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ABSTRACT

We propose a New Keynesian model with the shadow rate, which is the federal funds rate during normal times. At the zero lower bound, we establish empirically the negative shadow rate summarizes unconventional monetary policy with its resemblance to private interest rates, the Fed's balance sheet, and Taylor rule. Theoretically, we formalize our shadow rate New Keynesian model with QE and lending facilities. Our model generates the data-consistent result: a negative supply shock is always contractionary. It also salvages the New Keynesian model from the zero lower bound induced structural break.

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1 Introduction

After almost two decades of Japan's experience of zero interest rates, the Great Recession brought the US economy the same problem, followed by the UK and Euro area. The zero lower bound (ZLB) poses issues for advanced economies and consequently economic research. The ZLB invalidates the traditional monetary policy tool because central banks are unable to further lower policy rates. Subsequently, central banks around the world have introduced unconventional policy tools such as large-scale asset purchases (or QE), lending facilities, and forward guidance. How economic models accommodate the ZLB and unconventional monetary policy has become the new challenge for economic research. This paper proposes a novel New Keynesian model with the shadow rate to address this issue.

Policy makers and economists argue a similarity exists between the conventional and unconventional monetary policies in various contexts; for example, see Bullard (2012), Powell (2013), Blanchard (2016), and Wu and Xia (2016). Furthermore, Wu and Xia (2016) propose a shadow rate as the coherent summary of monetary policy: the shadow rate is the federal funds rate when the ZLB is not binding; otherwise, it is negative to account for unconventional policy tools. Altig (2014) of the Atlanta Fed, Hakkio and Kahn (2014) of the Kansas City Fed, and others have subsequently adopted Wu and Xia's (2016) shadow rate as the monetary policy stance for policy analyses.

The main contribution of the paper is introducing this shadow rate into the New Keynesian model. The shadow rate is the central ingredient. We investigate new empirical evidence to establish its relevance and motivate our new model. First, the shadow rate is highly correlated with the Fed's balance sheet, with the correlation being -0.94 throughout the QE phase. This finding validates the shadow rate as a summary for unconventional monetary policy. Second, at the ZLB, the shadow rate comoves almost perfectly with an overall financial conditions index and various private interest rates, which are the relevant interest rates for households and firms. This evidence suggests replacing the fed funds rate with the shadow rate in the New Keynesian model can summarize how the private economy factors in the additional stimulus from unconventional policy tools. Third, the shadow rate follows the same Taylor rule as the fed funds rate did prior to the ZLB. This result proposes the shadow rate Taylor rule, which extends the historical Taylor rule into the ZLB period with the shadow rate.

One contribution of the paper is to introduce a three-equation linear shadow rate New Keynesian model based on these empirical findings. This model proposes the shadow rate as a sensible and tractable summary for all unconventional policy tools, allowing the model to remain linear without a ZLB-induced structural break. The shadow rate replaces the policy rate entering the IS curve. The zero lower bound on the Taylor rule is lifted, which becomes a shadow rate Taylor rule. The Phillips curve remains the same. During normal times, this model is the same as the standard New Keynesian model. However, monetary policy remains active in our model when the ZLB prevails, which is not the case in the standard model.

The next contribution is to formalize the three-equation shadow rate system with agents' optimization problems where some major unconventional policy tools are implemented by the central bank. First, the negative shadow rate can be implemented through QE programs. The government's bond purchases lower bond yields without changing the policy rate, which works by reducing the risk premium. This risk-premium channel of QE is consistent with the empirical findings of Hamilton and Wu (2012) and Gagnon et al. (2011). We demonstrate the equivalence between the shadow rate and QE in our model, providing one microfoundation for the shadow rate IS curve. To achieve this equivalence, the model requires a linear relationship between log bond holdings by the Fed and the shadow rate. This relationship is verified in the data, with the correlation between these two variables being -0.92.

Second, we map lending facilities, which inject liquidity into the economy, into the shadow rate framework. The primary example of this policy is the Federal Reserve's Term Asset-Backed Securities Loan Facility. We model lending facilities by allowing the government to extend extra credit directly to the private sector; that is, the government can vary the loan-to-value ratio the borrowers face as a policy tool. The lending facilities are coupled with a tax policy on interest rate payments, which, according to Waller (2016) of the St. Louis Fed, is the nature of the recent negative interest rate policy in Europe and Japan. We then establish an equivalence between the shadow rate and the lending facilities – tax policy channel, which constitutes another microfoundation for the shadow rate IS curve.

Although we present our main model and the shadow rate equivalence for QE in the linearized form, the usefulness of the shadow rate goes beyond linearity. We demonstrate this point with the lending facilities – tax policy channel, where the equivalence is also established without linearization. Whether or not the model is linearized, the common theme is that the shadow rate serves as a summary statistic for various unconventional policy tools and does not introduce a structural break at the ZLB.

The standard New Keynesian model is associated with some distinctive modeling implications at the ZLB, some of which are counterfactual or puzzling. First, in such a model, a negative supply shock stimulates the economy. In contrast to this model implication, empirical evidence from Wieland (2015) and Garín et al. (2016) demonstrate a similar response of output to a supply shock during normal times and at the zero lower bound. We show this conterfactual implication of the standard model is due to the lack of policy interventions at the ZLB. Our model restores the data-consistent implication by introducing unconventional monetary policy through the shadow rate. A related issue is the size of the government-spending multiplier. This is still an on-going debate. In a standard model without unconventional monetary policy, this multiplier is much larger at the ZLB. This larger multiplier also disappears in our model.

Besides the benefit of sensible economic implications, the shadow rate also salvages the New Keynesian model from technical issues due to the structural break introduced by the ZLB. The ZLB imposes one of the biggest challenges for solving and estimating these models. Methods proposed in the literature to address this issue either produce economically uncompelling implications or are extremely computationally demanding. This challenge will not go away even after the economy lifts off from the zero lower bound. Our shadow rate model proposes a compelling solution to this challenge. It does not incur a structural break at the ZLB whether we work with a linear or non-linear model. Therefore, it restores the traditional solution and estimation methods' validity.

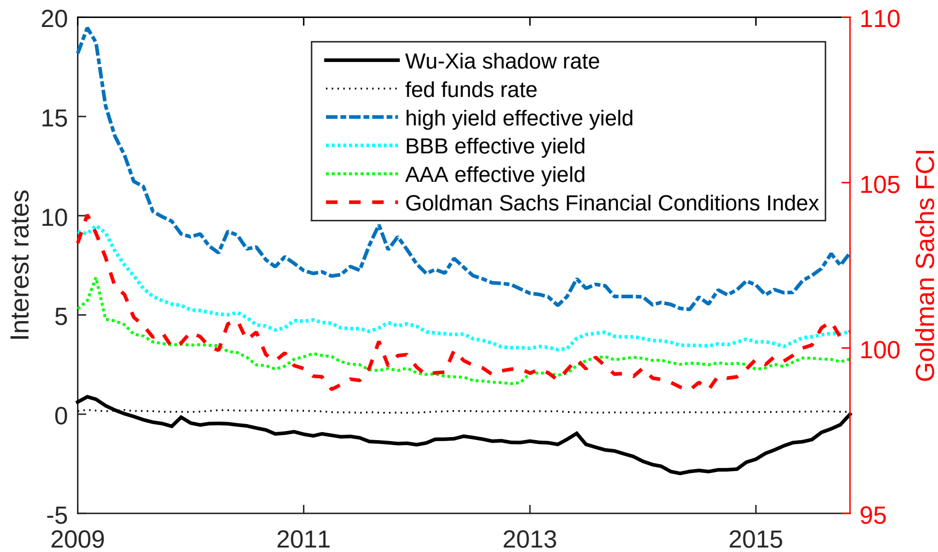
The rest of the paper after a brief literature review proceeds as follows. Section 2 provides new empirical evidence on the shadow rate. Section 3 proposes a three-equation linear shadow rate New Keynesian model. Subsequently, Sections 4 and 5 map QE and lending facilities into this model theoretically. Section 6 discusses quantitative analyses, and Section 7 concludes.

Related literature: Our paper contributes to the DSGE literature on unconventional monetary policy. Cúrdia and Woodford (2011), Chen et al. (2012), and Gertler and Karadi (2013) study asset purchases, that is, QE. Gertler and Karadi (2011), Williamson (2012), and Del Negro et al. (2016) evaluate central banks' liquidity provision, along the lines of lending facilities. McKay et al. (2014), Del Negro et al. (2015), and Kulish et al. (2016) focus on forward guidance. We model QE and lending facilities directly. Our paper also speaks to forward guidance in the sense that the shadow rate reflects changes in medium- or long-term yields due to forward guidance. A direct mapping between the two is in Wu and Xia (2016).

Our paper differs from the existing literature in the follow respects. First, we use the shadow rate to provide one coherent framework for the ZLB period as well as for normal times, whereas models in the literature are specifically targeted for the ZLB. Consequently, our framework provides a natural extension to models researchers developed prior to the ZLB, because the shadow rate is the same as the fed funds rate when the ZLB is not binding.

Second, rather than focus on a specific policy tool, we use the shadow rate as a summary for all unconventional monetary policy measures. Third, the shadow rate is not subject to a structural break at the ZLB, which makes the model tractable and alleviates numerical and computational issues.

Figure 1: Shadow rate and private interest rates



Notes: black solid line: the Wu-Xia shadow rate; black dotted line: the effective fed funds rate; blue dash-dotted line: the BofA Merrill Lynch US High Yield Effective Yield; cyan dotted line: the BofA Merrill Lynch US Corporate BBB Effective Yield; green dotted line: the BofA Merrill Lynch US Corporate AAA Effective Yield; red dashed line: the Goldman Sachs Financial Conditions Index. Left scale: interest rates in percentage points; right scale: the Goldman Sachs Financial Conditions Index. The ZLB sample spans from January 2009 to November 2015.

2 Shadow rate: new empirical evidence

We will propose using the shadow rate in a New Keynesian model in Sections 3 - 5 to conveniently summarize unconventional monetary policy in a tractable and plausible way. This section presents some new empirical evidence to establish this relationship and motivate the usefulness of the shadow rate, which is defined as follows:

$$r_t = \max(0,\ s_t), \tag{2.1}$$

where r_t is the policy rate, such as the fed funds rate, and s_t is the shadow rate.

Between January 2009 and November 2015, the effective fed funds rate is close to zero and does not move much, defining the ZLB; see the black dotted line in Figure 1. However, the Wu and Xia (2016) shadow rate in solid black still displays variation tracking unconventional monetary policy. It dropped 3% from the onset of the ZLB until mid-2014, representing an easing stance of the Fed. Subsequently, a 3% tightening was implemented between then and the time of the liftoff from the ZLB in November 2015.

2.1 A summary for unconventional monetary policy

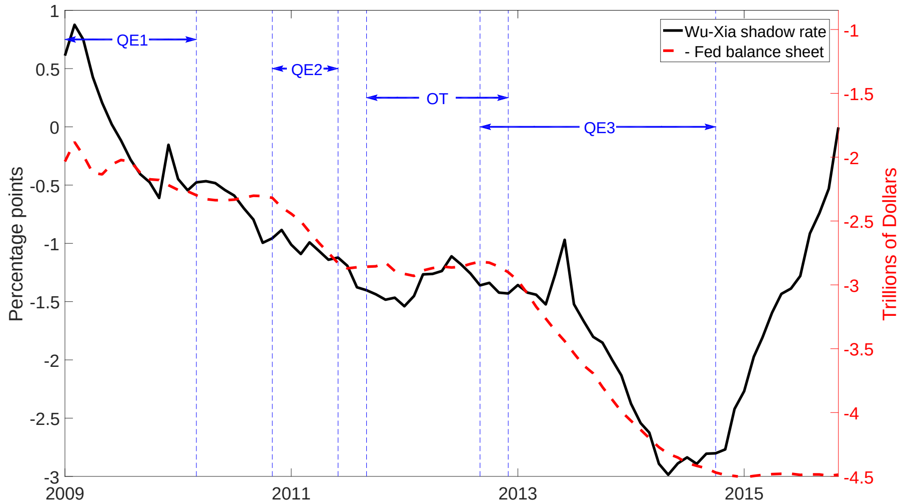
Private interest rates summarize the effect of monetary policy, whether conventional or unconventional, on the overall economy. An easing monetary policy intends to lower private interest rates, which disincentives saving, motivates agents to borrow and invest more, and altogether leads to a higher aggregate demand. The conventional monetary policy achieves this by lowering the policy rate. When the policy rate is stuck at zero, unconventional policy tools target to stimulate the economy by further lowering private interest rates.

We assess the comovement between the shadow rate and various private interest rates and financial conditions in the data to warrant the choice of the shadow rate as the summary for the overall effect of unconventional policy tools. Figure 1 demonstrates that various interest rates that private agents face comove with the shadow rate. The blue dash-dotted line is the high yield effective yield, the cyan dotted line is the BBB effective yield, and the green dotted line is the AAA effective yield. None of these corporate borrowing rates, whether investment grade or high yield, face the ZLB: they are at least 1.5% and display meaningful variations. They track the U-shape dynamics of the shadow rate. Consequently, they are highly correlated with the shadow rate, with correlations of 0.8, 0.8, and 0.6, respectively.

We also plot the Goldman Sachs Financial Conditions Index in red, which tracks broad financial markets including equity prices, the US dollar, Treasury yields, and credit spreads. It depicts the same story, and also has a high correlation with the shadow rate at 0.8. To obtain these correlations, the shadow rate's role is instrumental, and it cannot be replaced by the 10-year Treasury rate. For example, the correlation between the 10-year Treasury and the Goldman Sachs Financial Conditions Index is only 0.27.

Next, we compare the shadow rate with unconventional monetary policy directly. One popular measure of the overall unconventional monetary policy is the Federal Reserve's

Figure 2: Shadow rate and Fed's balance sheet



Notes: black solid line: the Wu-Xia shadow rate; red dashed line: the negative Fed's balance sheet. Left scale: interest rates in percentage points; right scale: negative Fed's balance sheet in trillions of dollars. QE1: the first round of QE from November 2008 to March 2010; QE2: the second round of QE from November 2010 to June 2011. OT: operation twist from September 2011 to December 2012. QE3: the third round of QE from September 2012 to October 2014.

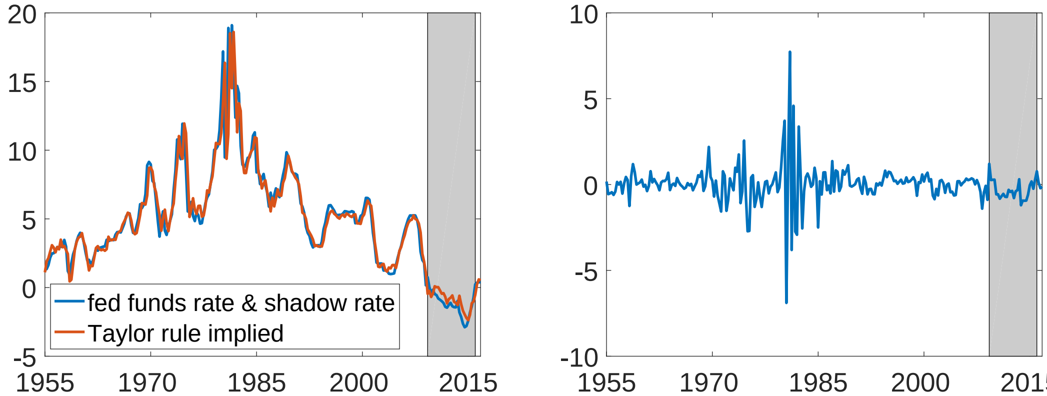
balance sheet. Figure 2 displays such a comparison. The Fed's assets in red grow from about \$2 trillion in 2009 to about \$4.5 trillion as of January 2015. The net expansion over this period reflects primarily the large-scale asset purchases (QE). The Wu and Xia (2016) shadow rate has a high correlation with the Fed's balance sheet at -0.74. The correlation is even higher throughout the QE phase, and the number is -0.94 up until the end of QE3.

2.2 Shadow rate Taylor rule

We have established the shadow rate as a tractable summary for unconventional monetary policy. Next, we assess whether it follows the same Taylor rule as the fed funds rate did prior to the ZLB episode. We begin by defining the shadow rate Taylor rule:

$$s_t = \phi_s s_{t-1} + (1 - \phi_s) \left[\phi_y (y_t - y_t^n) + \phi_\pi \pi_t + s \right], \tag{2.2}$$

Figure 3: Taylor rule



Notes: Left panel: blue line: observed fed funds rate and shadow rate; red line: Taylor rule implied rate. Right panel: monetary policy shock. Shaded area: ZLB. Data are quarterly from 1954Q4 to 2016Q3.

where y_t is output and π_t is inflation. y_t^n is potential output, which is equilibrium output under flexible prices. ϕ_s captures the persistence of the process, and ϕ_y and ϕ_π denote the responsiveness of the shadow rate to the output gap and inflation, respectively. The restriction $\phi_\pi > 1$ guarantees a unique, non-explosive equilibrium. s is the steady-state value of the shadow rate.

To evaluate whether the Taylor rule is a good description of the shadow rate dynamics, we estimate the shadow rate Taylor rule (2.2) empirically via regressing the shadow rate on the output gap and inflation. For the shadow rate, we take the Wu and Xia (2016) splined series of the fed funds rate during normal times and shadow rate at the ZLB. The output gap is the difference between GDP and potential GDP, measured in the 2009 chained dollar. Inflation is the GDP deflator. The data are quarterly from 1954Q1 to 2016Q3.

Figure 3 plots the regression results. In the left panel, we plot together the implemented monetary policy in blue and what the Taylor rule prescribes in red. The Taylor rule seems to be a good description of what actually happens, including the ZLB period. We also plot the regression residual, which can be interpreted as the monetary policy shock, in the right panel. Most prominently, the size of the monetary policy shock was much larger during the 1980s when interest rates were high. On the contrary, the shock during the ZLB period had a similar size to the rest of the sample.

To see more formally whether a structural break exists, we perform an F test: the F statistic of 2 is smaller than the 5% critical value 2.37, and we fail to reject the null of no structural break. This result is consistent with Wu and Xia's (2016) finding.

3 A shadow rate New Keynesian model (SRNKM)

In this and the next two sections, we propose a novel shadow rate New Keynesian model, which, according to the empirical evidence presented in Section 2, captures both the conventional interest rate rule and unconventional policy tools in a coherent and tractable way. This section presents the three-equation linear version of the model, and Sections 4 - 5 then micro-found this model with two popular unconventional policy tools: QE and lending facilities. Section 3.1 sets up the linear model, 3.2 introduces a potential extension that nests the standard New Keynesian model, and we then discuss our model's economic implications in 3.3 and computational advantages in 3.4.

3.1 Main model with shadow rate

The relevant interest rates affecting households' and firms' decisions are private interest rates r_t^B, through which conventional and unconventional monetary policies transmit into the economy. This notion argues for replacing r_t in the standard IS curve with private interest rates. A private interest rate can be represented by the sum of the shadow rate s_t and a constant wedge according to the evidence in Figure 1. Consequently, an IS curve with a private interest rate r_t^B is equivalent to an IS curve with the shadow rate, as the constant drops out.

This new IS curve, together with the shadow rate Taylor rule defined in (2.2), leads to the three-equation linear shadow rate New Keynesian model defined as follows:

Definition 1 *The shadow rate New Keynesian model consists of the shadow rate IS curve*

$$y_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - s) + \mathbb{E}_t y_{t+1}, \tag{3.1}$$

New Keynesian Phillips curve

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa(y_t - y_t^n), \tag{3.2}$$

and shadow rate Taylor rule (2.2).[1]

$\mathbb{E}$ is the expectation operator, lowercase letters are logs, and letters without t subscripts are either coefficients or steady-state values. All the coefficients are positive. Equation (3.1) describes that demand is a decreasing function of the real interest rate $rr_t = s_t - \mathbb{E}_t\pi_{t+1}$, where σ is the reciprocal of the intertemporal elasticity of substitution.

Sections 4 - 5 will micro-found the shadow rate IS curve in (3.1) by implementing two major unconventional policy tools. Note a negative shadow interest rate is not the actual borrowing or lending rate firms and households face. Rather, we propose to use it as a summary statistic for all the measures of the Fed's conventional and unconventional policies. It is the conventional monetary policy $r_t = s_t$ when $s_t \geq 0$. At the ZLB, the effects of unconventional policy tools on private rates can be summarized by $s_t < 0$.

Equation (3.2) is the New Keynesian Phillips curve, characterizing the relationship between inflation and output. β is the discount factor, and κ depends on the degree of nominal rigidity and other preference parameters. See Appendix A for further details of the model.

3.2 Extension: partially active and inactive monetary policy

The difference between our shadow rate model in Definition 1 and the standard New Keynesian model (see Galí (2008), for example) lies in the IS curve, which has the following form

[1]The potential output is a linear function of technology, which follows an exogenous process.

in the standard model:

$$y_t = -\frac{1}{\sigma}(r_t - \mathbb{E}_t\pi_{t+1} - r) + \mathbb{E}_t y_{t+1}, \tag{3.3}$$

and r_t relates to s_t through (2.1). The steady-state policy rate r equals the steady-state shadow rate s. The standard New Keynesian model consists of (2.1) - (2.2) and (3.2) - (3.3). Monetary policy is completely inactive at the ZLB in this model. So are most models in the literature.

We can extend our IS curve to nest the standard model as follows:

$$y_t = -\frac{1}{\sigma}(\mathcal{S}_t - \mathbb{E}_t\pi_{t+1} - s) + \mathbb{E}_t y_{t+1}, \tag{3.4}$$

where $\mathcal{S}_t = vr_t + (1-v)s_t$. Our shadow rate model corresponds to $v = 0$, and $v = 1$ is the standard New Keynesian model. This extension also allows the possibility that unconventional policy is partially active when $0 < v < 1$.

3.3 Economic implications

The standard New Keynesian model is associated with some distinctive modeling implications at the ZLB, some of which are counterfactual or puzzling. We focus on two such implications that are often discussed in the literature. First, a negative supply shock stimulates the economy, which is considered to be counterfactual.[2] Second, the government-spending multiplier is much larger than usual, and this is still an on-going debate. We demonstrate qualitative implications in this section, and leave the discussion of quantitative implications to Section 6.

Both a transitory negative shock on productivity or a positive government spending shock causes higher inflation. During normal times, in response to higher inflation, the interest

[2]Christiano et al. (2015) point out this implication depends on whether the shock is temporary or permanent. We refer to models in the literature with a transitory shock as the "standard" model.

rate increases more than one-for-one, implying a higher real interest rate, which in turn suppresses the demand. This implies lower output in response to the negative supply shock, and a government multiplier less than 1.

The standard model suggests opposite implications for both scenarios at the ZLB due to the lack of policy interventions. A constant policy rate in the standard New Keynesian model implies a lower real interest rate instead, which then stimulates private consumption, investment, and hence the overall economy. Therefore, the standard model implies a stimulative negative supply shock and larger government spending multiplier.

In contrast to the implication of the standard New Keynesian model, empirical evidence from Wieland (2015) and Garín et al. (2016) demonstrate a similar response of output to a supply shock during normal times and at the zero lower bound. Our model with the shadow rate capturing unconventional monetary policy is able to generate this data-consistent implication. The shadow rate reacts positively to higher inflation through unconventional monetary policy, which is how the central bank would react with a conventional monetary policy. Such a reaction increases the real rate private agents face, and implies a lower output in the model, which is consistent with the data. Moreover, the same model suggests that the fiscal multiplier is the same as usual, contributing to the ongoing debate. Our model implication is consistent with Braun et al. (2012), Mertens and Ravn (2014), Swanson and Williams (2014), and Wieland (2015).

The difference between our model with the shadow rate and the standard model is the existence of unconventional monetary policy. Unconventional monetary policy tools, such as large-scale asset purchases, lending facilities, and forward guidance, are designed to continue stimulating the economy when the traditional policy tool is unavailable. For example, QE programs purchase bonds to lower their interest rates, meaning households and firms face lower borrowing or lending rates, which subsequently boost the aggregate demand. These channels work similarly to the conventional interest rate rule if the Fed were able to lower the short-term interest rate further.

3.4 Computational advantages

Besides the benefit of sensible economic implications, the shadow rate model also salvages the New Keynesian model from the structural break introduced by the occasionally binding ZLB on the policy rate. The ZLB imposes one of the biggest challenges for solving and estimating these models.

To get around the zero lower bound, one strand of research linearizes the equilibrium conditions without considering the ZLB, and then assumes the ZLB is driven by some exogenous variables, such as preference, that follow a Markov-switching process with an absorbing state and known switching probabilities. These assumptions greatly simplify the solution. However, the cost of this shortcut is also substantial. First, it directly distorts model implications such as the fiscal multiplier; for example, see Fernández-Villaverde et al. (2015) and Aruoba et al. (2016). Second, many shocks are set to zero in this solution method, making it impossible to have the model match the data. Third, linearized equilibrium relations may hide nonlinear interactions between the ZLB and agents' decision rules; see the discussion in Braun et al. (2012).

Another strand of literature uses global projection methods to approximate agents' decision rules in a New Keynesian model with ZLB, such as Gust et al. (2012), Fernández-Villaverde et al. (2015), and Aruoba et al. (2016). As the model becomes nonlinear, estimating it becomes challenging. For linear models, the Kalman filter provides analytical expressions for the likelihood. With non-linear models, the Kalman filter is replaced by the particle filter. The non-linearity dramatically increases computing time and demands for more computing power.

This challenge does not go away after the economy lifts off from the ZLB. Instead, it becomes even more problematic as time goes on, because research can no longer discard the ZLB period, as it is not at the end of the sample anymore. The central tension is how we treat the seven-year period of the ZLB.

Our shadow rate model proposes a compelling solution to this challenge. Our model does not incur a structural break at the ZLB as the standard model does, and therefore, it restores traditional solution and estimation methods' validity.

4 Mapping QE into SRNKM

We have shown the relationship between the shadow rate and unconventional monetary policy empirically in Section 2. Next, we formalize this link. We micro-found the SRNKM introduced in Section 3 using two major programs: QE in this section and lending facilities in Section 5.

4.1 Model of QE

The first policy tool is large-scale asset purchases (QE). QE programs work through a risk premium channel: central banks' purchases of bonds lower their interest rates by reducing the risk compensation agents require to hold them. This channel is motivated by the empirical literature; see, for example, Gagnon et al. (2011) and Hamilton and Wu (2012). To keep the model to a minimum, we set it up with government bonds in this section to demonstrate the equivalence between QE and the shadow rate. The same equivalence holds when bonds are issued by firms as well; see Appendix B.1.

Households maximize their utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right), \tag{4.1}$$

where C_t is consumption, and L_t is labor supply. They face the following budget constraint:

$$C_t + \frac{B_t^H}{P_t} = \frac{R_{t-1}^B B_{t-1}^H}{P_t} + W_t L_t + T_t, \tag{4.2}$$

where B_{t-1}^H is the amount of nominal bond households hold from $t-1$ to t, and the corresponding gross return on this nominal asset is R_{t-1}^B. P_t is the price level, W_t is the real wage, and T_t is net lump-sum transfers and profits. The first-order condition with respect to real bond holdings $\tilde{B}_t^H \equiv B_t^H / P_t$ is

$$C_t^{-\sigma} = \beta R_t^B \mathbb{E}_t \left(\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}} \right), \tag{4.3}$$

where $\Pi_{t+1} \equiv P_{t+1}/P_t$ is inflation from t to $t+1$.

Linearizing the QE Euler equation and imposing the goods market clearing condition $Y_t = C_t$ yield to the QE IS curve:

$$y_t = -\frac{1}{\sigma} \left(r_t^B - \mathbb{E}_t \pi_{t+1} - r^B \right) + \mathbb{E}_t y_{t+1}, \tag{4.4}$$

where small letters are logs, and letters without t subscripts are steady-state values or parameters. The QE IS curve differs from the standard IS curve (3.3) in that it is the return on bonds rather than the fed funds rate that is the relevant interest rate households face.

Define

$$rp_t \equiv r_t^B - r_t, \tag{4.5}$$

where the policy rate r_t follows the Taylor rule during normal times as in (2.1) and (2.2). The wedge between the two rates rp_t is referred to as the risk premium. Empirical research, for example, Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2012), and Hamilton and Wu (2012), finds a larger amount of bonds the central bank holds through QE operations are associated with a lower risk premium, which suggests rp_t is a decreasing function of the total purchase of bonds by the government b_t^G:[3]

$$rp_t'(b_t^G) < 0. \tag{4.6}$$

This negative relationship, without additional assumptions about functional forms, then suggests the following in the linear model:

$$rp_t(b_t^G) = rp - \varsigma(b_t^G - b^G), \tag{4.7}$$

where $\varsigma > 0$.

During normal times, $b_t^G = b^G$, and $rp_t(b^G) = rp$. This leads to $r_t^B = r_t + rp$. In other words, the borrowing rate comoves with the policy rate with a constant wedge. The assumption of a constant risk premium during normal times can be relaxed to allow stochastic shocks. This extension does not change our results. An example with the shock to risk premium specified similar to Smets and Wouters (2007), which is then interpreted as the "liquidity preference shock" by Campbell et al. (2016), is in Appendix B.2.

When the ZLB binds $r_t = 0$, the central bank turns to large-scale asset purchases to increase its bond holdings b_t^G. The total supply of bonds is held by households and the government: $B_t = B_t^H + B_t^G$, where B_t can be time-varying and subject to exogenous shocks.

The linearized model incorporating QE is captured by the new Euler equation (4.4), the risk premium channel of bond purchase (4.5) and (4.7), and together with the usual Phillips curve (3.2), policy rule (2.1) and (2.2).

[3] A similar relationship between bond quantity and risk premium is also established in the international economic literature, see Uribe and Yue (2009) and Nason and Rogers (2006), for example, with the former motivating it by some cost associated with financial intermediaries who facilitate bond tractions. Risk premium is a well-established empirical fact in the term structure literature, see Wright (2011), Bauer et al. (2012), Bauer et al. (2014), and Creal and Wu (forthcoming).

4.2 Shadow rate equivalence for QE

Monetary policy enters the Euler equation (4.4) through the return on bond

$$r_t^B = r_t + rp - \varsigma(b_t^G - b^G). \tag{4.8}$$

During normal times, $b_t^G = b^G$, $r_t^B = r_t + rp$, and monetary policy operates through the usual Taylor rule on r_t, which is equal to the shadow rate s_t. At the zero lower bound, the policy rate no longer moves, $r_t = 0$, and the overall effect of monetary policy is $r_t^B = rp - \varsigma(b_t^G - b^G)$. If at the ZLB,

$$b_t^G = b^G - \frac{s_t}{\varsigma}, \tag{4.9}$$

then

$$r_t^B = s_t + rp \tag{4.10}$$

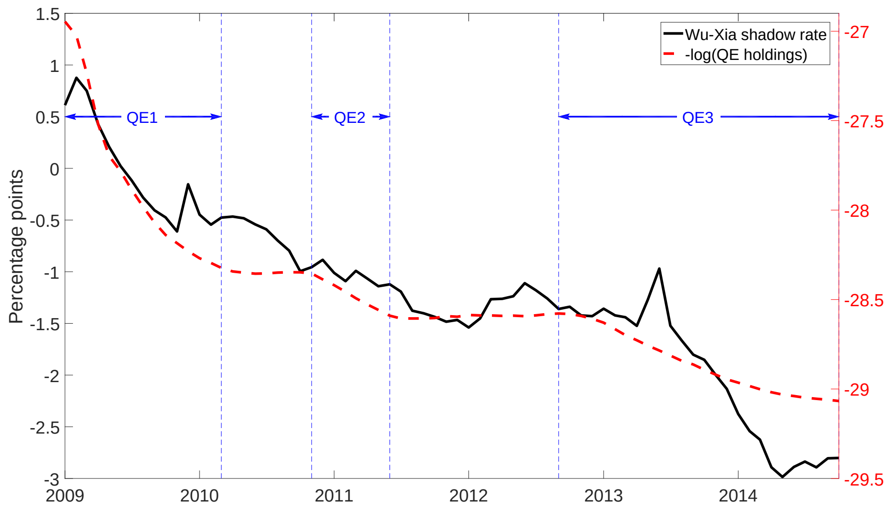
can capture both conventional and unconventional policies. Although the return on bond in (4.8) deviates from the conventional policy rate r_t with a time-varying wedge, the difference between the return on bond in (4.10) and s_t is a constant. This leads to the following proposition.

Proposition 1 *The shadow rate New Keynesian model represented by the shadow rate IS curve (3.1), New Keynesian Phillips curve (3.2), and shadow rate Taylor rule (2.2) nests both the conventional Taylor interest rate rule and QE operation that changes risk premium through (4.7) if*

$$\begin{cases} r_t = s_t,\ b_t^G = b^G & \textit{for } s_t \geq 0 \\ r_t = 0,\ b_t^G \textit{ follows (4.9)} & \textit{for } s_t < 0. \end{cases}$$

Proof: See Appendix C.

Figure 4: QE and shadow rate



Notes: black solid line: the Wu-Xia shadow rate; red dashed line: the negative of the log of the Fed's asset holdings through QEs (including Treasury securities, Federal agency debt securities, and mortgage-backed securities). Left scale: interest rates in percentage points; right scale: negative of log asset holdings. Data source: Federal Reserve Statistical Release H.4.1

Proposition 1 establishes QE as one microfoundation for (3.1). Note that at the ZLB, a negative shadow interest rate is not the actual borrowing or lending rate firms and households face, nor does the Fed sets it directly. Rather, Proposition 1 determines how much QE operation of bond purchases is needed to achieve the level of negative shadow rate prescribed by the Taylor rule.

An extension from government bonds to corporate bonds is in Appendix B.1. The equivalence holds regardless who issues bonds, as long as the relationship between risk premium, bond holdings, and shadow rate in Proposition 1 holds.

4.3 Quantifying assumptions in Proposition 1

Proposition 1 assumes a linear relationship between b_t^G and s_t with a negative correlation at the ZLB in (4.9). Figure 4 verifies this relationship in the data, where the shadow rate is in black and the negative of the log of the Fed's asset holdings through QE purchases is in red, including Treasury securities, Federal agency debt securities, and mortgage-backed securities. They comove with a high correlation of 0.92 from QE1 to QE3. The relation in the figure can also inform us about the coefficient ς and the effects of QE on the shadow rate. We estimate them by regressing the shadow rate s_t on log asset holdings of the Fed b_t^G. The slope coefficient is -1.83, which means when the Fed increases its bond holdings by 1%, the shadow rate decreases by 0.0183%. QE1 increases Fed's holdings on Treasuries, Federal agency debt securities and mortgage-backed securities from 490 billion to 2 trillion, mapping into about a 2.5% decrease in the shadow rate. This number is larger than the actual change in the shadow rate, and the difference can be explained by unwinding lending facilities. QE3 is another larger operation, changing Fed asset holdings from 2.6 trillion to 4.2 trillion. Although QE3 is as big an operation as QE1 in the dollar amount, the percentage change of QE3 is much smaller. Our model implies a 0.9% decrease in the shadow rate. The difference between this number and the actual change can be explained by the expansionary forward guidance at the time.

5 Mapping lending facilities into SRNKM

In this section, we map lending facilities into the SRNKM introduced in Section 3. These facilities inject liquidity into the economy by extending loans to the private sector. One prominent example is the Federal Reserve's Term Asset-Backed Securities Loan Facility. This channel has been assessed by, for example, Ashcraft et al. (2010) and Del Negro et al. (2016). Policies similar to lending facilities have been implemented by other central banks as well. For example, the Eurosystem's valuation haircuts vary the haircut schedule as a risk-management tool post financial crisis. The UK also has three decades of experience using credit controls.

5.1 Model of lending facilities

We extend the standard model characterized by (2.1) - (2.2) and (3.2) - (3.3) in the following respects. First, we introduce entrepreneurs to produce intermediate goods using capital and labor and then sell them in a competitive market to the retailers. Entrepreneurs maximize their lifetime utility. They have a lower discount factor and are less patient than households. They borrow from households using capital as collateral up to a constant loan-to-value ratio allowed by the households. Second, we allow the government to have two additional policy tools at the ZLB. First, it can loosen the borrowing constraint by directly lending to entrepreneurs through lending facilities, effectively making the loan-to-value ratio higher and time varying. Another policy the government employs at the zero lower bound is a tax on the interest rate income of households and a subsidy to entrepreneurs. Taxing interest rate income can be motivated by the recent phenomenon of negative interest rates in Europe and Japan, according to Waller (2016) of the St. Louis Fed. The pre-tax/subsidy private interest rate imposes a constant markup over the policy rate $R_t^B = R_t RP$, a simplified version of the setup in Section 4.

Entrepreneurs (denoted by a superscript E) produce intermediate good Y_t^E according to a Cobb-Douglas function,

$$Y_t^E = AK_{t-1}^{\alpha}(L_t)^{1-\alpha}, \tag{5.1}$$

where A is technology, K_{t-1} is physical capital used at period t and determined at $t-1$, and α is capital share of production. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate, and I_t is investment. Entrepreneurs sell the intermediate goods to retailers at price P_t^E, and the markup for the retailers is $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption C_t^E, investment on capital stock I_t, and labor input

L_t to maximize their utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \log C_t^E, \tag{5.2}$$

where the entrepreneurs' discount factor γ is smaller than households' β. The borrowing constraint is

$$\tilde{B}_t \leq M_t \mathbb{E}_t \left(\frac{K_t \Pi_{t+1}}{R_t^B} \right), \tag{5.3}$$

where $\tilde{B}_t$ is the amount of real corporate bonds issued by the entrepreneurs and M_t is the loan-to-value ratio. The entrepreneurs' budget constraint is

$$\frac{Y_t^E}{X_t} + \tilde{B}_t = \frac{R_{t-1}^B \tilde{B}_{t-1}}{\mathcal{T}_{t-1} \Pi_t} + W_t L_t + I_t + C_t^E, \tag{5.4}$$

where the tax schedule $\mathcal{T}_{t-1}$ is posted at $t-1$ and levied at t. The first-order conditions are labor demand and the consumption Euler equation:

$$W_t = \frac{(1-\alpha) A K_{t-1}^{\alpha} L_t^{-\alpha}}{X_t}, \tag{5.5}$$

$$\frac{1}{C_t^E} \left(1 - \frac{M_t \mathbb{E}_t \Pi_{t+1}}{R_t^B} \right) = \gamma \mathbb{E}_t \left[\frac{1}{C_{t+1}^E} \left(\frac{\alpha Y_{t+1}^E}{X_{t+1} K_t} - \frac{M_t}{\mathcal{T}_t} + 1 - \delta \right) \right]. \tag{5.6}$$

Households maximize their utility (4.1) subject to the budget constraint:

$$C_t + \tilde{B}_t^H = \frac{R_{t-1}^B \tilde{B}_{t-1}^H}{\mathcal{T}_{t-1} \Pi_t} + W_t L_t + T_t. \tag{5.7}$$

Hence, their consumption Euler equation is:

$$C_t^{-\sigma} = \beta R_t^B \mathbb{E}_t \left(\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1} \mathcal{T}_t} \right), \tag{5.8}$$

and labor supply satisfies:

$$W_t = C_t^{\sigma} L_t^{\eta}. \tag{5.9}$$

Households are willing to lend entrepreneurs $\tilde{B}_t^H$ with a constant loan-to-value ratio M:

$$\tilde{B}_t^H \leq M\mathbb{E}_t\left(\frac{K_t\Pi_{t+1}}{R_t^B}\right). \tag{5.10}$$

During normal times, $\tilde{B}_t = \tilde{B}_t^H$ and $M_t = M$. At the ZLB, the government can provide extra credit to firms through lending facilities allowing $M_t > M$, which take the form

$$\tilde{B}_t^G = (M_t - M)\mathbb{E}_t\left(\frac{K_t\Pi_{t+1}}{R_t^B}\right). \tag{5.11}$$

Consequently, the total credit firms obtain equals the households' bond holdings plus the government's bond holdings $\tilde{B}_t = \tilde{B}_t^H + \tilde{B}_t^G$.

The monopolistically competitive final goods producers, who face Calvo-stickiness, behave the same as in the benchmark model. Details can be found in Appendix A.3. The government still implements the Taylor rule during normal times. The goods market clears if

$$Y_t = C_t + C_t^E + I_t. \tag{5.12}$$

5.2 Shadow rate equivalence for lending facilities

The unconventional policy variables appear in pairs with the conventional monetary policy R_t in equilibrium conditions. In households' consumption Euler equation (5.8) and households' and entrepreneurs' budget constraints (5.7) and (5.4), government policy appears in the form $R_t/\mathcal{T}_t$. In the entrepreneurs' borrowing constraint (5.3) and first-order condition (5.6), it appears in the form R_t/M_t. Hence, to stimulate the economy by reducing $R_t/\mathcal{T}_t$ and R_t/M_t, the government can operate through the conventional monetary policy by lowering R_t, or through unconventional policy tools by losing the credit constraint (increasing M_t) and increasing tax and transfer $\mathcal{T}_t$. Moreover, $M_t/\mathcal{T}_t$ enters entrepreneurs' Euler equation (5.6), and moving both proportionally keeps this ratio constant.

Unconventional policy tools stimulate the economy through the following channels. First, a looser borrowing constraint allows entrepreneurs to secure more loans. Second, the tax benefit for entrepreneurs' interest rate payment effectively lowers their borrowing cost, encouraging them to borrow, consume, invest, and produce more. All together, these channels help the economy get out of the "liquidity trap," and boost the aggregate demand and hence output.

The following proposition formalizes the equivalence between conventional and unconventional policies, and this equivalence does not require a linearized model:

Proposition 2 *If*

$$\begin{cases} R_t = S_t,\ \mathcal{T}_t = 1,\ M_t = M & \text{for } S_t \geq 1 \\ \mathcal{T}_t = M_t/M = 1/S_t & \text{for } S_t < 1, \end{cases}$$

then $R_t/\mathcal{T}_t = S_t,\ R_t/M_t = S_t/M,\ M_t/\mathcal{T}_t = M\ \forall S_t$.

Proof: See Appendix C.

Proposition 2 suggests the dynamics of $R_t/\mathcal{T}_t$ and R_t/M_t can be captured by a single variable S_t. The equivalence in the non-linear model can also be extended to its linear version.

The linear system describing the equilibrium allocation $\{c_t, c_t^E, y_t, k_t, i_t, \tilde{b}_t\}_{t=0}^{\infty}$ and prices and policies $\{x_t, \pi_t, r_t, s_t, m_t, \tau_t\}_{t=0}^{\infty}$ consists of (2.1) and (2.2), policy rules for changing m_t

and τ_t, and

$$c_t = -\frac{1}{\sigma}(r_t - \tau_t - \mathbb{E}_t\pi_{t+1} - r) + \mathbb{E}_t c_{t+1}, \tag{5.13}$$

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(r_{t-1} + \tilde{b}_{t-1} - \tau_{t-1} - \pi_{t-1} + rp) - Ii_t + \Lambda_1, \tag{5.14}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m_t - r_t - rp), \tag{5.15}$$

$$\begin{aligned} 0 = &\left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) \\ &+ \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - r_t + m_t - rp) + \gamma M(\tau_t - m_t) + \Lambda_2, \end{aligned} \tag{5.16}$$

$$y_t = \frac{\alpha(1+\eta)}{\alpha+\eta}k_{t-1} - \frac{1-\alpha}{\alpha+\eta}(x_t + \sigma c_t) + \frac{1+\eta}{\alpha+\eta}a + \frac{1-\alpha}{\alpha+\eta}\log(1-\alpha), \tag{5.17}$$

$$k_t = (1-\delta)k_{t-1} + \delta i_t - \delta\log\delta, \tag{5.18}$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} - \lambda\left(x_t - x\right), \tag{5.19}$$

$$y_t = \frac{C}{Y}c_t + \frac{C^E}{Y}c_t^E + \left(1 - \frac{C}{Y} - \frac{C^E}{Y}\right)i_t, \tag{5.20}$$

where Λ_1 and Λ_2 are functions of steady-state values, defined in Appendix B.1. (5.13) linearizes the households' consumption Euler equation (5.8), and it differs from the standard Euler equation (3.3) mainly because of the tax. (5.14) is from the entrepreneurs' budget constraint (5.4) and labor demand first-order condition (5.5). (5.15) is the linear expression for the borrowing constraint (5.3) when it is binding. (5.16) linearizes the entrepreneurs' consumption Euler equation (5.6). (5.17) combines the production function (5.1) and labor supply first-order condition (5.9). (5.18) is the linearized capital accumulation process. (5.19) is the New Keynesian Phillips curve expressed with the price markup, which is equivalent to (3.2), and $\lambda = \kappa/(\sigma + \eta)$. (5.20) is the linearized version of the goods market-clearing condition (5.12).

Finally, the following proposition builds the equivalence between the shadow rate policy and lending facility – tax policy in the linear model:

Proposition 3 *The shadow rate New Keynesian model represented by the shadow rate IS* *curve*

$$c_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - s) + \mathbb{E}_t c_{t+1}, \tag{5.21}$$

the shadow rate Taylor rule (2.2), together with (5.17) - (5.20) and

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(s_{t-1} + rp + \tilde{b}_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, \tag{5.22}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m - s_t - rp), \tag{5.23}$$

$$\begin{aligned} 0 = &\left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) \\ &+ \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - s_t - rp + m) - \gamma M m + \Lambda_2, \end{aligned} \tag{5.24}$$

nests both the conventional Taylor interest rate rule and lending facility – tax policy in the model summarized by (2.1) - (2.2) and (5.13) - (5.20) if

$$\begin{cases} r_t = s_t,\ \tau_t = 0,\ m_t = m & \textit{for } s_t \geq 0 \\ \tau_t = m_t - m = -s_t & \textit{for } s_t < 0. \end{cases}$$

Proof: See Appendix C.

Hence, Proposition 3 establishes the lending facility – tax policy channel as another microfoundation for (3.1), because (5.21) is (3.1) without imposing the market clearing condition.

6 Quantitative analyses

The mechanism for how the shadow rate New Keynesian model works has been demonstrated qualitatively in Section 3. In this section, we study quantitative implications of this model. We first explain the model and methodology. Then we will discuss the consequence of a negative inflation shock at the ZLB and relate it to the economic implications discussed in Section 3.3.

6.1 Model and methodology

Shadow rate vs. standard model We analyze contrasts between our shadow rate model and the standard model. We term the standard model as the model that does not have unconventional monetary policy. Although the extended model has many more ingredients than the standard three-equation New Keynesian model, they share similar qualitative implications that are discussed in Section 3.3. In this model, it is $r_t = 0$ that enters the Euler equation, budget constraint, borrowing constraint, and so on at the ZLB. By contrast, the shadow model has unconventional monetary policy. It replaces r_t with the negative shadow rate s_t at the ZLB.

Extended model Many components are from Iacoviello's (2005) full model, including five sectors, of which two are households. Both types of households work, consume, and hold housing stocks. The difference is their discount factors. Patient households have a higher discount factor and save. Impatient households have lower discount factors and borrow from patient households using their existing housing as collateral. Entrepreneurs also have a lower discount factor than patient households, and hence borrow from them with collaterals as well. Entrepreneurs consume, invest, and hold houses. They use housing, capital, and labor as inputs to produce identical intermediate goods and sell them in a competitive market to retailers. Retailers are monopolistically competitive. They differentiate intermediate goods into final goods, and set prices with Calvo-type stickiness. The government implements a Taylor rule.

We have shown how this negative shadow rate can be implemented through various unconventional policy tools in Sections 4 and 5. These unconventional tools set our model apart from Iacoviello's (2005). First, we use a time-varying risk premium to capture QE discussed in Section 4. Second, we allow the loan-to-value ratio to be time-varying to model lending facilities. Additionally, lenders' (borrowers') bond returns (payments) are subject to a time-varying tax (subsidy) at the ZLB. These two policies together constitute the channel discussed in Section 5. We also differ from his model by allowing the government to adjust the aggregate demand through changing its expenditure so that we can study the government-spending multiplier. Last but not least, we introduce a preference shock to create the ZLB environment, similar to Christiano et al. (2011), Fernández-Villaverde et al. (2015), and many others.[4] The detailed model setup is in Appendix D.1. Many parameter values are taken from Iacoviello (2005) and Fernández-Villaverde et al. (2015), and more calibration details are in Appendix D.2,

Methodology For our shadow rate model capturing unconventional monetary policy, we work with a linear model where only the shadow rate enters the model representing all possible channels for monetary policy. In this case, the constraint of the ZLB for the policy rate does not impose any non-linearity in our model. Full details of the linear model are in Appendix D.4.1. After we solve the model, we then use the results from Propositions 1 - 3 to demonstrate how the negative shadow rate can be implemented with underlying unconventional policy tools discussed in Sections 4 and 5. The details are in Appendix D.4.2 - Appendix D.4.3.

As a comparison, we also analyze the standard model with the ZLB constraint. This model is piecewise linear and described in Appendix D.4.4. We apply the method of Guerrieri and Iacoviello (2015).

Zero lower bound environment To create a ZLB environment, we follow the literature to impose a series of positive preference shocks on the economy. The shocks last from period 1 to 15, with a total size of 4%. They cause people to save more, push the nominal policy rate r_t to zero at period 8, and keep it there until about period 20. The impulse responses

[4] Schorfheide et al. (2014), and Creal and Wu (2016) introduce preference shocks to study risk premium.

to this sequence of shocks are in Appendix E.[5]

6.2 Negative inflation shock at the ZLB

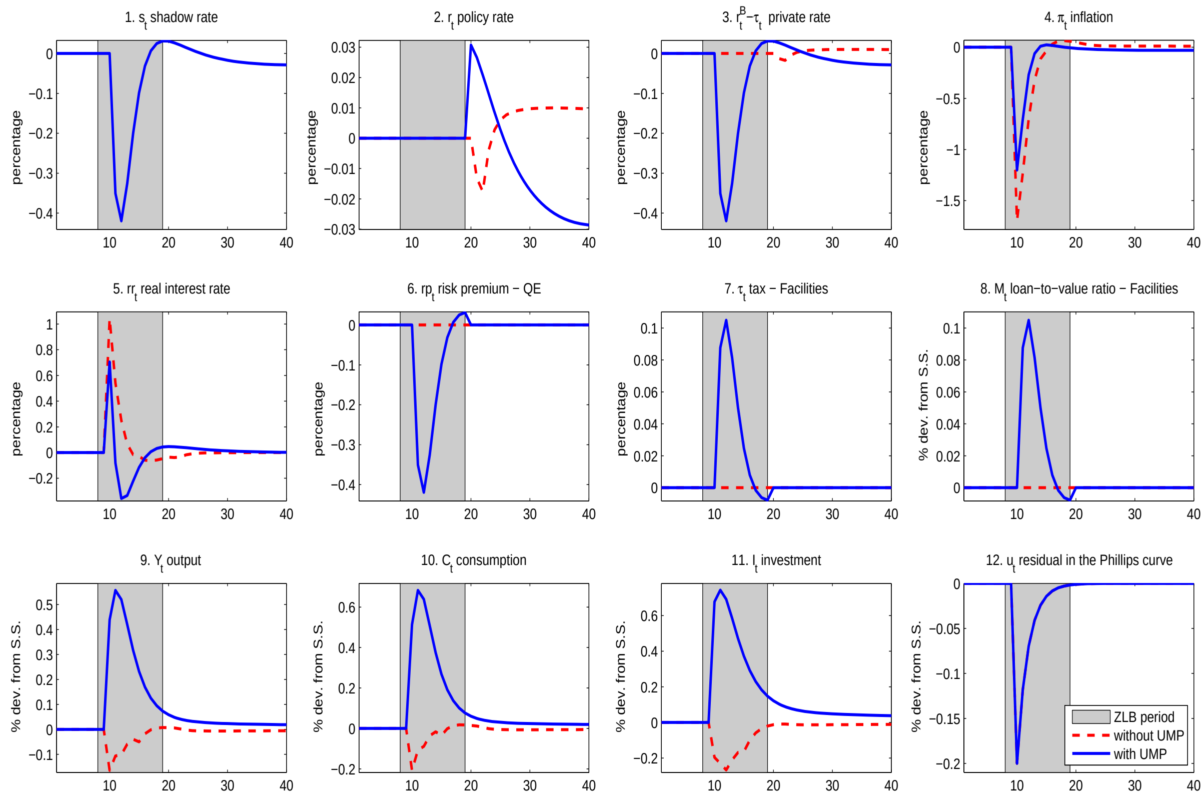
One of the major concerns of the ZLB is deflation. Once the economy encounters a deflationary spiral, the problem will exacerbate: a decrease in price leads to lower production, which in turn contributes to a lower wage and demand. Lower demand further decreases the price. In this section, we investigate the effect of unconventional monetary policy in fighting deflation at the ZLB through the lens of our shadow rate New Keynesian model.

On top of the positive preference shocks to create the ZLB environment, we introduce a negative inflation shock of the size 0.2% at period 10. To investigate its marginal impact on the economy, we take the difference between the total effect of both shocks and the effect of only preference shocks, and plot the difference in Figure 5. The red lines capture the impact of the negative inflation shock in a standard model without unconventional monetary policy. The blue lines represent the difference this inflation shock makes when unconventional monetary policy is present and summarized by the shadow rate. We also map the shadow rate into various unconventional policy tools: the risk premium in plot 6 captures the QE in Section 4, and the combination of the loan-to-value ratio in plot 8 and the tax rate in plot 7 capture the lending facilities – tax policy discussed in Section 5.

With unconventional monetary policy, inflation decreases less from the maximum decline of 1.7% in red to 1.2% in blue; that is, the responsiveness of unconventional monetary policy alleviates some of the deflationary concern. Inflation expectation shares a similar pattern with inflation. The policy rate does not respond in either case. However, the access to unconventional monetary policy allows the shadow rate and hence the private rate to drop further. Both of them drop by 0.4%. A lower shadow rate can be implemented either through a QE channel (blue line in plot 6) or a lending facility – fiscal policy (blue lines in plots 7 and 8) as soon as the ZLB hits in period 8. The drop in the risk premium from

[5] Our results in Sections 6.2 - 6.4 are robust to alternative shocks to create the ZLB environment, for example, inflation shocks.

Figure 5: Negative inflation shock



Notes: We hit the economy with two types of shocks. First, a series of positive preference shocks occurs in periods 1 - 15, and the total shock size is 4%. Second, a negative inflation shock happens in period 10 with a size of 0.2%. We difference out the effect of preference shocks, and only plot the additional effect of the inflation shock. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 4) or the lending facilities in plots 7 and 8 (per Section 5). The shaded area marks periods 8-19.

the steady-state level 3.6% to 3.2% can explain the 0.4% decrease in the shadow rate from zero. Alternatively, the loan-to-value ratio goes up by 0.1%, and the tax rate goes from 0 to 0.1%. Translating these numbers into the annual rate, $0.1\% \times 4 = 0.4\%$, can explain the same amount of change in the shadow rate. Note the tax is levied on total proceeds.

With unconventional monetary policy, a lower nominal rate and higher inflation expectation imply a less and more transitory increase in the real rate. As a consequence, less deflationary pressure provides firms more incentive to produce, and drives up demand as well. For example, the overall output increases by 0.6% rather than decreasing by 0.2%.

The differences in responses to the inflation shock provide the basic mechanism to explain the economic implications discussed in Section 3.3, which we will now turn to.

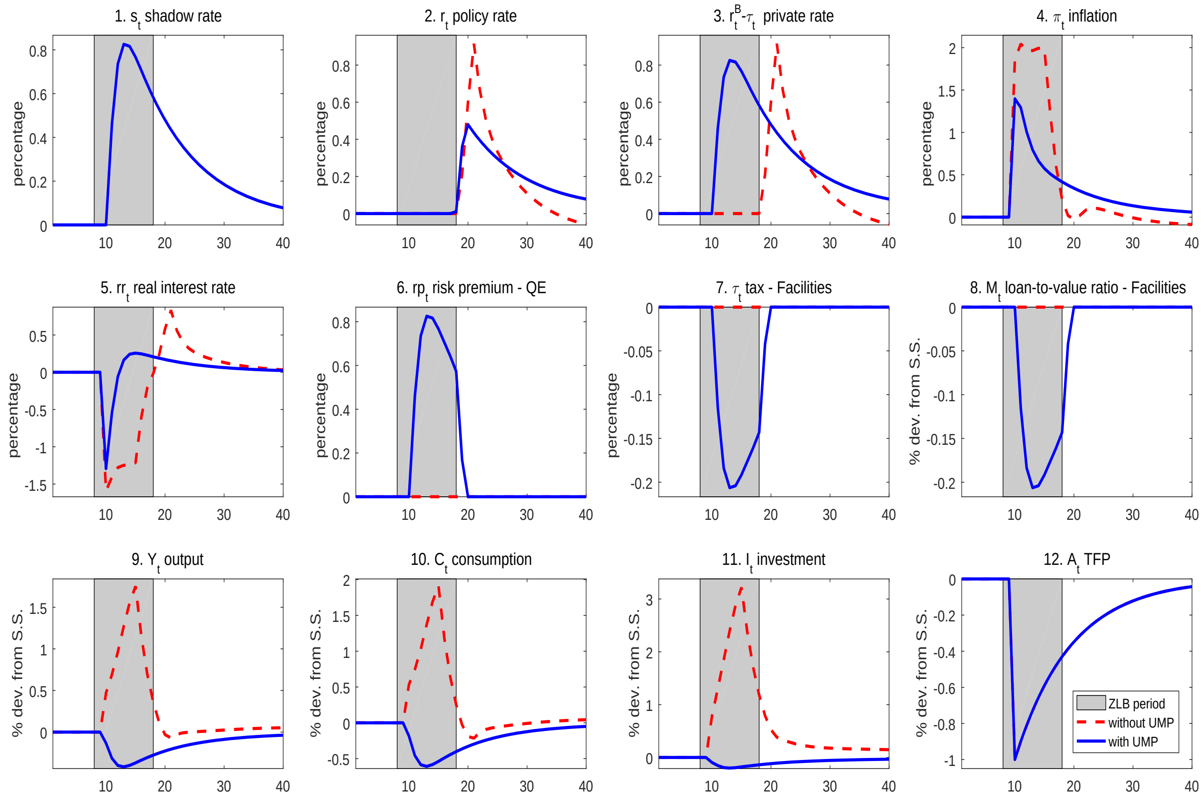
6.3 Negative supply shock at the ZLB

As discussed in Section 3.3 according to the standard New Keynesian model, during normal times, a negative supply shock produces a negative effect on output. By contrast, at the ZLB, the same shock produces a positive effect. The latter is counterfactual; for example, see Wieland (2015) and Garín et al. (2016). Our model with the shadow rate in Sections 3 - 5 reconciles the similarity between normal times and the ZLB found in the data and the contrast implied by the New Keynesian model. Although the policy rate still has a ZLB, a coherent shadow rate Taylor rule summarizes both the conventional and unconventional policy tools. Hence, it is able to produce the right implications for both time periods.

To demonstrate these implications, we add a negative TFP innovation of the size 1% at period 10 in addition to the preference shocks. We take the difference between the total effects of both shocks and the effect of only preference shocks, and plot the difference in Figure 6.

The red line in plot 9 shows a negative supply shock increases output at the ZLB. This finding is consistent with the implication of a standard New Keynesian model, and contradicts the empirical findings. By contrast, the blue line, where we introduce unconventional monetary policy through our shadow rate policy rule, produces a negative impact of such a shock. This result is data-consistent. The same contrast can be further extended to other real variables, consumption, and investment. More specifically, with the presence of unconventional monetary policy, output decreases by 0.4%, consumption by 0.6%, and investment by 0.2%.

Figure 6: Negative TFP shock



Notes: We hit the economy with two types of shocks. First, a series of positive preference shocks occurs in periods 1-15, and the total shock size is 4%. Second, a negative TFP shock happens in period 10 with a size of 1%. We difference out the effect of preference shocks and only plot the additional effect of the TFP shock. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 4) or the lending facilities in plots 7 and 8 (per Section 5). The shaded area marks periods 8-18.

The differences in impulse responses reflect whether monetary policy is active. This works through the same mechanism as explained in Section 6.2. The differences are the directions and magnitudes. In the case with active unconventional monetary policy, the shadow rate increases by 0.8%, and this can be done through either increasing the risk premium by the same amount or decreasing the loan-to-value ratio and tax rate by 0.2%, which amounts to

0.8% in annualized rates.

6.4 Government spending multiplier at the ZLB

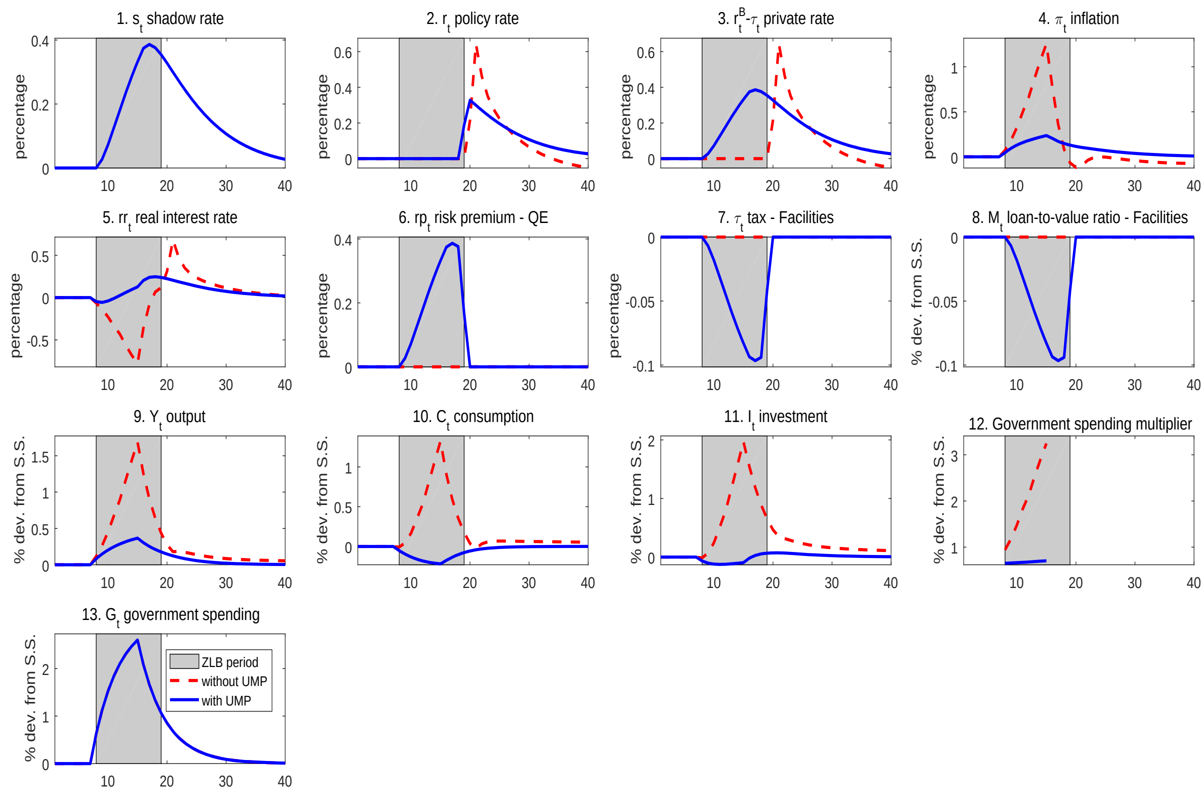
The government-spending multiplier is generally considered to be less than 1 during normal times. Whether this is the case at the ZLB is a heavily debated topic. Many studies, such as Christiano et al. (2011) and Eggertsson (2010), argue that at the ZLB, the multiplier is larger than 1. This finding is a standard result of the New Keynesian model as we mentioned in Section 3.3. By contrast, Braun et al. (2012) and Mertens and Ravn (2014) do not find much difference between the fiscal multiplier at the ZLB and during normal times. We have shown in Section 3 that their finding is consistent with a New Keynesian model accommodating unconventional monetary policy.

This section further provides some numerical evidence for this contrast. Our analyses are in Figure 7. In addition to the 15-period positive preference shocks that create the ZLB environment, we introduce another source of shocks that increase government spending from period 8 to 15 with a total size of 5%. The red lines capture the additional impact of government-spending shocks without unconventional monetary policy. The blue lines represent the differences these additional shocks make when unconventional monetary policy is present.

The red line in plot 12 shows the government-spending multiplier is mostly above 1 and peaks at around 3.2 when the policy rate is bounded at zero and the central bank takes no additional measures to smooth the economy. By contrast, the number is less than 0.85 in blue when the central bank monitors and adjusts the shadow rate through implementing unconventional monetary policy.

Positive government shocks push up the aggregate demand, which leads to a rising pressure on inflation. This again lands itself as another application of the mechanism explained in Section 6.2. A higher inflation without policy intervention boosts the private economy, yielding a multiplier greater than 1. By contrast, in our model, the shadow rate increases

Figure 7: Positive government spending shocks



Notes: We hit the economy with two types of shocks. First, a series of positive preference shocks occurs in periods 1 - 15, and the total shock size is 4%. Second, government-spending shocks occur from periods 8-15 with a total size of 5%. We difference out the effect of preference shocks, and only plot the additional effect of the government-spending shock. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 4) or the lending facilities in plots 7 and 8 (per Section 5). The shaded area marks periods 8-19.

by 0.4% in response to such a shock, crowding out private consumption by 0.2% and investment by 0.1%. Although output still increases by 0.4%, its change is less than the shocks themselves, producing a smaller multiplier. The change in the shadow rate in our model can be implemented through increasing the risk premium by 0.4% or reducing the loan-to-value ratio and tax rate by 0.1%.

7 Conclusion

We have built a New Keynesian model with the shadow rate, which coherently captures the conventional interest rate rule in normal times, and unconventional monetary policy at the ZLB. The model is the same as the standard New Keynesian model when the policy rate is above zero. When the policy rate is binding at zero, however, unlike the standard model with an inactive monetary policy, the central bank in our model continues to monitor and adjust the shadow rate following the shadow rate Taylor rule. A negative shadow rate prescribed by this Taylor rule can be implemented, for example, by QE and/or lending facilities. Our model restores the data-consistent result that a negative supply shock is always contractionary. Relatedly, the unusually large government-spending multiplier in the standard New Keynesian model at the ZLB also disappears. Besides incorporating unconventional policy tools in a sensible and tractable way, our model does not incur a structural break at the ZLB whether we work with a linear or non-linear model. Hence, it restores existing solution and estimation methods' validity, which addresses to technical challenges that come with the ZLB.

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Appendix A Shadow rate New Keynesian model

Appendix A.1 Households

A representative infinitely-living household seeks to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right) \tag{A.1}$$

subject to the budget constraint

$$C_t + \frac{B_t}{P_t} \leq \frac{R_{t-1}^B B_{t-1}}{P_t} + W_t L_t + T_t, \tag{A.2}$$

where $\mathbb{E}$ is the expectation operator, and C_t and L_t denote time t consumption and hours worked, respectively. The nominal gross interest rate R_{t-1}^B pays for bonds B_{t-1} carried from $t-1$ to t, determined at time $t-1$. P_t is the price level. W_t and T_t denote the real wage rate and firms' profits net of lump-sum taxes.

The optimal consumption-saving and labor supply decisions are given by the two first-order conditions below:

$$C_t^{-\sigma} = \beta R_t^B \mathbb{E}_t \left(\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}} \right) \tag{A.3}$$

$$W_t = \frac{L_t^{\eta}}{C_t^{-\sigma}}, \tag{A.4}$$

where $\Pi_t \equiv P_t / P_{t-1}$ is inflation from $t-1$ to t.

Appendix A.2 Wholesale firms

A continuum of wholesale firms exist, producing identical intermediate goods and selling them in a competitive market. All firms have the same production function:

$$Y_t^E = A L_t, \tag{A.5}$$

where A is the technology and is normalized to 1. The price for intermediate goods Y_t^E is P_t^E, and we define the price markup as $X_t = P_t / P_t^E$.

Firms maximize their profit by choosing labor:

$$\begin{aligned} \max_{L_t} \quad & Y_t^E / X_t - W_t L_t \\ s.t. \quad & Y_t^E = A L_t. \end{aligned}$$

The first-order condition is

$$\frac{1}{X_t} = \frac{W_t}{A}. \tag{A.6}$$

Appendix A.3 Retailers

A continuum of monopolistically competitive retailers of mass 1, indexed by z, differentiate one unit of intermediate goods into one unit of retail goods $Y_t(z)$ at no cost, and sell it at price $P_t(z)$. The final good Y_t is a CES aggregation of the differentiated goods, $Y_t = (\int_0^1 Y_t(z)^{1-\frac{1}{\epsilon}} dz)^{\frac{\epsilon}{\epsilon-1}}$.[6] We also refer to Y_t as output. Each firm may reset its price with probability $1-\theta$ in any given period, independent of when the

[6] $Y_t = (\int_0^1 Y_t(z)^{1-\frac{1}{\epsilon}} dz)^{\frac{\epsilon}{\epsilon-1}} \approx \int_0^1 Y_t(z) dz = Y_t^E$. The approximation is done through linearization in the neighborhood of the zero-inflation steady state. In what follows, we do not differentiate between Y_t and Y_t^E.

last adjustment happened. The remaining θ fraction of firms keep their prices unchanged. A retailer that can reset its price will choose price $P_t^*(z)$ to maximize the present value of profits while that price remains effective:

$$\max_{P_t^*(z)} \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \left(\frac{P_t}{P_{t+k}}\right) [P_t^*(z) Y_{t+k|t}(z) - P_{t+k}^E Y_{t+k|t}(z))], \tag{A.7}$$

where $Y_{t+k|t}(z)$ is the demand for goods z at time $t+k$ when the price of the good is set at time t at $P_t^*(z)$, which satisfies

$$Y_{t+k|t}(z) = \left(\frac{P_t^*(z)}{P_{t+k}}\right)^{-\epsilon} Y_{t+k}. \tag{A.8}$$

Because every firm faces the same optimization problem, we eliminate the index z. The first-order condition associated with the firm's optimization problem is:

$$\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \left(\frac{P_t}{P_{t+k}}\right) Y_{t+k|t} \left(P^* - \frac{\epsilon}{\epsilon - 1} P_{t+k}^E\right) = 0. \tag{A.9}$$

As a fraction θ of prices stay unchanged, the aggregate price dynamics follow the equation:

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon}. \tag{A.10}$$

Appendix A.4 Government

Monetary policy follows a shadow rate Taylor rule without the ZLB:

$$S_t = S_{t-1}^{\phi_s} \left[S \Pi_t^{\phi_\pi} \left(\frac{Y_t}{Y_t^n}\right)^{\phi_y} \right]^{1-\phi_s}, \tag{A.11}$$

where S and Y are the steady-state shadow rate and output, and Y_t^n is the potential output determined by the economy with flexible prices.[7] The bond return R_t^B equals the shadow rate multiplying a constant risk premium:

$$R_t^B = S_t RP. \tag{A.12}$$

Appendix A.5 Equilibrium

The goods market clears if

$$Y_t = C_t. \tag{A.13}$$

(A.3), (A.5), and (A.9) imply the following relationship between steady-state variables:

$$R = 1/\beta, \tag{A.14}$$
$$Y = AL = L, \tag{A.15}$$
$$1/X = \epsilon/(\epsilon - 1). \tag{A.16}$$

(3.1) is the linear version of (A.3) with (A.12) and (A.13) imposed. Log-linearizing (A.9) and (A.10) yields to (3.2), where the coefficient $\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}(\sigma + \eta)$. Taking logs of (A.11) gives us (2.2).

[7] We assume a zero-inflation steady state, implying $\Pi = 1$.

Appendix B Alternative specifications for QE

Appendix B.1 QE with corporate bonds

Entrepreneurs Bonds are issued by entrepreneurs (denoted by a superscript E) instead of the government. They produce intermediate good Y_t^E according to a Cobb-Douglas function,

$$Y_t^E = AK_{t-1}^{\alpha}(L_t)^{1-\alpha}, \tag{B.1}$$

where A is technology, K_{t-1} is physical capital used at period t and determined at $t-1$, L_t is labor supply, and α is the capital share of production. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate, and I_t is investment. Entrepreneurs sell the intermediate goods to retailers at price P_t^E, and the markup for the retailers is $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption C_t^E, investment on capital stock I_t, and labor input L_t to maximize their utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \log C_t^E, \tag{B.2}$$

where the entrepreneurs' discount factor γ is smaller than households' β. Their borrowing constraint is

$$\tilde{B}_t \leq M\mathbb{E}_t\left(\frac{K_t \Pi_{t+1}}{R_t^B}\right), \tag{B.3}$$

where $\tilde{B}_t$ is the amount of real corporate bonds issued by the entrepreneurs at t, and the gross return on this asset from t to $t+1$ is R_t^B. $\Pi_{t+1} \equiv P_{t+1}/P_t$ is inflation. M is the loan-to-value ratio. The entrepreneurs' budget constraint is

$$\frac{Y_t^E}{X_t} + \tilde{B}_t = \frac{R_{t-1}^B \tilde{B}_{t-1}}{\Pi_t} + W_t L_t + I_t + C_t^E, \tag{B.4}$$

The first-order conditions are labor demand and the consumption Euler equation:

$$W_t = \frac{(1-\alpha)AK_{t-1}^{\alpha}L_t^{-\alpha}}{X_t}, \tag{B.5}$$

$$\frac{1}{C_t^E}\left(1 - \frac{M\mathbb{E}_t \Pi_{t+1}}{R_t^B}\right) = \gamma \mathbb{E}_t\left[\frac{1}{C_{t+1}^E}\left(\frac{\alpha Y_{t+1}^E}{X_{t+1}K_t} - M + 1 - \delta\right)\right]. \tag{B.6}$$

Households and government The households' problem is the same as in Section 4.1. The central bank is also the same as in Section 4.1: it follows the Taylor rule (2.1) and (2.2) during normal times, and purchases bonds to lower risk premium at the ZLB according to (4.5) and (4.7). The goods market clearing condition is $Y_t = C_t + C_t^E + I_t$.

Equilibrium The linear system describing the equilibrium allocation $\{c_t, c_t^E, y_t, k_t, i_t, \tilde{b}_t, b_t^G\}_{t=0}^{\infty}$ and prices $\{x_t, \pi_t, r_t^B, r_t, rp_t, s_t\}_{t=0}^{\infty}$ consists of (2.1), (2.2), (4.5), (4.7), a policy rule for government purchases at the ZLB, and

$$c_t = -\frac{1}{\sigma}(r_t^B - \mathbb{E}_t\pi_{t+1} - r^B) + \mathbb{E}_t c_{t+1}, \tag{B.7}$$

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(r_{t-1}^B + \tilde{b}_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, \tag{B.8}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m - r_t^B), \tag{B.9}$$

$$0 = \left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) + \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - r_t^B) + \tilde{\Lambda}_2, \tag{B.10}$$

$$y_t = \frac{\alpha(1+\eta)}{\alpha+\eta}k_{t-1} - \frac{1-\alpha}{\alpha+\eta}(x_t + \sigma c_t) + \frac{1+\eta}{\alpha+\eta}a + \frac{1-\alpha}{\alpha+\eta}\log(1-\alpha), \tag{B.11}$$

$$k_t = (1-\delta)k_{t-1} + \delta i_t - \delta\log\delta, \tag{B.12}$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} - \lambda\left(x_t - x\right), \tag{B.13}$$

$$y_t = \frac{C}{Y}c_t + \frac{C^E}{Y}c_t^E + \left(1 - \frac{C}{Y} - \frac{C^E}{Y}\right)i_t, \tag{B.14}$$

where $\Lambda_1 = C^E\log C^E - \alpha\frac{Y}{X}\log\frac{Y}{X} - \tilde{B}\log\tilde{B} + R^B\tilde{B}\log R^B\tilde{B} + I\log I$, $\tilde{\Lambda}_2 = -\frac{\gamma\alpha Y}{XK}\log\frac{Y}{XK} + \frac{M}{R^B}\log R^B$. The Λ_2 in (5.16) is $\Lambda_2 = \tilde{\Lambda}_2 - \left(\frac{1}{R^B} - \gamma\right)M\log M$.

Equivalence Therefore, Proposition 1 becomes

Corollary 1 *The shadow rate New Keynesian model represented by the shadow rate IS curve*

$$c_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - s) + \mathbb{E}_t c_{t+1}, \tag{B.15}$$

the shadow rate Taylor rule (2.2), together with (B.11) - (B.14) and

$$C^E c_t^E = \alpha\frac{Y}{X}(y_t - x_t) + \tilde{B}\tilde{b}_t - R^B\tilde{B}(s_{t-1} + rp + \tilde{b}_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, \tag{B.16}$$

$$\tilde{b}_t = \mathbb{E}_t(k_t + \pi_{t+1} + m - s_t - rp), \tag{B.17}$$

$$0 = \left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) + \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - s_t - rp + m) + \Lambda_2, \tag{B.18}$$

nests both the conventional Taylor interest rate rule and QE operation that changes risk premium in the model summarized by (2.1), (2.2), (4.5), (4.7), and (B.7) - (B.14) if

$$\begin{cases} r_t = s_t,\ b_t^G = b^G & \textit{for } s_t \geq 0 \\ r_t = 0,\ b_t^G = b^G - \frac{s_t}{\varsigma} & \textit{for } s_t < 0. \end{cases}$$

Appendix B.2 Time-varying risk premium

We add an exogenous premium shock, similar to Smets and Wouters (2007): (4.7) becomes

$$rp_t(b_t^G) = rp - \varsigma(b_t^G - b^G) + \epsilon_{b,t}, \tag{B.19}$$

where $\epsilon_{b,t}$ is a white noise premium shock. With this extension, the risk premium is time-varying during normal times when $b_t^G = b^G$. Under the conditions imposed in Proposition 1, $r_t^B = s_t + rp + \epsilon_{b,t}$. Imposing the market clearing condition, the shadow rate IS curve in (3.1) becomes

$$y_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - s + \epsilon_{b,t}) + \mathbb{E}_t y_{t+1}. \tag{B.20}$$

None of the other equilibrium conditions change in the model. The New Keynesian Phillips curve (3.2) and shadow rate Taylor rule (2.2) remain the same. Therefore, Proposition 1 becomes

Corollary 2 *The shadow rate New Keynesian model represented by the shadow rate IS curve (B.20), the New Keynesian Phillips curve (3.2), and shadow rate Taylor rule (2.2) nests both the conventional Taylor interest rate rule and QE operation that changes risk premium through (B.19) if*

$$\begin{cases} r_t = s_t,\ b_t^G = b^G & \text{for } s_t \geq 0 \\ r_t = 0,\ b_t^G = b^G - \frac{s_t}{\varsigma} & \text{for } s_t < 0. \end{cases}$$

Appendix C Proof of Propositions

Proof for Proposition 1 During normal times $b_t^G = b^G, r_t^B = r_t + rp, r_t = s_t$, the Euler equation (4.4) becomes

$$\begin{aligned} y_t &= -\frac{1}{\sigma}\left(r_t + rp - \mathbb{E}_t\pi_{t+1} - r^B\right) + \mathbb{E}_t y_{t+1} \\ &= -\frac{1}{\sigma}\left(s_t - \mathbb{E}_t\pi_{t+1} - s\right) + \mathbb{E}_t y_{t+1}. \end{aligned} \tag{C.1}$$

At the ZLB $r_t = 0, b_t^G = b^G - \frac{s_t}{\varsigma}$, use the unconventional monetary policy in (4.7), and (4.4) becomes

$$\begin{aligned} y_t &= -\frac{1}{\sigma}\left(rp - \varsigma(b_t^G - b^G) - \mathbb{E}_t\pi_{t+1} - r^B\right) + \mathbb{E}_t y_{t+1} \\ &= -\frac{1}{\sigma}\left(s_t - \mathbb{E}_t\pi_{t+1} - s\right) + \mathbb{E}_t y_{t+1}. \end{aligned} \tag{C.2}$$

Proof for Proposition 2 During normal times, $R_t = S_t$, $\mathcal{T}_t = 1$, and $M_t = M$ imply $R_t/\mathcal{T}_t = S_t$, $R_t/M_t = S_t/M$, and $M_t/\mathcal{T}_t = M$. At the ZLB, $\mathcal{T}_t = M_t/M = 1/S_t$, and $R_t = 1$ imply $R_t/\mathcal{T}_t = S_t$, $R_t/M_t = S_t/M$, and $M_t/\mathcal{T}_t = M$.

Proof for Proposition 3 $r_t - \tau_t$ enters (5.13) and (5.14), and Lemma 2 have shown $r_t - \tau_t = log(R_t/\mathcal{T}_t) = s_t$. $r_t - m_t$ enters (5.15) and (5.16), and Lemma 2 have shown $r_t - m_t = log(R_t/M_t) = s_t - m$. $\tau_t - m_t$ enters (5.16), and Lemma 2 have shown $m_t - \tau_t = log(M_t/\mathcal{T}_t) = m$. Therefore, equations (5.13)-(5.16) can be expressed with the shadow rate as in (3.1) together with (B.16) - (B.18).

Appendix D Extended model

Appendix D.1 Setup

Appendix D.1.1 Patient households

Patient households (denoted with a superscript P) maximize their lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \left(\Pi_{i=1}^{t}\beta_i\right) \left[\log C_t^P + j \log H_t^P - (L_t^P)^{1+\eta}/(1+\eta) + \chi_{\mathcal{M}} \log(\mathcal{M}_t^P/P_t)\right],$$

where β_t is the discount factor fluctuating around mean β and following the process $\beta_t/\beta = (\beta_{t-1}/\beta)^{\rho_\beta}\, \varepsilon_{\beta,t}$. C_t^P is consumption, j indicates the marginal utility of housing, H_t^P is the holdings of housing, L_t^P is hours of work, and $\mathcal{M}_t^P/P_t$ is the real money balance.

Assume households lend in nominal terms at time $t-1$ with the amount of loan B_{t-1}^P, and receive $R_{t-1}^B R_{t-1}^B$ at time t. The bond return R_{t-1}^B is determined at time $t-1$ for bond-carrying between $t-1$ and t.

The bond return is higher than the policy rate R_t by a risk premium RP_t and $R_t^B = R_t RP_t$. The gross tax rate on bond return $\mathcal{T} - t - 1$ is assumed to be known $t-1$. The budget constraint of households follows:

$$C_t^P + Q_t \Delta H_t^P + \frac{B_t^P}{P_t} = \frac{R_{t-1}^B B_{t-1}^P}{\mathcal{T}_{t-1} P_t} + W_t^P L_t^P + \mathcal{D}_t + T_t^P - \Delta \mathcal{M}_t^P / P_t, \tag{D.1}$$

where Δ is the first difference operator. Q_t denotes the real housing price, W_t^P is the real wage, and $\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. $\mathcal{D}_t$ is the lump-sump profits received from the retailer, and T_t^P is the net government transfer.

The first-order conditions for consumption, labor supply, and housing demand are

$$\frac{1}{C_t^P} = \mathbb{E}_t \left(\frac{\beta_{t+1} R_t^B}{\mathcal{T}_t \Pi_{t+1} C_{t+1}^P} \right) \tag{D.2}$$

$$W_t^P = (L_t^P)^\eta C_t^P \tag{D.3}$$

$$\frac{Q_t}{C_t^P} = \frac{j}{H_t^P} + \mathbb{E}_t \left(\frac{\beta_{t+1} Q_{t+1}}{C_{t+1}^P} \right). \tag{D.4}$$

Appendix D.1.2 Impatient households

Impatient households (denoted with a superscript I) have a lower discount factor β^I than the patient ones, which guarantees the borrowing constraint for the impatient households binds in equilibrium. They choose consumption C_t^I, housing service H_t^I, and labor supply L_t^I to maximize lifetime utility given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} (\beta^I)^t \left[\log C_t^I + j \log H_t^I - (L_t^I)^{1+\eta}/(1+\eta) + \chi_{\mathcal{M}} \log(\mathcal{M}_t^I / P_t) \right].$$

The budget constraint and borrowing constraint are

$$C_t^I + Q_t \Delta H_t^I + \frac{R_{t-1}^B B_{t-1}^I}{\mathcal{T}_{t-1} P_t} = \frac{B_t^I}{P_t} + W_t^I L_t^I + T_t^I - \Delta \mathcal{M}_t^I / P_t \tag{D.5}$$

$$B_t^I / P_t \leq M_t^I \mathbb{E}_t (Q_{t+1} H_t^I \Pi_{t+1} / R_t^B). \tag{D.6}$$

The first-order conditions for labor supply and housing service can be summarized as follows:

$$W_t^I = (L_t^I)^\eta C_t^I \tag{D.7}$$

$$\frac{Q_t}{C_t^I} = \frac{j}{H_t^I} + \mathbb{E}_t \left[\beta^I \frac{Q_{t+1}}{C_{t+1}^I} \left(1 - \frac{M_t^I}{\mathcal{T}_t} \right) + \frac{M_t^I Q_{t+1} \Pi_{t+1}}{C_t^I R_t^B} \right]. \tag{D.8}$$

Appendix D.1.3 Entrepreneurs

Entrepreneurs (denoted by superscript E) produce intermediate good Y_t^E according to a Cobb-Douglas function:

$$Y_t^E = A_t K_{t-1}^\mu (H_{t-1}^E)^\nu (L_t^P)^{\alpha(1-\mu-\nu)} (L_t^I)^{(1-\alpha)(1-\mu-\nu)}, \tag{D.9}$$

where the technology A_t has a random shock $A_t/A = (A_{t-1}/A)^{\rho_a} \varepsilon_{a,t}$ and A is normalized to be 1. Both the housing input H_{t-1}^E and physical capital K_{t-1} used for the period t production are determined at time $t-1$. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate, and I_t is investment. Capital installation entails an adjustment cost: $\xi_{K,t} = \psi(I_t/K_{t-1} - \delta)^2 K_{t-1}/(2\delta)$. Entrepreneurs sell the intermediate goods to retailers at price P_t^E. The markup for the retailers is $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption c_t, investment on capital stock I_t, housing service H_t^E, and labor input L_t^P and L_t^I to maximize their utility $\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \log C_t^E$, where the entrepreneurs' discount factor γ is smaller than β. The borrowing constraint entrepreneurs face is

$$B_t^E/P_t \le M_t^E \mathbb{E}_t(Q_{t+1}H_t^E \Pi_{t+1}/R_t^B). \tag{D.10}$$

The budget constraint is

$$\frac{Y_t^E}{X_t} + \frac{B_t^E}{P_t} = C_t^E + Q_t \Delta H_t^E + \frac{R_{t-1}^B B_{t-1}^E}{\mathcal{T}_{t-1}P_t} + W_t^P L_t^P + W_t^I L_t^I + I_t + \xi_{K,t}. \tag{D.11}$$

The first-order conditions can be expressed in four equations:

$$\frac{Q_t}{C_t^E} = \mathbb{E}_t\left\{\frac{\gamma}{C_{t+1}^E}\left[\frac{\nu Y_{t+1}^E}{X_{t+1}H_t^E} + \left(1 - \frac{M_t^E}{\mathcal{T}_t}\right)Q_{t+1}\right] + \frac{1}{C_t^E}\frac{M_t^E Q_{t+1}\Pi_{t+1}}{R_t^B}\right\} \tag{D.12}$$

$$W_t^P = \frac{\alpha(1-\mu-\nu)Y_t^E}{X_t L_t^P} \tag{D.13}$$

$$W_t^I = \frac{(1-\alpha)(1-\mu-\nu)Y_t^E}{X_t L_t^I} \tag{D.14}$$

$$\frac{1}{C_t^E}\left[1 + \frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}} - \delta\right)\right] = \gamma \mathbb{E}_t\left\{\frac{1}{C_{t+1}^E}\left[\frac{\mu Y_{t+1}^E}{X_{t+1}K_t} + (1-\delta) - \frac{\psi}{2\delta}(\delta - \frac{I_{t+1}}{K_t})(2 - \delta + \frac{I_{t+1}}{K_t})\right]\right\}. \tag{D.15}$$

Appendix D.1.4 Retailers

A continuum of retailers of mass 1, indexed by z, buy intermediate goods Y_t^E from entrepreneurs at P_t^E in a competitive market, differentiate one unit of goods at no cost into one unit of retail goods $Y_t(z)$, and sell it at the price $P_t(z)$. Final goods Y_t are from a CES aggregation of the differentiated goods produced by retailers, $Y_t = (\int_0^1 Y_t(z)^{1-\frac{1}{\epsilon}}dz)^{\frac{\epsilon}{\epsilon-1}}$, the aggregate price index is $P_t = (\int_0^1 P_t(z)^{1-\epsilon}dz)^{\frac{1}{1-\epsilon}}$, and the individual demand curve is $Y(z) = \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} Y_t$, where ϵ is the elasticity of substitution for the CES aggregation.

They face Calvo-stickiness: the sales price can be updated every period with a probability of $1-\theta$. When retailers can optimize the price with a probability θ, they reset it at $P_t^*(z)$; otherwise, the price is partially indexed to the past inflation; that is,

$$P_t(z) = \begin{cases} P_{t-1}(z)\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\xi_p}\Pi^{1-\xi_p}, \\ P_t^*(z) \end{cases} \tag{D.16}$$

where Π is the steady-state inflation.

The optimal price $P_t^*(z)$ set by retailers that can change price at time t solves:

$$\max_{P_t^*(z)} \sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left[\Lambda_{t,k}(P_t/P_{t+k})\left(P_t^*(z)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\xi_p}\Pi^{(1-\xi_p)k}Y_{t+k}(z) - P_{t+k}^E Y_{t+k|t}(z)\right)\right],$$

where $\Lambda_{t,k} \equiv \beta^k(C_t^P/C_{t+k}^P)$ is the patient households' real stochastic discount factor between t and $t+k$, and subject to

$$Y_{t+k|t}(z) = \left(\frac{P_t^*(z)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\xi_p}\Pi^{(1-\xi_p)k}}{P_{t+k}}\right)^{-\epsilon} Y_{t+k}.$$

The first-order condition for the retailer's problem takes the form

$$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left[\Lambda_{t,k}P_t\left(\frac{(\epsilon-1)P_t^*(z)(P_{t+k-1}/P_{t-1})^{\xi_p}\Pi^{(1-\xi_p)k}}{P_{t+k}} - \frac{\epsilon}{X_{t+k}}\right)Y_{t+k|t}(z)\right] = 0. \tag{D.17}$$

The aggregate price level evolves as follows:

$$P_t = \left\{\theta\left[P_{t-1}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\xi_p}\Pi^{1-\xi_p}\right] + (1-\theta)(P_t^*)^{1-\epsilon}\right\}^{1/(1-\epsilon)}. \tag{D.18}$$

Appendix D.1.5 Government

The central bank adjusts policy rates following a Taylor rule bounded by 0:[8]

$$\frac{S_t}{R} = \left(\frac{S_{t-1}}{R}\right)^{\phi_s}\left[(\Pi_{t-1}/\Pi)^{\phi_\pi}(Y_{t-1}/Y)^{\phi_y}\right]^{1-\phi_s}, \tag{D.19}$$

$$R_t = \max\{S_t,\, 1\}, \tag{D.20}$$

where R, Π, and Y are steady-state policy rate, inflation, and output, respectively.

The net government transfer in households' sectors consists of two parts: one is to balance the change in real money balance, and the other is lump-sum taxes to finance government spending, bond purchases (QE), and lending to private sectors (lending facilities):

$$T_t^P = T_t^{P,1} + T_t^{P,2} \tag{D.21}$$

$$T_t^{P,1} = \Delta\mathcal{M}_t^P/P_t \tag{D.22}$$

$$T_t^{P,2} = -\alpha(G_t + B_t^G) \tag{D.23}$$

$$T_t^I = T_t^{I,1} + T_t^{I,2} \tag{D.24}$$

$$T_t^{I,1} = \Delta\mathcal{M}_t^I/P_t \tag{D.25}$$

$$T_t^{I,2} = -(1-\alpha)(G_t + B_t^G). \tag{D.26}$$

where $T_t^{P,1}(T_t^{I,1})$ is the transfer to patient (impatient) households to balance their changes in real money balance, and $T_t^{P,2}(T_t^{I,2})$ is a negative transfer or a lump-sum tax to patient (impatient) households to cover government spending and unconventional monetary policy. The share of the lump-sum tax of each sector is determined by its labor share, respectively. The government budget constraint is of the form

$$G_t + \frac{B_t^G}{P_t} - \frac{R_{t-1}^B B_{t-1}^G}{\mathcal{T}_{t-1}P_t} - T_t^{P,2} - T_t^{I,2} = 0, \tag{D.27}$$

where G_t is government spending, and follows the process:

$$\frac{G_t}{G} = \left(\frac{G_{t-1}}{G}\right)^{\rho_g}\varepsilon_{g,t}, \tag{D.28}$$

where $\varepsilon_{g,t}$ is the government-spending shock.

Appendix D.1.6 Equilibrium

The equilibrium consists of an allocation,

$$\{H_t^E, H_t^P, H_t^I, L_t^E, L_t^P, L_t^I, Y_t, C_t^E, C_t^P, C_t^I, B_t^E, B_t^P, B_t^I, B_t^G, G_t\}_{t=0}^\infty,$$

and a sequence of prices,

$$\{W_t^P, W_t^I, S_t, P_t, P_t^*, X_t, Q_t\}_{t=0}^\infty,$$

that solves the household and firm problems and market-clearing conditions:
$H_t^E + H_t^P + H_t^I = H$, $C_t^E + C_t^P + C_t^I + I_t + G_t = Y_t$, $B_t^P + B_t^G = B_t^E + B_t^I$.

[8] We follow Iacoviello (2005) to assume the Taylor rule depends on lagged output and inflation. Whether the variables are lagged or contemporaneous does not affect our results.

Appendix D.2 Calibration

Table D.1: Calibrated parameters in the extended model

para	description	source	value
β	discount factor of patient households	Iacoviello (2005)	0.99
β^I	discount factor of impatient households	Iacoviello (2005)	0.95
γ	discount factor of entrepreneurs	Iacoviello (2005)	0.98
j	steady-state weight on housing services	Iacoviello (2005)	0.1
η	labor supply aversion	Iacoviello (2005)	0.01
μ	capital share in production	Iacoviello (2005)	0.3
ν	housing share in production	Iacoviello (2005)	0.03
δ	capital depreciation rate	Iacoviello (2005)	0.03
X	steady state gross markup	Iacoviello (2005)	1.05
θ	probability that cannot re-optimize	Iacoviello (2005)	0.75
α	patient households' wage share	Iacoviello (2005)	0.64
M^E	loan-to-value ratio for entrepreneurs	Iacoviello (2005)	0.89
M^I	loan-to-value ratio for impatient households	Iacoviello (2005)	0.55
r_R	interest rate persistence	Iacoviello (2005)	0.73
r_Y	interest rate response to output	Iacoviello (2005)	0.27
r_Π	interest rate response to inflation	Iacoviello (2005)	0.13
$\frac{G}{Y}$	steady-state government-spending-to-output ratio	Fernández-Villaverde et al. (2015)	0.20
ρ_a	autocorrelation of technology shock	Fernández-Villaverde et al. (2015)	0.90
ρ_g	autocorrelation of government-spending shock	Fernández-Villaverde et al. (2015)	0.80
ρ_β	autocorrelation of discount rate shock	Fernández-Villaverde et al. (2015)	0.80
σ_a	standard deviation of technology shock	Fernández-Villaverde et al. (2015)	0.0025
σ_g	standard deviation of government-spending shock	Fernández-Villaverde et al. (2015)	0.0025
σ_β	standard deviation of discount rate shock	Fernández-Villaverde et al. (2015)	0.0025
ξ_p	price indexation	Smets and Wouters (2007)	0.24
Π	steady-state inflation	2% annual inflation	1.005
B^G	steady-state government bond holdings	no gov. intervention in private bond market	0
$\mathcal{T}$	steady-state tax (subsidy) on interest rate income (payment)	no tax in normal times	1
rp	steady-state risk premium	3.6% risk premium annually	1.009

Table D.1 presents the calibrated parameters. Many of them are from Iacoviello (2005), Fernández-Villaverde et al. (2015), and Smets and Wouters (2007). For other parameters, we match the following empirical moments. The steady-state gross inflation is set to 1.005, which implies a 2% annual inflation rate. Steady-state government bond holding is 0, implying the government does not intervene in the private bond market during normal times. The steady-state tax on the gross interest rate income is set to 1 to imply zero tax on net interest rate income during normal times. The net quarterly risk premium is set to 0.9% to match the 3.6% average historical annual risk premium.

Appendix D.3 Steady state

The patient households' Euler equation gives us the steady-state private borrowing rate, shadow rate, and the real private borrowing rate:

$$R^B = \Pi/\beta \tag{D.29}$$
$$SR = R^B/RP \tag{D.30}$$
$$RR^B = 1/\beta. \tag{D.31}$$

Capital accumulation and entrepreneurs' first-order condition on investment together result in the investment-output ratio:

$$\frac{I}{Y} = \frac{\gamma\mu\delta}{[1-\gamma(1-\delta)]X}. \tag{D.32}$$

Entrepreneurs' first-order condition on housing, the borrowing constraint, and budget constraint give their real estate share, debt-to-output, and consumption-to-output ratio:

$$\frac{QH^E}{Y} = \frac{\gamma\nu}{X(1-\gamma^e)} \tag{D.33}$$
$$\frac{B^E}{Y} = \beta m \frac{QH^E}{Y} \tag{D.34}$$
$$\frac{C^E}{Y} = \left[\mu + \nu - \frac{\delta\gamma\mu}{1-\gamma(1-\delta)} - (1-\beta)mX\frac{QH^E}{Y}\right]\frac{1}{X}, \tag{D.35}$$

where $\gamma^e = \gamma - m\gamma + m\beta$.

Impatient households' budget constraint, borrowing constraint, and first-order condition on housing give their real estate share, debt-to-output, and consumption-to-output ratio:

$$\frac{QH^I}{C^I} = j/[1-\beta''(1-M^I) - M^I/(RR^B)] \tag{D.36}$$
$$\frac{B^I}{QH^I} = M^I\Pi/(R^B) \tag{D.37}$$
$$\frac{T^I - \Delta M^I/P}{Y} = -(1-\alpha)\frac{G}{Y} \tag{D.38}$$
$$\frac{C^I}{Y} = \frac{s^I + \frac{T^I - \Delta M^I/P}{Y}}{1 + \frac{QH^I}{C^I}(RR^B-1)\frac{B^I}{QH^I}}, \tag{D.39}$$

where $s^I = \frac{(1-\alpha)(1-\mu-\nu)}{X}$ is the income share of impatient households.

The bond-market-clearing condition, patient households' budget constraint, and first-order condition with respect to housing imply

$$\frac{B^P}{Y} = \frac{B^E}{Y} + \frac{B^I}{Y} \tag{D.40}$$

$$\frac{T^P - \Delta M^P/P}{Y} = -\alpha\frac{G}{Y} \tag{D.41}$$

$$\frac{C^P}{Y} = s^P + \frac{T^P - \Delta M^P/P}{Y} + (RR^B - 1)\frac{B^P}{Y} \tag{D.42}$$

$$\frac{QH^P}{C^P} = j/(1-\beta) \tag{D.43}$$

$$\frac{QH^P}{Y} = \frac{QH^P}{C^P}\frac{C^P}{Y}, \tag{D.44}$$

where

$$s^P = [\alpha(1-\mu-\nu) + X - 1]/X$$

is the income shares of patient households.

Housing shares of different sectors follows:

$$\frac{H^E}{H^P} = \frac{QH^E}{Y}/\frac{QH^P}{Y} \tag{D.45}$$

$$\frac{H^I}{H^P} = \frac{QH^I}{Y}/\frac{QH^E}{Y}. \tag{D.46}$$

Appendix D.4 Log-linear model

Propositions 1 - 3 describe the conditions under which the conventional and two unconventional policy tools are equivalent and can be coherently summarized by the shadow rate. We present the linear model with the shadow rate representation first in Appendix D.4.1. Then, we map it into specific policy tools in Appendix D.4.2 - Appendix D.4.3. Appendix D.4.4 explains the implementation of the model without unconventional monetary policy.

Appendix D.4.1 Shadow rate representation

In this representation, $R_t^B = S_t RP$, $M_t^I = M^I, M_t^E = M^E$, and $\mathcal{T}_t = \mathcal{T}$. Let hatted variables in lower case denote percentage changes from the steady state. The model can be expressed in the following blocks of equations:

1. Aggregate demand:

$$\widehat{y}_t = \frac{C^E}{Y}\widehat{c}_t^E + \frac{C^P}{Y}\widehat{c}_t^P + \frac{C^I}{Y}\widehat{c}_t^I + \frac{I}{Y}\widehat{i}_t + \frac{G}{Y}\widehat{g}_t \tag{D.47}$$

$$\widehat{c}_t^P = \mathbb{E}_t(\widehat{c}_{t+1}^P - \widehat{r}_t^B + \widehat{\pi}_{t+1} - \widehat{\beta}_{t+1}) \tag{D.48}$$

$$\widehat{i}_t - \widehat{k}_{t-1} = \gamma\left(\mathbb{E}_t\widehat{i}_{t+1} - \widehat{k}_t\right) + \frac{1-\gamma(1-\delta)}{\psi}\left(\mathbb{E}_t\left[\widehat{y}_{t+1} - \widehat{x}_{t+1}\right] - \widehat{k}_t\right) + \frac{1}{\psi}\left(\widehat{c}_t^E - \mathbb{E}_t\widehat{c}_{t+1}^E\right) \tag{D.49}$$

2. Housing/consumption margin:

$$
\begin{aligned}
\widehat{q}_t = & \ \gamma^e \mathbb{E}_t \widehat{q}_{t+1} + (1-\gamma^e)\left(\mathbb{E}_t \widehat{y}_{t+1} - \mathbb{E}_t \widehat{x}_{t+1} - \widehat{h}_t^E\right) + \left(1 - M^E\beta\right)\left(\widehat{c}_t^E - \mathbb{E}_t \widehat{c}_{t+1}^E\right) \\
& + M^E \beta \left(\mathbb{E}_t \widehat{\pi}_{t+1} - \widehat{s}_t\right) \qquad \text{(D.50)}
\end{aligned}
$$

$$
\widehat{q}_t = \gamma^h \mathbb{E}_t \widehat{q}_{t+1} - \left(1-\gamma^h\right)\widehat{h}_t^I + M^I\beta\left(\mathbb{E}_t\widehat{\pi}_{t+1} - \widehat{s}_t\right) + \left(1 - \frac{M^I\beta}{\mathcal{T}}\right)\widehat{c}_t^I - \beta^I\left(1 - M^I\right)\mathbb{E}_t \widehat{c}_{t+1}^I \quad \text{(D.51)}
$$

$$
\widehat{q}_t = \beta \mathbb{E}_t(\widehat{q}_{t+1} + \widehat{\beta}_{t+1}) + \left(\widehat{c}_t^P - \beta\mathbb{E}_t\widehat{c}_{t+1}^P\right) + (1-\beta)\frac{H^E}{H^P}\widehat{h}_t^E - (1-\beta)\frac{H^I}{H^P}\widehat{h}_t^I, \quad \text{(D.52)}
$$

where

$$
\begin{aligned}
\gamma^e &= M^E\beta + (1-M^E)\gamma \\
\gamma^h &= M^I\beta + \left(1 - M^I\right)\beta^I
\end{aligned}
$$

3. Borrowing constraints:

$$
\begin{aligned}
\widehat{b}_t - \widehat{p}_t &= \mathbb{E}_t\widehat{q}_{t+1} - (\widehat{s}_t - \mathbb{E}_t\widehat{\pi}_{t+1}) + \widehat{h}_t^E & \text{(D.53)} \\
\widehat{b}_t'' - \widehat{p}_t &= \mathbb{E}_t\widehat{q}_{t+1} - (\widehat{s}_t - \mathbb{E}_t\widehat{\pi}_{t+1}) + \widehat{h}_t^I & \text{(D.54)}
\end{aligned}
$$

4. Aggregate supply:

$$
\begin{aligned}
\widehat{y}_t &= \frac{1+\eta}{\eta+\nu+\mu}(\widehat{a}_t + \nu\widehat{h}_{t-1}^E + \mu\widehat{k}_{t-1}) - \frac{1-\nu-\mu}{\eta+\nu+\mu}(\widehat{x}_t + \alpha\widehat{c}_t^P + (1-\alpha)\widehat{c}_t^I) & \text{(D.55)} \\
\widehat{\pi}_t &= \frac{\beta}{1+\beta\xi_p}\mathbb{E}_t\widehat{\pi}_{t+1} + \frac{\xi_p}{1+\beta\xi_p}\widehat{\pi}_{t-1} - \frac{1}{1+\beta\xi_p}\lambda\widehat{x}_t + \widehat{e}_{\pi,t}, & \text{(D.56)}
\end{aligned}
$$

where

$$
\lambda = (1-\theta)(1-\beta\theta)/\theta
$$

5. Flows of funds/evolution of state variables:

$$
\begin{aligned}
& \widehat{k}_t = \delta\widehat{i}_t + (1-\delta)\widehat{k}_{t-1} & \text{(D.57)} \\
& \frac{B^E}{Y}(\widehat{b}_t^E - \widehat{p}_t) = \frac{C^E}{Y}\widehat{c}_t^E + \frac{QH^E}{Y}(\widehat{h}_t^E - \widehat{h}_{t-1}^E) + \frac{I}{Y}\widehat{i}_t + RR^B\frac{B^E}{Y}(\widehat{s}_{t-1} - \widehat{\pi}_t + \widehat{b}_{t-1}^E - \widehat{p}_{t-1}) \\
& \quad -(1 - s^P - s^I)(\widehat{y}_t - \widehat{x}_t) & \text{(D.58)} \\
& \frac{B^I}{Y}(\widehat{b}_t^I - \widehat{p}_t) = \frac{C^I}{Y}\widehat{c}_t^I + \frac{QH^I}{Y}(\widehat{h}_t^I - \widehat{h}_{t-1}^I) + RR^e\frac{B^I}{Y}(\widehat{r}_{t-1}^B - \widehat{\pi}_t + \widehat{b}_{t-1}^I - \widehat{p}_{t-1}) \\
& \quad -s^I(\widehat{y}_t - \widehat{x}_t) - \frac{(1-\alpha)G}{Y}\widehat{g}_t & \text{(D.59)}
\end{aligned}
$$

6. Monetary policy rule and shock processes:

$$
\begin{aligned}
\widehat{s}_t &= (1-r_R)[(1+r_\Pi)\widehat{\pi}_{t-1} + r_Y\widehat{y}_{t-1}] + r_R\widehat{s}_{t-1} & \text{(D.60)} \\
\widehat{a}_t &= \rho_a\widehat{a}_{t-1} + \widehat{e}_{a,t} & \text{(D.61)} \\
\widehat{\beta}_t &= \rho_\beta\widehat{\beta}_{t-1} + \widehat{e}_{\beta,t} & \text{(D.62)} \\
\widehat{g}_t &= \rho_g\widehat{g}_{t-1} + \widehat{e}_{g,t}. & \text{(D.63)}
\end{aligned}
$$

Appendix D.4.2 QE

Use the decomposition in (4.5),

$$
R_t^B = R_t RP_t. \quad \text{(D.64)}
$$

During normal times, the central bank varies R_t, whereas at the ZLB, it lowers RP_t through purchasing bonds from impatient households' and entrepreneurs' to decrease the bond supply to patient households. Both actions can mimic the dynamics in the shadow rate S_t. In this case, we keep the following policy variables constant: $M_t^I = M^I$, $M_t^E = M^E$, and $\mathcal{T}_t = \mathcal{T}$.

Proposition 1 implies

$$\begin{cases} \widehat{r}_t = \widehat{s}_t, \widehat{rp}_t = 0 \rightarrow \widehat{r}_t^B = \widehat{s}_t & \text{for } s_t \geq 0 \\ \widehat{rp}_t = \widehat{s}_t + s \rightarrow \widehat{r}_t^B = \widehat{s}_t & \text{for } s_t < 0. \end{cases}$$

Appendix D.4.3 Lending facilities

In this case, risk premium is kept at a constant $R_t^B = R_t RP$. At the ZLB, the government can increase the loan-to-value ratio so that impatient households and entrepreneurs can borrow more money for consumption and production, whereas the patient households still lend according to the borrowing constraints with constant loan-to-value ratios. Moreover, a tax is placed on interest rate income, which is then transferred to the borrowers.

Proposition 3 implies

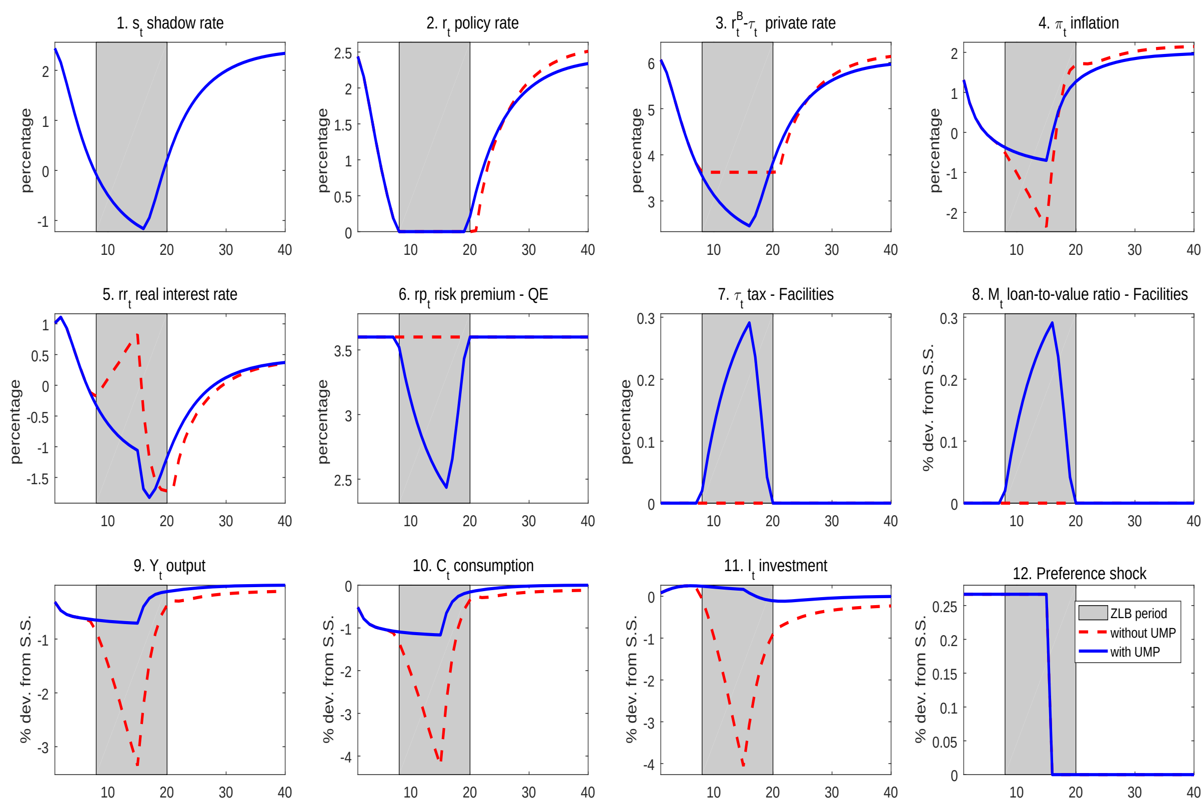
$$\begin{cases} \widehat{r}_t = \widehat{s}_t, \widehat{\tau}_t = \widehat{m}_t^I = \widehat{m}_t^E = 0 \rightarrow \widehat{r}_t^B - \widehat{\tau}_t = \widehat{s}_t & \text{for } s_t \geq 0 \\ \widehat{\tau}_t = \widehat{m}_t^I = \widehat{m}_t^E = -(\widehat{s}_t + s) \rightarrow \widehat{r}_t^B - \widehat{\tau}_t = \widehat{s}_t & \text{for } s_t < 0. \end{cases}$$

Appendix D.4.4 No unconventional monetary policy

For the model without unconventional monetary policy, replace $\widehat{s}_t$ with $\widehat{r}_t$ in (D.47) - (D.59), and augment the monetary policy in (D.60) with (2.1).

Appendix E ZLB environment

Figure E.1: Positive preference shocks



Notes: We hit the economy with a series of positive preference shocks, which occurs in periods 1 - 15, and the total shock size is 4%. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-7 are levels in percentage, among which plots 1-6 are annualized. Plots 8-12 are measured in percentage deviation from the steady state. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 4) or the lending facilities in plots 7 and 8 (per Section 5). The shaded area marks the ZLB period from 8-20.