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STEPPING ON A RAKE:
REPLICATION AND DIAGNOSIS

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Stepping on a Rake: Replication and Diagnosis
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ABSTRACT

The fiscal theory of the price level can describe monetary policy. Governments can set interest rate targets and thereby affect inflation, with no change in fiscal surpluses. The same basic mechanism describes interest rate targets, forward guidance, open market operations, and quantitative easing. It does not require any monetary, pricing, or other frictions. In the presence of long-term debt, higher interest rates lead to temporarily lower inflation, a challenging sign. I derive and replicate the results of the Sims (2011) “stepping on a rake” model, which first produced this negative sign, and produces realistic impulse-response functions. I show that Sims' result is robust to many model features, but essentially requires long-term debt.

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1. Introduction

The fiscal theory of the price level does not just describe inflation and deflation driven by fiscal events. The fiscal theory of the price level also offers a cogent and unified description of *monetary* policy: how governments can set interest rate targets; and how governments can affect inflation and the real economy, by interest rate targets, by forward guidance about interest rate targets, and by varying the quantity and maturity structure of government debt in open-market and quantitative-easing operations, all without changes in fiscal surpluses.

In the presence of long-term debt, an interest rate rise can produce a temporarily lower inflation rate in this fiscal theory. Higher nominal interest rates mean lower nominal bond prices, which lower the nominal market value of outstanding government debt. If expected future surpluses do not change, as I assume of "monetary policy," and at the existing price level, the market value of government debt is then less than its real value to consumers and investors. People then try to buy government debt, and to buy less goods and services, i.e. aggregate demand declines. The price level declines, until the real value of nominal debt again matches the real present value of primary surpluses. Since the inflation-lowering effect of higher rates depends on long-term bond prices, and thus centrally on expected future short term rates, this mechanism is also a model of forward guidance.

Similarly, in the presence of long-term debt, buying back some long term debt, with no change in surpluses, means less debt coming due at future dates, backed by unchanged surpluses. This reduces the future price level. The extra money from today's bond purchase drives up today's price level, an inflationary stimulus. The higher current and lower future price level imply a lower long-term interest rate. This is a version of quantitative easing.

Eventually, however, the Fisher effect wins out, and persistently higher nominal interest rates mean higher inflation. Sims (2011) calls this pattern of responses to a persistent interest rate rise "stepping on a rake." He offers it as an explanation of the 1970s: Attempts to lower inflation by raising interest rates, without fiscal reform, temporarily lowered inflation, but then inflation came back even more strongly.

This fiscal theory of monetary policy requires no frictions at all – no money demand, special liquid asset, sticky prices, financial frictions, liquidity constraints, irrational expectations, and so forth. Yes, one may add frictions and elaborations to obtain realistic dynamics. In particular, pricing frictions deliver output effects of monetary policy. But we can now start with a simple supply and demand benchmark, which delivers basic signs and intuitions, and then add frictions to match dynamics, as we do elsewhere in economics, rather than *require* frictions to even get going, i.e. to determine the price level, to describe the central bank's ability to control interest rates and inflation, or to produce the basic sign of monetary policy.

Sims (2011) is a ready-made example. Sims specifies an interest rate target which rises with inflation and output; fiscal surpluses that respond to output, with larger deficits in recessions; a standard forward-looking sticky-price Phillips curve; a habit-like preference for smooth consumption; and long-term government debt. His policy parameters are in the "passive-money / active-fiscal" region of the Leeper (1991) categorization: Interest rates respond less than one for one to inflation, and fiscal surpluses do not respond to validate revisions in the value of government debt coming from arbitrary changes in the price level.

Sims shows that a rise in interest rates produces a transitory decline in inflation (see the top left panel of figure 2 below). If the interest rate rise is persistent, inflation eventually rises. Sims also generates smooth impulse response functions with signs and magnitudes typical of the active-money New-Keynesian literature.

1.1. Context

One may be surprised that I focus on the temporary inflation decline rather than the eventual long-term inflation rise in response to persistently higher interest rates. Common intuition suggests that higher interest rates lower inflation, so the short-run negative response seems easy, and the long run positive response seems a novelty. Indeed, Sims' analysis of the 1970s was novel. Conventional wisdom says that inflation grew out of control in the 1970s because rates were not raised high enough or for long enough, not because they were raised too much or for too long.

However, it has since become clear that this theoretical presumption should be reversed. Models with forward-looking people, including standard new-Keynesian models, produce higher long-run inflation in response to persistently higher interest rates. They have great trouble to produce even a temporarily lower inflation. The intuition is simple: In these models, the Fisher relation, that nominal interest rate equals real rate plus expected inflation, is a *stable* steady state. Real rates settle down eventually, so inflation is attracted to the nominal rate.

In fact, other than by this fiscal-theory mechanism, we do not have a simple, modern (rational agents, market clearing) economic model that robustly produces even a short-run negative effect of interest rates on inflation. (The qualifiers simple, modern, economic, and robust are important to this statement. See Cochrane (2017a) for extensive discussion.) Thus, the ability to produce a negative inflation response fills a gaping hole for monetary theory in general, and not just for the fiscal theory.

(Some confusion on this point comes down to the difference between interest rates and monetary policy shocks. New-Keynesian models can have a negative response of inflation to monetary policy shocks, but a positive response to interest rates, since interest rates also respond negatively to the shock. In policy rule such as $i_t = \phi_\pi \pi_t + v_t^i$, with $\phi_\pi > 1$, when both i an π rise by the same amount, v^i declines.)

The standard intuition that higher interest rates lead to lower inflation comes from adaptive expectations models. In these models, the Fisher equation is an *unstable* steady state, so a persistent rise in the nominal rate sends inflation spiraling off negatively. Real rates rises by more than the nominal rate. Adaptive expectations thinking remains strong in policy circles, so the short run negative sign seems natural and the long run positive sign seems puzzling in that context. But current economic theory in forward-looking models goes the opposite way.

That a negative sign is a gaping hole, and Sims' success in filling it, is also not obvious from reading either Sims or the standard active-money / passive-fiscal new-Keynesian literature. Superficially, Sims' model and impulse-response functions look similar to those of standard medium-scale New-Keynesian models. For example, Smets and Wouters (2003) generate inflation that declines and interest rates that rise after a monetary policy shock. After 5 months, inflation turns around and is higher. Interest rates and inflation also go uniformly in the same direction in response to an inflation objective shock. (See their figure 11, p. 1159 and figure 12, p. 1160.) Rotemberg and Woodford (1997) produce a negative sign. Christiano, Eichenbaum and Trabandt (2016) add a search and matching model, among other ingredients, to a new-Keynesian model and obtain a nice movement of inflation opposite to an interest rate rise. (See their figure 1.) These models, like Sims', also produce output declines ("c" in figure 2).

Medium-scale new-Keynesian models thus seem already to provide the desired negative sign, and the subsequent stepping on a rake rise, though the latter is frequently regarded as a bug not a feature. It seems that Sims just says that there is a fiscal-theoretic alternative; that one can produce roughly similar impulse-response functions from an active-fiscal / passive-money regime in an otherwise fairly standard new-Keynesian model. Uniting both papers, the fiscal theory of the price level is ready to compete directly with standard active-money new-Keynesian models as a framework for understanding the macroeconomy.

That is already a large contribution, that was Sims' other point in writing the model, and it is important if one accepts the theoretical difficulties of the standard new-Keynesian approach (e.g. Cochrane (2011a)) and one wishes for an alternative. But it does not show gaping holes in the standard approach, nor a particular empirical advantage of the fiscal framework.

However, the negative effect of interest rates on inflation in these standard new-Keynesian models is fragile. It is sensitive to model complexities, such as the search and matching model in Christiano, Eichenbaum and Trabandt (2016). It relies on transitory interest rate movements. Interest rate changes all die out within 6 months in the above papers. In standard new-Keynesian models, long-lasting interest rate rises lead to immediately higher inflation. Thus their negative sign predictions fail the above qualifiers "simple" and "robust."

The inflation decline from the fiscal stepping on a rake mechanism, by contrast, is *stronger* for permanent interest rate rises, as they have larger effects on long-term bond prices. It is robust to the monetary policy shock process, and to all of the other model elaborations. We can see the basic effect in a completely frictionless model, as explored in the first section of this paper.

The contribution of this paper, then, is the architectural adage that "less is more." You don't see this robustness, or the basic economic story of the inflation decline, in Sims' full model. (Sims also does not explore the main topic of the first half of this paper, how the government targets nominal interest rates, or how the fiscal theory describes monetary policy in general.) Stripping away the other ingredients – procyclical fiscal policy, policy rule reactions to output and inflation, the preference for smooth consumption, even price stickiness – the basic negative sign remains. Long-term debt and an unexpected shock are essential – the negative sign disappears without these. But we didn't know that either.

I also show that the model has a smooth frictionless limit, unlike many new-Keynesian models (see Cochrane (2017b)). The hump-shaped responses, with no price-level jumps, in Sims' model smoothly approach the price-level jumps of the frictionless model. Thus, the frictionless model with price-level jumps does provide useful guidance to how a model without jumps behaves.

Sims' paper is therefore really the second step in a natural research program. It shows that one can add common new-Keynesian elaborations to simple frictionless models, preserve the negative effect of interest rates on inflation, and achieve the same sort of reasonable impulse response functions as the rest of the new-Keynesian literature.

Sims also does not state the model, he does not derive the equilibrium conditions, and he does not explain how to compute impulse-response functions. I fill that gap, and confirm Sims' results. Sims' paper is methodologically as well as historically important, so showing how to solve it is useful.

1.2. Limits and Literature

The stepping on a rake mechanism does not revive all classic monetary doctrines, beliefs, and mechanisms surrounding a negative effect of interest rates on inflation. In particular, it only gives a temporary inflation decline. It does not produce the view that steady high rates will slowly grind down inflation, a common view of the 1980s. Rational expectations new-Keynesian models share this feature.

Inflation declines when the rate rise is announced, not when it happens. A fully anticipated rate rise has no effect. The inflation decline is *stronger* when prices are *less* sticky, counter to traditional intuition. The aggregate demand reduction leading to lower inflation comes from a wealth effect of long-term bonds, not from either intertemporal substitution or static IS shifts induced by a higher contemporaneous interest rate. It can occur with no change in the contemporaneous rate. Thus, it has nothing to do with the classic story that high nominal rates mean high real rates, which reduce aggregate demand and via a Phillips curve reduce inflation. For these reasons, the stepping on a rake mechanism also does not justify conventional policy conclusions based on adaptive expectations ISLM thinking.

Finally, the response to "monetary" policy – a change in interest rates – depends crucially on the associated fiscal policy – the maturity structure of outstanding debt, and how people expect the treasury to adjust surpluses in reaction to economic events and monetary policy actions. More long-term debt implies a larger effect of interest rates on inflation, an important and potentially testable implication.

This paper is about theory: *Can* a simple, rational, economic model describe monetary policy, and can such a model produce a negative response of inflation to interest rates in the short or long run? Neither sign may true in the data. The evidence for a negative effect of monetary policy on inflation is weak, and the evidence on long run signs even weaker. But it's important to know that monetary policy *can* produce this hallowed view in a simple rational model and how. Then, the sign in the data is a matter of calibration, not a deep test of whole classes of theories. Similarly, perhaps people are irrational. Perhaps super-rational central banks exploit irrational expectations, via a mechanism whose sign has no rational economic foundation. Perhaps, thereby, the sign is negative in the long and short run. I search for rational expectations models not for some rigid insistence that only they can describe the world, but to find out if they even exist and what they can do.

This paper is somewhat integrative. Thinking about monetary policy within the fiscal theory of the price level occurs scattered about in many sources, including my own, Cochrane (1998, 2005, 2011b, 2014, 2017a), as well as Leeper and Leith (2016), (see especially figure 6, with the stepping on a rake pattern) Leeper and Walker (2013), Leeper and Zhou (2013), Leeper (2016), Jacobson, Leeper and Preston (2017), other Sims papers, and those of other authors. The main novelty here is to integrate interest rate targets with long-term debt, to integrate interest rate targets and quantity operations, and to see how long-term debt creates a negative sign. And, it is perhaps a novelty relative to many readers' perceptions, to point out that the fiscal theory does *not* consign central banks to irrelevance, that inflation is not just a fiscal phenomenon, and that the fiscal theory does provide a cogent model of *monetary* policy.

In particular, stepping on a rake works by the same mechanism as the analysis of long-term debt in Cochrane (2001). Section 5 of that paper shows how long-term debt sales can lower inflation today, raise interest rates, and raise future inflation, quantitative tightening or easing. Stepping on a rake is the mirror image, in which higher interest rates lower inflation today, raise future inflation, and result in greater debt sales. Expressing the same policy in terms of interest rate targets makes contact with the empirical literature and the standard monetary policy literature. The connection to quantities, however, shows that the same mechanism lies behind interest rate targets, open market operations, forward guidance, and quantitative easing, unifying monetary policy in a single framework.

2. Monetary policy in a frictionless fiscal theory

This section sets out how monetary policy operates in a simple frictionless fiscal theory environment. I focus on long-term debt and the stepping on a rake dynamics, which are more novel.

2.1. A frictionless rake

A rise in interest rates first causes inflation to fall, and then to rise, in a frictionless fiscal theory model with long-term debt.

Use a constant real discount factor

$$\beta = 1/(1+r)$$

where r denotes the real rate of interest. Then the nominal interest rate i_t and inflation π_t follow the Fisher relationship,

$$\frac{1}{1+i_t} = \frac{1}{1+r} E_t \left(\frac{P_t}{P_{t+1}} \right) \tag{1}$$

$$i_t \approx r + E_t \pi_{t+1} \tag{2}$$

where P_t denotes the price level. A rise in the nominal interest rate implies an immediate rise in expected inflation. But the price level can still jump down when the interest rate rise is announced. (Throughout, I do not verbally distinguish between a rise in expected inflation and a decline in $E(P_t/P_{t+1})$.)

At the beginning of period t, the government has outstanding $B_{t-1}^{(t+j)}$ discount bonds of maturity j, each of which pays \$1 at time $t+j$. Then, the government debt valuation equation, which states that the real value of nominal government debt equals the real present value of primary surpluses, generalizes from the familiar version with one-period debt

$$\frac{B_{t-1}^{(t)}}{P_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}, \tag{3}$$

to

$$\frac{\sum_{j=0}^{\infty} Q_t^{(t+j)} B_{t-1}^{(t+j)}}{P_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}. \tag{4}$$

Here, s_{t+j} denotes the real primary surplus, and $Q_t^{(t+j)}$ denotes the time t nominal price of a j period discount bond. The maturing bond has price $Q_t^{(t)} = 1$. For $j > 0$, the bond price is, in this risk neutral constant real rate world,

$$Q_t^{(t+j)} = \beta^j E_t \left(\frac{P_t}{P_{t+j}} \right). \tag{5}$$

Now, take innovations $(E_t - E_{t-1})$ of (4). Define "monetary policy" as a change in current and expected future interest rates, and hence bond prices, that involves no change in fiscal surpluses, so $(E_t - E_{t-1})\, s_{t+j} = 0$. (I generalize this definition considerably below.) We have

$$(E_t - E_{t-1}) \frac{\sum_{j=0}^{\infty} Q_t^{(t+j)} B_{t-1}^{(t+j)}}{P_t} = (E_t - E_{t-1}) \sum_{j=0}^{\infty} \beta^j s_{t+j} = 0. \tag{6}$$

Debt $B_{t-1}^{(t+j)}$ is predetermined. The real present value of surpluses does not change by assumption. If an interest rate rise lowers bond prices $Q_t^{(t+j)}$, then the price level P_t must also fall. *The price level P_t must jump by the same proportional amount as the change in nominal market value of government debt.*

If the price level does not change, then the real value of government debt to investors is greater than its real market value. People try to buy more government debt, and thus less goods and services. This lack of "aggregate demand" pushes the price level down. The deflationary force is the same as that which occurs if the real present value of primary surpluses $\{s_{t+j}\}$ increases. It is a "wealth effect" of government debt.

The size of this short-term inflationary or disinflationary effect of monetary policy depends exactly on how much the nominal market value of the debt changes. It is larger for greater bond-price changes, and larger when more long-term debt is outstanding. Both restrictions may be useful econometrically, in understanding episodes, and in practice.

From the Fisher equation (2), however, higher interest rates mean higher expected future inflation. So, after the one-period price level drop, inflation rises – the stepping on a rake effect.

In the case of one-period debt, $B_{t-1}^{(t+j)} = 0$, $j > 0$, (6) reads

$$(E_t - E_{t-1}) \frac{B_{t-1}^{(t)}}{P_t} = (E_t - E_{t-1}) \sum_{j=0}^{\infty} \beta^j s_{t+j} = 0. \tag{7}$$

In this case, the price P_t does not change unless surpluses change. Long-term debt is crucial to the temporary inflation decline.

2.2. Example

A geometric maturity structure $B_{t-1}^{(t+j)} = \theta^j B_{t-1}$ is analytically convenient. A perpetuity is $\theta = 1$, and one-period debt is $\theta = 0$. Suppose the interest rate $i_{t+j} = i$ is expected to last forever, and suppose surpluses are constant s. The bond price is then $Q_t^{(t+j)} = 1/(1+i)^j$. The valuation equation (4) becomes

$$\frac{\sum_{j=0}^{\infty} Q_0^{(j)} \theta^j B_{-1}}{P_0} = \sum_{j=0}^{\infty} \frac{\theta^j}{(1+i)^j} \frac{B_{-1}}{P_0} = \frac{1+i}{1+i-\theta} \frac{B_{-1}}{P_0} = \frac{1+r}{r} s. \tag{8}$$

Start at a steady state $B_{-1} = B$, $P_{-1} = P$, $i_{-1} = r$. In this steady state we have

$$\frac{1+r}{1+r-\theta} \frac{B}{P} = \frac{1+r}{r} s. \tag{9}$$

Therefore, we can express (8) as

$$P_0 = \frac{(1+i)}{(1+r)} \frac{(1+r-\theta)}{(1+i-\theta)} P. \tag{10}$$

The price level path for $t \geq 0$ then displays greater inflation,

$$P_t = \left(\frac{1+i}{1+r} \right)^t P_0. \tag{11}$$

These formulas are prettier in continuous time, though keeping track of which variables can and can't jump is trickier. Sims' model is in continuous time, and this is its frictionless limit. The valuation equation (4) becomes

$$\frac{\int_{j=0}^{\infty} Q_t^{(t+j)} B_t^{(t+j)} dj}{P_t} = E_t \int_{j=0}^{\infty} e^{-rj} s_{t+j} dj.$$

With maturity structure $B_t^{(t+j)} = \vartheta e^{-\vartheta j} B_t$, and a constant interest rate $i_t = i$,

$$\vartheta \int_{j=0}^{\infty} e^{-ij} e^{-\vartheta j} dj \; \frac{B_t}{P_t} = \frac{\vartheta}{i+\vartheta} \frac{B_t}{P_t} = \frac{s}{r}. \tag{12}$$

Here $\vartheta = 0$ is the perpetuity and $\vartheta = \infty$ is instantaneous debt. B_t is predetermined. P_t can jump. Starting from the $i_t = r$, $t < 0$ steady state, if i_0 jumps to a new permanently higher value i, we now have

$$P_0 = \frac{r+\vartheta}{i+\vartheta} P \tag{13}$$

in place of (10). After that,

$$P_t = P_0 e^{(i-r)t}$$

in place of (11).

In the case of one-period debt, $\theta = 0$ or $\vartheta = \infty$, $P_0 = P$ and there is no downward jump. In the case of a perpetuity, $\theta = 1$ or $\vartheta = 0$, (10) becomes

$$P_0 = \frac{1+i}{1+r} \frac{r}{i} P. \tag{14}$$

and (13) becomes

$$P_0 = \frac{r}{i} P. \tag{15}$$

The price level P_0 jumps down as the interest rate rises, and proportionally to the interest rate rise. This is potentially a large effect; a rise in interest rates from $r = 3\%$ to $i = 4\%$ occasions a 25% price level drop. However, our governments maintain much shorter maturity structures, and monetary policy changes in interest rates are not permanent, reducing the size of the effect.

The "$\log(P_t)$" line of figure 1 plots inflation and the interest rate in the discrete-time version of this example, (10)-(11), using $\theta = 0.8$, which roughly approximates the maturity structure of U.S. federal debt. At time 0, the interest rate rises permanently and unexpectedly from 3% to 4%. The log price level $\log(P_0)$ jumps down by 3.3%. Thereafter, the price level grows by 1% per period, mirroring the rise in nominal rate. Sims' model in figures 3, 4 and 7 below gives this sort of dynamics, smeared out by the elaborations of his model.

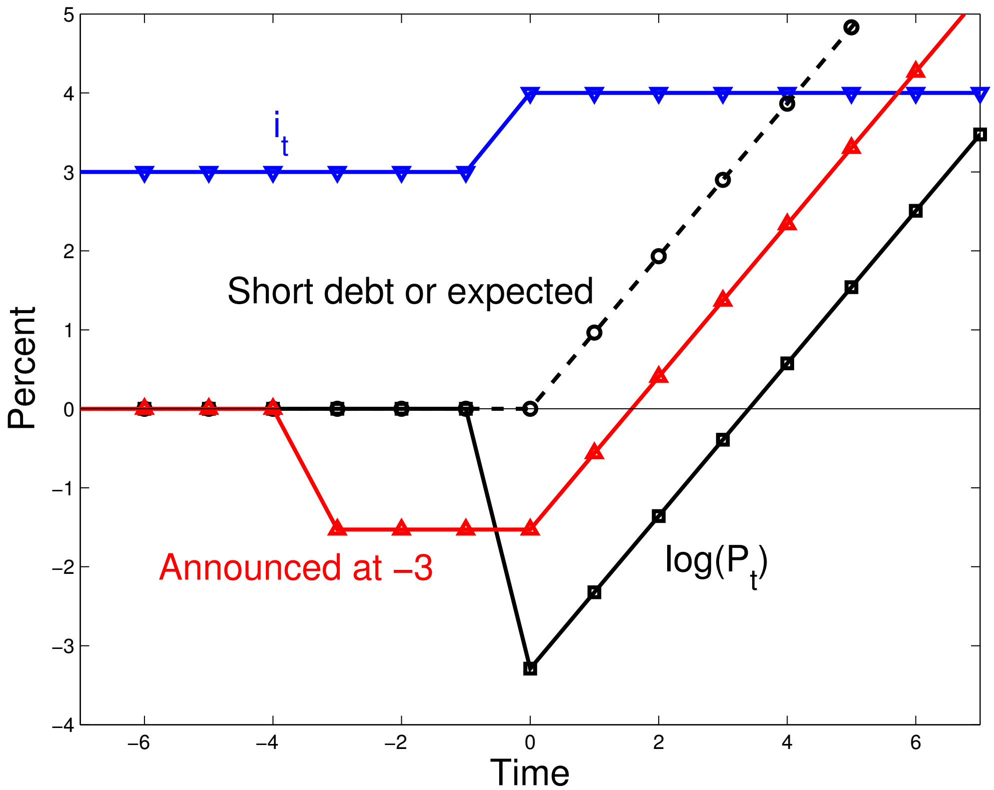

Figure 1: Response of log price level to an interest rate rise. $\theta = 0.8$.

The dashed line marked "short debt or expected" in figure 1 plots inflation in the $\theta = 0$ case of only one-period debt. In this case, inflation starts one period after the interest rate rise, with no downward jump.

2.3. Expected interest rates and forward guidance

In this model, the expected path of interest rates matters more than the current rate in determining a deflationary force. Looking at the basic innovation equation (6), a credible, persistent interest rate rise that lowers long term bond prices a lot has a stronger disinflationary effect than a tentative or transitory rate rise that induces smaller changes to long-term bond prices. In this way, this model gives an opposite picture from standard new-Keynesian models, that produce larger inflation declines for transitory AR(1) interest rate movements than for persistent interest-rate movements.

The short-term interest rate need not move at all. This model captures "forward guidance." If the central bank can credibly commit to higher or lower interest rates in the future, that announcement will change long-term bond prices, and it will have an immediate inflationary or deflationary impact, even if it has no effect on the current interest rate.

The inflationary or deflationary force in this model has really nothing to do with the contemporaneous interest rate. There is no variation in real interest rates , no reduction in aggregate demand due to a currently higher real interest rate, no Phillips curve, and so forth. The time-zero disinflation is entirely a "wealth effect" stemming from the value of government debt.

However, an announcement today of a future interest rate change only affects the value of debt whose maturity exceeds the time interval before rates change. Therefore, forward guidance has a smaller effect on current inflation than the same expectations coupled with a current rate rise. Forward guidance eventually loses its power altogether once the guidance period exceeds the longest outstanding bond maturities.

Equivalently, interest rate rises have disinflationary effects on the date of their announcement, not the date of the interest rate rise. Their effects are smaller, since a smaller range of bond prices is affected, and their effects disappear once they are long-enough anticipated.

For example, suppose that at time 0, the government announces that interest rates will rise starting at time T onward. Now, inflation starts in period $T+1$, and only bond prices of maturity $T+1$ or greater are affected. Splitting the numerator of (4),

$$\frac{\sum_{j=0}^{T} Q_0^{(j)} B_{-1}^{(j)} + \sum_{j=T+1}^{\infty} Q_0^{(j)} B_{-1}^{(j)}}{P_0} = E_0 \sum_{j=0}^{\infty} \beta^j s_j, \tag{16}$$

only the second term in the numerator on the left hand side is affected. Furthermore, for given interest rate rise, bond price declines in that second term are smaller: For a permanent rise from r to i starting at time T, the prices of bonds that mature at $j \leq T$ are unaffected, and the the prices of bonds that mature at $T+j$ are

$$Q_0^{(T+j)} = \frac{1}{(1+r)^T} \frac{1}{(1+i)^j} > \frac{1}{(1+i)^{T+j}}.$$

The downward price jump happens at time 0 when the interest rate rise is announced, not at time T when interest rates actually rise. When the time T of the interest rate rise exceeds the maturity of debt outstanding – if $B_{-1}^{(j)} = 0$ for $j > T$ – the price-level jump disappears. In this sense, a fully expected interest rate rise has no negative price effect.

Continuing the geometric maturity example, when the government announces at time 0 that interest rates will rise from r to i starting at time T, equation (16) reads

$$\left[\sum_{j=0}^{T} \frac{\theta^j}{(1+r)^j} + \sum_{j=T+1}^{\infty} \frac{\theta^T}{(1+r)^T} \frac{\theta^{(j-T)}}{(1+i)^{(j-T)}} \right] \frac{B_{-1}}{P_0} = \frac{s}{1-\beta}$$

and with a bit of algebra

$$\frac{P_0}{P} - 1 = \left(\frac{\theta}{1+r}\right)^T \left[\frac{(1+i)}{(1+r)}\frac{(1+r-\theta)}{(1+i-\theta)} - 1\right],$$

generalizing (10). In continuous time, we have

$$\left[\vartheta \int_0^T e^{-rj} e^{-\vartheta j} dj + \vartheta \int_T^\infty e^{-rT-i(j-T)} e^{-\vartheta j} dj\right] \frac{B_0}{P_0} = \frac{s}{r},$$

leading to

$$\frac{P_0}{P} - 1 = e^{-(r+\vartheta)T} \left(\frac{r+\vartheta}{i+\vartheta} - 1\right),$$

generalizing (13).

The price level P_0 still jumps – forward guidance works. Longer T or shorter maturity structures — lower θ or larger ϑ – give a smaller price-level jump for a given interest rate rise. As $T \to \infty$, the downward price level jump goes to zero.

figure 1 includes the discrete-time version of this case, in which the interest rate rise is announced at time -3. The price level jumps down at time -3, but that jump is smaller. The price level stays at the new lower level until the interest rate rises. On the date 0 that the interest rate rises there is no further jump. Inflation then rises following the higher nominal rate.

The line "short debt or expected" also presents the case that the interest rate rise is completely expected, before any long-term debt was sold, the $T \to \infty$ limit. A fully expected rate rise has no deflationary effect, even with long-term debt.

2.4. Quantities and mechanisms

To understand just how the government can set interest rates, the forces behind price-level determination, and how the interest-rate setting mechanism is related to quantitative easing, forward guidance, open market operations, and other debt operations, and how all this might be called "monetary policy," it is useful to follow the underlying bond purchases and sales.

It helps to tell a story about the sequence of events during a period. In the morning of period t, coupons $B_{t-1}^{(t)}$ come due. The government prints up fresh cash to redeem the coupons. At the end of the day, the government soaks up the newly printed cash by levying taxes net of spending $P_t s_t$, and by selling new debt.

In equilibrium, all the cash is soaked up in this way, as nobody wants to hold non-interest bearing cash overnight. Thus, we have the *flow* equilibrium condition

$$B_{t-1}^{(t)} = P_t s_t + Q_t^{(t+1)} B_t^{(t+1)} \tag{17}$$

for one-period debt, and

$$B_{t-1}^{(t)} = P_t s_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} \left(B_t^{(t+j)} - B_{t-1}^{(t+j)} \right) \tag{18}$$

with long-term debt. We can iterate (18) forward, with the consumer's transversality condition that the real value of nominal debt does not grow too fast, to obtain the present-value equilibrium condition (4), and we can use (4) at two adjacent dates to obtain (18).

This flow condition helps us to understand fiscal price level determination. If the government prints up more money in the morning to pay off maturing debt than it soaks up in the evening from tax payments and debt sales, then as the evening approaches, people try to get rid of unneeded money by buying goods and services. They bid up the price level, until larger net nominal tax payments $P_t s_t$ and bond sales soak up the money.

Printing money in the morning and soaking it up in the afternoon is a useful story, but not necessary. There is no transactions demand for money and no cash in advance constraint. Transactions in a frictionless economy can be handled with maturing government debt directly, or inside claims to maturing government debt. The "day" can collapse to a single moment. Government debt that promises to pay a dollar is valued even if there are no dollars, because it gives one the right to be relieved of one dollar's worth of tax liability. The dollar can remain a unit of account even if it is not a medium of exchange. (The model also extends straightforwardly to zero nominal interest rates, where cash and bonds are perfect substitutes, and to the case of a money demand by which people want to hold some cash overnight despite its interest cost.)

2.5. Interest rate targets

Now, as above I defined "monetary policy" as a change in interest rates with no change in fiscal surpluses, define here "monetary policy" as debt sales with no change in fiscal surpluses.

We can now answer, just *how* monetary policy can target the nominal interest rate, even with no monetary, financial, or pricing frictions, and no money demand in particular. This is a frictionless model, not just a frictionless limit in which the last dollar of money demand and money supply determine the price level.

It is easiest to see the mechanism with one-period debt, and then see that it survives in the presence of long-term debt. Consider what happens at the end of the day, if the government decides to change the amount of debt it sells, without changing current and future surpluses. Using $Q_t^{(t+1)} = 1/(1+i_t) = \beta E_t(P_t/P_{t+1})$ and the valuation formula (3) at time $t+1$, the real revenue the government obtains from selling such debt – the right hand term in the flow equation (17) – is

$$\frac{1}{1+i_t}\frac{B_t^{(t+1)}}{P_t} = \frac{Q_t^{(t+1)}B_t^{(t+1)}}{P_t} = \beta E_t\left(\frac{B_t^{(t+1)}}{P_{t+1}}\right) = E_t\sum_{j=0}^{\infty}\beta^{j+1}s_{t+1+j}. \tag{19}$$

The real revenue the government obtains by selling debt without changing surpluses is constant, independent of the amount of debt it sells, and equal to the real present value of future surpluses backing the debt. In this one-period debt case, all debt is rolled over every period, so the end of day debt sale is a claim to all future surpluses. The bond price $Q_t^{(t+1)}$, interest rate i_t, and expected inflation $E_t(P_t/P_{t+1})$ all move proportionally as the government sells more debt $B_t^{(t+1)}$. Selling additional debt is like a share split, which changes the number of shares without changing expected future dividends, which moves prices one for one, and which does not raise any revenue.

Alternatively, the government can announce the interest rate i_t, and offer to sell any amount of debt $B_t^{(t+1)}$ at that price, again fixing surpluses. Now (19) describes the quantity of debt that people will buy at the set price.

This is the key observation for an interest rate target. One might worry that in a frictionless economy, an attempt by the government to set its interest rate with a flat debt supply curve would lead to potentially infinite demands. This is not the case. The demand curve is unit-elastic, not infinitely elastic. Furthermore, quantities are not large. The constant-revenue demand curve is unit-elastic. To engineer a 1 percent higher interest rate, the government only needs to sell one percent more nominal debt.

The government can target its nominal rate by this mechanism, but not its *real* rate. An attempt to do that would, in this model, lead to infinite demands. Moreover, if the government promised (even implicitly) to raise surpluses in response to debt sales, then the demand would be indeterminate. Both cases can leave the impression that an interest rate target in a frictionless economy is infeasible, but these are not the cases we are studying.

These operations have some of the feel of traditional money supply and demand stories. Selling more debt $B_t^{(t+1)}$, in return for time t cash, raises interest rates, just as a traditional open market operation is said to do, though by a completely different mechanism. Flat or vertical debt supply curves work like flat or vertical money supply curves. A vertical debt supply curve sets the interest rate, or a horizontal supply curve – an interest rate target – will determine the quantity of debt.

Nonetheless, both debt sales and interest rate targets with constant surpluses initially feel distant from current institutions. Our treasuries, not central banks, issue debt. Treasuries sell more debt in order to fund current deficits, negative s_t, with greater revenues. To do so they must promise implicitly or explicitly to raise future surpluses. Our treasuries issue fixed quantities of debt at auction, they do not fix the interest rate and let the market determine the size of the issue. Our treasuries conduct the equivalent of equity offerings, which raise revenue, do not depress prices, and promise higher total dividends; not the equivalent of share splits, which raise no revenue, lower prices, and promise no change in dividends.

However, on closer look, this mechanism can be read as a model of our central banks and treasuries, taken to the frictionless limit. The central bank sets the short-term interest rate. It does so by setting the interest rate on reserves, the discount rate, or by setting a corridor for short-term borrowing. Our central banks allow free conversion of cash to interest-paying reserves, which are short-term government debt, and they fix the rate on reserves. Thus, the interest on reserves regime really is quite close to the fixed interest rate, horizontal supply regime described above. That people still hold cash overnight makes little difference to the model. However, in this context the central bank could as well be a committee that just announces the short-term rate. Practically a defining feature of central banks is that they may *not* engage in policies with direct fiscal consequences, such as helicopter money. They must always buy or sell one kind of debt in exchange for another.

This interest rate, and its expected future values, determine bond prices. The treasury then decides how much debt to sell at the new bond prices in order to finance its deficits. Given $Q_t^{(t+1)}$, P_t, and the surplus or deficit s_t the treasury must finance, (19) describes how much nominal debt $B_t^{(t+1)}$ the treasury must sell to finance the deficit. Therefore, the treasury can set a quantity to sell, and not a price, as it does. If the central bank raises interest rates one percent, the treasury will see one percent lower bond prices. The treasury will then raise the face value of debt it sells by one percent, in order to sell enough debt to roll over existing debt and to cover the current surplus or deficit. The government overall is really selling any quantity of debt at a fixed interest rate, though neither treasury nor central bank may be aware of that fact.

This institutional separation between treasury and central bank is important. Since expectations of future surpluses are somewhat nebulous, and treasury issues do not come with specific tax streams, it is important to have one institutional structure for selling more debt without raising revenue, without changing expected surpluses, and in order to affect interest rates and inflation; and a distinct institutional structure for selling debt that does raise revenue, does change expected future surpluses, and does not affect interest rates and inflation. By analogy, a share split and a secondary offering both increase the number of shares outstanding, and so look identical in analogous asset pricing equations. But they are conducted in very different institutional structures. One institutional structure – a share split – communicates no change in expected dividends, it changes the stock price, and it raises no revenue. The other institutional structure – a secondary offering – communicates a proportionate change in expected dividends, does not affect the stock price (beyond implicit information revelation), and raises revenue.

However, for the purposes here, it is not important to match closely current central bank and treasury operating procedures. This is the abstract, totally frictionless benchmark model. The point here is to understand that such a simplified model *can* work; that the government *can* set interest rates, expected inflation, and the price level in such a model; that the mechanics of such a model make economic sense. Models that wish to really mimic the details of current or historic operating procedures may well need to incorporate monetary, pricing, or financial frictions.

So, I continue to call "monetary policy" the setting of interest rates or bond prices, or changing the amount or maturity structure of debt $\{B_t^{(t+j)}\}$, with no change in fiscal surpluses $\{s_t\}$ (generalized below), and I continue to call changes in those surpluses "fiscal policy", potentially accompanied by bond sales, both henceforth without quotes.

(This fiscal policy can create inflation too, which in the presence of price stickiness may increase output. So, fiscal theory also contains a description of fiscal stimulus. It is also a completely different mechanism than the usual multiplier, however. By credibly lowering *future* surpluses, it encourages people to sell government debt, and thus try to buy goods and services. The decline in expected future surpluses rather than the current deficit is the key to any stimulative effect.)

In sum, with one-period debt and in this frictionless model, a clean separation occurs. From (7),

$$\frac{B_{t-1}^{(t)}}{P_{t-1}}\left(E_t - E_{t-1}\right)\left(\frac{P_t}{P_{t-1}}\right) = \left(E_t - E_{t-1}\right)\sum_{j=0}^{\infty}\beta^j s_{t+j}, \tag{20}$$

so unexpected inflation comes only from innovations to fiscal policy. From (19), monetary policy can entirely control nominal interest rates, Furthermore, by setting interest rates, monetary policy here directly controls expected inflation. So despite the fact that this is the "fiscal" theory of the price level, and in a model with no monetary, pricing, or financial frictions, there is plenty for "monetary" policy to do.

2.6. Long-term interest rates and quantitative easing

Monetary policy can also control long-term interest rates. Again, it can specify a vertical or horizontal supply curve: Expected future debt sales determine expected future one-period interest rates, or, expected future interest rate targets determine expected future debt sales. Then, the standard term structure of interest rates connects expected future one-period interest rates to long-term bond prices. In the perfect foresight case,

$$Q_t^{(t+j)} = \prod_{k=0}^{j-1} \frac{1}{1+i_{t+k}}.$$

Monetary policy can also control long-term bond prices directly, by buying and selling long-term bonds, in a policy that begins to look like quantitative easing. As a very simple example, suppose only one-period debt $B_{-1}^{(0)}$ is outstanding at the beginning of period 0, so P_0 is determined by

$$\frac{B_{-1}^{(0)}}{P_0} = E_0 \sum_{j=0}^{\infty} \beta^j s_j. \tag{21}$$

At the end of period 0, the government sells long-term debt for all periods j in the future, $\{B_0^{(j)}\}$, and then never buys, sells, or rolls over debt again so $B_{j-1}^{(j)} = B_0^{(j)}$. At period j, the government pays off the maturing debt from time j surpluses, so the time j price level is set by

$$\frac{B_0^{(j)}}{P_j} = s_j. \tag{22}$$

Now long-term bond prices are

$$\frac{Q_0^{(j)}}{P_0} = E_0\left(\beta^j \frac{1}{P_j}\right) = \beta^j \frac{E_0(s_j)}{B_0^{(j)}}. \tag{23}$$

Surpluses s_j are split among bond holders $B_0^{(j)}$. The more bonds sold, the lower the price of each bond. Equation (23) either describes how greater bond sales $B_0^{(j)}$ of each maturity lower bond prices $Q_0^{(j)}$ and raise long-term yields at that maturity, or it describes how many bonds $B_0^{(j)}$ the government will sell if it sets a fixed price $Q_0^{(j)}$.

Combinations of the two policies – expectations that the government will buy or sell debt in the future, in a state-contingent way, along with changes in the maturity structure of the debt today – offer a rich set of possibilities for the management of interest rates and expected inflation. "Rich" also means hard to analyze, however. This section stops with simple examples, as otherwise we are soon drowned in algebra. However, state-contingent debt sales and repurchases, such as rolling over more debt when a recession shocks surpluses, are a crucial way the government smooths surplus shocks across time and states, and thereby smooths inflation. For empirical application or policy analysis, we cannot stop with simple examples.

2.7. Stepping on a rake: debt view

In the last example, the government could control long-term bond prices, but since there was no long-term debt outstanding at period 0, changing interest rates or bond prices did not affect the initial price level P_0. Now, let us add outstanding long-term debt, to see the quantity side of the stepping on a rake mechanism.

To see the mechanism in the simplest example, consider a two-period version of the model. At time 0, there is long-term debt outstanding, coming due at time 1, $B_{-1}^{(1)} > 0$. At the end of time 0, the government may sell additional time 1 debt, in amount $\mathbf{B}_0^{(1)} - B_{-1}^{(1)}$. ($\mathbf{B}_0^{(1)}$ is the total amount outstanding at the end of time 0, and $\mathbf{B}_0^{(1)} - B_{-1}^{(1)}$ is the amount sold at time 0. I highlight $\mathbf{B}_0^{(1)}$ with boldface to focus on its influence.) The present value equation (4) determines prices at period 0 and 1 by

$$\frac{B_{-1}^{(0)} + Q_0^{(1)} B_{-1}^{(1)}}{P_0} = s_0 + \beta E_0 s_1 \tag{24}$$

$$\frac{\mathbf{B}_0^{(1)}}{P_1} = s_1. \tag{25}$$

Again, we fix surpluses s_0, s_1, pre-existing debt $B_{-1}^{(0)}, B_{-1}^{(1)}$, and look for effects on the price level, P_0, P_1 and bond prices and interest rates

$$Q_0^{(1)} = \frac{1}{1+i_0} = \beta E_0 \left(\frac{P_0}{P_1} \right). \tag{26}$$

Equation (25) directly determines P_1. Substituting (26) and (25) into (24), P_0 is given by

$$\frac{B_{-1}^{(0)}}{P_0} = s_0 + \left(\frac{\mathbf{B}_0^{(1)} - B_{-1}^{(1)}}{\mathbf{B}_0^{(1)}} \right) \beta E_0 \left(s_1 \right). \tag{27}$$

With outstanding long-term debt $B_{-1}^{(1)} > 0$, the price level P_0 is now affected by debt sales $\mathbf{B}_0^{(1)} - B_{-1}^{(1)}$ at time 1, with no change in surpluses – by monetary policy in the form of quantitative easing or tightening.

Examining (25) and (27), we see that raising $\mathbf{B}_0^{(1)}$ raises P_1 and lowers P_0 – the stepping on a rake effect. It thus raises the interest rate i_0. Thus, by controlling the amount of debt to be sold $\mathbf{B}_0^{(1)}$, monetary policy can still control the nominal interest rate and the expected inflation rate. Conversely, the government can announce an interest rate target i_0, and offer to buy and sell debt $\mathbf{B}_0^{(1)}$ freely at that rate. The model then tells us the demand for that debt.

The stepping on a rake effect happens because new long-term debt sales dilute existing long-term debt as a claim to future surpluses. We can split (25) between existing and newly-sold debt, as

$$\frac{(\mathbf{B}_0^{(1)} - B_{-1}^{(1)}) + B_{-1}^{(1)}}{P_1} = s_1.$$

The surpluses s_1 are divided between new sales and existing bonds. The new sales transfer to new bondholders resources that were expected by existing long-term bond holders. Selling such bonds generates revenue at time 0, that soaks up money and pushes down the time 0 price level. To see this fact, look the period 0 flow equation, the specialization of (18), which says that maturing bonds must be paid by surpluses or by new bond sales,

$$\frac{B_{-1}^{(0)}}{P_0} = s_0 + \frac{Q_0^{(1)}(\mathbf{B}_0^{(1)} - B_{-1}^{(1)})}{P_0}. \tag{28}$$

Using (25) and (26) in this flow equation, we can rewrite its second term, which represents the real revenue raised by bond sales at the end of time 0, so (28) becomes

$$\frac{B_{-1}^{(0)}}{P_0} = s_0 + \left(\frac{\mathbf{B}_0^{(1)} - B_{-1}^{(1)}}{\mathbf{B}_0^{(1)}}\right)\beta E\left(s_1\right).$$

When there is long-term debt outstanding $B_{-1}^{(1)}$, then selling new debt without changing future surpluses provides revenue in period 0. Fixing surpluses s_0, this revenue soaks up money and drives down the price level P_0.

2.8. Stepping on a rake: dynamic examples

Here, I consider the debt quantity side of the simple stepping on a rake examples plotted in figure 1. With one-period debt, there is a unique debt policy that generates the interest rate rise. With long-term debt, multiple debt policies produce the same interest rate and inflation path. I examine three of them: I show how the government can implement the rate rise with future short-term debt sales, with future long-term debt sales, and by a version of quantitative easing that rearranges the maturity structure at time 0 only.

A general rule describes interest rate targets: With constant surpluses s and with perfect foresight, the government debt valuation equation (4) implies that for interest rates $\{i_t\}$, debt must follow

$$\frac{\sum_{j=0}^{\infty} Q_{t+1}^{(t+1+j)} B_t^{(t+1+j)}}{\sum_{j=0}^{\infty} Q_t^{(t+j)} B_{t-1}^{(t+j)}} = \frac{P_{t+1}}{P_t} = \frac{1+i_t}{1+r} \tag{29}$$

or, *the market value of nominal debt must grow at the inflation rate.*

In the case of one-period debt, this rule is simple as usual,

$$\frac{B_t^{(t+1)}}{B_{t-1}^{(t)}} = \frac{P_{t+1}}{P_t} = \frac{1+i_t}{1+r}.$$

For the interest rate to rise from r to i permanently, at time 0, with initial condition $B_{-1}^{(0)}$, debt must simply start to grow at the inflation rate,

$$B_t^{(t+1)} = \left(\frac{1+i}{1+r}\right)^t B_{-1}^{(0)}.$$

An interest rate target will lead to this path of debt; a quantity rule must have this path to generate the desired path of interest rates.

With long-term debt, there are many debt policies consistent with a given interest rate path. As long as the nominal market value of the debt grows at the interest rate, the maturity structure of the debt is irrelevant to the impulse response function after date 0. (The maturity structure matters to later shocks, of course.) *All* that maters to generating the interest rate path is that the total nominal market value grow at the desired inflation rate.

As one example, suppose the economy starts at the steady state with perpetuities B^p outstanding, and additional one-period debt $\tilde{B}$. (I use a tilde, $\tilde{B}$, because there is also a maturing coupon in the perpetuity, so total one-period debt coming due at date t is $B_{t-1}^{(t)} = \tilde{B}_{t-1}^{(t)} + B^p$.) Suppose surpluses are constant $s_t = s$.

Then, suppose first that monetary policy is implemented by buying and selling this one-period debt only, in amounts $\tilde{B}_t^{(t+1)}$, as central banks traditionally did, and suppose the government does not buy and sell any more long-term debt $B_t^p = B^p$. The government debt valuation equation (4) now reads

$$\frac{\tilde{B}_{t-1}^{(t)} + \sum_{j=0}^{\infty} \frac{1}{(1+i)^j} B^p}{P_t} = \frac{\tilde{B}_{t-1}^{(t)} + \frac{1+i}{i} B^p}{P_t} = \frac{1+r}{r} s = \frac{\tilde{B} + \frac{1+r}{r} B^p}{P}. \tag{30}$$

(The central term with s is the present value of surpluses, held constant here. The equality to the right expresses the valuation equation at the initial steady state with $i_t = r$. The equalities to the left express the valuation equation at the new interest rate $i_t = i$.) To produce an interest rate i, or if the government fixes that rate and freely sells debt $\tilde{B}_t^{(t+1)}$ at that price, the overall nominal value of debt must grow at the inflation rate, from (29) and (30),

$$\frac{\tilde{B}_t^{(t+1)} + \frac{1+i}{i} B^p}{\tilde{B}_{t-1}^{(t)} + \frac{1+i}{i} B^p} = \frac{1+i}{1+r}.$$

With initial condition $\tilde{B}_{-1}^{(0)} = \tilde{B}$, the path of one-period debt $\tilde{B}_t^{(t+1)}$ must follow

$$\tilde{B}_t^{(t+1)} + \frac{1+i}{i} B^p = \left(\frac{1+i}{1+r}\right)^t \left(\tilde{B} + \frac{1+i}{i} B^p\right).$$

Here, monetary policy produces a persistent interest rate rise, by open market operations of one-period debt. Expected *future* one-period debt sales lead to higher expected future interest rates and expected future inflation.

The price-level jump at time 0 follows from the preexisting debt. At time 0, the condition that the present value of surpluses is unchanged (30) implies

$$P_0 = \frac{\tilde{B} + \frac{1+i}{i} B^p}{\tilde{B} + \frac{1+r}{r} B^p} P,$$

which leads to $P_0 = P$ in the one-period case $B^p = 0$, and to the downward jump proportional to the interest rate rise (14) in the perpetuity case $\tilde{B} = 0$.

The debt sales $\{\tilde{B}_t^{(t+1)}\}$ only set interest rates, they do not directly control the P_0 jump. Higher interest rates and a consequent lower market value of debt cause the price level P_0 to jump. There is no downward jump in debt. The current and expected future one-period debt sales devalue the outstanding perpetuities as claims to future surpluses.

To generate forward guidance, an announcement at 0 that interest rates will rise starting at time T, we follow the same idea. The market value of nominal debt must be expected to start rising at time T.

Secondly, the government could target interest rates $\{i_t\}$ by buying and selling perpetuities B_t^p, so $\tilde{B}_t = 0$. The condition that the nominal value of the debt rises at the inflation rate would be even simpler,

$$B_t^p = B^p \left(\frac{1+i}{1+r} \right)^t . \tag{31}$$

Governments typically do not do this. They typically accomplish monetary policy with short-term debt, and they typically accomplish fiscal policy, which implies changes in surpluses, with long-term debt. This separation may help to communicate different expectations of surpluses.

These examples rely on expected future debt sales or expected future interest rate targets to drive the yield curve today. Third, as above, the government can set long-term interest rates directly by transacting in long-term debt at time zero, in something like quantitative easing (QE) operations. In the presence of outstanding long-term debt, long-term debt sales will raise rates and lower the initial price level, and vice versa. By taking action immediately, the debt operation may communicate a change in central bank intentions more effectively than promising a change in future interest rate targets or future debt purchases and sales.

To generate a simple example, we can follow the same idea as in equation (23) – just sell debt at each maturity to give the desired price path, with no future debt sales. Again, suppose that at the end of time 0, the government sets the maturity structure of the debt $\{B_0^{(j)}\}$, and thereafter neither buys or sells any debt, $B_{j-1}^{(j)} = B_0^{(j)}$, simply paying off this debt as it comes due. Surpluses are constant. With no future sales or purchases, the price level at time j is set as in (22), $B_0^{(j)}/P_j = s$. Then, to produce the interest rate i at all dates in the future, the maturity structure must be

$$\frac{B_0^{(j)}}{B_0^{(j-1)}} = \frac{P_j}{P_{j-1}} = \frac{1+i}{1+r}. \tag{32}$$

Debt *across maturities* at time 0 must grow at the inflation rate.

$$B_0^{(j)} = \left(\frac{1+i}{1+r}\right)^j B_0^{(1)}. \tag{33}$$

We find $B_0^{(1)}$ from the initial condition (29) stating that the market value of debt from 0 to 1 grows at the rate of inflation to produce $i_0 = i$,

$$\frac{1+i}{1+r} = \frac{\sum_{j=0}^{\infty} \frac{1}{(1+i)^j} B_0^{(j)}}{\sum_{j=0}^{\infty} \frac{1}{(1+i)^j} B^p} = \frac{\sum_{j=0}^{\infty} \frac{1}{(1+i)^j} \left(\frac{1+i}{1+r}\right)^j B_0^{(1)}}{\frac{1+i}{i} B^p} = \frac{\frac{1+r}{r} B_0^{(1)}}{\frac{1+i}{i} B^p}. \tag{34}$$

Uniting (33) and (34), to start with a perpetuity B^p, and create interest rates that rise from r to i at time 0, promising no future debt sales, the date 0 maturity structure must be

$$\frac{B_0^{(j)}}{B^p} = \frac{r}{i} \left(\frac{1+i}{1+r}\right)^{j+1}.$$

For small j, this number is less than one, while for large j, the number is greater than one. Thus, to engineer the interest rate rise and consequent stepping on a rake inflation pattern, the government buys back short-term debt, and sells long-term debt. Conversely, to produce an upward price-level jump at time 0 (stimulus), and lower long-term interest rates, the government buys back long-term debt and issues short-term debt, quantitative easing.

In sum, we see how the same mechanism and result lies behind interest rate policy, forward guidance, and quantitative easing. The government can implement monetary policy by targeting short term or long-term interest rates, or by buying and selling bonds.

2.9. A constraint

How far can monetary policy go with such debt operations, or interest rate targets, with constant surpluses?

Using the bond price from (5), we can write the government debt valuation equation (4)

$$B_{-1}^{(0)} \frac{1}{P_0} + \sum_{j=1}^{\infty} \beta^j B_{-1}^{(j)} E_0 \left(\frac{1}{P_j}\right) = E_0 \sum_{j=0}^{\infty} \beta^j s_j. \tag{35}$$

With one-period debt, the second term is absent, ad surplus expectations alone drive shocks to the price level P_0. With long-term debt, surplus expectations at time 0 drive a moving average of current and future inverse price levels, weighted by debt outstanding at time 0. In the presence of outstanding long-term debt, the government can, by varying debt at time 0 or thereafter, or equivalently by varying interest rate targets, achieve any price level path consistent with (35), and only those levels. (This is equation (31) in Cochrane (2001), which has additional discussion.)

This equation makes precise an observation seen in the examples, and offers a general version of Sims' "stepping on a rake" observation. Monetary policy with constant surpluses can only lower the price level P_0 by raising prices at some other time. It cannot raise or lower the overall price level at every date $t \geq 0$. This part of the separation result (20) remains. Furthermore, for dates j with no debt outstanding, $B_{-1}^{(j)} = 0$, there is no trade-off. Monetary policy can still freely pick the expected price level $E_0(1/P_j)$ for such periods, but doing so has no effect on the initial price level P_0.

Likewise, using $Q_0^{(j)} = E_0(\beta^j P_0/P_j)$ we can write

$$\frac{B_{-1}^{(0)}}{P_0} + \frac{\sum_{j=1}^{\infty} \beta^j B_{-1}^{(j)} Q_0^{(j)}}{P_0} = E_0 \sum_{j=0}^{\infty} \beta^j s_j. \tag{36}$$

This equation acts as a similar constraint linking bond prices and the initial price level. Monetary policy can set any bond prices $Q_0^{(j)} > 0$, either directly by offering debt at fixed prices, via debt sales and purchases, or by expectations of future interest rates. Equation (36) then expresses the effect of these bond prices on the time-0 price level.

2.10. Continuous time and sticky prices

Sims' analysis seems to be quite different, in that it operates in continuous time and the price level P_t cannot jump. Continuous time is not an important difference. If interest rates jump up with no change in surplus, the price level P_0 jumps down, and then starts to rise. The absence of a price-level jump is an important difference.

In Sims' model, with no price-level jump, a rise in interest rates sets off a period of deflation, which cumulatively lowers the price level. As I show below, this apparent difference is not central. As one removes price stickiness, Sims' short period of deflation gets stronger, smoothly approaching the downward jump predicted by the frictionless model. Thus, the price-level jump in this frictionless model, which may seem artificial, is in fact a useful guide to drawn out disinflations of models with sticky prices.

The continuous-time setup with no price-level jumps is an important framework, and works a bit differently from the discrete-time model presented above. It's worth seeing here the basic mechanism before stepping in to the full model. Simplifying to either a perpetuity or to instantaneous debt, the risk-neutral (discounting at a constant real rate) government debt valuation equation (4) is

$$\frac{Q_t B_t}{P_t} = E_t \int_{\tau=t}^{\infty} e^{-\int_{v=t}^{\tau}(i_v - \pi_v)dv} s_\tau d\tau. \tag{37}$$

For short-term debt, $Q_t = 1$ always. In discrete time, or if prices can jump, innovations in $\{s_\tau\}$ induce a jump in P_t. That channel disappears in continuous time with sticky-price models that preclude price-level jumps. However, the present value relation (37) still selects equilibria. For given $\{s_t\}$ and $\{i_t\}$, there are typically multiple paths that equilibrium inflation $\{\pi_t\}$ can follow. Only one of those paths is consistent with (37).

A discount rate effect on the right hand side operates in place of a price-level jump on the left. Since $Q_t = 1$, outstanding B_t, and by assumption P_t all cannot jump when there is a jump to information about future s_τ, then real discount rates $\{i_v - \pi_v\}$ must change. If future s decline, for example, the discount rates $\{i_v - \pi_v\}$ must also decline so that the present value on the right hand side of (37) is unchanged. Therefore, a sticky-price model with short-term debt, subject to a fiscal shock, will substitute a period of higher inflation π for the immediate jump upward P_t of a frictionless model.

With long-term debt, the nominal bond price Q_t in (37) can jump down when monetary policy raises interest rates. If the price level P_t cannot jump, the path $\{\pi_t\}$ on the right hand side must adjust to produce a higher real discount rate and a lower present value of surpluses. At a majority of dates on the path, π_t must rise less than i_t so that real discount rates rise. Thus, the downward price-level jump of the frictionless model becomes a period of lower inflation when the price level cannot jump.

2.11. A last word on surpluses

For these simple examples, I have defined "monetary policy" as changes in nominal debt or interest rate targets, holding surpluses fixed. While convenient for working out examples, this definition is needlessly restrictive, and will need to be generalized for serious applied work.

Surpluses do not need to be fixed, or exogenous in the fiscal theory. For the fiscal theory to work, the minimum requirement is that the surplus does not respond to alternative price levels in a way that automatically validates the government debt valuation equation (4) for any price level.

In simplest terms, the supply and demand curves may not lie on top of each other. That is a weak requirement. It is a specification about off-equilibrium beliefs, as there is no way to test this assumption by data from a given equilibrium: The government debt valuation equation (4) holds in all models, active-fiscal or passive-fiscal. (Cochrane (2017a) sections 6.1-6.3 expand on this point, with examples.)

Other endogenous surplus responses can, and likely should, be included. The question is, which of these fiscal changes do we want to consider as fiscal policy and which do we want to consider as endogenous responses to monetary policy?

For example, Sims includes the fact that surpluses rise and fall with GDP, due to procyclical tax revenues, automatic stabilizers, and the predictable tendency of governments to engage in countercyclical fiscal policy. One might well consider that endogenous response to be part of the effects of monetary policy, as Sims does. The central bank cannot *directly* change expenditures or tax rates. But if the central bank, by changing interest rates, increases output, and that increased output raises fiscal surpluses, one would want to include the secondary effects of those surpluses on the price level as part of the "effects of monetary policy," at least for a quantitative analysis. One might also include the fiscal effects of imperfect indexation and seigniorage.

One may want to exclude, on the other hand, independent fiscal responses, changes that require action by fiscal authorities, and changes taken in response directly to shocks that would not be taken in response to GDP, unemployment, and inflation that result from monetary policy actions. This is a tricky distinction, both conceptually and in empirical applications. Fiscal policy typically responds to the same events that occasion monetary policy. The recession of 2008 brought zero rates and QE, but it also brought on deliberate fiscal stimulus. To evaluate the effects of monetary policy in isolation, one wants the former but not the latter.

The delicate question of how fiscal authorities respond to monetary policy is important as well. In the stepping on a rake mechanism, I assumed no response. But do fiscal authorities not respond at all to interest rates or inflation? If, for example, fiscal authorities change surpluses so as to keep constant the present value of government debt, then the stepping on a rake mechanism disappears entirely. While one may not want to consider such responses as economic "effects of monetary policy," a central bank wanting to know the effects of its actions should definitely consider adverse or cooperative responses from the treasury and congress.

These are issues beyond the current paper. The point here: The definition of "monetary policy" going forward need not and should not be that there is no change at all in fiscal surpluses. Applied work will, as always, have to think hard about fiscal-monetary interactions.

3. Sims' model

The completely frictionless model of the last section is transparent, but it is far from realistic. To make a fiscal theory of monetary policy useful, we must embed it in a more realistic macroeconomic setting. The most natural setting adds sticky prices and other frictions and elaborations of modern DGSE models, designed to produce reasonable dynamics, but replacing fiscal theory of the price level with "active" monetary policy to select equilibria. Sims (2011) has to some extent already taken this step. But, as explained in the introduction, the connection of Sims' model to the analysis of the previous section is not at all obvious. It is not obvious that the frictionless model captures the essence of the full model, which of Sims' frictions are important for the stepping on a rake sign, or how the frictionless limit works. I undertake that connection here.

I derive and solve Sims' model in the appendix. The model is

$$di_t = [-\gamma(i_t - \bar{r}) + \phi_\pi \pi_t + \phi_c \dot{c}_t]\, dt + d\varepsilon_{mt} \tag{38}$$

$$d\pi_t = (\rho\pi_t - \kappa c_t)\, dt \; + d\delta_{\pi t} \tag{39}$$

$$dp_t = \pi_t dt \tag{40}$$

$$dy_t = y_t\,(y_t - i_t)\, dt + d\delta_{yt} \tag{41}$$

$$ds_t = \omega \dot{c}_t dt + d\varepsilon_{st} \tag{42}$$

$$db_t = [b_t(i_t - \pi_t) - s_t]\, dt - \frac{b_t}{y_t} d\delta_{yt} \tag{43}$$

$$d\lambda_t = -\lambda_t\,(r_t - \bar{r}\;)\, dt + d\delta_{\lambda t} \tag{44}$$

$$dc_t = \dot{c}_t dt \tag{45}$$

$$d\dot{c}_t = \left[\frac{\lambda_t}{\psi} e^{c_t} - \frac{1}{\psi} e^{c_t} e^{-\sigma c_t} + \bar{r}\dot{c}_t\right] dt + d\delta_{\dot{c}t} \tag{46}$$

$$\pi_t = i_t - r_t. \tag{47}$$

Equation (38) is the policy rule and $d\varepsilon_{mt}$ is the monetary policy shock. The nominal interest rate is i_t, r is the real interest rate with steady state and consumer discount rate $\bar{r}$, π is inflation, c is consumption equal to output. Equation (39) is the continuous-time Phillips curve. It is the obvious analogue of the discrete-time curve $\pi_t = \beta E_t \pi_{t+1} + \kappa c_t$. By (40), this Phillips curve allows a jump in the inflation rate but not in the (log) price level p. The perpetuity yield is y, ($1/y$ is the price), and equation (41) is the expectations-hypothesis relation between long y and short i rates –

there are no price-level jumps and no risk premiums. Equation (42) describes a primary surplus s that rises and falls with consumption growth. It contains a shock, so we can calculate the economy's response to fiscal shocks. Equation (43) tracks the evolution of the real market value of debt b, which consists of nominal perpetuities. The surplus does not respond to inflation-induced changes in the value of government debt, and the debt in (43) grows at the real rate of interest. Thus, for all but one equilibrium, debt will explode and violate the consumer's transversality condition. This is the "active fiscal" specification. Debt and yield share a shock $d\delta_{yt}$. A shock to long bond yields also shocks the market value of the debt. Equations (44)-(46) describe marginal utility λ and consumption c with a "habit" term that values a smooth consumption path: The utility function adds a penalty for the derivative of log consumption growth,

$$U = E\int_{t=0}^{\infty} e^{-\bar{r}t}\left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{1}{2}\psi\left(\frac{1}{C}\frac{dC}{dt}\right)^2\right]dt.$$

Equation (47) is the Fisher equation defining the real rate of interest.

Some equations only specify expected values. For example, the Phillips curve (47) is conventionally written $E_t(d\pi_t) = (\rho\pi_t - \kappa c_t)dt$. I include expectational shocks $d\delta_t$ to capture this fact. They satisfy $E_t d\delta_t = 0$. I use δ to distinguish them from structural shocks $d\varepsilon_t$. For each such expectational equation, we need one explosive eigenvalue of the system to uniquely determine the corresponding expectational shock $d\delta_t$ by the rule that the system must not be expected to explode.

Like Sims, however, I only study perfect-foresight solutions with a single probability-zero jump at time zero. This restriction also simplifies many of the model's equations. In a fully stochastic model, these equations would need additional terms such as risk premiums.

I linearize the model, and solve it in the usual way, sending unstable eigenvalues forward and stable eigenvalues backward.

The fiscal block (41) - (43) operates independently of the rest of the model – other variables enter here, but the variables $y, s,\ b$ determined here do not feed back on the rest of the system. As in other new-Keynesian models, the model without this block and passive monetary policy has multiple equilibria. But all but one of those equilibria lead to an explosive path for the real value of debt $\tilde{b}_t$. Therefore, the fiscal block selects equilibria.

4. Impulse-response functions

Sims uses parameters $\gamma = 0.5$; $\phi_\pi = 0.4$; $\phi_c = 0.75$; $\sigma = 2$; $\bar{r} = 0.05$; $\bar{s} = 0.1$; $\rho = 0.1$; $\delta = 0.2$; $\omega = 1.0$; $\psi = 2.0$. Here, $\phi_\pi < \gamma$ so we are in the fiscal theory of the price level region of passive

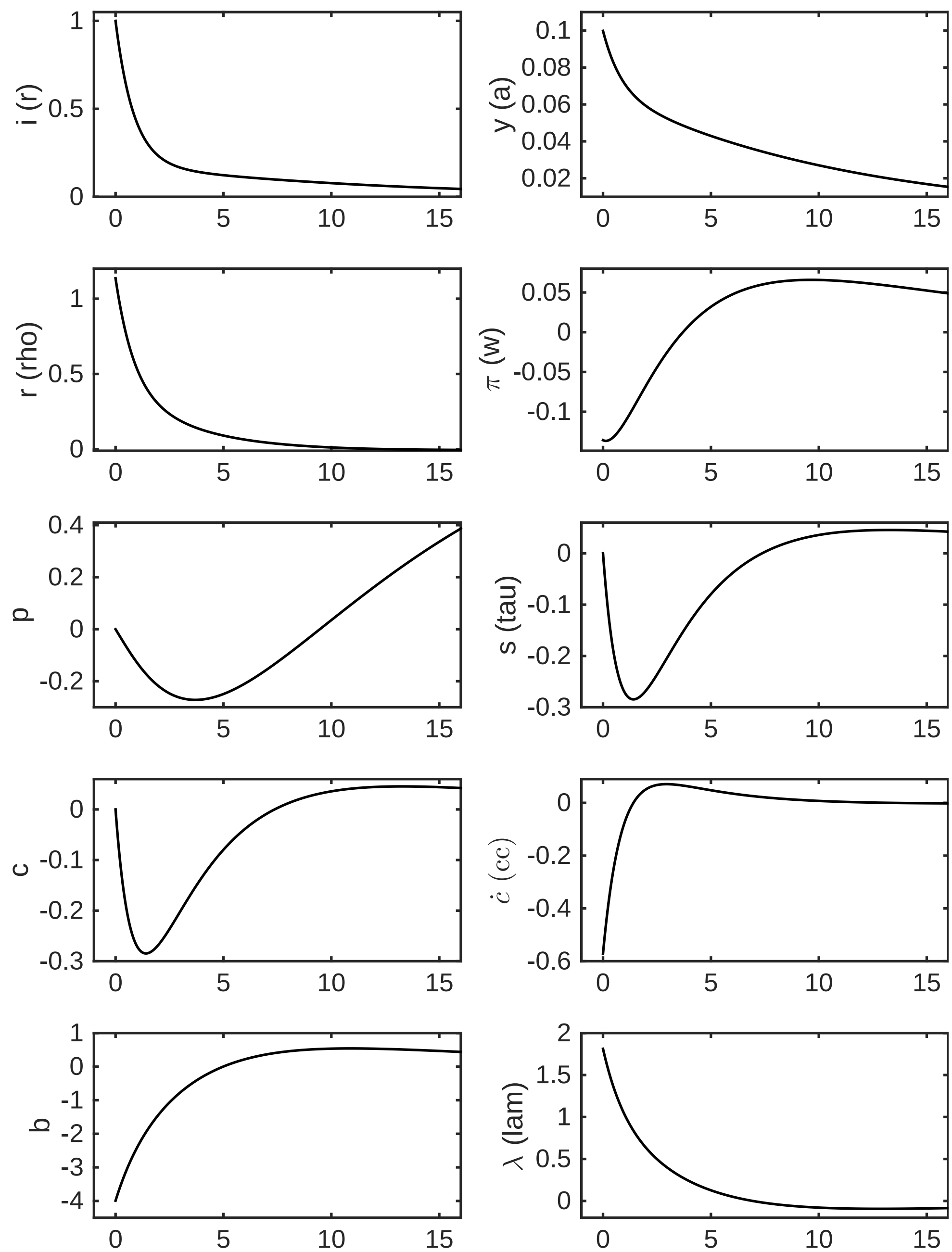

Figure 2: Responses to a monetary policy shock. Replication of Sims (2011) figure 3. Sims' variable labels are in parentheses.

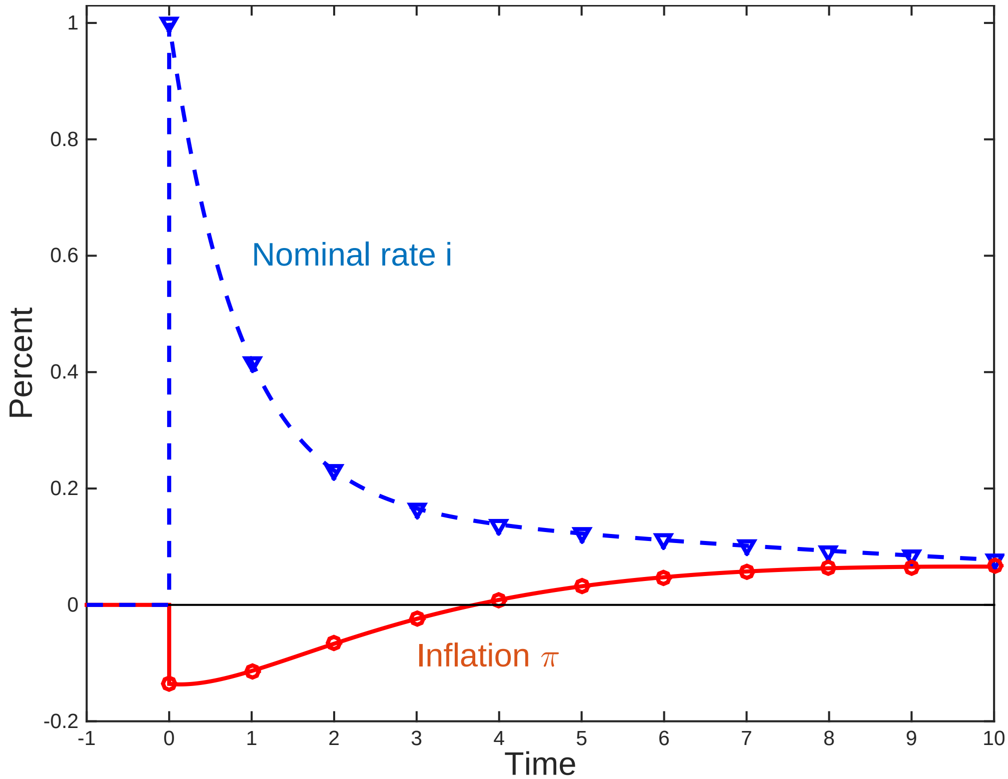

Figure 3: Response of nominal rate and inflation to a monetary policy shock in the Sims (2011) model.

monetary policy and active fiscal policy.

figure 2 presents the response of all variables to the monetary policy shock $d\varepsilon_{m0}$. This figure is visually identical to Sims (2011) figure 3. The price level and consumption do not jump at time zero. All the other variables jump.

figure 3 shows the response of interest rates and inflation to the monetary policy shock. You see the jump down in inflation, followed by its slow rise. The price *level* cannot jump, but *inflation* can and does jump.

4.1. Habits, Taylor rules, and fiscal responses

How many of Sims' ingredients are *necessary* to deliver a negative response of inflation to the interest rate rise? How many ingredients are useful to match dynamics, but not essential to the basic sign?

It turns out that the habit ψ, the Taylor rule γ, ϕ_c, ϕ_π, and the fiscal policy response ω do not matter for the negative response of inflation to the interest rate rise. Figure 4 presents the impulse response function for the case $\gamma = 0$, a permanent rise in rates; $\phi_c = \phi_\pi = 0$, an interest rate peg that does not respond to inflation or output; $\omega = 0$, surpluses do not respond to output; and $\psi = 0$, no habits.

The remaining model is, in place of (38)-(47),

$$dc_t = \frac{1}{\sigma}(i_t - \pi_t)dt + d\delta_{ct} \tag{48}$$

$$d\pi_t = (\rho\pi_t - \kappa c_t)\, dt \; + d\delta_{\pi t} \tag{49}$$

$$di_t = d\varepsilon_{mt} \tag{50}$$

$$ds_t = d\varepsilon_{st} = 0 \tag{51}$$

$$dy_t = y_t\,(y_t - i_t)\, dt + d\delta_{yt} \tag{52}$$

$$db_t = [b_t(i_t - \pi_t) - s_t]\, dt - \frac{b_t}{y_t} d\delta_{yt}. \tag{53}$$

This is the standard continuous-time new-Keynesian model (first two equations) with a non-responsive interest rate target, so passive-money active-fiscal, and with long-term debt. I refer to (48)-(53) as the "simple model" below.

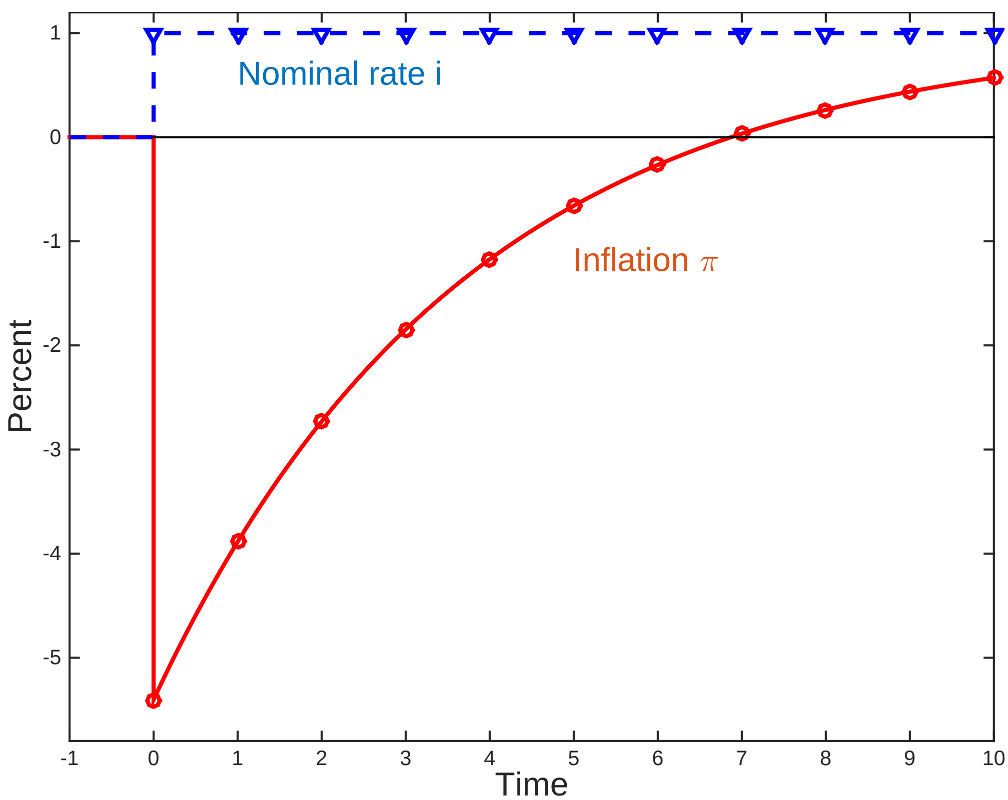

Figure 4: Response to a step-function rise in interest rates, in the simple model.

The short-run negative response of inflation to the interest rate rise is still there. It is stronger

– most of Sims' extra ingredients, which make the model more realistic, *reduce* the size of the basic effect. The same 1% nominal interest rate rise as in figure 3 now produces a 5% fall in inflation, not an 0.15% fall.

The largest reason for this difference is the permanent interest rate shock. A longer-lasting nominal interest rate rise has a greater effect on nominal bond prices $Q_t^{(t+j)}$ and so requires a greater jump in the price level P_t.

4.2. Response to expected monetary policy

As in the frictionless analysis, to produce a decline in inflation, the interest rate rise must be unexpected.

The top panel of figure 5 presents the response of inflation and interest rates of the full Sims model to a fully expected monetary policy shock $d\varepsilon_{m0}$. In this case, the interest rate response is Fisherian – inflation rises smoothly through the episode.

Interest rates respond in advance of the monetary policy shock at $t = 0$, because inflation and output move in advance of the shock and the interest rate rule responds to inflation and output.

The bottom panel of figure 5 plots the response of the simplified model (48)-(53) to a fully anticipated shock. The inflation rate rises smoothly throughout, just as in the discrete-time versions of this calculation presented in Cochrane (2017a).

4.3. Short-term debt

Long-term debt is also necessary for the negative response of inflation to interest rates.

The top panel of figure 6 presents the response function for the full Sims model to an unexpected monetary policy shock, with short-term debt in the place of long-term debt. (In a continuous-time model, short-term debt means fixed value, floating-rate debt. The price is always one, and it pays $i_t dt$ interest.) Inflation jumps *up* and is positive throughout.

The bottom graph shows the same exercise in the simple model, with only price-stickiness left. Here we see a perfectly Fisherian response to unexpected monetary policy – inflation rises instantly to match the rise in interest rates. Yes, this is the standard two-equation new - Keynesian model, with prices that are sticky and cannot jump. But *inflation* can jump. Recall equation (37),

$$\frac{Q_t B_t}{P_t} = E_t \int_{\tau=t}^{\infty} e^{-\int_{v=t}^{\tau}(i_v - \pi_v)dv} s_\tau d\tau \tag{54}$$

In the short-term debt case, the bond price $Q_t = 1$ cannot move. So inflation π_v moves exactly as much as the nominal interest rate i_v, leaving no change in present value on the right hand side and

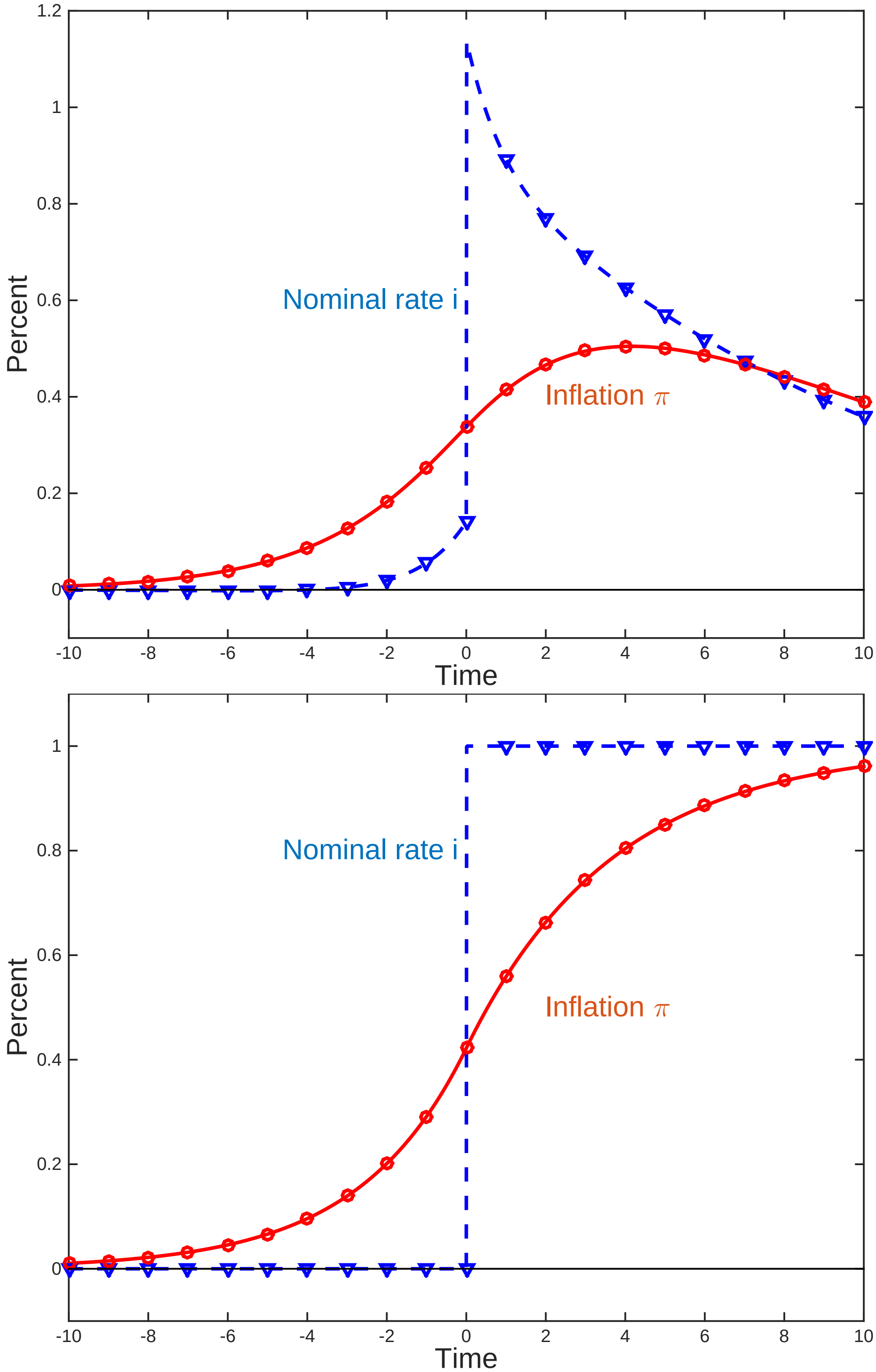

Figure 5: Response to expected monetary policy shocks. Top: Sims (2011) model. Bottom: simple model.

thus no need for the price level to jump on the left hand side.

With short-term debt, the responses to the expected shock are exactly the same as they are for long-term debt, as already shown in figure 5. Hence, the *only* effect of long-term debt in this model is that an unexpected shock can lower the value of long-term debt.

4.4. Varying price stickiness

How does price stickiness affect the responses? This question is interesting on its own. In addition, we want to verify that the frictionless limit is sensible. Many Keynesian and new-Keynesian models blow up as one reduces frictions, even when the frictionless limit points are well-behaved. (Cochrane (2017b).) Even when the frictionless limit is the same as the frictionless limit point, it is useful to see if the basic sign and mechanisms hold in the frictionless limit, as analyzed in the first part of this paper, leaving frictions to fill out dynamics and magnitudes, or whether the frictions are essential to the basic signs and mechanisms of the model.

The top panel of figure 7 shows the response of the price level to the step-function interest rate rise, in the simple model, as we reduce price stickiness. In this model, larger values of κ, the coefficient on consumption in the Phillips curve (39), correspond to less price stickiness. As price stickiness is reduced, the model steadily approaches the frictionless result of equation (15), a 20% downward jump in the price level ($r = 5\%$, $i = r + 1\%$, $P_0/P = r/i$), followed by steady growth that is 1% higher, due to the 1% higher nominal rate.

Thus, both desirable properties hold. The frictionless limit equals the frictionless limit point. The model does not suffer the frictionless-limit paradoxes. The central point – a temporary negative inflation response to higher interest rates – holds in the frictionless limit. Price stickiness, like habits, Taylor responses, and the fiscal response, is useful for producing realistic impulse-response functions, but price stickiness is not necessary for the basic point.

Going in the other direction, we see that the reduction in inflation from an interest rate rise is *reduced* as prices become *more* sticky. As price stickiness becomes absolute, $\kappa = 0$, the disinflationary effect vanishes entirely.

Like many other results, this one violates common intuition. But since the mechanism for a negative sign is utterly different from ISLM intuition, we should not expect that intuition to guide us. Price stickiness is not important for generating the sign, so there is no reason to expect more price stickiness to generate a greater effect.

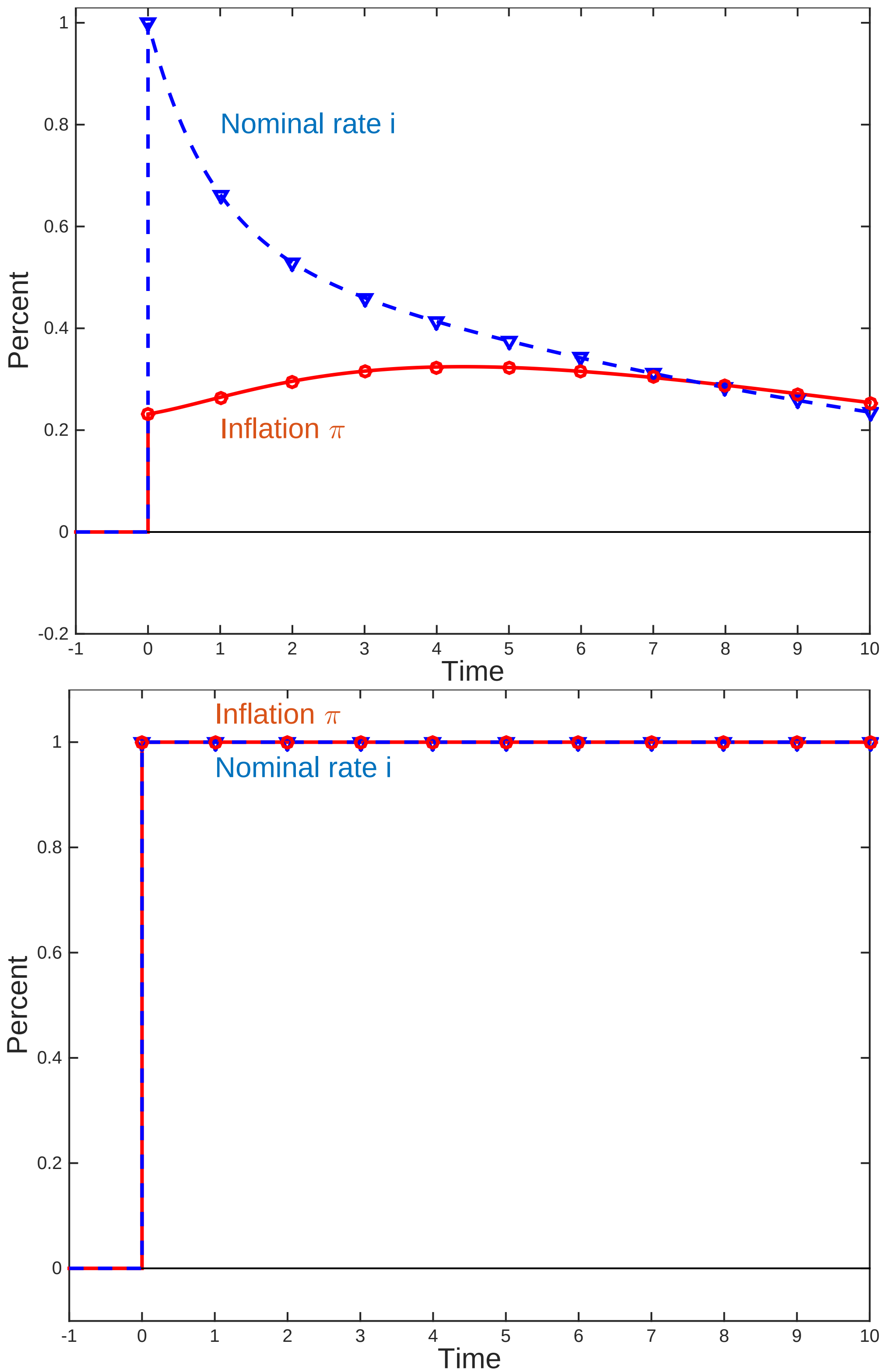

Figure 6: Top: Responses of the Sims model (top) and the simple model (bottom) to an unexpected monetary policy shock, with short-term debt. (The inflation and nominal rate lines lie on top of each other in the bottom panel.)

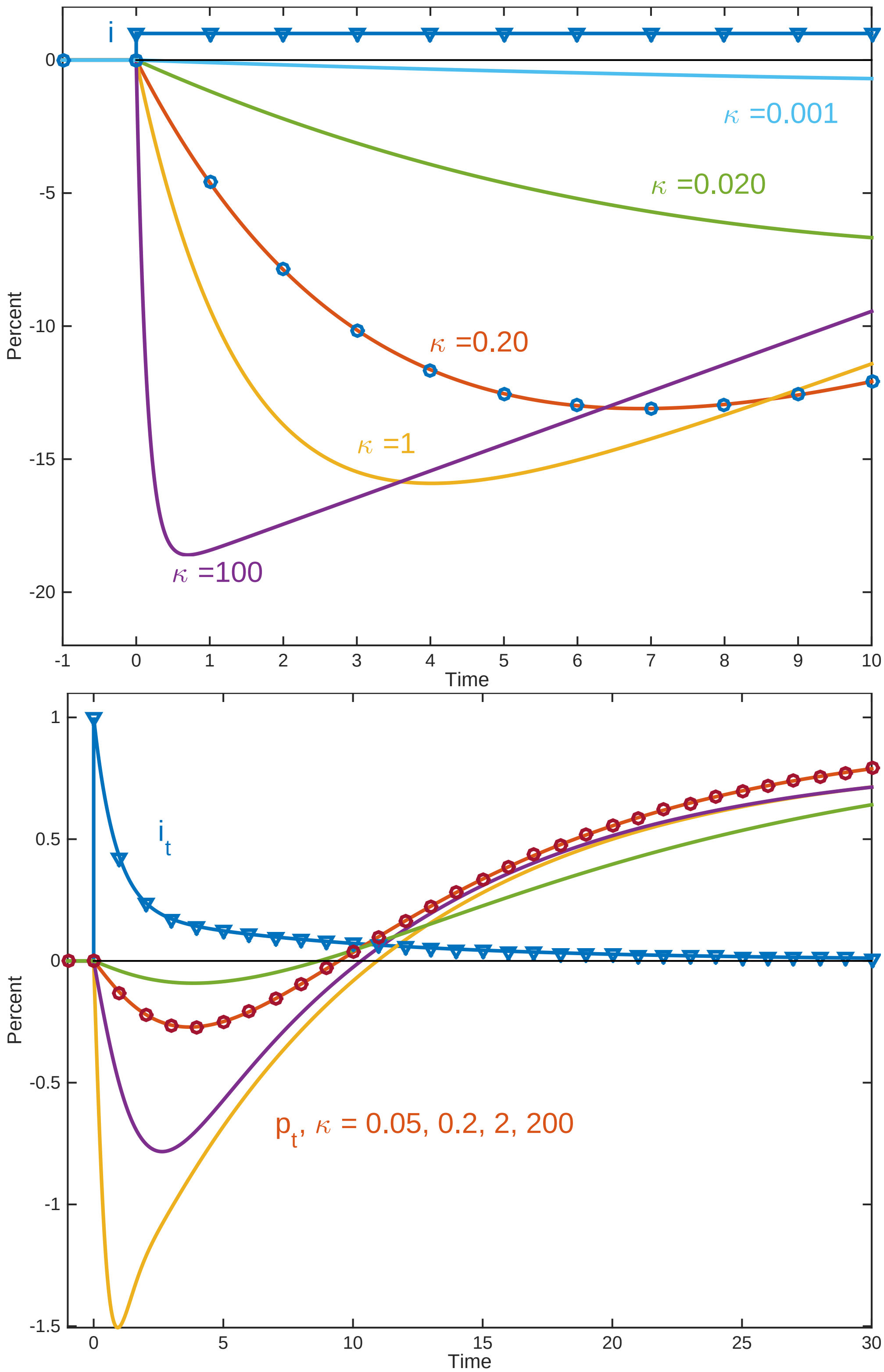

Figure 7: Response of the price level as price stickiness κ varies. Top: simple model. Bottom: full model. $\kappa = 0.2$ is the baseline value shown in other figures.

Equation (37),

$$\frac{Q_t B_t}{P_t} = E_t \int_{\tau=t}^{\infty} e^{-\int_{v=t}^{\tau}(i_v-\pi_v)dv} s_\tau d\tau, \tag{55}$$

helps to understand why stickier prices lead to a smaller effect. As prices become stickier, inflation π_v moves less. Higher nominal interest rates mean higher real interest rates. Higher real interest rates then discount future surpluses more heavily. Now the present value of surpluses on the right hand side declines just as the nominal bond price Q_t on the left hand side declines. In the limit that prices are perfectly sticky, inflation does not change at all, the nominal and real rates are the same, so the real present value of the debt on the right hand side falls exactly by the same amount as the nominal present value of the debt on the left hand side, and no disinflationary force remains.

The bottom panel of figure 7 shows the effect of greater or lesser price stickiness in the full Sims model. The general pattern is the same. Less price stickiness leads to a limit in which the price level jumps down by about 1.5%, and then inflation mirrors the nominal interest rate. As price stickiness increases, the pattern is the same, but the magnitude of the disinflation *decreases*. Again, this is the opposite of the usual sign, but again price stickiness is not generating the disinflation, but merely smoothing it out.

5. Conclusions

The fiscal theory of the price level can provide a cogent description of monetary policy, uniting the inflationary and disinflationary effects of interest rate policies, open market operations, forward guidance, and quantitative easing. In the presence of long-term debt, the simplest fiscal theory model produces a temporary inflation decline as a result of an interest rate rise. Sims (2011) shows how to extend this structure to include price stickiness and the absence of price-level jumps, and a form of habit persistence preferences, that generate reasonable impulse response functions broadly similar to those of standard active-money/passive-fiscal new-Keynesian models.

Though it seems promising for matching experience, however, the resulting model is quite different from standard monetary intuition. The decline in inflation is stronger for *less* price stickiness. It only occurs for unexpected interest rate rises. It really has nothing to do with the current interest rate; expectations of future interest rates reflected in the yield curve are the center of the mechanism. The model does not capture standard intuition that high nominal rates raise real rates, which reduce aggregate demand and through a Phillips curve lower inflation. The fact that it works at all in a completely frictionless model is proof of that fact. Inflation comes entirely from a "wealth effect" – as people try to hold more or fewer government bonds they lower or raise their demands for goods and services. Permanent interest rate rises eventually raise inflation.

Furthermore, the fiscal foundations of a fiscal theory of monetary policy remain are important. The disinflationary effect only happens in the presence of long-term government debt, and is driven by the decline in market value of that debt. The endogenous reaction of surpluses matters a lot for the sign and magnitude of the effects of monetary policy. Fiscal policy changes contemporaneous with a monetary policy shock will contaminate empirical measurement of the effects of that shock.

All this remains a foundation. Sims' model is a start, but one needs to develop a fully stochastic model. One needs to compare impulse-responses and correlations to data, in the style of standard active-money/passive-fiscal models. The full historical experience of the 1970s and 1980s remains unexplored. Stepping on a rake is a good story for the 1970s, but not an econometric test. Just how inflation declined in the 1980s remains a puzzle to rational expectations models, including the standard active-money models. The suggestion here that the 1980s represent a joint monetary fiscal stabilization remains to be fleshed out. Optimal policy, which has to trade off distorting taxation for inflation, remains an interesting fiscal-theory question.

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Appendix to "Stepping on a rake: The fiscal theory of monetary policy."

This Appendix derives, decodes, and solves the Sims (2011) model. Sims does not present either a derivation or a solution, so this appendix fills that gap.

1. Model statement

The model as presented by Sims (2011), starting with his equation (15) on p. 52, is

$$\dot{(i)}_t = -\gamma(i_t - \bar{r}) + \phi_\pi \dot{p}_t + \phi_c \dot{c}_t + \varepsilon_{mt} \tag{56}$$

$$i_t = r_t + \dot{p}_t \quad (*) \tag{57}$$

$$r_t = -\frac{\dot{\lambda}_t}{\lambda_t} + \bar{r} \quad (*) \tag{58}$$

$$\dot{b}_t = -b_t \dot{p}_t - b_t \frac{\dot{y}_t}{y_t} + y_t b_t - s_t \tag{59}$$

$$i_t = y_t - \frac{\dot{y}_t}{y_t} \quad (*) \tag{60}$$

$$\ddot{p}_t = \rho \dot{p}_t - \kappa c_t \ (*) \tag{61}$$

$$\dot{s}_t = \omega \dot{c}_t + \varepsilon_{st} \tag{62}$$

$$\lambda_t = e^{-\sigma c_t} + \psi \left[\ddot{c}_t - \bar{r}\dot{c}_t \right] e^{-c_t} (*). \tag{63}$$

Here, i_t is the nominal interest rate (I do not need the cumbersome notation $\dot{(i)}_t = di/dt$ below), r is the real interest rate with steady state and consumer discount rate $\bar{r}$, $p = \log(P_t)$ is the log price level, c is consumption which equals output, λ is the marginal utility of consumption, b is the real market value of government debt, which consists of nominal perpetuities, y is the perpetuity yield ($1/y$ is the price), and s_t is the real primary surplus. The last equation differs from Sims' by a typo in Sims' paper, that does not affect the calculations.

Here I translate from Sims' notation to a more standard notation. Sims uses r instead of i, ρ instead of r, $\bar{\tau} + \tau_t$ instead of s_t, a instead of y. Sims also uses parameters θ instead of ϕ_π, ϕ instead of ϕ_c, β instead of ρ, δ instead of κ.

Our goal is to calculate responses of this model to unexpected jumps in the shocks, ε_{mt} and ε_{st}.

2. The model derived and restated

We need to state the underlying model and derive these equilibrium conditions. We then need to linearize the model, transform the model to $dx/dt = Ax_t + \varepsilon_t$ form, and then we can solve it as a first order linear differential equation. We need to understand jumps and "forward-looking" equations marked by $(*)$. The impulse response functions (Sims' figure 3 and 4, my figure 2) feature jumps in all variables except p_t and c_t. So, we have to understand how variables respond to the ε_{mt} or ε_{st} jumps, and what the rules about jumps are.

My first step is to derive the model, and write the equations as

$$di_t = [-\gamma(i_t - \bar{r}) + \phi_\pi \pi_t + \phi_c \dot{c}_t]\, dt + d\varepsilon_{mt} \tag{64}$$

$$d\pi_t = (\rho\pi_t - \kappa c_t)\, dt \ + d\delta_{\pi t} \tag{65}$$

$$dy_t = y_t\,(y_t - i_t)\, dt + d\delta_{yt} \tag{66}$$

$$ds_t = \omega \dot{c}_t dt + d\varepsilon_{st} \tag{67}$$

$$db_t = [b_t(i_t - \pi_t) - s_t]\, dt - \frac{b_t}{y_t} d\delta_{yt} \tag{68}$$

$$d\lambda_t = -\lambda_t\,(r_t - \bar{r}\ )\, dt + d\delta_{\lambda t} \tag{69}$$

$$dc_t = \dot{c}_t dt \tag{70}$$

$$d\dot{c}_t = \left[\frac{\lambda_t}{\psi} e^{c_t} - \frac{1}{\psi} e^{c_t} e^{-\sigma c_t} + \bar{r}\dot{c}_t\right] dt + d\delta_{\dot{c}t} \tag{71}$$

with two side conditions,

$$dp_t = \pi_t dt \tag{72}$$

$$\pi_t = i_t - r_t. \tag{73}$$

Sims' starred (*) equations (57), (58), (60), (61) and (63) are "forward-looking." They specify the expectation of a forward-looking differential that may jump, or in a more general model, may have a diffusion component. I express the same idea more conventionally with differential notation dx_t and expectational shocks $d\delta_{xt}$ with $E_t(d\delta_{xt}) = 0$. Like Sims, however, I only study perfect-foresight solutions with a single probability-zero jump at time zero. This restriction also simplifies many of the model's equations. In a fully stochastic model, these equations would need additional terms such as risk premiums.

To understand these issues, consider the simplest discrete-time new-Keynesian model consisting only of a Fisher equation $i_t = E_t\pi_{t+1}$ and a Taylor rule $i_t = \phi_\pi \pi_t + w_t$ with a serially correlated disturbance $w_{t+1} = \theta w_t + \varepsilon_{t+1}$. Eliminating i_t, equilibria follow

$$E_t\pi_{t+1} - \pi_t = (\phi_\pi - 1)\pi_t + w_t.$$

This equation is "forward-looking" like the starred equations in Sims' model. Since it only ties down expected inflation, not actual inflation, it admits multiple equilibria: Any path

$$\pi_{t+1} - \pi_t = (\phi_\pi - 1)\pi_t + w_t + \delta_{t+1}$$

with $E_t(\delta_{t+1}) = 0$ is an equilibrium. This δ_{t+1} is the discrete-time equivalent of the expectational jumps $d\delta_{xt}$ above.

The conventional new-Keynesian model specifies $\phi_\pi > 1$ so the dynamics are explosive. Then the unique locally bounded equilibrium is

$$\pi_t = -E_t \sum_{j=0}^{\infty} \phi_\pi^{-(j+1)} w_{t+j} = -\frac{1}{\phi_\pi - \theta} w_t.$$

This solution amounts to a unique choice of the expectational shock δ_t in terms of the structural monetary policy shock ε_t,

$$\delta_t = -\frac{1}{\phi_\pi - \theta}\varepsilon_t. \tag{74}$$

This general principle applies to Sims' model: In order to produce a unique equilibrium, for each "forward-looking" or expectational difference equation, i.e. for each expectational shock δ, we need one explosive eigenvalue and one variable that can jump, in order to determine one expectational error δ in terms of structural shocks ε.

Now we can derive each equation in turn.

Policy Rule. Equation (64),

$$di_t = [-\gamma(i_t - \bar{r}) + \phi_\pi \pi_t + \phi_c \dot{c}_t]\, dt + d\varepsilon_{mt},$$

and its equivalent (56), are the monetary policy rule. The nominal interest rate mean-reverts, and rises with inflation and consumption growth. The rule allows a jump $d\varepsilon_{mt}$, which is the monetary policy shock. By examining the steady state $di_t = 0$, you can see that $\phi_\pi > \gamma$ is the Taylor rule "active" region in which interest rates respond more than one-for-one to inflation, and $\phi_\pi < \gamma$ is the "passive money" region.

All the variables on the right hand side of the monetary policy rule can jump, so in principle one should specify whether di_t is driven by pre-jump or by post-jump values (right or left limits). But since these variables are all multiplied by dt, and $d\varepsilon_{mt}$ is a jump, it does not matter which one specifies. For the same reason, when there is a jump $d\varepsilon_{mt}$, i_t jumps by the same amount $di_t = d\varepsilon_{mt}$, even though the other variables on the right hand side also respond to the jump.

Phillips. Equation (65),

$$d\pi_t = (\rho\pi_t - \kappa c_t)\, dt \; + d\delta_{\pi t}$$

defines the forward-looking continuous-time Phillips curve. It is the analogue of the discrete-time curve

$$\pi_t = \beta E_t \pi_{t+1} + \kappa c_t$$

which can be written in the form

$$E_t\pi_{t+1} - \pi_t = \left(\frac{1-\beta}{\beta}\right)\pi_t - \frac{\kappa}{\alpha}c_t$$

from which (65) follows immediately. Since (65) is a "forward-looking" equation, describing $E_t d\pi_t$, it includes a jump to expectations $d\delta_{\pi t}$.

This Phillips curve allows a jump in the *inflation rate* but not in the *price level.* The Phillips curve comes from a Calvo fairy who allows a fraction (constant) $\times$ dt of firms to change prices at any date. Since no mass of firms can change prices in an instant, prices cannot jump. This fact is reflected in (72), $dp_t = \pi_t dt$, which has no jump term.

Sims' Phillips curve, equation (61), also specifies a second derivative. The solution method is a first-order differential equation, so when there are second derivatives, here and in the consumer first order condition (63), I add an extra state variable to write the system in terms of first derivatives only, and keep track of the definition of the first derivative in (72) and (70).

Fisher. Equation (73),

$$\pi_t = i_t - r_t$$

and Sims' version (57), is the Fisher equation defining the real rate of interest. Sims introduces a structural shock ε_{it}, but he does not use it, so I leave it out.

This version of the Fisher equation stems from the first order conditions for intertemporal maximization. It adds no risk premium and imposes the absence of price-level jumps stemming from the Phillips curve, as follows.

The generic asset pricing equation for a security whose real value process is v_t and hence return is $dR_t = dv_t/v_t$ is

$$E_t dR_t = r_t dt - E_t\left(\frac{d\lambda_t}{\lambda_t}dR_t\right)$$

where λ_t is the marginal utility of consumption. However, the assumption that jumps are probability zero means that the second, risk aversion, terms disappear from asset pricing formulas, leaving us $E_t(dR_t) = r_t dt$.

In turn, the real return dR_t on the nominal riskfree asset, whose nominal value process is $dV_t/V_t = i_t dt$ and real value is $v_t = V_t/P_t$, is

$$dR_t = \frac{dv_t}{v_t} = i_t dt + \frac{d(1/P_t)}{(1/P_t)}.$$

Therefore, the continuous-time risk-neutral Fisher relation is

$$i_t dt + E_t\left(\frac{d(1/P_t)}{(1/P_t)}\right) = r_t dt.$$

This Fisher equation is forward-looking, and allows for price-level jumps, so in general it should have an expectational error. However, the Phillips curve does not allow for price-level jumps. Therefore we can write $d(1/P_t)/(1/P_t) = -dp_t = \pi_t dt$, leaving only (73).

Each of the inflation rate π_t, the nominal interest rate i_t and the real interest rate r_t can jump. This Fisher equation (73) means they must jump together, however, an example of a tie between jumps in variables.

Term Structure. Equation (66) and Sims' version (60) are the term structure relation between long and short rates. The perpetuity has nominal yield y_t, nominal price $1/y_t$ and pays a constant coupon $1dt$. Thus, the condition that the expected nominal perpetuity return should equal the riskfree nominal rate (there are no price-level jumps and no risk premiums) is

$$i_t dt = \frac{1dt + E_t d\left(1/y_t\right)}{1/y_t} \approx y_t dt - E_t \frac{dy_t}{y_t}.$$

There are jumps in y_t, so the second equality is a linearization. The next step will be to linearize the model anyway. However, if one wishes to extend Sims' model by solving the nonlinear version, or by including nonzero shock probability and hence risk premiums, one should keep the nonlinear version. Rearranging, we have equation (66),

$$dy_t = y_t\left(y_t - i_t\right) dt + d\delta_{yt}.$$

Surplus. Equation (67),

$$ds_t = \omega \dot{c}_t dt + d\varepsilon_{st},$$

and Sims' version (62) describe a primary surplus that rises and falls with consumption growth, with surpluses in booms and deficits in recessions. The surplus can also jump, so we can plot the economy's response to fiscal shocks.

Debt. Equation (68),

$$db_t = \left[b_t(i_t - \pi_t) - s_t\right] dt - \frac{b_t}{y_t} d\delta_{yt}$$

tracks the evolution of the real market value of debt b. With probability zero jumps, the expected return on the perpetuity is the same as the expected return on short-term debt, $i_t - \pi_t$, hence the first term. Jumps to the perpetuity yield y induce jumps in the market value of the debt through the last term.

By definition, $b_t \equiv B_t/(y_t P_t)$ is the real market value of government debt, where B_t is the number of perpetuities outstanding. Start from the flow condition that the government must sell new perpetuities at price $1/y_t$ to cover the difference between coupon payments $\$1 \times B_t$ and primary surpluses s_t,

$$\frac{1}{y_t P_t} dB_t = \frac{B_t}{P_t} dt - s_t dt. \tag{75}$$

B_t does not jump. Now note

$$db_t = d\left(\frac{B_t}{y_t P_t}\right) = \frac{1}{y_t P_t} dB_t + b_t \frac{d(1/y_t)}{1/y_t} - b_t dp_t,$$

Here I use the fact that there are no price-level jumps. Substituting into (75), with $\pi_t dt = dp_t$, and solving for db_t, we obtain

$$db_t = [(y_t - \pi_t) b_t - s_t] \, dt + b_t \frac{d(1/y_t)}{1/y_t}.$$

The market value of debt b_t can jump, because the bond price can jump. However, this is an ex-post equation, restricting how any jump in db_t is induced by the jump in bond prices dy_t. It does not just describe the expected change $E_t(db_t)$, so it does not require an expectational error or an extra explosive eigenvalue.

To connect the jump in debt to the jump in bond prices, I use the same linearization of bond prices,

$$\frac{d(1/y_t)}{1/y_t} \approx -\frac{dy_t}{y_t},$$

giving Sims' version (59),

$$db_t = [b_t(y_t - \pi_t) - s_t] \, dt - \frac{b_t}{y_t} dy_t.$$

I substitute for dy_t from (66), which leads to (68).

Consumption. Equations (69)-(71), and Sims' version (58) and (63), describe marginal utility with a "habit" term that values a smooth consumption path. The utility function adds a penalty for the derivative of log consumption growth,

$$U = E \int_{t=0}^{\infty} e^{-\bar{r}t} \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{1}{2} \psi \left(\frac{1}{C} \frac{dC}{dt} \right)^2 \right] dt.$$

To derive marginal utility, set this up as a Hamiltonian with a constraint that wealth grows at the real interest rate

$$\dot{W}_t = r_t W_t - C_t.$$

The state variables are $x_t = [C_t \ W_t]$ and the control variable is $u_t = dC_t/dt$. The current value Hamiltonian is then

$$H \;=\; \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{1}{2}\psi\left(\frac{1}{C_t}\frac{dC_t}{dt}\right)^2 + \lambda_t\left(r_t W_t - C_t\right) + \gamma_t \frac{dC_t}{dt}.$$

The first order conditions are

$$\frac{\partial H}{\partial u} = -\psi \frac{1}{C_t^2}\frac{dC_t}{dt} + \gamma_t = 0 \tag{76}$$

$$\frac{\partial H}{\partial C} = C_t^{-\sigma} + \psi\frac{1}{C_t^3}\left(\frac{dC_t}{dt}\right)^2 - \lambda_t = -\dot{\gamma}_t + \bar{r}\gamma_t \tag{77}$$

$$\frac{\partial H}{\partial W} = \lambda r_t = -\dot{\lambda}_t + \bar{r}\lambda_t. \tag{78}$$

From (78), we have Sims' expression (58),

$$r_t = -\frac{\dot{\lambda}_t}{\lambda_t} + \bar{r}. \tag{79}$$

This is a forward-looking expectational equation in which marginal utility λ can jump. I add an expectational shock to produce (69),

$$d\lambda_t = -\lambda_t\left(r_t - \bar{r}\ \right)dt + d\delta_{\lambda t}.$$

Differentiating (76), and dropping t subscripts,

$$\dot{\gamma} = -2\psi\frac{1}{C^3}\left(\frac{dC}{dt}\right)^2 + \psi\frac{1}{C^2}\frac{d^2C}{dt^2}. \tag{80}$$

Substituting (80) and (76) into (77),

$$\lambda = C_t^{-\sigma} - \psi\left[\frac{1}{C^2}\left(\frac{dC}{dt}\right)^2 - \frac{1}{C}\frac{d^2C}{dt^2} + \bar{r}\frac{1}{C}\frac{dC}{dt}\right]\frac{1}{C}. \tag{81}$$

Note with $c = \log(C)$,

$$\begin{aligned} \frac{dc}{dt} &= \frac{1}{C}\frac{dC}{dt} \\ \left(\frac{dc}{dt}\right)^2 &= \frac{1}{C^2}\left(\frac{dC}{dt}\right)^2 \\ \frac{d^2c}{dt^2} + \left(\frac{dc}{dt}\right)^2 &= \frac{1}{C}\frac{d^2C}{dt^2}. \end{aligned}$$

Substituting in to (81),

$$\lambda = C_t^{-\sigma} - \psi \left[\left(\frac{dc}{dt} \right)^2 - \frac{d^2c}{dt^2} - \left(\frac{dc}{dt} \right)^2 + \bar{r}\frac{dc}{dt} \right] \frac{1}{C}$$

$$\lambda = C_t^{-\sigma} - \psi \left[- \frac{d^2c}{dt^2} + \bar{r}\frac{dc}{dt} \right] \frac{1}{C}.$$

This gives us equation (63),

$$\lambda = e^{-\sigma c} + \psi \left[\ddot{c} - \bar{r}\dot{c} \right] e^{-c}. \tag{82}$$

Sims gives the corresponding equation (his equation (22)) as

$$\lambda = e^{-\sigma c} + \psi \left[\ddot{c} - \dot{c}^2 \right] e^{-c} \tag{83}$$

The presence of $\dot{c}^2$ in place of $\bar{r}\dot{c}$ is a typo, confirmed by Sims. I verify that the typo does not affect Sims' calculations.

The penalty on the second derivative of log consumption means that consumption cannot jump. Therefore, as with inflation, I introduce a state variable $\dot{c}_t$ of the first derivative of consumption, adding (70), $dc_t = \dot{c}_t dt$. I then specify the second-order differential equation (82) containing $\ddot{c}$, $\dot{c}$, and c as a paired first-order differential equation consisting of (70) and (71),

$$d\dot{c}_t = \left[\frac{\lambda_t}{\psi} e^{c_t} - \frac{1}{\psi} e^{c_t} e^{-\sigma c_t} + \bar{r}\dot{c}_t \right] dt + d\delta_{\dot{c}t}.$$

The first derivative of consumption can jump, so (82) implies a a forward-looking expectational equation in $E_t \left[d\dot{c}_t \right]$. I add the corresponding expectational shock $d\delta_{\dot{c}t}$.

3. Linearization

The next step is to linearize the model (64)-(73) around the steady state. The steady state occurs where all time derivatives are zero, so the left hand sides of (64)-(72) are all equal to zero and all shocks are zero. I use bars, $\bar{x}$ to denote steady state values. Solving, we find that all rates of return are equal, $\bar{i} = \bar{r} = \bar{y}$. Taxes pay for the coupons on debt, $\bar{y}\bar{b} = \bar{s}$. The Phillips curve (65) means $\bar{c} = 0$, and then the marginal value of wealth is one, $\bar{\lambda} = 1$.

Linearizing around this steady state, and using tilde notation for differences from the steady state for variables $i,\ y,\ r,\ s,\ b,\ \lambda$ that are not zero at that state, $\tilde{x}_t \equiv x_t - \bar{x}_t$ the linearized version of (64)-(71) is

$$d\tilde{i}_t = \left[-\gamma\tilde{i}_t + \phi_\pi \pi_t + \phi_c \dot{c}_t\right] dt + d\varepsilon_{mt} \tag{84}$$

$$d\pi_t = \left(\rho\pi_t - \kappa c_t\right) dt \ + d\delta_{\pi t} \tag{85}$$

$$d\tilde{y}_t = \bar{r}(\tilde{y}_t - \tilde{i}_t)dt + d\delta_{yt} \tag{86}$$

$$d\tilde{s}_t = \omega \dot{c}_t dt + d\varepsilon_{st} \tag{87}$$

$$d\tilde{b}_t = \left[\bar{b}\left(\tilde{i}_t - \pi_t\right) + \bar{r}\tilde{b}_t - \tilde{s}_t\right] dt - \frac{\bar{b}}{\bar{r}} d\delta_{yt} \tag{88}$$

$$d\tilde{\lambda}_t = -\left(\tilde{i}_t - \pi_t\right) dt + d\delta_{\lambda t} \tag{89}$$

$$dc_t = \dot{c}_t dt \tag{90}$$

$$d\dot{c}_t = \left[\frac{1}{\psi}\tilde{\lambda}_t + \frac{\sigma}{\psi} c_t + \bar{r}\dot{c}_t\right] dt + d\delta_{\dot{c}_t}. \tag{91}$$

Here, I used the linearized (73), $\tilde{r}_t = \pi_t - \tilde{i}_t$ to eliminate the real interest rate $\tilde{r}_t$. The price level p_t does not enter the model, so we do not need (72). We can just add $d\tilde{p}_t = \pi_t dt$ to calculate the price level when needed.

The linearization of (68) gives

$$d\tilde{b}_t = \left[\bar{r}\tilde{b}_t + \bar{b}\left(\tilde{i}_t - \pi_t\right) - \tilde{s}_t\right] dt - \left[\frac{\bar{b}}{\bar{y}} + \frac{\tilde{b}_t}{\bar{y}} - \frac{\bar{b}}{\bar{y}^2}\tilde{y}_t\right] d\delta_{yt}. \tag{92}$$

However, the impulse response function takes place when variables are at steady states, so I eliminate the state-dependent shock response in (92) and simplify to (88).

4. Solution

Expressing the model in matrix notation, and reordering the equations with the structural shocks first,

$$d\begin{bmatrix} \tilde{i}_t \\ \tilde{s}_t \\ \pi_t \\ \tilde{y}_t \\ \tilde{b}_t \\ \tilde{\lambda}_t \\ \dot{c}_t \\ c_t \end{bmatrix} = \begin{bmatrix} -\gamma & 0 & \phi_\pi & 0 & 0 & 0 & \phi_c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & -\kappa \\ -\bar{r} & 0 & 0 & \bar{r} & 0 & 0 & 0 & 0 \\ \bar{b} & -1 & -\bar{b} & 0 & \bar{r} & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\psi & \bar{r} & \sigma/\psi \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} dt + \begin{bmatrix} d\varepsilon_{mt} \\ d\varepsilon_{st} \\ d\delta_{\pi t} \\ d\delta_{yt} \\ -(\bar{b}/\bar{r})\ d\delta_{yt} \\ d\delta_{\lambda t} \\ d\delta_{\dot{c}t} \\ 0 \end{bmatrix}$$

$$dx_t = Ax_t dt + d\varepsilon_t.$$

I solve the differential equation, and then use the shocks and jumps to set up a set of initial conditions x_0. Without the shock term, we have

$$\frac{dx}{dt} = Ax_t = Q\Lambda Q^{-1}x_t$$
$$\frac{dQ^{-1}x_t}{dt} = \Lambda Q^{-1}x_t$$

or

$$\frac{dz_t}{dt} = \Lambda z_t$$

$$z_t = Q^{-1}x_t; x_t = Qz_t$$

where Q is a matrix of eigenvectors, and Λ a diagonal matrix of eigenvalues $\{\nu_i\}$ of A. To rule out explosions, we must have $z_{it} = 0$ for each element i of z_t corresponding to an explosive eigenvalue $\nu_i \geq 0$. Since the z are linear combinations of the x, this condition imposes a set of linear restrictions on x_t and x_0 in particular,

$$\left[Q^{-1}\right]_{i,:} x_0 = \left[Q^{-1}\right]_{i,:} d\varepsilon_0 = 0.$$

where $\left[Q^{-1}\right]_{i,:}$ denotes the ith row of Q^{-1}. This is a set of linear restrictions on the shocks $d\varepsilon_0$. In turn, this set of linear restrictions allows us to determine the expectational errors $d\delta_0$ as a function of the underlying shocks $d\varepsilon_{m0}$, $d\varepsilon_{s0}$ just as in the simple case of equation (74). This system has four undefined expectational errors, so we need exactly four non-negative eigenvalues for the model to be uniquely determined, which is the case.

Here the active-fiscal passive-money assumption is important. With active fiscal policy, debt explodes for all but one value of the initial conditions $d\delta_0$. With active monetary policy, interest rates and inflation explode for all but one value of the initial conditions $d\delta_0$.

To find the instantaneous response to the shocks, then, we must solve

$$\begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \\ \left[Q^{-1}\right]_{4,:} \end{bmatrix}_{4\times 8} \begin{bmatrix} d\varepsilon_{mt} \\ d\varepsilon_{st} \\ d\delta_{\pi t} \\ d\delta_{yt} \\ -\bar{b}/\bar{r}\, d\delta_{yt} \\ d\delta_{\lambda t} \\ d\delta_{\dot{c}t} \\ 0 \end{bmatrix}_{8\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times 1} \tag{93}$$

for $d\delta_{\pi t}$, $d\delta_{yt}$, $d\delta_{\lambda t}$, $d\delta_{ct}$ where $1, 2, 3, 4$ denote the indices of the explosive ($\nu_i > 0$) eigenvalues. Break up the ε and δ parts of the shock vector to write

$$\begin{bmatrix} d\varepsilon_{mt} \\ d\varepsilon_{st} \\ d\delta_{\pi t} \\ d\delta_{yt} \\ -\bar{b}/\bar{r}\, d\delta_{yt} \\ d\delta_{\lambda t} \\ d\delta_{\dot{c}t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\bar{b}/\bar{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d\delta_{\pi t} \\ d\delta_{yt} \\ d\delta_{\lambda t} \\ d\delta_{ct} \end{bmatrix} + \begin{bmatrix} d\varepsilon_{mt} \\ d\varepsilon_{st} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{94}$$

Then, we can solve (93),

$$\begin{bmatrix} d\delta_{\pi t} \\ d\delta_{yt} \\ d\delta_{\lambda t} \\ d\delta_{ct} \end{bmatrix} = - \left(\begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \\ \left[Q^{-1}\right]_{4,:} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\bar{b}/\bar{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \\ \left[Q^{-1}\right]_{4,:} \end{bmatrix} \begin{bmatrix} d\varepsilon_{mt} \\ d\varepsilon_{st} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using (94) again, we now have the full jump shock vector $d\varepsilon_0$, and therefore the time-zero value x_0 of all variables.

It's easiest to solve the differential equation forward using the transformed z variables, $z_0 = Q^{-1}x_0$. Finally, the impulse-response function is given by $z_{jt} = e^{-\nu_j t} z_{0i}$; $x_t = Qz_t$.

5. The model without habits

To calculate the $\psi = 0$ limit point, in which consumption can jump, we have to solve it separately for that case, as $1/\psi$ terms show up in the regular model solution, and equations disappear.

In the model without habits, consumption can jump. So we have in place of (64)-(73),

$$dc_t = \frac{1}{\sigma}(i_t - \pi_t)dt + d\delta_{ct} \tag{95}$$

$$di_t = [-\gamma i_t + \phi_\pi \pi_t]\, dt + \phi_c dc_t + d\varepsilon_{mt} \tag{96}$$

$$ds_t = \omega dc_t + d\varepsilon_{st}. \tag{97}$$

The remaining equations are unchanged. For completeness, they are

$$d\pi_t = (\rho\pi_t - \kappa c_t)\, dt \ + d\delta_{\pi t} \tag{98}$$

$$dy_t = y_t\,(y_t - i_t)\, dt + d\delta_{yt} \tag{99}$$

$$db_t = [b_t(i_t - \pi_t) - s_t]\, dt - \frac{b_t}{y_t} d\delta_{yt} \tag{100}$$

$$\pi_t = i_t - r_t. \tag{101}$$

I solve this model and verify that the $\psi = 0$ limit of the full model approaches the solution of this model. One might worry that consumption can jump at $\psi = 0$ and cannot jump for any $\psi > 0$, no matter how small ψ. However, the fast hump-shaped responses smoothly approach a jump, as they do when we remove price stickiness.

In the paper, I present results of a model (48)-(53) that further simplifies with $\gamma = 0$, $\phi_\pi = 0$, $\phi_c = 0$, $\omega = 0$, retaining only price stickiness $\kappa < \infty$ and long-term debt.

For the $\psi = 0$ case, i.e. standard power utility, instead of (69) - (71), we have

$$d\lambda_t = -\lambda_t\,(r_t - \bar{r}\ )\, dt + d\delta_{\lambda t} \tag{102}$$

$$\lambda_t = e^{-\sigma c_t}. \tag{103}$$

We linearize to

$$d\tilde{\lambda}_t = -\tilde{r}_t dt + d\delta_{\lambda t} \tag{104}$$

$$\tilde{\lambda}_t = -\sigma c_t \tag{105}$$

We can eliminate $\tilde{\lambda}$, so we have

$$dc_t = \frac{1}{\sigma}\tilde{r}_t dt + d\delta_{ct} = \frac{1}{\sigma}(\tilde{i}_t - \pi_t)dt + d\delta_{ct}.$$

$\tilde{\lambda}$ does not appear elsewhere.

Next, we must adapt the other appearances of the state variable $\dot{c}_t$ in the original model, and the fact that the level of consumption may now jump. To allow a response of fiscal policy to consumption, in place of

$$ds_t = \omega \dot{c}_t dt + d\varepsilon_{st}$$

we have

$$ds_t = \omega dc_t + d\varepsilon_{st} = \frac{\omega}{\sigma}(i_t - \pi_t)dt + \omega d\delta_{ct} + d\varepsilon_{st}$$

When consumption jumps, so do taxes.

The monetary policy rule

$$d\tilde{i}_t = \left[-\gamma\tilde{i}_t + \phi_\pi \pi_t + \phi_c \dot{c}_t\right] dt + d\varepsilon_{mt}$$

becomes

$$d\tilde{i}_t = \left[-\gamma\tilde{i}_t + \phi_\pi \pi_t\right] dt + \phi_c dc_t + d\varepsilon_{mt}$$
$$d\tilde{i}_t = \left[-\gamma\tilde{i}_t + \phi_\pi \pi_t\right] dt + \phi_c \left[\frac{1}{\sigma}(\tilde{i}_t - \pi_t)dt + d\delta_{ct}\right] + d\varepsilon_{mt}$$
$$d\tilde{i}_t = \left\{\left(\frac{\phi_c}{\sigma} - \gamma\right)\tilde{i}_t + \left(\phi_\pi - \frac{\phi_c}{\sigma}\right)\pi_t\right\} dt + \phi_c d\delta_{ct} + d\varepsilon_{mt}$$

The system is then

$$d\begin{bmatrix} \tilde{i}_t \\ \tilde{s}_t \\ \pi_t \\ \tilde{y}_t \\ \tilde{b}_t \\ c_t \end{bmatrix} = \begin{bmatrix} \frac{\phi_c}{\sigma} - \gamma & 0 & \phi_\pi - \frac{\phi_c}{\sigma} & 0 & 0 & 0 \\ \omega/\sigma & 0 & -\omega/\sigma & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & -\kappa \\ -\bar{r} & 0 & 0 & \bar{r} & 0 & 0 \\ b & -1 & -b & 0 & \bar{r} & 0 \\ 1/\sigma & 0 & -1/\sigma & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{i}_t \\ s_t \\ \pi_t \\ \tilde{y}_t \\ \tilde{b}_t \\ c_t \end{bmatrix} dt + \begin{bmatrix} d\varepsilon_{mt} + \phi_c d\delta_{ct} \\ d\varepsilon_{st} + \omega d\delta_{ct} \\ d\delta_{\pi t} \\ d\delta_{yt} \\ -\bar{b}/\bar{r}\ d\delta_{yt} \\ d\delta_{ct} \end{bmatrix}.$$

With three undetermined shocks $d\delta_t$, we need three explosive eigenvalues. The shocks now solve

$$\begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \end{bmatrix}_{3\times 6} \begin{bmatrix} d\varepsilon_{mt} + \phi_c d\delta_{ct} \\ d\varepsilon_{st} + \omega d\delta_{ct} \\ d\delta_{\pi t} \\ d\delta_{yt} \\ -\bar{b}/\bar{r}\; d\delta_{yt} \\ d\delta_{ct} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $d\delta_{\pi t}$, $d\delta_{bt}$, $d\delta_{ct}$ given $d\epsilon_{mt}$, $d\epsilon_{st}$. The matrix carpentry:

$$\begin{bmatrix} d\varepsilon_{mt} + \phi_c d\delta_{ct} \\ d\varepsilon_{st} + \omega d\delta_{ct} \\ d\delta_{\pi t} \\ d\delta_{yt} \\ -\bar{b}/\bar{r} d\delta_{yt} \\ d\delta_{ct} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \phi_c \\ 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\bar{b}/\bar{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\delta_{\pi t} \\ d\delta_{yt} \\ d\delta_{ct} \end{bmatrix} + \begin{bmatrix} d\varepsilon_{mt} \\ d\varepsilon_{st} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{106}$$

$$\begin{bmatrix} d\delta_{\pi t} \\ d\delta_{yt} \\ d\delta_{ct} \end{bmatrix} = -\left(\begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \end{bmatrix} \begin{bmatrix} 0 & 0 & \phi_c \\ 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\bar{b}/\bar{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \end{bmatrix} \begin{bmatrix} d\varepsilon_{mt} \\ d\varepsilon_{st} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This calculation produces a response to the chosen $d\varepsilon_t$ shocks. The actual interest rate move $di_t = d\varepsilon_{mt} + \phi_c d\delta_{ct}$ is different when the policy rule responds to consumption growth; we no longer have $di_t = d\varepsilon_{mt}$.

6. Calculating the response to expected rate rises

When the monetary policy shock $d\varepsilon_{mt}$ is expected, all the expectational errors $d\delta_t$ are equal to zero. That makes solving the model a lot easier. I posit a single jump at time 0. The system is

$$dx_t = Ax_t dt + d\varepsilon_t$$

with

$$d\varepsilon_t = \begin{bmatrix} d\varepsilon_{mt} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'.$$

The bounded solutions are then:

$$\nu_i > 0:$$
$$z_{it} = -\left[\left[Q^{-1}\right]_{i,:} d\varepsilon_0\right] e^{\nu_i t}; t \le 0;$$
$$z_{it} = 0;\ t > 0$$

$$\nu_i < 0:$$
$$z_{it} = \left[\left[Q^{-1}\right]_{i,:} d\varepsilon_0\right] e^{\nu_i t}; t \ge 0;$$
$$z_{it} = 0;\ t < 0.$$

In words, each state variable z_{it} jumps by an amount $\left[Q^{-1}\right]_{i,:} d\varepsilon_0$ at time 0. The state variables corresponding to explosive eigenvalues trend down until they hit $-\left[Q^{-1}\right]_{i,:} d\varepsilon_0$ at time $t = 0$, then jump up to 0 at time $t = 0 + \Delta$. The state variables corresponding to stable eigenvalues are zero until time $t = 0$. They jump up to $\left[Q^{-1}\right]_{i,:} d\varepsilon_0$ at time $t = 0 + \Delta$, then decay exponentially.

7. Model with short-term debt

The maturity structure only matters to the db_t equation. To derive the db_t equation in the case of short-term debt, start with the definition that the real value of the debt is $b_t \equiv B_t/P_t$. Here B_t is the quantity of instantaneous, floating-rate debt. I do not divide by y_t as the price of such debt is always one.

Then,

$$db_t = \frac{dB_t}{P_t} + \frac{B_t}{P_t}\frac{d(1/P_t)}{1/P_t}.$$

The flow condition now states that interest must be paid from surpluses or new debt issues,

$$B_t i_t dt = P_t s_t dt + dB_t$$

$$b_t i_t dt = s_t dt + db_t - b_t \frac{d(1/P_t)}{1/P_t}$$

$$db_t = (i_t b_t - s_t)dt + b_t \frac{d(1/P_t)}{1/P_t}$$

The instantaneous value of short-term debt can only jump if there is a price-level jump. Sims' sticky-price model rules out such jumps, so the last term is

$$\frac{d(1/P_t)}{1/P_t} = -\pi_t dt.$$

With $i_t = r_t + \pi_t$ we then have

$$db_t = [b_t (i_t - \pi_t) - s_t] \, dt$$

whereas with long-term debt it was (68),

$$db_t = [b_t (i_t - \pi_t) - s_t] \, dt - \frac{b_t}{y_t} d\delta_{yt}$$

The only difference between short and long-term debt in this model is that the instantaneous response of the value of debt to a yield shock is absent for short-term debt.