\mathbf{R}/\partial A_{jt} \to 0$ *as* $A_{jt} \to \infty$.

Proof. I begin by using the implicit function theorem on the two-dimensional system obtained from equations (12) and (13). Rewriting previous expressions for G_j and G_k to hold s_{jt} fixed at some value s, the two-dimensional system becomes:

$$1 = \kappa\,\nu_j A_Y^{\frac{\epsilon-1}{\epsilon}}\left[\frac{Y_t\,(R_{jt}, R_{kt}, s_{jt}=s)}{Y_{jt}\,(R_{jt}, s_{jt}=s)}\right]^{1/\epsilon}\left[\frac{Y_{jt}\,(R_{jt}, s_{jt}=s)}{R_{jt}}\right]^{1/\sigma}\left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \triangleq H_j(R_{jt}, R_{kt}; s_{jt}=s),$$
$$1 = \kappa\,(1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}}\left[\frac{Y_t\,(R_{jt}, R_{kt}, s_{jt}=s)}{Y_{kt}\,(R_{kt}, s_{jt}=s)}\right]^{1/\epsilon}\left[\frac{Y_{kt}\,(R_{kt}, s_{jt}=s)}{R_{kt}}\right]^{1/\sigma}\left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \triangleq H_k(R_{jt}, R_{kt}; s_{jt}=s).$$

Fixing $s_{jt} = s$ makes A_{jt} a parameter. I analyze the following:

$$\begin{aligned}
\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} =& \frac{R_{jt}}{R_{kt}} \left\{ \frac{\partial R_{jt}}{\partial A_{jt}} \frac{1}{R_{jt}} - \frac{\partial R_{kt}}{\partial A_{jt}} \frac{1}{R_{kt}} \right\} \\
=& \frac{R_{jt}}{R_{kt}} \left\{ \frac{1}{R_{jt}} \frac{-\frac{\partial H_j}{\partial A_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}} \frac{\partial H_k}{\partial A_{jt}}}{det(H)} - \frac{1}{R_{kt}} \frac{-\frac{\partial H_k}{\partial A_{jt}} \frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}} \frac{\partial H_j}{\partial A_{jt}}}{det(H)} \right\} \\
=& \frac{R_{jt}}{R_{kt}} \frac{1}{det(H)} \left\{ -\frac{\partial H_j}{\partial A_{jt}} \left[\frac{1}{R_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_k}{\partial R_{jt}} \right] + \frac{\partial H_k}{\partial A_{jt}} \left[\frac{1}{R_{jt}} \frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_j}{\partial R_{jt}} \right] \right\}.
\end{aligned} \tag{A-15}$$

Differentiation and algebraic manipulations (including applying relationships from Lemma A-3) yield:

$$-\frac{\partial H_j}{\partial A_{jt}} = -H_j \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \right\} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} \frac{1}{A_{jt}},$$

$$\frac{\partial H_k}{\partial A_{jt}} = H_k \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} \frac{1}{A_{jt}},$$

$$\begin{aligned}
\frac{1}{R_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_k}{\partial R_{jt}} =& \frac{H_k}{R_{jt} R_{kt}} \left\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{kt}, X_{kt}} \left[1 - \Sigma_{X,R} \right] \right. \\
&\left. + \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{Y_{jt}, X_{jt}} - \Sigma_{Y_{kt}, X_{kt}} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{R_{jt}} \frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_j}{\partial R_{jt}} =& \frac{H_j}{R_{jt} R_{kt}} \left\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{jt}, X_{jt}} \left[1 - \Sigma_{X,R} \right] \right. \\
&\left. + \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{Y_{kt}, X_{kt}} - \Sigma_{Y_{jt}, X_{jt}} \right] \right\}.
\end{aligned}$$

Using these in equation (A-15), we obtain:

$$\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} = \frac{1}{A_{jt}} \frac{1}{det(H)} \frac{R_{jt}}{R_{kt}} \frac{H_j H_k}{R_{jt} R_{kt}} \Sigma_{X,A} \left(\frac{1}{\sigma} - \frac{1}{\epsilon} \right) \Sigma_{Y_{jt}, X_{jt}} \left(\frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{kt}, X_{kt}} [1 - \Sigma_{X,R}] \right). \tag{A-16}$$

Now consider $det(H)$. It follows from our analysis of $det(G)$ with $\Sigma_{s,R} = 0$. Make this change in equation (A-6):

$$
\begin{aligned}
det(H) = \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(& \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}}\Bigg]\Bigg\} \\
& \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{kt},R_{kt}|X_{kt}} - \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}}\Bigg]\Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}}\Bigg[- \Bigg(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}}\Bigg)\Bigg] \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\Bigg[- \Bigg(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}}\Bigg)\Bigg] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma}\Bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}}\Bigg]\Bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}}\Bigg]\Bigg).
\end{aligned}
$$

Now analyze, using relations in Lemma A-3:

$$
\begin{aligned}
det(H) = \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(& \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\Bigg[1 - \Sigma_{X,R}\Bigg]\Bigg\}\Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\Bigg[1 - \Sigma_{X,R}\Bigg]\Bigg\} \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}}\Bigg(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X,R}\Bigg) \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\Bigg(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X,R}\Bigg) \\
& + \frac{1}{\epsilon}\frac{1}{\sigma}\Bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X,R}\Bigg]\Bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X,R}\Bigg]\Bigg) \\
= \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(& \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\Bigg[1 - \Sigma_{X,R}\Bigg]\Bigg\}\Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\Bigg[1 - \Sigma_{X,R}\Bigg]\Bigg\} \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}}\Bigg[1 - \Sigma_{Y_{kt},X_{kt}}(1 - \Sigma_{X,R})\Bigg] \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma}\Bigg\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\Bigg[1 - \Sigma_{Y_{jt},X_{jt}}(1 - \Sigma_{X,R})\Bigg] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{jt},X_{jt}}(1 - \Sigma_{X,R})\Bigg]\Bigg[1 - \Sigma_{Y_{kt},X_{kt}}(1 - \Sigma_{X,R})\Bigg]\Bigg).
\end{aligned}
$$

From Lemma A-3, $1 - \Sigma_{X,R} = \frac{\sigma}{\psi}\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}$. Substituting $det(H)$ into equation (A-16), we have:

$$
\begin{aligned}
\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} =&\frac{1}{A_{jt}}\frac{R_{jt}}{R_{kt}}\Sigma_{X,A}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\Sigma_{Y_{jt},X_{jt}}\left(\frac{1}{\psi}+\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}[1-\Sigma_{X,R}]\right)\\
&\left(\left\{\frac{1}{\psi}+\frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[1-\Sigma_{X,R}\right]\right\}\left\{\frac{1}{\psi}+\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\left[1-\Sigma_{X,R}\right]\right\}\right.\\
&+\left\{\frac{1}{\psi}+\frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}}\left[1-\Sigma_{Y_{kt},X_{kt}}(1-\Sigma_{X,R})\right]\\
&+\left\{\frac{1}{\psi}+\frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\left[1-\Sigma_{Y_{jt},X_{jt}}(1-\Sigma_{X,R})\right]\\
&\left.+\frac{1}{\epsilon}\frac{1}{\sigma}\left[1-\Sigma_{Y_{jt},X_{jt}}(1-\Sigma_{X,R})\right]\left[1-\Sigma_{Y_{kt},X_{kt}}(1-\Sigma_{X,R})\right]\right)^{-1} \qquad \text{(A-17)}\\
&>0.
\end{aligned}
$$

We have established the first part of the lemma. To establish the second part, use Lemma A-3 in equation (A-17).

□

Lemma A-7. *Fix $s_{jt} = s$. If $\sigma > 1$ or σ is not too much smaller than 1, then Π_{jt}/Π_{kt} increases in $A_{j(t-1)}$. As $A_{j(t-1)} \to \infty$, Π_{jt}/Π_{kt} decreases in $A_{j(t-1)}$ for all $\sigma < 1$.*

Proof. To a first-order approximation, we have, with s_{jt} fixed at s,

$$
\begin{aligned}
&\frac{\mathrm{d}\ln[\Pi_{jt}/\Pi_{kt}]}{\mathrm{d}A_{j(t-1)}}\\
\approx&\frac{1}{A_{j(t-1)}}\left[1-\frac{1}{\sigma+\alpha(1-\sigma)}\right]+\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}\frac{\partial A_{jt}}{\partial A_{j(t-1)}}\frac{\partial[R_{jt}/R_{kt}]}{\partial A_{jt}}\frac{R_{kt}}{R_{jt}}\\
=&\frac{1}{A_{j(t-1)}}\left[1-\frac{1}{\sigma+\alpha(1-\sigma)}\right]+\frac{1}{\psi}\frac{\psi+\sigma}{\sigma+\alpha(1-\sigma)}(1+\eta\gamma s)\frac{\partial[R_{jt}/R_{kt}]}{\partial A_{jt}}\frac{R_{kt}}{R_{jt}}\\
=&\frac{1}{A_{j(t-1)}}\frac{(1-\alpha)(\sigma-1)}{\sigma+\alpha(1-\sigma)}+\frac{1}{\psi}\frac{\psi+\sigma}{\sigma+\alpha(1-\sigma)}(1+\eta\gamma s)\frac{\partial[R_{jt}/R_{kt}]}{\partial A_{jt}}\frac{R_{kt}}{R_{jt}}.
\end{aligned}
$$

The first term is positive if and only if $\sigma > 1$ and, using Lemma A-6, the second term is positive. Therefore the whole expression is positive if $\sigma > 1$. The first term becomes small for σ close to 1. Therefore the second term dominates (and the whole expression is positive) for σ not too much smaller than 1. Finally, Lemma A-6 shows that the second term goes to 0 as $A_{j(t-1)} \to \infty$ if $\sigma < 1$. Therefore the whole expression is negative if $\sigma < 1$ and $A_{j(t-1)} \to \infty$.

□

Finally, consider the evolution of relative extraction and thus of market size effects. From equation (14), R_{jt}/R_{kt} increases in s_{jt}. Define $\hat{s}_{t+1}$ as the unique value of $s_{j(t+1)}$ such that sector j's share of resource extraction increases from time t to $t+1$ if and only if $s_{j(t+1)} \geq \hat{s}_{t+1}$. Lemma A-6 implies that $\hat{s}_{t+1} \in (0,1)$.

Lemma A-8. *If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \geq [\Psi_j/\Psi_k]^{1/[(1-\alpha)(1+\psi)]}$. If $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \leq [\Psi_j/\Psi_k]^{1/[(1-\alpha)(1+\psi)]}$.*

Proof. The change in R_{jt}/R_{kt} from time t to $t+1$ is

$$\begin{aligned}\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} =& \frac{(R_{j(t+1)} - R_{jt})R_{kt} - (R_{k(t+1)} - R_{kt})R_{jt}}{R_{k(t+1)}R_{kt}} \\ \propto& \frac{R_{j(t+1)} - R_{jt}}{R_{jt}} - \frac{R_{k(t+1)} - R_{kt}}{R_{kt}},\end{aligned}$$

where the first equality adds and subtracts $R_{jt}R_{kt}$ in the numerator and the second line factors $R_{jt}/R_{k(t+1)}$. To a first-order approximation, this is proportional to

$$\frac{1}{R_{jt}}\left(\frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right) - \frac{1}{R_{kt}}\left(\frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right),$$

with the derivatives evaluated at the time t allocation. Note that s_{jt} is included in A_{jt} when differentiating with respect to A_{jt}, which reflects that we will seek the allocation of scientists that holds R_{jt}/R_{kt} constant. Defining $H_j(R_{jt}, R_{kt}; s_{jt} = s)$ and $H_k(R_{jt}, R_{kt}; s_{jt} = s)$ as in the proof of Lemma A-6 and using the implicit function theorem, the previous expression becomes:

$$\begin{aligned}&\frac{1}{R_{jt}}\left(\frac{-\frac{\partial H_j}{\partial A_{jt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}}\frac{\partial H_k}{\partial A_{jt}}}{det(H)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial H_j}{\partial A_{kt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}}\frac{\partial H_k}{\partial A_{kt}}}{det(H)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ &- \frac{1}{R_{kt}}\left(\frac{-\frac{\partial H_k}{\partial A_{jt}}\frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}}\frac{\partial H_j}{\partial A_{jt}}}{det(H)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial H_k}{\partial A_{kt}}\frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}}\frac{\partial H_j}{\partial A_{kt}}}{det(H)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ \propto& \left[-\frac{\partial H_j}{\partial A_{jt}}s_{j(t+1)}A_{jt} - \frac{\partial H_j}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial H_k}{\partial R_{jt}}\right] \\ &+ \left[\frac{\partial H_k}{\partial A_{jt}}s_{j(t+1)}A_{jt} + \frac{\partial H_k}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial H_j}{\partial R_{jt}}\right],\end{aligned} \tag{A-18}$$

where the second expression factors $\eta\gamma/det(H)$, which is readily seen to be positive by altering the proof of Lemma A-5 to set the $\Sigma_{s,R}$ terms to zero. Differentiation and algebraic manipulations (including applying relationships from Lemma A-3) yield:

$$-\frac{\partial H_j}{\partial A_{jt}} s_{j(t+1)} A_{jt} - \frac{\partial H_j}{\partial A_{kt}} s_{k(t+1)} A_{kt} = - H_j \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \right\} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} s_{j(t+1)} \\ - H_j \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \Sigma_{Y_{kt}, X_{kt}} \Sigma_{X_{kt}, A_{kt}} (1 - s_{j(t+1)}),$$

$$\frac{\partial H_k}{\partial A_{jt}} s_{j(t+1)} A_{jt} + \frac{\partial H_k}{\partial A_{kt}} s_{k(t+1)} A_{kt} = H_k \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \right\} \Sigma_{Y_{kt}, X_{kt}} \Sigma_{X_{kt}, A_{kt}} (1 - s_{j(t+1)}) \\ + H_k \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} s_{j(t+1)}.$$

Substitute these and expressions derived in the proof of Lemma A-6 into (A-18) and factor

$\Sigma_{X,A} H_j H_k/[R_{jt} R_{kt}]$:

$$\begin{aligned}
&\Bigg\{ - s_{j(t+1)} \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \bigg\} \Sigma_{Y_{jt},X_{jt}} - (1 - s_{j(t+1)}) \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \Bigg\} \\
&\qquad \Bigg\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} \bigg[1 - \Sigma_{X,R} \bigg] \Bigg\} \\
&+ \Bigg\{ (1 - s_{j(t+1)}) \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \bigg\} \Sigma_{Y_{kt},X_{kt}} + s_{j(t+1)} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} \Bigg\} \\
&\qquad \Bigg\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{jt},X_{jt}} \bigg(1 - \Sigma_{X,R} \bigg) \Bigg\} \\
&+ \frac{1}{\epsilon} \bigg[1 - \Sigma_{X,R} \bigg] \bigg[\Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}} \bigg] \\
&\qquad \Bigg\{ - s_{j(t+1)} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \bigg] \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} - (1 - s_{j(t+1)}) \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \bigg\} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \Bigg\} \\
&- \frac{1}{\epsilon^2} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_t,Y_{kt}} \bigg[1 - \Sigma_{X,R} \bigg] \bigg[\Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}} \bigg] \bigg\{ (1 - s_{j(t+1)}) \Sigma_{Y_{kt},X_{kt}} + s_{j(t+1)} \Sigma_{Y_{jt},X_{jt}} \bigg\} \\
=& s_{j(t+1)} \Sigma_{Y_{jt},X_{jt}} \Bigg\{ \frac{1}{\psi} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \bigg] \\
&\qquad + \frac{1}{\sigma} \bigg(1 - \Sigma_{X,R} \bigg) \bigg[\frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} \bigg] \Bigg\} \\
&- (1 - s_{j(t+1)}) \Sigma_{Y_{kt},X_{kt}} \Bigg\{ \frac{1}{\psi} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \bigg] \\
&\qquad + \frac{1}{\sigma} \bigg(1 - \Sigma_{X,R} \bigg) \bigg[\frac{1}{\sigma} \Sigma_{Y_{jt},X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \bigg] \Bigg\} \\
&+ \frac{1}{\epsilon} \bigg[1 - \Sigma_{X,R} \bigg] \bigg[\Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}} \bigg] \\
&\qquad \Bigg\{ - s_{j(t+1)} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \bigg] \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} - (1 - s_{j(t+1)}) \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \bigg\} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \\
&\qquad - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_t,Y_{kt}} \bigg[(1 - s_{j(t+1)}) \Sigma_{Y_{kt},X_{kt}} + s_{j(t+1)} \Sigma_{Y_{jt},X_{jt}} \bigg] \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
=&s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\right]\Sigma_{Y_{kt},X_{kt}}\right\}\\
&-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}}\right]\Sigma_{Y_{jt},X_{jt}}\right\}\\
&-s_{j(t+1)}\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,Y_{jt}}\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}+(1-s_{j(t+1)})\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\right]\\
&+\frac{1}{\sigma^2}\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)-\frac{1}{\sigma}\frac{1}{\epsilon}\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\left(2s_{j(t+1)}-1\right)\\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\right]+\frac{1}{\sigma}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)
\end{aligned}
$$

Substituting for $\Sigma_{X,R}$ and rearranging, we obtain

$$
\begin{aligned}
\frac{1}{\psi}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\Bigg[&s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{kt},X_{kt}}\right)\\
&-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{jt},X_{jt}}\right)\Bigg].
\end{aligned}
\tag{A-19}
$$

This expression is positive if and only if the term in brackets is positive. Define $\hat{s}_{t+1}$ as the $s_{j(t+1)}$ such that $R_{jt}/R_{kt}=R_{j(t+1)}/R_{k(t+1)}$. Then $\hat{s}_{t+1}$ is the root of the term in brackets. Solving for that root, we have:

$$
\hat{s}_{t+1}=\frac{\Sigma_{Y_{kt},X_{kt}}C_{jt}}{\Sigma_{Y_{jt},X_{jt}}C_{kt}+\Sigma_{Y_{kt},X_{kt}}C_{jt}},
\tag{A-20}
$$

where $\Sigma_{w,z}$ is the elasticity of w with respect to z and where

$$
\begin{aligned}
C_{jt}\triangleq&1+\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}\left[\psi-\frac{\alpha}{1-\alpha}\right]\Sigma_{Y_{jt},X_{jt}}>0,\\
C_{kt}\triangleq&1+\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}\left[\psi-\frac{\alpha}{1-\alpha}\right]\Sigma_{Y_{kt},X_{kt}}>0.
\end{aligned}
$$

Thus,

$$
\left\{\hat{s}_{t+1}\geq\frac{1}{2}\right\}\Leftrightarrow\left\{\Sigma_{Y_{kt},X_{kt}}\geq\Sigma_{Y_{jt},X_{jt}}\right\},
$$

where the right-hand side is evaluated at $\hat{s}_{t+1}$. Using the explicit expressions for the elasticities, for intermediate-good production, and for X_{jt} and X_{kt} (see equation (A-2)), we have:

$$\begin{aligned}
\Sigma_{Y_{kt},X_{kt}} \geq & \Sigma_{Y_{jt},X_{jt}} \\
\Leftrightarrow 0 \leq & \frac{(1-\kappa)X_{kt}^{\frac{\sigma-1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}Y_{kt}^{\frac{\sigma-1}{\sigma}}}{Y_{kt}^{\frac{\sigma-1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}}} \qquad \text{(A-21)} \\
\Leftrightarrow 0 \leq & X_{kt}^{\frac{\sigma-1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}} - X_{jt}^{\frac{\sigma-1}{\sigma}}Y_{kt}^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 0 \leq & \kappa R_{jt}^{\frac{\sigma-1}{\sigma}}X_{kt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}X_{kt}^{\frac{\sigma-1}{\sigma}} - \kappa R_{kt}^{\frac{\sigma-1}{\sigma}}X_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa)X_{kt}^{\frac{\sigma-1}{\sigma}}X_{jt}^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq & \left(\frac{R_{jt}\left[\frac{1-\kappa}{\kappa}\left(\frac{R_{kt}}{\Psi_k}\right)^{1/\psi}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}}\left[\frac{R_{kt}}{A_{kt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}}A_{kt}}{R_{kt}\left[\frac{1-\kappa}{\kappa}\left(\frac{R_{jt}}{\Psi_j}\right)^{1/\psi}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}}\left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}}A_{jt}} \right)^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq & \left[\left(\frac{\Psi_j}{\Psi_k}\right)^{\frac{\alpha\,\sigma/\psi}{\sigma(1-\alpha)+\alpha}} \left(\frac{R_{jt}}{R_{kt}}\right)^{\frac{\sigma(1-\alpha-\alpha/\psi)}{\sigma(1-\alpha)+\alpha}} \left(\frac{A_{kt}}{A_{jt}}\right)^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq & \left(\frac{\Psi_j}{\Psi_k}\right)^{\chi\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]} \left(\frac{1+\eta\gamma s_{jt}}{1+\eta\gamma s_{kt}}\right)^{-\chi\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]} \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{\chi(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi)-(1+\sigma/\psi)]}, \qquad \text{(A-22)}
\end{aligned}$$

where the final line substitutes for R_{jt}/R_{kt} from equation (10) (which must hold for $\hat{s}_{t+1}$ interior) and where

$$\chi \triangleq \frac{\sigma-1}{[\sigma(1-\alpha)+\alpha][1+\sigma/\psi]} < 0 \text{ iff } \sigma < 1.$$

The right-hand side of inequality (A-22) is increasing in s_{jt} if and only if $\sigma < 1$. Therefore, if $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the strict version of the inequality does not hold at $s_{jt} = 0.5$, and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the inequality holds at $s_{jt} = 0.5$. If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \geq \left[\frac{\Psi_j}{\Psi_k}\right]^{\theta},$$

and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \leq \left[\frac{\Psi_j}{\Psi_k}\right]^{\theta},$$

where

$$\theta \triangleq \frac{-\frac{1}{\psi}[\alpha + \sigma(1-\alpha)]}{(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi) - (1+\sigma/\psi)]} = \frac{1}{(1-\alpha)(1+\psi)} > 0.$$

□

Proof of Proposition A-1

The tâtonnement adjustment process generates, to constants of proportionality, the following system for finding the equilibrium within period t:

$$\dot{R}_{jt} = h\Big(G_j(R_{jt}, R_{kt}) - 1\Big),$$
$$\dot{R}_{kt} = h\Big(G_k(R_{jt}, R_{kt}) - 1\Big),$$

where dots indicate time derivatives (with the fictional time for finding an equilibrium here flowing within a period t), $h(0) = 0$, and $h'(\cdot) > 0$. The system's steady state occurs at the equilibrium values, which I denote with stars. Linearizing around the steady state, we have

$$\begin{bmatrix} \dot{R}_{jt} \\ \dot{R}_{kt} \end{bmatrix} \approx h'(0) \begin{bmatrix} \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}} \\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}} \end{bmatrix} \begin{bmatrix} R_{jt} - R^*_{jt} \\ R_{kt} - R^*_{kt} \end{bmatrix} = h'(0)\, G \begin{bmatrix} R_{jt} - R^*_{jt} \\ R_{kt} - R^*_{kt} \end{bmatrix},$$

where G is the 2×2 matrix of derivatives, each evaluated at (R^*_{jt}, R^*_{kt}). Lemma A-4 implies that the trace of G is strictly negative, in which case at least one of the two eigenvalues must be strictly negative. Lemma A-5 shows that $det(G) > 0$, which means that both eigenvalues must have the same sign. Therefore both eigenvalues are strictly negative. The linearized system is therefore globally asymptotically stable, and, by Lyapunov's Theorem of the First Approximation, the full nonlinear system is locally asymptotically stable around the equilibrium.

Proof of Corollary A-2

Now treat equations (12) and (13) as functions of R_{jt}, R_{kt}, and s_{jt} (recognizing that $s_{kt} = 1 - s_{jt}$):

$$1 = \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{Y_{jt}(R_{jt}, s_{jt})}\right]^{1/\epsilon} \left[\frac{Y_{jt}(R_{jt}, s_{jt})}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \triangleq \hat{G}_j(R_{jt}, R_{kt}; s_{jt}),$$

$$1 = \kappa\,(1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{Y_{kt}(R_{kt}, s_{jt})}\right]^{1/\epsilon} \left[\frac{Y_{kt}(R_{kt}, s_{jt})}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \triangleq \hat{G}_k(R_{jt}, R_{kt}; s_{jt}).$$

This system of equations implicitly defines R_{jt} and R_{kt} as functions of the parameter s_{jt}. Define the matrix $\hat{G}$ analogously to the matrix G. Using the implicit function theorem, we have

$$\frac{\partial R_{jt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} \quad \text{and} \quad \frac{\partial R_{kt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})}.$$

Interpreting equation (10) as implicitly defining s_{jt} as a function of R_{jt} and R_{kt}, we have:

$$\frac{\partial s_{jt}}{\partial R_{jt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}} \quad \text{and} \quad \frac{\partial s_{jt}}{\partial R_{kt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}},$$

and thus

$$\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \quad \text{and} \quad \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}.$$

Using these expressions, consider how the right-hand side of equation (A-1) changes in s_{jt}:

$$\begin{aligned}
\frac{\mathrm{d}[\Pi_{jt}/\Pi_{kt}]}{\mathrm{d}s_{jt}} =& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}\frac{\partial R_{jt}}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}\frac{\partial R_{kt}}{\partial s_{jt}} \\
=& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} \\
& - \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} - \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})} \\
\propto& -\frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} \\
& - \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}} - \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}} \\
=& -\left(\frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\right) \\
& + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \\
=& -det(G).
\end{aligned}$$

The third expression factored $det(\hat{G})$, which is positive by the proof of Proposition A-1 for a corner solution in s_{jt}, and it also factored $\partial[\Pi_{jt}/\Pi_{kt}]/\partial s_{jt}$, which is negative. The final equality recognizes that the only difference between the equations with a hat and the equations without a hat are that the equations without a hat allow s_{jt} to vary with R_{jt} and R_{kt}. Lemma A-5 showed that $det(G) > 0$. Thus the right-hand side of equation (A-1) strictly decreases in s_{jt}.

Proof of Lemma 1

Under the given assumption that $\nu = 0.5$ and $\Psi_j = \Psi_k$, we have $R_{jt} = R_{kt}$ when $A_{j(t-1)} = A_{k(t-1)}$ and $s_{jt} = 0.5$. Therefore, it is easy to see that $\Pi_{jt}/\Pi_{kt} = 1$ at $s_{jt} = 0.5$ when $A_{j(t-1)} = A_{k(t-1)}$. By Lemma A-7, increasing $A_{j(t-1)}$ increases Π_{jt}/Π_{kt} if either $\sigma > 1$ or σ is not too much smaller than 1. In those cases, Corollary A-2 gives us that $A_{j(t-1)} > A_{k(t-1)}$ implies $s^*_{jt} > 0.5$. The lemma follows from observing that $A_{j(t-1)} > A_{k(t-1)}$ and $\Psi_j = \Psi_k$ imply that $A_{j(t-1)}/A_{k(t-1)} > (\Psi_j/\Psi_k)^{1/[(1-\alpha)(1+\psi)]}$.

Proof of Proposition 2

First, consider a case in which $\sigma > 1$ and in which Assumption 1 holds. From Lemma A-8, $\hat{s}_{t+1} < 0.5$. Therefore $s_{jt_0} > \hat{s}_{t+1}$. Assume that $s_{j(t_0+1)} < s_{jt_0}$. From equation (10), $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)}$ increases in A_{jt_0}/A_{kt_0} for any given $s_{j(t_0+1)}$ if $\sigma > 1$. Therefore, for the equilibrium to have $s_{j(t_0+1)} < s_{jt_0}$, it must be true that $R_{jt_0}/R_{kt_0} > R_{j(t_0+1)}/R_{k(t_0+1)}$ and thus $s_{j(t_0+1)} < \hat{s}_{t_0+1}$. From Corollary A-2 and $s_{jt_0} > \hat{s}_{t_0+1}$, it must be true that $\Pi_{jt_0}/\Pi_{kt_0} > 1$ when evaluated at $\hat{s}_{t_0+1}$. Because $R_{jt_0}/R_{kt_0} = R_{j(t_0+1)}/R_{k(t_0+1)}$ if $s_{j(t_0+1)} = \hat{s}_{t_0+1}$ and $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$ by $s_{jt_0} > 0.5$, it therefore must be true that $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)} > 1$ when evaluated at $\hat{s}_{t_0+1}$. By Corollary A-2, it then must be true that $s_{j(t_0+1)} > \hat{s}_{t_0+1}$. We have a contradiction. It must be true that $s_{j(t_0+1)} \geq s_{jt_0}$.

Because $s_{j(t_0+1)} \geq s_{jt_0} > 0.5 > \hat{s}_{t+1}$, it follows that $R_{jt_0}/R_{kt_0} \leq R_{j(t_0+1)}/R_{k(t_0+1)}$ and $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$. Therefore Assumption 1 still holds at time $t_0 + 1$. Proceeding by induction, sector j's shares of research and extraction increase forever: resource j is locked-in from time t_0 if $\sigma > 1$ and Assumption 1 holds at time t_0. We have established the first part of the proposition.

Next, consider a case in which $\sigma < 1$ and in which Assumption 1 holds. Let time $w \geq t_0$ be the first time after t_0 at which sector j's share of extraction begins decreasing, so that $R_{jx}/R_{kx} \leq R_{j(x+1)}/R_{k(x+1)}$ for all $x \in [t_0, w-1]$ and $R_{jw}/R_{kw} > R_{j(w+1)}/R_{k(w+1)}$, which in turn requires $s_{jx} \geq \hat{s}_x$ for all $x \in [t_0+1, w]$ and $s_{j(w+1)} < \hat{s}_{w+1}$. Note that $s_{jt_0} > 0.5$ implies that $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$. Assume that sector j's share of research begins declining sometime after its share of extraction does, so that $s_{jx} \leq s_{j(x+1)}$ for all $x \in [t_0, w]$. Then we have $A_{jx}/A_{kx} > A_{j(x-1)}/A_{k(x-1)}$ for all $x \in [t_0, w+1]$, and thus $A_{jx}/A_{kx} > [\Psi_j/\Psi_k]^\theta$ for all $x \in [t_0, w+1]$. Using this with Lemma A-8 and $\sigma < 1$ then implies $\hat{s}_{x+1} \geq 0.5$ for all $x \in [t_0, w+2]$. Combining this with the requirement that $s_{jw} \geq \hat{s}_w$, we have $s_{jw} \geq 0.5$. From equation (10) and $\sigma < 1$, we then have $s_{j(w+1)} \geq s_{jw}$ only if $R_{jw}/R_{kw} \leq R_{j(w+1)}/R_{k(w+1)}$. But that contradicts the definition of w, which required $R_{jw}/R_{kw} > R_{j(w+1)}/R_{k(w+1)}$. Sector j's share of research must have begun declining no later than time w. We have shown that a transition in extraction occurs only after a transition in research.

We now have two possibilities. We will see that the first one implies that $s_{jx} \geq 0.5$ at all times $x \in [t+1, w]$ and the second one generates a contradiction.

First, we could have $A_{j(x-2)}/A_{k(x-2)} \geq [\Psi_j/\Psi_k]^\theta$ at all times $x \in [t_0+1, w]$. Then by Lemma A-8, $\hat{s}_x \geq 0.5$ at all times $x \in [t_0+1, w]$. The definition of time w then requires $s_{jx} \geq 0.5$ at all times $x \in [t_0+1, w]$.

Second, we could have $A_{j(x-2)}/A_{k(x-2)} < [\Psi_j/\Psi_k]^\theta$ at some time $x \in [t_0+1, w]$. In order for this to happen, it must be true that $s_{jx} < 0.5$ at some times $x \in [t_0+2, w]$.[31] Let z be the first time at which $s_{jx} < 0.5$. $A_{j(t_0-1)}/A_{k(t_0-1)} > [\Psi_j/\Psi_k]^\theta$ and $s_{jx} \geq 0.5$ for all $x \in [t_0, z-1]$ imply that $A_{j(z-2)}/A_{k(z-2)} > [\Psi_j/\Psi_k]^\theta$, which implies by Lemma A-8 and $\sigma < 1$ that $\hat{s}_z \geq 0.5$. So we have $s_{jz} < \hat{s}_z$, which means that $R_{j(z-1)}/R_{k(z-1)} > R_{jz}/R_{kz}$. But this contradicts the definition of time w as the first time at which sector j's share of extraction begins decreasing.

Therefore, we must have $A_{j(x-2)}/A_{k(x-2)} \geq [\Psi_j/\Psi_k]^\theta$ and $s_{jx} \geq 0.5$ at all times $x \in [t_0+1, w]$. Observe that $s_{jx} \geq 0.5$ at all times $x \in [t_0, w]$ implies $A_{jx}/A_{kx} \geq A_{j(x-1)}/A_{k(x-1)}$ at all times $x \in [t_0, w]$. We have shown that a transition in technology happens only after a transition in extraction.

Finally, consider the first time $z > t_0$ at which $R_{jz} < R_{kz}$. Assume that $\Psi_j \geq \Psi_k$ and that $s_{jx} \geq 0.5$ for $x \in [t_0, z]$. Assumption 1, $\Psi_j \geq \Psi_k$, and $s_{jx} \geq 0.5$ imply $A_{jx} \geq A_{kx}$ for $x \in [t_0, z]$. Using $\sigma < 1$, we see that $A_{j(z-1)} \geq A_{k(z-1)}$, $\Psi_j \geq \Psi_k$, and $R_{jz} < R_{kz}$ imply that the right-hand side of equation (A-1) is < 1 when evaluated at $s_{jz} = 0.5$. So by Corollary A-2, time z equilibrium scientists must be less than 0.5. But $s_{jz} < 0.5$ contradicts $s_{jx} \geq 0.5$ for $x \in [t_0, z]$. Therefore, if $\Psi_j \geq \Psi_k$, then there must be some time $x \in [t_0, z]$ at which $s_{jx} < 0.5$. We have shown that if $\Psi_j \geq \Psi_k$, then sector k must begin dominating research before it begins dominating extraction.

Proof of Proposition 3

First consider $\sigma > 1$. When $s^*_{jt} = 1$, only $A_{j(t-1)}$ changes over time, increasing by $\eta\gamma A_{j(t-1)}$ at each time t. By Lemma A-7, $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)} > \Pi_{jt_0}/\Pi_{kt_0}$ if $s_{j(t_0+1)} \geq s_{jt_0}$. If $s_{jt_0} = 1$, then $\Pi_{jt_0} > \Pi_{kt_0}$, in which case $\Pi_{j(t_0+1)} > \Pi_{k(t_0+1)}$ if $s_{j(t_0+1)} = s_{jt_0}$. It is then an equilibrium for s^*_{jt} to equal 1 for all $t \geq t_0$. An analogous proof covers the case where $s^*_{jt} = 0$.

Now consider $\sigma < 1$. If $s^*_{jt} = 1$ for all $t \geq t_0$, then $A_{j(t-1)} \to \infty$ as $t \to \infty$ and, by Lemma A-6, R_{jt}/R_{kt} goes to a constant. In that case, from equation (10), Π_{jt}/Π_{kt} goes to zero for all s_{jt}. But Π_{jt}/Π_{kt} cannot be zero if $s^*_{jt} = 1$ because $s^*_{jt} = 1$ implies that $\Pi_{jt}/\Pi_{kt} \geq 1$. We have contradicted the assumption that $s^*_{jt} = 1$ for all $t \geq t_0$. Analogous arguments show that it cannot be true that $s^*_{kt} = 1$ for all $t \geq t_0$. It therefore must be true that, for all t_0, there exists some $t > t_0$ such that $s^*_{jt} \in (0, 1)$.

[31] Recall that $s_{jt} \geq 0.5$ and $s_{j(t+1)} \geq s_{jt}$ imply $s_{j(t+1)} \geq 0.5$.

Proof of Proposition 4

We know that $\Pi^*_{jt}/\Pi^*_{kt} = 1$ when $s^*_{jt} \in (0,1)$. Assume that $s^*_{jt} \in (0.5,1)$. By Lemma A-7, $\Pi_{j(t+1)}/\Pi_{k(t+1)} > 1$ when evaluated at s^*_{jt}. Therefore, by Corollary A-2, $s^*_{j(t+1)} > s^*_{jt}$. Analogous arguments apply when $s^*_{jt} \in (0,0.5)$.

Proof of Proposition 5

Assume $\sigma < 1$. We know from Proposition 3 that a corner research allocation cannot persist indefinitely. Therefore, A_{jt} and A_{kt} both become arbitrarily large as t becomes large. From equations (8), (9), and (2), we have

$$\begin{aligned} X_{jt} &= \left\{ \left[\left(\frac{R_{jt}}{\Psi_j} \right)^{1/\psi} \frac{1-\kappa}{\kappa} \right]^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}} \right\}^{\frac{\alpha}{1-\alpha}} A_{jt} \\ &= \left[\Psi_j^{-1/\psi} \frac{1-\kappa}{\kappa} \right]^{\frac{\sigma\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} R_{jt}^{\frac{\alpha(1+\sigma/\psi)}{\sigma(1-\alpha)+\alpha}}. \end{aligned}$$

X_{jt} and X_{kt} thus also become arbitrarily large as t becomes large. This in turn implies that $Y_{jt} \to \kappa^{\frac{\sigma}{\sigma-1}} R_{jt}$ and $Y_{kt} \to \kappa^{\frac{\sigma}{\sigma-1}} R_{kt}$ as t becomes large. From equation (14), we have:

$$\left[\frac{R_{jt}}{R_{kt}} \right]^{\frac{1}{\sigma}+\frac{1}{\psi}} \to \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \left[\frac{R_{jt}}{R_{kt}} \right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}}$$

as t becomes large. Therefore, as $t \to \infty$,

$$\frac{R_{jt}}{R_{kt}} \to \left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}}. \tag{A-23}$$

Define $\Omega_t \triangleq A_{jt}/A_{kt}$, so that

$$\Omega_t = \frac{1+\eta\gamma s_{jt}}{1+\eta\gamma(1-s_{jt})} \Omega_{t-1}. \tag{A-24}$$

From Proposition 3, $\Pi^*_{jt}/\Pi^*_{kt} = 1$ for some t sufficiently large. Using this and equation (A-23) in equation (10), we have:

$$\frac{1+\eta\gamma s^*_{jt}}{1+\eta\gamma(1-s^*_{jt})} = \Omega_{t-1}^{-(1-\sigma)(1-\alpha)} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi}.$$

Therefore, from equation (A-24),

$$\Omega_t = \Omega_{t-1}^{1-(1-\sigma)(1-\alpha)} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi}.$$

Define $\tilde{\Omega}_t \triangleq \ln[\Omega_t]$. We then have:

$$\tilde{\Omega}_t = [1-(1-\sigma)(1-\alpha)]\tilde{\Omega}_{t-1} + \ln \left[\left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi} \right].$$

This is a linear difference equation. For $\sigma < 1$, the coefficient on $\tilde{\Omega}_{t-1}$ is strictly between 0 and 1. The linear difference equation is therefore stable. The system approaches a steady state in $\tilde{\Omega}_t$ and therefore in Ω_t. From equation (A-24), any steady state in Ω_t must have $s_{jt}^* = 0.5$. Therefore as $t \to \infty$, $s_{jt}^* \to 0.5$. We have established the first result.

Equation (A-23) implies that if $\nu_j = 0.5$ and $\Psi_j = \Psi_k$ then $R_{jt}^* = R_{kt}^*$. Further, if $\nu_j \geq 0.5$ and $\Psi_j \geq \Psi_k$ with at least one inequality being strict, then $R_{jt}^* > R_{kt}^*$. Now substitute into equation (10) and use $s_{jt} = 0.5$:

$$\begin{aligned}
\frac{\Pi_{jt}}{\Pi_{kt}} &\to \left(\frac{A_{j(t-1)}}{A_{k(t-1)}} \right)^{\frac{-(1-\sigma)(1-\alpha)}{\sigma+\alpha(1-\sigma)}} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k} \right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \\
&= \left(\frac{A_{j(t-1)}}{A_{k(t-1)}} \right)^{\frac{-(1-\sigma)(1-\alpha)}{\sigma+\alpha(1-\sigma)}} \left(\frac{\nu_j}{1-\nu_j} \right)^{\frac{\sigma+\psi}{\sigma+\alpha(1-\sigma)} \frac{\epsilon}{\epsilon+\psi}} \left(\frac{\Psi_j}{\Psi_k} \right)^{\frac{\epsilon-\sigma}{\sigma+\alpha(1-\sigma)} \frac{1}{\epsilon+\psi}},
\end{aligned}$$

and this must equal 1 because $s_{jt}^* = 0.5$. Therefore, if $\nu_j = 0.5$ and $\Psi_j = \Psi_k$ then $A_{jt} = A_{kt}$, and if $\nu_j \geq 0.5$ and $\Psi_j \geq \Psi_k$ with at least one inequality being strict, then $A_{jt} > A_{kt}$. We have established the second and third results.

Finally, as t becomes large along a path with $s_{jt}^* = 0.5$, using previous results in equation (12) yields:

$$\begin{aligned}
\left[\frac{R_{jt}}{\Psi_j}\right]^{1/\psi} \rightarrow & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma} \\
= & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{\kappa^{\frac{\sigma}{\sigma-1}} R_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
= & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{\kappa^{\frac{\sigma}{\sigma-1}} R_{jt}}{A_Y Y_{jt}\left(\nu_j + (1-\nu_j)\left(\frac{Y_{kt}}{Y_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
= & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{1}{A_Y \left(\nu_j + (1-\nu_j)\left(\frac{R_{kt}}{R_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
= & \nu_j \kappa^{\frac{\sigma}{\sigma-1}} A_Y \left[\nu_j + (1-\nu_j)\left(\frac{R_{kt}}{R_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{1}{\epsilon-1}}.
\end{aligned} \tag{A-25}$$

From equation (A-23), R^*_{jt}/R^*_{kt} becomes constant as t becomes large. Then from (A-25), R^*_{jt} approaches a constant. An analogous derivation establishes that R^*_{kt} approaches a constant. We have established the final result.