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STEPPING ON A RAKE:
REPLICATION AND DIAGNOSIS

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ABSTRACT

This paper replicates Sims (2011) "stepping on a rake" model. It derives the model, shows how to solve it, offers some extensions, and boils the paper down to its central ingredient. Sims' article is important: it is a simple modern economic model that produces a temporary decline in inflation when the central bank persistently raises interest rates. Inflation then rises. The model's essential feature is long term debt. When interest rates increase, the nominal market value of long-term government debt falls. If fiscal surpluses are unchanged, the price level must fall so that the real value of government debt matches the unchanged real present value of surpluses. The model offers a unified treatment of interest rate targets, quantitative easing and forward guidance that works even in a frictionless setup.

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1 Introduction

This paper replicates Sims (2011), derives the model, shows how to solve it, offers some extensions, and boils the paper down to its central ingredient.

Sims' article is important: it is a simple modern economic model that produces a temporary decline in inflation when the central bank persistently raises interest rates. Cochrane (2016) surveys the literature and finds that new-Keynesian rational expectations models predict an *increase* in inflation, both in the short and long run, in response to a persistent rise in interest rates. It also avoids the troublesome new-Keynesian assumption that the central bank uses a never-observed threat of instability to produce determinacy, relying on the fiscal theory of the price level instead. In old Keynesian and monetarist models, a rise in interest rates sends inflation on an unstable downward spiral, so both the short-run and long-run inflation effect is negative. However, such models rely crucially on irrational adaptive expectations, and they are inconsistent with the observed stability of inflation in the U.S. and Europe's decade, and Japan's two decades, at the zero bound.

Sims' paper is also methodologically useful. It adopts a simple, tractable continuous-time specification with sticky prices, which is a convenient framework for further exploration.

However, Sims does not state the model, he does not derive the equilibrium conditions, and he does not explain how to compute impulse-response functions. This paper fills that gap, and confirms Sims' results.

Sims also does not explore what the minimum set of ingredients is to deliver a temporary negative sign. He's after a bigger result, namely an impulse response function that delivers an entire path consistent with VAR estimates, not just the basic sign. He also does not offer much economic intuition for that sign.

I first explain the central, fiscal-theoretic, story for the temporary negative inflation effect in a frictionless models. In words, higher nominal interest rates lower the nominal value of long-term debt. If we define "monetary policy" as a change in nominal interest rates that does not affect primary surpluses, then such a change does not change the real value of primary surpluses. For a lower nominal value of nominal debt to cor-

respond to an unchanged real value of primary surpluses, the price level must fall. The mechanism is simple "aggregate demand." People want more government bonds and fewer goods and services. This basic mechanism survives when the real interest rate variation of the full sticky-price model is included, and changes the real present value of unchanging surpluses.

I then derive Sims' model and explain how to solve it and calculate impulse responses. I verify Sims' calculations. I verify that the complications of Sims' model, habits, Taylor rule, and procyclical fiscal policy, do not matter for that central result. They are useful for producing a realistic impulse-response function, and thus useful ingredients for applied modeling here, as in the standard new-Keynesian tradition such as Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2003). For example, habits give a hump shape response. On the other hand, I show that long-term debt and an unexpected shock are crucial to the negative sign. An expected monetary policy tightening produces a rise in inflation throughout. Similarly, Sims' model with short-term debt produces an instant rise in inflation. The negative inflation response is centrally a fiscal-theoretic phenomenon.

That is, I think, the most important lesson going forward. The response to "monetary" policy – a change in interest rate target – depends crucially on the associated fiscal policy – the maturity structure of outstanding debt, and how people expect the Treasury will adjust surpluses in reaction to economic events and monetary policy actions. Furthermore, Sims' basic mechanism works in an entirely frictionless model, so has nothing to do with monetary or pricing distortions. Even the mechanism for targeting interest rates requires no monetary or pricing distortions.

Sims' negative sign does not justify conventional policy conclusions, such as the desirability of the Taylor principle, or raising interest rates to *permanently* lower inflation as in the 1980s. Since the short-term negative response of inflation only occurs for unexpected interest rate rises, it does not work for systematic policy, the $\phi_\pi \pi$ in the Taylor rule $i = \phi_\pi \pi + \pi_y y$. And the stepping on a rake mechanism says that as raising rates eventually raises inflation, as happened in the 1970s, so raising rates does not eventually lower inflation as occurred in the 1980s. In this framework, that permanent disinflation requires fiscal policy cooperation.

2 A frictionless rake

Here is how a totally frictionless model delivers the result that a rise in interest rates first causes inflation to fall, and then to rise.

Use risk-neutral valuation at a constant real factor $\beta = 1/(1+r)$. Then interest rates i_t and inflation follow

$$\frac{1}{1+i_t} = \frac{1}{1+r} E_t \frac{P_t}{P_{t+1}}$$

where P_t denotes the price level. A rise in nominal interest rates implies an immediate rise in expected inflation. But the price level can still jump down when interest rates increase unexpectedly.

At the beginning of period t, the government has outstanding $B_{t-1}^{(t+j)}$ discount bonds of maturity j, each of which pays \$1 at time $t+j$. Then, the government debt valuation equation stating that the real value of nominal government debt equals the real present value of primary surpluses is

$$\frac{\sum_{j=0}^{\infty} Q_t^{(t+j)} B_{t-1}^{(t+j)}}{P_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}. \tag{1}$$

Here, s_{t+j} denotes the real primary surplus. The symbol $Q_t^{(t+j)}$ denotes the time t nominal price of a j period discount bond, which pays \$1 at time $t+j$. $Q_t^{(t)} = 1$ for the maturing bond $j = 0$. For $j > 0$, the bond price is, in this risk neutral constant real rate world,

$$Q_t^{(t+j)} = E_t \left(\prod_{k=1}^{j} \frac{1}{1+i_{t+k-1}} \right) = \frac{1}{(1+r)^j} E_t \left(\frac{P_t}{P_{t+j}} \right). \tag{2}$$

Higher current or expected future interest rates lower bond prices.

Now, take innovations $(E_t - E_{t-1})$ of (1). Define "monetary policy" as a change in current and expected future interest rates, and hence bond prices, that involves no change in fiscal policy, so $(E_t - E_{t-1})\, s_{t+j} = 0$. We have

$$(E_t - E_{t-1}) \frac{\sum_{j=0}^{\infty} Q_t^{(t+j)} B_{t-1}^{(t+j)}}{P_t} = (E_t - E_{t-1}) \sum_{j=0}^{\infty} \beta^j s_{t+j} = 0. \tag{3}$$

Debt B_{t-1} is predetermined. The real value of surpluses does not change by assumption. So any innovation to bond prices must have a corresponding innovation to the price level. If an interest rate rise, including expectations of higher interest rates in the future, lowers bond prices $Q_t^{(t+j)}$, then the price level must also fall. *The price level P_t must jump by exactly the same proportional amount as the change in nominal market value of the debt.*

The mechanism is just as if the real present value of primary surpluses $\{s_{t+j}\}$ had increased. The real value of government debt is greater than its nominal value. People try to buy more government debt, and thus less goods and services. It feels like a lack of "aggregate demand." That force pushes the price level down.

In the case of one-period debt, $B_t^{(t+j)} = 0, \quad j > 0$,

$$(E_t - E_{t-1}) \frac{B_{t-1}^{(t)}}{P_t} = (E_t - E_{t-1}) \sum_{j=0}^{\infty} \beta^j s_{t+j} = 0.$$

so the price P_t does not change unless surpluses change. Inflation rises when interest rates rise, with no price level drop. The presence of long-term debt is crucial to to the temporary price decline.

The deflationary force in this model depends entirely on how much the price of long term bonds, and thus the nominal market value of the debt, declines. Bond prices are determined by the path of expected future interest rates. Thus, in this model, the *expected path* of interest rates matters far more than the current rate in determining a deflationary force.

Therefore, this model gives a very simple role and explanation of "forward guidance." If the central bank can make an announcement that credibly commits to higher or lower interest rates in the future, that announcement will change long-term bond prices and have an immediate inflationary or deflationary impact.

The model also suggests a restriction useful econometrically and in practice. Monetary policy actions and announcements exploit this mechanism to temporarily raise and lower inflation only and to the extent that they lower the nominal market value of government debt.

A different view of Sims' mechanism help intuition. Using the bond price from (2), we can write the government debt valuation equation (1)

$$\frac{B_{t-1}^{(t)}}{P_t} + \sum_{j=1}^{\infty} \beta^j B_{t-1}^{(j)} E_t \left(\frac{1}{P_{t+j}} \right) = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j}. \tag{4}$$

By writing out the first term separately, we have before us the short-term debt case, in which the second term is absent. With one-period debt, surplus expectations drive shocks to the price level P_t. With long-term debt, surplus expectations at time 0 drive the debt-weighted moving average of current and future price levels instead. Within that constraint, in the presence of long-term debt, the government can choose a different path of price levels without changing surpluses. A rise in nominal interest rates means that price levels in the far future must rise. As a result, price levels in the near future, to satisfy (4) must fall. (Cochrane (2001) explores this mechanism in detail.)

This formulation emphasizes the fiscal foundation of Sims' effect. When the government chooses higher nominal interest rates, and hence higher inflation and a higher future price level, it thereby devalues the long-dated coupons. This is great news for the Treasury – it does not have to raise as many real taxes s_{t+j} to pay off coupons. But in Sims' exercise, the Treasury stubbornly refuses: The central bank says, you can pay off the \$1 coupons with (say) half as many real resources. B_{t+j}/P_{t+j} falls by half so you can, if you wish lower s_{t+j} by half. But the Treasury says, no, we're going to insist on paying off the coupons B_{t+j} with exactly the same real resources. How can this loggerhead be solved? Well, by fixing interest rates, the central bank here only controls the time path of prices P_{t+1}/P_t. It does not control the initial price level P_t. So the initial price level must jump down, so that overall bondholders *are* paid back the same amount as before, just with a different time pattern. Long-term debt holders lose, short-term debt holders gain.

Here, you can see the central assumption of the analysis. Why does the Treasury stubbornly refuse to reduce surpluses when the central bank wants to inflate away long-date coupons? Why does the Treasury not reduce future surpluses instead? That's not a question to answer here. We have *defined* "monetary policy" as a change in interest rates without a change in surpluses, as central banks are not allowed to directly

change fiscal surpluses. If actual treasuries respond to central bank actions, then one would see different effects. If actual fiscal policies responded to the same events that induce central banks to raise interest rates, then one would see different effects. This discussion all emphasizes Sims' (and my) main point: In order to understand the effects of interest rate changes, the central question is how *fiscal policy* behaves. Fixed surpluses are a textbook, problem-set, assumption, worth working out but not the final answer for policy or historical analysis.

And nothing else matters. By stripping Sims' effect down to a simple frictionless model, we see that Sims' decline in inflation does *not* involve sticky prices, habits in preferences, money, manipulation of real interest rates, IS and Philips curves, real interest rates lowering investment or "aggregate demand," or anything else remotely "monetary."

To gain more intuition and connect the point to Sims' analysis, consider a very simple example: At time 0, interest rates rise unexpectedly and permanently from $i_t = i$ to $i_t = i^*$. Again there is no fiscal policy shock. Inflation immediately rises to

$$\frac{P^*_{t+1}}{P^*_t} = \frac{1+i^*}{1+r};\ t \geq 0.$$

where P^*_t denotes the price level after the interest rate change. But the price level P_0 may jump down.

Suppose government debt consists of nominal perpetuities, and surpluses are constant s. Now we can write $B^{(j)}_{-1} = B_{-1}$, since the coupon is the same for all dates. Bond prices follow

$$Q^{(j)}_0 = \frac{1}{(1+i^*)^j}.$$

Then, (1) becomes

$$\frac{\sum_{j=0}^{\infty} Q^{(j)}_0 B_{-1}}{P_0} = \frac{1+i^*}{i^*}\frac{B_{-1}}{P_0} = \frac{s}{1-\beta}.$$

If the price and interest rate had been expected to be P and i, giving the same relation between unstarred variables, we can divide and write

$$\frac{P_0}{P} = \frac{1+i^*}{1+i}\frac{i}{i^*}.$$

In continuous time, it's even simpler,

$$\frac{P_0}{P} = \frac{i}{i^*}. \tag{5}$$

Thus, *If the government is funded by perpetuities, a positive permanent shock to interest rates implies a equal proportionate jump down in the price level.* A rise of interest rates from 5% to 6% occasions a 20% price-level drop, before inflation increases by one percentage point. Sims' model in Figure 7 below gives this sort of dynamics, smeared out by the frictions of his model.

In the context of this example, it is algebraically easy to see *how* the government raises interest rates, and we can see that Sims' interest rate rise is equivalent to an inverse quantitative easing operation. To raise interest rates, the central bank sells long-term bonds. Selling long-term bonds, in the face of constant surpluses, devalues existing bondholders' claims, and thus raises future inflation. Buying and extinguishing short-term bonds raises the value of existing bondholders' claims, which lowers near-term price levels.

Therefore, Sims' mechanism describes quantitative easing, interest rate policy, and forward guidance all in one breath, again needing no frictions.

To see how this works, consider again the case of a perpetuity and a one-time unexpected shock from i to i^* at time 0. (1) is

$$\frac{\sum_{j=0}^{\infty} Q_0^{(j)} B_{-1}}{P_0} = \frac{1+i^*}{i^*}\frac{B_{-1}}{P_0} = \frac{1+r}{r}s. \tag{6}$$

The point is to determine P_0 in terms of predetermined B_{-1} and the shock to interest rates i^*.

$$\frac{1}{P_0} = \frac{1+r}{1+i^*}\frac{i^*}{r}\frac{s}{B_{-1}}. \tag{7}$$

Now, consider the same equation one period in the future,

$$\frac{1+i^*}{i^*}\frac{B_0}{P_1} = \frac{1+r}{r}s \tag{8}$$

With

$$\frac{1}{P_1} = \frac{1+r}{1+i^*}\frac{1}{P_0}$$

and (7), we have

$$B_0 = \frac{1+i^*}{1+r}B_{-1} \tag{9}$$

Similarly, to support interest rates that are $i_t = i^*$ further in the future,

$$B_t = \left(\frac{1+i^*}{1+r}\right)^{t-1} B_{-1}$$

Equation (9) can be used and interpreted in two ways. If the government sells more debt B_0 at time 0, without changing surpluses, the value of that debt declines. Selling more debt without changing surpluses is a lot like a share split, which changes the number of shares without changing dividends or earnings. Thus, by selling more debt B_0, the government raises nominal interest rate i^*, and vice versa. This is the "quantitative easing" interpretation. In QE operations, central banks bought back more long maturity debt, in a more complicated pattern, and thus lowered long-term interest rates without changing the one period rate, but the mechanism is the same.

Second, the government can target interest rates $i_t = i^*$ and offer to sell as many perpetuities as people want at that price. If the government can commit to keep surpluses unchanged, equation (9) describes how many perpetuities the government will sell at the price.

This equation thus answers just *how* the government can implement an interest rate target, even in a completely frictionless model with no money, no reserve requirements, and so forth. One might worry, for example, that if the government announces and interest rate and says it will sell any amount of bonds at that rate, it will face a horizontal demand curve and be swamped. This equation reassures us that it will not. Cochrane (2014a) argues that this mechanism in fact can describe our current institutional arrangements in which a central bank sets an interest rate and Treasuries auction an apparently fixed number of securities.

In sum, Sims' mechanism operates even in a completely frictionless model – no monetary frictions, no pricing frictions. If interest rates rise unexpectedly, or if ex-

pected future rates rise unexpectedly so that long-term bond prices fall, prices will first fall, and then rise. The crucial ingredient is outstanding long-term debt, and fiscal policy that does not fully absorb the inflationary impact of the interest rate change. The mechanism treats interest rate targets, forward guidance about future rate changes, and quantiative easing operations in the same breath. However, it only operates for unexpected interest rate changes, and it operates on the day of announcement, not on the day of interest rate change. Fully expected interest rate changes raise inflation uniformly.

2.1 Continuous time and sticky prices

Sims' analysis seems to be quite different, in that it operates in continuous time and the price level P_t cannot jump. A rise in interest rates sets off a period of deflation, which cumulatively lowers the price level. However, as I show below, this apparent difference is not central. As one removes price stickiness, Sims' short period of deflation gets stronger and stronger, smoothly approaching the downward jump predicted by the frictionless model.

The continuous time setup with no price level jumps is an important framework, and works a bit differently from the discrete time model presented above. Simplifying to either a perpetuity or to instantaneous debt, the government debt valuation equation is

$$\frac{Q_t B_t}{P_t} = E_t \int_{\tau=t}^{\infty} e^{-\int_{v=t}^{\tau}(i_v - \pi_v)dv} s_\tau d\tau$$

where Q is the bond price, B the number (face value) of bonds, P is the price level, i is the nominal interest rate, π is the inflation rate and s is the real primary surplus.

For short-term debt, $Q_t = 1$ always. In discrete time, or if prices can jump, innovations in s_τ can induce a jump in P_t. That channel disappears in continuous time with sticky-price models such as Sims' that preclude price-level jumps. However, this fiscal relation can still select equilibria. For given $\{s_t\}$ and $\{i_t\}$, this relation implies a restriction on what path $\{\pi_t\}$ may follow, and still picks a unique equilibrium $\{\pi_t\}$ from the set of multiple equilibria allowed by sticky-price models.

Now, a discount rate effect must operate. If Q_t, B_t, and P_t all cannot jump when

there is a jump to information about future s_τ, then the discount rates $i_v - \pi_v$ must change. If future s decline, for example, the discount rates must also decline so that the present value is unchanged. Therefore, we anticipate that a sticky-price model with one-period debt will substitute a period of higher inflation π for the immediate jump upward P_t of a frictionless model in response to a fiscal shock

When the central bank raises expected interest rates $\{i_t\}$, with no change in surpluses, in a model that disallows a jump in P_t, the path $\{\pi_t\}$ must rise so that the present value on the right side is unchanged. The pure Fisherian result obtained in discrete time will work, $\pi_t = i_t - r$ leaves discount rates unchanged. Models with price frictions may have more complex dynamics, trading more inflation at some dates and less at others, but the path of inflation must still produce no change in present value of the surplus.

With long-term debt, however, the nominal bond price Q_t can jump down when the central bank raises interest rates. If the price level P_t cannot jump, the path $\{\pi_t\}$ on the right hand side must therefore adjust, now to produce a higher real discount rate and a lower present value of surpluses. At a majority of dates on the path, π_t must rise less than i_t so that real discount rates rise. Relative to short-term debt, we produce a path with less inflation. Thus, the downward price jump of the frictionless model becomes a period of lower inflation when the price level cannot jump.

3 Sims' model

The model as presented by Sims (2011), starting with equation (15) on p. 52, is

$$\dot{r}_t = -\gamma(r_t - \bar{\rho}) + \theta\dot{p}_t + \phi\dot{c}_t + \varepsilon_{mt} \tag{10}$$

$$r_t = \rho_t + \dot{p}_t \ (*) \tag{11}$$

$$\rho_t = -\frac{\dot{\lambda}_t}{\lambda_t} + \bar{\rho} \ (*) \tag{12}$$

$$\dot{b}_t = -b_t\dot{p}_t - b_t\frac{\dot{a}_t}{a_t} + a_t b_t - \tau_t - \bar{\tau} \tag{13}$$

$$r_t = a_t - \frac{\dot{a}_t}{a_t}(*) \tag{14}$$

$$\ddot{p}_t = \beta\dot{p}_t - \delta c_t \ (*) \tag{15}$$

$$\dot{\tau}_t = \omega\dot{c}_t + \varepsilon_{\tau t} \tag{16}$$

$$\lambda_t = e^{-\sigma c_t} + \psi\left[\ddot{c}_t - \bar{\rho}\dot{c}_t\right]e^{-c_t}(*) \tag{17}$$

Here I use Sims' notation, r instead of i for the nominal interest rate, ρ instead of r for the real interest rate, and $\bar{\tau} + \tau_t$ instead of s for the real primary surplus. The other symbols are p for the log price level, c for log consumption, λ for marginal utility, b for the real value of government debt, a for the nominal perpetuity yield. I also use Sims' nonstandard notation for parameters. The last equation differs from Sims' by two typos in Sims' paper, that do not affect the calculations. Details in the derivation below.

Our goal is to calculate responses of this model to unexpected jumps in the shocks, ε_{mt} and $\varepsilon_{\tau t}$.

We need to state the underlying model and derive these equilibrium conditions. We then need to linearize the model, transform the model to to $dx/dt = Ax_t + \varepsilon_t$ form, and then solve it as a first order linear differential equation. We need to understand jumps and "forward - looking" equations. The impulse response functions (Sims' Figure 3 and 4) feature jumps in all variables except p_t and c_t. So, we have to understand how variables respond to the ε_{mt} or $\varepsilon_{\tau t}$ jumps, and what the rules about jumps are.

3.1 The model derived and restated

Sims' model is a perfect-foresight continuous-time model, but allowing a probability-zero jump in some variables. (Probability zero, because otherwise risk aversion terms would show up in asset pricing formulas.)

Reordering the equations, and writing them in a more standard form,

$$dr_t = [-\gamma(r_t - \bar{\rho}) + \theta\pi_t + \phi\dot{c}_t]\,dt + d\varepsilon_{mt} \tag{18}$$

$$E_t\,(dp_t) = (r_t - \rho_t)\,dt \quad (*) \tag{19}$$

$$dp_t = \pi_t dt \tag{20}$$

$$E_t\,(d\pi_t) = (\beta\pi_t - \delta c_t)\,dt \quad (*) \tag{21}$$

$$E_t\,(da_t) = a_t\,(a_t - r_t)\,dt \quad (*) \tag{22}$$

$$d\tau_t = \omega\dot{c}_t dt + d\varepsilon_{\tau t} \tag{23}$$

$$db_t = (a_t b_t - b_t\pi_t - \tau_t - \bar{\tau})\,dt - \frac{b_t}{a_t}da_t \tag{24}$$

$$E_t\,(d\lambda_t) = -\lambda_t\,(\rho_t - \bar{\rho}\,)\,dt \;\; (*) \tag{25}$$

$$dc_t = \dot{c}_t dt \tag{26}$$

$$E_t\,[d\dot{c}_t] = \left(\frac{\lambda_t}{\psi}e^{c_t} - \frac{1}{\psi}e^{c_t}e^{-\sigma c_t} + \bar{\rho}\dot{c}_t\right)dt \;\; (*) \tag{27}$$

I use differential notation dx rather than derivative notation $\dot{x}$ for variables that can jump.

The starred equations are "forward-looking," they specify the expectation of a forward-looking differential. To understand the issue, consider the simplest discrete-time new-Keynesian model consisting only of a Fisher equation $i_t = E_t\pi_{t+1}$ and a Taylor rule $i_t = \phi\pi_t + w_t$. The equilibrium is

$$E_t\pi_{t+1} = \phi\pi_t + w_t$$

$$w_{t+1} = \rho w_t + \epsilon_{t+1}$$

This equation is "forward-looking" like the starred equations in Sims' model. It admits

multiple equilibria: Any path

$$\pi_{t+1} = \phi\pi_t + w_{t+1} + \delta_{t+1}$$

with $E_t\delta_{t+1} = 0$ is an equilibrium. This form with an expectational shock is useful for solutions, as you don't have then to do anything special about expectations. It also helps to keep track of how δ jumps in one variable are reflected in similar jumps to other variables. Therefore, I reexpress Sims' model with such expectational shocks in the next step.

The conventional model specifies $\phi > 1$ so the dynamics are explosive. Then the unique non-explosive equilibrium is

$$\pi_t = -E_t \sum_{j=0}^{\infty} \phi^{-(j+1)} w_{t+j} = -\frac{1}{\phi - \rho} w_t$$

This solution amounts to a unique choice of δ_t. This general principle applies to Sims' model: For each "forward-looking" or expectational difference equation, we need to have one explosive eigenvalue and one variable that can jump to the non-explosive saddle-path equilibrium, or equivalently one expectational error. This consideration motivates several discussions in the derivation of the models' equations.

Taylor. Equation (18) is the monetary policy rule. The nominal interest rate mean-reverts, and rises with inflation and consumption growth. The rule allows a jump $d\varepsilon_{mt}$, which generates the monetary policy shock. By examining the steady state $dr_t = 0$, you can see that $\theta > \gamma$ is the Taylor rule region in which interest rates respond more than one for one to inflation, and $\theta < \gamma$ is the "passive money" region.

All the variables on the right hand side of the monetary policy rule can jump, so in principle one should specify whether dr_t is driven by pre-jump or post-jump values (right or left limits). But since these variables are all multiplied by dt it does not matter which one specifies. For the same reason, when there is a jump $d\varepsilon_{mt}$, r jumps by the same amount $dr_t = d\varepsilon_{mt}$, even though the other variables also respond to the jump, and when there is a jump $d\varepsilon_{\tau t}$, we still have $dr_t = 0$ even though the variables on the right hand side may jump.

Phillips. Equation (20) and (21) define the forward-looking Phillips curve. It is the analogue of the discrete-time curve

$$\pi_t = \alpha E_t \pi_{t+1} + \kappa c_t$$

which can be written in the form

$$E_t \pi_{t+1} - \pi_t = \left(\frac{1-\alpha}{\alpha} \right) \pi_t - \frac{\kappa}{\alpha} c_t.$$

from which (21) follows immediately. I use (20) to connect price changes and inflation changes. The solution method is a first-order differential equation, so when there are second derivatives involved, I add an extra state variable to write the system in terms of first derivatives only.

Since this is a "forward-looking" equation, I write the Phillips curve below in the form

$$d\pi_t = (\beta \pi_t - \delta c_t)\, dt + d\delta_{\pi t}$$

where $d\delta_{\pi t}$ is an arbitrary expectational jump.

Fisher. Equation (19) is the Fisher equation defining the real rate of interest. It is "forward looking" and allows a price level jump. In discrete time, this equation would read

$$r_t = \rho_t + E_t \left(p_{t+\Delta} - p_t \right) = \rho_t + E_t \left(\pi_{t+\Delta} \right).$$

Sims introduces a structural shock ε_{rt}, but he does not use it, so I leave it out.

The generic asset pricing equation for a security whose real value process is v_t and hence return is $dR_t = dv_t / v_t$ is

$$E_t dR_t = \rho_t dt - E_t \left(\frac{d\lambda_t}{\lambda_t} dR_t \right)$$

where λ_t is the marginal utility of consumption. Sims avoids the second risk aversion term by specifying an infinitesimal probability jump as the only source of randomness.

In the presence of a potential price level jump, the real return on the nominal short

term bond is

$$dR_t = r_t dt + \frac{d(1/P_t)}{(1/P_t)}$$

so the risk-neutral Fisher relation is really

$$r_t dt + E_t\left(\frac{d(1/P_t)}{(1/P_t)}\right) = \rho_t dt$$

Replacing the term in the expectation on the left with $-E_t(dp_t)$ is a linearization or approximation.

However, while this Fisher equation and (19) allow for price-level jumps, in Sims' specification the Phillips curve does not allow for such jumps - inflation can jump, but the price level cannot jump. The Phillips curve comes from a Calvo fairy who allows a fraction $(\text{constant})dt$ of firms to change prices at any date. Since no mass of firms can change prices in an instant, prices cannot jump.

Without price level jumps or (diffusion terms), we can write $d(1/P_t)/(1/P_t) = -dp_t$ and with (20) $dp_t = \pi_t dt$ the Fisher equation becomes simply

$$\pi_t = r_t - \rho_t.$$

I use this form below.

In sum, with no price level jumps, the Fisher equation is no longer "forward-looking." We lose one expectational error, so we need one less an explosive eigenvalue.

Term Structure. Equation (22) is the term structure relation between long and short rates. The perpetuity has nominal yield a_t, nominal price $1/a_t$ and pays a constant coupon $1dt$. Thus, the condition that the expected nominal perpetuity return should equal the riskfree nominal rate (there are no price level jumps and no risk premiums) is

$$r_t dt = \frac{1dt + E_t d\,(1/a_t)}{1/a_t} \approx a_t dt - E_t \frac{da_t}{a_t}.$$

Equation (22) follows. There are jumps in a_t, so and thus the second equality is a linearization or approximation. The next step will be to linearize the model anyway. However, if one wishes to extend Sims' model by solving the nonlinear version, or including nonzero shock probability and hence risk premiums, one should keep the nonlinear

version.

This is a forward-looking equation, so I introduce the corresponding expectational error

$$da_t = a_t(a_t - r_t)dt + d\delta_{at} \tag{28}$$

Debt. Equations (23) and (24) describe government finances. Equation (23) describes a primary surplus that rises and falls with consumption growth, and can jump. Equation (24) is the government budget constraint. By definition, $b_t \equiv B_t/(a_t P_t)$ is the real market value of government debt, where B_t is the number of perpetuities outstanding and P_t is the price level. Sims models the real value of government debt because the consumer's transversality condition states that this real value may not explode. That condition is a key "forward-looking" condition which forces variables to jump when shocks occur.

To derive this equation, start from the observation that the government must sell new perpetuities at price $1/a_t$ to cover the difference between coupon payments $\$1 \times B_t$ and primary surpluses $\tau_t + \bar{\tau}$, ($\bar{\tau}$ is the steady state, τ_t the deviation from steady state)

$$\frac{1}{a_t P_t} dB_t = \frac{B_t}{P_t} dt - (\tau_t + \bar{\tau})dt. \tag{29}$$

B_t does not jump.

Now note

$$db_t = d\left(\frac{B_t}{a_t P_t}\right) = \frac{1}{a_t P_t} dB_t + b_t \frac{d(1/a_t)}{1/a_t} - b_t dp_t.$$

Here I have used the fact that there are no price level jumps. Substituting into (29), and with $\pi_t dt = dp_t$, and solving for db_t,

$$db_t = b_t \frac{d(1/a_t)}{1/a_t} + \left[(a_t - \pi_t)b_t - (\tau_t + \bar{\tau})\right] dt.$$

The face value of debt B_t does not jump. The market value can jump, because the bond price can jump. This is an ex-post equation, restricting the actual change db_t not just the expected change $E_t db_t$, so it does not require an expectational error or an extra explosive eigenvalue. (Forward differences and "forward-looking" are not the same thing.) Its jump is entirely induced by the jump in bond prices.

To connect the jump in debt to the jump in bond prices, I use the same linearization of the latter, giving (24),

$$db_t = [a_t b_t - (\tau_t + \bar{\tau}) - b_t \pi_t]\, dt - \frac{b_t}{a_t} da_t.$$

In the next step, I split da_t on the right hand side to

$$da_t = E_t da_t + (da_t - E_t da_t) = a_t\,(a_t - r_t)\, dt + d\delta_{at}$$

Then we can write

$$db_t = -\frac{b_t}{a_t}\left[a_t\,(a_t - r_t)\, dt + d\delta_{at}\right] + [a_t b_t - b_t \pi_t - (\tau_t + \bar{\tau})]\, dt.$$

$$db_t = -\left[b_t\,(\pi_t - r_t) + \tau_t + \bar{\tau}\right] dt - \frac{b_t}{a_t} d\delta_{at} \tag{30}$$

I use this form below.

Consumption. Equations (25)-(27) describe marginal utility with a "habit" term that values a smooth consumption path. The utility function adds a penalty for the derivative of log consumption growth,

$$U = E \int_{t=0}^{\infty} e^{-\bar{\rho} t} \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{1}{2}\psi \left(\frac{1}{C}\frac{dC}{dt}\right)^2\right] dt.$$

To derive marginal utility, set this up as a Hamiltonian with a constraint that wealth grows at the interest rate

$$\dot{W}_t = \rho_t W_t - C_t.$$

The state variables are $x_t = [C_t\ W_t]$ and the control variable is $u_t = dC_t/dt$. The current value Hamiltonian is then

$$H\ = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{1}{2}\psi \left(\frac{1}{C_t}\frac{dC_t}{dt}\right)^2 + \lambda\,(\rho_t W_t - C_t) + \gamma \frac{dC_t}{dt}$$

The first order conditions are

$$\frac{\partial H}{\partial u} = 0 : -\psi \frac{1}{C_t^2}\frac{dC_t}{dt} + \gamma = 0 \tag{31}$$

$$\frac{\partial H}{\partial C} = C_t^{-\sigma} + \psi \frac{1}{C_t^3}\left(\frac{dC_t}{dt}\right)^2 - \lambda = -\dot{\gamma} + \bar{\rho}\gamma \tag{32}$$

$$\frac{\partial H}{\partial W} = \lambda \rho_t = -\dot{\lambda} + \bar{\rho}\lambda \tag{33}$$

From (33),

$$\rho_t = -\frac{\dot{\lambda}}{\lambda} + \bar{\rho}.$$

From (31), dropping t subscripts,

$$\gamma = \psi \frac{1}{C^2}\frac{dC}{dt}$$
$$\dot{\gamma} = -2\psi \frac{1}{C^3}\left(\frac{dC}{dt}\right)^2 + \psi \frac{1}{C^2}\frac{d^2C}{dt^2},$$

so, from (32),

$$\lambda = C_t^{-\sigma} + \psi \frac{1}{C^3}\left(\frac{dC}{dt}\right)^2 + \dot{\gamma} - \bar{\rho}\gamma$$

$$\lambda = C_t^{-\sigma} - \psi \frac{1}{C^3}\left(\frac{dC}{dt}\right)^2 + \psi \frac{1}{C^2}\frac{d^2C}{dt^2} - \bar{\rho}\psi \frac{1}{C^2}\frac{dC}{dt}.$$

Note with $c = \log(C)$,

$$\frac{dc}{dt} = \frac{1}{C}\frac{dC}{dt}$$
$$\left(\frac{dc}{dt}\right)^2 = \frac{1}{C^2}\left(\frac{dC}{dt}\right)^2$$
$$\frac{d^2c}{dt^2} = -\frac{1}{C^2}\left(\frac{dC}{dt}\right)^2 + \frac{1}{C}\frac{d^2C}{dt^2}$$
$$\frac{d^2c}{dt^2} + \left(\frac{dc}{dt}\right)^2 = \frac{1}{C}\frac{d^2C}{dt^2}$$

so

$$\lambda = C_t^{-\sigma} - \psi\left[\frac{1}{C^2}\left(\frac{dC}{dt}\right)^2 - \frac{1}{C}\frac{d^2C}{dt^2} + \bar{\rho}\frac{1}{C}\frac{dC}{dt}\right]\frac{1}{C}$$

$$\lambda = C_t^{-\sigma} - \psi\left[\left(\frac{dc}{dt}\right)^2 - \frac{d^2c}{dt^2} - \left(\frac{dc}{dt}\right)^2 + \bar{\rho}\frac{dc}{dt}\right]\frac{1}{C}$$

$$\lambda = C_t^{-\sigma} - \psi\left[-\frac{d^2c}{dt^2} + \bar{\rho}\frac{dc}{dt}\right]\frac{1}{C}$$

$$\lambda = e^{-\sigma c} + \psi\left[\ddot{c} - \bar{\rho}\dot{c}\right]e^{-c}. \tag{34}$$

Sims gives the corresponding equation (his equation (22)) as

$$\lambda = e^{-\sigma c} + \psi \left[\ddot{c} - \dot{c}^2\right] e^{-c} \tag{35}$$

The final $\bar{\rho}\dot{c}$ term is missing in Sims' paper, a typo confirmed by Sims. To keep track of it I will use $\hat{\rho}$ in its place, and then we can choose $\hat{\rho} = \bar{\rho}$ or $\hat{\rho} = 0$. I verify that the typo does not extend to Sims' calculations. Sims includes a $\dot{c}^2$ term, which I believe to be a typo or algebra mistake. (It can result from omitting the second term in (32).) However Sims' subsequent linearization procedure drops this squared term, so its inclusion or omission makes no difference to the calculations.

The marginal utility λ is as usual a forward-looking expectational equation which can both jump, and for which we have to tie down an expectational error.

$$E_t\left(d\lambda_t\right) = -\lambda_t\left(\rho_t - \bar{\rho}\,\right) dt(*)$$

The penalty on the second derivative of log consumption means that consumption cannot jump. Therefore, as with inflation, I introduce a state variable $\dot{c}_t$ of the first derivative of consumption, and specify the second-order differential equation containing $\ddot{c}$, $\dot{c}$, and c as a paired first-order differential equation. Finally, the first derivative of consumption can jump, so we (34) implies a a forward-looking expectational equation,

$$E_t\left[d\dot{c}_t\right] = \left(\frac{\lambda_t}{\psi}e^{c_t} - \frac{1}{\psi}e^{(1-\sigma)c_t} + \hat{\rho}\dot{c}_t\right) dt(*). \tag{36}$$

I add a corresponding expectational error $d\delta_{\dot{c}t}$ below

3.2 Linearization

The model is now

$$dr_t = [-\gamma(r_t - \bar{\rho}) + \theta\pi_t + \phi\dot{c}_t]\, dt + d\varepsilon_{mt} \tag{37}$$

$$\pi_t = r_t - \rho_t \tag{38}$$

$$d\pi_t = (\beta\pi_t - \delta c_t)\, dt \; + d\delta_{\pi t} \tag{39}$$

$$da_t = a_t\,(a_t - r_t)\, dt + d\delta_{at} \tag{40}$$

$$d\tau_t = \omega\dot{c}_t dt + d\varepsilon_{\tau t} \tag{41}$$

$$db_t = -\left[b_t\,(\pi_t - r_t) + \tau_t + \bar{\tau}\right] dt - \frac{b_t}{a_t} d\delta_{at} \tag{42}$$

$$d\lambda_t = -\lambda_t\,(\rho_t - \bar{\rho}\,)\, dt + d\delta_{\lambda t} \tag{43}$$

$$dc_t = \dot{c}_t dt \tag{44}$$

$$d\dot{c}_t = \left[\frac{\lambda_t}{\psi} e^{c_t} - \frac{1}{\psi} e^{c_t} e^{-\sigma c_t} + \hat{\rho}\dot{c}_t\right] dt + d\delta_{\dot{c}t} \tag{45}$$

where I have introduced the expectational errors $d\delta_t$.

Since the price level does not enter the model, I drop the definition $dp_t = \pi_t dt$ from the model solution. We can use it later to compute the price level.

The steady state is where all time derivatives are zero. All rates of return equal $\bar{\rho}$,

$$r = \bar{\rho} = \rho = a.$$

I use variables without t subscripts to denote steady state values. Taxes pay for the coupons,

$$ab = \bar{\tau}.$$

The Phillips curve means $c = 0$, and then the marginal value of wealth is one.

$$0 = c;\ \lambda = 1 \tag{46}$$

Linearizing around this steady state, working to $dx_t = Ax_t dt + d\varepsilon_t$ representation, and using tilde notation for differences to the steady state for variables that are not zero

at that state,

$$\tilde{r}_t \equiv r_t - \bar{\rho}$$
$$\tilde{\rho}_t \equiv \rho_t - \bar{\rho}$$
$$\tilde{a}_t \equiv a_t - \bar{\rho}$$
$$\tilde{b}_t \equiv b_t - b$$
$$\tilde{\lambda}_t \equiv \lambda_t - 1,$$

the linearized model is

$$d\tilde{r}_t = \left[-\gamma\tilde{r}_t + \theta\pi_t + \phi\dot{c}_t\right] dt + d\varepsilon_{mt} \tag{47}$$
$$d\pi_t = \left(\beta\pi_t - \delta c_t\right) dt \ + d\delta_{\pi t} \tag{48}$$
$$d\tilde{a}_t = \bar{\rho}(\tilde{a}_t - \tilde{r}_t)dt + d\delta_{at} \tag{49}$$
$$d\tau_t = \omega\dot{c}_t dt + d\varepsilon_{\tau t} \tag{50}$$
$$db_t = -\left[b\left(\pi_t - \tilde{r}_t\right) - \bar{\rho}\tilde{b}_t + \tau_t\right] dt - \frac{b}{\bar{\rho}} d\delta_{at} \tag{51}$$
$$d\tilde{\lambda}_t = -\left(\tilde{r}_t - \tilde{\pi}_t\right) dt + d\delta_{\lambda t} \tag{52}$$
$$dc_t = \dot{c}_t dt \tag{53}$$
$$d\dot{c}_t = \left[\frac{1}{\psi}\tilde{\lambda}_t + \frac{\sigma}{\psi}c_t + \hat{\rho}\dot{c}_t\right] dt + d\delta_{\dot{c}t} \tag{54}$$

Here, I used

$$\pi_t = \tilde{r}_t - \tilde{\rho}_t$$

to eliminate the real interest rate $\tilde{\rho}_t$. Also, the linearization of (42) gives in fact

$$db_t = -\left[b\left(\pi_t - \tilde{r}_t\right) - \bar{\rho}\tilde{b}_t + \tau_t\right] dt - \left[\frac{b}{a} + \frac{\tilde{b}_t}{a} - \frac{b}{a^2}\tilde{a}_t\right] d\delta_{at}. \tag{55}$$

However, the impulse response function takes place when variables are at steady states, so I eliminate the state-dependent shock response in (55) and simplify to (51).

The model is, at last, in the standard form $dx_t = Ax_t dt + d\epsilon_t$.

The fiscal block (49), (50), (51) operates independently of the rest of the model –

other variables enter here, but the variables a, τ, b determined here do not feed back on the rest of the system. As in other new-Keynesian models, the model without this block and passive monetary policy is indeterminate, it has multiple equilibria. But all but one of those equilibria lead to an explosive path for the real value of debt b_t. Therefore, the fiscal block selects equilibria.

3.3 Solution

Expressing the model in matrix notation

$$d\begin{bmatrix} \tilde{r}_t \\ \pi_t \\ \tilde{a}_t \\ \tau_t \\ \tilde{b}_t \\ \tilde{\lambda}_t \\ c_t \\ \dot{c}_t \end{bmatrix} = \begin{bmatrix} -\gamma & \theta & 0 & 0 & 0 & 0 & 0 & \phi \\ 0 & \beta & 0 & 0 & 0 & 0 & -\delta & 0 \\ -\bar{\rho} & 0 & \bar{\rho} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ b & -b & 0 & -1 & \bar{\rho} & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1/\psi & \sigma/\psi & \hat{\rho} \end{bmatrix} \begin{bmatrix} \tilde{r}_t \\ \pi_t \\ \tilde{a}_t \\ \tau_t \\ \tilde{b}_t \\ \tilde{\lambda}_t \\ c_t \\ \dot{c}_t \end{bmatrix} dt + \begin{bmatrix} d\varepsilon_{mt} \\ d\delta_{\pi t} \\ d\delta_{at} \\ d\varepsilon_{\tau t} \\ -(b/\bar{\rho})\, d\delta_{at} \\ d\delta_{\lambda t} \\ 0 \\ d\delta_{\dot{c}t} \end{bmatrix}$$

$$dx_t = Ax_t dt + d\varepsilon_t.$$

It's easiest to solve the differential equation, and then use the shocks and jumps to set up a set of initial conditions x_0. Without the shock term, we have

$$\begin{aligned} \frac{dx}{dt} &= Ax_t = Q\Lambda Q^{-1}x_t \\ \frac{dQ^{-1}x_t}{dt} &= \Lambda Q^{-1}x_t \\ \frac{dy_t}{dt} &= \Lambda y_t \end{aligned}$$

$$y_t = Q^{-1}x_t; x_t = Qy_t$$

where Q is a matrix of eigenvectors, and Λ a diagonal matrix of eigenvalues of A. To rule out explosions, we must have $y_{it} = 0$ for each element i of y_t corresponding to an explosive eigenvalue $\lambda_{it} \geq 0$. Since the y are linear combinations of the x, this condition

imposes a set of linear restrictions on x_t and x_0 in particular,

$$\left[Q^{-1}\right]_{i,:} x_0 = 0$$

where $\left[Q^{-1}\right]_{i,:}$ denotes the ith row of Q^{-1}. Thus, also,

$$\left[Q^{-1}\right]_{i,:} d\varepsilon_0 = 0.$$

This is a set of linear restrictions on the shocks $d\varepsilon_0$. In turn, this set of linear restrictions allows us to determine the expectational errors δ as a function of the underlying shocks ε. This system has four undefined expectational errors, so we need exactly four non-negative eigenvalues for the model to be uniquely determined, which is the case.

To find the instantaneous response to the shocks, then, we must solve

$$\begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \\ \left[Q^{-1}\right]_{4,:} \end{bmatrix}_{4\times 8} \begin{bmatrix} d\varepsilon_{mt} \\ d\delta_{\pi t} \\ d\delta_{at} \\ d\varepsilon_{\tau t} \\ -b/\bar{\rho} d\delta_{at} \\ d\delta_{\lambda t} \\ 0 \\ d\delta_{\dot{c}t} \end{bmatrix}_{8\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times 1} \tag{56}$$

for $d\delta_{\pi t}$, $d\delta_{at}$, $d\delta_{\lambda t}$, $d\delta_{ct}$ where $i = 1, 2, 3, 4$ denotes the indices of the explosive eigenval-

ues. Break up the ϵ and δ parts of the shock vector in (56) to write

$$\begin{bmatrix} d\varepsilon_{mt} \\ d\delta_{\pi t} \\ d\delta_{at} \\ 0 \\ -b/\bar{\rho} d\delta_{at} \\ d\delta_{\lambda t} \\ 0 \\ d\delta_{\dot{c}t} \end{bmatrix}_{8\times 1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -b/\bar{\rho} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\delta_{\pi t} \\ d\delta_{at} \\ d\delta_{\lambda t} \\ d\delta_{ct} \end{bmatrix}_{4\times 1} + \begin{bmatrix} d\varepsilon_{mt} \\ 0 \\ 0 \\ d\varepsilon_{\tau t} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{8\times 1} \tag{57}$$

Then, we can solve (56),

$$\begin{bmatrix} d\delta_{\pi t} \\ d\delta_{at} \\ d\delta_{\lambda t} \\ d\delta_{ct} \end{bmatrix}_{4\times 1} = - \left(\begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \\ \left[Q^{-1}\right]_{4,:} \end{bmatrix}_{4\times 8} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -b/\bar{\rho} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \\ \left[Q^{-1}\right]_{4,:} \end{bmatrix} \begin{bmatrix} d\varepsilon_{mt} \\ 0 \\ 0 \\ d\varepsilon_{\tau t} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{8\times 1}.$$

Using (57) again, we now have the full jump shock vector $d\varepsilon_0$, and therefore the time-zero value x_0 of all variables.

It's easiest to solve the differential equation forward using the transformed y variables, $y_0 = Q^{-1}x_0$ This should produce $[y_0]_i = 0$ for all nonzero eigenvalues, but it is numerically safer to impose that fact, constructing instead$[y_0]_j = \left[Q^{-1}\right]_{j,:} x_0$ only for the non-positive eigenvalues j.

Finally, the impulse response function is given by $y_{jt} = e^{-\lambda_{jt}} y_{0i}$; $x_t = Qy_t$.

4 Impulse-response functions

Sims uses parameters $\gamma = 0.5$; $\theta = 0.4$; $\phi = 0.75$; $\sigma = 2$; $\bar{\rho} = 0.05$; $\bar{\tau} = 0.1$; $\beta = 0.1$; $\delta = 0.2$; $\omega = 1.0$; $\psi = 2.0$. Here, $\theta < \gamma$ so we are in the fiscal theory of the price level region of passive monetary policy and active fiscal policy, in the Leeper (1991) categorization.

Figure 1 shows the response of interest rates and inflation to the monetary policy shock. You see the jump down in inflation, followed by its slow rise.

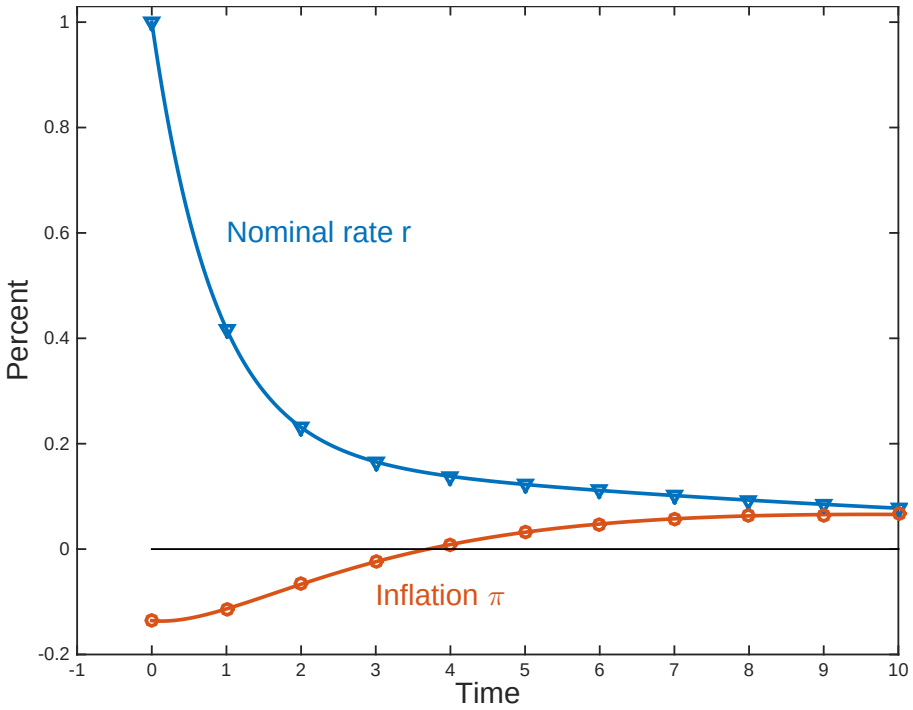

Figure 1: Response to a monetary policy shock in the Sims (2011) model.

Figure 2 presents the response of all variables to the monetary policy shock $d\varepsilon_m$. This figure is visually identical (to my eyes) to Sims (2011) Figure 3.

The price level and consumption do not jump at time zero. All the other variables do jump downward, including inflation. Inflation could, like consumption, start at zero and then build up in a hump-shaped pattern. It does not. The jump in inflation is important. The model does not produce a decline in inflation without this jump.

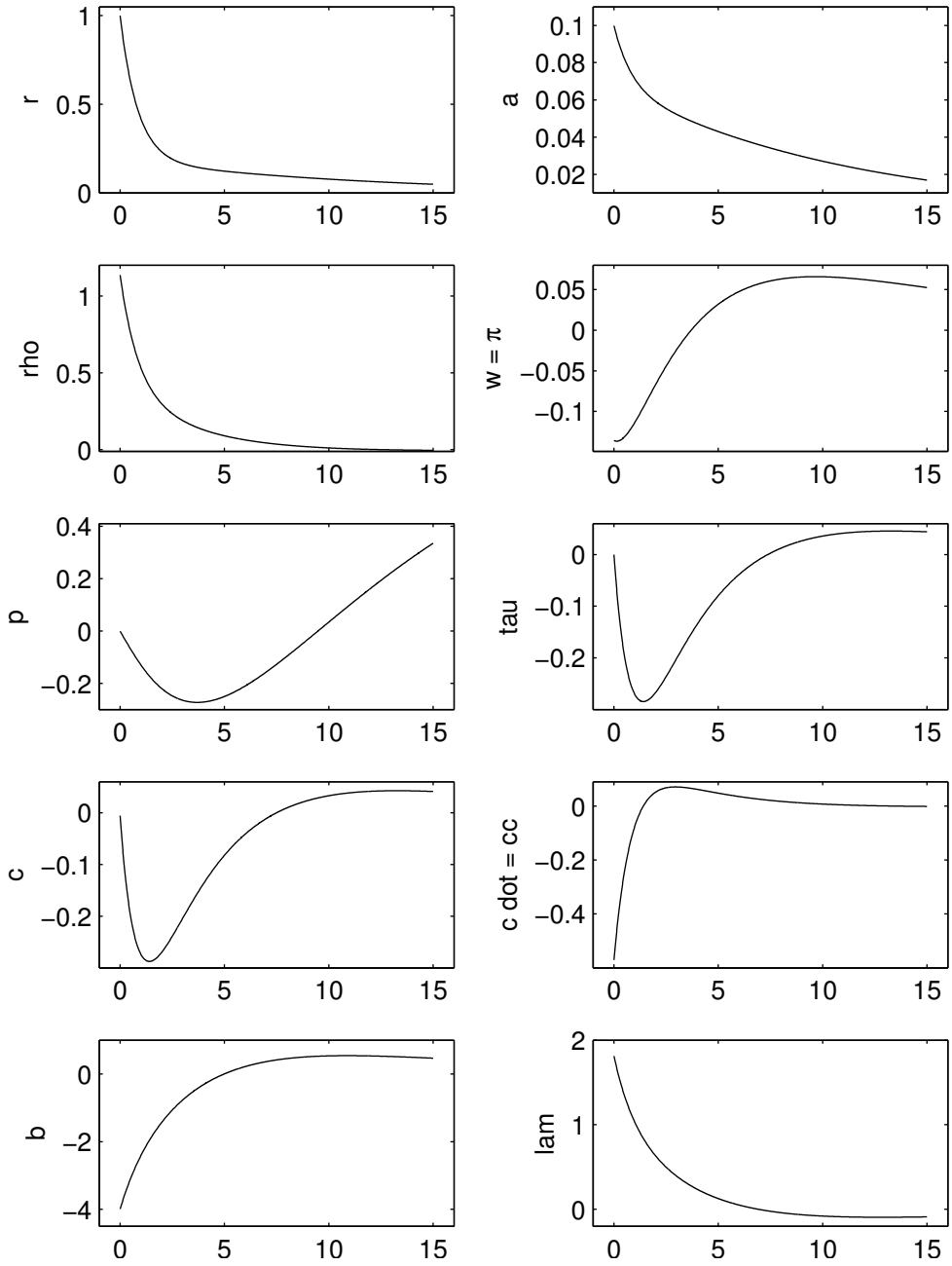

Figure 2: Responses to a monetary policy shock. Replication of Sims (2011) Figure 3.

4.1 Habits, Taylor rules, and fiscal responses

How many of Sims' ingredients are *necessary* to deliver a negative response of inflation to the interest rate rise? How many ingredients are useful to match dynamics, but not essential to the basic sign?

It turns out that the habit ψ, the Taylor rule γ, ϕ, θ, and the fiscal policy response ω do not matter for the negative response of inflation to the interest rate rise. Figure 3 presents the impulse response function for the case $\gamma = 0$, a permanent rise in rates; $\phi = \theta = 0$, an interest rate peg that does not respond to inflation or output; $\omega = 0$, surpluses do not respond to output; and $\psi = 0$, no habits. (Not shown, the limit of the response functions as $\psi \to 0$ is well-behaved. One might worry that consumption can jump at $\psi = 0$ and cannot jump for any $\psi > 0$, no matter how small ψ. However, the fast hump-shaped responses smoothly approach a jump.)

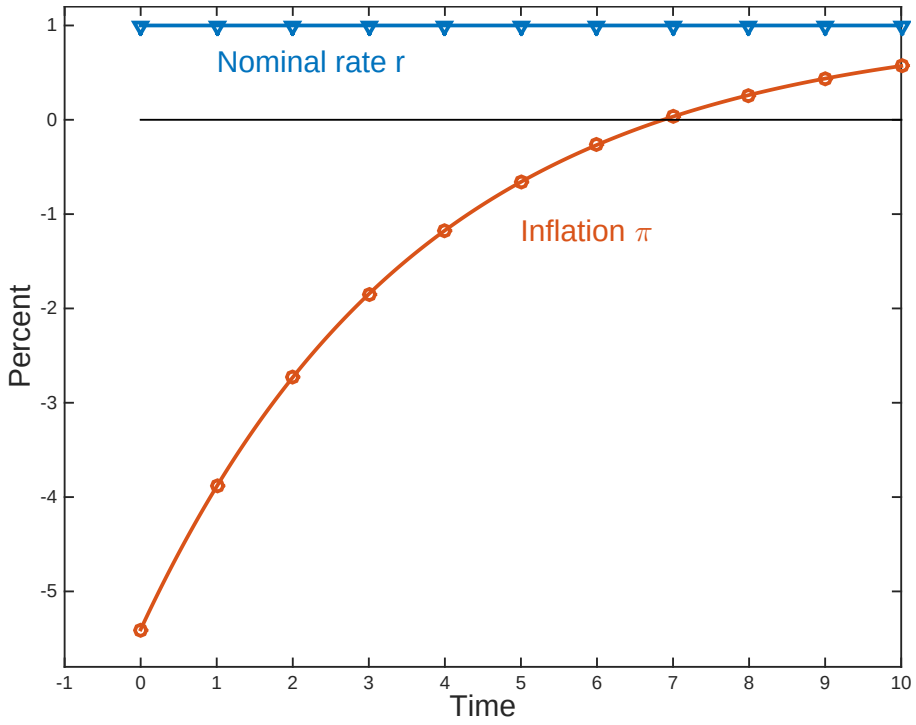

Figure 3: Response to a step-function rise in interest rates, in the simple model. The policy rule does not respond to output or inflation $\phi = \theta = 0$, fiscal policy does not respond to output $\omega = 0$, and there are no habits, $u(c) = c^{-\sigma}$; $\psi = 0$.

The short-run negative response of inflation to the rise in interest rates is still there, in fact stronger than ever. The same 1% nominal interest rate rise as in Figure 1 now produces a 5% fall in inflation, not an 0.1% fall, and consequently a 6% rise in the real rate of interest. This magnitude is driven by the duration of the interest rate shock, permanent in this case. The longer-lasting the shock, the greater its effect on long term bond prices.

4.2 Response to expected monetary policy

Two parts of Sims' specification are necessary for the negative sign result: that the interest rate shock is unexpected, and that debt is long term.

The top panel of Figure 4 presents the response of the full Sims model to an expected monetary policy shock. In this case, the interest rate response is fully Fisherian – inflation rises smoothly through the episode. (The shock only happens at time $t = 0$. However, the endogenous responses of the interest rate rule to output and inflation mean that interest rates move a bit ahead of the shock and move more than the shock on its day.)

The bottom panel of Figure 4 plots the response of the simplified model with no Taylor rule $\gamma = \psi = \phi = 0$, no fiscal response $\omega = 0$ and no habits $\psi = 0$ to a fully anticipated shock. The inflation rate rises smoothly throughout, just as in the discrete-time versions of this calculation presented in Cochrane (2016).

The negative response of inflation to an interest rate rise depends crucially on that rise being unexpected, and therefore triggering a revision in the present value of future surpluses.

4.2.1 Calculating the response to expected rate rises

When the monetary policy shock ε_{mt} is expected, all the expectational errors δ_t are equal to zero. That makes solving the model a lot easier. I'll posit a single jump at time 0. The system is

$$dx_t = Ax_t dt + d\varepsilon_t$$

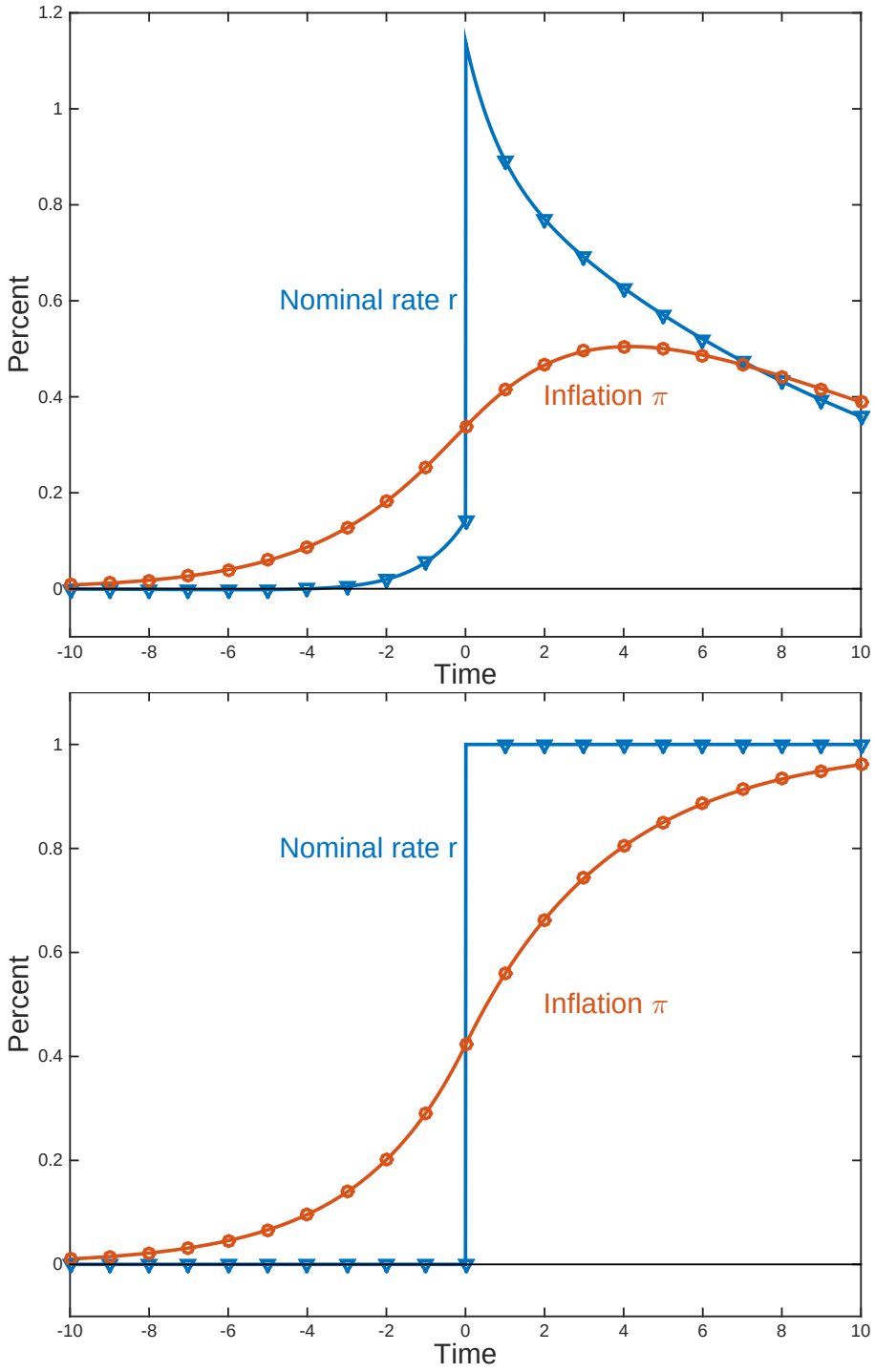

Figure 4: Response to expected monetary policy shocks. Top: l Sims (2011) model. Bottom: Simple model with no habit, Taylor rule, or fiscal response.

$$d\varepsilon_t = \begin{bmatrix} \varepsilon_{mt} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

The bounded solutions are then:

$$\begin{aligned} &\lambda_i > 0: \\ &y_{it} = -\left[\left[Q^{-1}\right]_{i,:} d\varepsilon_0\right] e^{\lambda_i t}; t \leq 0; \\ &y_{it} = 0; t > 0 \end{aligned}$$

$$\begin{aligned} &\lambda_i < 0: \\ &y_{it} = \left[\left[Q^{-1}\right]_{i,:} d\varepsilon_0\right] e^{\lambda_i t}; t \geq 0; \\ &y_{it} = 0; t < 0 \end{aligned}$$

In words, each state variable y_{it} jumps by an amount $\left[Q^{-1}\right]_{i,:} d\varepsilon_0$ at time 0. The state variables corresponding to explosive eigenvalues trend down until they hit $-\left[Q^{-1}\right]_{i,:} d\varepsilon_0$ at time $t = 0$, then jump up to 0 at time $t = 0 + \Delta$. The state variables corresponding to stable eigenvalues are zero until time $t = 0$. They jump up to $\left[Q^{-1}\right]_{i,:} d\varepsilon_0$ at time $t = 0 + \Delta$, then decay exponentially.

4.3 Short-term debt

Long-term debt is also necessary for the negative response of inflation to interest rates.

Figure 5 presents the response function for the full Sims model to unexpected and expected monetary shocks, with short-term debt in the place of long-term debt. (In a continuous time model, short-term debt means fixed value, floating-rate debt. The price is always one, and it pays $r_t dt$ interest.) For the unexpected shock, inflation jumps up and is positive throughout. The response to the expected shock is exactly the same as it was for long-term debt. Hence, the *only* effect of long-term debt in this model is that an unexpected shock lowers the value of debt.

Figure 6 presents the response function of the simple model, with no Taylor rule, habits, or fiscal responses, to an unexpected and expected permanent monetary policy shock.

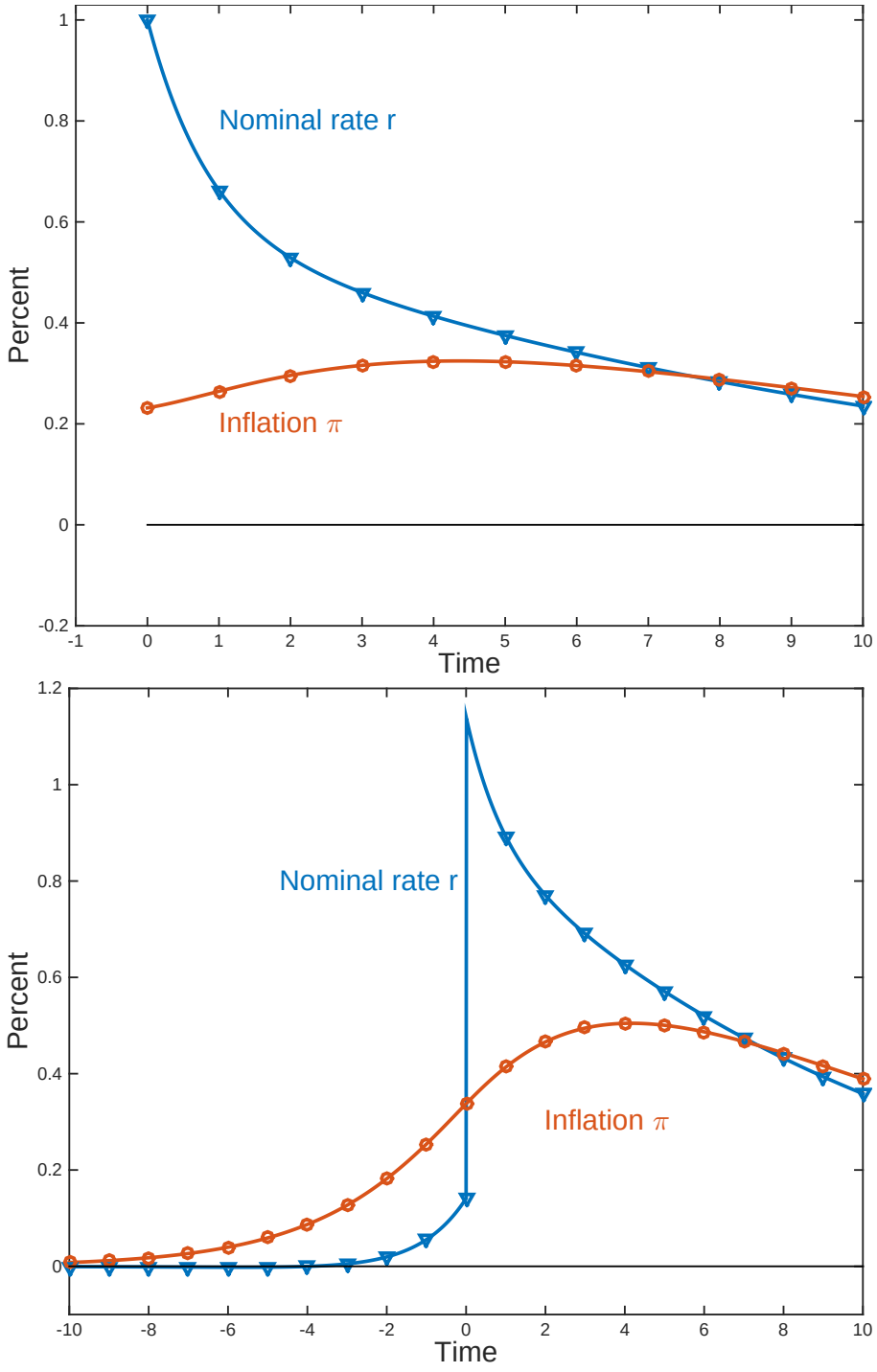

Figure 5: Responses of the Sims model to a monetary policy shock, with short-term debt. Top: response to an unexpected interest rate rise. Bottom: response to an expected interest rate rise.

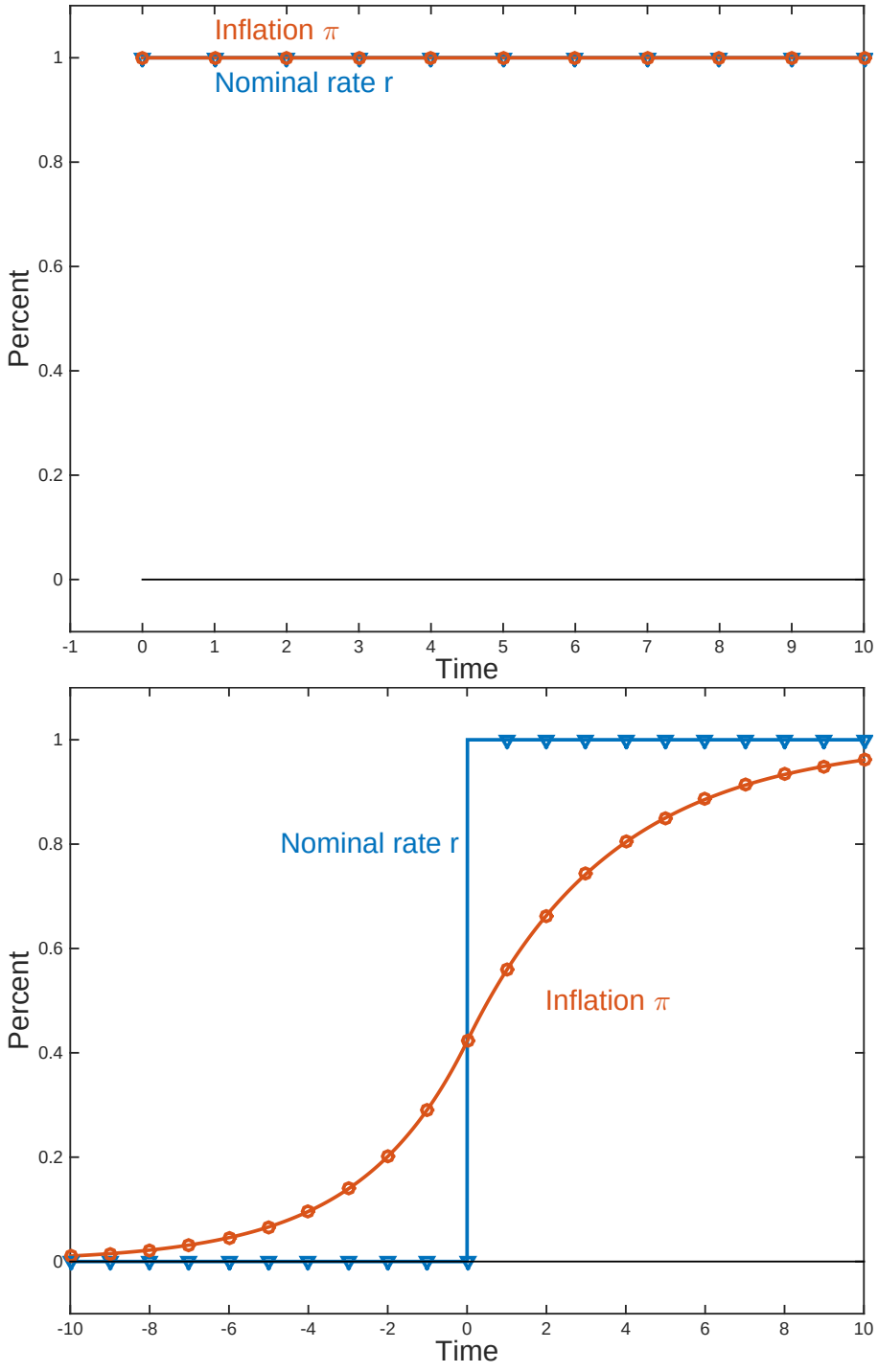

Figure 6: Responses of the simple model to a monetary policy shock, with short-term debt. Top: response to an unexpected interest rate rise. Bottom: response to an expected interest rate rise.

In the top graph, we see a perfectly Fisherian response to unexpected monetary policy. Yes, this is the standard two-equation new - Keynesian model, with sticky prices and prices cannot jump. But *inflation* can jump in this model, and with short term debt it does. If inflation jumps to equal the jump in interest rate, then there is no change to the present value of unchanged surpluses. Then B/P need not change, which is fortunate since B is predetermined and P can't jump.

The corresponding exercise in discrete time, the response to an unexpected interest-rate shock with no change in surpluses, presented in Cochrane (2016) does not produce a pure Fisherian response. Instead, inflation jumps up to a path that looks like the path shown here for the expected case. In discrete time, the shortest bonds are one period, and unexpected inflation also implies a price level jump, which affects the real value of debt. The lesson is that predictions of this sort of model are sensitive to the maturity structure of debt, even the difference between one year and zero.

The response to an expected interest rate rise is exactly as it was with long-term debt. Long vs. short term debt affects the results only by inducing a change in the value of debt at the time of the shock.

4.3.1 Model with short-term debt

The maturity structure only matters to the dB_t equation. To derive the db_t equation in the case of short term debt, start with the definition that the real value of the debt is

$$b_t \equiv B_t/P_t$$

Here B_t is the quantity of instantaneous, i.e. floating rate debt. I do not divide by a_t as the price of such debt is always one.

Then,

$$db_t = \frac{dB_t}{P_t} + \frac{B_t}{P_t}\frac{d(1/P_t)}{1/P_t}.$$

The flow budget constraint now states that interest must be paid from surpluses or new debt issues,

$$B_t r_t dt = P_t(\tau_t + \bar{\tau})dt + dB_t$$

$$b_t r_t dt = (\tau_t + \bar{\tau})dt + db_t - b_t \frac{d(1/P_t)}{1/P_t}$$

$$db_t = b_t r_t dt - (\tau_t + \bar{\tau})dt + b_t \frac{d(1/P_t)}{1/P_t}$$

The instantaneous value of short term debt can only jump if there is a price level jump. Sims' sticky-price model rules out such jumps, so the last term is

$$\frac{d(1/P_t)}{1/P_t} = -\pi_t dt.$$

With $r_t = \rho_t + \pi_t$ we then have

$$db_t = \left[b_t \left(r_t - \pi_t\right) - \left(\tau_s + \bar{\tau}\right)\right] dt$$

whereas with long term debt before it was

$$db_t = \left[b_t \left(r_t - \pi_t\right) - \left(\tau_t + \bar{\tau}\right)\right] dt - \frac{b_t}{a_t} d\delta_{at}$$

The only difference between short and long term debt in this model is that the instantaneous response of the value of debt to a yield shock is absent for short term debt.

4.4 Less price stickiness

In any model, we want to verify that the frictionless limit is sensible. Many Keynesian and new-Keynesian models blow up as one reduces frictions, though the frictionless limit point is sensible. (See Cochrane (2014b).) When the frictionless limit is well-behaved, it is useful see whether the basic sign and mechanisms hold in the frictionless limit point, leaving frictions to fill out dynamics and magnitudes, or whether the frictions are essential to the basic point. Both properties hold here. The frictionless limit is smooth, and the central point – a temporary negative inflation response to higher interest rates – holds in the frictionless limit and frictionless model. Price stickiness, like habits, Taylor responses, and the fiscal response, is useful for producing realistic impulse-response functions, but not necessary for the basic point.

Figure 7 shows the response of inflation (top) and of the price level (bottom) to the

step-function interest rate rise, in the simple model, as we reduce price stickiness. In this model, larger values of δ, the coefficient on consumption in the Phillips curve (21),

$$E_t\left(d\pi_t\right) = \left(\beta\pi_t - \delta c_t\right) dt$$

correspond to less price stickiness. As δ rises, consumption varies less for a given variation in inflation; as $\delta \to \infty$, inflation is independent of consumption, which is the frictionless model.

The response of inflation at the top of Figure 7 seems worrisome: as we reduce stickiness, the negative response of inflation to interest rate rises gets bigger and bigger. This behavior also occurs in the full Sims model. This starts to look like one of the new-Keynesian model pathologies.

But disinflation gets bigger and bigger for a shorter and shorter time. When we plot the response of the price level to the interest rate shock, at the bottom of Figure 7, a clearer picture emerges. The path of inflation approaches a 20% jump down in the price level, followed by steady inflation at the 1% higher inflation rate corresponding to the 1% higher nominal interest rate.

And that limit is also the limit point: A frictionless model with long term debt produces that result, as we saw following equation (21).

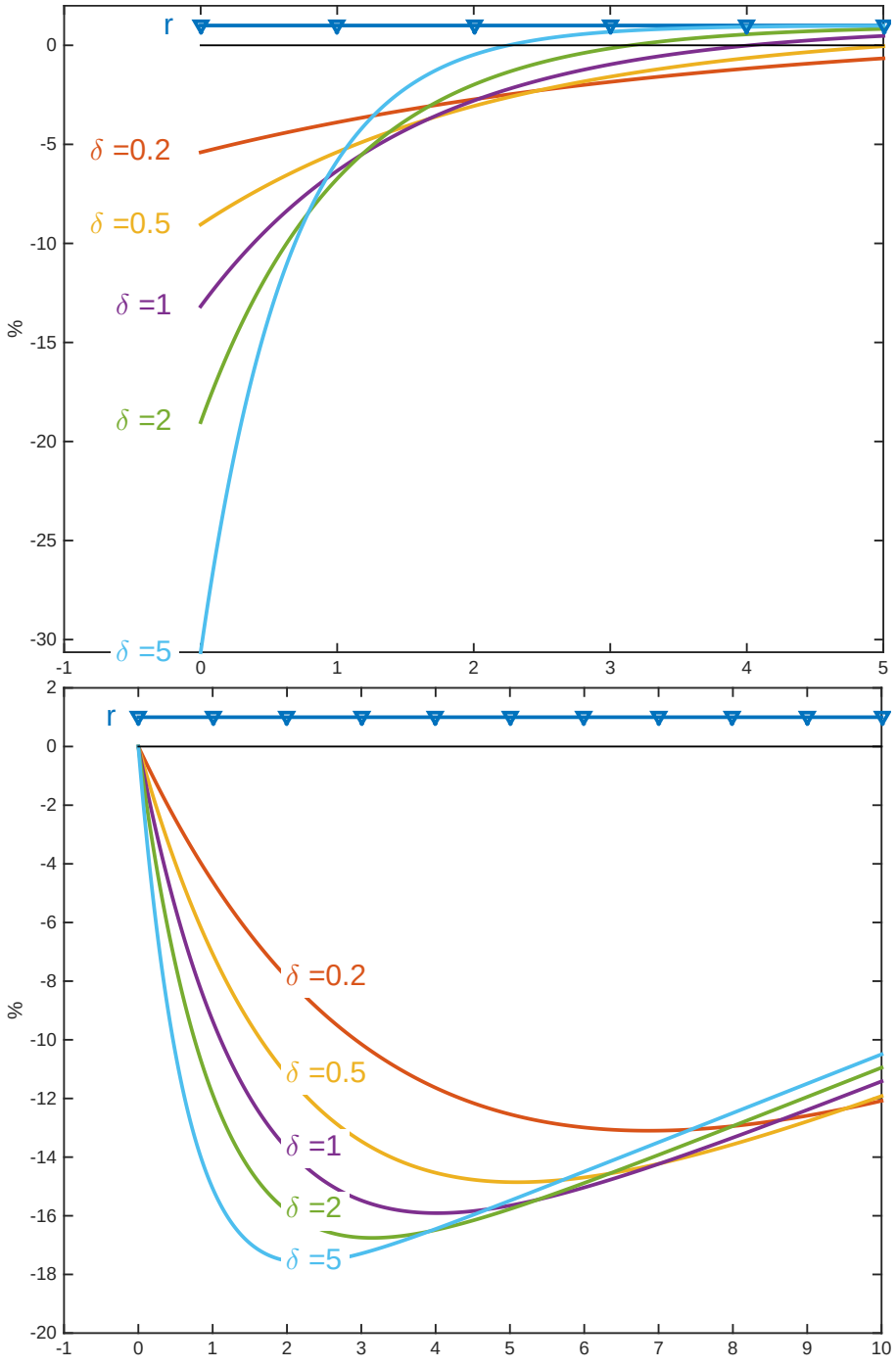

Figure 7: Response of inflation (top) and price level (bottom) to a surprise step function in interest rates, in the simple model with long term debt, as price stickiness is reduced.

5 Conclusions

If you want to understand how monetary policy appears to lower interest rates – and then often struggles with subsequent inflation – as happened in the 1970s, then Sims' model is the basis for elaboration.

It is far from a model of "monetary policy" however. All the action comes from the fiscal theory of the price level. Without a surprise, and a surprise change in the value of government debt, and unless the fiscal authorities keep surpluses constant as inflation devalues their long-term commitments, the model does not produce even the desired negative sign.

That is, however, its most important point. We are used to thinking of fiscal underpinnings as a vague requirement that government finances not go totally off kilter, and then monetary policy can do its job. No. Sims' article points that the fiscal underpinnings are central to understand the sign and dynamics of "monetary" policy.

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6 Appendix. Solving the model without habits

To calculate the $\psi = 0$ limit point, in which consumption can jump, we have to solve it separately for that case, as $1/\psi$ terms show up in the regular model solution. For the $\psi = 0$ case, instead of

$$d\lambda_t = -\lambda_t \left(\rho_t - \bar{\rho}\right) dt + d\delta_{\lambda t} \tag{58}$$

$$dc_t = \dot{c}_t dt \tag{59}$$

$$d\dot{c}_t = \left[\frac{\lambda_t}{\psi} e^{c_t} - \frac{1}{\psi} e^{c_t} e^{-\sigma c_t} + \hat{\rho}\dot{c}_t\right] dt + d\delta_{\dot{c}t} \tag{60}$$

we have

$$d\lambda_t = -\lambda_t \left(\rho_t - \bar{\rho}\right) dt + d\delta_{\lambda t} \tag{61}$$

$$\lambda_t = e^{-\sigma c_t}. \tag{62}$$

We linearize to

$$d\tilde{\lambda}_t = -\tilde{\rho}_t dt + d\delta_{\lambda t} \tag{63}$$

$$\tilde{\lambda}_t = -\sigma c_t \tag{64}$$

We can eliminate λ, so we have

$$dc_t = \frac{1}{\sigma}\tilde{\rho}_t dt + d\delta_{ct} = \frac{1}{\sigma}(\tilde{r}_t - \pi_t)dt + d\delta_{ct}.$$

λ does not appear elsewhere. Next, we must adapt the other appearances of $\dot{c}_t$. To allow a response of fiscal policy to consumption, in place of

$$d\tau_t = \omega \dot{c}_t dt + d\varepsilon_{\tau t}$$

we have

$$d\tau_t = \omega dc_t + d\varepsilon_{\tau t} = \frac{\omega}{\sigma}(\tilde{r}_t - \pi_t)dt + \omega d\delta_{ct} + d\varepsilon_{\tau t}$$

When consumption jumps, so do taxes.

The linearized monetary policy rule

$$d\tilde{r}_t = [-\gamma\tilde{r}_t + \theta\pi_t + \phi\dot{c}_t]\,dt + d\varepsilon_{mt}$$

becomes

$$d\tilde{r}_t = [-\gamma\tilde{r}_t + \theta\pi_t]\,dt + \phi dc_t + d\varepsilon_{mt}$$

$$d\tilde{r}_t = [-\gamma\tilde{r}_t + \theta\pi_t]\,dt + \phi\left[\frac{1}{\sigma}(\tilde{r}_t - \pi_t)dt + d\delta_{ct}\right] + d\varepsilon_{mt}$$

$$d\tilde{r}_t = \left\{\left(\frac{\phi}{\sigma} - \gamma\right)\tilde{r}_t + \left(\theta - \frac{\phi}{\sigma}\right)\pi_t\right\}dt + \phi d\delta_{ct} + d\varepsilon_{mt}$$

The system is then

$$d\begin{bmatrix}\tilde{r}_t\\ \pi_t\\ \tilde{a}_t\\ \tau_t\\ \tilde{b}_t\\ c_t\end{bmatrix} = \begin{bmatrix}\frac{\phi}{\sigma}-\gamma & \theta-\frac{\phi}{\sigma} & 0 & 0 & 0 & 0\\ 0 & \beta & 0 & 0 & 0 & -\delta\\ -\bar{\rho} & 0 & \bar{\rho} & 0 & 0 & 0\\ \omega/\sigma & -\omega/\sigma & 0 & 0 & 0 & 0\\ b & -b & 0 & -1 & \bar{\rho} & 0\\ 1/\sigma & -1/\sigma & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{r}_t\\ \pi_t\\ \tilde{a}_t\\ \tau_t\\ \tilde{b}_t\\ c_t\end{bmatrix}dt + \begin{bmatrix}d\varepsilon_{mt}+\phi d\delta_{ct}\\ d\delta_{\pi t}\\ d\delta_{at}\\ d\varepsilon_{\tau t}+\omega d\delta_{ct}\\ -b/\bar{\rho}d\delta_{at}\\ d\delta_{ct}\end{bmatrix}$$

With three undetermined shocks δ, we need three explosive eigenvalues. The shocks now solve

$$\begin{bmatrix}\left[Q^{-1}\right]_{1,:}\\ \left[Q^{-1}\right]_{2,:}\\ \left[Q^{-1}\right]_{3,:}\end{bmatrix}_{3\times 6}\begin{bmatrix}d\varepsilon_{mt}+\phi d\delta_{ct}\\ d\delta_{\pi t}\\ d\delta_{at}\\ d\varepsilon_{\tau t}+\omega d\delta_{ct}\\ -b/\bar{\rho}d\delta_{at}\\ d\delta_{ct}\end{bmatrix}_{6\times 1} = \begin{bmatrix}0\\0\\0\end{bmatrix}_{3\times 1}$$

for $d\delta_{\pi t}, d\delta_{at}, d\delta_{\lambda t}, d\delta_{ct}$. The matrix carpentry:

$$\begin{bmatrix} d\varepsilon_{mt} + \phi d\delta_{ct} \\ d\delta_{\pi t} \\ d\delta_{at} \\ d\varepsilon_{\tau t} + \omega d\delta_{ct} \\ -b/\bar{\rho} d\delta_{at} \\ d\delta_{ct} \end{bmatrix}_{6\times 1} = \begin{bmatrix} 0 & 0 & \phi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \\ 0 & -b/\bar{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\delta_{\pi t} \\ d\delta_{at} \\ d\delta_{ct} \end{bmatrix}_{3\times 1} + \begin{bmatrix} d\varepsilon_{mt} \\ 0 \\ 0 \\ d\varepsilon_{\tau t} \\ 0 \\ 0 \end{bmatrix}_{6\times 1} \tag{65}$$

$$\begin{bmatrix} d\delta_{\pi t} \\ d\delta_{at} \\ d\delta_{ct} \end{bmatrix}_{4\times 1} = - \left(\begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \end{bmatrix} \begin{bmatrix} 0 & 0 & \phi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \\ 0 & -b/\bar{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} \left[Q^{-1}\right]_{1,:} \\ \left[Q^{-1}\right]_{2,:} \\ \left[Q^{-1}\right]_{3,:} \end{bmatrix} \begin{bmatrix} d\varepsilon_{mt} \\ 0 \\ 0 \\ d\varepsilon_{\tau t} \\ 0 \\ 0 \end{bmatrix}_{8\times 1}$$

Note this will produce a response to the chosen $d\varepsilon_t$ shocks. The actual interest rate move will be larger.