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HABITS AND LEVERAGE

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ABSTRACT

Many stylized facts of leverage, trading, and asset prices follow from a frictionless general equilibrium model that features agents' heterogeneity in endowments and habit preferences. Our model predicts that aggregate debt increases in good times when stock prices are high, return volatility is low, and levered agents enjoy a "consumption boom." Our model is consistent with poorer agents borrowing more and with recent evidence on intermediaries' leverage being a priced factor of asset returns. In crisis times, levered agents strongly deleverage by "fire selling" their risky assets as asset prices drop. Yet, consistently with the data, their debt-to-wealth ratios increase because their wealth decline faster due to higher discount rates.

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1. Introduction

The financial crisis has elicited much research into the understanding of the dynamics of aggregate leverage and its impact on asset prices and economic growth. Recent empirical and theoretical research has produced a variety of results that, as argued by many, should inform a reconsideration of existing frictionless models. Amongst these we have (i) the evidence that excessive credit supply may lead to financial crises;[1] (ii) the growth in household debt and the causal relation between the deleveraging of levered households and their low future consumption growth;[2] (iii) the idea that active leveraging and deleveraging of households and financial institutions directly contributes to the rise and fall of asset prices;[3] (iv) the evidence that the aggregate leverage ratio of financial institutions is a risk factor in asset pricing;[4] (v) the view that balance sheet recessions are critical components of business cycle fluctuations;[5] and many others. Most of these explanations rely on some form of market friction, behavioral bias or both, and propose a causal effect for the effects of leverage on aggregate economic and financial phenomena. In this paper we put forward a simple frictionless general equilibrium model with endogenous leverage that offers a coherent explanation of most of these relations between agents' leverage, their consumption, and asset prices.

We posit an economy populated with agents whose preferences feature external habits. Specifically, agents' utilities are determined by the distance between their own level of consumption and the level of aggregate endowment, appropriately scaled; roughly agents care about consumption inequality. How much agents care about this distance varies across agents and over the business cycle. In particular, agents care more about their relative standing in bad times than in good times and there are some agents who care more than others about this comparison between their own level of consumption and habits. This cross sectional heterogeneity introduces motives for risk sharing and asset trading in general. Agents also differ in their level of endowment, which is also an important determinant of their risk bearing capacity. The model aggregates nicely to standard external habit models such as Campbell and Cochrane (1999) and Menzly, Santos and Veronesi (2004) and thus inherits the asset pricing properties of these models and in particular the dynamics of risk and return that were their original motivation.

External habit models feature strong discount effects, which, as shown by Hansen and

[1] See for instance Jordà, Schularick and Taylor (2011).
[2] See Justiniano, Primiceri and Tambalotti (2013) and Mian and Sufi (2015).
[3] See e.g. Shleifer and Vishny (2011), Geanakoplos (2010).
[4] See He and Krishnamurthy (2013) and Adrian, Etula and Muir (2014).
[5] See Huo and Ríos-Rull (2013) and Mian, Rao and Sufi (2013).

Jagannathan (1991), are required to explain the Sharpe ratios observed in financial markets. We argue that these strong discount effects are also important to understand the dynamics of risk sharing. Standard risk sharing arguments require that agents with large risk bearing capacity insure those with low risk bearing capacity. In models where, for instance agents have CRRA preferences, such as Dumas (1989) and Longstaff and Wang (2013), this means the agents who provide the insurance consume a large share of aggregate consumption when this is large and a low share when instead aggregate consumption is low. This is obviously also the case in our framework, but in addition the share of consumption also depends on whatever state variable drives discount effects, which introduces additional sources of non-linearities in the efficient risk sharing arrangement. The reason is that in our model risk aversion changes depending on the actual realization of the aggregate endowment and thus so do the efficiency gains associated with risk sharing.

We decentralize the efficient allocation by allowing agents to trade in a claim to the aggregate endowment process and debt that is in zero net supply and provide a full characterization of the corresponding competitive equilibrium. We show that agents with higher initial endowment and/or weaker habit preferences have higher risk tolerance and thus provide insurance by issuing risk-free debt to agents with lower endowment and/or stronger habit preferences. The latter agents are more risk averse and hence want to hold risk-free debt to insure against fluctuations in their marginal utility of consumption.

A striking property of the competitive equilibrium is that the aggregate leverage, defined as the total debt-to-output ratio, is procyclical, an intuitive result but one that does not obtain in standard models.[6] The reason hinges on the decrease in aggregate risk aversion in good times, which makes agents with high risk tolerance willing to take on a larger fraction of the aggregate risk by issuing more risk-free debt to agents with lower risk tolerance. Thus, procyclical leverage emerges naturally as the result of the optimal trading of utility maximizing agents in an equilibrium that implements an optimal risk sharing allocation.

Besides procyclical aggregate leverage, our model has several additional predictions that are consistent with numerous stylized facts. First, a mild habit heterogeneity induces agents with low endowment to leverage in equilibrium. That is, unlike most of the previous literature, our model is consistent with the empirical evidence in that poorer agents borrow more than richer agents to increase consumption. Intuitively, habit heterogeneity allows for a large number of low risk averse agents among those with low endowments.

[6]Most models with heterogeneous agents feature only two types of agents. Thus, leverage is necessarily inverse-U shaped in the wealth share, as it must be zero when wealth is mostly in the hands of one or the other agent. Moreover, in such models, lower aggregate risk – typical in good times – tend to reduce leverage due to lower risk-sharing needs.

Second, higher aggregate leverage should correlate with (*i*) higher valuation ratios, (*ii*) lower return volatility, (*iii*) lower future excess returns, and (*iv*) a "consumption boom" of those agents who lever up, who then should experience a consumption slump relative to others, on average. The reason is that as explained above, in good times leverage increases as aggregate risk aversion declines. Lower risk aversion implies high valuation ratios and lower stock return volatility, as well as lower future excess returns, explaining (*i*) through (*iii*). In addition, levered agents who took up levered positions do especially well when stock market increases, implying higher consumption in good times. Mean reversion, however, implies that these same agents should also expect a relatively lower future consumption growth after their consumption binge, explaining (*iv*).

Our model also implies active trading. For instance, a series of negative aggregate shocks induces deleveraging of levered agents through the active sales of their positions in risky stocks. It follows that stock price declines occur exactly at the time when levered agents actively sell their risky positions to reduce leverage. This commonality of asset sales and stock price declines give the impression of a "selling pressure" affecting asset prices, when in fact equilibrium prices and quantities comove due to the variation in aggregate risk aversion, but there is no causal relation between trading and price movements. Indeed, in our model the representative agent is *independent* of agents' heterogeneity and thus the same asset pricing implications result even with identical agents and hence no trading.

While our model implies that during bad times aggregate leverage declines, levered agents' debt-to-wealth ratios increase, as wealth declines faster than debt due to severe discount-rate effects. Hence, while the aggregate level of debt is pro-cyclical, the debt-to-wealth ratio of levered agents is countercyclical, which is broadly consistent with the empirical evidence. For instance, during 2007 - 2009 crisis the debt-to-wealth ratio of levered households increased considerably due to the decline in the value of their assets, especially housing.

Our model's predictions about leverage dynamics sheds some light on recent empirical results in the intermediary asset pricing literature. High net-worth agents lever up to invest in risky securities, as intermediaries do in much of this literature. Because the leverage of these agents correlates with the aggregate economy risk aversion, our model implies that leverage is a priced risk factor in cross-sectional regressions. However, the sign of the price of risk depends on whether we measure leverage using market prices (e.g. debt-to-wealth ratios) or not (e.g. debt-to-output ratio), which is consistent with recent empirical evidence (Adrian, Etula, and Muir (2014) and Kelly, He, and Manela (2016)).

Finally, our model has predictions about the source of the variation in wealth inequality.

Heterogeneity in endowments make wealth inequality increase in good times, as agents with large endowment borrow and thus enjoy capital gains in those times. In contrast, heterogeneity in habits make poor agents borrow, who then enjoy an increase in their wealth in good times and lead to a lower dispersion in wealth shares. These two different sources of heterogeneity thus imply a complex dynamics of wealth dispersion over the business cycle. Once again, the model emphasizes that while asset prices affect wealth inequality, the converse does not hold, as asset prices are identical with homogeneous agents, and hence in the same model without wealth dispersion.

Our model has the considerable advantage of simplicity: All formulas for asset prices, portfolio allocation, and leverage are in closed form, no numerical solutions are required, and their intuition follows from basic economic principles. Moreover, because our model aggregates to the representative agent of Menzly, Santos, and Veronesi (2004), except that we allow for time varying aggregate uncertainty, we can calibrate its parameters to match the properties of aggregate return dynamics. Our model thus, unlike most of the literature, has clears quantitative implications, not just qualitative ones.

Clearly many explanations have been put forth to explain the growth of leverage and of household debt in particular during the run up to the crisis. For instance, Bernanke (2005) argues that the global savings glut, the excess savings of East Asian nations in particular, is to blame for the ample liquidity in the years leading up to the Great Recession, which reduced rates and facilitated the remarkable rise in household leverage; Shin (2012) shows how regulatory changes, the adoption of Basel II, led European banks to increase lending in the US; Pinto (2010), Wallison (2011) and Calomiris and Haber (2014) argue that the Community Reinvestment Act played a pivotal role in the expansion of mortgage lending to risky households (but see Bhutta and Ringo (2015)); Mendoza and Quadrini (2009) show how world financial integration leads to an increase in net credit. The list goes on.

When the crisis came, the crash in prices and the rapid deleveraging of households and financial intermediaries was interpreted appealing to classic inefficient runs arguments a la Diamond and Dybvig (1983) as in Gorton and Metrick (2010) or contagion. He and Krishnamurthy (2008) connect the fall in asset prices to the shortage of capital in the intermediation sector. Finally, much research has focused on the impact that the crisis had on the consumption of households. For instance Mian and Sufi (2014) argue that debt overhang is to blame for the drop in consumption in counties where households were greatly levered.

Our point here is not to claim that these frictions are not important but simply to offer an alternative explanation that is consistent with complete markets and that matches what we know from the asset pricing literature. We highlight that leverage is an endogenous quantity and thus cannot be used as an independent variable to explain other facts. For instance, when debt overhang is put forth as an explanation for low consumption patterns amongst levered households the alternative hypothesis of efficient risk sharing cannot be dismissed outright. Both explanations operate in the same direction and thus assessing the quantitative plausibility of one requires controlling for the other.

This paper is related to the literature on optimal risk sharing, starting with Borch (1962). Much of this literature is concerned with assessing to what extent consumers are effectively insured against idiosyncratic shocks to income and wealth.[7] Our model does not feature idiosyncratic income shocks but there is still a motive for risk sharing that is linked to different sensitivities of habits to aggregate shocks. Our paper is more closely related to Dumas (1989), Wang (1996), Bolton and Harris (2013), Longstaff and Wang (2013), and Bhamra and Uppal (2014). These papers consider two groups of agents with constant risk aversion, and trading and asset prices are generated by aggregate shocks through the variation in the wealth distribution. While similar in spirit, our model generates several novel results that do not follow from this previous work, such as procyclical leverage, countercyclical debt-to-wealth ratios, higher leverage amongst poorer agents, procyclical wealth dispersion, consistency with asset pricing facts, and so on.

Our model is closely related to Chan and Kogan (2002), who also consider a continuum of agents with habit preferences and heterogeneous risk aversion. In their setting, however, the risk aversions of individual agents are constant, while in our setting they are time varying in response to business cycle variation, a crucial ingredient in our model. Moreover, Chan and Kogan (2002) do not investigate the leverage dynamics implied by their model, which is instead our focus. Finally, our paper also connects to the recent literature on the determinants of the supply of safe assets (Barro and Mollerus (2014) and Caballero and Fahri (2014)), though the focus here is on the implications of safe asset shortage for economic activity.

The paper is structured as follows. The next section presents the model. Section 3 characterizes the optimal risk sharing arrangement. Decentralization of the efficient allocation and characterization of the competitive equilibrium are covered in Section 4. Section 5 evaluates the model quantitatively and Section 6 concludes. All proofs are in the Appendix.

[7] See for instance Dynarski and Sheffrin (1987), Cochrane (1991), Mace (1991) and Townsend (1994).

2. The model

Preferences. There is a continuum of agents endowed with log utility preferences defined over consumption C_{it} in excess of agent-specific external habit indices X_{it}:

$$u\left(C_{i,t}, X_{i,t}, t\right) = e^{-\rho t} \log\left(C_{it} - X_{it}\right)$$

Agents are heterogeneous in the habit indices X_{it}, which are given by

$$X_{it} = g_{it}\left(D_t - \int X_{jt} dj\right) \tag{1}$$

That is, the habit level X_{it} of agent i is proportional to the difference between aggregate output D_t and the average habit $\int X_{jt} dj$, which we call the *excess output* henceforth. A higher excess output decreases agent i's utility, an effect that captures a notion of "Envy the Joneses." The excess output $\left(D_t - \int X_{jt} dj\right)$ is in fact an index of the "happiness" of the Joneses – their utility is higher the higher the distance of D_t from average habit $\int X_{jt} dj$ – a fact that makes agent i less happy as it pushes up his habit level X_{it} and thus reduces his utility. Our model is thus an external habit model defined on utility – as opposed to consumption – in that other people happiness is negatively perceived by agent i.

The sensitivity of agent i's habit X_{it} to aggregate excess output $(D_t - \int X_{jt} dj)$ depends on the agent-specific proportionality factor g_{it}, which is heterogeneous across agents and depends linearly on a state variable, to be described shortly, Y_t:

$$g_{it} = a_i Y_t + b_i \tag{2}$$

where $a_i > 0$ and b_i are heterogeneous across agents and such that

$$\int a_i di = 1 \ .$$

Endowment. Aggregate endowment – which we also refer to as dividends or output – follows the process

$$\frac{dD_t}{D_t} = \mu_D \ dt \ + \ \sigma_D(Y_t) \ dZ_t \tag{3}$$

where the drift rate μ_D is constant.[8] The volatility $\sigma_D(Y_t)$ of aggregate endowment – which we refer to as *economic uncertainty* – depends on the state variable Y_t, which follows

$$dY_t = k \ (\overline{Y} - Y_t) \ dt - v \ Y_t \ \left[\frac{dD_t}{D_t} - \mu_D dt\right] \tag{4}$$

[8] As will be shown below the drift μ_D does not play any role into any of relevant formulas, except for the risk-free rate. The main results of the paper are thus consistent with a richer specification of the drift μ_D.

That is, Y_t increases after bad aggregate shocks, $\frac{dD_t}{D_t} < \mu_D dt$, and it hovers around its central tendency $\overline{Y}$. It is useful to interpret Y_t as a *recession indicator*: During good times Y_t is low and during bad times Y_t is high. We assume throughout that Y_t is bounded below by a constant $\lambda \geq 1$. This technical restriction is motivated by our preference specification above and it can be achieved by assuming that $\sigma_D(Y_t) \to 0$ as $Y_t \to \lambda$ (under some technical conditions). We otherwise leave the diffusion terms $\sigma_D(Y_t)$ in (3) unspecified for now, although we normally assume that economic uncertainty is higher in bad times, i.e. $\sigma'_D(Y_t) > 0$.

At time 0 each agent is endowed with a fraction w_i of the aggregate endowment process D_t. The fractions w_i satisfy $\int w_i di = 1$, and the technical condition

$$w_i > \frac{a_i(\overline{Y} - \lambda) + \lambda - 1}{\overline{Y}} \tag{A1}$$

which ensures that each agent has sufficient wealth to ensure positive consumption over habit in equilibrium, and hence well defined preferences. A1 is assumed throughout.

Discussion. Our preference specification differs from the standard external habit model of Campbell and Cochrane (1999) and Menzly, Santos and Veronesi (2004, MSV henceforth). In particular, our model is one *without* consumption externalities as habit levels depend only on exogenous processes and not on consumption choices. This modeling choice allows the application of standard aggregation results which considerably simplifies the analysis.

Second our model features two relevant sources of variation across agents: Initial endowments, as summarized by the distribution of ω_i, and the sensitivity of individual habits X_{it} to excess output, as summarized by g_{it}, which results in differences in attitudes towards risk. These two dimensions seem a natural starting point to investigate optimal risk sharing as well as portfolio decisions.[9]

Notice though that our model features no idiosyncratic shocks to individual endowment as agents simply receive a constant fraction w_i of the aggregate endowment process. Individual endowment processes are thus perfectly correlated and thus they are not the driver of risk sharing motives. Instead in our model risk sharing motives arise exclusively because agents are exposed differently to business cycle fluctuations through their sensitivity to habits. Indeed how sensitive agents are to shocks in excess output depend on the state variable Y_t. Economically, assumption (2) implies that in bad times (after negative output shocks) the habit loadings g_{it} increase, making habit preferences become more important on average.

[9]For instance, two recent theoretical contributions that consider these two sources of cross sectional variation are Longstaff and Wang (2012) and Bolton and Harris (2013). Empirically these sources of variation have been investigated by, for example, Chiappori and Paeilla (2011) and Calvet and Sodini (2014), though the results in these two papers are rather different.

However, different sensitivities a_i imply that changes in Y_t differentially impact the external habit index as g_{it} increase more for agents with high a_i than for those with low a_i. We set $b_i = \lambda(1 - a_i) - 1$, which ensures $g_{it} > 0$ for every i and for every t (as $Y_t > \lambda$), and allows for a simple aggregation below. This assumption does not affect the results.

Finally, we note that the case of homogeneous preferences ($a_i = 1$ for all i) and/or homogeneous endowments ($w_i = 1$ for all i) are special cases, as is the case in which habits are constant ($v = 0$ in (4)). We investigate these special cases as well below.

3. Optimal risk sharing

As already mentioned, markets are complete and therefore standard aggregation results imply that a representative agent exists, a planner, that solves the program

$$U(D_t, \{X_{it}\}, t) = \max_{C_{it}} \int \phi_i u(C_{it}, X_{it}, t)\, di \qquad \text{subject to} \qquad \int C_{it} di = D_t \tag{5}$$

where all Pareto weights $\phi_i > 0$ are set at time zero, renormalized such that $\int \phi_i di = 1$ and are consistent with the initial distribution of wealth in a way to be described shortly. The first order condition implies that

$$u_C(C_{it}, X_{it}, t) = \frac{\phi_i e^{-\rho t}}{C_{it} - X_{it}} = M_t \qquad \text{for all } i, \tag{6}$$

where M_t is the Lagrange multiplier associated with the resource constraint in (5).[10] Straightforward calculations[11] show that

$$M_t = \frac{e^{-\rho t}}{D_t - \int X_{jt} dj} \qquad \text{and} \qquad C_{it} = (g_{it} + \phi_i)\left(D_t - \int X_{jt} dj\right). \tag{7}$$

The optimal consumption of agent i increases if the excess output, $D_t - \int X_{jt} dj$, increases or if the habit loading g_{it} increases. This is intuitive, as such agents place relatively more weight on excess output and thus want to consume relatively more. In addition, agents with a higher Pareto weight ϕ_i also consume more as they are favored by the social planner.

We finally aggregate total optimal consumption and impose market clearing to obtain

$$D_t = \int C_{it} di = \left[\int (g_{it} + \phi_i)\, di\right]\left(D_t - \int X_{it} di\right). \tag{8}$$

[10]This result was first derived by Borch (1962, equation (1) p. 427).

[11]It is enough to solve for C_{it} in (6), integrate across agents (recall $\int \phi_i di = 1$), and use the resource constraint to yield M_t. Plugging this expression in (6) yields C_{it}.

Using $\int \phi_i di = 1$, we can solve for the equilibrium excess output as

$$D_t - \int X_{it} di = \frac{D_t}{\int g_{it} di + 1} > 0. \tag{9}$$

This intermediate result also shows that individual excess consumption $C_{it} - X_{it}$ is positive for all i, which ensures all agents' utility functions are well defined.[12] Notice also an important implication of (9) and that is that preferences can be expressed as

$$u\left(C_{i,t}, X_{i,t}, t\right) = e^{-\rho t} \log\left(C_{it} - \psi_{it} D_t\right) \qquad \text{with} \qquad \psi_{it} \equiv \frac{g_{it}}{\int g_{it} di + 1}.$$

Individual agents compare their own consumption to aggregate endowment properly scaled by ψ_{it}, which is agent specific and dependent on Y_t. Roughly agents care about their relative standing in society, which is subject to fluctuations. It is these fluctuations what introduces motives for risk sharing. The next proposition solves for the Pareto weights and the share of the aggregate endowment that each agent commands.

Proposition 1 *(Efficient allocation). Let the economy be at its stochastic steady state at time 0, $Y_0 = \overline{Y}$, and normalize $D_0 = \rho$. Then (a) the Pareto weights are*

$$\phi_i = a_i \lambda + (w_i - a_i)\,\overline{Y} + 1 - \lambda \tag{10}$$

(b) The share of the aggregate endowment accruing to agent i is given by

$$C_{it} = \left[a_i + (w_i - a_i)\frac{\overline{Y}}{Y_t}\right] D_t \qquad \text{or} \qquad s_{it} \equiv \frac{C_{it}}{D_t} = a_i + (w_i - a_i)\frac{\overline{Y}}{Y_t} \tag{11}$$

Pareto weights (10) are increasing in the fraction of the initial aggregate endowment w_i and decreasing in habit sensitivity a_i. The first result is standard. To understand the second, given optimal consumption (7), agents with higher sensitivity a_i have a higher habit loading $g_{it} = a_i(Y_t - \lambda) + \lambda - 1$ and thus would like to consume more. Given (7), for given initial endowment w_i, the Pareto weight ϕ_i must then decline to ensure that such consumption can be financed by the optimal trading strategy.

Equation (11) captures the essential properties of the optimal risk sharing rule, that is, agents with high endowment w_i or low habit sensitivity a_i enjoy a high consumption share $s_{it} = C_{it}/D_t$ during good times, that is, when the recession indicator Y_t is low, and vice versa. To grasp the intuition consider first the curvature of the utility function of an individual agent, which we refer to as "risk aversion" for simplicity:

$$Curv_{it} = -\frac{C_{it} u_{cc}(C_{it}, X_{it}, t)}{u_c(C_{it}, X_{it}, t)} = 1 + \frac{a_i(Y_t - \lambda) + \lambda - 1}{w_i \overline{Y} - a_i(\overline{Y} - \lambda) - \lambda + 1}. \tag{12}$$

[12] To see this, substitute the excess output into (7) and use (1). Given g_{it} in (2), we have $\int g_{it} di + 1 = Y_t$.

Expression (12) shows that agents with higher endowment w_i or lower habit sensitivity a_i have lower risk aversion. Moreover, an increase in recession indicator Y_t increases the curvature of every agent, but more so for agents with a high habit sensitivity a_i or low endowment w_i. These variations in curvature generates the need for risk sharing as embedded in the sharing rule (11).

Preference heterogeneity and business cycle variation combine to determine the planner's transfer scheme needed to support the optimal allocation. Let $\tau_{it} > 0$ be the transfer received by agent i at time t above her endowment $w_i D_t$; if instead the agent consumes below her endowment then $\tau_{it} < 0$. Trivial computations prove the next corollary.[13]

Corollary 2 *The transfers that implement the efficient allocation are given by*

$$\tau_{it} = -\left(w_i - a_i\right)\left(1 - \frac{\overline{Y}}{Y_t}\right) D_t. \tag{13}$$

Notice that agents for whom $w_i - a_i > 0$ receive transfers, $\tau_{it} > 0$, when $Y_t < \overline{Y}$, that is in good times, and pay $\tau_{it} < 0$ in bad times, when $Y_t > \overline{Y}$. The opposite is the case for the agents for whom $w_i - a_i < 0$. In effect, optimal risk sharing requires agents with $w_i - a_i > 0$ to insure agents with $w_i - a_i < 0$.

We emphasize an important attribute of our model and that is that habits are key to deliver all the results in our paper. Indeed, assume that $Y_t = \overline{Y}$ for all t (i.e. $v = 0$ in (4)). In this case our model collapses to an economy populated with agents with log preferences, the share of consumption of each agent is simply $s_{it} = w_i$ and, as it will be shown below, no trading occurs amongst agents. Thus, our model does not deliver risk sharing motives beyond what is induced by the habit features of our preference specification.

4. Competitive equilibrium

4.1. Decentralization

Financial markets. Having characterized the optimal allocation of risk across agents in different states of nature we turn next to the competitive equilibrium that supports it. Clearly we can introduce a complete set of Arrow-Debreu markets at the initial date, let agents trade and after that simply accept delivery and make payments. It was Arrow's (1964) original insight that decentralization can be achieved with a sparser financial market structure. There

[13] Simply subtract from the optimal consumption allocation (11) the consumption under autarchy, $w_i D_t$.

are obviously many ways of introducing this sparser financial market structure but here we follow many others and simply introduce a stock market and a market for borrowing and lending. Specifically we assume that each of the agents i is endowed with an initial fraction w_i of a claim to the aggregate endowment D_t. We normalize the aggregate number of shares to one and denote by P_t the price of the share to the aggregate endowment process, which is competitively traded. Second, we introduce a market for borrowing and lending between agents. Specifically we assume that there is an asset in zero net supply, a bond, with a price B_t, yielding an instantaneous rate of return of r_t, so that $B_t = e^{\int_0^t r_u du}$. Both P_t and r_t are determined in equilibrium. Because all quantities depend on one Brownian motion (dZ_t), markets are dynamically complete.

The portfolio problem. Armed with this we can introduce the agents' problem. Indeed, given prices $\{P_t, r_t\}$ agents choose consumption C_{it} and portfolio allocations in stocks N_{it} and bonds N_{it}^0 to maximize their expected utilities

$$\max_{\{C_{it}, N_{it}, N_{it}^0\}} E_0 \left[\int_0^{\infty} e^{-\rho t} \log \left(C_{it} - X_{it} \right) dt \right]$$

subject to the budget constraint equation

$$dW_{it} = N_{it}(dP_t + D_t dt) + N_{it}^0 B_t r_t dt - C_{it} dt$$

with initial condition $W_{i,0} = w_i P_0$.

Definition of a competitive equilibrium. A competitive equilibrium is a series of stochastic processes for prices $\{P_t, r_t\}$ and allocations $\{C_{it}, N_{it}, N_{it}^0\}_{i \in \mathcal{I}}$ such that agents maximize their intertemporal utilities and markets clear $\int C_{it} di = D_t$, $\int N_{it} di = 1$, and $\int N_{it}^0 di = 0$. The economy starts at time 0 in its stochastic steady state $Y_0 = \overline{Y}$. Without loss of generality, we normalize the initial output $D_0 = \rho$ for notational convenience.

The competitive and the decentralization of the efficient allocation. We are now ready to describe the competitive equilibrium and show that it indeed supports the efficient allocation. We leave the characterization of the equilibrium for the next section.

Proposition 3 *(Competitive equilibrium). Define the surplus consumption ratio as in Campbell and Cochrane (1999) and Menzly, Santos, and Veronesi (2004) as*

$$S_t = \frac{D_t - \int X_{it} di}{D_t} = \frac{1}{Y_t}, \tag{14}$$

where the last equality stems from (9), and denote with some mild abuse of notation $\sigma_D(Y_t) = \sigma_D(S_t)$*. Then the following price processes and allocations support the efficient allocation (11) as a competitive equilibrium outcome:*

1. *Stock prices and interest rates*

$$P_t = \left(\frac{\rho + k\overline{Y}S_t}{\rho(\rho+k)}\right) D_t \tag{15}$$

$$r_t = \rho + \mu_D - (1-v)\sigma_D^2(S_t) + k\left(1 - \overline{Y}S_t\right) \tag{16}$$

2. *The position in bonds $N_{it}^0 B_t$ and stocks N_{it} of agent i at time t are, respectively,*

$$N_{it}^0 B_t = -v\,(w_i - a_i)\,H(S_t)\,D_t \tag{17}$$

$$N_{it} = a_i + (\rho + k)(1+v)\,(w_i - a_i)\,H(S_t) \tag{18}$$

where

$$H(S_t) = \frac{\overline{Y}S_t}{\rho + k(1+v)\overline{Y}S_t} > 0 \tag{19}$$

4.2. Asset prices

The stock price in Proposition 3 is identical to the one found in MSV, which obtained in a representative consumer model. The reason is that our model does indeed aggregate to yield a representative consumer similar to the one in that paper. Indeed, having solved for the Pareto weights (10) and the individual consumption allocations we can substitute back in the objective function in (5) and obtain the equilibrium state price density.

Proposition 4 *(The stochastic discount factor). The equilibrium state price density is*

$$M_t = e^{-\rho t} D_t^{-1} S_t^{-1}. \tag{20}$$

Given the risk-free rate r_t in (16), the stochastic discount factor follows

$$\frac{dM_t}{M_t} = -r_t dt - \sigma_{M,t} dZ_t \qquad \textit{with} \qquad \sigma_{M,t} = (1+v)\sigma_D(S_t), \tag{21}$$

The state price density in (20) is similar to the one in Campbell and Cochrane (1999) and MSV. Equation (14) shows that the recession indicator Y_t is the inverse surplus consumption ratio of MSV. Indeed, as in this earlier work, Y_t can be shown to be linearly related to the aggregate risk aversion of the representative agent (see footnote 4 in MSV).

We are now ready to discuss the asset prices in Proposition 3. Start, briefly, with the risk free rate r_t. The terms $\rho + \mu_D - \sigma_D^2(S_t)$ in (16) are the standard log-utility terms, namely, time discount, expected aggregate consumption growth, and precautionary savings. The additional two terms, $k(1-\overline{Y}S_t)$ and $v\ \sigma_D(S_t)$, are additional intertemporal substitution and precautionary savings terms, respectively, associated with the external habit features of the model (see MSV for details).

As for the stock price (15), the intuition for this expression is by now standard (Campbell and Cochrane (1999) and MSV). A negative aggregate shock $dZ_t < 0$ decreases the price directly through its impact on D_t, but it also increases the risk aversion Y_t and hence reduces $S_t = 1/Y_t$, which pushes down the stock price P_t further. External habit persistence models thus generate variation in prices that are driven not only by cash-flow shocks but also discount effects. Indeed, we show in the Appendix the volatility of stock returns is

$$\sigma_P(S_t) = \sigma_D(S_t)\left(1 + \frac{vk\overline{Y}S_t}{\rho + k\overline{Y}S_t}\right). \tag{22}$$

In addition, as shown in (21), the market price of risk also is time varying, not only because of the variation in consumption volatility ($\sigma_D(S_t)$) but also because of the variation in the volatility of aggregate risk aversion, given by $v\sigma_D(S_t)$. In MSV, a lower surplus consumption ratio S_t increases the average market price of risk and makes it time varying. This generates the predictability of stock returns. Indeed, denoting the total stock return as $dR_P = (dP_t + D_t dt)/dt$, the risk premium

$$E_t\left[dR_P - r_t dt\right] = \sigma_M(S_t)\sigma_P(S_t)dt \tag{23}$$

increases compared to the case with log utility both because the aggregate amount of risk $\sigma_P(S_t)$ increases and because the market price of risk $\sigma_M(S_t)$ increases.

An important property of asset prices (P_t and r_t) in our model is the following:

Corollary 5 *Asset prices are independent of the endowment distribution across agents as well as the distribution of preferences. In particular the model has identical asset pricing implications even if all agents are identical, i.e.* $a_i = 1$ *and* $w_i = 1$ *for all* i.

The asset pricing implications of our model are thus "orthogonal" to its cross sectional implications: P_t in equation (15) and r_t in (16) are independent of the distribution of either current consumption or wealth in the population. This property distinguishes our model from the existing literature such as Longstaff and Wang (2012) or Chan and Kogan (2002). Importantly, in this earlier literature the variation in risk premia is driven by endogenous changes in the cross-sectional distribution of wealth. Roughly more risk-tolerant agents hold a higher proportion of their wealth in stocks. A drop in stock prices reduces the fraction of aggregate wealth controlled by such agents and hence their contribution to the aggregate risk aversion. The conditional properties of returns thus rely on strong fluctuations in the cross sectional distribution of wealth.

In contrast, in the present paper agents' risk aversions change, which in turn induces additional variation in premia and puts less pressure on the changes in the distribution of wealth to produce quantitatively plausible conditional properties for returns. Indeed, Corollary 5 asserts exactly that the asset pricing implications are identical even when agents are homogeneous and thus there is no variation in cross-sectional distribution of wealth. Corollary 5 thus allows us to separate cleanly the asset pricing implications of our model from its implications for trading, leverage and risk sharing, which we further discuss below. In particular, the corollary clarifies that equilibrium prices and quantities do not need to be causally related to each other, but rather comove with each other because of fundamental state variables, such as S_t in our model.

4.3. Leverage and risk sharing

We turn next to the characterization of the portfolio strategies in Proposition 3.

Corollary 6 *(Individual leverage). (a) The position in bonds is* $N_{it}^0 B_t < 0$ *if and only if* $w_i - a_i > 0$*. That is, agents with* $w_i > a_i$ *take on leverage.*

(b) The investment in stock of agent i *in proportion to wealth is*

$$\frac{N_{it}P_t}{W_{it}} = \frac{1 + v\left(1 - \frac{\rho}{\rho + \overline{Y}[k + (\rho + k)(w_i - a_i)/a_i]S_t}\right)}{1 + v\left(1 - \frac{\rho}{\rho + \overline{Y}kS_t}\right)} > 1 \quad \textit{if and only if} \quad w_i - a_i > 0. \tag{24}$$

Recall that, as shown in equation (13), optimal risk sharing requires transfers from agents with $w_i - a_i > 0$ to those with $w_i - a_i < 0$ when Y_t is high (or S_t is low) and the opposite when Y_t is low (or S_t is high). Equations (17) and (18) show the portfolios of stocks and bonds needed to implement the efficient allocation. This is achieved by having the agents with large risk bearing capacity, agents with $w_i - a_i > 0$, issue debt in order to insure those agents with lower risk bearing capacity, $w_i - a_i < 0$. Part (b) of Corollary 6 shows that indeed agents with $w_i - a_i > 0$ lever up to achieve a position in stocks that is higher than 100% of their wealth.

Expression (24) shows that for given level of habit sensitivity a_i, agents with higher wealth w_i invest comparatively more in stocks, a result that finds empirical support in Wachter and

Yogo (2010). Indeed, as in their paper, our habit preferences imply that utility is not homothetic in wealth (due to habit), thereby implying that agents with a higher endowment invest comparatively more in the risky asset.

Expressions (17) and (18) show that the amount of leverage and asset allocation depend on the function $H(S_t)$, which is time varying as the recession indicator $Y_t = S_t^{-1}$ moves over time. We discuss the dynamics of leverage in the next section.

4.4. The supply of safe assets: Leverage dynamics

A particular feature of our model is that that the risk attitudes of the agents in the economy fluctuate with the recession indicator Y_t (see equation (12)). As Y_t increases, for instance, the risk bearing capacity of the agents for whom $w_i - a_i > 0$ decreases precisely when the demand for insurance by the agents with $w_i - a_i < 0$ increases. The supply of safe assets, to use the term that has become standard in the recent literature, may decrease precisely when it is most needed,[14] an issue explored by some recent papers.[15] In this section we focus on the dynamics of the aggregate leverage, which we define as total debt scaled by total output:

$$L(S_t) \equiv \frac{-\int_{i:N_{it}^0<0} N_{it}^0 B_t di}{D_t}$$

where the negative sign is to make this number positive. We find that aggregate leverage is

$$L(S_t) = vK_1 H(S_t) \quad \text{where} \quad K_1 \equiv \int_{i:(w_i - a_i)>0} (w_i - a_i)\, di > 0 \tag{25}$$

and the function $H(S_t)$ is in (19). It is immediate to see that $H(S_t)$ is strictly increasing in S_t, yielding the following corollary:

Corollary 7 *(Aggregate leverage). Aggregate leverage $L(S_t)$ is procyclical, increasing in good times (high S_t) and decreasing in bad times (low S_t).*

To gain intuition, note first that risk sharing and leverage are two related but distinct concepts. While the amount of risk sharing is due to agents' preferences and their need

[14]In our model the debt issued by the agents with the largest risk bearing capacity is safe because they delever as negative shocks accumulate in order to maintain their marginal utility bounded away from infinity.

[15]See for instance Barro and Mollerus (2014), who propose a model based on Epstein-Zin preferences to offer predictions about the ratio of safe assets to output in the economy. Gorton, Lewellen and Mettrick (2012) and Krishnamurthy and Vissing-Jorgensen (2012) provide empirical evidence regarding the demand for safe assets. In all these papers the presence of "outside debt" in the form of government debt plays a critical role in driving the variation of the supply of safe assets by the private sector, a mechanism that is absent in this paper.

to equalize their marginal rates of substitutions across states and times (i.e. the planner's problem), in the decentralized economy the amount of trading and leverage critically depends on the types of securities that are available in the market.

In this paper we only allow agents to trade a single stock that is a claim to total future dividends, and risk-free bonds in zero net supply. The *equilibrium* properties of the assets used to achieve optimal risk sharing are critical. To understand the intuition about procyclical leverage, consider first the value of a contingent claim that pays the consumption of agent i, C_{it}, as its dividend rate. We denote such contingent claim P_{it}. The value of such contingent claim, if it existed, would be (see Appendix):

$$P_{it} = E_t\left[\int_t^{\infty} \frac{M_\tau}{M_t} C_{i\tau} d\tau\right] = \frac{\rho a_i + (\rho(w_i - a_i) + k w_i)\overline{Y} S_t}{\rho(\rho + k)} D_t. \tag{26}$$

If this asset was traded, agent i would just purchase it at time $t = 0$ and live happily thereafter, as this security pays his/her optimal consumption for each future state and time. While this security is not traded, because of dynamically complete markets agent i can "manufacture it" through a proper trading strategy of the available assets. That is, agent i would like his/her portfolio allocation in stocks and bonds to be such that for every t:[16]

$$N_{it} P_t + N_{it}^0 B_t = P_{it} \tag{27}$$

For this to be satisfied for every t (and pay C_{it} as dividend), we must have that the portfolio and the security have the same sensitivity to shocks dZ_t. Denoting by $\sigma_{Pi}(S_t)$ the volatility of P_{it}, the portfolio allocation N_{it} and N_{it}^0 must then satisfy

$$N_{it} = \frac{P_{it}\sigma_{Pi}(S_t)}{P_t \sigma_P(S_t)} \qquad \text{and} \qquad N_{it}^0 B_t = P_{it} - N_{it} P_t = P_{it}\left(1 - \frac{\sigma_{Pi}(S_t)}{\sigma_P(S_t)}\right). \tag{28}$$

That is, the bond position, $N_{it}^0 B_t$, crucially depends on the ratio of volatilities $\frac{\sigma_{Pi}(S_t)}{\sigma_P(S_t)}$: If this ratio is greater than one, the agent is leveraging his/her investment in the stock market. The volatility of the contingent claim is

$$\sigma_{Pi}(S_t) = \sigma_D(S_t)\left(1 + \frac{v\left(k + (\rho + k)(w_i - a_i)/a_i\right)\overline{Y} S_t}{\rho + \left(k + (\rho + k)(w_i - a_i)/a_i\right)\overline{Y} S_t}\right). \tag{29}$$

Comparing this expression with $\sigma_P(S_t)$ in (22), we see that $\sigma_{Pi}(S_t) > \sigma_P(S_t)$ if and only if $w_i - a_i > 0$. That is, agents with $w_i - a_i > 0$ leverage their portfolio. Intuitively, from

[16] This argument follows Cox and Huang (1989).

the optimal risk sharing rule (11), agents with a high $w_i - a_i > 0$ have a high consumption share in good times, when S_t is high, and a low consumption share in bad times, when S_t is low. This particular consumption profile implies that the value of the contingent claim P_{it} is more sensitive to discount rate shocks than the stock price P_t. As a result the "replicating" portfolio requires some leverage to match such sensitivity.

Equation (28) also highlights the reason why aggregate leverage, $L(S_t)$, increases in good times (high S_t). This is due to a "level effect": from (29) and (22) the ratio of volatilities actually declines as S_t increases. This is intuitive as the hypothetical contingent claim pays out more in good times and hence becomes less sensitive to discount rate shocks then. However, from (26) the value of the hypothetical contingent claim P_{it} increases in good times because the discount rate declines and more than overcomes the decline in the ratio of volatilities. As a result, the aggregate leverage increases in good times.

While an aggregate procyclical leverage may seem intuitive, it is not normally implied by, for instance, standard CRRA models with differences in risk aversion. In such models, less risk averse agents borrow from more risk averse agents, who want to hold riskless bonds rather than risky assets. As aggregate wealth becomes more concentrated in the hands of less risk-averse agents, the need of borrowing and lending declines, which in turn decreases aggregate leverage. Moreover, a decline in aggregate uncertainty – which normally occur in good times – actually decreases leverage in such models, as it reduces the risk-sharing motives of trade. In our model, in contrast, the decrease in aggregate risk aversion in good times make agents with high-risk bearing capacity even more willing to take on risk and hence increase their supply of risk-free assets to those who have a lower risk bearing capacity.

Finally notice that good times, periods when S_t is high, also periods when expected excess returns are low as both the market price of risk $\sigma_M(S_t)$ and aggregate uncertainty $\sigma_D(S_t)$ are low.[17] Thus high aggregate leverage $L(S_t)$ should predict low future excess returns.

4.5. Individual leverage and consumption

The following corollary follows immediately from Proposition 1 and Corollary 6.

Corollary 8 *Agents with higher leverage enjoy higher consumption share during good times.*

After a sequence of good economic shocks aggregate risk aversion declines. Thus, agents with positive $(w_i - a_i)$ increase their leverage and experience a consumption "boom". The

[17]Note that we have not made any assumptions yet on $\sigma_D(S_t)$, except that it vanishes for $S_t \to \lambda^{-1}$.

two effects are not directly related, however. The increase in consumption is due to the higher investment in stocks that have higher payoffs in good times. Because good times also have lower aggregate risk aversion, moreover, these same agents also increase their leverage at these times. Hence, our model predicts a positive comovement of leverage and consumption at the household level. An implication of this result is that agents who took on higher leverage during good times are those that suffer a bigger drop in consumption growth as S_t mean reverts. In particular, we have the following corollary:

Corollary 9 *Agent i's consumption growth satisfies*

$$E\left[\frac{dC_{it}}{C_{it}}\right]/dt = \mu_D + \frac{(w_i - a_i)\overline{Y}S_t}{a_i + (w_i - a_i)\overline{Y}S_t}F(S_t) \tag{30}$$

with

$$F(S_t) = k(1 - \overline{Y}S_t) + (1+v)v\ \sigma_D^2(S_t) \tag{31}$$

If $\sigma_D(S_t)$ is decreasing in S_t with $\sigma_D(\lambda^{-1}) = 0$, then the function $F(S)$ has $F'(S) < 0$ and $F(0) > 0$ and $F(\lambda^{-1}) = k(1 - \lambda^{-1}\overline{Y}) < 0$. Thus, there exists a unique solution S^* to $F(S^*) = 0$ such that for all i and j with $w^i - a^i > 0$ and $w^j - a^j < 0$ we have

$$E\left[\frac{dC_{it}}{C_{it}}\right] < \mu_D < E\left[\frac{dC_{jt}}{C_{jt}}\right] \qquad \text{for} \qquad S_t > S^* \tag{32}$$

$$E\left[\frac{dC_{it}}{C_{it}}\right] > \mu_D > E\left[\frac{dC_{jt}}{C_{jt}}\right] \qquad \text{for} \qquad S_t < S^* \tag{33}$$

This corollary shows that cross-sectionally agents with high $w_i - a_i > 0$ have a lower expected growth rate of consumption when S_t is high. We know that these are also times when such agents are heavily leveraged. It follows then that agents who are heavily leveraged enjoy both a high consumption boom in good times, but a lower *future* expected consumption growth. These agents also expect a higher consumption growth when S_t is low. Therefore, Corollaries 7 and 9 imply the following:

Corollary 10 *Periods with high aggregate leverage $L(S_t)$ forecast lower consumption growth for highly levered agents compared to those with lower leverage.*

That is, according to Corollary 10, periods of very high aggregate leverage should follow on average by periods in which levered agents "retrench" and experience consumption growth that is comparatively lower than those agents who did not take on leverage.

This implication of our model speaks to some of the recent debate regarding the low consumption growth of levered households following the Great Recession. Some argue that the observed drop in consumption growth was purely due to a wealth effect, as levered households tend to live in counties that experienced big drops in housing values, whereas others have emphasized the critical role of debt in explaining this drop.[18] Clearly these effects are important but our contribution is to show that high leverage followed by low consumption growth is precisely what arises from risk sharing arguments in models that can address the observed conditional properties of asset returns, as external habit models do.

Indeed, suppose that such cross-sectional differences have a spatial nature, e.g. counties with richer agents or agents with lower risk aversion. Then, such counties should experience a credit boom during good times with high consumption growth, followed by a relative consumption slump during bad times. Corollary 10 then highlights the crucial role of proper identification strategies when asserting that higher leverage *casually* make levered counties (or countries) suffer comparatively slower growth in the future. If there is any residual correlation between the instrument employed in the identification strategy and S_t in previous equations, then the causality interpretation of standard instrumental variable or diff-in-diff estimators is undermined.

Finally, comparing the function $F(S)$ in Equation (31) with the interest rate expression in Equation (16), we see that the term $k(1-\overline{Y}S_t)$ enters both expressions. As explained in Campbell and Cochrane (1999) and Menzly, Santos, and Veronesi (2004), this term reflects the impact on interest rates of agents' preferences for intertemporal consumption smoothing due to the predictability of the surplus consumption ratio. Thus, in a deterministic model in which $\sigma_D = 0$ but $\overline{Y}S_t \neq 1$, this common term affects both the interest rate and expected consumption of agents. Specifically, an increase in interest rate due to this term induces agents with positive $w_i - a_i$ to increase future consumption growth. That is, such agents have a low elasticity of intertemporal substitution. Conversely, agents with $w_i - a_i < 0$ tend to have an expected consumption growth that reacts negatively to changes in interest rates, thereby showing a high elasticity of intertemporal substitution.

4.6. Active trading in stocks and bonds

Corollary 11 *(Active trading) (a) Agents with positive leverage (i.e. with $w_i - a_i > 0$) increase their stock position in good times (S_t high) and decrease it in bad times (S_t low.) Agents with negative levrage (i.e. with $w_i - a_i < 0$) do the opposite.*

[18]See for instance Mian and Sufi (2014, in particular pages 39-45) for a nice exposition of this debate.

(b) Agents with higher absolute difference $|w_i - a_i|$ trade more in response to changes to the aggregate surplus consumption ratio S_t.

Corollary 11-a says that agents with positive leverage increase the number of units of stocks purchased in good times, and decrease them in bad times. That is, these agents *actively trade* in stocks. In a model with passive investors, an agent who is long stocks may mechanically find himself with a higher allocation in stocks during good times because the stock yields good returns in good times. Corollary 11-a instead says that an agent who is leveraged ($w_i - a_i > 0$) actively *increases leverage* in good times to buy more shares of stocks in such times. Such agent acts as a trend chaser, as he increases his stock positions after good market news. Conversely, agents with $w_i - a_i < 0$ do the opposite and hence act as contrarian investors. Corollary 11-b predicts that there is heterogeneity in trading, as some agents trade more than others.

Corollary 11-a also implies that levered traders *actively deleverage* as times are getting worse (S_t declines) by actively *selling* the risky assets. In fact, such deleveraging is especially strong during "crisis" times, as shown next:

Corollary 12 *("Panic deleveraging") The function $H(S_t)$ in (19) is concave in S_t. Therefore, both leverage and asset holdings of levered agents decrease by an increasing larger amount as time get worse, i.e. as S_t declines.*

Corollary 12 shows that $H(S_t)$ is not only increasing in S_t but it is concave in it. Such concavity has an important additional economic implication: during good times (S_t high) we should observe higher aggregate leverage and higher asset holdings of levered agents, but less variation of both compared to bad times (S_t low). This implies that as S_t declines, levered agents decrease their leverage by an increasingly larger amount, giving the impression of a "panic deleveraging" during bad times.

Because deleveraging occurs as both the stock price plunges and the wealth of levered investors drops, an observer may be tempted to conclude that the "selling pressure" of deleveraging agents is the cause of the drop in the stock price. While in reality such effects may occur, in our model the joint dynamics of deleveraging and price drop happens for the simple reason that during bad times aggregate risk aversion increases. Indeed, as shown in Corollary 5, the same asset pricing implications obtain even without heterogeneity and hence no trade. Our model then should caution against the excessive reliance on the simple intuition of price declines due to the "price pressure" of some agents in the economy.

4.7. The dynamics of wealth and wealth dispersion

In our model, all wealth is financial in nature, as it is composed by positions in stocks and bonds $W_{it} = N_{it}P_t + N_{it}^0 B_t$. It immediately follows from (27) that each agent's wealth equals the contingent claim P_{it} discussed in (26), i.e. $W_{it} = P_{it}$. Next proposition follows:

Proposition 13 *(a) The wealth-output ratio of agent i is given by*

$$\frac{W_{it}}{D_t} = \frac{1}{\rho}\left[\frac{\rho}{\rho+k}a_i\left(1-\overline{Y}S_t\right) + w_i\overline{Y}S_t\right] \tag{34}$$

(b) The wealth-share of agent i is given by

$$\frac{W_{it}}{\int_j W_{jt}dj} = \frac{W_{it}}{P_t} \quad = \quad a_i + (w_i - a_i)\frac{(\rho+k)\overline{Y}S_t}{\rho + k\overline{Y}S_t} \tag{35}$$

Expression (34) shows that the wealth to output ratio depends on agents' share of aggregate endowment w_i and their average habit sensitivity a_i. Higher w_i increases agents' wealth in good times because they take on more leverage and thus reap the gains of an increase in stock market prices. For given w_i, however, agents with higher a_i have wealth that increases less or even decrease in good times compared to agents with lower a_i. As discussed earlier, the latter type of agents tends to take on more leverage to increase their stock holdings, which increase their wealth when stock market increases, and vice versa.

Finally, when the economy is at the aggregate steady state, i.e. $\overline{Y}S_t = 1$, then heterogeneity in preference does not matter. The reason is that we determined the Pareto weights at time 0 under the assumption that $\overline{Y}S_0 = 1$. The Pareto weights thus "undo" the heterogeneity in preferences at the steady state.

While expression (34) characterizes the wealth of agent i compared to aggregate output, expression (35) characterizes the wealth share, that is, the wealth of agent i compared to the aggregate wealth in the economy. By market clearing, we must have that the aggregate wealth in the economy equals the value of financial assets, that is, $\int_j W_{jt}dj = P_t$. The same discount effects that affect the wealth of each agent also affects the aggregate wealth. Expression (35) clearly shows that agents with high leverage (those with $w_i - a_i > 0$) will enjoy an increase in their wealth share during good times (S_t high) but a decrease during bad times (S_t low). This effect is not surprising, as leverage amplifies the impact of discount rate shocks on each agent's wealth compared to their impact on the aggregate stock market.

These results imply the following properties of the cross-sectional dispersion of wealth.

Proposition 14 *Let $Var^{CS}(a_i)$, $Var^{CS}(w_i)$, and $Covar^{CS}(a_i, w_i)$ denote the cross-sectional variance of preference characteristics a_i and in share w_i of aggregate endowment, and their covariance, respectively. Then, the cross-sectional variance of wealth/output ratio is*

$$
\begin{aligned}
Var_t^{CS}\left(\frac{W_{it}}{D_t}\right) &= Var^{CS}(a_i)\left(\frac{1-\overline{Y}S_t}{\rho+k}\right)^2 + Var^{CS}(w_i)\left(\frac{\overline{Y}S_t}{\rho}\right)^2 \\
&\quad + 2Cov^{CS}(a_i, w_i)\frac{\left(1-\overline{Y}S_t\right)\left(\overline{Y}S_t\right)}{(\rho+k)\rho}
\end{aligned} \tag{36}
$$

and the cross-sectional variance of wealth shares $W_{it}/\int_j W_j dj$ is

$$
Var_t^{CS}\left(\frac{W_{it}}{\int_j W_j dj}\right) = Var_t^{CS}\left(\frac{W_{it}}{D_t}\right)\left(\frac{\rho(\rho+\kappa)}{\rho+k\overline{Y}S_t}\right)^2 \tag{37}
$$

To understand the intuition behind (36), recall first that when $\overline{Y}S_t = 1$, the economy is at its stochastic steady state, which is the initial condition at time 0 when agents' wealth is $W_{i0} = w_i$, their initial endowment. Thus, (36) shows that when the system is at its stochastic steady state, the wealth dispersion is given by the dispersion in endowments w_i.

Consider now the case in which the cross-sectional covariance between endowment and preferences is zero, $Cov^{CS}(a_i, w_i) = 0$. During good times the surplus consumption ratio S_t increases. Whether this variation brings about an increase or decrease in wealth distribution, however, depends on the importance of the heterogeneity in preferences $Var^{CS}(a_i)$ relative to the dispersion in shares of aggregate endowment across the population. For instance, if $Var^{CS}(a_i) = 0$, then during good times (high S_t) the dispersion in wealth increases, while it decreases during bad times. Intuitively, when $Var^{CS}(a_i) = 0$, all agents differ from each other only in shares of aggregate endowment. Thus, agents with higher endowments take on a more leveraged position and their wealth increase during good times, and so does the dispersion of wealth.

However, if $Var^{CS}(w_i) = 0$, then the dispersion in wealth is null at the aggregate stochastic steady state $\overline{Y}S_t = 1$, but it otherwise increases, both in good or in bad times, due to heterogeneous preferences. The intuition stems immediately from (34) and the discussion in previous section: Differential stock holdings across agents induced by differential preferences generate an increase in dispersion both during good times, as agents heavily invested in stocks outperform, and symmetrically in bad times, when they underperform.

The dispersion of wealth share in (37) is proportional to the dispersion of wealth/output ratio in (36), except that the proportionality factor decreases in good times. This is due to the increase in aggregate wealth $\int W_{jt} dj = P_t$ in good times. Thus, even if the dispersion of wealth/output increases as S_t increase, the wealth share may still decline if discount rate effects are strong enough.

In sum, (36) and (37) show that the variation in the distribution of wealth as S_t changes is not straightforward and it depends the differential impact of cross-sectional differences in endowment versus preferences. We further discuss these effects in the calibration section.

We conclude this section with a characterization of agents' returns on investments:

Proposition 15 *The expected return on wealth portfolio of agent i's is*

$$E_t[dR_{W,i} - r_t dt] = \beta_i(S_t)E[dR_P - r_t dt] \tag{38}$$

where

$$\beta_i(S_t) = \frac{Cov_t(dR_{W,i}, dR_P)}{Var_t(dR_P)} = \frac{1 + v \left(1 - \frac{\rho}{\rho + [k + (\rho + k)(w_i - a_i)/a_i]\overline{Y}S_t}\right)}{1 + v \left(1 - \frac{\rho}{\rho + k\overline{Y}S_t}\right)}$$

In particular, $\beta_i(S_t) > 1$ if and only if $w_i > a_i$.

Proposition 15 shows that independently on whether times are good or bad, agents with a higher leverage enjoy higher average return on wealth than agents with lower leverage. Indeed, $\beta(S_t) > 1$ for all S_t if $w_i > a_i$. This result does not imply that on average, such agents will be infinitely wealthy in the infinite future – a standard result in models with agents with heterogeneous risk aversion – as we already know that the wealth distribution is stationary. The resolution of the puzzle is simply that such agents also take on more risk ($\sigma_{W,i} = \sigma_{P,i}$ in (29) is higher for $w_i - a_i$ higher) which implies larger losses than others during bad times. This argument shows that even if some agents enjoy higher average return on capital (wealth) all the time, this fact per se' does not lead the conclusion of a permanently more concentrated wealth distribution.

4.8. Intermediary asset pricing and the leverage risk price

Our model also sheds light on recent empirical findings in the "intermediary asset pricing" literature (Adrian, Etula and Muir (2014) and He, Kelly and Manela (2016)), which is in turn inspired by some recent theoretical advances (He and Krishnamurthy, 2013). This literature emphasizes that households access markets for risky securities largely through financial intermediaries. Intermediary capital is needed to facilitate this access and capital ratios are priced risk factors. Importantly, intermediaries lever up, issuing the safe securities that households (and other agents) use to substitute intertemporally as well as manage their risk exposures. Because households are not allowed to directly invest in the risky asset, intermediaries therefore effectively transform the safe assets held by households into investments in the risky asset and effectively price the risky asset.

This is also the case here. Indeed in our model, agents who take on leverage to purchase the risky assets also supply risk-free assets to those agents who want to limit their risk exposure (see discussion in Section 4.4.) and thus they are akin to financial intermediaries. The only difference with the intermediary asset pricing literature is that *all* agents can invest in the risky asset themselves and therefore the marginal valuation of the risky asset is the same for both leveraged and unleveraged agents.

The intermediary asset pricing literature finds that measures of capital equity ratio of financial intermediaries are predictors of returns in the cross section. A debate in this literature is whether there is a *negative* or *positive* price of risk associated with shocks to the capital ratio of the financial intermediaries (see Adrian, Etula and Muir (2014) and He, Kelly and Manela (2016), respectively). Our model sheds light on this debate by showing first that agents' leverage is a priced factor and that the leverage risk price has a different sign depending on whether we measure intermediaries' leverage using market prices or not.

Formally, in our model, the conditional CAPM holds, as it is apparent from Equation (38). If we could easily measure S_t in the data, we could compute expected returns off the conditional CAPM. However, suppose, reasonably, that the surplus S_t is not observable, but we rather observe a monotonic transformation $\ell_t = Q(S_t)$ of it. Let $d\ell_t = \mu_{\ell,t}dt + \sigma_{\ell,t}dZ_t$ where $\mu_{\ell,t}$ and $\sigma_{\ell,t}$ can be derived from Ito's lemma. In this case, we can write the state price density equivalently as

$$M_t = e^{-\rho t} D_t^{-1} S_t^{-1} = e^{-\rho t} D_t^{-1} q(\ell_t)^{-1}$$

where $S_t = q(\ell_t) = Q^{-1}(\ell_t)$. We thus obtain that $\sigma_{M,t} = \sigma_{D,t} + \frac{q'(\ell_t)}{q(\ell_t)}\sigma_{\ell,t}$ and therefore the risk premium for *any* asset with return $dR_{i,t}$ can be written as

$$E_t[dR_{it} - r_t dt] = Cov_t\left(\frac{dD_t}{D_t}, dR_{it}\right) + \frac{q'(\ell_t)}{q(\ell_t)} Cov_t\left(d\ell_t, dR_{it}\right)$$

The first term corresponds to the usual log-utility, consumption-CAPM term, while the second term corresponds to the additional risk premium due to shocks to ℓ_t.[19]

Consider now a highly leveraged agent i in our economy, i.e. one with $w_i > a_i$. As argued above, such agent issues risk-free bonds to other agents and use the proceeds to purchase

[19] This decomposition is for illustrative purposes only. All shocks are perfectly correlated in our model and so there is only one priced of risk factor.

risky securities. We can consider such agent an intermediary. Consider now the leverage of such agent. We have two potential measures, namely, its debt-to-output ratio,

$$\ell_t = Q_{it}^{D/O}(S_t) = -\frac{N_{it}^0 B_t}{D_t} = v\left(w_i - a_i\right) H\left(S_t\right);$$

or is debt-to-wealth ratio

$$\ell_t = Q_{it}^{D/W}(S_t) = -\frac{N_{it}^0 B_t}{W_{it}} = \frac{\sigma_{Wi}(S_t)}{\sigma_P(S)} - 1.$$

These two measures of leverage have different properties. In particular, $Q_{it}^{D/O}(S_t)$ is monotonically increasing in S_t while $Q_{it}^{D/W}(S_t)$ is monotonically decreasing in S_t. We then obtain the following corollary:

Corollary 16 *(price of leverage risk) (a) The price of leverage risk is positive,* $\lambda_t^{D/O} = \frac{q^{D/O\,\prime}(\ell_t)}{q^{D/O}(\ell_t)} > 0$*, when leverage is measured as the debt-to-output ratio ("book leverage").*

(b) The price of leverage risk is negative, $\lambda_t^{D/W} = \frac{q^{D/W\,\prime}(\ell_t)}{q^{D/W}(\ell_t)} < 0$*, when leverage is measured as the debt-to-wealth ratio ("market leverage").*

The economics behind this corollary is important: Our model generates strong discount effects that affect the valuation of securities. While intuitively our model generates a deleveraging during bad times – which coincide with high marginal utility – the strong increase in discount rates pushes market prices even lower, which in turn increase leverage ratios computed off market prices. The sign of leverage risk prices therefore critically depends on the type of leverage that is being considered, and especially on whether market prices are used or not in the computation.[20]

To link these results to the empirical evidence in Adrian, Etula and Muir (2014) and He, Kelly and Manela (2016), one could equate the levered agent's debt-to-output ratio to the "book leverage" of financial intermediaries, as it measures the agent's *amount* of debt; this leverage measure does not use market prices, and it is in fact procyclical. In contrast, a levered agent's debt-to-wealth ratio is akin to a measure of "market leverage" for financial intermediaries, as wealth is computed from market prices, which are affected by discount effects and is in fact countercyclical. These two different measures imply prices of "leverage risk" of opposite signs. Finally, we also note that $q^{D/O}(\ell_t)$ and $q^{D/W}(\ell_t)$ are non-linearly related with each other, and therefore the results of cross-sectional tests would not be the exact opposite, as found in the literature (e.g. He, Kelly, and Manela (2016)).

[20]Clearly, the loadings also have opposite signs for the two cases. Because $\sigma_{\ell,t} = Q'(S_t) S_t v \sigma_D(S_t)$, then $\sigma_{\ell,t} > 0$ if leverage is the debt-to-output ratio and $\sigma_{\ell,t} < 0$ when it is the debt-to-wealth ratio. Thus, $Cov_t\left(d\ell_t, dR_{it}\right) > 0$ in the former case and $Cov_t\left(d\ell_t, dR_{it}\right) < 0$ in the latter case.

5. Quantitative implications

We now provide a quantitative assessment of the effects discussed in previous sections. While the results in previous sections do not depend on the specific form of $\sigma_D(Y_t)$, we now make a specific reasonable assumption in order to make the model comparable with previous research. In particular, we assume

$$\sigma_D(Y_t) = \sigma^{max}\left(1 - \lambda Y_t^{-1}\right) \tag{39}$$

This assumption implies that dividend volatility increases when the recession index increases, but it is also bounded between $[0, \sigma^{max}]$.[21] This assumption about output volatility is consistent with existing evidence that aggregate uncertainty increases in bad times (see e.g. Jurado, Ludvigson, and Ng (2015)), it satisfies the technical condition $\sigma_D(Y) \to 0$ as $Y_t \to \lambda$, and it also allows us to compare our results with previous literature, as we obtain

$$dY_t = k(\overline{Y} - Y_t)dt - (Y_t - \lambda)\overline{v}dZ_t$$

with $\overline{v} = v\sigma^{max}$ which is similar to the one in MSV.[22]

For the calibration we use the same parameters as in MSV Table 1 to model the dynamics of Y_t. These are are reported in Panel A of Table 1. The only additional parameter is σ^{max}, which we choose to match the average consumption volatility $E[\sigma_D(S_t)] = std[\Delta \log(C_t^{data})]$, where the expectation can be computed from the stationary density of Y_t.[23]

Figure 1 reports the conditional moments implied by the model as a function of the surplus-consumption ratio S_t. As in MSV Figure 1, Panel A reports the stationary distribution of the surplus-consumption ratio S_t and shows that most of the probability mass is around $\overline{S} = 0.0294$, although S_t drops considerably below occasionally. The price-dividend ratio is increasing in S_t (panel B), while volatility, risk premium and interest rates decline with S_t (panel C). Finally, the Sharpe ratio is also strongly time varying, and it is higher in bad times (low S_t) and lower in good times (high S_t). This figure is virtually identical to Figure 1 in MSV, which highlights that our mild calibration of consumption volatility (with a maximum of only 6.4%) is such to have a minor on impact on the level of asset prices.

[21] The alternative of assuming e.g. $\sigma_D(Y)$ as linear in Y_t would result in $\sigma_D(Y)$ potentially diverging to infinity as Y_t increases.

[22] Technically, we also impose $\sigma_D(S_t)$ converges to zero for $S_t \leq \epsilon$ for some small but strictly positive $\epsilon > 0$ to ensure integrability of stochastic integrals. This faster convergence to zero for a strictly positive number can be achieved through a killing function, as in Cheriditto and Gabaix (2008). We do not specify such functions explicitly here, for notational convenience.

[23] See the Appendix in MSV. In addition, note that in MSV, $\alpha = \overline{v}/\sigma$ and therefore we compute $\overline{v} = \alpha\sigma$.

Table 1: Parameters and Moments

Panel A. Parameter Estimates

ρ	k	$\overline{Y}$	λ	$\overline{v}$	μ	σ^{max}
0.0416	0.1567	34	20	1.1194	0.0218	0.0641

Panel B. Moments (1952 – 2014)

	$E[R]$	$Std(R)$	$E[r_f]$	$Std(r_f)$	$E[P/D]$	$Std[P/D]$	SR	$E[\sigma_t]$	Std(σ_t)
Data	7.13%	16.55%	1.00%	1.00%	38	15	43%	1.41%	0.52%
Model	7.00%	24.58%	1.68%	5.84%	30.37	5.90	28.48%	1.42%	1.20%

Panel C. P/D Predictability R^2

	1 year	2 year	3 year	4 year	5 year
Data	5.12%	8.25%	9.22%	9.59%	12.45%
Model	9.22%	14.47%	17.65%	20.13%	21.87%

Given the parameters in Panel A of Table 1, we simulate 10,000 years of quarterly data and report the aggregate moments in Panel B. As in MSV, Table 1, the model fits well the asset pricing data, though both the volatilities of stock returns and of the risk free rate are higher than the empirically observed one.[24] Still, the model yields a respectable Sharpe ratio of 28.48%. Finally, the simulated model generates an average consumption volatility of 1.42% with a standard deviation of 1.20%. This latter variation is a bit higher than the variation of consumption volatility in the data (0.52%), where the latter is computed fitting a GARCH(1,1) model to quarterly consumption data, and then taking the standard deviation of the annualized GARCH volatility. Our calibrated number is however lower than the standard deviation of dividend growth' volatility, which is instead around 1.50%.

The calibrated model also generates a strong predictability of stock returns (Panel C), with R^2 ranging between 9.22% at one year to 21.87% at 5 year. This predictability is stronger than the one generated in MSV and also the one in the data. This is due to the combined effect of time varying economic uncertainty (*i.e.* the quantity of risk) and time

[24]The volatility of the risk free rate can be substantially reduced by making the natural assumption that expected dividend growth μ_D decreases in bad times, i.e. when the recession indicator Y_t is high. Indeed, in the extreme, by assuming $\mu_D(Y_t) = \overline{\mu}_D + (1-v)\sigma_D(Y)^2 - k(1-\overline{Y}Y_t^{-1})$, which is decreasing in Y_t, we would obtain a constant interest rates $r = \rho + \overline{\mu}_D$. No other result in the paper depend on $\mu_D(Y_t)$ and thus all the other results would remain unaltered by the change.

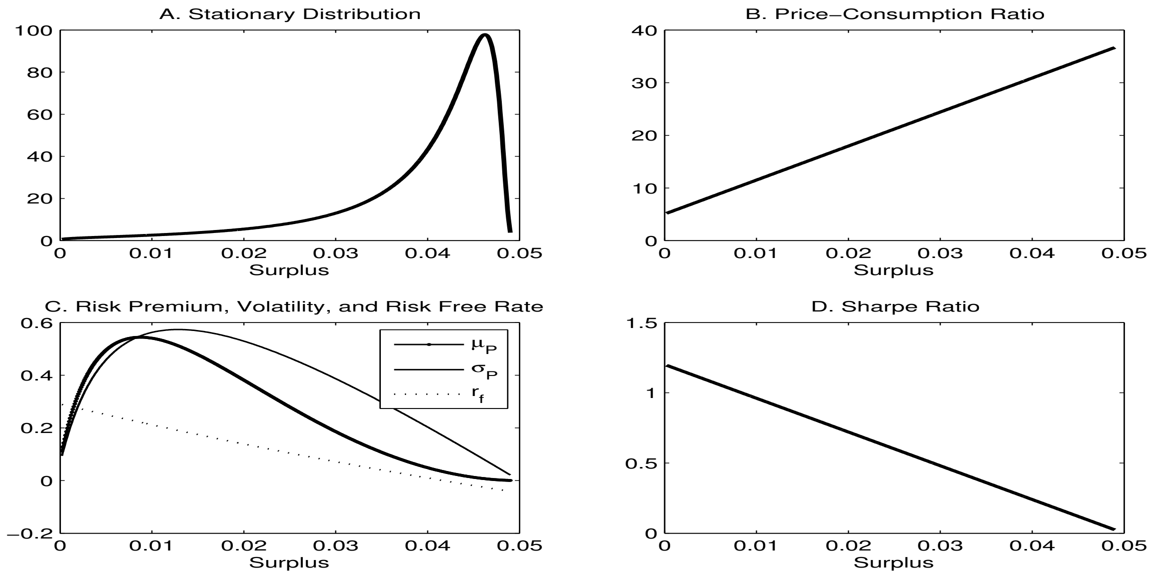

Figure 1: Conditional Moments. Panel A shows the stationary probability density function of the surplus consumption ratio S_t. Panel B shows the P/D ratio as a function of S_t. Panel C plot the expected excess return $E_t\left[dR_P - r_t dt\right]$, the return volatility $\sigma_P(S_t)$ and the interest rate $r(S_t)$ as functions of S_t. Finally, Panel D shows the Sharpe ratio $E_t\left[dR_P - r_t dt\right]/\sigma_P(S_t)$ against S_t.

varying risk aversion (*i.e.* the market price of risk), which move in the same direction.

5.1. The cross-section of agents' behavior: Who levers?

We now make some assumptions about the dispersion of initial endowments w_i and of preferences a_i. A full micro-founded "calibration" is clearly problematic in our setting, given the types of preference specification. We resort to illustrate the model's prediction through a reasonable numerical illustration which yields sensible quantities for some observables, such as debt levels and consumption. For the habit loading parameters a_i we simply assume they are uniformly distributed between $\underline{a} = 0.5$ and $\overline{a} = 1.5$, so as $\int a_i di = 1$. Endowments w_i must meet assumption A1. While distributions can be found such that a_i and w_i are independent, A1 severely restricts the dispersion of such distributions. We instead assume that Pareto weights ϕ_i are distributed independently of preferences a_i and obtain the endowments by inverting (10):

$$w_i = \frac{\phi_i + a_i(\overline{Y} - \lambda) + \lambda - 1}{\overline{Y}}. \tag{40}$$

To ensure a skewed distribution of wealth, we assume

$$\phi_i = e^{-\sigma_w \varepsilon_i - \frac{1}{2}\sigma_w^2}$$

with $\varepsilon_i \sim N(0,1)$ and $\sigma_w = 2$. Thus, $\int_i \phi_i di = E^{CS}[\phi_i] = 1$. This procedure ensures that the Pareto weights are positive and hence all the constraints are satisfied. While all agents have random Pareto weights, and therefore contribute to the representative agent in a random manner, the procedure implies that agents with higher habit sensitivity a_i also have a higher endowment, a required condition to have well defined preferences in equilibrium.

Panel A and B of Figure 2 shows the resulting distribution of preferences and endowment in a simulation of 200,000 agents. In particular, Panel B shows a markedly skewed distribution of endowments (the extreme right tail of the distribution is omitted to provide a better visual impression). Because of the restriction $\int w_i di = 1$, the distribution shows a large mass of agents with $w_i < 1$ to allow for some agents with a very large endowment. Panel C shows the relation between endowments on the x-axis and preference on the y-axis. The white area in the top-left corner is due to restriction A1: Agents with high habit loading a_i must have high initial endowment w_i to ensure a feasible consumption plan.

Finally, Panel D shows the relation between endowment w_i and leverage, namely, $w_i - a_i$. Indeed, recall that only agents with $w_i - a_i > 0$ lever up (see Corollary 6). Leverage is thus "U-shaped" in our calibration of the cross section in that two types of agents lever up, those with very low endowment but with also very low sensitivity to habit and those agents with very high endowment. The group with intermediate endowment, in contrast, are heterogeneous in that some leverage and some purchase the risk-free asset.

Our assumption on the joint distribution of preferences a_i and endowments w_i, plotted in Figure 2, yields a cross section of debt-to-assets that matches well its empirical counterpart. Panel A of Figure 3 plots the distribution of debt-to-assets of agents who take on debt in simulations during three types of periods: Booms (S_t high), recessions (S_t low), and crisis (S_t very low). First, in general, agents with lower net worth (W_t) take on more debt as a fraction of assets ($N_{it}P_t$). The reason is that in the calibration above, these types of agents are less risk averse, as their a_i is on average lower. This is the effect of the constraint A1, also shown in Panels C and D of Figure 2: Agents with low endowment may have low risk aversion parameter a_i.

The second important effect of Panel A, however, is that the debt-to-asset ratio substantially increases in crisis periods, that is, those rare times in which S_t is on the left-hand-side of its distribution (see Panel A of Figure 1). This an important channel in our model: While agents who borrow deleverage when S_t decline (Corollary 12), and hence reduce their amount of debt, the debt-to-asset ratio actually increases, because the value of assets declines by even more. That is, active deleveraging and increasing debt-to-asset ratios are perfectly

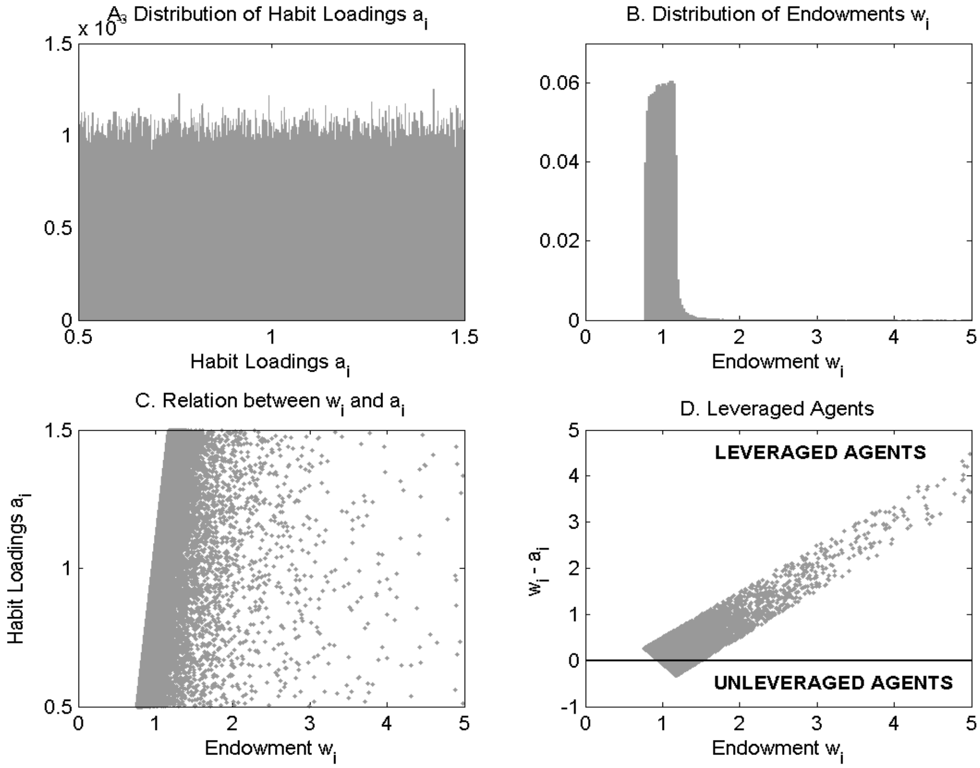

Figure 2: Preference and Endowment Distribution . Panel A plots the simulated distribution of preference parameters a_i from a uniform [0.5,1.5]. Panel B plots the simulated endowment distribution $w_i = \frac{\phi_i + a_i(\overline{Y} - \lambda) + \lambda - 1}{\overline{Y}}$ where $\phi_i = e^{-\sigma_w \varepsilon_i - \frac{1}{2}\sigma_w^2}$ are lognormally distributed. Panel C shows the relation between endowments w_i and preferences a_i. Panel D shows the relation between endowments w_i and $w_i - a_i$, where we recall that agents with $w_i - a_i > 0$ take on debt.

compatible events when assets are valued at market values.

Panel B of Figure 3 shows that similar effects occur at the household level in the data. We use the Surveys of Consumer Finances conducted in 2007 and 2009. This last survey was conducted on the same sample of households as the 2007 survey, and thus it reflects a panel of agents. The debt-to-asset ratio of households is decreasing in their net worth. Interestingly, the debt-to-asset ratio increased substantially between 2007 and 2009 for the same agents that are ranked as of their 2009 net worth. We rank households on their net worth in 2009 to highlight how the increase in debt-to-asset ratio for these agents between 2007 and 2009 was especially due to a decline in asset value, which decreases net worth. Indeed, agents who suffered larger losses due to declining asset values will be moving to the left of the net worth distribution, and for given debt, would have a higher debt-to-asset ratio. The figure clearly indicates how the variation in assets generate an increase in debt-to-assets, in line with our model. Notice though that the model is not able to match the observed *level*

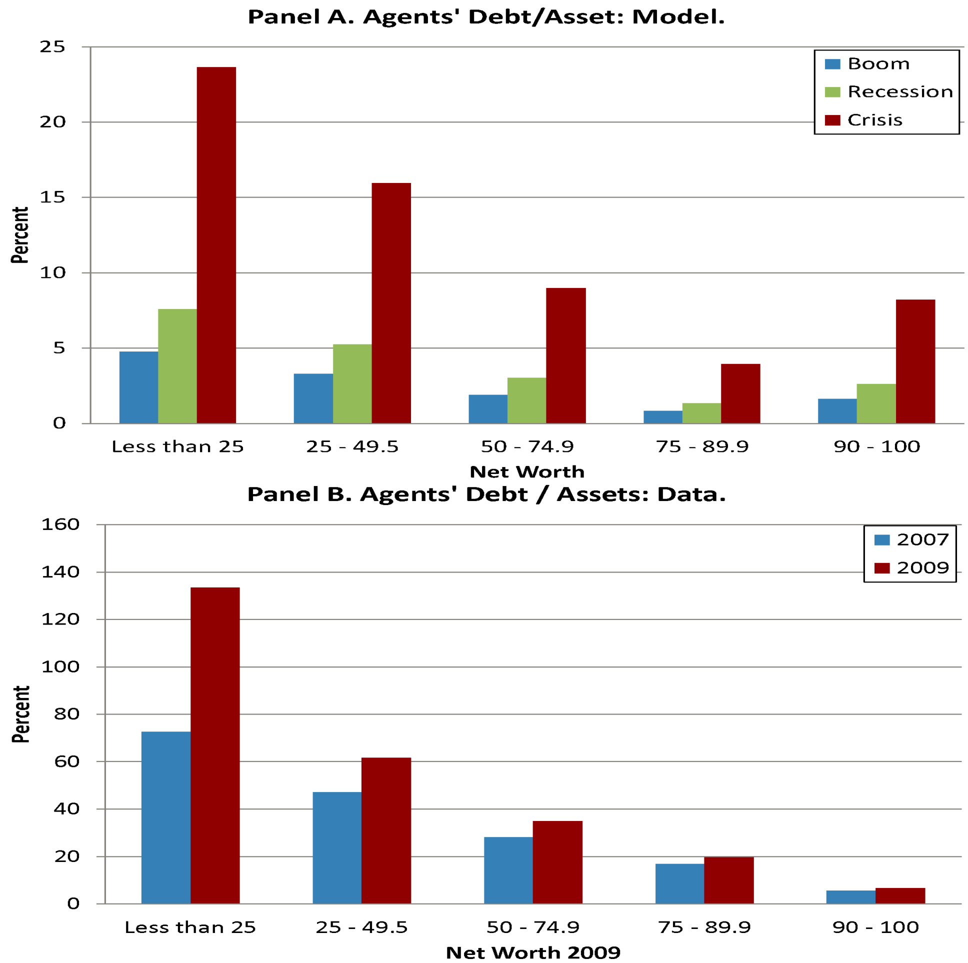

Figure 3: Debt-to-assets ratios across the wealth distribution. Model and data . Panel A plots the distribution of debt-to-asset ratios of agents who take on debt in simulations during three types of periods: Booms (S_t high), recessions (S_t low) and crisis (S_t very low). Panel B plots the distribution of debt-to-asset ratios from the Survey of Consumer Finances in 2007 and in 2009, which were conducted on the same sample of agents.

of debt-to-assets amongst the poorest agents.

In sum, our model is able to capture an important fact in the cross section, namely that the less wealthy lever more. This feature of our model stands in contrast with most models with heterogeneous agents, such as, for example, Dumas (1982) and Longstaff and Wang (2013). In these models less risk averse agents lever up, invest in risky stocks, and become richer as a result. These models thus imply counterfactually that leverage is more pronounced amongst richer agents. In contrast, in our model the two different sources of heterogeneity, combined with the implicit assumption that agents with low endowment have lower habit preferences a_i, imply that poor agents lever up more, consistently with the data.

Figure 4 plots the unconditional average consumption and wealth characteristics of each agent i with initial endowment w_i. In particular, Panel A shows the unconditional average diffusion $\sigma_{C,i}(S_t)$ of each agent i, which can be easily computed from the unconditional distribution of the surplus consumption ratio S_t in Panel A of Figure 1. The volatility of individual agents' consumption growth is higher than the aggregate 1.42% in Table 1, but not extraordinarily high compared to the available data. Indeed, most of the observations are below 10% volatility, which is in line with the estimates reported by e.g. Brav, Constantinides, and Geczy (2002, Table 1). Some of the averages are higher than 10%, especially for agents with a large endowment w_i, who are leveraged agents (see Panel D of Figure 2). Finally, some of the average diffusions are negative, indicating that such agents decrease consumption following positive aggregate shocks to the economy. Indeed, risk sharing entails some agents insuring others, which imply a different behavior of consumption choices across agents in response to aggregate shocks. Panel C of Figure 4 shows the unconditional average expected consumption growth of agents i, $E[dC_{i,t}/C_{i,t}]$, which ranges between 2% and 6%. In particular, agents with a high endowment have a higher expected consumption growth, as they also have a higher consumption volatility.

Panels B and D of Figure 4 show the average characteristics of agents' wealth W_{it}. First, Panel B shows that that the average volatility of agents' wealth is far larger than their consumption growth due to the large discount effects that affect the valuation of wealth. Second, agents with higher initial endowment have not only a higher volatility of wealth, but also a higher expected return on wealth, on average. Such agents are those that are leveraged and thus these results are in fact quite intuitive. We emphasize that Figure 4 plots unconditional averages for agents with a given endowment w_i. That is, agents with a large endowment (at time 0) would enjoy a high expected excess return on wealth on average. Yet, the distribution of wealth is stationary in our model because such agents also take on much more risk, which re-equilibrates the economy when the surplus consumption ratio drops.

5.2. The aggregate behavior of levered agents

Given the parameter distributions in the previous section, we now look at the behavior of leverage in aggregate. Inspection of the formulas derived throughout the paper reveals that the aggregate behavior of levered agents depends on two quantities:

$$K_0 = \int_{i:w_i - a_i > 0} a_i di; \qquad K_1 = \int_{i:w_i - a_i > 0} (w_i - a_i) di \tag{41}$$

Given the parameters in discussed in Figure 2, we obtain $K_0 = 0.2483$ and $K_1 = 0.1526$.

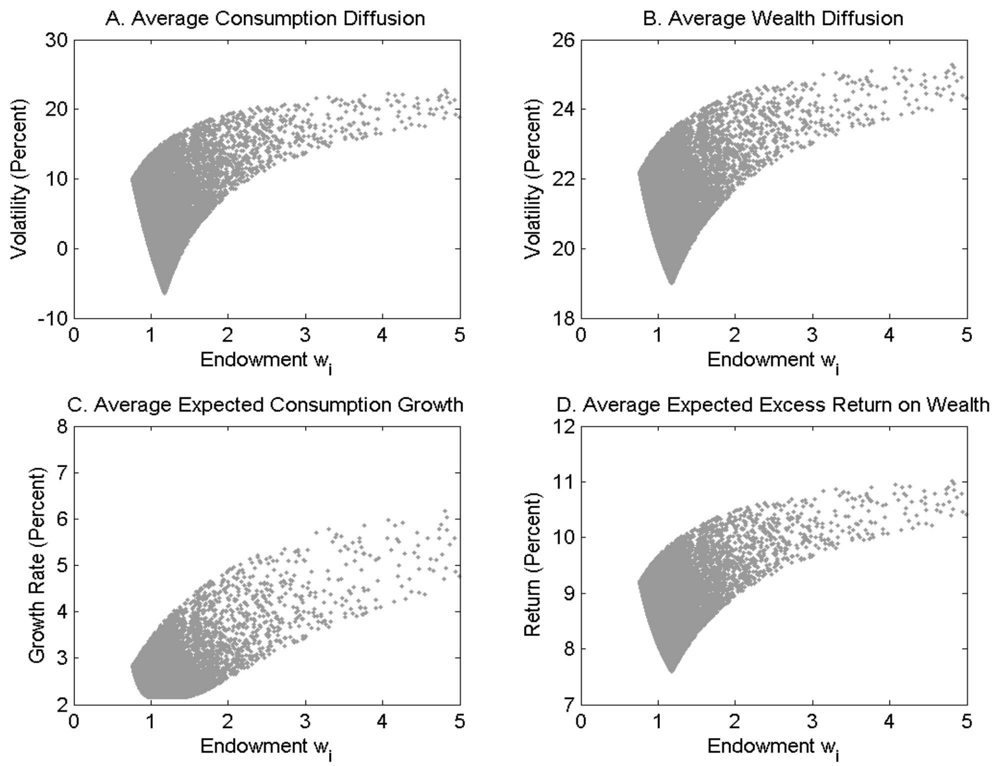

Figure 4: The Cross-Section of Consumption and Wealth Processes . Panel A plots the unconditional average consumption diffusion $E[\sigma_{C,i}(S_t)]$ for each agent i with endowment w_i. Panel B plots the unconditional average wealth diffusion $E[\sigma_{W,i}(S_t)]$. Panel C plots the unconditional average expected consumption growth $E[dC_i/C_i]$ of each agent i with endowment w_i, while Panel C plots the annualized unconditional expected return on wealth $E[dW_i/W_i]/dt - r_t$ for each agent i. The x-axis has been truncated at 5 for better visual impression.

5.2.1. Leverage and stock holdings in good and bad times

Equation (25) shows that aggregate leverage is

$$L(S_t) = v\ K_1\ H(S_t) \tag{42}$$

Similarly, from (18) the aggregate stock holding of levered agents can be written as

$$N^{Lev}(S_t) = K_0 + (\rho + k)\,(1 + v)\,K_1 H(S_t) \tag{43}$$

where recall $H(S_t)$ is given by (19).

As discussed in Corollaries 7 and 12, $H(S_t)$ is increasing and concave in S_t. That is, leverage and aggregate allocation to stocks are not only procyclical, but they also decline increasingly faster as times get worse, i.e. as S_t decline. Panels A and B of Figure 5 shows

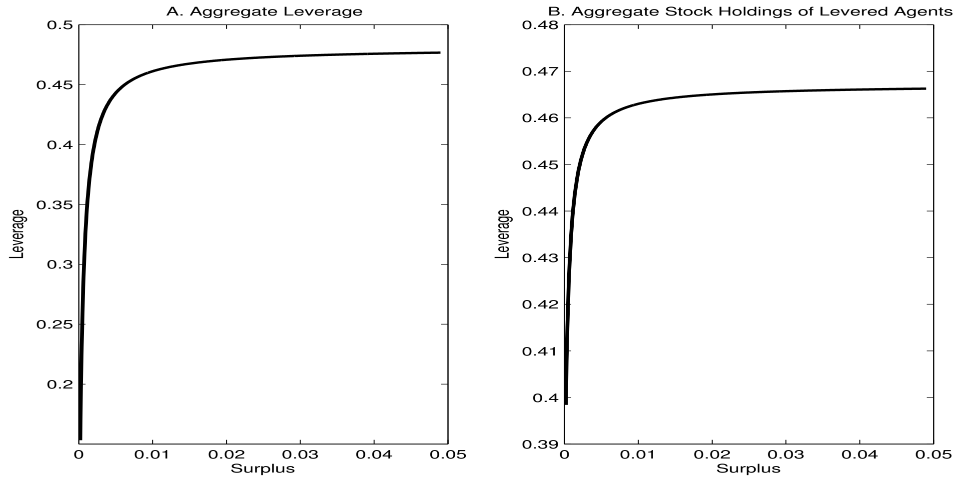

Figure 5: Aggregate Leverage and Aggregate Stock Holdings of Levered Agents Panel A plots the aggregate leverage in the economy as a function of the surplus consumption ratio S_t. Panel B reports the aggregate holdings of stocks from the agents who have leveraged positions (i.e. those in Panel A). The parameters used are those in Table 1 and Figure 2.

the patterns of $L(S_t)$ and the aggregate stock holdings of the levered agents, $N^{Lev}(S_t)$, under the parameter choices in Table 1. The concavity of $H(S_t)$ is especially strong for very low levels of S_t: Deleverage accelerates rapidly as bad times turn into severe distress. It is useful to recall that the function $H(S_t)$ is independent of the assumptions about the aggregate volatility $\sigma_D(S_t)$, and thus the strong concavity displayed in Figure 5 stems from the increase in aggregate risk aversion in bad times, with the implied decrease of differential sensitivity of stock prices and agents' wealth to discount rate shocks.

As already mentioned, this non-linear behavior of leverage and risky asset holdings of levered agents with respect to the surplus consumption ratio suggests that levered agents "fire sell" risky assets to decrease leverage in bad times. This is shown in the simulated path illustrated in Figure 6. Panel A shows 100 years of artificial quarterly data of the surplus consumption ratio S_t, while panel B reports the corresponding economic uncertainty $\sigma_D(S_t)$. Panel C shows the variation in the price-dividend ratio due to variation in the surplus consumption ratio, with a visible drop of the stock price from 30 to less than 10 in the middle of the simulated sample. Panel D shows the stock return volatility, which increases dramatically during bad times, as it increases to almost 60% during the "crisis".

Panel E demonstrates the impact of the variation of the surplus consumption ratio on aggregate leverage, i.e. the aggregate debt-to-output ratio, and the aggregate stock holdings of levered agents. As it is apparent, the variation of both quantities is rather limited most of the time, except during the extreme bad event visible in the middle of the sample. In

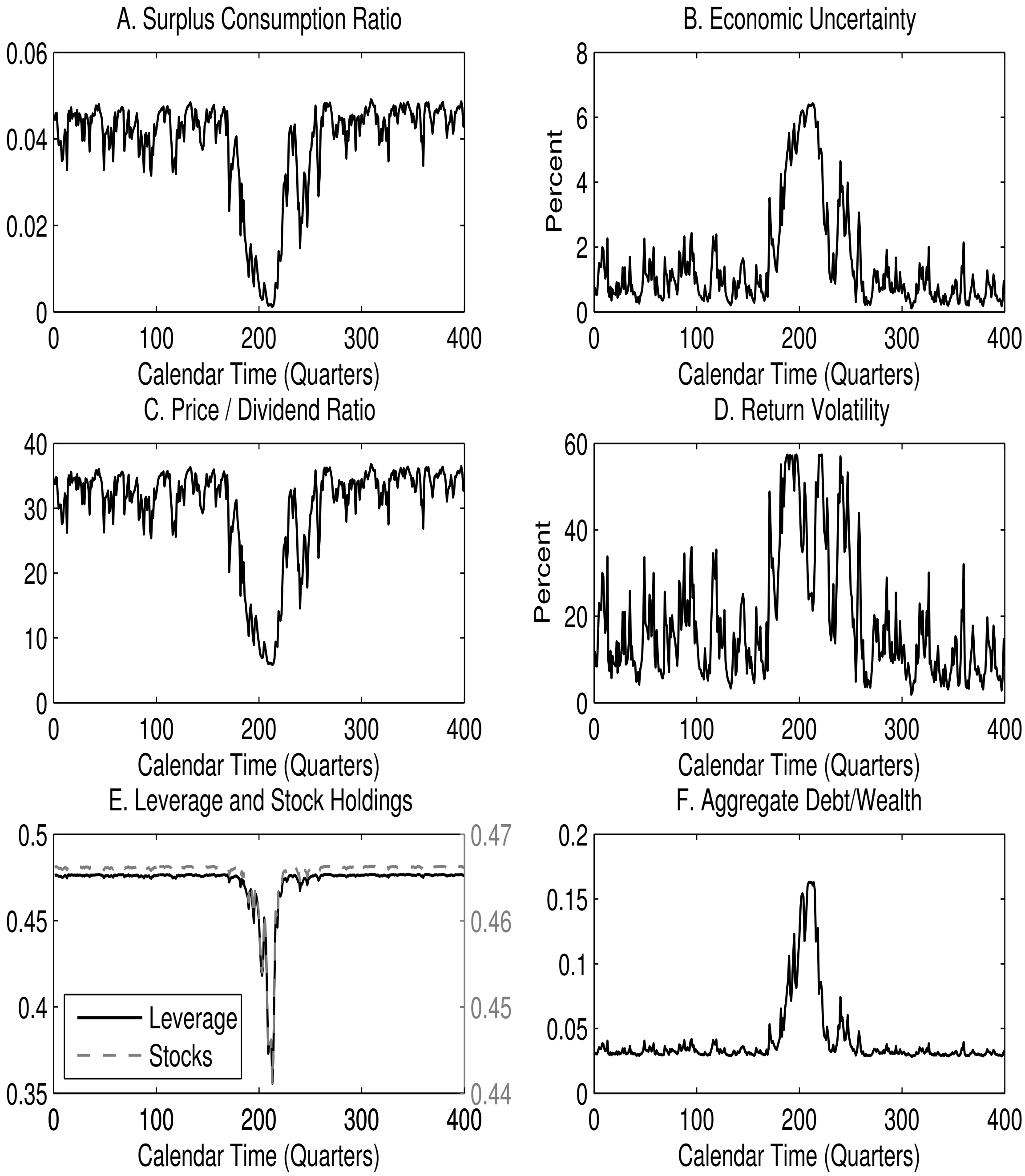

Figure 6: "Fire Sales" in a Simulation Run . This figure plots the time series of several quantities in 100 years of quarterly artificial data. Panel A reports the surplus consumption ratio S_t. Panel B reports the consumption volatility $\sigma_D(S_t)$. Panel C and D report the price-dividend ratio and the stock return volatility, respectively. Panel E reports the aggregate leverage, defined as debt-to-output ratio (solid black line, left axis), and the aggregate position in risky stock of levered agents (grey dashed line, right axis). Panel F reports the aggregate debt-to-wealth ratio of levered agents. This simulated sample was selected to highlight the effect of a crisis, visible in the middle of the sample.

this occasion, as the surplus consumption ratio drops and economic uncertainty increases, levered agents decrease their indebtness and liquidate their stock positions.

Finally, Panel F shows the debt-to-wealth ratio of the levered agents, and it highlights that the model is consistent with the observation that the efforts of all levered agents to delever simultaneously results in an increase in leverage *ratios*, i.e. debt-to-wealth ratios. Indeed, while Panel E shows that aggregate debt declines during bad times, Panel F shows that the aggregate debt-to-wealth ratio actually increases, as levered agents' wealth declines faster then the decline in debt leverage.

In sum thus, as economic conditions deteriorate (a drop in S_t) prices fall but agents only delever and liquidate stock positions slowly. As bad times turn into severely distressed conditions, deleveraging and stock liquidation accelerates, creating the impression of a panic selling episode. Leverage *ratios*, debt-to-wealth, increase sharply as prices drop faster than the deleveraging. In addition, as shown in Corollary 10 and discussed further below, the consumption of highly levered agent falls. These results obtain in the absence of any contagion effects, liquidity dry ups or debt overhang considerations. They are the result of the optimal trading of utility maximizing agents in an equilibrium that in fact implements an optimal risk sharing allocation. Our claim, again, is not that these particular frictions do not matter but rather to argue that the dynamics in quantities and prices observed in crises obtain naturally in risk sharing models that feature the strong discount effects needed to obtain reasonable asset pricing implications. Tests aimed at uncovering the aforementioned frictions have to control for the component of these dynamics that are the result of optimal risk sharing.

5.2.2. The consumption of levered agents

Similarly to leverage and the stock-holdings of levered agents, the aggregate consumption of levered agents can be easily computed from the individual consumption shares as $C_t^{Lev}/D_t = K_0 + K_1\overline{Y}S_t$, where again K_0 and K_1 are in (41). The aggregate consumption share of unlevered agents is simply $C_t^{Unlev}/D_t = 1 - C_t^{Lev}/D_t$.

Panel A of Figure 7 shows the aggregate consumption share of levered agents (solid line) and unlevered agents (dashed line). As established in Proposition 1 and Corollary 6, the consumption share of levered agents increases in the surplus consumption ratio.

Panel B of Figure 7 shows the aggregate expected consumption growth of levered agents

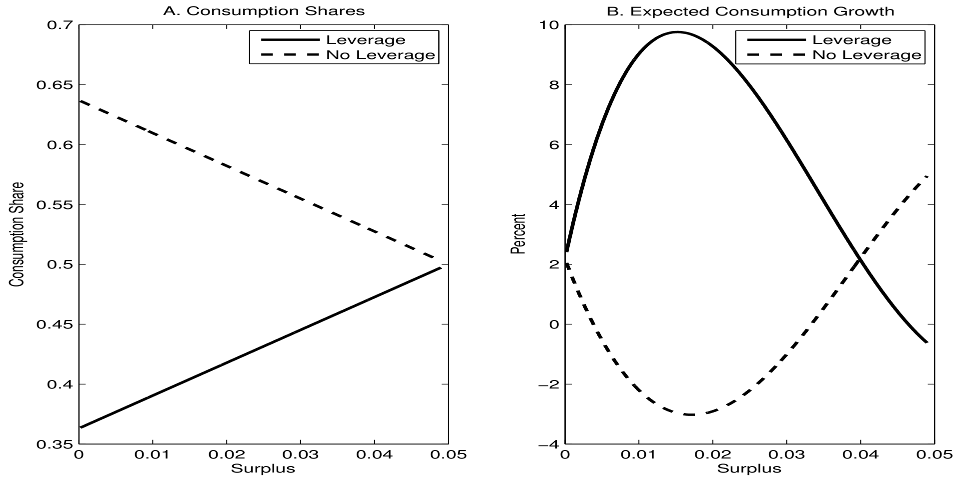

Figure 7: The Consumption of Levered Agents. Panel A plots the aggregate consumption shares of leveraged agents and unleveraged agents in the economy as a function of the surplus consumption ratio S_t. Panel B plot their expected consumption growth $E[dC/C]$. The parameters used are those in Table 1 and Figure 2.

(solid line) and unlevered agents (dashed line), computed as

$$E\left[\frac{dC_t^{Lev}}{C_t^{Lev}}\right]dt = \mu_D + \frac{K_1\overline{Y}S_t}{K_0 + K_1\overline{Y}S_t}F(S_t) \tag{44}$$

and $F(S_t)$ is in (31). Consistently with Panel A, during good times levered agents are at the peak of their consumption share and therefore, they should expect lower consumption growth going forward. Conversely, during good times unlevered agents are at the bottom of their consumption share, and therefore should expect higher consumption growth going forward.

5.2.3. The wealth dynamics of levered agents

From (34) the wealth-to-output ratio of levered agents is[25]

$$\frac{W_t^{Lev}}{D_t} = \frac{1}{\rho}\left[\frac{\rho}{\rho+k}K_0\left(1-\overline{Y}S_t\right) + (K_0+K_1)\overline{Y}S_t\right] \tag{45}$$

The wealth-output ratio of unlevered agents is $W_t^{Unlev}/D_t = P_t/D_t - W^{Lev}/D_t$. From here, we can also compute the expected return and volatility of wealth, obtaining expressions similar to those in Proposition 15 .

Panel A of Figure 8 plots the wealth/output ratio of levered and unlevered agents. Not surprisingly, both the wealth/output ratios increase with the surplus consumption ratio,

[25]Note that $\int_{i:w_i-a_i>0} w_i di = \int_{i:w_i-a_i>0} a_i di + \int_{i:w_i-a_i>0}(w_i - a_i)di = K_0 + K_1$.

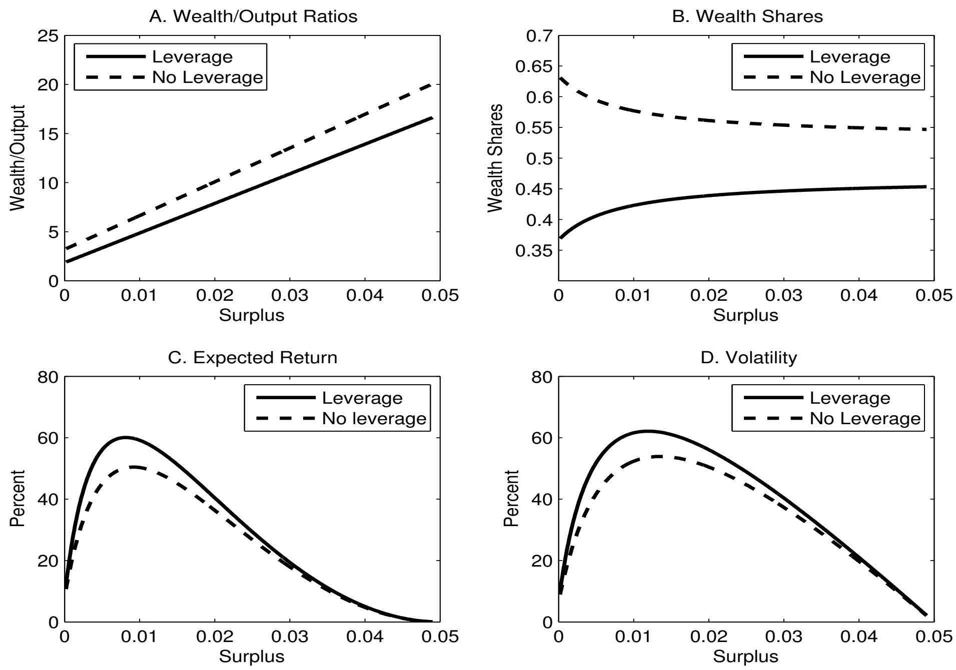

Figure 8: The Wealth of Levered and Unlevered Agents Panel A plots the wealth/output ratio for levered and unlevered agents. Panel B plot the wealth shares for levered and unlevered agents, where wealth share equal the wealth of each group of agents divided by total wealth. Panel C plots the expected return on wealth for the levered and unlevered agents as two aggregate groups. Panel D plots the volatility of wealth for levered and unlevered agents. The parameters used are those in Table 1 and Figure 2.

as the aggregate economy become wealthier. Panel B, however, shows that the share of aggregate wealth in the hands of levered agents increases with S_t, while unlevered agents see a reduction of their wealth share in such times. In addition, the increase in the wealth share of levered agents is concave, flattening out at high levels of S_t.

Panel C of Figure 8 shows that levered agents enjoy a uniformly higher expected return than unlevered agents. That is, as a group, those agents with $w_i - a_i > 0$, obtain higher average returns on wealth than unlevered agents. In many models with heterogeneous agents, this higher return would tend to generate an accelerated accumulation of capital of levered agents, who eventually would own the whole economy (see e.g. Dumas (1989)). This feature does not hold here because levered agents also take on more risk, as shown in Panel D of Figure 8. That is, even if agents with $w_i - a_i > 0$ have a higher average return on wealth, they also hold riskier portfolios, which leads to severe losses during downturns (S_t declining). As a consequence, the wealth share fluctuates as shown in Panel B.

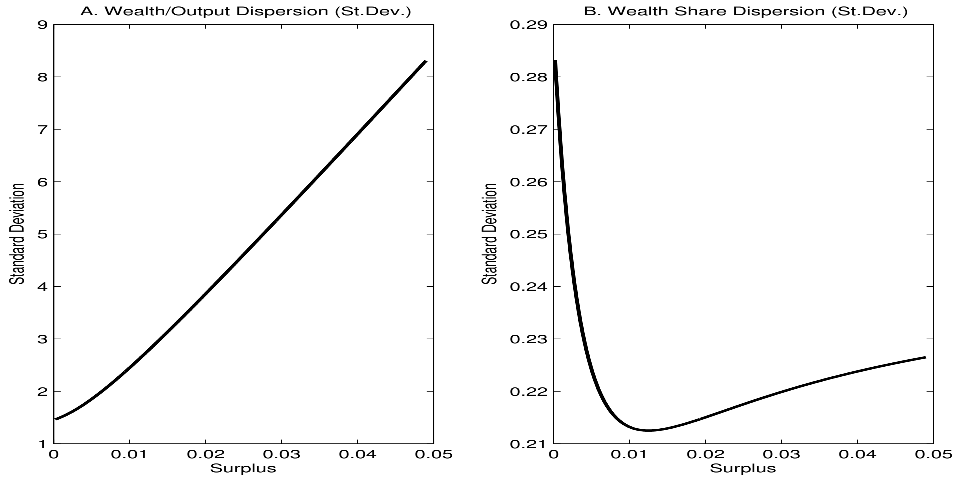

Figure 9: Wealth Dispersion. Panel A plots the cross-sectional dispersion – in standard deviation units – of the wealth to output of individual agents, plotted against the surplus consumption ratio (S_t). Panel B plots the cross-sectional dispersion – in standard deviation units – of the wealth shares, i.e. $W_{it}/\int W_{jt}dj$, against S_t. The parameters used are those in Table 1 and Figure 2.

5.2.4. Wealth dispersion

We finally consider the wealth dispersion implied by the model in good and bad times. There are two measures of wealth dispersion. Panel A of Figure 9 plots the cross-sectional standard deviation of wealth-output ratios, as already introduced in equation (36). The plot shows a strongly increasing dispersion of wealth as times get better. This is a level effect: as the aggregate wealth increase, the *level* difference of wealth-output ratio increases. This pattern was in fact evident already in Panel A of Figure 8, as the wealth-output ratios of levered and unlevered agent diverges as times get better (S_t increases).

However, a second measure of wealth dispersion is the dispersion compared to aggregate wealth. Indeed, we know from Panel B of Figure 1 that the wealth increases seven-fold from very bad times to very good times, and so the question is how this increase in wealth is shared across agents in the model. Panel B of Figure 9 shows that the cross-sectional standard deviation of wealth normalized by aggregate wealth actually declines for part of the range, to then increase for high S_t. This is consistent with the finding in Panel B of Figure 8 which shows the convergence of wealth shares between levered and unlevered agents.

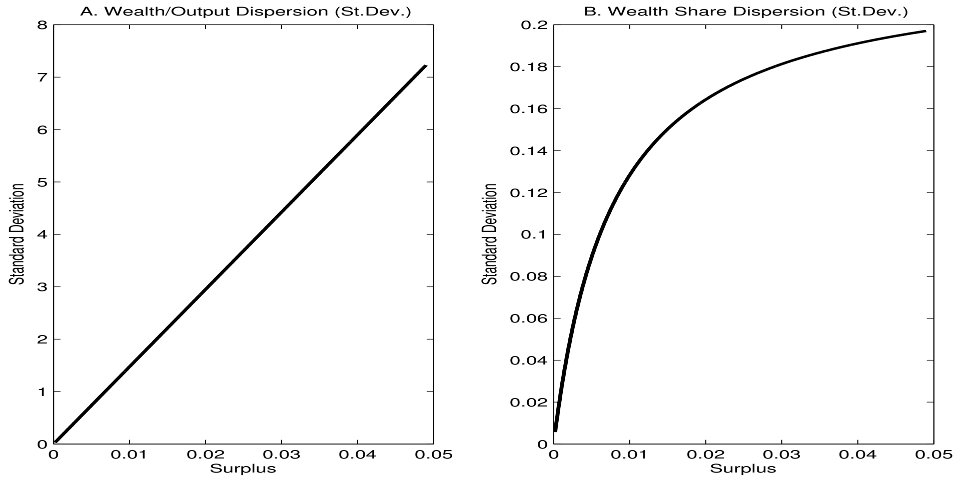

Figure 10: Wealth Dispersion with No Preference Heterogeneity. . Panel A plots the cross-sectional dispersion – in standard deviation units – of the wealth to output of individual agents, plotted against the surplus consumption ratio (S_t). Panel B plots the cross-sectional dispersion – in standard deviation units – of the wealth shares, i.e. $W_{it}/\int W_{jt}dj$, against S_t. The parameters used are those in Table 1 and Figure 2 except that we set $a_i = 1$ for all i.

5.2.5. The role of the dispersion in endowments

We conclude this section with an illustration of the impact of initial endowments on the wealth inequality over the business cycle. Consider the case in which all agents have the same preferences, that is, $a_i = 1$ for all i but the distribution of Pareto weights is as above. This results in log-normally distributed initial endowments, although this distributional assumption is irrelevant for the result.

Figure 10 is the analog of Figure 9 under the new assumptions. As is evident, Panel A is very similar to Panel A of the previous case due to the "level effect" on the wealth distribution. Panel B is instead markedly different: The model with homogeneous agents who have though differential endowment at time 0 generate a procyclical relative wealth inequality. That is, richer agents not only become richer during good times because of the level effect, but they become relatively richer compared to the rest of the economy. This is due to the fact that under this parametrization, rich agents borrow while poor agents lend. Thus, during good times, levered agents do better and become wealthier, even relative to the economy. The case discussed in Figure 9 had both very poor and very rich agents borrowing. Thus, the two extreme groups of the economy would do well in good times, reducing the wealth inequality relative to the wealth of the economy.

6. Conclusions

Our general equilibrium model with heterogeneous agents, habits, and countercyclical uncertainty, is able to tie together several stylized facts related to leverage, consumption, and asset prices. For instance, our model predicts that aggregate leverage should be procyclical, it should correlate with high valuation ratios, low volatility, and with a "consumption boom" of levered agents. Agents actively trade in risky assets, moreover, and delever in bad times by "fire selling" their risky positions as their wealth decline and debt-to-wealth increase.

An important message of the paper is to re-emphasize the perhaps obvious point that leverage is an *endogenous* quantity and thus that some caution must be taken when making causal statements about the impact of leverage on other economic quantities. For instance, in our model agents who increased leverage during good times will suffer low consumption growth in bad times. There is nothing inefficient of this allocation: Those agents who *decide* to take on higher leverage are implicitly providing insurance to the other agents who instead would like to buy safe assets. Similarly, the increase in leverage in good times is the result of an optimal, efficient risk-sharing allocation, and should predict low future asset pricing returns. Once again, it is not high leverage that implies that future return are low (because it increases the chance of a financial crisis, for instance), but rather the fact that lower risk premia due to subsided discount rate shocks induce agents with higher risk bearing capacity to take higher leverage to achieve their optimal consumption profile.

Admittedly, our model is simple in that it only has one state variable and all quantities are driven by only one shock. Our simplifying assumptions thus imply that all quantities move in lock-step and there is a likely unrealistic perfect (positive or negative) correlation between leverage, prices, volatility, expected return, consumption, and so on. These simplifying assumptions allow us to obtain closed form solutions for all quantities in the model, and thus obtain a better understanding of the various economic forces that affect leverage and asset prices. Future research may attempt to generalize our simple setting to obtain more realistic dynamics.

Appendix: Proofs

Proof of Proposition 1. The Lagrangean

$$\mathcal{L}(C_i) = \int \phi_i u(C_{it}, X_{it}, t)\, di - M_t \left(\int C_{it} di - D_t \right)$$

implies that agents' marginal utilities satisfy

$$\phi_i u_c(C_{it}, X_{it}, t) = M_t. \tag{46}$$

Thus, consumption satisfies

$$C_{it} - X_{it} = \phi_i e^{-\rho t}\ M_t^{-1} \tag{47}$$

The individual excess consumption is inversely related to the Lagrange multiplier M_t. To obtain the equilibrium value of M_t, we integrate across agents

$$\int C_{it} di - \int X_{it} di = \left(\int \phi_i di \right)\ e^{-\rho t} M_t^{-1} = e^{-\rho t} M_t^{-1}$$

Using the market clearing condition $D_t = \int C_{it} di$ we find that the Lagrangean multiplier is

$$M_t = e^{-\rho t} \frac{1}{D_t - \int X_{it} di} \tag{48}$$

Finally, plugging this expression into (47) we obtain that agent i's consumption is given by

$$C_{it} - X_{it} = \phi_i \left(D_t - \int X_{jt} dj \right) \tag{49}$$

Each agent's excess consumption over habit is proportional to aggregate excess output. This condition also implies that in equilibrium, the ratio of any two agents' marginal utilities is constant (and equal to the ratio of Pareto weights), a standard result with complete markets. Substituting X_{it} from (1) and using (2) we obtain the optimal consumption of agent i in Proposition 1. □

The proof of Proposition 3 follows after proof of Proposition 4, to which we first turn.

Proof of Proposition 4. The Lagrange multiplier at time t in equation (48) provides the marginal utility of the representative agent. Using (9) we find:

$$M_t = e^{-\rho t} D_t^{-1} Y_t$$

The interest rate and SDF can be found by applying Ito's lemma to M_t. □

The pricing function for the consumption claim is

$$\begin{aligned}
P_t &= E_t \left[\int_t^\infty \frac{M_\tau}{M_t} D_\tau d\tau \right] && (50)\\
&= D_t Y_t^{-1} E_t \left[\int_t^\infty e^{-\rho(\tau - t)} D_\tau^{-1} Y_\tau D_\tau d\tau \right] && (51)\\
&= D_t S_t E_t \left[\int_t^\infty e^{-\rho(\tau - t)} Y_\tau d\tau \right] && (52)\\
&= D_t S_t \int_t^\infty e^{-\rho(\tau - t)} E_t [Y_\tau]\, d\tau && (53)\\
&= D_t S_t \int_t^\infty e^{-\rho(\tau - t)} (\overline{Y} + (Y_t - \overline{Y}) e^{-k(\tau - t)}) d\tau && (54)\\
&= D_t S_t \left(\frac{\overline{Y}}{\rho} + \frac{(Y_t - \overline{Y})}{\rho + k} \right) && (55)\\
&= D_t S_t \left(\frac{\rho Y_t + k\overline{Y}}{\rho(\rho + k)} \right) && (56)
\end{aligned}$$

Ito's lemma on P_t gives the further results about stock return volatility and expected return.

Finally, from market completeness, the wealth of agent i is always equal to the discounted value of his/her optimal consumption, which can be written as

$$
\begin{aligned}
C_{i,t} &= (g_{it} + \phi_i)\left(D_t - \int X_{jt} dj\right) & (57)\\
&= (a_i(Y_t - \lambda) + \lambda - 1 + \phi_i) S_t D_t & (58)
\end{aligned}
$$

We then have

$$
\begin{aligned}
W_{i,t} &= E_t\left[\int_t^\infty \frac{M_\tau}{M_t} C_{i,\tau} d\tau\right]\\
&= D_t S_t E_t\left[\int_t^\infty e^{-\rho(\tau-t)} D_\tau^{-1} S_\tau^{-1} C_{i,\tau} d\tau\right]\\
&= D_t S_t E_t\left[\int_t^\infty e^{-\rho(\tau-t)} (a_i(Y_\tau - \lambda) + \lambda - 1 + \phi_i) d\tau\right]\\
&= D_t S_t E_t\left[\int_t^\infty e^{-\rho(\tau-t)} (a_i Y_\tau - a_i\lambda + +\lambda - 1 + \phi_i) d\tau\right]\\
&= D_t S_t\left[a_i \frac{(Y_t - \overline{Y})}{\rho + k} + \frac{a_i(\overline{Y} - \lambda) + \lambda - 1 + \phi_i}{\rho}\right] & (59)
\end{aligned}
$$

where we used the fact that $E_t[Y_\tau] = \overline{Y} + (Y_t - \overline{Y})e^{-k(\tau-t)}$. At time 0, the economy starts at its stochastic steady state, $Y_0 = \overline{Y}$, which implies $S_0 = \overline{S} = 1/\overline{Y} = 1/Y_0$. In addition, assume $D_0 = \rho$. Agent i's endowment is w_i. Therefore, we obtain that the budget constraint implies

$$
\begin{aligned}
w_i = W_{i,0} &= D_0 S_0\left[a_i \frac{(Y_0 - \overline{Y})}{\rho + k} + \frac{a_i(\overline{Y} - \lambda) + \lambda - 1 + \phi_i}{\rho}\right]\\
&= D_0 S_0\left[\frac{a_i(\overline{Y} - \lambda) + \lambda - 1 + \phi_i}{\rho}\right]\\
&= \overline{S}\left[a_i(\overline{Y} - \lambda) + \lambda - 1 + \phi_i\right]
\end{aligned}
$$

$$w_i/\overline{S} = \left[a_i(\overline{Y} - \lambda) + \lambda - 1 + \phi_i\right]$$

or

$$\phi_i = w_i\overline{Y} - \left[a_i(\overline{Y} - \lambda) + \lambda - 1\right] \ .$$

as in Proposition 1.

The curvature of the utility function can be obtained from the definition of curvature and by substituting C_{it} and ϕ_i in the resulting expression.

The consumption/output ratio (58) can then be written as

$$
\begin{aligned}
\frac{C_{i,t}}{D_t} &= (a_i(Y_t - \lambda) + \lambda - 1 + \phi_i) S_t\\
&= (a_i(Y_t - \lambda) + w_i\overline{Y} - a_i(\overline{Y} - \lambda)) S_t\\
&= (a_i(Y_t - \overline{Y}) + w_i\overline{Y}) S_t\\
&= a_i(1 - \overline{Y} S_t) + w_i \overline{Y} S_t
\end{aligned}
$$

Proof of Proposition 3. Given the results of Propositions 1 and 4, and the standard result that the efficient allocation maximize agents' utility, the only part left to show is the optimal allocation to stocks and bonds. From Cox and Huang (1989), the dynamic budget equation can be written as the present value of future consumption discounted using the stochastic discount factor. The optimal allocation can be found by finding the "replicating" portfolio, that is, the position in stocks and bonds that satisfies the static budget equation.

We denote for simplicity

$$\sigma_Y(Y) = v\sigma_D(Y) \tag{60}$$

First, note that the process for surplus consumption ratio is

$$\begin{aligned}
dS_t &= -Y_t^{-2}dY_t + Y_t^{-3}dY_t^2 \\
&= -Y_t^{-2}k(\overline{Y} - Y_t)dt + Y_t^{-1}\sigma_Y(Y)dZ_t + Y_t^{-1}\sigma_Y(Y)^2dt \\
&= Y_t^{-1}k(1 - \overline{Y}/Y_t)dt + Y_t^{-1}\sigma_Y(Y)dZ_t + Y_t^{-1}\sigma_Y(Y)^2dt \\
&= Y_t^{-1}\left(k(1 - \overline{Y}/Y_t) + \sigma_Y(Y)^2dt\right)dt + Y_t^{-1}\sigma_Y(Y)dZ_t
\end{aligned}$$

Consider now agents' wealth obtained in (59) which we can write as

$$\begin{aligned}
W_{i,t} &= D_t\frac{1}{\rho}\left[a_i\frac{\rho}{\rho+k}(1 - \overline{Y}S_t) + w_i\overline{Y}S_t\right] &(61)\\
&= D_t\frac{1}{\rho(\rho+k)}\left[a_i\rho + (w_i(\rho+k) - a_i\rho)\overline{Y}S_t\right] &(62)
\end{aligned}$$

From Ito's lemma, the diffusion of wealth process $dW_{i,t}/W_{i,t}$ is

$$\sigma_{W,i}(S_t) = \sigma_D(S_t) + \frac{(w_i(\rho+k) - a_i\rho)\overline{Y}Y_t^{-1}\sigma_Y(Y_t)}{a_i\rho + (w_i(\rho+k) - a_i\rho)\overline{Y}Y_t^{-1}} \tag{63}$$

By market completeness (Cox and Huang (1989)), agent i's wealth is always equal to his/her allocation to stocks and bonds

$$W_{it} = N_{i,t}P_t + N_{i,t}^0B_t$$

From this latter expression, N_{it} must be chosen to equate the diffusion of the portfolio to the diffusion of wealth. That is, such that

$$N_{it}P_t\sigma_P(S_t) = W_{i,t}\sigma_{W,i}(S_t)$$

Solving for N_{it} gives

$$
\begin{aligned}
N_{it} &= \frac{W_{it}\sigma_{Wi}(Y)}{P_t\sigma_P(Y)} \\
&= \frac{\left(\rho a_i + \left(w_i(\rho+k) - \rho a_i\right)\overline{Y}/Y_t\right)}{\left(\rho + k\overline{Y}/Y_t\right)}\left(\frac{\sigma_D(Y) + \frac{\left(w_i(\rho+k)-\rho a_i\right)\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho a_i + \left(w_i(\rho+k)-\rho a_i\right)\overline{Y}/Y_t\right)}}{\sigma_D(Y) + \frac{k\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho+k\overline{Y}/Y_t\right)}}\right) \\
&= \frac{\left(\rho a_i + \left(w_i(\rho+k) - \rho a_i\right)\overline{Y}/Y_t\right)}{\left(\rho + k\overline{Y}/Y_t\right)}\left(\frac{\frac{\sigma_D(Y)\left(\rho a_i + \left(w_i(\rho+k)-\rho a_i\right)\overline{Y}/Y_t\right) + \left(w_i(\rho+k)-\rho a_i\right)\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho a_i + \left(w_i(\rho+k)-\rho a_i\right)\overline{Y}/Y_t\right)}}{\frac{\sigma_D(Y)\left(\rho+k\overline{Y}/Y_t\right) + k\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho+k\overline{Y}/Y_t\right)}}\right) \\
&= \left(\frac{\sigma_D(Y)\left(\rho a_i + \left(w_i(\rho+k)-\rho a_i\right)\overline{Y}/Y_t\right) + \left(w_i(\rho+k)-\rho a_i\right)\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\sigma_D(Y)\left(\rho+k\overline{Y}/Y_t\right) + k\overline{Y}Y_t^{-1}\sigma_Y(Y)}\right) \\
&= \frac{\sigma_D(Y)\rho a_i + \sigma_D(Y)\overline{Y}/Y_t\left(w_i(\rho+k)-\rho a_i\right) + w_i(\rho+k)\overline{Y}Y_t^{-1}\sigma_Y(Y) - \rho a_i\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\sigma_D(Y)\left(\rho+k\overline{Y}/Y_t\right) + k\overline{Y}Y_t^{-1}\sigma_Y(Y)} \\
&= a_i + (\rho+k)\frac{\sigma_D(Y)\overline{Y}/Y_t + \overline{Y}Y_t^{-1}\sigma_Y(Y)}{\sigma_D(Y)\left(\rho+k\overline{Y}/Y_t\right) + k\overline{Y}Y_t^{-1}\sigma_Y(Y)}(w_i - a_i) \\
&= a_i + (\rho+k)\frac{\sigma_D(Y)\overline{Y}/Y_t + \overline{Y}Y_t^{-1}\sigma_Y(Y)}{\sigma_D(Y)\rho + k\left(\sigma_D(Y)\overline{Y}/Y_t + \overline{Y}Y_t^{-1}\sigma_Y(Y)\right)}(w_i - a_i) \\
&= a_i + (\rho+k)\frac{\overline{Y}/Y_t\left[\sigma_D(Y) + \sigma_Y(Y)\right]}{\sigma_D(Y)\rho + k\overline{Y}/Y_t\left[\sigma_D(Y) + \sigma_Y(Y)\right]}(w_i - a_i) \\
&= a_i + (\rho+k)\frac{\overline{Y}/Y_t\sigma_M(Y)}{\sigma_D(Y)\rho + k\overline{Y}/Y_t\sigma_M(Y)}(w_i - a_i)
\end{aligned}
$$

where

$$\sigma_M(Y) = \sigma_D(Y) + \sigma_Y(D)$$

Finally, substituting $\sigma_Y(Y) = v\sigma_D(Y)$ from definition (60) and deleting $\sigma_D(Y)$ throughout, the result follows.

Similarly, we have that the amount in bonds is

$$
\begin{aligned}
N_{it}^0 B_t &= W_{it} - N_{it}P_t \\
&= D_t\frac{1}{\rho}\left(\frac{\rho}{\rho+k}a_i + \left(w_i - \frac{\rho}{\rho+k}a_i\right)\overline{Y}/Y_t\right) - N_{it}D_t\frac{\left(\rho+k\overline{Y}/Y_t\right)}{\rho(\rho+k)} \\
&= D_t\frac{1}{\rho(\rho+k)}\left[\left(\rho a_i + \left(w_i(\rho+k)-\rho a_i\right)\overline{Y}/Y_t\right) - N_{it}\left(\rho+k\overline{Y}/Y_t\right)\right] \\
&= D_t\frac{1}{\rho(\rho+k)}\left[a_i\left(\rho+k\overline{Y}/Y_t\right) + w_i(\rho+k)\overline{Y}/Y_t - a_i(\rho+k)\overline{Y}/Y_t - N_{it}\left(\rho+k\overline{Y}/Y_t\right)\right] \\
&= D_t\frac{1}{\rho(\rho+k)}\left[a_i\left(\rho+k\overline{Y}/Y_t\right) + (w_i-a_i)(\rho+k)\overline{Y}/Y_t - N_{it}\left(\rho+k\overline{Y}/Y_t\right)\right] \\
&= D_t\frac{1}{\rho}\left[\overline{Y}/Y_t - \frac{\overline{Y}/Y_t\sigma_M(Y)}{\sigma_D(Y)\rho + k\overline{Y}/Y_t\sigma_M(Y)}\left(\rho+k\overline{Y}/Y_t\right)\right](w_i - a_i) \\
&= D_t\frac{1}{\rho}\left[\frac{\overline{Y}/Y_t\left[\sigma_D(Y)\rho + k\overline{Y}/Y_t\sigma_M(Y)\right] - \overline{Y}/Y_t\sigma_M(Y)\left(\rho+k\overline{Y}/Y_t\right)}{\sigma_D(Y)\rho + k\overline{Y}/Y_t\sigma_M(Y)}\right](w_i - a_i) \\
&= -D_t\left[\frac{\overline{Y}/Y_t\left(\sigma_M(Y) - \sigma_D(Y)\right)}{\sigma_D(Y)\rho + k\overline{Y}/Y_t\sigma_M(Y)}\right](w_i - a_i) \\
&= -D_t\left[\frac{\overline{Y}/Y_t\left(\sigma_M(Y)/\sigma_D(Y) - 1\right)}{\rho + k\overline{Y}/Y_t\sigma_M(Y)/\sigma_D(Y)}\right](w_i - a_i)
\end{aligned}
$$

Finally, substituting $\sigma_Y(Y) = v\sigma_D(Y)$ from definition (60) and deleting $\sigma_D(Y)$ throughout, the result follows. $\square$

Proof of Corollary 1. Part (a) is immediate from the expression for N_{it}^0 in Proposition 2. Part (b) can be shown as follows:

$$
\begin{aligned}
\frac{N_{it}P_t}{W_{it}} &= \frac{\sigma_{Wi}(Y)}{\sigma_P(Y)} \\
&= \frac{\sigma_D(Y) + \frac{\left(w_i - \frac{\rho}{\rho+k}a_i\right)\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\frac{\rho}{\rho+k}a_i + \left(w_i - \frac{\rho}{\rho+k}a_i\right)\overline{Y}/Y_t\right)}}{\sigma_D(Y) + \frac{k\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho + k\overline{Y}/Y_t\right)}} \\
&= \frac{\sigma_D(Y) + \frac{(w_i(\rho+k) - \rho a_i)\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho a_i + (w_i(\rho+k) - \rho a_i)\overline{Y}/Y_t\right)}}{\sigma_D(Y) + \frac{k\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho + k\overline{Y}/Y_t\right)}} \\
&= \frac{\sigma_D(Y) + \sigma_Y(Y)\left(\frac{(w_i(\rho+k) - \rho a_i)\overline{Y}Y_t^{-1}}{\left(\rho a_i + (w_i(\rho+k) - \rho a_i)\overline{Y}/Y_t\right)}\right)}{\sigma_D(Y) + \sigma_Y(Y)\left(\frac{k\overline{Y}Y_t^{-1}}{\left(\rho + k\overline{Y}/Y_t\right)}\right)} \\
&= \frac{\sigma_D(Y) + \sigma_Y(Y)\left(1 - \frac{\rho}{\rho + [k + (\rho+k)(w_i - a_i)/a_i]\overline{Y}/Y_t}\right)}{\sigma_D(Y) + \sigma_Y(Y)\left(1 - \frac{\rho}{\left(\rho + k\overline{Y}/Y_t\right)}\right)}
\end{aligned}
$$

Finally, substituting $\sigma_Y(Y) = v\sigma_D(Y)$ from definition (60) and deleting $\sigma_D(Y)$ throughout, the result follows. Q.E.D.

Proof of Corollary 2. Immediate from the expression of $H(S_t)$. Q.E.D.

Proof of Corollary 3. Immediate from the fact $L(S_t)$ is increasing and the fact that agents with $w_i - a_i > 0$ are leveraged and have C_{it}/D_t that is increasing in S_t. Q.E.D.

Proof of Corollary 4. The expression of $E[dC_{it}/C_{it}]$ stems from the application of Ito's lemma to the consumption $C_{it} = D_t[a_i + (w_i - a_i)\overline{Y}S_t]$. The remaing part is immediate from the statement in the corollary. Q.E.D.

Proof of Corollary 5. Immediate from Corollary 2 and 4. Q.E.D.

Proof of Corollary 6. Immediate from Corollary 2. Q.E.D.

Proof of Corollary 7. Immediate from Proposition 2 and 3. The state price density and the price of stocks are independent of cross-sectional quantities. Q.E.D.

Proof of Proposition 8 Substituting ϕ_i into the $W_{i,t}$

$$
\begin{aligned}
\frac{W_{i,t}}{D_t} &= S_t\left[a_i\frac{(Y_t - \overline{Y})}{\rho + k} + \frac{a_i(\overline{Y} - \lambda) + \lambda - 1 + \phi_i}{\rho}\right] && (64) \\
&= \frac{1}{\rho}\left[a_i\frac{\rho}{\rho + k}(1 - S_t/\overline{S}) + w_i S_t/\overline{S}\right] && (65)
\end{aligned}
$$

which is the expression of wealth/output ratios in Proposition 8. The expression for consumption/wealth ratio follows from these last two results.

Proof of Proposition 9. Immediate from the definition of cross-sectional variance and the result in Proposition 8. Q.E.D.

Proof of Proposition 10. This proposition follows from an application of Ito's lemma to the wealth process in (65). Q.E.D.

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