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Self-Fulfilling Debt Crises, Revisited: The Art of the Desperate Deal
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ABSTRACT

We revisit self-fulfilling rollover crises by introducing an alternative equilibrium selection that involves bond auctions at depressed but strictly positive equilibrium prices, a scenario in line with observed sovereign debt crises. We refer to these auctions as "desperate deals," the defining feature of which is a price schedule that makes the government indifferent to default or repayment. The government randomizes at the time of repayment, which we show can be implemented in pure strategies by introducing stochastic political payoffs or external bailouts. Quantitatively, auctions at fire-sale prices are crucial for generating realistic spread volatility.

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1 Introduction

In this paper, we explore a novel class of self-fulfilling sovereign debt crisis equilibria. We build on the familiar Cole and Kehoe (2000) framework in which a coordination failure can lead to a "failed auction" and subsequent default.[1] We extend this to incorporate self-fulfilling equilibria in which the sovereign auctions bonds at fire-sale – but strictly positive – prices. Such "desperate deals" are consistent with the experiences of emerging markets and recent European crisis countries, in which spreads are high and volatile but default remains relatively rare. The standard Cole-Kehoe equilibrium has difficulty explaining such episodes given the stark assumption that a crisis results in a price of zero for new issuances and default with probability one. We explore quantitatively the differences between our framework and the canonical model and show that including fire-sale auctions as part of the equilibrium path is crucial for understanding the high volatility of spreads.

The framework we explore builds on the standard Eaton and Gersovitz (1981) model and the recent quantitative versions beginning with Aguiar and Gopinath (2006) and Arellano (2008). In particular, the government of a small open economy faces endowment risk and issues non-contingent (but defaultable) bonds to a pool of competitive foreign investors. As in Cole and Kehoe (2000), our timing convention allows the sovereign to default in the same period as a successful auction. Cole and Kehoe used this timing to support an equilibrium price of zero for any amount of bonds sold at auction, which in turn is supported by immediate default due to the inability to roll over maturing bonds. Cole and Kehoe considered an equilibrium selection in which bonds are auctioned at positive prices in non-crisis periods but, conditional on the realization of a sunspot, creditors coordinate on the zero-price equilibria, triggering default.

As mentioned, the failed auctions of the standard Cole-Kehoe model shed light on how creditor beliefs can play a role in generating defaults and how this prospect affects government policy ex ante. However, in practice, sovereigns in crisis frequently escape default by issuing a minimal amount of bonds at low prices. As a motivating example, consider the case of

[1] There are two main traditions in the self-fulfilling debt crisis literature, one associated with Calvo (1988) and the other with Cole and Kehoe (2000). Loosely speaking, the former tradition focuses on the link between prices today and budget sets (and incentives to default) tomorrow. See Lorenzoni and Werning (2013) and Ayres, Navarro, Nicolini, and Teles (2015) for recent papers in the Calvo tradition. The Cole and Kehoe (2000) model features multiple pairs of prices and contemporaneous default decisions that satisfy equilibrium conditions, with multiplicity reminiscent of a bank run. Recent papers in this tradition include Conesa and Kehoe (2011) and Aguiar, Amador, Farhi, and Gopinath (2015).

Portugal.[2] Yields on Portugal's bonds increased in 2010. By the start of 2011, Portugal was in distress and having difficulty rolling over its maturing bonds. In January 2011, it issued one billion euros in a "private placement" that was reportedly purchased by China.[3] This was not sufficient to stem the crisis, and in May of that year Portugal began to draw on emergency funding from the European Union (EU). In late 2012, the prospect of bonds maturing in 2013 loomed. In anticipation, the Portuguese debt agency repurchased bonds maturing in September 2013 while issuing bonds maturing in 2015. This swap was accomplished not through default, negotiation, and restructuring but rather was implemented via a dual auction.[4] The *OECD Sovereign Borrowing Outlook 2013*[5] referred to this type of transaction as "market-friendly solutions to resume market access and to ease near-term redemption pressures." A benefit of the operation was avoiding the risk of a failed auction in 2013 when the original bonds matured.[6] As it turned out, Portugal did successfully auction bonds in 2013 but did so without the threat of a rollover crisis due to the maturity swap.

This narrative gives a sense of the rich menu of possibilities, even in the absence of outright default and renegotiation, that is observed in many sovereign debt crises, both in Europe and emerging markets. We capture some of this richness in a tractable manner by incorporating desperate deals as part of the equilibrium outcome during a coordination failure. In particular, we follow Cole and Kehoe and introduce a sunspot that coordinates creditor beliefs between a relatively high equilibrium price schedule and a crisis price schedule. However, rather than the latter involving zero prices and immediate default, we consider an equilibrium price schedule that makes the government indifferent to default or repayment immediately after the auction. In our quantitative model, such prices typically imply spreads that are roughly 500 basis points higher than they are during non-crisis periods, which is in line with many real-world episodes. This price schedule is rationalized by allowing the government to play a mixed strategy over post-auction default, with the probability of default consistent with that period's equilibrium price schedule.

[2] We are grateful for conversations with Pedro Teles regarding Portugal's debt management during the crisis.

[3] See http://ftalphaville.ft.com/2011/01/11/453471/p-p-p-ick-up-a-portuguese-private-placement/ and http://uk.reuters.com/article/portugal-bonds-idUSLDE7061QG20110107.

[4] See http://www.igcp.pt/fotos/editor2/2013/Relatario_Anual/Financiamento_Estado_Port_uk.pdf, page 6.

[5] http://dx.doi.org/10.1787/sov_b_outlk-2013-en, page 120.

[6] The Portuguese debt agency annual report for 2012 (http://www.igcp.pt/gca/?id=108) notes that "the management of the debt portfolio takes into account the refinancing profile of (IGCP) the debt, so as to avoid an excessive concentration of redemptions..." Its 2013 report states that its various operations "enabled the IGCP to accumulate levels of liquidity," which it used in part to reduce additional future commitments.

We extend the model to discuss how the desperate deals price schedule can be supported in an equilibrium in which the government plays pure strategies. In the spirit of Harsanyi (1973), we introduce a shock to the default payoff at the time of settlement that is orthogonal to income and debt. A natural interpretation of this shock is the incumbent party's political payoff to default versus repayment conditional on fundamentals. An alternative extension considers bailouts from a supranational institution, such as the International Monetary Fund (IMF) or EU, that has an unpredictable component. From the perspective of the lenders, the bonds sold during a crisis are lottery tickets whose payoff depends on short-run political outcomes, either domestically or in foreign capitals. The movement in these external forces may be small, but they have a large impact on bond prices when the government is close to indifference, providing a potential source of high-frequency volatility in secondary market prices. In this sense, our approach corresponds to a worldview that debt crises push a sovereign to the brink of default, but whether default is actually realized is a random outcome that is independent of fundamentals and, from the creditors' perspective, a matter of luck.

A few features of this approach are worthy of note. The equilibrium price schedule and the government's mixed-strategy response are part of a competitive equilibrium. Although bargaining and renegotiation are important aspects of sovereign default,[7] many emerging markets and all European crisis countries other than Greece have managed their crises without resorting to outright default. The auctions we consider are arm's-length transactions involving competitive prices. Moreover, as the prices are competitive they do not involve implicit transfers.

Although desperate deals do not involve bargaining or transfers, they do benefit legacy bondholders (compared with default) and deliver the default value to the government without the associated deadweight costs of default. In this sense, conditional on the occurrence of a crisis, the deals raise the efficiency of bond markets. Given the competitive nature of the bond market, the sovereign reaps this gain ex ante through better prices. We show that this has important implications for welfare as well as the willingness of the government to borrow despite the prospect of crises. An important analytical insight of the Cole-Kehoe (2000) model is that the potential of a crisis and the associated ex ante equilibrium price schedule induce the government to deleverage in order to avoid being vulnerable to a self-fulfilling run. Replacing failed auctions with desperate deals mitigates this tendency. In fact, one can construct a scenario in which the government rolls over bonds at fire-sale

[7] See Benjamin and Wright (2008), for example.

prices for a protracted period that ends in default at a random point in time. This is a perspective on prolonged crises that is an alternative to Conesa and Kehoe's (2011) "gambling for redemption" and Lorenzoni and Werning's (2013) "slow moving" debt crises.

The paper introduces and discusses the theoretical concepts buttressing the desperate deals equilibria in a simple one-period bond framework similar to Eaton and Gersovitz (1981). We then enrich the model to include longer maturity bonds and risk-averse lenders to evaluate the quantitative implications of desperate deals. We calibrate the model to Mexico and quantitatively contrast our benchmark model with desperate deals to the canonical Cole-Kehoe framework in which crises generate certain default. With desperate deals, we match key bond market regularities, including the average and standard deviation of bond spreads, average debt-to-income ratios, and a default frequency of twice every one hundred years, the latter being consistent with broad historical samples. In the Cole-Kehoe version of the model, the standard deviation of bond spreads is a factor of *twenty-five* times too small. While that model generates frequent enough defaults, the sovereign never borrows into high spreads. In our benchmark model, the government is more willing to accumulate debt and, more important, to issue bonds at fire-sale prices when faced with the crisis price schedule.

Using our benchmark model, we also contrast defaults due to a coordination failure versus "fundamental" defaults in which the government defaults despite the creditors coordinating on the better equilibrium price schedule. The latter have a distinct boom-bust pattern, in which default is preceded by abnormally high growth followed by a large negative growth realization. The high growth generates high bond prices, inducing the government to leverage up. The relatively high level of debt leaves the sovereign unwilling to repay when an abnormally low growth outcome is realized. This pattern shares something in common with the data, but recent empirical work suggests booms followed by large recessions represent only a fraction of debt crises in practice.[8] Moreover, fundamental defaults do not generate an ex ante spike in spreads, as the low growth realization is largely unanticipated given the persistence of the endowment process.

The model's defaults associated with coordination failures do not have an anticipatory boom and coincide with a relatively moderate contraction of endowment. Relatively high debt levels are also a necessary component of default, but in our benchmark model these are frequently observed in the ergodic distribution due to the reasons previously discussed. Given this vulnerability, a coordination failure generates a spike in spreads as the government

[8]See the handbook chapter of Aguiar, Chatterjee, Cole, and Stangebye (2016) for more details.

issues bonds at desperate deal prices. The benchmark simulations rationalize why large recessions may not yield large jumps in spreads, while smaller recessions can be associated with extremely adverse outcomes, a pattern observed in many sovereign debt crises.

We also consider crisis equilibria in which the government *repurchases* non-maturing bonds at fire-sale prices. As legacy bondholders prefer this outcome to default, ex ante prices are higher when such events are likely to occur. We show that this has ambiguous effects on ex ante government welfare. On the one hand, a better outcome in the event of a crisis increases the efficiency of bond markets, which the government captures via prices due to competitive markets for its bonds. On the other hand, more efficient markets encourage the sovereign to borrow more, raising the likelihood of a crisis going forward, which may lower prices.

The idea that some factor other than domestic fundamentals, such as creditor beliefs about the equilibrium behavior of other lenders, is compelling. Aguiar, Chatterjee, Cole, and Stangebye (2016) document a number of supporting facts regarding emerging market and European bonds. First, as is well known, emerging market spreads over benchmark risk-free bonds are volatile. Second, while large spikes in spreads are correlated with declines in output, the correlation is relatively weak. In fact, a sizable proportion of such spikes occur when growth is positive and in line with historical means. The same holds in the shorter sample of European crisis countries (Portugal, Ireland, Italy, Spain, and Greece). While the literature has shown some of the variation in spreads can be explained by shifts in measures of global risk premia, there remains a large and time-varying unexplained residual component. One possible interpretation of this residual source of risk is shifts in creditors beliefs about the behavior of other creditors. Recently, Bocola and Dovis (2016) performed an accounting exercise on the Italian debt crisis and found that shifts in the probability of a self-fulfilling crisis played a non-negligible role in explaining the spike in spreads.

The rest of the paper is organized as follows. Section 2 lays out a simple model to introduce the concept of desperate deals; Section 3 augments and calibrates the model to explore its quantitative implications; Section 4 presents the quantitative results; and Section 5 concludes.

2 Model

We present the model in two phases. In this section, we consider a simple model that will allow a clear exposition of the paper's novel crisis equilibrium. In Section 3, we extend the model for quantitative analysis.

2.1 Environment

We consider a single-good, discrete-time environment. There is a small open economy that receives a stochastic endowment and (initially) has access to international capital markets. We assume that the economy's aggregate consumption and saving decisions are made by a sovereign government.[9] The economy is small in the sense that its endowment realizations and its fiscal policy do not affect the world risk-free interest rate.

The economy receives a stochastic endowment $Y_t \in \mathbb{Y} \equiv [\underline{Y}, \overline{Y}]$, with $0 < \underline{Y} < \overline{Y}$. For this section, we assume Y_t is *iid* over time. The sovereign's preferences over the sequence of aggregate consumption $\{C_t\}_{t=0}^{\infty}$ is given by:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t),$$

where $\beta \in (0, 1)$ and $u : \mathbb{R}^+ \to \mathbb{R}$ is continuous, strictly increasing, and concave.

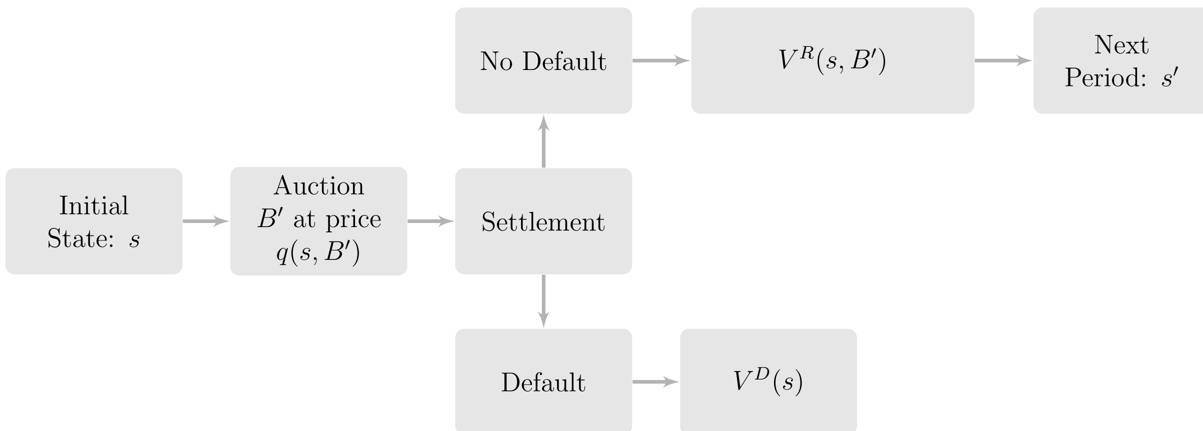
The rest of the world is populated by risk-neutral lenders who discount at the rate $R^{-1} = (1 + r^*)^{-1}$. In this section, financial markets are restricted to a one-period, non-contingent discount bond. Let B denote the outstanding stock of bonds at the start of a period; note that $B > 0$ indicates the government is a net debtor, and $B < 0$ a net creditor. To rule out Ponzi schemes, we place an upper bound on debt: $B \in (-\infty, \overline{B}]$.

The timing of events within a period is depicted in Figure 1. The government enters with a debt payment due in the current period of B. At the start of the period, the endowment Y is realized. A sunspot that coordinates creditor beliefs, ρ, is also realized. The sunspot is *iid* over time and will be discussed in detail in Section 2.2. After observing the exogenous states (Y, ρ), the government decides how much debt to issue (or assets to buy), denoted by

[9] That is, we assume that the sovereign has enough instruments to implement any feasible consumption sequence as a domestic competitive equilibrium and therefore abstract from the problem of individual residents of the domestic economy. This does not mean that the government necessarily shares the preferences of its constituents but rather that it is the relevant decision maker viz-à-viz international financial markets.

B'. After issuing new debt, the government decides whether to repay B at "settlement." If it defaults, the creditors receive zero, and the government is excluded from financial markets going forward. In the quantitative model, we allow for stochastic re-entry.

Figure 1: Timing Within a Period



Note that the default decision takes place *after* new debt is issued. This timing follows Cole and Kehoe (2000) and differs from Eaton and Gersovitz (1981) and the existing quantitative literature. An important implication of our timing is that holders of newly issued bonds are not fully compensated if the government defaults immediately after the auction. Different from Cole and Kehoe, if $B' > 0$ and the government defaults, we assume the government does not receive the auction revenue. For this section, we assume that the period's auction revenue is lost to both creditors and the government.[10] In the quantitative model, this revenue is split pro rata among the government's creditors (old and new). For $B' \leq 0$, the government is purchasing assets, which are risk free.[11]

Let $s = (Y, \rho, B)$ denote the vector of exogenous and endogenous states at the start of the period, and S the set of all possible states. We shall consider equilibria such that the equilibrium outcome is a function of s and the credit history. If the government has not defaulted in the past, it faces an equilibrium price schedule for new debt B' given by $q(s, B')$. The government takes the schedule as given but recognizes that the price of its debt may vary with B'. In that sense, the government is large in its own debt market.

[10] One interpretation of this assumption is that the legal battles following a default result in the dissipation of any revenue raised in the auction or any foreign assets held by the government.

[11] In equilibrium, it will always be the case that the government does not default if it purchases assets at auction. However, off equilibrium there may an asset level that is not feasible given the endowment. We assume that these carry the risk-free price, which can be rationalized by having the assets written off if the government fails to pay for them at settlement.

After the auction, the government decides whether to default or repay. If the government defaults, it simply consumes its endowment every period thereafter. Let $V^D(s)$ denote this value:

$$V^D(s) = u(Y) + \frac{\beta \mathbb{E} u(Y')}{1-\beta}. \tag{1}$$

Note that in the model without re-entry, V^D does not depend on the rest of the equilibrium, and for clarity we will write $V^D(Y)$ in place of $V^D(s)$ for the remainder of this section. If the government repays, it obtains value:

$$V^R(s, B') = u(Y + q(s, B')B' - B) + \beta \mathbb{E}\left[V(s')|B' \in s'\right] \tag{2}$$

if $Y + q(s, B')B' - B \geq 0$. If $Y + q(s, B')B' - B < 0$, then repayment is infeasible, and we set $V^R(s, B') = -\infty$; obviously, default will always dominate this value. In (2), the value function $V(s')$ denotes the value at the start of next period. In particular,

$$V(s) = \max \left\langle \max_{B' \leq \overline{B}} V^R(s, B'), V^D(s) \right\rangle. \tag{3}$$

Note that the expectation in (2) incorporates that B' is in the government's information set at the time of repayment in the current period. The expectation is not explicitly conditional on s due to the *iid* assumption for Y and ρ.

Given an equilibrium price schedule q, equations (1),(2), and (3) characterize the government's problem. Let $\mathcal{B} : S \to (-\infty, \overline{B}]$ denote the government's optimal debt-issuance policy function[12] and $\mathcal{D} : S \times (-\infty, \overline{B}] \to [0, 1]$ the default policy function. Note that $\mathcal{D}$ depends on B' as well as s, as discussed, and takes values in the *interval* $[0, 1]$ to allow for randomization over the default decision, which will play a role in our crisis equilibrium.

Given the small open economy and risk-neutral lenders' assumptions, prices must satisfy a break-even condition. Specifically, given the government's policy functions, the price schedule $q : S \times (-\infty, \overline{B}] \to [0, R^{-1}]$ must equate the expected return on sovereign debt to the risk-free

[12] As we shall see, the government may be indifferent across alternative levels of debt issuance; that is, its optimal policy is a correspondence. However, prices depend on which element of the correspondence is chosen; therefore, equilibrium outcomes are sensitive to how we break ties. Hence, equilibrium selection involves a rule for debt issuance when indifferent. This will be discussed in detail in our quantitative implementation.

rate:

$$q(s,B') = \begin{cases} R^{-1}(1-\mathcal{D}(s,B'))\mathbb{E}\left[1-\mathcal{D}(s',\mathcal{B}(s'))|B' \in s'\right] \text{ if } B' \in (0,\overline{B}] \\ R^{-1} \text{ if } B' \leq 0. \end{cases} \tag{4}$$

In the first line, note that our timing implies that bondholders are vulnerable to two default decisions, one in the current period immediately after auction and one after next period's auction. Since next period's default probability depends on next period's debt choice, the pricing also depends on the government's borrowing policy function.[13] The second line states that assets are always sold at the risk-free price.

The definition of equilibrium is standard:

Definition 1. *An* equilibrium *consists of a price schedule q and government policy functions $\mathcal{B}$ and $\mathcal{D}$ such that: (i) $\mathcal{B}$ and $\mathcal{D}$ and the induced value functions solve the government's problem given q; and (ii) q satisfies the break-even condition* (4) *given $\mathcal{D}$ and $\mathcal{B}$.*

2.2 Equilibrium Selection

Other than the timing of the auction and default decisions, the environment described has the same elements as the simplest version of the Eaton-Gersovitz model. The one-period Eaton-Gersovitz model has a unique equilibrium (Auclert and Rognlie, 2016). However, under the Cole-Kehoe timing, the fact that the government cannot commit to repaying outstanding bonds at the time it auctions new bonds opens the door to multiplicity. We now discuss how we select an equilibrium and construct crises that involve desperate deals.

For expositional purposes, we will construct equilibria as the fixed point of an iterative scheme.[14] Consider a candidate equilibrium $(q_0, \mathcal{B}_0, \mathcal{D}_0)$ with associated value functions V_0 and V_0^R. We take V_0^R to be decreasing in debt due, a property that we prove to be true in equilibirium at the end of the section. From next period onward, agents assume these functions will describe equilibrium behavior going forward. Given this continuation equilibrium, we will construct alternative prices and policies that satisfy the equilibrium conditions, with the alternatives indexed by the sunspot ρ. In this sense, we focus on a

[13]The fact that the break-even condition is imposed for all B' – even those that occur off equilibrium – is a perfection requirement: It ensures that if the government were to deviate from $\mathcal{B}$ and issue a suboptimal amount of debt, these bonds would be priced in a manner consistent with equilibrium behavior going forward.

[14]This approach also tracks how the quantitative model is solved.

"static" multiplicity, by which we mean that we can support multiple equilibrium outcomes in the current period holding constant equilibrium behavior going forward. We shall consider three possible beliefs: $\rho \in \{EG, CK, DD\}$, where EG stands for "Eaton-Gersovitz," CK for "Cole-Kehoe," and DD for "Desperate Deals." The assumption that ρ is *iid* over time allows us to hold constant expectations of future equilibrium behavior as we vary the current period's beliefs.

Eaton-Gersovitz Beliefs

Given the continuation equilibrium, suppose that in the current period the government could commit not to default at the time of the auction. This intraperiod commitment is what distinguishes the Eaton-Gersovitz timing from the Cole-Kehoe timing. To be precise, suppose that the government has such commitment only in the *current* period and then follows the candidate equilibrium behavior going forward. We will later verify if and when this is credible absent commitment. Let q_{EG} denote the price schedule that would satisfy the lenders' break-even condition under this scenario:

$$q_{EG}(B') \equiv \begin{cases} R^{-1}\mathbb{E}\left[1 - \mathcal{D}_0(s', \mathcal{B}_0(s'))|B' \in s'\right] \text{ if } B' \in (0, \overline{B}] \\ R^{-1} \text{ if } B' \leq 0. \end{cases} \tag{5}$$

Contrasting (5) with (4), we see the difference is the assumption that the current period's default policy is set to zero for all B'. This makes q_{EG} an upper bound on the equilibrium price schedule conditional on the continuation equilibrium. Given the *iid* assumptions and the fact that q_{EG} is determined by next period's default decisions, q_{EG} is a function only of B'. With this price schedule, we can define the government's associated repayment value:

$$V_{EG}^R(Y, B, B') \equiv \begin{cases} u\left(Y + q_{EG}(B')B' - B\right) + \beta\mathbb{E}\left[V_0(s')|B' \in s'\right] \text{ if } Y + q_{EG}(B')B' \geq B \\ -\infty \text{ otherwise.} \end{cases}$$

We can now construct an equilibrium price schedule and value function that are consistent with the Eaton-Gersovitz beliefs. To aid in the exposition, we introduce Figure 2. Panel (a) depicts the value of repayment as a function of B' holding constant (B, Y), which is suppressed in the figure's notation. The solid hump-shaped line depicts $V_{EG}^R(Y, B, B')$ as we vary B'. The peak of this curve indicates the optimal issuance policy given q_{EG}.[15] For

[15] We cannot state analytically that the function is single peaked (although it is in the quantitative model). The non-monotonicity comes from the fact that as we increase B' current consumption increases (subject to

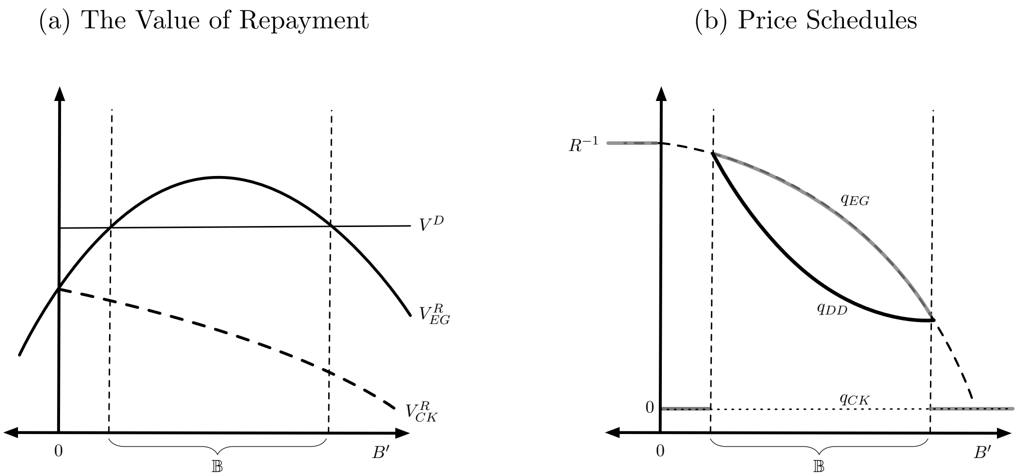
reference, we include the value of default given Y, which is the horizontal line labeled V^D in the figure.

The set $\mathbb{B}(Y, B)$ consists of debt issuances B' such that the government finds it optimal to repay at settlement:

$$\mathbb{B}(Y, B) \equiv \left\{ B' \in (0, \overline{B}] | V^R_{EG}(Y, B, B') \geq V^D(Y) \right\}. \tag{6}$$

When $B' \in \mathbb{B}(Y, B)$, then $V^R_{EG}(Y, B, B') \geq V^D(Y)$, and the belief that the government will not default in the current period can be made self-confirming. That is, q_{EG} can be rationalized as an equilibrium outcome.[16]

Figure 2: Values and Prices



Note: Panel (a): Heuristic diagram of value of repayment when prices are q_{EG} (solid line denotes V^R_{EG}) and q_{CK} (dashed line denotes V^R_{CK}). Functions are drawn for a fixed (Y, B). $\mathbb{B}$ is short-hand for $\mathbb{B}(Y, B)$, and V^D is short-hand for $V^D(Y)$. The desperate deals value of repayment is identical to V^D on the domain $\mathbb{B}$. Panel (b): Price schedules. The downward sloping dashed line is q_{EG}, the horizontal dotted line at 0 is q_{CK}, and the intermediate solid line on domain $\mathbb{B}$ is q_{DD}. The shaded line that equals R^{-1} for $B' \leq 0$, q_{EG} for $B' \in \mathbb{B}$, and zero otherwise traces out the equilibrium price schedule when $\rho = EG$. The equilibrium price schedule for $\rho = DD$ differs from $\rho = EG$ only on the domain $\mathbb{B}$.

In particular, let $\rho = EG$ indicate that agents believe the government will not default

being on the upward part of the debt Laffer curve; that is, $d[q(B')B']/dB' > 0$), but the continuation value declines. Once B' reaches the downward sloping part of the Laffer curve, V^R_{EG} unambiguously declines in B'.

[16] Of course, for some (Y, B) it will be the case that $\mathbb{B}(Y, B) = \emptyset$. In Figure 2, that case would have V^R_{EG} lying strictly below V^D for all B'.

at the current period's settlement whenever $B' \in \mathbb{B}(B, Y)$. Specifically, define a new price schedule q_1 for $\rho = EG$ by:

$$q_1([Y, EG, B], B') \equiv \begin{cases} q_{EG}(B') \text{ if } B' \in \mathbb{B}(Y, B) \cup (-\infty, 0] \\ 0 \text{ otherwise.} \end{cases} \tag{7}$$

The first line states that the price schedule is equal to q_{EG} when the government can credibly promise not to default in the current period. This occurs when the government purchases assets ($B' \leq 0$) or when $B' \in \mathbb{B}(Y, B)$. Otherwise, the government will default immediately after the auction, and hence the only equilibrium price that can be supported is zero. The $\rho = EG$ price schedule therefore tracks the intraperiod commitment price schedule when such a price is time consistent given the lack of commitment within the period; otherwise, the price is zero.

We depict prices in Panel (b) of Figure 2. The downward sloping dashed line is q_{EG}, which equals R^{-1} for $B' < 0$ and decreases in B' for $B' > 0$ (see Section 2.3 Proposition 1). The equilibrium price for $\rho = EG$ is the shaded line that equals q_{EG} for $B' \in \mathbb{B}$, R^{-1} for $B' \leq 0$, and zero otherwise.

Facing this price schedule, the government's repayment value is:

$$V_1^R([Y, EG, B], B') \equiv \begin{cases} V_{EG}^R(Y, B, B') \text{ if } B' \in \mathbb{B}(Y, B) \cup (-\infty, 0] \\ u(Y - B) + \beta \mathbb{E}\left[V_0(s') | B' \in s'\right] \text{ if } B' \notin \mathbb{B}(Y, B) \cup (-\infty, 0] \text{ \& } B \leq Y \\ -\infty \text{ otherwise.} \end{cases}$$

The first line states that the value is V_{EG}^R whenever $q_1 = q_{EG}$. The second line states that, if the government were to issue debt at zero price, then it pays outstanding debt out of its current endowment. The final line assumes an infinite loss if the government attempts to repay bonds when infeasible.

The associated policy functions are $\mathcal{B}_1([Y, EG, B]) \in \operatorname{argmax}_{B' \leq \overline{B}} V_1^R([Y, EG, B], B')$ and

$$\mathcal{D}_1([Y, EG, B], B') = \begin{cases} 0 \text{ if } B' \in \mathbb{B}(Y, B) \\ 1 \text{ otherwise.} \end{cases} \tag{8}$$

Cole-Kehoe Beliefs

Given the continuation equilibrium, the price schedule q_1 for $\rho = EG$ is the highest price that can be supported in the current period as it rules out immediate default whenever possible. We can also consider the worst possible price as a candidate equilibrium, which will form the basis of our Cole-Kehoe equilibrium. In particular, assume that the government will default with probability one at settlement if $B' \geq 0$. Let

$$q_{CK}(B') \equiv \begin{cases} 0 \text{ if } B' \in (0, \overline{B}] \\ R^{-1} \text{ if } B' \leq 0. \end{cases} \tag{9}$$

That is, debt is issued at zero price, but assets are purchased at the risk-free rate. We depict q_{CK} in Figure 2 Panel (b) with the dotted horizontal line equal to 0 for $B' > 0$.

As before, we can define the associated repayment value:

$$V^R_{CK}(Y, B, B') \equiv \begin{cases} u(Y - B) + \beta \mathbb{E}\left[V(s') | B' \in s'\right] \text{ if } B' \in (0, \overline{B}] \text{ \& } B \leq Y \\ V^R_{EG}(Y, B, B') \text{ if } B' \leq 0 \\ -\infty \text{ if } B' \geq 0 \text{ \& } B > Y. \end{cases} \tag{10}$$

The first line indicates that all debt issuances occur at price zero, and hence inherited debt must be paid out of the current endowment when feasible. If the government purchases assets ($B' < 0$), then it pays the risk-free price, which is the same as under the Eaton-Gersovitz beliefs. The final line concerns the case when repayment is not feasible under zero prices.

V^R_{CK} is depicted in Figure 2 by the dashed line. For $B' < 0$, V^R_{CK} tracks V^R_{EG} as both assume risk-free prices for purchasing assets. For $B' > 0$, V^R_{CK} is strictly decreasing and less than V^R_{EG} (on the domain such that $q_{EG} > 0$). The strict monotonicity in this case follows from the fact that additional debt lowers the continuation value (a property proved later), but does not increase current consumption.

If $V^R_{CK}(Y, B, B') \leq V^D(Y)$, then a price of zero can be supported as an equilibrium outcome. We denote the beliefs of the worst price schedule by $\rho = CK$. In particular, define an equilibrium price schedule under these beliefs by:

$$q_1([Y, CK, B], B') \equiv \begin{cases} q_{CK}(B') \text{ if } V^R_{CK}(Y, B, B') \leq V^D(Y) \\ q_{EG}(B') \text{ otherwise.} \end{cases} \tag{11}$$

The first line states that prices are zero if default is weakly preferable to repayment when faced with zero prices. However, if the government prefers to repay even when debt has a zero price (or buys assets), then prices are the same as under the Eaton-Gersovitz beliefs. The associated repayment value function is:

$$V_1^R([Y, CK, B], B') \equiv \begin{cases} V_{CK}^R(Y, B, B') \text{ if } V_{CK}^R(Y, B, B') \leq V^D(Y) \\ V_1^R([Y, EG, B], B') \text{ otherwise.} \end{cases} \tag{12}$$

The default policy rationalizes the price schedule and is consistent with whether $V_1^R \gtrless V^D$:

$$\mathcal{D}_1([Y, CK, B], B') = \begin{cases} 1 \text{ if } V_1^R([Y, CK, B], B') \leq V^D(Y) \\ 0 \text{ otherwise.} \end{cases} \tag{13}$$

Comparing q_1 for $\rho = EG$ with $\rho = CK$, we see that there are pairs (Y, B) in which beliefs matter as to whether the government will default in the current period. In particular, we define the "crisis zone" as:

$$\mathbb{C} \equiv \left\{ (Y, B) \in \mathbb{Y} \times (0, \overline{B}] \Big| \max_{B' \leq \overline{B}} V_{CK}^R(Y, B, B') \leq V^D(Y) \leq \max_{B' \leq \overline{B}} V_{EG}^R(Y, B, B') \right\}. \tag{14}$$

If the initial states are in this set, when the government can issue debt at the $\rho = EG$ price it will not default. However, if it faces the $\rho = CK$ price, it will default. For fundamentals $(Y, B) \in \mathbb{C}$, the government is vulnerable to a Cole-Kehoe self-fulfilling rollover crisis.

The scenario depicted in Figure 2 is one in which $(Y, B) \in \mathbb{C}$. This can be seen by the fact that at $B' = 0$ the value of repayment is less than V^D, while the peak of V_{EG}^R is above V^D. If the intercept at $B' = 0$ were above V^D, then the government prefers to repay even at zero prices, while if the peak of V_{EG}^R were below V^D, then the government prefers to default even when facing the best price schedule.

Desperate Deal Beliefs

We now turn to our third set of equilibrium beliefs, the desperate deal beliefs. These beliefs are relevant in the crisis zone $\mathbb{C}$; that is, when (Y, B) are such that creditor beliefs determine whether the sovereign repays or defaults. Given $(Y, B) \in \mathbb{C}$, we construct a price for each $B' \in \mathbb{B}(Y, B)$. By definition of $\mathbb{B}$, the value V_{EG}^R is weakly greater than V^D on this set. By the definition of $\mathbb{C}$, the value V_{CK}^R is below V^D on $\mathbb{B}$ (and for any $B' > 0$).

Thus, the value of repayment is weakly greater than default when facing q_{EG} but weakly less when facing $q_{CK} = 0$. By continuity of the utility function, there is a price that makes the government indifferent to repayment and default. Specifically, for a given (Y, B, B'), with $(Y, B) \in \mathbb{C}$ and $B' \in \mathbb{B}(Y, B)$, we can define $q_{DD}(Y, B, B') \in [0, q_{EG}(Y, B, B')]$ as the unique solution to:

$$u\left(Y + q_{DD}(Y, B, B')B' - B\right) + \beta\mathbb{E}[V_0(s')|B' \in s'] = V^D(Y). \tag{15}$$

Note that for $B' \in \mathbb{B}$ such that the inequality in (6) is strict and $V^R_{CK}(Y, B, B') < V^D(Y)$, we have $0 < q_{DD}(Y, B, B') < q_{EG}(Y, B, B')$. In particular, prices are non-zero but strictly below the $\rho = EG$ price.

To see how this works, consider the hump-shaped V^R_{EG} curve in Figure 2 Panel (a). On the domain $B' \in \mathbb{B}(Y, B)$, the value of repayment is weakly greater than V^D when bonds are issued at price q_{EG}. A lower price, holding constant the amount issued, lowers the value of repayment. As we vary the price from q_{EG} to $q_{CK} = 0$, we lower the value of repayment from V^R_{EG} toward V^R_{CK} in a continuous fashion. The desperate deals price is reached when the value of repayment equals V^D.

In Figure 2 Panel (b), we include q_{DD} on the domain $B'\mathbb{B}(B, Y)$. At the boundaries of $\mathbb{B}$, $V^R_{EG} = V^D$, and hence $q_{DD} = q_{EG}$. For B' such that $V^R_{EG} > V^D > V^R_{CK}$, we have $q_{DD} \in (0, q_{EG})$. As we will discuss, q_{DD} is not necessarily monotonic in B'. We depict it as such for expositional ease.

We use q_{DD} as the basis for our third "static" equilibrium. In particular,

$$q_1([Y, DD, B], B') \equiv \begin{cases} q_{DD}(Y, B, B') \text{ if } (Y, B) \in \mathbb{C} \ \& \ B' \in \mathbb{B}(Y, B) \\ q_1([Y, EG, B], B') \text{ otherwise.} \end{cases} \tag{16}$$

The value of repayment follows directly from the construction of q_{DD}:

$$V^R_1([Y, DD, B], B') \equiv \begin{cases} V^D(Y) \text{ if } (Y, B) \in \mathbb{C} \ \& \ B' \in \mathbb{B}(Y, B) \\ V^R_1([Y, EG, B], B') \text{ otherwise.} \end{cases} \tag{17}$$

For $(Y, B) \notin \mathbb{C}$ or for $B' \notin \mathbb{B}$, prices and repayment values are the same as under the Eaton-Gersovitz beliefs. On this domain, either the government will default even when facing the highest possible prices or the government will repay even when facing a zero price. In both cases, prices are independent of beliefs.

To rationalize the desperate deals price, we need to ensure that the lenders' break-even condition is satisfied. We do this by allowing the government to randomize over default and repayment at settlement when indifferent. We will provide our motivation for randomization in Section 2.5, but to define the equilibrium, we have:

$$\mathcal{D}_1([Y,DD,B],B') \equiv \begin{cases} 1 - \frac{q_{DD}(Y,B,B')}{q_{EG}(B')} \text{ if } (Y,B) \in \mathbb{C} \ \& \ B' \in \mathbb{B}(Y,B) \\ \mathcal{D}_1([Y,EG,B],B') \text{ otherwise.} \end{cases} \tag{18}$$

By definition of $\mathbb{B}$, the first line is always between zero and one, and strictly so when $V^R_{EG}(Y,B,B') > V^R_{CK}(Y,B,B')$. The second line is the counterpart to the corresponding price schedule and repayment value when outside of the crisis zone or when $B' \notin \mathbb{B}$.

To complete the equilibrium construction, we have $\mathcal{B}_1(s) \in \text{argmax}_{B' \leq \overline{B}} V_1^R(s,B')$ for all $s \in S$. For desperate deals, there may be a range of B' along which the government is indifferent; that is, all $B' \in \mathbb{B}$ generate a repayment value of $V^D(Y)$ by construction (but with different prices and default probabilities). In our quantitative implementation, we will look at various alternative selections from the optimal policy correspondence. As the desperate deal price schedule and the associated default probability vary with B', the selection will have significance for ex ante equilibrium prices and policies.[17]

Given the initial continuation equilibrium $\{q_0, V_0^R, \mathcal{D}_0, \mathcal{B}_0\}$, we have now constructed $\{q_1, V_1^R, \mathcal{D}_1, \mathcal{B}_1\}$ that also satisfy the equilibrium conditions for the current period. A fixed point of this mapping is therefore an equilibrium.

2.3 Properties of the Equilibrium

For $\rho \neq DD$, the equilibrium shares many of the intuitive properties of the canonical Eaton-Gersovitz model. We will state these properties, and then discuss to what extent they carry over to the desperate deals scenario. All proofs are in the Appendix.

Proposition 1. *In an equilibrium, for any $s = [Y,\rho,B] \in S$ such that $\rho \neq DD$ and for any $B' \in (-\infty, \overline{B}]$:*

[17] We have not shown analytically that V^R has a unique maximum for $\rho \neq DD$, although this is the case in our quantitative model. However, if there were multiple optimums, which B' that is chosen does not change the default probabilities for $\rho \neq DD$. Hence, simple tie-breaking rules are without loss.

(i) The repayment value $V^R(s, B')$ is weakly increasing in Y and weakly decreasing in B, with strict monotonicity in both cases if repayment is feasible (that is, $Y + q(s, B')B' > B$);

(ii) The default policy function $\mathcal{D}(s, B')$ is weakly increasing in B and weakly decreasing in Y; and

(iii) The price schedule $q(s, B')$ is weakly decreasing in B and weakly increasing in Y.

The first statement (i) follows immediately from the budget set. The monotonicity of default in debt due follows immediately from part (i) plus the fact that V^D is invariant to B. The monotonicity of default in endowment depends more sensitively on how $V^D(Y)$ varies with Y; in the simple case of *iid* shocks and no re-entry, the result of Arellano (2008) carries through almost directly. The price schedule monotonicity result mirrors that of default policy through the break-even condition.

We now turn to the case of $\rho = DD$, which in some instances operates differently than the canonical model. First, from our discussion of Figure 2, we have an ordering of V^R, default policy, and q across belief states:

Proposition 2. *For any $Y \in \mathbb{Y}$ and $(B, B') \in (-\infty, \overline{B}]^2$, we have:*

(i) $V^R([Y, CK, B], B') \leq V^R([Y, DD, B], B') \leq V^R([Y, EG, B], B')$;

(ii) $\mathcal{D}([Y, CK, B], B') \leq \mathcal{D}([Y, DD, B], B') \leq \mathcal{D}([Y, EG, B], B')$; and

(iii) $q([Y, CK, B], B') \leq q([Y, DD, B], B') \leq q([Y, EG, B], B')$.

In terms of comparative statics with respect to inherited debt B and endowment Y, the value of repayment is monotonic, all else equal:

Proposition 3. *For any $s \in S$ such that $\rho = DD$, $V^R(s, B')$ is weakly increasing in Y and weakly decreasing in B. The monotonicity is strict in Y whenever $Y + q(s, B')B' > B$.*

Note that the monotonicity is not strict in B even when consumption is interior, as desperate deal prices keep V^R equal to V^D, which is invariant to B.

In contrast to Proposition 1, default policies and prices are not weakly monotonic in Y and B when $\rho = DD$. Recall that q_{DD} is the price that makes the government indifferent to repayment or default. If B increases, holding all else constant (including $B' > 0$), then the desperate deal price must be higher to maintain indifference. Similarly, an increase in the endowment raises the value of repayment, and thus q_{DD} must be lower to maintain indifference. That is, to generate a crisis when debt is low or endowment is high requires a relatively low price of debt.

2.4 Notable Features of Desperate Deals

The defining feature of the Cole-Kehoe crisis is the inability to raise any money at auction. In contrast, the desperate deals price schedule is positive for debt issuances but less than the price under the non-crisis beliefs for the same fundamentals. From the government's perspective, the price schedule leaves them indifferent to default. From the lenders' perspective, they are purchasing a lottery ticket that immediately loses value with some positive probability but otherwise pays off the value that the bond would trade at under non-crisis beliefs (q_{EG}). In this way, the equilibrium reflects the situation in which bond prices are positive but depressed, pushing the government up against its indifference condition. It then becomes a random outcome whether the bonds retain their value into the next period.

Another feature of desperate deals is that they occur at equilibrium prices. This is not a bargaining outcome in which creditors or the government threaten to "walk away." The lenders have no incentive to hold out from the auction, as at the margin they are indifferent to participating or not. Thus, while this results in a positive price for legacy lenders who are selling their bonds at the same time, this is not a partial default or haircut in the usual sense.

While the government is indifferent to the amount issued in a crisis at the "indifference" price schedule, the legacy lenders are not. In particular, they would like the government to choose an amount that maximizes the probability of repayment. However, given these are arm's-length transactions, there is no market mechanism to induce the government to select the surplus maximizing policy. This is the natural counterpart of the competitive assumption that there is no way for the government to "select" the Eaton-Gersovitz equilibrium price schedule that does not involve a rollover crisis.

The ability to issue bonds in a crisis reduces the deadweight losses associated with rollover crises. In particular, depending on the realization of the mixed-strategy randomization, some rollover crises are not followed with immediate default. Given a certain probability of a rollover crisis next period, the ability to engage in desperate deals will therefore raise bond prices ex ante. We shall see in the quantitative implementation that this encourages the government to borrow more in equilibrium and makes the bond market a more efficient provider of intertemporal smoothing and insurance.

In our equilibrium construction, we assumed that ρ is *iid* over time. There are natural ways to enrich the process for ρ. For example, suppose that $\rho = DD$ were an absorbing state. Then the government would simply continue to issue bonds at desperate deal prices until, eventually, it defaulted at settlement. This follows from the fact that, when faced with desperate deal prices, the government has no incentive to reduce its level of debt once the crisis has begun. This speaks to episodes in which bond market participants feel that the level of debt is unsustainable, yet the government neither defaults nor attempts a fiscal correction over an extended period while continuing to roll over debt. This provides an alternative narrative to Conesa and Kehoe's (2011) "gambling for redemption" in which participants anticipate a possible economic recovery. It also contrasts with Lorenzoni and Werning's (2013) "slow moving" debt crises in which debt builds gradually over time.

2.5 Purification: Politics and Bailouts

At the core of the desperate deals equilibrium is the government's indifference to default and the subsequent randomization at the time payment is due. However, this is not as "knife edge" as it may first appear. The strict indifference is not necessary with a straightforward extension in the spirit of Harsanyi (1973). We view the mixed strategy as capturing other facets of the government's repayment decision that generate default risk. A primary example is the political payoff to the current incumbent party regarding the popularity of repayment versus default, which may vary stochastically and generate uncertainty between an auction and the repayment of maturing bonds. An alternative source of uncertainty is whether an international institution, such as the IMF or EU, bails out the sovereign at settlement. These sources of risk can serve as rationales for desperate deals even in environments where the government plays pure strategies.

Consider an extension of the model in which the payoff to default is $V^D(Y)$ plus an additional shock ϵ. This latter shock only becomes known when the government is making its default decision (and after the auction).[18] A natural interpretation of ϵ is the realization

[18] There is a long history of such random payoffs to default. See the handbook chapters of Eaton and Fernandez (1995) and Aguiar and Amador (2014) as well as more recent papers by Aguiar, Amador, Hopenhayn, and Werning (2016) and Chatterjee, Corbae, Dempsey, and Rios-Rull (2016).

of political payoffs to default that are orthogonal to output and the quantity of debt due. Let ϵ be distributed *iid* over time with continuous cdf F on support $[\underline{\epsilon}, \bar{\epsilon}]$.

Given (Y, B, B'), suppose there exists a threshold ϵ^* such that the government strictly prefers to default at settlement if $\epsilon > \epsilon^*$ and strictly prefers to repay if $\epsilon < \epsilon^*$. Hence, the lenders' break-even condition requires the auction price to be $F(\epsilon^*)q_{EG}(B')$. This is consistent with ϵ^* being the indifference threshold as long as:

$$u\left(Y - B + F(\epsilon^*)q_{EG}(B')B'\right) + \mathbb{E}\left[V(s')|B' \in s'\right] = V^D(Y) + \epsilon^*. \tag{19}$$

Now consider (Y, B, B') such that $V^R_{EG}(Y, B, B') > V^D(Y) > V^R_{CK}(Y, B, B')$, which is the relevant scenario for our desperate deals equilibrium. Suppose the support of ϵ is small enough that

$$V^R_{CK} - V^D < \underline{\epsilon} < \bar{\epsilon} < V^R_{EG} - V^D,$$

where we omit the arguments of the value functions for clarity. Then there exists an $\epsilon^* \in [\underline{\epsilon}, \bar{\epsilon}]$ that satisfies (19). To see this, note that the left-hand side minus the right-hand side of (19) is continuous in ϵ^*; is strictly positive when $\epsilon^* = \bar{\epsilon}$, as the left-hand side becomes V^R_{EG} at $F(\bar{\epsilon}) = 1$ and $V^R_{EG} - V^D - \bar{\epsilon} > 0$; and is strictly negative when $\epsilon^* = \underline{\epsilon}$, as the left-hand side becomes V^R_{CK} at $F(\underline{\epsilon}) = 0$ and $V^R_{CK} - V^D - \underline{\epsilon} < 0$. Therefore, there exists a threshold $\epsilon^* \in [\underline{\epsilon}, \bar{\epsilon}]$ that satisfies (19).[19]

An alternative interpretation of ϵ is bailouts from a third party. Many debt crises end either in default or by some sort of outside intervention, typically involving an international institution such as the IMF or a supranational institution such as the European Central Bank. In the Mexican debt crisis of 1994-95, it was the U.S. government that extended emergency credit. Bailouts and self-fulfilling crises were recently analyzed by Roch and Uhlig (2016).

Such a bailout would imply that resources at settlement are the endowment Y plus third-party funds ϵ. We assume the bailout is received only if the government repays. Again, there is a threshold for the size of bailout ϵ^* such that the government repays if $\epsilon > \epsilon^*$ and defaults otherwise. The price at auction is therefore $(1-F(\epsilon^*))q_{EG}(B')$, and the indifference condition

[19] Our mixed-strategy equilibria can be considered the limit of a pure-strategy equilibrium as we shrink the support $[\underline{\epsilon}, \bar{\epsilon}]$.

becomes:

$$u\left(Y+\epsilon^{*}+(1-F(\epsilon^{*}))q_{EG}(B')B'-B\right)+\mathbb{E}\left[V(s')|B'\in s'\right]=V^{D}(Y).$$

Again, for an appropriate choice of support, there exists an indifference threshold $\epsilon^* \in [\underline{\epsilon}, \bar{\epsilon}]$. In this case, desperate deals are bets on the size of the intervention.

One nice feature of either version of the extended model is it can explain why one might see large, high-frequency changes in the secondary market price of the government's debt during a crisis. To see this, imagine that news about the realization of ϵ could become public between the auction and the government's default decision. This news would change creditors' posterior regarding ϵ, which would shift the price of the outstanding debt that had just been auctioned as well as the old debt that is coming due. Note that with a small support for ϵ, this news could be minor in its impact on the government's incentives but nevertheless have a large impact on the spread.

3 Quantitative Model

In this section, we demonstrate that an augmented quantitative version of our model generates equilibrium outcomes that match the data. A now well-established puzzle in the quantitative sovereign debt literature is the difficulty in replicating the observed volatility of spreads due to the government's willingness to avoid extremely high interest rates through small adjustments in bond issuances or outright default. The challenge of generating realistic spread volatility is discussed thoroughly in our recent handbook chapter (Aguiar, Chatterjee, Cole, and Stangebye, 2016). The chapter discusses the limitations of existing fixes, such as combining nonlinear default costs with highly volatile trend-stationary endowment processes.[20] We show in this section how desperate deals generate realistic volatility in sovereign spreads combined with an empirically plausible probability of default. The link between desperate deals and the existing approaches to spread volatility is discussed as well.

To illustrate the ease with which the model can match the data, we posit a stochastic growth process for the endowment, calibrate it to Mexico – a medium-volatility country – and dispense with nonlinear default costs. We do this because Mexico's crisis in 1995 was the

[20] For example, see Aguiar and Gopinath (2006), Arellano (2008), Hatchondo and Martinez (2009), and Chatterjee and Eyigungor (2012).

motivation for Cole and Kehoe (2000). We choose stochastic growth rather than a trend-stationary process because it is a more realistic process for developing countries (Aguiar and Gopinath, 2007). We dispense with nonlinear default costs because Mexico entered a decade of depressed economic conditions after its debt crisis in 1982, despite starting out in a recession, suggesting that the costs of default are severe in recessions as well as in booms.[21]

The major changes relative to the model of Section 2 are (i) a persistent stochastic growth process for the endowment; (ii) longer maturity bonds; (iii) risk-averse creditors; (iv) a richer default state, including re-entry and output loss; and (v) no destruction of auction revenues in the case of default. The longer maturity bonds allow us to consider buybacks, which have been an important element in many crises. The addition of risk-averse lenders allows us to separate movements in risk premia from default risk. As the elements of the quantitative model are straightforward extensions of the model described in Section 2, we keep the exposition streamlined and include most of the details in the Appendix.

3.1 Overview of Additional Features

For the quantitative analysis, we allow the endowment to follow a stochastic trend plus cycle process as in Aguiar and Gopinath (2007). In particular, $Y_t = G_t e^{z_t}$, where $\ln G_t \equiv \sum_{s=1}^{t} g_s$ is the cumulation of period growth rates g_t and z_t represents fluctuations around trend growth. We assume the growth rate process is governed by

$$g_{t+1} = (1 - \rho_g)\bar{g} + \rho_g g_t + \varepsilon_{t+1},$$

where $\varepsilon \sim N(0, \sigma_g^2)$. The transitory component of output z_t is assumed to be *iid*, orthogonal to ε_t and to have mean zero and variance σ_z^2.

We incorporate longer maturity bonds in a tractable manner by introducing random maturity bonds, as in Leland (1994).[22] In particular, each bond matures next period with a constant hazard rate $\lambda \in [0, 1]$.[23] The expected maturity of a bond is $1/\lambda$ periods, and so $\lambda = 0$ is a console and $\lambda = 1$ is one-period debt. We also assume bonds pay a coupon

[21]See `http://users.econ.umn.edu/~tkehoe/papers/mexico-chile.pdf`.

[22]See also Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012), and Arellano and Ramanarayanan (2012).

[23]We let the unit of a bond be infinitesimally small and let maturity be *iid* across individual bonds, such that with probability one a fraction λ of any non-degenerate portfolio of bonds matures each period. The constant hazard of maturity implies that all bonds are symmetric before the realization of maturity at the start of the period, regardless of when they were purchased.

every period up to and including the period of maturity, which, without loss of generality, we normalize to the risk-free rate r^*. With this normalization, a risk-free bond will have an equilibrium price of one.

If the government defaults, we assume that in the period of default the sovereign's payoff is:

$$V^D(s) \equiv u(Y^D) + \beta \mathbb{E} V^E(s'), \tag{20}$$

where $Y^D = Ge^{\bar{z}}$ and V^E denotes the continuation value while in the default (exclusion) state:

$$V^E(s) = u((1-\phi)Y) + \beta(1-\xi)\mathbb{E}\left[V^D(s')|s\right] + \beta\xi\mathbb{E}\left[V(s')|s, B' = 0\right]. \tag{21}$$

With probability ξ the sovereign regains access to debt markets with zero debt and a clean credit history. Note that in the period of default, we evaluate Y^D at the mean *iid* endowment shock $z = \bar{z}$. This is done for computational reasons and makes V^D (but not V^R) independent of the current z realization. Given the low variance of our calibrated z, this is not a crucial assumption. For expositional reasons, we do not impose the output punishment ϕ until the period after default. This simplifies the comparison of endowment growth in periods of default with repayment and is innocuous given that ϕ can be scaled accordingly.

In the simple model of Section 2, we assumed that if the government defaults at settlement, all revenues raised at the prior auction are lost. In the quantitative model, we treat this scenario in a more realistic manner, laid out in detail in the Appendix. In brief, auction revenue is held in escrow until settlement. At settlement, the government can use proceeds to repay maturing debt. However, in the event of default, the proceeds are distributed across all bondholders, both old and new. The assumption that bondholders receive payments (if any) in proportion to the face value of their claims reflects the *pari passu* and acceleration clauses typically included in sovereign bond contracts. The important element carried over from the model of the previous section is that bonds purchased in that period's auction are at risk immediately, as existing bondholders can claim a fraction of the auction revenue in the event of within-period default.

For the quantitative model, we introduce risk-averse lenders. This allows us to separately target the average spread and the probability of default. In particular, we assume that financial markets are segmented and that only a subset of foreign agents participate in the sovereign debt market. This assumption allows us to introduce risk premia on sovereign bonds while treating the risk-free rate as parametric. For tractability, we assume that a set of lenders has access to the sovereign bond market for one period and then exits, to be replaced by a new set of identical lenders. The short horizon of the specialist lenders is for tractability, avoiding the need to solve an infinite horizon portfolio problem and carry another endogenous state variable.

Specifically, each period a unit measure of identical lenders enters the sovereign debt market. Let W denote the aggregate wealth of the agents that can participate in the current period's bond market. The entering "young" lenders allocate their wealth across sovereign bonds and a risk-free asset that yields $1 + r^*$. As noted previously, the risk-free rate is pinned down by the larger world financial market, and specialists in the sovereign bond market can freely borrow and lend at this rate. Let $\tilde{R}$ denote the realized one-period return on sovereign bonds. As in the one-period bond model, the return is determined in part by the government's default policies. However, with longer maturity debt, future realizations of (Y, ρ) as well as future debt-issuance decisions induce capital gains and losses on non-maturing debt. The expression for $\tilde{R}$ in terms of these factors is detailed in the Appendix.

Lenders have preferences over wealth when old, W_o, given by:

$$v(W_o) = \frac{W_o^{1-\gamma}}{1-\gamma}.$$

The young lenders' problem is to allocate a fraction μ of their wealth in sovereign bonds, and the remainder in risk-free bonds. Given the homogeneity of preferences, the optimal decision conditional on s and B' is defined by:

$$\mu^*(s, B') = \underset{\mu}{\operatorname{argmax}} \, \mathbb{E}\left[v\left((1-\mu)(1+r^*) + \mu\tilde{R} \right) \middle| s, B' \right]. \tag{22}$$

In equilibrium, the market for bonds must clear. In particular,

$$\mu^*(s, B')W = q(s, B')B'. \tag{23}$$

As B' increases, lenders devote more of their wealth to sovereign bonds, and therefore prices must fall to generate the appropriate risk premium to clear the market.

The definition of an equilibrium in the augmented model is the natural extension of the one given for the simple model of Section 2:

Definition 2 (Equilibrium)**.** *An* equilibrium *consists of a price schedule q, government policy functions $\mathcal{B}$ and $\mathcal{D}$, and a lender portfolio policy function μ^* such that: (i) $\mathcal{B}$ and $\mathcal{D}$ solve the government's problem, conditional on q and μ^*; (ii) μ^* solves the representative lender's problem* (22) *conditional on q and the government's policy functions; and (iii) market clearing: equation* (23) *holds for all (s, B') such that $s \in S$ and $B'/Y \leq \overline{B}$.*

The functions $\{q_{EG}, q_{DD}, V^R_{EG}, V^R_{CK}\}$ and the sets $\mathbb{C}$ and $\mathbb{B}$ defined in Section 2 can be redefined for the augmented model in a direct fashion. See the Appendix for formal definitions. The fact that the endowment is persistent implies that Y and g are relevant for forecasting future equilibrium behavior. Thus, we need to add them as arguments to the functions and include them as elements of the set $\mathbb{C}$ and the index of $\mathbb{B}$. For expositional simplicity, we simply include s as the additional argument when the endowment states are relevant arguments.

3.2 Calibration

To calibrate the model, we set a number of parameters prior to simulation and then select others to match simulated moments with their empirical counterparts.

The pre-set parameters are reported in Table 1. We estimate the endowment process using Mexican data, the details of which are in the Appendix. We assume the government has power utility with a coefficient of relative risk aversion of 2. We set the lenders' risk aversion coefficient to 2 as well. The risk premium will depend on the lenders' wealth relative to the size of the bond market. We calibrate W by matching moments in a procedure discussed later. We set the risk-free interest rate at 1 percent quarterly (hence 4 percent annually). The average maturity length is set to 8 quarters; that is, $\lambda = 1/8$, which implies a Macaulay duration of 6.4 quarters.[24] We set the re-entry probability after default to 0.125 quarterly; that is, the average duration of default is 2 years.[25]

[24]This is shorter than the average maturity (or duration) observed in many emerging markets. However, maturity length is not constant over time and tends to shorten when the probability of a crisis is high (Arellano and Ramanarayanan, 2012, Broner, Lorenzoni, and Schmukler, 2013). Moreover, much of a country's short-term debt is issued domestically (whether in dollars or local currency) and thus is not reflected in the average maturity of external debt. For simplicity, our model only has external debt of constant maturity, raising the question of how to accurately capture a world in which maturity varies over time and the amount due (and to whom) in any given quarter is not uniform. Given our focus on crises, we set the average maturity length to a value that is relatively short.

[25]This is in the range documented by Gelos, Sahay, and Sandleris (2011) for the 1990s but lower than Tomz and Wright's (2013) median of 6.5 years using a much longer sample. Again, this is not crucial given

As noted in the previous section, the sovereign is indifferent over a range of debt issuances when facing the desperate deal price schedule. In our benchmark model, we set the debt-issuance policy to be half the value of maturing debt:

$$\mathcal{B}(s) = \left(1 + \frac{\lambda}{2}\right) B,$$

when $\rho = DD$ and fundamentals are in the crisis zone. Later, we also explore an alternative in which the government repurchases non-maturing debt in a crisis.

In our benchmark model, we consider beliefs that fluctuate between $\rho = EG$ and $\rho = DD$ and put zero mass on $\rho = CK$. We will also report results for an alternative that shifts the mass from $\rho = DD$ to $\rho = CK$ and puts zero probability mass on $\rho = DD$. The former model is referred to as "benchmark" and the latter as "CK."

Table 1: Parameters I: Set Prior to Simulation

Parameter	Value	Source
Endowments:		
$\bar{g}$	0.0034	Mexico GDP Data 1980Q1–2015Q1
ρ_g	0.445	
σ_g	0.012	
σ_z	0.003	
Preferences:		
Sovereign CRRA (σ)	2	Standard
Creditor CRRA (γ)	2	Standard
Financial Markets:		
Quarterly Risk-Free Rate (r^*)	0.01	Standard
Reciprocal of Avg. Maturity (λ)	0.125	N/A
Default Re-entry Prob (ξ)	0.125	Gelos et al (2011)

Note: Pre-set parameters for calibrated model. CRRA refers to coefficient of relative risk aversion and GDP refers to gross domestic product.

We calibrate the remaining parameters by simulating the model and matching targeted empirical moments. In particular, the remaining parameters of the benchmark model are the

that we scale the value of default by choosing the parameter ϕ.

probability of transiting from $\rho = EG$ to $\rho = DD$, and vice versa; the government's discount factor; the wealth of the creditors, which we assume is a constant proportion of endowment $w = \frac{W}{Y}$; and the proportion of output lost during default. We assume that beliefs follow an *iid* process over time; that is, $\Pr(\rho' = DD|\rho = EG) = \Pr(\rho' = DD|\rho = DD)$. This leaves one transition probability and three parameters. We set these to match the average debt-to-income ratio, the average spread defined by (24), and the standard deviation of spreads, using Mexico as our empirical counterpart, as well as an average default rate of 2 percent per annum, which is in line with the estimates of Tomz and Wright (2013) using a broad sample of countries over a relatively long time period.

Specifically, we match the average external Mexican debt to annual GDP (both in U.S. dollars) for the period 2002Q1 through 2014Q3. The average over this period is 16.4 percent, which translates into a quarterly debt-to-income ratio of 65.6 percent. This measure of debt includes external debt by the government as well as banks. A longer time series exists for a narrower stock of debt issued by the federal government. This series suggests that debt levels were higher in the 1990s and have been falling in the 2000s and 2010s. Hence, our measure of 65 percent may be an understatement.

We also match the mean and standard deviation of the spread over U.S. bonds for Mexican debt. The average Emerging Market Bond Index (EMBI) spread for Mexico over the entire period is 3.4 percent, with a standard deviation of 2.5 percent. For the simulated model, we compute the spread implied by the equilibrium price. Specifically, we denote by $r(s, B')$ the implicit yield of a risk-free bond paying r^* each period and maturing with probability λ that is purchased at a price $q(s, B')$:[26]

$$r(s, B') = \frac{r^* + \lambda}{q(s, B')} - \lambda. \tag{24}$$

The implied quarterly spread is then $r(s, B') - r^*$, which we annualize before comparing with the EMBI data.[27]

While moment matching involves simultaneously matching four moments using four vary-

[26]That is, $q(s, B') = \sum_{k=1}^{\infty}(1 + r(s, B'))^{-k}(1 - \lambda)^{k-1}\left[r^* + \lambda\right]$.

[27]In the simulated model, the mean debt-to-income ratio and the spread are conditional on not being in the default state. More specifically, we compute the mean conditional on being out of the default state for at least 25 quarters. The reason we condition on being in good credit standing for a significant period of time is that the government exits default status with zero debt. Zero debt after default is not a realistic feature of the model, and hence we focus on the ergodic distribution conditional on having sufficient time to rebuild debt.

ing parameters, we can provide a heuristic guide regarding which moments are particularly important for determining which parameter based on how the model behaves when we have varied parameters. In particular, the debt-to-income ratio is sensitive to the choice of the punishment ϕ. Given a level of debt-to-income, the propensity to default is sensitive to the discount factor β. As we will discuss in detail later, the risk of a rollover crisis is important for generating the empirical volatility of spreads, which pins down the probability $\rho = DD$. Finally, given the risk of default and spread volatility, the average spread reflects an average risk premium that is sensitive to lenders' wealth.

These four targets, the model counterparts (under the column "Benchmark Model"), and the associated parameters are reported in Table 2. The model is simulated 1.5 million times, as default is a rare event. The model is able to hit all four targets precisely. To do so, the probability of a rollover crisis is 10 percent. As we shall see, this does not mean that a crisis occurs every ten quarters on average. In particular, a crisis requires that the debt is high enough and other fundamentals bad enough that a rollover crisis can be supported in equilibrium; that is, $s \in \mathbb{C}$. As debt is an endogenous state variable, the government has the ability to avoid a rollover crisis. This will be a key element of the discussion to follow. Given the vulnerability to default in general and a rollover crisis in particular, we need a fairly low discount factor (0.84 quarterly) to ensure the government accumulates the target debt levels. Upon default, the government loses 6.8 percent of its endowment. The final parameter, creditor wealth to GDP, is a factor of 3.75.

4 Discussion of Quantitative Results

With the calibrated model in hand, we can discuss how the introduction of desperate deals changes equilibrium outcomes. It is useful to contrast our model of crises with a model in which rollover crises end in immediate default, as in Cole and Kehoe (2000). In Table 2, we add the moments from the alternative model in the column labeled "CK." This model has the same parameters as the benchmark but with a bond issuance policy of zero during crises (that is, $B' = (1 - \lambda)B$). Recall that the government is indifferent to the amount issued (or defaulting) when faced with the rollover crisis price schedule, and thus zero issuance and default are also an equilibrium outcome of the model, as discussed in Section 2. In the "CK" equilibrium selection, conditional on a crisis the equilibrium price for issuances is zero, and the sovereign defaults with probability one at settlement.

Table 2: Parameters II: Simulated Method of Moments

Target Moment	Data	Benchmark Model	CK (No Deals)
Debt-to-Income (Quarterly)	65.6%	65.6%	63.9%
Mean Annualized Spread $r(s, B') - r^*$	3.4%	3.4%	3.5%
Quarterly Std Dev of Annualized Spread	2.5%	2.5%	0.1%
Default Frequency (Annually)	2.0%	2.0%	2.3%

Parameter	Value
Crisis Probability $\Pr(\rho = DD)$	10%
Discount Factor (β)	0.84
Default Cost (ϕ)	6.8%
Creditor Wealth Relative to Y ($w = \frac{W}{Y}$)	3.75

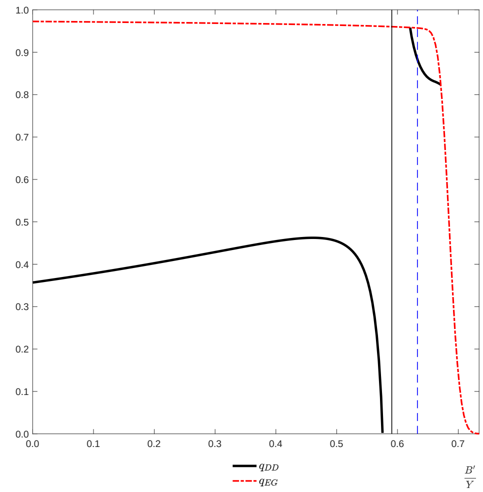
Note: The top panel reports the empirical moments and the model counterparts for our benchmark model and the alternative without desperate deals. The bottom panel shows the values of the four parameters calibrated from matching the benchmark moments in the top panel with their empirical counterparts.

4.1 Spread Volatility

Contrasting the moments of the two models reported in Table 2, the major difference is the volatility of spreads. In the benchmark model, we match the empirical standard deviation of 2.5 percent, while the CK model has a standard deviation of only 0.1 percent. While Table 2 does not recalibrate the CK model to target the volatility of spreads, we have searched over a large parameter space seeking values that generate spread volatility similar to that observed in the data while also matching the other empirical moments. This exercise demonstrated that a well-calibrated CK model does not generate volatile spreads despite having frequent rollover crises. Indeed, the difference in volatility reported in Table 2 is striking given that average spreads and default frequency are similar across the two models.

To understand this difference in spread volatility as well as how the model works more generally, we begin with the price schedule, depicted in Figure 3. The figure is the quantitative counterpart to Figure 2 Panel (b). The figure is drawn for a specific $s \in \mathbb{C}$, with B' on the horizontal axis and q on the vertical axis. The solid vertical line identifies the amount of non-maturing debt, $(1 - \lambda)B$, which indicates the B' associated with zero issuances today.

Figure 3: Equilibrium Price Schedule: Crisis



Note: Crisis bond price schedule, q_{DD} (solid), and non-crisis schedule, q_{EG} (dashed), as a function of B'/Y evaluated at $g = -0.0212$, $z = -0.0008$, and $B/Y = 0.652$. The solid vertical line is $(1-\lambda)B/Y$, and the dashed vertical line is $(1-\lambda/2)\,B/Y$.

To the left of the this line, the sovereign is repurchasing bonds. To the right, the sovereign is issuing bonds. The dashed vertical line is $B' = (1-\lambda/2)B$, which is our benchmark equilibrium issuance policy.

The nonlinear, downward sloping dashed line is q_{EG}. The solid line depicts q_{DD}, that is, the price that makes the government indifferent to repayment and default at settlement. Different from Figure 2 and the model of Section 2, with longer term bonds we have a region that represents the repurchasing of non-maturing bonds. In particular, to the left of the vertical line the government is repurchasing non-matured debt. We postpone discussion of the buyback region until Section 4.5.

As is familiar from the quantitative sovereign debt literature, the q_{EG} price schedule is highly nonlinear. This plays an important role in debt dynamics. The government, as a monopolist in its own debt, internalizes the impact of bond issuance on prices. The impact of reducing debt issuances at the margin is the slope of q_{EG} times the quantity issued. That is, any increase in price the government secures by reducing debt at the margin applies to all inframarginal debt issuances. In equilibrium, therefore, the government remains on the flatter portion of the price schedule, with impatience pushing debt issuances toward the start of the "bend" in the price schedule. In the Appendix, we report the ergodic distribution of debt-to-income, which is tightly clustered around this point.

This behavior generates stable prices during non-crisis periods. Any shift in q_{EG} due to shocks to the endowment are matched by small movements in debt issuances to prevent a sharp decline in prices. This generates stable spreads absent crises. In Figure A2 in the Appendix, we report the ergodic distribution of non-crisis spreads, which is tightly clustered around the targeted average spread of 3.4 percent.

Figure 3 sheds light on how the model generates sharp spikes in spreads during a crisis. The gap between q_{DD} and q_{EG} potentially represents a large increase in spreads (and a decline in prices). Differently than during non-crisis periods, the government has no incentive to counter this decline by reducing debt issuances. Recall that the government is indifferent across all B' along the q_{DD} price schedule. Therefore, it is willing to pay the high spreads without altering its level of debt. In the CK model, the government defaults during a crisis. Thus, we never observe in equilibrium the government issuing bonds at high (but finite) spreads.

Figure 4 considers spreads during crisis events in the benchmark model. Panel (a) plots the distribution conditional on a rollover crisis. We separate the events that result in repayment from those in which the randomization comes up as default. As required by equilibrium, the crises with a higher spread are more likely to generate a subsequent default. Moreover, the distribution covers a range of spreads that encompass magnitudes observed during events such as Mexico 1994 and Greece 2012. Panel (b) depicts the distribution of the randomization probabilities during crises. The mean of this distribution is 0.20; that is, conditional on a rollover crisis, the sovereign defaults 20 percent of the time.

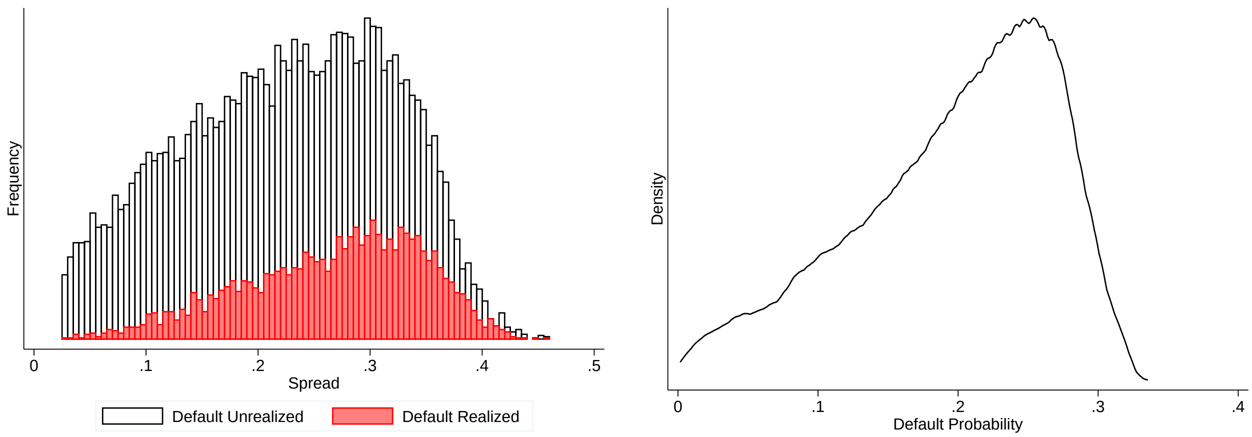
A crisis in the benchmark model generates a positive price for non-maturing bonds. This contrasts with the CK model, in which a crisis generates prices of zero. The government captures this through better ex ante prices in the benchmark model (see Appendix Figure A1). In this sense, the bond market with desperate deals is more efficient as it economizes on the deadweight costs of default.

There are other ways to improve the efficiency of the bond market in models of this type. Looking at equilibria absent rollover crises is one approach, but this removes the high volatility at the same time. For example, Aguiar and Gopinath (2006) explore default in a model with a similar endowment process but no self-fulfilling crises and find a very stable

Figure 4: Crisis Spreads

(a) Annualized Spreads

(b) Randomization Probabilities $\mathcal{D}(s, B')$



Note: Panel (a) depicts the histogram of spreads in the benchmark model conditional on a rollover crisis. The unfilled bars denote episodes that did not result in a default, while the shaded bars depict the distribution conditional on a subsequent within-period default at settlement. Panel (b) depicts the simulated distribution of the government's mixed-strategy probability of default, $\mathcal{D}$, during rollover crises.

spread.

Contrasting with this is the approach taken by Arellano (2008), who introduces nonlinear default costs. Specifically, defaults occurring during low endowment realizations are not associated with an output loss, reducing the deadweight costs of default. Moreover, this makes non-contingent bonds better insurance, as default is partially "forgiven" when output is low (which is the typical default scenario). The sovereign is thus willing to borrow at fairly high spreads, generating empirical volatility of spreads. However, this requires a volatile output process (Arellano calibrates to Argentina). Aguiar, Chatterjee, Cole, and Stangebye (2016) calibrate an Arellano-type nonlinear default cost using Mexican data and find much of the volatility in spreads disappears. Moreover, as default costs are sensitive to output, this increases the importance of income in spread fluctuations, contrasting with the modest role of fundamentals in explaining spreads (see Aguiar, Chatterjee, Cole, and Stangebye, 2016).

Efficiency could also be enhanced by renegotiating under the threat of default. If bargaining is efficient, this eliminates the deadweight costs of default. Yue (2010) explores such a model using an endowment calibration similar to Aguiar and Gopinath (2006) and generates volatile spreads. Yue's renegotiations happen immediately, and thus the sovereign is never punished for default on the equilibrium path. However, Benjamin and Wright (2008) document that in practice it takes many years for defaults to be resolved. Thus introducing bargaining during default may not spare the sovereign the costs of default. However, in practice defaults are resolved through partial repayment, which compensates creditors. Our desperate deals scenario shares this aspect, although the partial payments occur through competitive secondary market trades rather than default resolution. This feature is compelling, as many crisis episodes are not associated with default.

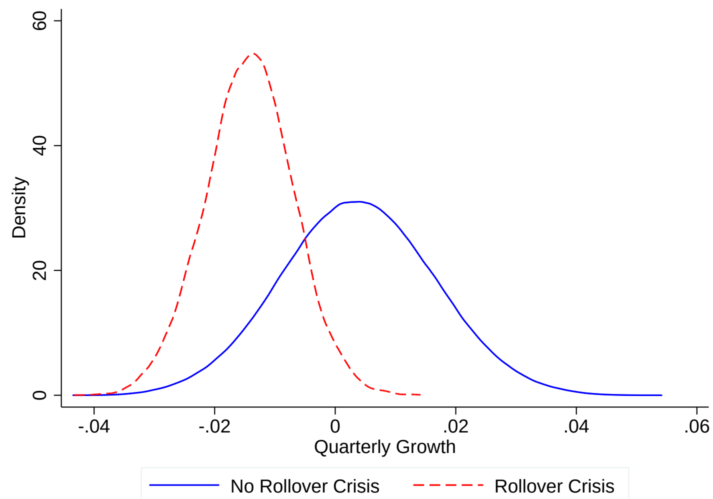
This discussion of making bond markets more efficient highlights a key aspect of the models. While adding a sunspot and self-fulfilling crises conceivably could generate a volatile bond market, the sovereign always has the option of avoiding the drama by not borrowing. This is what happens in our CK alternative model. While the government defaults in response to a combination of high debt, low output, and a run on its bonds, the sovereign never ventures into the region of the state space in which spreads are particularly volatile. Impatience is not enough, as the nonlinear prices essentially ration government debt. Viewed in this way, the volatility associated with real-life markets must be relatively benign to support creditors willing to lend and governments willing to borrow. In this study, the mitigating factor is the ability to issue bonds at fire-sale prices in crisis episodes.

The higher $\rho = EG$ prices in the benchmark model relative to the CK alternative model imply that the government's welfare is higher in the benchmark. Although in the midst of the crisis the government is indifferent to default or issuance at desperate deal prices, the legacy lenders are not. As prices are competitive, the government captures this difference when it issues bonds in non-crisis times. We have computed welfare gains at various points in the state space. The gains are positive for the benchmark model over the CK model, although the magnitudes are very small (on the order of one-quarter of one percent additional consumption in perpetuity). Such small welfare gains are not unusual in models of uninsurable business cycle risk. What is perhaps more striking is that the dramatic increase in spread volatility represents a net positive gain in welfare.

4.2 Frequency of Crises

The preceding discussion highlights that the government controls its vulnerability to a rollover crisis through its debt dynamics. In the benchmark, the sovereign is in a crisis

Figure 5: Growth and Rollover Crises



quarter (i.e., $\rho = DD$ and $s \in \mathbb{C}$) only 1.3 percent of the time. This is despite the fact that the exogenous probability of $\rho = DD$ is calibrated to be 10 percent. The rarity of rollover crises therefore reflects the fact that the government avoids the crisis zone. In particular, the sovereign is in the crisis zone only 13.0 percent of the non-excluded quarters. Exposing itself to a rollover crisis is costly ex ante due to the equilibrium price schedule. A rollover crisis therefore requires both a high level of debt and a relatively negative growth shock.

To see this, Figure 5 depicts the distribution of growth conditional on a rollover crisis; that is, conditional on $s \in \mathbb{C}$ and $\rho = DD$. The crisis distribution is shifted to the left, indicating that self-fulfilling crises in our model involve a combination of a shift in beliefs and a negative output realization. The mean growth conditional on a crisis is -1.4 percent, compared with 0.4 percent for non-crisis quarters. Moreover, 97 percent of rollover crises are associated with negative growth.

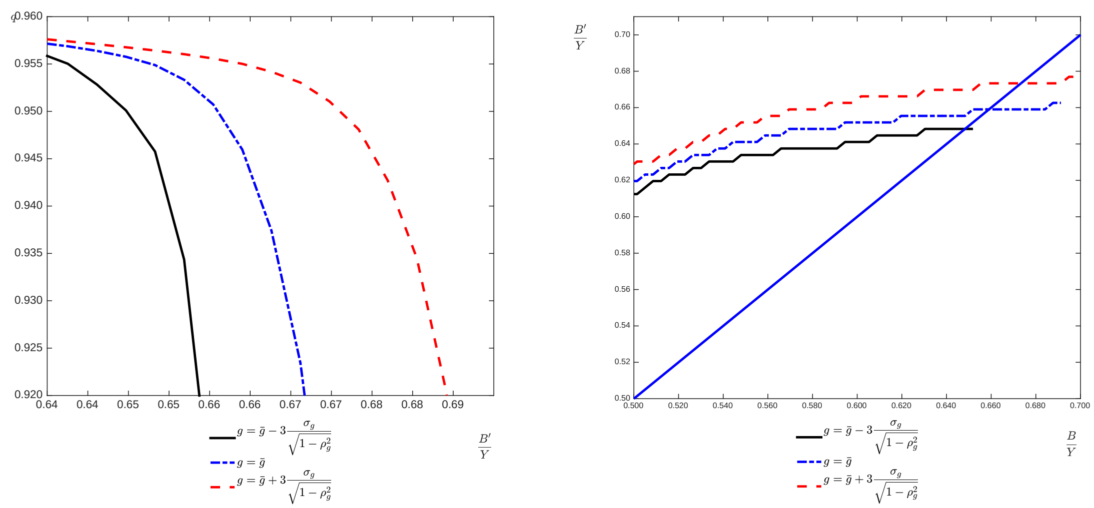
4.3 Non-crisis Dynamics

As previously noted, the interesting spread dynamics in our quantitative model are generated by desperate deals. To obtain a better sense of dynamics during non-crisis periods, Figure 6 Panel (a) depicts q_{EG} for three values of g, namely, the mean endowment realization and plus or minus three standard deviations of the unconditional distribution of g. The figure zooms into the relevant debt levels surrounding the ergodic distribution's mean level of debt-to-income. A high realization of g bodes well for future endowment, and thus future bond prices. Thus, the price schedule shifts up and out for high realizations of g. This reflects that, in an incomplete markets environment, default is relatively attractive during low output realizations (see Proposition 1).

Figure 6: Non-crisis ($\rho = EG$) Behavior: Response to g

(a) Price Schedule

(b) Debt-Issuance Policy



Note: Panel (a) depicts the benchmark $q_{EG}(g, B')$ as a function of B' for different values of g. The top (dashed) schedule is the highest g in our discretization, specifically, three standard deviations of the unconditional g distribution above the mean. The lowest (solid line) is three standard deviations below the mean, and the middle schedule corresponds to the mean g. The schedule is evaluated at $z = 0$, $\rho = EG$, and $B/Y = 0.656$, the ergodic mean. Panel (b) depicts the benchmark model's bond issuance policy function $\mathcal{B}$, normalized by Y, as a function of B/Y for various realizations of g, evaluated at $z = 0$ and $\rho = EG$.

The fact that the price schedule shifts up and out in a boom generates pro-cyclical debt issuances in equilibrium. From Figure 6 Panel (a), we see that the nonlinear portion shifts to the right in response to high g realizations, as default is less likely going forward. This encourages additional borrowing, given the government's impatience. Panel (b) of the figure depicts the policy function for the same three values of g as Panel (a). The high-g policy lies above the mean-g policy, which in turn lies above the low-g policy. Moreover, near the 45-degree line, the policy functions are very flat. This indicates that the government leverages up and down very quickly in this region in response to shocks to g. Finally, the nonlinear price schedule results in the government not borrowing enough to raise spreads substantially, as high spreads (low q) are associated with regions in which the price is highly elastic. This discourages borrowing and lowers the volatility of spreads. This generates a fairly low volatility of spreads absent desperate deals.

4.4 Default Postmortems

We now turn to default episodes in the benchmark model. Table 3 reports simulated moments conditional on default in the current quarter. For the benchmark model, we can see that spreads spike during a default episode and growth is relatively low. Moreover, we see that two-thirds of defaults coincide with a rollover crisis. In the CK alternative, this fraction is nearly 94 percent.

To obtain a better sense of the nature of rollover crisis defaults, in Figure 7 we perform an event study analysis in the benchmark model's simulation. In particular, we normalize $t = 0$ as the quarter of default and then explore mean behavior in the preceding five quarters. The solid line depicts default events that occur with a rollover crisis ($\rho = DD$ and $s \in \mathbb{C}$), which we label "self-fulfilling" defaults. The dashed line depicts defaults that occur outside a rollover crisis ($\rho = EG$ or $s \notin \mathbb{C}$). We label the latter defaults as "fundamental" defaults, as the default would occur that period regardless of the realization of ρ; of course, the fact that future crises could occur play a role in the default decision today, as these events are embedded in the value of repayment.

Panel (a) of Figure 7 plots the mean growth leading up to a default event. For fundamental defaults, we see a boom-bust pattern. Two quarters prior to default tends to be associated with high growth, which is then followed by a mediocre growth realization the period before default. The default itself coincides with a large negative growth realization. This pattern is the classic fundamental driven default; the high growth induces the government to borrow, and then if a large negative growth shock occurs while the economy is so highly leveraged, the sovereign defaults. For self-fulfilling defaults, ex ante growth is not particularly elevated, and default itself coincides with a mildly negative growth realization. The self-fulfilling defaults are thus associated with relatively minor recessions.

Panel (b) depicts the trajectory of debt before default. We see the increase in debt typical before a fundamental default, which reflects the boom period just discussed. In particular, debt-to-income ratios are relatively high once lower growth rates are realized. A relatively

Table 3: Defaults

	Benchmark	CK
Conditional Mean $r - r$	26.8	NA
Conditional Mean $\Delta(r - r^*)$	23.6	NA
Conditional Mean Δy	-1.8	-1.8
Conditional Fraction $\Delta y < 0$	95.7%	98.8%
Conditional Mean $\frac{B}{Y}$	67.0%	65.0%
Conditional Fraction $\rho = DD$	62.8%	93.9%

high debt level is also necessary to sustain a self-fulfilling crisis in equilibrium, although the average level at the time of default is less than that associated with a fundamental default.

Finally, Panel (c) depicts spreads. For fundamental defaults, there is hardly any increase in spreads prior to the default, and spreads are undefined in the period of default. The fundamental defaults combine the shift up in the price schedule during the boom period and the sovereign's best response of adding debt in response, keeping spreads largely unchanged. The default then occurs because an unusually large negative growth rate is realized after a relatively large positive growth rate; as low growth is relatively unlikely to follow high growth, spreads do not anticipate the fundamental default (other than the unconditional risk in all quarters). This contrasts with self-fulfilling crises, in which spreads spike in the quarter of the default as the government issues debt at very low prices. Creditors understand the risk of imminent default and charge accordingly.

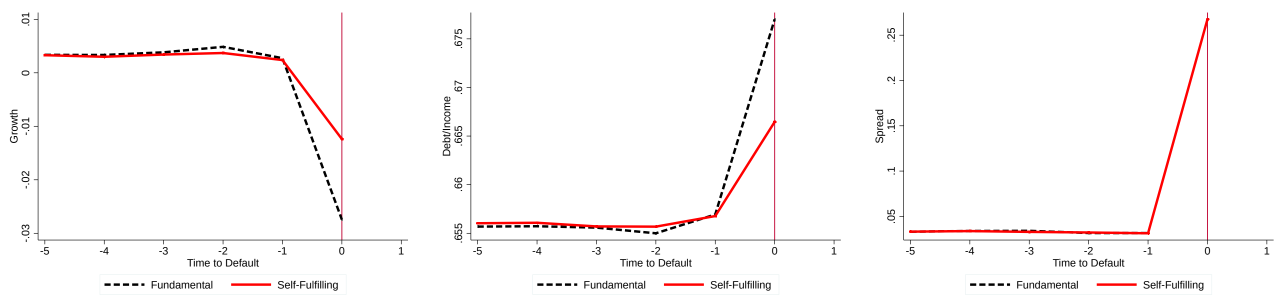
4.5 Debt Buybacks, Revisited

As noted in the introduction, Portugal repurchased bonds during the crisis. Greece also repurchased outstanding debt in 2012. A well-known critique by Bulow and Rogoff (1988) argued that such buybacks are welfare-reducing for the sovereign. We now explore buybacks in two variants on our benchmark model. Both alternatives have the sovereign *buying back* debt when default is imminent.

We first discuss prices during debt buybacks, which were not possible in the one-period bond model of Section 2. Consider the domain to the left of $B' = (1 - \lambda)B$ in Figure

Figure 7: Default Event Studies

(a) Growth (b) Debt/Income (c) Spreads



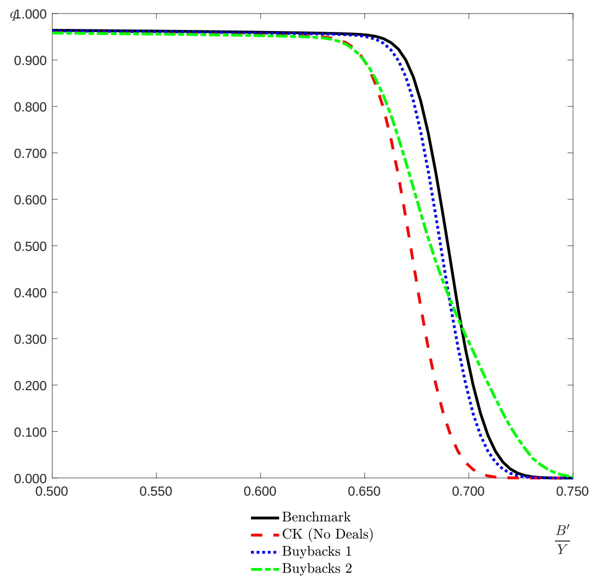
3. The cost of repurchases is reduced consumption, and the benefit is reduced debt going forward. At $B' = (1 - \lambda)B$, the government prefers to default as $s \in \mathbb{C}$. Therefore, for small repurchases, the sovereign needs a very low price to make it indifferent. For arbitrarily small buybacks, the price would need to be negative, which is therefore not sustainable in equilibrium. The only price sustainable in equilibrium in the neighborhood to the left of $B' = (1-\lambda)B$ is zero (regardless of beliefs), and the government defaults at settlement with probability one. Of course, if $\rho = EG$, the government would never choose to issue B' in this region.

For large enough buybacks, there is a price low enough that the government is indifferent between default and repurchasing its debt at fire-sale prices. This price is q_{DD}. Note that on this domain ($0 < B' < (1 - \lambda)B$), q_{EG} cannot be supported in equilibrium given our timing convention. Repurchases at a price higher than q_{DD}, by definition, generate default at settlement. Therefore, q_{EG} is not credible absent commitment. On this domain, q_{DD} is also the equilibrium price schedule for $\rho = EG$, although, again, the sovereign would never choose to issue on this domain. Note, as well, for buybacks we have a non-monotonic q_{DD}.

We now consider buybacks under two scenarios. In the first alternative, we assume the government repurchases non-maturing debt during a desperate deal crisis. Like the benchmark model's issuances, these repurchases take place at the q_{DD} price schedule that leaves the government indifferent. We posit that the government repurchases 10 percent of its non-maturing debt.[28]

[28]Specifically, consider a period in which $\rho = DD$ and $s \in \mathbb{C}$. The government then sets $B' = 0.9(1-\lambda)B$ as long as $0.9(1-\lambda)B \in \mathbb{B}(s)$. If $0.9(1-\lambda)B \notin \mathbb{B}(s)$, the government defaults.

Figure 8: Equilibrium Price Schedule: Alternative Models



Note: This figure depicts $q_{EG}(s, B')$ as a function of B'/Y, with $g = \bar{g}$, $z = 0$, $\rho = EG$ and $B/Y = 0.656$. As discussed in the text, the Eaton-Gersovitz price schedule assumes no default in the current period and $\rho = EG$. The solid line is the benchmark schedule, the dashed line is the CK model, the dotted line indicates the alternative with repurchases during rollover crises, and the dash-dotted line indicates repurchases during fundamental defaults.

The second alternative concerns behavior during a fundamental default. Recall that a fundamental default is defined as a default that occurs even if the government faces the Eaton-Gersovitz price schedule, q_{EG}; that is, s such that $\mathbb{B}(s) = \emptyset$. These defaults occur when debt is relatively high and a low endowment is realized. Such defaults are the focus of the quantitative sovereign debt literature. For a fundamental default, there is no level of issuances that is sustainable in equilibrium at positive prices, as the government strictly prefers to default rather than issue at the best possible price schedule q_{EG}. For this exercise, we explore what happens when the government repurchases debt at fire-sale prices. In particular, we extend $\rho = EG$ prices into the buyback region using the same mixed-strategy approach as is the case for desperate deals, having the government repurchase debt at q_{DD}. Like in the first buyback alternative, we assume the sovereign repurchases 10 percent of the non-maturing bonds during fundamental default episodes.

Figure 8 shows the government's price schedules under various scenarios. As we can see from the figure, debt issuances or buybacks through desperate deals during rollover crises shift up the equilibrium price schedule fairly uniformly relative to the no-deals scenario (over the range in which there is some risk of default). However, desperate buybacks during rollover crises do so by less than debt issuances.

In contrast, desperate buybacks during *fundamental* crises have very little impact on the price schedule for the low-risk portions of the schedule (where q is only moderately below the risk-free level). However, it shifts it out quite a bit for high-risk portions. As a result, it actually twists the price schedule relative to the benchmark desperate deals scenario.

The reason that desperate buybacks during rollover crises do not impact the price schedule as much as desperate issuances can be understood from Figure 3. Buybacks do not raise the price as much as issuances and hence result in smaller gains to the legacy lenders.

The reason for twisting of the schedule with buybacks during fundamental crises is two-fold: (i) these crises occur only in fairly extreme portions of the debt-output space, and (ii) the fact that they reduce the incentive to deleverage in these events means that they encourage the country to borrow slightly more during adverse regions of the state space. The first factor is salient in extreme portions of the state space where the debt level is high, and this pushes out the price schedule. However, in more moderate portions of the state space, the second factor and the threat of future dilution are key, which explains why the buybacks-during-fundamental-crises schedule is actually slightly below the no-deals schedule right where the two schedules bend.

We have computed the welfare gains or losses across these various alternatives, comparing the respective value functions evaluated at zero debt and mean growth. The welfare ordering of desperate buybacks versus issuances during rollover crises tracks the ordering of the price schedules. The desperate issuance benchmark has the best price schedule of these three and has the largest ex ante welfare level, the environment with desperate buybacks during rollover crises has the next best schedule and welfare, and the no-deals CK model has the lowest prices and welfare.

The fact that the price schedule for desperate buybacks during fundamental crises is twisted relative to the benchmark makes welfare gains or losses less transparent. However, because the extreme portion of the price schedule where prices are higher under this scenario is rarely reached, welfare ends up lower than either of the other desperate deals scenarios. The surprise here is that welfare in the model with desperate buybacks during fundamental crises is lower than the CK model, albeit the difference is small in magnitude.

Our results on implications and efficacy of debt buybacks have interesting implications for a long-standing debate on their benefits. Buybacks of debt emerged as a potential policy tool during the Latin American crisis of the late 1980s, with Brazil, Chile, and Mexico undertaking large billion-dollar-plus repurchases. In a classic article, Bulow and Rogoff (1988) pointed out that the very low prices on sovereign debt before a buyback was proposed did not reflect the price at which the buyback would actually occur. They argued that the debt would be bought back at the equilibrium price after the buyback, and thus the lenders would receive most or all of the gains from the buyback.

In our model, the government is indifferent in the midst of a crisis between issuances, default, or buybacks. However, dynamically speaking, the prospect of future transfers to creditors under these scenarios does generate equilibrium effects (fundamentally, the transfers occur because the government avoids the deadweight losses from default with some probability). In particular, they have the potential to be welfare-improving, where this improvement comes via a more favorable price schedule for government debt. However, a better price schedule also exacerbates the time-consistency problem of the government: With a better price schedule, investors rationally expect the government to borrow more in the future, which works to push down the price schedule via the dilution effect. What we see in our quantitive results is exactly the sort of mixed results that this calculus suggests, with buybacks during rollover crises being ex ante welfare-improving and those during fundamental crises welfare-reducing.

5 Conclusion

In this paper, we extended the nature of self-fulfilling crises to include bond issuances and buybacks at fire-sale prices during a rollover crisis. This was motivated by the fact that crises in practice are often associated with positive issuances (and occasional repurchases) at abnormally high spreads. The addition of these desperate deals changes the nature of equilibrium spreads in the quantitative model, primarily increasing the volatility of spreads. Absent such deals, the volatility of spreads is an order of magnitude too small, despite the presence of self-fulfilling crises and defaults as frequent as in the benchmark model. In the no-deals model, the sovereign either deleverages or defaults in response to adverse credit conditions. With deals, the government is willing to endure the high and volatile spreads associated with crises as it is indifferent to repayment and default in such situations.

However, creditors strictly prefer the positive prices of such deals, conditional on a crisis, and thus are willing to purchase bonds ex ante at more favorable prices for the issuer. This latter effect induces more borrowing on the part of the government as well as higher ex ante welfare, despite the volatility of spreads. The nature of desperate deals in the model and the associated equilibrium behavior provide a lens to interpret the interest rate spikes and debt dynamics observed in recent sovereign debt crises.

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Appendix

A1 Proofs

Proof of Proposition 1

Throughout, consider (Y_0, B_0) and (Y_1, B_1), both in $\mathbb{Y} \times (-\infty, \overline{B}]$, with $Y_0 \leq Y_1$ and $B_0 \geq B_1$, and with at least one of the inequalities strict.

We begin with some properties of V^R_{EG} and V^R_{CK}.

Lemma A1. *For any $B' \leq \overline{B}$, we have (i) $V^R_{EG}(Y_0, B_0, B') \leq V^R_{EG}(Y_1, B_1, B')$; and (ii) $V^R_{CK}(Y_0, B_0, B') \leq V^R_{CK}(Y_1, B_1, B')$. Moreover, the inequality in part (i) is strict if $q_{EG}(B')B' > B_1 - Y_1$, and the inequality in part (ii) is strict if either (a) $B' > 0$ and $B_1 < Y_1$ or (b) $B' \in (R(B_1 - Y_1), 0]$.*

Proof. Part (i): Recall that $q_{EG}(B')$ is independent of (Y, B). Hence, for any $B' \leq \overline{B}$, the definition of V^R_{EG} immediately implies the weak monotonicity result. If $q_{EG}(B')B' > B_1 - Y_1$, then consumption upon repayment is interior and the strict inequality follows immediately from the definition of V^R_{EG}. Part (ii): Recall that $q_{CK}(B')$ is also independent of (Y, B). For $B' \leq 0$, $V^R_{CK} = V^R_{EG}$, and hence Part (i) implies the weak monotonicity in general, the strict monotonicity if $q_{EG}(B')B' = R^{-1}B' > B_1 - Y_1$, where the first equality follows from the definition of q_{EG} for $B' \leq 0$. For $B' > 0$, V^R_{CK} depends on (Y, B) only through $u(Y - B)$. This is weakly increasing, and the inequality is strict as long as $Y_1 > B_1$. □

Another useful property that is a straightforward extension of Arellano (2008) is:

Lemma A2. *For any $B' \leq \overline{B}$, we have: (i) If $V^R_{EG}(Y_1, B_1, B') < V^D(Y_1)$, then $V^R_{EG}(Y_0, B_0, B') < V^D(Y_0)$, and (ii) If $V^R_{CK}(Y_1, B_1, B') \leq V^D(Y_1)$, then $V^R_{CK}(Y_0, B_0, B') \leq V^D(Y_0)$.*

Proof. We first prove that, if default is optimal, then the sovereign is making a net payment at settlement. Specifically, for $B' \leq \overline{B}$, if $u(Y - x) + \beta\mathbb{E}[V(s')|B' \in s'] \leq V^D(Y)$, then $x \geq 0$, and if $u(Y - x) + \beta\mathbb{E}[V(s')|B' \in s'] < V^D(Y)$, then $x > 0$. To see this, recall that $V^D(Y) = u(Y) + \beta\mathbb{E}u(Y')/(1-\beta) = u(Y) + \beta\mathbb{E}V^D(Y')$. Moreover, $\mathbb{E}[V(s')|B' \in s'] \geq \mathbb{E}V^D(Y')$ as it is always feasible to default next period. Thus, $u(Y - x) + \beta\mathbb{E}[V(s')|B' \in s'] \leq V^D(Y)$ implies $u(Y - x) \leq u(Y)$. If the first inequality is strict, then so is the second. As u is strictly increasing, we have the result that $x \geq (>)0$.

To apply this result to the lemma, suppose $V^R_{EG}(Y_1, B_1, B') < V^D(Y_1)$. Rearranging we have:

$$u\left(Y_1 - B_1 + q_{EG}(B')B'\right) - u(Y_1) < \beta \mathbb{E}u(Y')/(1-\beta) - \mathbb{E}[V(s')|B' \in s']. \tag{25}$$

Let $x = B_1 - q_{EG}(B')B'$, and the above implies $x > 0$. Taking the case of $Y_0 < Y_1$ and $\Delta B \equiv B_0 - B_1 \geq 0$, we have:

$$\begin{aligned} & u\left(Y_0 - B_0 + q_{EG}(B')B'\right) - u(Y_0) \\ & = u(Y_0 - x - \Delta B) - u(Y_0) \\ & \leq u(Y_0 - x) - u(Y_0) \\ & < u(Y_1 - x) - u(Y_1), \end{aligned}$$

where the last inequality follows from the strict concavity of u and the fact that $Y_0 < Y_1$ and $x > 0$. If $Y_0 = Y_1$ and $B_0 > B_1$, then $\Delta B > 0$ and the second-to-last inequality is strict. This plus (25) implies:

$$u\left(Y_0 - B_0 + q_{EG}(B')B'\right) - u(Y_0) < \beta \mathbb{E}u(Y')/(1-\beta) - \mathbb{E}[V(s')|B' \in s']. \tag{26}$$

Hence, $V^R_{EG}(Y_0, B_0, B') < V^D(Y_0)$. This proves Part (i) of the lemma.

Part (ii) is proved in an identical fashion with $x = B_1$ if $B' > 0$. If $B' \leq 0$, then $V^R_{CK} = V^R_{EG}$ and the results of Part (i) apply. □

Lemmas A1 and A2 have the following corollary, which follows directly from the definition of $\mathbb{B}$:

Corollary A3. *Given $Y_0 \leq Y_1$ and $B_0 \geq B_1$, we have $\mathbb{B}(Y_0, B_0) \subset \mathbb{B}(Y_1, B_1)$.*

We are now ready to prove Proposition 1. We begin with the case $\rho = EG$:

Proof. Recall that by definition we have $V^R = V_{EG}$ if $\rho = EG$ and $B' \leq 0$, where we omit the arguments for simplicity. Hence, for $B' \leq 0$, the monotonicity of V^R in (Y, B) follows from Lemma A1 Part (i). Now suppose $B' > 0$ and $B' \in \mathbb{B}(Y_0, B_0) \subset \mathbb{B}(Y_1, B_1)$, where the subset follows from Corollary A3. Then by definition $V^R = V_{EG}$ and monotonicity follows from Lemma A1 Part (i). If $B' \notin \mathbb{B}(Y_0, B_0)$ and $B_0 > Y_0$, then repayment is not feasible at (Y_0, B_0, B') and $V^R([Y_0, EG, B_0], B') = -\infty$. Weak monotonicity follows immediately. If repayment is feasible for $B' \notin \mathbb{B}(Y_0, B_0)$, then $V^R([Y_0, EG, B_0], B') = u(Y_0 - B_0) + \beta E[V(s')|B' \in s'] < u(Y_1 - B_1) + \beta E[V(s')|B' \in s'] \leq V^R_{EG}(Y_1, B_1, B')$. The latter two expressions define $V^R([Y_1, EG, B_1], B')$ depending on whether B' is not or is a member of $\mathbb{B}(Y_1, B_1)$, respectively. This completes the proof of Part (i) of the proposition for $\rho = EG$.

Part (ii) follows immediately from Corollary A3. Part (iii) follows immediately from Part (ii) and the lenders' break-even condition.

□

Proof of Proposition 1 for $\rho = CK$:

Proof. To see the monotonicity of V^R when $\rho = CK$, we appeal to the definition from Section 2.2 under Cole-Kehoe beliefs. Suppose $V^R_{CK}(Y_1, B_1, B') \leq V^D(Y)$. Then Lemma A2 implies that $V^R_{CK}(Y_0, B_0, B') \leq V^D(Y)$ as well. Thus $V^R = V^R_{CK}$ at both points. Lemma A1 Part (ii) then implies monotonicity. If $V^R_{CK}(Y_0, B_0, B') > V^D(Y)$, then Lemma A1 Part (ii) implies $V^R_{CK}(Y_1, B_1, B') > V^D(Y)$ as well. In this case, $V^R = V^R_{EG}$ for both points. Lemma A1 Part (i) implies monotonicity in this case. The remaining case is $V^R_{CK}(Y_1, B_1, B') > V^D(Y) \geq V^R_{CK}(Y_0, B_0, B')$. In this case, $V^R([Y_1, CK, B_1], B') = V^R_{EG}(Y_1, B_1, B') > V^D \geq V^R_{CK}(Y_0, B_0, B') = V^R([Y_0, CK, B_0], B')$. Hence, V^R is monotone in (Y, B) for $\rho = CK$.

To see the weak monotonicity of $\mathcal{D}$, we need to show that, if $\mathcal{D}([Y_1, CK, B_1], B') = 1$, then $\mathcal{D}([Y_0, CK, B_0], B') = 1$. The former condition implies that $V^R_{CK}(Y_1, B_1, B') \leq V^D(Y_1)$. Lemma A2 then implies that $V^R_{CK}(Y_0, B_0, B') \leq V^D(Y_0)$ as well. By definition of V^R for $\rho = CK$, this implies $\mathcal{D}([Y_0, CK, B_0], B') = 1$. The monotonicity of the price schedule then follows from the lenders' break-even condition.

□

Proof of Proposition 2

Proof. Part (iii) of the proposition follows immediately from the construction of prices under alternative beliefs laid out in Section 2.2. Part (i) follows immediately from this fact and the implications for consumption conditional on repayment. Part (ii) follows immediately from Part (i) and the tie-breaking assumptions under the different belief regimes when indifferent to default. □

Proof of Proposition 3

Proof. Consider $Y_0 \leq Y_1$ and $B_0 \geq B_1$, with one inequality strict. Suppose $(Y_0, B_0) \in \mathbb{C}$ and $B' \in \mathbb{B}(Y_0, B_0)$. By definition of the desperate deals value function, we have $V^R([Y_0, DD, B_0], B') = V^D(Y_0)$. From Corollary A3, we also have $B' \in \mathbb{B}(Y_1, B_1)$. If $(Y_1, B_1) \in \mathbb{C}$, we have $V^R([Y_1, DD, B_1], B') = V^D(Y_1)$; otherwise, $V^R_{CK}(Y_1 B_1], B') > V^D(Y_1)$. In either case,

$$V^R([Y_1, DD, B_1], B') \geq V^R([Y_0, DD, B_0], B'),$$

with strict inequality if $Y_1 > Y_0$. □

A2 Details of Quantitative Extension and Calibration

In this Appendix, we flesh out the details of the quantitative model and the calibration. In particular, we provide details on (i) the calibration of the endowment process, (ii) a fuller discussion of the default value, (iii) a detailed description of settlement in the quantitative model, and (iv) a detailed description of the lenders' problem.

Endowment

We assume the growth rate process is governed by

$$g_{t+1} = (1 - \rho_g)\bar{g} + \rho_g g_t + \varepsilon_{t+1},$$

where $\varepsilon \sim N(0, \sigma_g^2)$. The transitory component of output z_t is assumed to be *iid*, orthogonal to ε_t and to have mean zero and variance σ_z^2. The implied growth rate of log output is

$$y_{t+1} - y_t = g_{t+1} + z_{t+1} - z_t + \varepsilon_{t+1}.$$

We estimate this model using quarterly Mexican constant-price GDP for the period 1980Q1 through 2015Q1. The estimated parameter vector is reported in Table 1. The estimates suggest that the stochastic trend is the primary driver of GDP fluctuations for Mexico, consistent with Aguiar and Gopinath (2007).

With these parameters in hand, we discretize the process for g using Tauchen's method using 50 grid points spanning $\pm 3\sigma_g/\sqrt{1 - \rho_g^2}$. The *iid* z shock is drawn from a continuous Normal distribution truncated at $\pm 3\sigma_z$. When taking expectations, we numerically integrate over z's continuous distribution by evaluating at 11 grid points. For computational reasons, we approximate $e^z \approx 1 + z$ and exploit the linearity of z in the budget set. Given the small variance of z, this is not a bad approximation.

Given the non-stationarity of income, our No-Ponzi upper bound on end-of-period debt is now expressed as a ratio of current income: $\frac{B_{t+1}}{Y_t} \leq \bar{B}, \forall t$. Our state vector s is augmented to include g as well as Y in order to correctly forecast future income.

To compute the model, we exploit the homogeneity of the government and lender preferences and normalize by G_t to render the problem stationary. We place normalized debt on a grid of 350 points distributed uniformly over the domain $\frac{B}{G} \in [0, 1.25]$. The boundaries are not binding in equilibrium given the government's impatience and the incentive to default at high debt levels (and hence $q \to 0$ as $B' \to \infty$).

Default

As noted in the text, the value of default in the quantitive model is given by:

$$V^D(s) \equiv u(Y^D) + \beta \mathbb{E} V^E(s'),$$

where $Y^D = Ge^{\bar{z}}$, and V^E denotes the continuation value while in the default (exclusion) state:

$$V^E(s) = u((1-\phi)Y) + \beta(1-\xi)\mathbb{E}\left[V^D(s')|s\right] + \beta\xi\mathbb{E}\left[V(s')|s, B' = 0\right].$$

Note that the cost of default is linear in output, as in Aguiar and Gopinath (2006). In contrast, Arellano (2008) introduced a non-linear cost of default, which made default disproportionately more costly in good endowment states and "forgiven" – at least in terms of output costs – in low endowment states. The Arellano specification amplifies the impact of endowment fluctuations in the decision to default while also making default a better insurance option in low endowment states. This helps the model generate additional volatility of spreads and frequency of default but does so by making endowment risk more important rather than less. The empirical facts outlined in work like Tomz and Wright (2007) suggest that this pulls the model in the wrong direction relative to the data.[29]

[29] Moreover, Aguiar, Chatterjee, Cole, and Stangebye (2016) show the Arellano non-linear cost requires a very volatile endowment process to generate volatile spreads. In particular, the endowment process calibrated to Mexico used here is not sufficiently volatile. This indicates that the typical calibration to Argentina's more volatile output process is not representative of other emerging markets.

The government's budget set conditional on repayment becomes:

$$C \leq Y + q(s, B') \left[B' - (1-\lambda)B\right] - (r^* + \lambda)B. \tag{27}$$

The government's problem, the associated value functions $\{V(s), V^R(s, B'), V^D(s)\}$, and policy functions $\{\mathcal{B}(s, B'), \mathcal{D}(s, B')\}$ are the immediate extensions of those from Section 2 to the augmented environment. We assume $u(C) = C^{1-\sigma}/(1-\sigma)$, with $\sigma = 2$. The government's discount factor is set through the moment matching procedure described in the text.

Auction Proceeds

In the simple model of Section 2, we assumed that if the government defaults at settlement all revenues raised at the prior auction are lost. We now treat this scenario in a more realistic manner.

It is useful to define $x(s, B')$ as the equilibrium amount raised at auction per endowment, if positive, given an amount auctioned B' and a price schedule $q(s, B')$:

$$x(s, B') \equiv \max\left\{q(s, B')(B' - (1-\lambda)B), 0\right\}. \tag{28}$$

The proceeds from auction are held in escrow until the government makes a repayment decision. The government can use these funds to pay its outstanding liabilities but cannot draw on them for consumption unless all such payments are made. In particular, given outstanding debt B, the government is contractually obligated to pay λB in principal and r^*B in interest payments. These payments are financed through current endowment as well as the revenue raised by the auction of new debt. If the government makes its contracted payments, it consumes according to (27) and continues on to the next period with the new debt state implied by B'.

If the government defaults, the amount in settlement $x(s, B')$ is disbursed to all claimants on the basis of the face value of their claims. In particular, there are holders of current liabilities, totaling $(r^* + \lambda)B$, as well as holders of future liabilities, with a face value B'. In the period of default, each unit of such claims receives a payout R^D:

$$R^D(s, B') = \frac{x(s, B')}{B' + (r^* + \lambda)B}. \tag{29}$$

If $B' < (1-\lambda)B$, then the government has repurchased bonds on net. In this case, we assume that the original holders of the repurchased bonds receive their payment at the time of the auction and that there are no funds left in escrow at the time of default. In this case, $R^D(s, B') = 0$.

Risk-Averse Lenders

Recall that each period a unit measure of identical lenders enters the sovereign debt market with wealth W and decides to invest a fraction μ in sovereign bonds. The realized return on sovereign bonds is $\tilde{R}$. To see how $\tilde{R}$ is determined in equilibrium, recall the timing of Figure 1. "Old" lenders enter a period with B units of debt. A fraction λ of the representative portfolio matures, which is to be paid at settlement. We also assume that all coupon payments (on both maturing and non-maturing bonds) are to be paid at settlement. The remaining (ex-coupon) non-matured bonds, $(1 - \lambda)B$, are sold to "young" lenders at the time of auction. In particular, new lenders purchase from the legacy lenders the stock of non-maturing bonds plus any new bonds the government auctions at the same time.[30] At the end of the auction, new/young lenders hold all non-maturing bonds.

With this timing, we can compute the return on bonds purchased in the current period by young lenders in state s when the government's end-of-auction stock of debt is B'. In particular, consider a young lender that purchases a unit-measure portfolio today, paying $q(s, B')$ at auction. If the government defaults in the current period, the young lender receives $R^D(s, B')$, where R^D is defined by (29). As the lender is still young, it can invest this amount in risk-free bonds. If the government does not default this period, the young lender holds the sovereign bonds into the next period.

Next period, the lender is now "old." It sells $1-\lambda$ at auction and receives $q(s', B'')$, where B'' reflects next period's debt-issuance decisions. In equilibrium, this will be $B'' = \mathcal{B}(s')$. The lender receives $q(s', B'')(1-\lambda)$ for these bonds regardless of the government's subsequent default decision. In addition, if the government does not default, the lender receives $r^* + \lambda$ at settlement. Otherwise, it receives $R^D(s', B'')(r^* + \lambda)$ at settlement.

Let δ and δ' denote indicator functions that take the value of one if the government defaults in the current or next period, respectively, and zero otherwise. The preceding implies that the realized return on a sovereign bond, denoted $\tilde{R}$, purchased at price $q(s, B')$

[30]Our auction assumption is that payments due within the period, that is, coupons and matured bonds, are not sold to new lenders. This is done for tractability, as currently due payments will sell at a different price than new bonds.

is given by:

$$\tilde{R} = \frac{1}{q(s, B')} \Big[(1-\delta) q(s', B'')(1-\lambda) \\ + \delta R^D(s, B')(1+r^*) \\ + (1-\delta)\delta' R^D(s', B'')(r^* + \lambda) \\ + (1-\delta)(1-\delta')(r^* + \lambda) \Big]. \tag{30}$$

The first term on the right is the sale of non-maturing bonds at next period's auction, which occurs only if there is no default this period; the second term is the payment at settlement in case of immediate default, which is then invested at the risk-free rate; the third term is the payment at settlement next period in case of default, scaled by the claims on coupons and matured principal; and the final term is the payment of coupon and principal absent default in either period. In the first line, note that, while next period's price incorporates the government's default policy that period, the sale takes place before next period's default decision is made. Hence, it is not multiplied by $1 - \delta'$.

In forming expectations over $\tilde{R}$, the lender uses the equilibrium policy functions of the government:

$$\delta = 1 \text{ with probability } \mathcal{D}(s, B'); \\ \delta' = 1 \text{ with probability } \mathcal{D}(s', B'') \text{ in state } s'; \text{ and} \\ B'' = \mathcal{B}(s').$$

The first-order condition for the lender's problem is the usual condition:

$$\mathbb{E} M(\tilde{R} - (1 + r^*)) = 0,$$

where $M = v'((1-\mu^*)(1+r^*) + \mu^* \tilde{R})$ is the stochastic discount factor. If lenders are risk-neutral, then $\mathbb{E}\tilde{R} = 1 + r^*$. When $\gamma > 0$, we will have a positive risk premium. In particular, $q(s, B')$ will be such that lenders receive the appropriate compensation for the probability of default plus any additional risk premium required to bear such risk. Note that the stochastic discount factor depends on μ^*. Note as well that the government has an incentive to adjust the level of debt to manipulate the risk premium via the term $\tilde{R}$. This is the risk premium complement to the government's incentive to manipulate the risk-neutral price by adjusting the probability of default at the margin.

A3 Additional Quantitative Results

In this appendix, we report additional results from the quantitative model. In Figure A1 Panel (a) we plot $q_{EG}(s, B')$ as a function of B'/Y. Recall from Section 2 that q_{EG} is the price schedule assuming the government does not default in the current period, given the continuation equilibrium behavior. Given that the growth shock is persistent, g is relevant for forecasting future states, and hence q_{EG} is also a function of g. We evaluate q_{EG} at the mean of g.

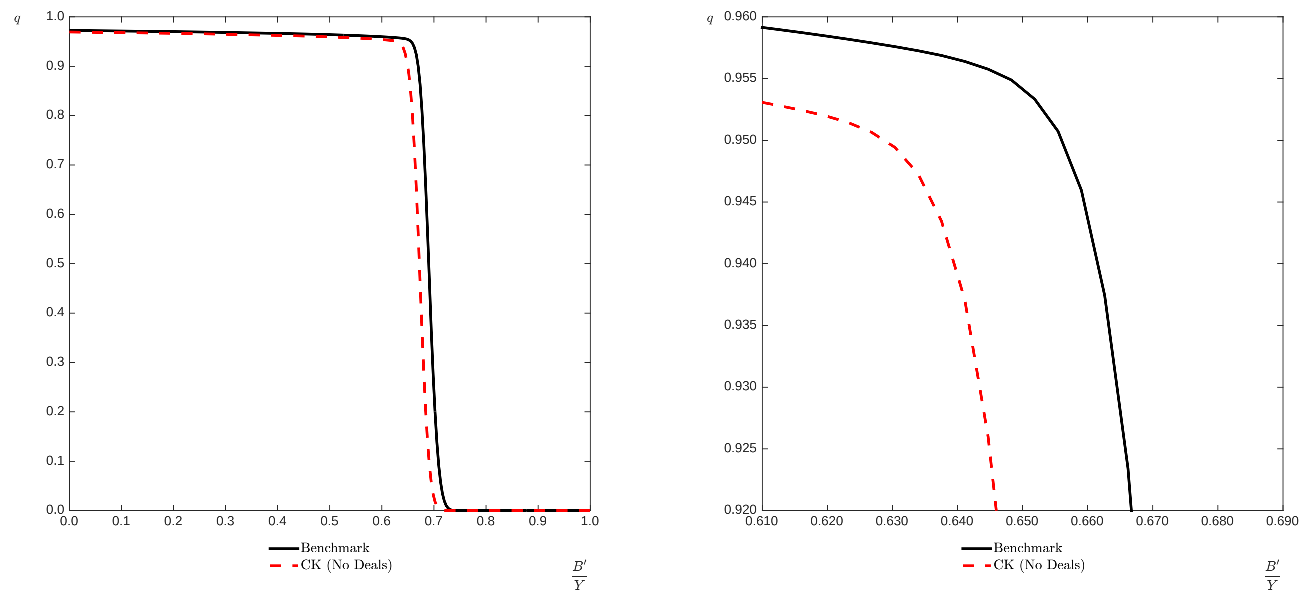
As is usual in these models, the price schedule is highly nonlinear. The relevant region is in the neighborhood of the mean debt-to-income level of 65.6 percent. Figure A2 depicts the ergodic distribution of debt-to-income in our simulated model, conditional on at least 25 quarters having passed since the most recent default. The figure indicates a fairly tight distribution around the calibrated mean, a point we discuss below. In Panel (b) of Figure A1, we plot the price schedule over the tighter range relevant for the equilibrium debt distribution.

Figure A3 depicts debt policy functions in non-crisis periods evaluated at the mean growth rate.

The government's policy function, the equilibrium price schedule, and the stochastic processes for endowment combine to generate a spread distribution. The ergodic distribution is depicted in Panel (a) of Figure A4. Most of the distribution is concentrated around the mean spread of 3.4 percent, with a long right tail during rollover crises. Panel (b) zooms in on the non-crisis part of the distribution by plotting the distribution conditional on $q(s, B') = q_{EG}(s, B')$, that is, no crisis. Absent a crisis, there is a fairly tight distribution of spreads. In Panel (b) we also plot the spread distribution of the CK alternative. Absent deals, the figure depicts the full distribution of spreads absent default, as crisis periods always generate defaults and a price of zero. The CK distribution is similar to the conditional distribution of the benchmark model, indicating the importance of crisis deals in generating the volatility of the spread in the benchmark model.

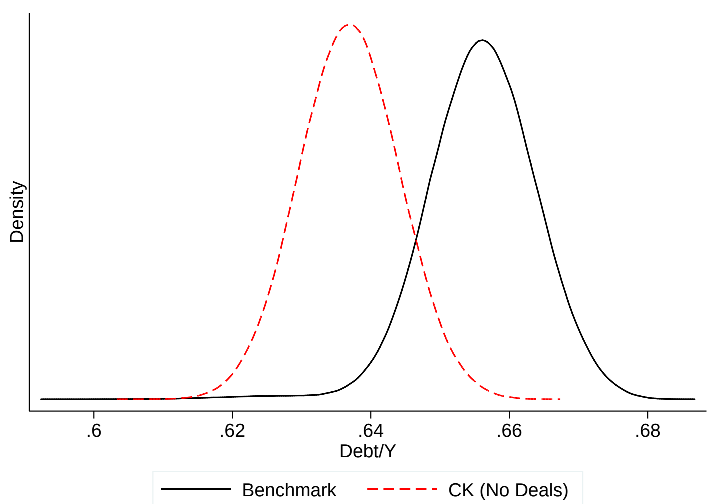
Figure A1: Equilibrium Price Schedule: No Crisis

(a) Entire Debt Domain (b) Relevant Debt Domain



Note: This figure depicts $q_{EG}(s, B')$ as a function of B'/Y, with $g = \bar{g}$, $z = 0$, $\rho = EG$, and $B/Y = 0.656$. As discussed in the text, the Eaton-Gersovitz price schedule assumes no default in the current period and $\rho = EG$. The solid line is the benchmark schedule and the dashed line is the CK alternative. Panel (a) depicts the entire debt domain, while Panel (b) zooms into the domain that is relevant in the ergodic distribution.

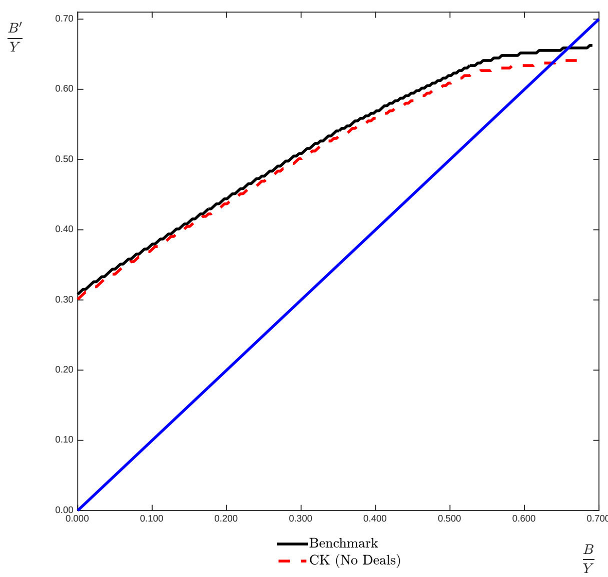
Figure A2: Ergodic Distribution of Debt



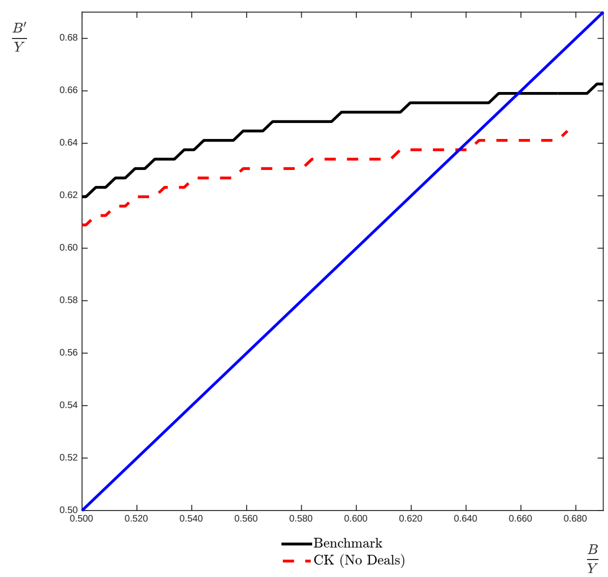
Note: This figure depicts the kernel density of debt-to-income for the benchmark model simulation (solid line) and the CK alternative. The distributions are conditional on no default within the last 25 quarters.

Figure A3: Debt-Issuance Policy Functions

(a) Full Domain



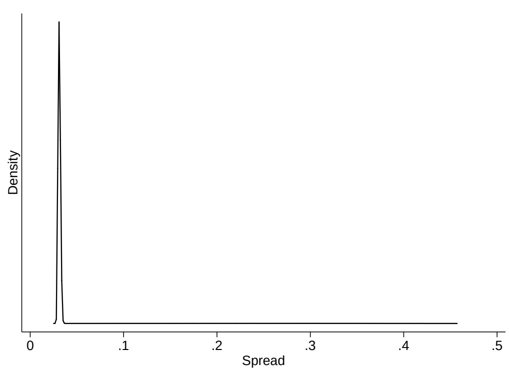
(b) Relevant Domain



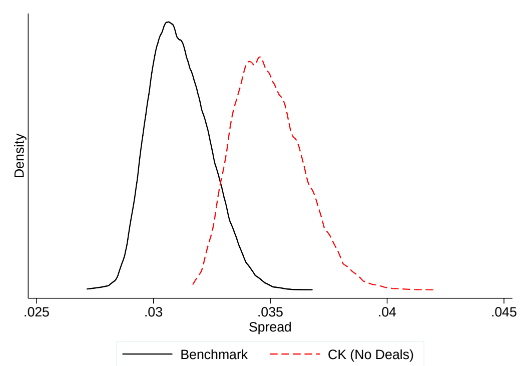
Note: This figure depicts the bond-issuance policy function $\mathcal{B}$, normalized by Y, as a function of B/Y. The solid line is the benchmark policy, and the dashed line is the CK alternative. The schedules are evaluated at the mean values of g and z and for $\rho = EG$. Panel (a) depicts the policy function over the entire debt domain, while Panel (b) focuses on the part of the domain relevant for the ergodic distribution.

Figure A4: Ergodic Distribution of Annualized Spreads

(a) Unconditional



(b) Conditional on No Crisis



Note: This figure depicts the simulated distribution of spreads. Panel (a) depicts the distribution of spreads in the benchmark model including rollover crises. Panel (b) depicts the distribution of spreads conditional on no rollover crisis for the benchmark model (solid line) and the CK alternative (dashed line).