NBER WORKING PAPER SERIES

MACRO, MONEY AND FINANCE: A CONTINUOUS TIME APPROACH

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Working Paper 22343
http://www.nber.org/papers/w22343

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
June 2016

This article was written for the Handbook of Macroeconomics. We are grateful to comments by various participants of the Princeton Initiative: Macro, Money and Finance, and especially to Zhiguo He, Lunyang Huang, Ji Huang, Falk Mazelis, Sebastian Merkel, Greg Phelan, and Christian Wolf, as well as to the editors John Taylor and Harald Uhlig. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

At least one co-author has disclosed a financial relationship of potential relevance for this research. Further information is available online at http://www.nber.org/papers/w22343.ack

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NBER Working Paper No. 22343
June 2016
JEL No. C63,E32,E41,E44,E51,G01,G11,G20

ABSTRACT

This paper puts forward a teaching manual for how to set up and solve a continuous time model that allows one to analyze endogenous (1) level and risk dynamics. The latter includes (2) tail risk and crisis probability as well as (3) the Volatility Paradox. Concepts such as (4) illiquidity and liquidity mismatch, (5) endogenous leverage, (6) the Paradox of Prudence, (7) undercapitalized sectors (8) time-varying risk premia, and (9) the external funding premium are part of the analysis. Financial frictions also give rise to an endogenous (10) value of money.

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1 Introduction

The recent financial crisis in the US and the subsequent Euro Crisis are vivid reminders of the importance of financial frictions in understanding macroeconomic trends and cycles. While financial markets are self-stabilizing in normal times, economies become vulnerable to a crisis after a run up of (debt) imbalances and (credit) bubbles. In particular, debt, leverage, maturity and liquidity mismatch tend to rise when measured volatility is low. Vulnerability risk tends to build up in the background, and only materializes when crises erupt, a phenomenon referred to as the "Volatility Paradox."

Adverse feedback loops can make the market spiral out of balance. The dynamics of an economy with financial frictions are highly non-linear. Small shocks lead to large economic dislocations. In situations with multiple equilibria, runs on financial institutions or sudden stops on countries can occur even absent any fundamental trigger. Empirically, these phenomena show up as fat tails in the distribution of real economic variables and asset price returns.

Our research proposes a continuous time method to capture the whole *endogenous risk dynamics* and hence goes beyond studying simply the persistence and amplification of an individual adverse shock. Instead of focusing only on levels, the first moments, the second moments, and movements of risk variables are all an integral part of the analysis, as they drive agents' consumption, (precautionary) savings and investment decisions. After a negative shock, we do not assume that the economy returns to the steady state deterministically, but rather uncertainty might be heightened making the length of the slump stochastic. As agents respond to the new situation, they affect both the risk and the risk premia.

Endogenous risk is time-varying and depends on illiquidity. Liquidity comes in three flavors. Technological illiquidity refers to the irreversibility of physical investment. Instead of undoing the initial investment, another option is to sell off the investment. This is only reasonable when market liquidity is sufficiently high. Finally, with sufficient funding liquidity one can issue claims against the payoffs of the assets. Incentive problems dictate that these claims are typically short-term debt claims. Debt comes with the drawback that risk is concentrated in the indebted sector. In addition, short-term debt leads to liquidity risk exposure. Agents may be forced to fire-sell their assets if they cannot undo the investment, market liquidity is low and funding is restricted, e.g. very short term. In short, when there is a liquidity mismatch between technological and market liquidity on the asset side and funding liquidity on the liability side of the balance sheet, the economy is vulnerable to

instability.

Models with financial frictions necessarily have to encompass multiple sectors. Financial frictions prevent funds from flowing to undercapitalized sectors, create debt overhang problems, and/or preclude optimal ex-ante risk sharing. This is in contrast to a world with perfect financial markets in which only aggregate risk matters, as all agents' marginal rate of substitutions are equalized in equilibrium and consequently aggregation to a single representative agent is possible. In models with financial frictions and heterogeneous agents the wealth distribution matters.

Importantly, financial frictions also give rise to the value of money. Money is a liquid store of value and safe asset. This approach provides not only a complementary perspective to New Keynesian models, in which price and wage rigidities are the primary drivers of money value, but also enables the revival of the traditional literature on "money and banking".[1]

Ultimately, economic analysis should guide policy. It is important to go beyond partial equilibrium analysis since general equilibrium effects can be subtle and counterintuitive. A model has to be tractable enough to conduct a meaningful welfare analysis to evaluate various policy instruments. A welfare analysis lends itself to study the interaction of various policy instruments.

In sum, the goal of this chapter is to put forward a manual for how to set up and solve a continuous time macro-finance model. The tractability that continuous time offers allows us to study a host of new properties of fully solved equilibria. This includes the full characterization of endogenous (1) level and risk dynamics. The latter includes (2) tail risk and crisis probability as well as (3) the Volatility Paradox. In addition, it should help us think about (4) illiquidity and liquidity mismatch, (5) endogenous leverage, (6) Paradox of Prudence, (7) undercapitalized sectors, (8) time-varying risk premia, and (9) the external funding premium. From a welfare perspective we would like to ask normative questions about the (10) inefficiencies of financial crises and (11) the effects of policies using various instruments.

We start with a brief history of macro and finance research since the Great Depression in the 1930s. We then put forward arguments in favor of continuous time modeling before surveying the ongoing continuous time literature. The main part of the paper builds up a step by step outline how to solve continuous time models starting with the simplest benchmark and enriching the model by adding more building blocks.

[1] See e.g. Chandler (1948).

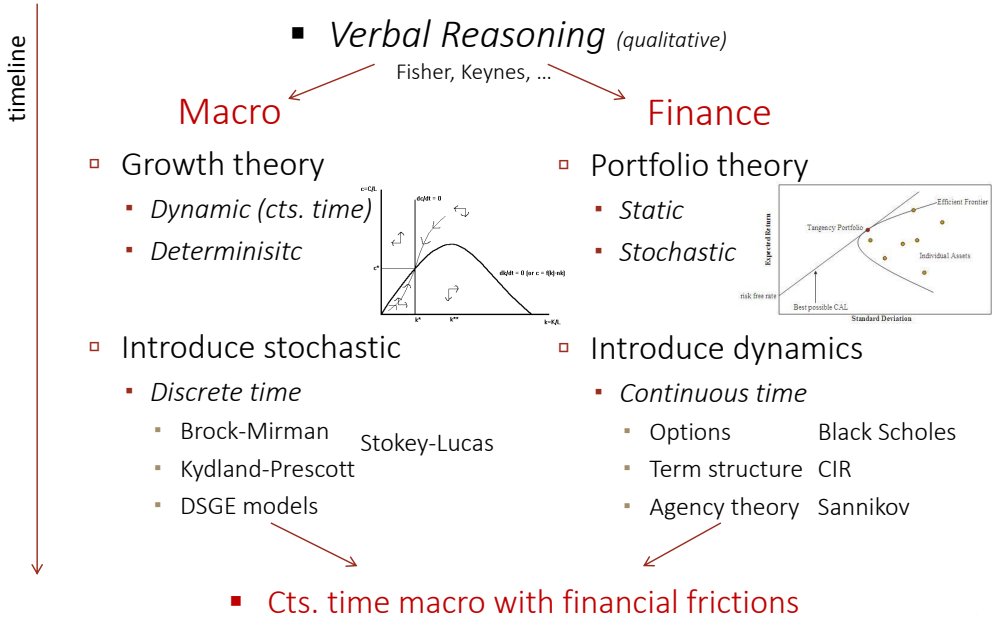

Figure 1: Methods in Macroeconomic and Financial Research since the Great Depression

1.1 A Brief History of Macroeconomics and Finance

Macroeconomics as a field in economics was born during the great depression in the 1930s. At that time economists like Fisher (1933), Keynes (1936), Gurley and Shaw (1955), Minsky (1957) and Kindleberger (1978) stressed the importance of the interaction between financial instability and macroeconomic aggregates. In particular, certain sectors in the economy including the financial sector can become balance sheet impaired and can drag down parts of the economy. Patinkin (1956) and Tobin (1969) also emphasized that financial stability and price stability are intertwined and hence that macroeconomics, monetary economics and finance are closely linked.

As economics became more analytical and model-based, macroeconomics and finance went into different directions. See Figure 1. Hicks's (1937) IS-LM Keynesian macro model is both static and deterministic. Macroeconomic growth models, most prominently the Solow (1956) growth model, are dynamic and many of them are in continuous time. However, they exclude stochastic elements: risk and uncertainty play no role. In contrast, the formal finance literature starting with Markowitz (1952) portfolio theory focused exclusively on risk. These models are static models and ignore the time dimension.

In the 1970s and early 1980s macroeconomists introduced stochastic elements into their

dynamic models. Early "fresh water" models that included time and stochastic elements were Brock and Mirman's (1972) Brock and Mirman's (1972) stochastic growth model and real business cycle models à la Kydland and Prescott (1982). The influential graduate text book of Stokey and Lucas (1989) provided the necessary toolkit for a fully microfounded dynamic and stochastic analysis. The "salt water" New Keynesian branch of macro introduced price rigidities and studied countercyclical policy in rational expectations models, Taylor (1979) and Mankiw and Romer (1991). The two branches merged and developed DSGE models which were both dynamic, the D in DSGE, and stochastic, the S in DSGE. However, unlike in many of the earlier growth models, time is discrete in real business cycle and New Keynesian DSGE models à la Woodford (2003). Most DSGE models capture only the log-linearized dynamics around the steady state. The log-linearized theoretical analysis squared nicely with its empirical counterpart, the linear Vector Autoregression Regression (VAR) estimation technique pioneered by Sims (1980).

Finance also experienced great breakthroughs in the 1970s. Stochastic Calculus (Ito calculus), which underlies the Black and Scholes (1973) option pricing model, revolutionized finance. Besides option pricing, term structure of interest rate models like Cox et al. (1985) were developed. More recently, Sannikov (2008) developed continuous time tools for financial contracting, which allow one to capture contracting frictions in a tractable way.

Our line of research is the next natural step. It essentially merges macroeconomics and finance using continuous time stochastic models. In terms of financial frictions, it builds on earlier work by Bernanke et al. (1999) (BGG), Kiyotaki and Moore (1997) (KM), Bianchi (2011), Mendoza (2010) and others. Our approach replicates two important results from the linearized versions of classic models of BGG and KM, that (1) temporary macro shocks can have a *persistent* effect on economic activity by making borrowers "undercapitalized" and (2) price movements *amplify* shocks. In KM, the leverage is limited by an always binding collateral constraint. In Bianchi (2011) and Mendoza (2010) it is occasionally binding. Our approach focuses mostly on incomplete market frictions, where the leverage of potentially undercapitalized borrowers is usually endogenous. In particular, it responds to the magnitude of fundamental (exogenous) macroeconomic shocks and the level of financial innovations that enable better risk management. Interestingly, leverage responds to a much lesser extent to the presence of endogenous tail risk. Equilibrium leverage in normal times is a key determinant of the probability of crises.

1.2 The Case for Continuous Time Macro Models

As economists we have no hesitation in assuming a continuous action space in order to ensure nice first order optimality conditions that are free of integer problems. In the same vain we typically assume a continuum of agents to guarantee an environment with perfect competition and (tractable) price taking behavior.

Assuming a continuous time framework has two advantages: it is often more tractable and might conceptually be a closer representation of reality. In terms of tractability, continuous time allows one to derive more analytical steps and more closed form characterizations of the equilibrium before resorting to a numerical analysis. For example, in our case one can derive explicit closed form expressions for amplification terms. The reason is that only the slope of the price function, i.e. the (local) derivative w.r.t. state variables, is necessary to characterize amplification. In contrast, in discrete time settings the whole price function is needed, as the jump size may vary. Also, instantaneous returns are essentially log-normal, which makes it easy to take expectations. It is also easy to derive the portfolio choice problem and to link returns to net worth dynamics via the budget constraint. In discrete-time models this feature can only be achieved through a (Campbell-Shiller) log-linear approximation. It is therefore not surprising that the term structure literature uses continuous time models. Admittedly, some of these features are due to the continuous nature of certain stochastic processes, like Brownian Motions and other Ito Processes. Hereby, one implicitly assumes that agents can adjust their consumption or portfolio continuously as their wealth changes. The feature that their wealth never jumps beyond a specific point, e.g. the insolvency point, greatly simplifies the exposition.

Conceptually, in certain dimensions a continuous time representation might also square better with reality. People do not consume only at the end of the quarter, even though data come in quarterly. Discrete time models implicitly assume linear time aggregation within a quarter and a non-linear one across quarters. In other words, the intertemporal elasticity of consumption within a quarter is infinite while across quarters it is given by the curvature of the utility function. Continuous time models treat every time unit the same. Similarly, it is well-known that for multivariate models mixing data with different degrees of smoothness (such as consumption data and financial data) can seriously impair inference.

The biggest advantage of our continuous time approach is that it allows a full characterization of the whole dynamical system including the risk dynamics instead of simply a log-linearized representation around the steady state. Note that impulse response functions capture only the expected path after a shock that starts at the steady state. Also,

the stationary distribution can be bi-modal and exhibit large swings, unlike stable normal distributions that log-linearized models imply.

1.3 The Nascent Continuous Time Macro-finance Literature

This chapter builds on Brunnermeier and Sannikov (2014).[2] It extends this work by allowing for more general utility functions, precautionary savings and for endogenous equity issuance. Work by Basak and Cuoco (1998) and He and Krishnamurthy (2012), (2013) on intermediary asset pricing are part of the core papers in this literature. Isohätälä et al. (2014) study a partial equilibrium model. DiTella (2013) introduces exogenous uncertainty shocks that can lead to balance sheet recessions even when contracting based on aggregate state variables is possible.

Phelan (2014) considers a setting in which banks issue equity and leverage can be procyclical. Adrian and Boyarchenko (2012) achieve procyclical leverage by introducing liquidity preference shocks. Adrian and Boyarchenko (2013) consider the interaction between two types of intermediaries: banks and non-banks. Huang (2014) studies shadow banks, which circumvent regulatory constraints but are subject to an endogenous enforcement constraint. In Moreira and Savov (2016)'s macro model shadow banks issue money-like claims. In downturns they scale back their activity. This slows down the recovery and creates a scarcity in collateral. Klimenko et al. (2015) show that regulation that prohibits dividend payouts is typicially superior to very tight capital requirements. In Moll (2014) capital is misallocated since productive agents are limited by collateral constraints to lever up.

Several papers also tried to calibrate continuous time macro-finance models to recent events. For example, He and Krishnamurthy (2014) do so by including housing as a second form of capital. Mittnik and Semmler (2013) employ a multi-regime vector autoregression approach to capture the non-linearity of these models.[3]

In international economics, these methods are employed in Brunnermeier and Sannikov (2015b). In a two good, two country model, the overly indebted country is vulnerable to sudden stops, and hence capital controls might improve welfare. Maggiori (2013) models risk sharing across countries which are at different stages of financial development.

[2]For an alternative survey on continuous time macro models, see e.g. Isohätälä et al. (2015).

[3]Note that in the estimation of DSGE models, Fernandez-Villaverde and Rubio-Ramiro (2010) show that parameter estimates and the moments generated by the model depend quite sensitively on whether a linearized DSGE is estimated via Kalman filtering or whether the true DSGE model is estimated via particle filtering.

Models with financial frictions also open up an avenue for new models in monetary economics thereby reviving the field "money and banking". In "The I Theory of Money" money is a bubble like in Samuelson (1958) or Bewley (1977). Inside money is created endogenously by the intermediary sector, and monetary policy and macroprudential policy interact. Achdou et al. (2015) provide a solution algorithm for Bewley models with uninsurable endowment risk in a continuous time setting. In Drechsler et al. (2014) banks are less risk averse and monetary policy affects risk premia. Werning (2012) studies the zero lower bound problem in a tractable deterministic continuous time New Keynesian model.

Rappoport and Walsh (2012) set up a discrete time macro model, which has similar economic results, and which converges in the continuous-time limit to the model of Brunnermeier and Sannikov (2014).

2 A Simple Real Economy Model

We start first with a particularly simple model to illustrate how equilibrium conditions - utility maximization and market clearing - translate into an equilibrium characterization. This simple model trivializes most of the issues we are after, e.g. the model has no price effects or endogenous risk. We do get some interesting takeaways, such as that risk premia spike in crises. After establishing the conceptual framework for what an equilibrium is, we move on to tackle more complex models.

2.1 Model Setup

This model is a variation of Basak and Cuoco (1998). The economy has a risky asset in positive net supply and a risk-free asset in zero net supply. There are two types of agents - experts and households. Only experts can hold the risky asset - households can only lend to experts at the risk-free rate r_t, determined endogenously in equilibrium. The friction is that experts can finance their holdings of the risky asset only through debt - by selling short the risk-free asset to households. That is, experts cannot issue equity. We assume that all agents are small, and behave as price-takers. That is, unlike in microstructure models with noise traders, agents have no price impact.

Technology. Net of investment, physical capital, k_t, generates consumption output at the rate of

$$(a - \iota_t)k_t \, dt,$$

where a is a productivity parameter and ι_t is the reinvestment rate per unit of capital. The production technology is constant returns to scale.

The productive asset (capital), k_t, evolves according to

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) \, dt + \sigma \, dZ_t, \tag{2.1}$$

where $\Phi(\iota_t)$ is an investment function with adjustment costs, such that $\Phi(0) = 0$, $\Phi' > 0$ and $\Phi'' \leq 0$. Thus, in the absence of investment, capital simply depreciates at rate δ. The concavity of $\Phi(\cdot)$ reflects decreasing returns to scale, and for negative values of ι_t, corresponds to *technological illiquidity* - the marginal cost of capital depends on the rate of investment/disinvestment.

The aggregate amount of capital is denoted by K_t, and q_t is the price of capital. Hence, the aggregate net worth in the economy is $q_t K_t$. If N_t is the aggregate net worth of experts, then the aggregate net worth of households is $q_t K_t - N_t$.

Experts' wealth share is denoted by

$$\eta_t = \frac{N_t}{q_t K_t} \in [0, 1].$$

Preferences. For *tractability*, all agents are assumed to have logarithmic utility with discount rate ρ, of the form

$$E\left[\int_0^\infty e^{-\rho t} \log c_t \, dt\right],$$

where c_t is consumption at time t.

2.2 A Step-by-Step Approach

Definition. *An equilibrium is a map from histories of macro shocks $\{Z_s, \ s \leq t\}$ to the price of capital q_t, risk-free rate r_t, as well as asset holdings and consumption choices of all agents, such that*

1. *agents behave to maximize utility and*

2. markets clear.

To find an equilibrium, we need to write down equations that the processes q_t, r_t, etc. have to satisfy, and that characterize how these processes evolve with the realizations of shocks Z. It will be convenient to express these relationships using a state variable. Here the relevant state variable, which describes the distribution of wealth, is the fraction of wealth owned by the experts, η_t. When η_t drops, experts become more balance sheet constrained.

We solve the equilibrium in three steps. First, we postulate some endogenous processes. As a second step, we use the equilibrium conditions, i.e. utility maximization and market clearing, to write down restrictions q_t and r_t need to satisfy. In this simple model, we will be able to express q_t and r_t as functions of η_t in closed form. Third, we need to derive the law of motion of the state variable, the wealth share η_t.

Step 1: Postulate Equilibrium Processes. The first step is to postulate certain endogenous price processes. For example, suppose that the price per unit of capital q_t follows an Ito process

$$\frac{dq_t}{q_t} = \mu_t^q \, dt + \sigma_t^q \, dZ_t, \tag{2.2}$$

which, of course, is endogenous in equilibrium.

An investment in capital generates, in addition to the dividend rate $(a-\iota)k_t dt$, the capital gains at rate

$$\frac{d(k_t q_t)}{k_t q_t}.$$

Ito's Lemma for the product of two stochastic processes can be used to derive this process.

ITO'S FORMULA FOR PRODUCT. Suppose two processes X_t and Y_t follow

$$\frac{dX_t}{X_t} = \mu_t^X \, dt + \sigma_t^X \, dZ_t \quad \text{and} \quad \frac{dY_t}{Y_t} = \mu_t^Y \, dt + \sigma_t^Y \, dZ_t.$$

Then the product of two processes follows

$$\frac{d(X_t Y_t)}{X_t Y_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y) \, dt + (\sigma_t^X + \sigma_t^Y) \, dZ_t. \tag{2.3}$$

Using Ito's Lemma, the investment in capital generates capital gains at rate

$$\frac{d(k_t q_t)}{k_t q_t} = (\Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t.$$

Then capital earns the return of

$$dr_t^k = \underbrace{\frac{a - \iota_t}{q_t}\, dt}_{\text{dividend yield}} + \underbrace{(\Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t}_{\frac{d(k_t q_t)}{k_t q_t}\text{, the capital gains rate}}. \tag{2.4}$$

Thus, generally a part of the risk from holding capital is fundamental, $\sigma\, dZ_t$, and a part is endogenous, $\sigma_t^q\, dZ_t$.

Remarks

- For general utility functions one also has to postulate the stochastic discount factor process or equivalently a process for the marginal utility or the consumption process dc_t/c_t. For details see Section 3.1.
- Note that in monetary models like Brunnermeier and Sannikov (2015, 2016) one also has to postulate a process p_t for the value of money which can be stochastic due to inflation risk. In Section 4 we present a simple monetary model.

Step 2: The Equilibrium Conditions. Equilibrium conditions come in two flavors: Optimality conditions and market clearing conditions.

Optimal internal investment rate. Note that the rate of internal investment ι_t does not affect the risk of capital. The optimal investment rate that maximizes the expected return satisfies the first-order condition

$$\Phi'(\iota_t) = \frac{1}{q_t}. \tag{2.5}$$

Optimal consumption rate. Logarithmic utility has two convenient properties, which we derive formally for a more general case in Section 3.1. These two properties help reduce the number of equations that characterize equilibrium. First, for agents with log utility

$$\text{consumption} = \rho \cdot \text{net worth} \tag{2.6}$$

that is, they always consume a fixed fraction of wealth (permanent income) regardless of the risk-free rate or risky investment opportunities. The consumption Euler equation reduces to

a particularly simple form.

Optimal portfolio choice. The optimal risk exposure of a log-utility agent in the optimal portfolio choice problem depends on the attractiveness of risky investment, measured by the Sharpe ratio, defined as expected excess returns divided by the standard deviation. Formally, the equilibrium condition is

$$\text{Sharpe ratio of risky investment} = \text{volatility of net worth}, \tag{2.7}$$

where the volatility is relative (measured as percentage change per unit of time).[4]

Goods Market clearing. We use equations (2.6) and (2.7) to formalize equilibrium conditions, and characterize equilibrium. First, from condition (2.6), the aggregate consumption of all agents is $\rho q_t K_t$, and aggregate output is $(a-\iota(q_t))K_t$, where investment ι is an increasing function of q defined by (2.5). From market clearing for consumption goods, these must be equal, and so

$$\rho q_t = a - \iota(q_t) \tag{2.8}$$

is the equilibrium price of the risky capital. The aggregate consumption of experts must be $\rho N_t = \rho \eta_t q_t K_t$, and the aggregate consumption of households is $\rho(1-\eta_t)q_t K_t$. Condition (2.8) alone leads to a *constant* value of the *price of capital* q. That is, $\mu_t^q = \sigma_t^q = 0$.

Example 1. Suppose the investment function takes the form

$$\Phi(\iota) = \frac{\log(\kappa\iota + 1)}{\kappa},$$

where κ is the adjustment cost parameter. Then $\Phi'(0) = 1$. Higher κ makes function Φ more concave, and as $\kappa \to 0$, $\Phi(\iota) \to \iota$, a fully elastic investment function with no adjustment costs. Then the optimal investment rate is $\iota = (q-1)/\kappa$, and the market-clearing condition (2.8) leads to the price of

$$q = \frac{1+\kappa a}{1+\kappa\rho}.$$

The price converges to 1 as $\kappa \to 0$, i.e. the investment technology is fully elastic. The price q converges to a/ρ as $\kappa \to \infty$.

[4]For example, if the annual volatility of S&P 500 is 15% and the risk premium is 3% (so that the Sharpe ratio is 3%/15% = 0.2), then a log utility agent wants to hold a portfolio with volatility 0.2 = 20%. This corresponds to a weight of 1.33 on S&P 500, and -0.33 on the risk-free asset.

Second, we can use condition (2.7) for experts to figure out the equilibrium **risk-free rate**. We first look at the return on risky and risk-free assets to compute the Sharpe ratio of risky investments. We then look at balance sheets of experts to compute the volatility of their wealth. Finally, we use equation (2.7) to get the risk-free rate.

Because q is constant, the risky asset earns a return of

$$dr_t^k = \underbrace{\frac{a-\iota}{q}\,dt}_{\rho\text{, dividend yield}} + \underbrace{(\Phi(\iota)-\delta)\,dt + \sigma\,dZ_t}_{\text{capital gains rate}},$$

and the risk-free asset earns r_t. Note that the dividend yield equals ρ by the goods market clearing condition. Hence, the Sharpe ratio of risky investment is

$$\frac{\rho + \Phi(\iota) - \delta - r_t}{\sigma}.$$

Note that since the price-dividend ratio is constant any change in the risk premium must come from the variation in the risk free rate r_t.

Because experts must hold all the risky capital in the economy, with value $q_t K_t$ (households cannot hold capital), and absorb risk through net worth N_t, the volatility of their net worth is

$$\frac{q_t K_t}{N_t}\sigma = \frac{\sigma}{\eta_t}.$$

Using (2.7),

$$\frac{\sigma}{\eta_t} = \frac{\rho + \Phi(\iota) - \delta - r_t}{\sigma} \quad \Rightarrow \quad r_t = \rho + \Phi(\iota) - \delta - \frac{\sigma^2}{\eta_t}. \tag{2.9}$$

Step 3: The Law of Motion of η_t. To finish deriving the equilibrium, we need to describe how shocks Z_t affect the state variable $\eta_t = N_t/(q_t K_t)$. First, since η_t is a ratio, the following formula will be helpful for us:

ITO'S FORMULA FOR RATIO. Suppose two processes X_t and Y_t follow

$$\frac{dX_t}{X_t} = \mu_t^X\,dt + \sigma_t^X\,dZ_t \quad \text{and} \quad \frac{dY_t}{Y_t} = \mu_t^Y\,dt + \sigma_t^Y\,dZ_t.$$

Then ratio of two processes follows

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + (\sigma_t^Y)^2 - \sigma_t^X\sigma_t^Y)\,dt + (\sigma_t^X - \sigma_t^Y)\,dZ_t. \tag{2.10}$$

Second, it is convenient to express the laws of motion of the numerator and denominator of η_t in terms of total risk and the Sharpe ratio given by (2.9). Specifically,

$$\frac{dN_t}{N_t} = r_t\,dt + \underbrace{\frac{\sigma}{\eta_t}}_{\text{risk}}\underbrace{\frac{\sigma}{\eta_t}}_{\text{Sharpe}}\,dt + \frac{\sigma}{\eta_t}\,dZ_t - \underbrace{\rho\,dt}_{\text{consumption}} \quad \text{and}$$

$$\frac{d(q_tK_t)}{q_tK_t} = r_t\,dt + \underbrace{\sigma}_{\text{risk}}\underbrace{\frac{\sigma}{\eta_t}}_{\text{Sharpe}}\,dt + \sigma\,dZ_t - \underbrace{\rho\,dt}_{\text{dividend yield}}.$$

In the latter equation, we subtract the dividend yield from the total return on capital to obtain the capital gains rate.

Using the formula for the ratio,

$$\frac{d\eta_t}{\eta_t} = (r_t + \sigma^2/\eta_t^2 - \rho - r_t - \sigma^2/\eta_t + \rho + \sigma^2 - \sigma^2/\eta_t)\,dt + (\sigma/\eta_t - \sigma)\,dZ_t$$

$$= \frac{(1-\eta_t)^2}{\eta_t^2}\sigma^2\,dt + \frac{1-\eta_t}{\eta_t}\sigma\,dZ_t. \tag{2.11}$$

Step 4: Expressing $q(\eta)$ as a function of η is not necessary in this simple model, since q is a constant.

2.3 Observations

Several key observations about equilibrium characteristics are worth pointing out. Variable η_t fluctuates with macro shocks - a positive shock increases the wealth share of experts. This is because experts are levered. A negative shock erodes η_t, and experts require a higher risk premium to hold risky assets. Experts must be convinced to keep holding risky assets by the increasing Sharpe ratio

$$\frac{\sigma}{\eta_t} = \frac{\rho + \Phi(\iota) - \delta - r_t}{\sigma},$$

which goes to ∞ as η_t goes to 0. Strangely, this is achieved due to the risk-free rate $r_t = \rho + \Phi(\iota) - \delta - \sigma^2/\eta_t$ going to $-\infty$, rather than due to a depressed price of the risky asset, as illustrated in the top right panel of Figure 2.

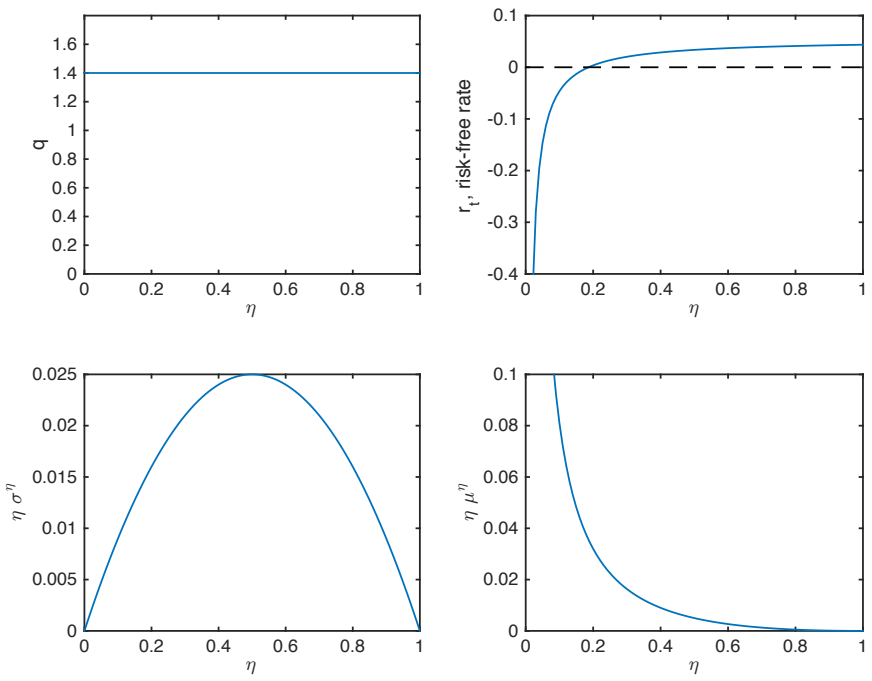

Figure 2: Equilibrium in the simple real model,
$a = .11$, $\rho = 5\%$, $\sigma = .1$, and $\Phi = \log(\kappa\iota + 1)/\kappa$ with $\kappa = 10$.

Because q_t is constant, as illustrated in the top left panel, there is no endogenous risk, no amplification and no volatility effects. Therefore, in this model, assumptions that allow for such a simple solution also eliminate any price effects that we are so interested in. We have to work harder to get those effects.

Besides the absence of price effects, in this model it is also the case that in the long run the expert sector becomes so large that it overwhelms the whole economy. To see this, note that the drift of η_t is always positive. This feature is typical of models in which one group of agents has an advantage over another group - in this case only experts can invest in the risky asset. It is possible to prevent the expert sector from becoming too large through an additional assumption. For example, Bernanke et al. (1999) assume that experts are randomly hit by a shock that makes them households. Alternatively, if experts have a higher discount rate than households, then a greater consumption rate prevents the expert sector from becoming too large.

The main purpose of this section was to show how equilibrium conditions can be translated into formulas that describe the behavior of the economy. Next, we can consider more complicated models, in which the price of the risky asset q_t reacts to shocks. We also develop a methodology that allows for agents to have more complicated preferences and for a nontrivial distribution of assets among agents.

3 A Model with Price Effects and Instabilities

We now illustrate how our step approach can be used to solve a more complex model, which we borrow and extend from Brunnermeier and Sannikov (2014). We will be able to get a number of important takeaways from the model:

1. Equilibrium dynamics are characterized by a relatively stable steady state, where the system spends most of the time, and a crisis regime. In the steady state, experts are adequately capitalized and risk premia fall. The experts' consumption offsets their earnings - hence the steady state is formed. Experts have the capacity to absorb most macro shocks, hence prices near the steady state are quite stable. However, an unusually long sequence of negative shocks causes experts to suffer significant losses, and pushes the equilibrium into a crisis regime. In the crisis regime, experts are undercapitalized and constrained. Shocks affect their demand for assets - market liquidity at the macro level can dry up -, and thus affect prices of the assets that experts hold.

This creates feedback effects, which generate fire-sales and *endogenous risk*. Volatility is endogenous and also feeds back in agents' behavior.

2. High volatility during crisis times may push the system into a very depressed region, where experts' net worth is close to 0. If that happens, it takes a long time for the economy to recover. Thus, the system spends a considerable amount of time far away from the steady state. The stationary distribution may be bimodal.

3. Endogenous risk during crises makes assets more correlated.

4. There is a "*Volatility Paradox*" , because risk-taking is endogenous. If the aggregate risk parameter σ becomes smaller, the economy does not become more stable. The reason is that experts allow greater leverage, and pay out profits sooner, in response to lower fundamental risk. Due to greater leverage, the economy is prone to crises even when exogenous shocks are smaller. In fact, endogenous risk during crises may actually be higher when σ is lower.

5. Financial innovations, such as securitization and derivatives hedging, that allow for more efficient risk-sharing among experts, may make the system less stable in equilibrium. The reason, again, is that risk-taking is endogenous. By diversifying idiosyncratic risks, experts tend to increase leverage, amplifying systemic risks.

Before going into details how we can extend our simple real economy model from section 2 to display these additional features, we take a detour to discuss the classic problem of optimal consumption and portfolio choice in continuous time.

3.1 Optimal Portfolio Choice with General Utility Functions

We start with a brief description of how to extend the optimal consumption and portfolio choice conditions (such as (2.6) and (2.7)) to the case of a general utility function. The key result is that any asset, which an agent can hold, can be priced from the agent's marginal utility of wealth θ_t. The first-order condition for optimal consumption is $\theta_t = u'(c_t)$, so the marginal utility of wealth is also the marginal utility of consumption (unless the agent is "at the corner").[5]

[5]If the agent is risk-neutral, then his marginal utility of consumption is always 1, but the agent may choose to not consume if his marginal utility of wealth is greater than 1.

If the agent has discount rate ρ then $\xi_t = e^{-\rho t}\theta_t$ is the stochastic discount factor (SDF) to price assets. We can write

$$\frac{d\xi_t}{\xi_t} = -r_t\,dt - \varsigma_t\,dZ_t, \tag{3.1}$$

where r_t is the (shadow) risk-free rate and ς_t is the price of risk dZ_t.

For any asset A that the agent can invest in, with return

$$dr_t^A = \mu_t^A\,dt + \sigma_t^A\,dZ_t,$$

we must have

$$\mu_t^A = r_t + \varsigma_t\sigma_t^A. \tag{3.2}$$

Equations (3.1) and (3.2) are simple, yet extremely powerful.

Martingale Method. To derive equation (3.2) consider a trading strategy of investing 1 dollar into asset A at time 0 and keep on reinvesting any dividends the asset might pay out. Denote the value of this strategy at time t by v_t (then $v_0 = 1$, obviously). Clearly, its capital gains rate is

$$\frac{dv_t}{v_t} = dr_t^A.$$

For an arbitrary $s \le t$ consider an investor who can only trade at s and t. That is, he faces a simple two-period portfolio problem. The Euler equation for the standard two-period portfolio problem is

$$v_s = E_s\left[\frac{\xi_t}{\xi_s}v_t\right] \Rightarrow \xi_s v_s = E_s[\xi_t v_t].$$

That is, $\xi_t v_t$ must be a martingale on the time domain $\{s,t\}$. For an investor who can trade continuously $\xi_t v_t$ must be a martingale for any t, since we picked s,t arbitrarily. Next, by Itô's formula

$$\frac{d(\xi_t v_t)}{\xi_t v_t} = (\mu_t^\xi + \mu_t^v + \sigma_t^\xi\sigma_t^v)dt + (\sigma_t^\xi + \sigma_t^v)dZ_t = (-r_t + \mu_t^A - \varsigma_t\sigma_t^A)dt + (\sigma_t^A - \varsigma_t)dZ_t.$$

This is a martingale if and only if the drift vanishes, i.e. equation (3.2) holds.

Derivation via Stochastic Maximum Principle. One can also derive the pricing equations and consumption rule using the stochastic maximum principle. Let us consider an agent who maximizes

$$E\left[\int_0^\infty e^{-\rho t}u(c_t)\,dt\right],$$

and whose net worth follows

$$dn_t = n_t \left(r_t \, dt + \sum_A x_t^A ((\mu_t^A - r_t) \, dt + \sigma_t^A \, dZ_t) \right) - c_t \, dt,$$

with initial wealth $n_0 > 0$ and where x_t^A are portfolio weights on various assets A. Investment opportunities are stochastic and exogenous, i.e. they do not depend on the agent's strategy.

The stochastic maximum principle allows us to derive first-order conditions for maximization from the Hamiltonian. Introducing a multiplier ξ_t on n_t (i.e. marginal utility of wealth) and denoting the volatility of ξ_t by $-\varsigma_t \xi_t$, the Hamiltonian is written as

$$H = e^{-\rho t} u(c) + \xi_t \underbrace{\{(r_t + \sum_A x^A (\mu_t^A - r_t)) n_t - c\}}_{\text{drift of } n_t} - \varsigma_t \xi_t \underbrace{\sum_A x^A \sigma_t^A n_t}_{\text{volatility of } n_t}.$$

By differentiating the Hamiltonian with respect to controls, we get the first-order conditions, and by differentiating it with respect to the state n_t, we get the law of motion of the multiplier ξ_t.

The first-order condition with respect to c is

$$e^{-\rho t} u'(c_t) = \xi_t,$$

which implies that the multiplier on the agent's wealth is his discounted marginal utility of consumption. The first-order condition with respect to the portfolio weight x^A is

$$\xi_t (\mu_t^A - r_t) - \varsigma_t \xi_t \sigma_t^A = 0,$$

which implies (3.2).

In addition, the drift of ξ_t is

$$-H_n = -\xi_t r_t,$$

where we already used the first-order conditions with respect to x^A to perform cancellations. It follows that the law of motion of ξ_t is

$$d\xi_t = -\xi_t r_t \, dt - \varsigma_t \xi_t \, dZ_t,$$

which corresponds to (3.1).

Value Function Derivation for CRRA Utility. Macroeconomists are most familiar with this method. With CRRA utility, the agent's value function takes a power form

$$\frac{u(\omega_t n_t)}{\rho}. \tag{3.3}$$

This form comes from the fact that if the agent's wealth changes by a factor of x, then his optimal consumption at all future states changes by the same factor - hence ω_t is determined so that $u(\omega_t)/\rho$ is the value function at unit wealth. Marginal utility of consumption and marginal utility of wealth are equated if $c_t^{-\gamma} = \omega_t^{1-\gamma} n_t^{-\gamma}/\rho$, or

$$\frac{c_t}{n_t} = \rho^{1/\gamma} \omega_t^{1-1/\gamma}. \tag{3.4}$$

For log utility, $\gamma = 1$ and this equation implies that $c_t/n_t = \rho$ as we claimed in (2.6).

For $\gamma \neq 1$, by expressing ω_t as a function of the consumption rate c_t/n_t, we find that the agent's continuation utility is

$$\frac{c_t^{-\gamma} n_t}{1-\gamma}. \tag{3.5}$$

This remarkable expression shows that the agent's net worth and consumption rate are sufficient to compute the agent's welfare, and no additional information about the agent's stochastic investment opportunities is needed.

Given the agent's (postulated) consumption process of

$$\frac{dc_t}{c_t} = \mu_t^c \, dt + \sigma_t^c \, dZ_t,$$

by Ito's Lemma, marginal utility $c^{-\gamma}$ follows

$$\frac{d(c_t^{-\gamma})}{c_t^{-\gamma}} = \left(-\gamma \mu_t^c + \frac{\gamma(\gamma+1)}{2} (\sigma_t^c)^2 \right) \, dt - \gamma \sigma_t^c \, dZ_t. \tag{3.6}$$

Substituting this into (3.2), we obtain the following relationship for the pricing of any risky asset relative to the risk-free asset:

$$\frac{\mu_t^A - r_t}{\sigma_t^A} = \gamma \sigma_t^c = \varsigma_t. \tag{3.7}$$

Recall that $\xi_t = e^{-\rho t} u'(c_t)$ and hence $\frac{d\xi_t}{\xi_t} = -\rho - \frac{d(c_t^{-\gamma})}{c_t^{-\gamma}}$. Minus the drift of the SDF is the

risk-free rate, i.e.

$$r_t = \rho + \gamma\mu_t^c - \frac{\gamma(\gamma+1)}{2}(\sigma_t^c)^2. \tag{3.8}$$

Two special cases with particularly nice analytical solutions deserve special attention.

Example with CRRA and Constant Investment Opportunities. With constant investment opportunities, then ω_t is a constant, hence (3.4) implies that $\sigma_t^c = \sigma_t^n$, just like in the logarithmic case. Hence, (3.7) implies that

$$\underbrace{\frac{\mu_t^A - r}{\sigma_t^A}}_{\varsigma} = \gamma\sigma_t^n,$$

i.e. the volatility of net worth is the Sharpe ratio divided by the risk aversion coefficient γ. Note that this property also holds when ω_t is not a constant as long as it evolve deterministically.

Now, the agent's net worth follows

$$\frac{dn_t}{n_t} = r\,dt + \frac{\varsigma^2}{\gamma}\,dt + \frac{\varsigma}{\gamma}\,dZ_t - \frac{c_t}{n_t}\,dt,$$

and, since consumption is proportional to net worth, (3.8) implies that

$$r = \rho + \gamma\left(r + \frac{\varsigma^2}{\gamma} - \frac{c_t}{n_t}\right) - \frac{\gamma(\gamma+1)}{2}\frac{\varsigma^2}{\gamma^2} \quad \Rightarrow \quad \frac{c_t}{n_t} = \rho + \frac{\gamma-1}{\gamma}\left(r - \rho + \frac{\varsigma^2}{2\gamma}\right).$$

Hence, consumption ratio increases with better investment opportunities when $\gamma > 1$ and falls otherwise.

Example with Log Utility. We can verify that the consumption and asset-pricing relationships for logarithmic utility of equation. Note from (3.4) follows directly (2.6),

$$c_t = \rho n_t.$$

Since the SDF is $\xi_t = e^{-\rho t}/c_t = e^{-\rho t}/(\rho n_t)$ (for any ω_t) it follows that $\sigma_t^n = \sigma_t^c = \varsigma_t$ (i.e. minus the volatility of ξ_t). Hence, (3.2) implies that

$$\frac{\mu_t^A - r_t}{\sigma_t^A} = \sigma_t^n,$$

where the left hand side is the Sharpe ratio, and the right hand side is the volatility of net worth.

3.2 Model with Heterogeneous Productivity Levels and Preferences

In order to study endogenous risk, market illiquidity, fire-sales etc., we now assume that the household sector can also hold physical capital, but households are assumed to be less productive. Specifically, their productivity parameter $\underline{a} < a$, and hence their willingness to pay for capital, is lower than that of experts. In this generalized setting, experts now have only two ways out when they become less capitalized and want to scale back their operation: fire-sell the capital to households at a possibly large price discount (market illiquidity) or "uninvest'' and suffer adjustment costs (technological illiquidity).

Less productive households earn a return of

$$d\underline{r}_t^k = \underbrace{\frac{\underline{a} - \iota_t}{q_t}\, dt}_{\text{dividend yield}} + \underbrace{(\Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t}_{\frac{d(q_t k_t)}{q_t k_t}\text{, the capital gains rate}} \tag{3.9}$$

when they manage the physical capital. The households' return differs from that of experts, (2.4), only in the dividend yield that they earn.

We generalize the model in several other ways. (i) We enable experts to issue some (outside) equity, even though they cannot be 100% equity financed. Specifically, we suppose that experts must retain at least a fraction $\underline{\chi} \in (0, 1]$ of equity. (ii) We generalize the model by including a force that prevents experts from "saving their way out" away from the constraints. In particular, we assume that experts could have a higher discount rate ρ than that of households, $\underline{\rho}$. (iii) Equipped with the results derived in Subsection 3.1 we generalize experts' and households' utility functions from log to CRRA with risk aversion coefficient γ.[6]

To summarize, experts and households maximize, respectively

$$E\left[\int_0^\infty e^{-\rho t} u(c_t)\right] dt, \quad \text{and} \quad E\left[\int_0^\infty e^{-\underline{\rho} t} u(\underline{c}_t)\right] dt.$$

We denote the fraction of capital allocated to experts by $\psi_t \leq 1$ and the fraction of equity retained by experts by $\chi_t \geq \underline{\chi}$.

[6] Brunnermeier and Sannikov (2014) explicitly consider the case of risk-neutral experts and households. Experts are constrained to consume nonnegative quantities, but households can consume both positive and negative amounts. This assumption leads to the simplification that the risk-free rate in the economy r_t always equals the households' discount rate $\underline{\rho}$.

We want to characterize how any history of shocks $\{Z_s, s \leq t\}$ maps to equilibrium prices q_t and r_t, asset allocations ψ_t and χ_t, and consumption so that (1) all agents maximize utility through optimal consumption and portfolio choices and (2) markets clear. Agents optimize portfolios subject to constraints (no short-selling of capital and a bound on equity issuance by experts). For example, households can invest in capital, the risk-free asset, and experts' equity, and optimize over portfolio weights on these three assets (with a nonnegative weight on capital). Thus, the solution is based on a classic problem in asset pricing. Note also that because the required returns are different between households and experts, the experts' inside equity will generally earn a different return from the equity held by households - experts will earn "management fees" that households do not earn.[7]

3.3 The 4-Step Approach

We can solve for the equilibrium in four steps. First, postulate processes for prices and stochastic discount factor. Second, write down the consumption-portfolio optimization and market-clearing conditions. These conditions imply a stochastic law of motion of the price q_t, the required risk premia for experts and households ς_t and $\underline{\varsigma}_t$, together with variables ψ_t and χ_t. Third, focusing on the experts' balance sheets we write down the law of motion of expert's wealth share

$$\eta_t = \frac{N_t}{q_t K_t},$$

as a percentage of the whole wealth in the economy. As before, K_t is the total amount of capital in the economy. Fourth, we look for a Markov equilibrium, and characterize equations for q_t, ψ_t, etc., as functions of η_t. We solve these equations numerically either as a system of ordinary differential equations (using the shooting method) or as a system of partial differential equations in time, via a procedure analogous to value function iteration in discrete time.

Step 1: Postulating Equilibrium Processes. As before we postulate the equilibrium prices process for physical capital.

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t.$$

[7] This is not a universal assumption in the literature. For example, He and Krishnamurthy (2013) assume that returns are equally split between experts and households, so that rationing is required to prevent households from demanding more expert equity than the total supply of expert equity.

Furthermore, as experts and household have different investment opportunities, we postulate two stochastic discount factor (SDF) processes, one for experts and one for households.

$$\frac{d\xi_t}{\xi_t} = -r_t\,dt - \varsigma_t\,dZ_t, \quad \text{and} \quad \frac{d\underline{\xi}_t}{\underline{\xi}_t} = -\underline{r}_t\,dt - \underline{\varsigma}_t\,dZ_t,$$

respectively. Note that since both can trade the risk-free asset the drift of both SDF processes has to be the same.

Step 2: Equilibrium Conditions. Note that since both experts and households can trade the risk-free asset the drift of both SDF processes has to be the same, i.e. $\underline{r}_t = r_t$. Moreover, (3.2) implies the following asset-pricing relationship for capital held by experts:

$$\frac{\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t}{\sigma + \sigma_t^q} = \chi_t\varsigma_t + (1-\chi_t)\underline{\varsigma}_t, \tag{3.10}$$

where χ_t is the inside equity share, i.e. the fraction of risk held by experts.

The required return on capital held by experts depends on the equilibrium capital structure that experts use. If experts require a higher risk premium than households, then $\chi_t = \underline{\chi}$, i.e. experts will issue the maximum equity they can. Thus, we have[8]

$$\chi_t = \underline{\chi} \;\text{ if }\; \varsigma_t > \underline{\varsigma}_t, \quad \text{otherwise} \quad \varsigma_t = \underline{\varsigma}_t.$$

Under this condition, we can replace χ_t with $\underline{\chi}$ in (3.10).

An asset-pricing relationship for capital held by households is

$$\frac{\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r_t}{\sigma + \sigma_t^q} \le \underline{\varsigma}_t, \tag{3.11}$$

with equality if $\psi_t < 1$, i.e. households hold capital in positive amounts. Note that households may choose not to hold any capital, and if so, then the Sharpe ratio they would earn from capital could fall below that required by the asset-pricing relationship.

It is useful to combine (3.10) and (3.11), eliminating μ_t^q and r_t, to obtain

$$\frac{(a-\underline{a})/q_t}{\sigma + \sigma_t^q} \ge \underline{\chi}(\varsigma_t - \underline{\varsigma}_t), \tag{3.12}$$

[8] We can rule out the case that $\varsigma_t < \underline{\varsigma}_t$ and $\chi_t = 1$: experts cannot face lower risk premia than households if households hold zero risk.

with equality if $\psi_t < 1$.

The required risk premia can be tied to the agents' consumption processes via (3.24) in the CRRA case and to the agents' net worth processes in the special logarithmic case. Under the baseline risk-neutrality assumptions of Brunnermeier and Sannikov (2014), $\underline{\varsigma} = 0$ when households are risk-neutral and financially unconstrained - i.e. they can consume negatively.

We will use these conditions to characterize q_t, ψ_t, χ_t, etc. as functions of η_t. Before we do that, though, we must derive an equation for the law of motion of $\eta_t = N_t/(q_t K_t)$.

Step 3: The Law of Motion of η_t. It is convenient to express the laws of motion of the numerator and denominator of η_t by focusing on risks and risk premia. Specifically, the experts' net worth follows

$$\frac{dN_t}{N_t} = r_t\,dt + \underbrace{\frac{\chi_t\psi_t}{\eta_t}(\sigma+\sigma_t^q)}_{\text{risk}}\Big(\underbrace{\varsigma_t}_{\text{risk premium}}\,dt + dZ_t\Big) - \frac{C_t}{N_t}\,dt.$$

To derive the evolution of $q_t K_t$, note that the capital gains rate is the same for both type of agents. Thus, we can just aggregate the individual laws of motion to an aggregate law of motion. After replacing the term $\Phi(\iota_t) - \delta + \mu_t^q - \sigma\sigma_t^q - r_t$ using (3.10), we obtain

$$\frac{d(q_tK_t)}{q_tK_t} = r_t\,dt + (\sigma+\sigma_t^q)\left(\left(\underline{\chi}\varsigma_t + (1-\underline{\chi})\,\underline{\varsigma}_t\right)dt + \,dZ_t\right) - \frac{a-\iota_t}{q_t}\,dt.$$

This is the total return on capital (e.g. that held by experts) minus the dividend yield.

Using the already familiar formula (2.10) for a ratio of two stochastic processes, we have

$$\frac{d\eta_t}{\eta_t} = \mu_t^\eta\,dt + \sigma_t^\eta\,dt = \left(\frac{a-\iota_t}{q_t} - \frac{C_t}{N_t}\right)dt + \frac{\chi_t\psi_t - \eta_t}{\eta_t}(\sigma+\sigma_t^q)\left((\varsigma_t - \sigma - \sigma_t^q)\,dt + \,dZ_t\right) +$$
$$(\sigma+\sigma_t^q)(1-\underline{\chi})(\varsigma_t - \underline{\varsigma}_t)\,dt. \tag{3.13}$$

Step 4: Converting the Equilibrium Conditions and Laws of Motion (3.13) into Equations for $q(\eta)$, $\theta(\eta)$, $\psi(\eta)$, $\chi(\eta)$ etc. The procedure to convert the equilibrium conditions and the law of motion of η_t into numerically solvable equations for $q(\eta)$, $\psi(\eta)$, etc., depends on the underlying assumptions on the agents' preferences. (The log-utility case is the easiest to solve.) In each case, we have to use Ito's Lemma, which allows us to replace terms such as σ_t^q, σ_t^θ, μ_t^q etc. with expressions containing the derivatives of q and θ, in order to arrive at solvable differential equations for these functions in the end.

For example, using Ito's Lemma we can tie the volatility of q_t with the first derivative of $q(\eta)$ as follows

$$\sigma_t^q q(\eta) = q'(\eta) \underbrace{(\chi_t \psi_t - \eta_t)(\sigma + \sigma_t^q)}_{\eta \sigma_t^\eta}. \tag{3.14}$$

Rewriting equation (3.14) yields a closed form solution for the amplification mechanism.

$$\sigma_t^\eta = \frac{\frac{\chi_t \psi_t}{\eta_t} - 1}{1 - [\frac{\chi_t \psi_t}{\eta_t} - 1] \frac{q'(\eta_t)}{q(\eta_t)/\eta_t}} \sigma \tag{3.15}$$

The numerator $\frac{\chi_t \psi_t}{\eta_t} - 1$ captures the leverage ratio of the expert sector. The amplification increases with the leverage ratio, the leverage effect. The denominator captures the "loss spiral". Mathematically, it reflects an infinite geometric series. The impact of the loss spiral increases with the product of the leverage ratio and price elasticity, $\frac{q'}{q/\eta}$. The latter measures "market illiquidity", the percentage price impact due to a percentage decline in η_t. Market illiquidity arises from the technological specialization of capital, measured here by the difference $a - \underline{a}$ between the experts' and households' productivity parameters. Market illiquidity interacts with technological illiquidity, captured by the curvature of $\Phi(\cdot)$.

There are various methods to solve the equilibrium equations. Below, we discuss two methods that have been used in practice. One method involves ordinary differential equations (ODE) - we refer to it as the "shooting method" and illustrate it using the risk-neutral preferences of Brunnermeier and Sannikov (2014). The second method involves partial differential equations, and is reminiscent of value function iteration in discrete time.

3.4 Method 1: The Shooting Method

This method involves converting the equations above into a system of ODEs. Before we dive into this, in order to understand how this can be done, we review a very simple and well-known model to illustrate the gist of what we have to do. The model illustrates the pricing of a perpetual American put.

Example 3. Consider the problem of pricing a perpetual option to abandon an asset for an amount K. Given a risk-free rate of r and volatility σ, if the asset pays no dividends, its value follows a geometric Brownian motion

$$\frac{dV_t}{V_t} = r\,dt + \sigma\,dZ_t \tag{3.16}$$

under the risk-neutral measure.

Under the risk-neutral measure, the expected return of any security must be r. Thus, if the put value P_t follows $dP_t = \mu_t^P P_t\,dt + \sigma_t^P P_t\,dZ_t$, then we must have

$$r = \mu_t^P. \tag{3.17}$$

Suppose we would like to calculate how the put value P_t depends on the value of the assets V_t. Then we face a problem that is completely analogous to the model with financial frictions we described in this section. We have a law of motion of the state variable V_t and a relationship (3.17) that the stochastic evolution of P_t has to satisfy, and we would like to characterize P_t as a function of V_t.

How can we do this? Easy. Using Ito's Lemma

$$\mu_t^P P_t = rV_tP'(V_t) + \frac{1}{2}\sigma^2 V_t^2 P''(V_t),$$

and so (3.17) becomes

$$r = \frac{rVP'(V) + \frac{1}{2}\sigma^2 V^2 P''(V)}{P(V)}. \tag{3.18}$$

If function $P(V)$ satisfies this equation, then the process $P_t = P(V_t)$ will satisfy (3.17). We are able to go from an equation like (3.17) to a differential equation (3.18) by assuming that the value of the put is a *function* of the value of the asset.

We can solve the second-order ordinary differential equation (ODE) (3.18) if we have two boundary conditions. We have $P(V) \to 0$ as $V \to \infty$ since the put becomes worthless if it is never exercised. We also have $P(V) - (K - V) \geq 0$, since $P(V)$ must equal the intrinsic value at the point where the put is exercised.

Our problem is similar: we have an equation for the stochastic law of motion of the state variable (3.13), as well as the equilibrium conditions that processes $q(\eta_t)$, $\psi(\eta_t)$, etc. must satisfy. Certainly, the equations are more complicated than those of the put-pricing problem, and the law of motion of η_t is endogenous. However, the mechanics of solving these

equations is the same - we have to use Ito's Lemma.

Here we illustrate the derivation of an appropriate set of ordinary differential equations, as well as the "shooting" method for solving them, using the risk-neutral model of Brunnermeier and Sannikov (2014). Assume that experts and households are risk neutral, and while experts must consume non-negatively, households can have both positive and negative consumption. Then the required risk premium of households is $\underline{\varsigma}_t = 0$. The required risk premium of experts is $-\sigma_t^\theta$, where θ_t is the marginal utility of the experts' wealth that follows

$$\frac{d\theta_t}{\theta_t} = \mu_t^\theta \, dt + \sigma_t^\theta \, dZ_t.$$

We would like to construct differential equations to solve for the functions $q(\eta)$, $\theta(\eta)$ and $\psi(\eta)$. The equations will be of second order in $q(\eta)$ and $\theta(\eta)$, i.e. we will design a procedure to compute $q''(\eta)$ and $\theta''(\eta)$, as well as $\psi(\eta)$, from η, $q(\eta)$, $q'(\eta)$ and $\theta(\eta)$, $\theta'(\eta)$. Note also that, since households demand no risk premium, i.e. $\underline{\varsigma}_t = 0$, experts will issue the maximum allowed fraction of equity to households, so $\chi_t = \underline{\chi}$ at all times.

In this case $q(\eta)$ is an increasing function that satisfies the boundary condition

$$q(0) = \max_\iota \frac{\underline{a} - \iota}{r - \Phi(\iota) + \delta},$$

the Gordon growth formula for the value of capital when it is permanently managed by households. Any expert can get infinite utility if he can buy capital at the price of $q(0)$, so

$$\lim_{\eta \to 0} \theta(\eta) = \infty. \tag{3.19}$$

Function $\theta(\eta)$ is decreasing: the marginal value of the experts' net worth is declining as η rises, and investment opportunities become less valuable. Experts refrain from consumption whenever $\theta(\eta) > 1$, and consume only at point η^* where $\theta(\eta^*) = 1$, i.e. the marginal value of the experts' net worth is exactly 1. That point becomes the reflecting boundary of the system. That is, the system does not go beyond the reflecting boundary and is rather thrown back. In addition, at the reflecting boundary η^* functions $q(\eta)$ and $\theta(\eta)$ must satisfy

$$q'(\eta^*) = \theta'(\eta^*) = 0.$$

Now to the differential equations. Equation (3.14) implies that

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\underline{\chi}\psi_t - \eta_t)}, \tag{3.20}$$

and by Ito's Lemma,

$$\sigma_t^\theta = \frac{\theta'(\eta)}{\theta(\eta)} \frac{(\underline{\chi}\psi_t - \eta_t)\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\underline{\chi}\psi_t - \eta_t)}. \tag{3.21}$$

Therefore, plugging these expressions into the asset-pricing equation (3.12), we obtain

$$\frac{a - \underline{a}}{q(\eta)} \geq -\underline{\chi}\frac{\theta'(\eta)}{\theta(\eta)} \frac{(\underline{\chi}\psi - \eta)\sigma^2}{\left(1 - \frac{q'(\eta)}{q(\eta)}(\underline{\chi}\psi - \eta)\right)^2}. \tag{3.22}$$

Assuming that $q'(\eta) > 0$ and $\theta'(\eta) < 0$, the right-hand side is increasing from 0 to ∞ as $\underline{\chi}\psi - \eta$ rises from 0 to $q(\eta)/q'(\eta)$. Thus, we have to set $\psi = 1$ whenever it is possible to do so (i.e. $\underline{\chi} - \eta < q(\eta)/q'(\eta)$) and this is consistent with inequality (3.22). Otherwise we determine ψ by solving the quadratic equation (3.22), in which we replace the $\geq$ sign with equality.

After that, we can find σ_t^q from (3.20), σ_t^θ from (3.21), μ_t^η and σ_t^η from (3.13) (where we set $C_t = 0$ since experts consume only at the boundary η^*), μ_t^q from the asset-pricing condition

$$\frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q - r = \underline{\chi}(\sigma + \sigma_t^q)(-\sigma_t^\theta),$$

μ_t^θ from the pricing condition for the risk-free asset

$$\mu_t^\theta = \rho - r,$$

and $q''(\eta)$ as well as $\theta''(\eta)$ from Ito's formula,

$$\mu_t^q q(\eta) = \mu_t^\eta \eta q'(\eta) + \frac{1}{2}(\sigma_t^\eta)^2\eta^2 q''(\eta) \quad \text{and} \quad \mu_t^\theta \theta(\eta) = \mu_t^\eta \eta \theta'(\eta) + \frac{1}{2}(\sigma_t^\eta)^2\eta^2\theta''(\eta).$$

Solving the system of ODEs numerically. We can use an ODE solver in Matlab, such as ode45, to solve the system of equations. We need to perform a search, since our boundary conditions are defined at two endpoints of $[0, \eta^*]$, and we also need to deal with a singularity at $\eta = 0$. The following algorithm performs an appropriate search and deals with the singularity issue, effectively, by solving the system of equations with the boundary

condition $\theta(0) = M$, for a large constant M, instead of (3.19):[9]

Algorithm. *Set*

$$q(0) = \max_{\iota} \frac{\underline{a} - \iota}{r - \Phi(\iota) + \underline{\delta}}, \quad \theta(0) = 1 \quad \text{and} \quad \theta'(0) = -10^{10}.$$

Perform the following procedure to find an appropriate boundary condition $q'(0)$. *Set* $q_L = 0$ *and* $q_H = 10^{15}$. *Repeat the following loop 50 times. Guess* $q'(0) = (q_L + q_H)/2$. *Use Matlab function ode45 to solve for* $q(\eta)$ *and* $\theta(\eta)$ *on the interval* $[0, ?)$ *until one of the following events is triggered, either (1)* $q(\eta)$ *reaches the upper bound*

$$q_{\max} = \max_{\iota} \frac{a - \iota}{r - \Phi(\iota) + \delta},$$

(2) the slope $\theta'(\eta)$ *reaches 0 or (3) the slope* $q'(\eta)$ *reaches 0. If integration has terminated for reason (3), we need to increase the initial guess of* $q'(0)$ *by setting* $q_L = q'(0)$. *Otherwise, we decrease the initial guess of* $q'(0)$, *by setting* $q_H = q'(0)$.

At the end, $\theta'(0)$ *and* $q'(0)$ *reach 0 at about the same point, which we denote by* η^*. *Divide the entire function* θ *by* $\theta(\eta^*)$.[10] *Then plot the solutions.*

Properties of the Solution. Let us interpret the solution of the risk-neutral model. Point η^* plays the role of the steady state of our system. The drift of η_t is positive everywhere on the interval $[0, \eta^*)$, because the expert sector, which is more productive than the household sector, is growing in expectation. Thus, the system is pushed towards η^* by the drift.

It turns out that the steady state is relatively stable, because volatility is low near η^*. To see this, recall that the amount of endogenous risk in asset prices, from (3.14), is given by

$$\sigma_t^q = \frac{q'(\eta)}{q(\eta)} \frac{(\underline{\chi}\psi_t - \eta_t)\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\underline{\chi}\psi_t - \eta_t)}.$$

From the boundary conditions, $q'(\eta^*) = 0$, so there is no endogenous risk near η^*.

However, below η^*, endogenous risk increases as $q'(\eta)$ becomes larger. As prices react to shocks, fundamental risk becomes amplified. As we see from the expression for σ_t^q, this amplification effect is nonlinear, since $q'(\eta)$ enters not only the numerator, but also the

[9] Footnote 10 below explains why it actually does not to matter what exact value one sets for $\theta(0)$.

[10] We can do this because whenever functions θ and q satisfy our system of equation, so do functions $\Theta\theta$ and q for any constant Θ. Because of that, also, it is immaterial what we set $\theta(0)$ to 1.

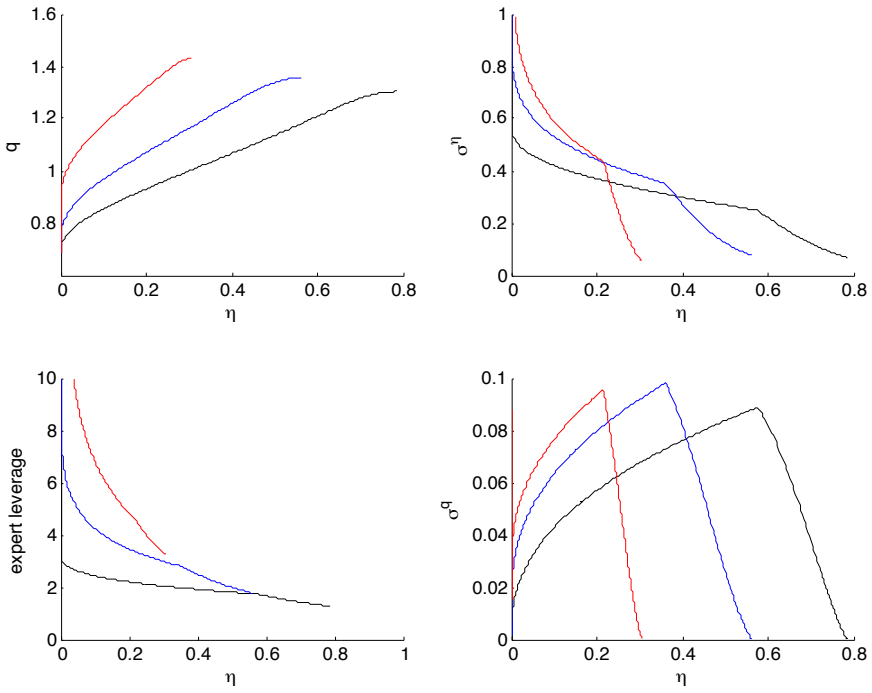

Figure 3: Equilibrium with $\sigma = 25\%$ (black), 10% (blue) and 2.5% (red).

denominator. This happens due to the feedback effect: an initial shock causes η_t to drop, which leads to a drop in q_t, which hurts experts who are holding capital and leads to a further decrease in η_t, and so on.

Of course, far in the depressed region the volatility of η_t, $\sigma_t^\eta \eta_t$, becomes low again in this model. This leads to a bimodal stationary distribution of η_t in equilibrium.[11]

Volatility paradox refers to the phenomenon that systemic risk can build up in quiet environments. We can illustrate this phenomenon through comparative statics on σ or the degree of the experts' equity constraint $\underline{\chi}$. One may guess that the system becomes much more stable as σ or $\underline{\chi}$ decline.

This is not the case, as illustrated in Figure 3 for parameters $\rho = 6\%$, $r = 5\%$, $a = 11\%$, $\underline{a} = 5\%$, $\delta = 3\%$ and an investment function of the form $\Phi(\iota) = \frac{1}{\kappa}(\sqrt{1 + 2\kappa\iota} - 1)$, $\kappa = 10$, $\underline{\chi} = 1$ and various values of σ. (The investment technology in this example has quadratic adjustment costs: an investment of $\Phi + \kappa\Phi^2/2$ generates new capital at rate Φ.)

The volatility paradox shows itself in a number of metrics. As exogenous risk declines,

[11] One can prove that the stationary distribution is bimodal analytically by analyzing the asymptotic properties of the solutions near $\eta = 0$ and using the Kolmogorov forward equations that characterize the stationary density - see Brunnermeier and Sannikov (2014) for details.

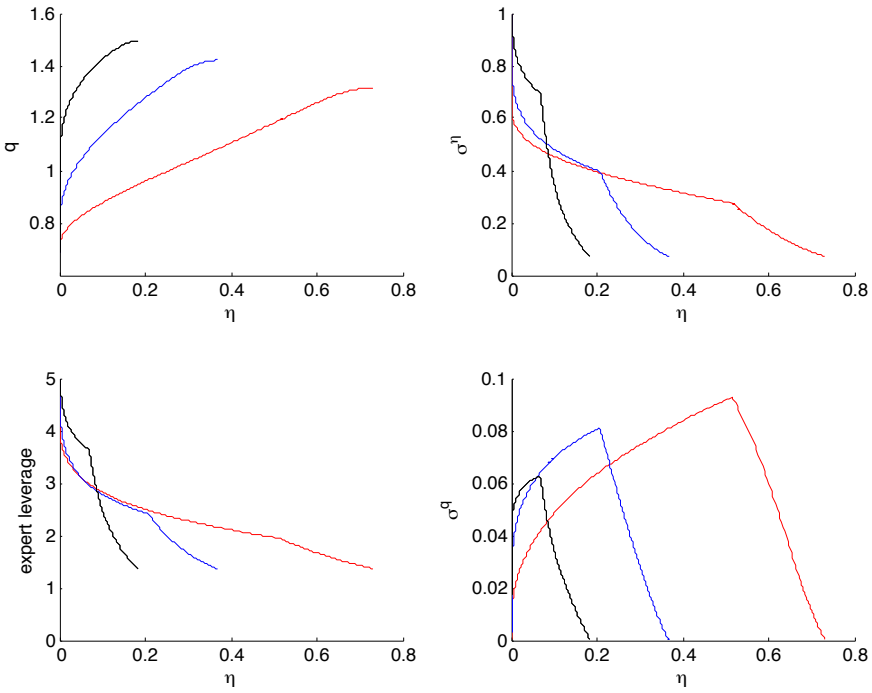

Figure 4: Equilibrium with $\underline{\chi} = 1$ (red), 0.5 (blue) and 0.25 (black).

- maximal endogenous risk σ_t^q may increase (as σ drops from 25% to 10% in Figure 3)
- the volatility σ_t^η near $\eta = 0$ rises (and this result can be proved analytically)
- from the steady state η^* it takes less time for volatility $\sigma + \sigma_t^q$ to double
- from the steady state, it may take less time to reach the peak of the crisis η^ψ, where experts start selling capital to households.[12]

Figure 4 takes the same parameters and $\sigma = 20\%$, but varies $\underline{\chi}$. As $\underline{\chi}$ falls, expert net worth at the steady state η^* drops significantly, and the volatility σ_t^η in the crisis regime rises.

3.5 Method 2: The Iterative Method

Here we describe the iterative method of finding the equilibrium, by solving a system of partial differential equations back in time away from a terminal condition. Specifically, imagine

[12] As σ declines, the system spends less time in the crisis region, so some measures of stability improve, but the amount of time spent in crisis does not converge to 0 as $\sigma \to 0$.

an economy that lasts for a finite time horizon $[0, T]$. Given a set of terminal conditions at time T, we would like to compute the equilibrium over the time horizon $[0, T]$. The iterative method is based on the premise that as we let $T \to \infty$, behavior at time 0 should converge to the equilibrium of the infinite-horizon economy. Computation uses the equilibrium conditions that express the drifts of various processes, and uses those drifts to obtain time derivatives for the corresponding functions of the state space. The iterative method is analogous to value function iteration in discrete time.

We illustrate the method here based on a model with CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Equilibrium conditions (3.10) and (3.8) provide two equations that express the drift of the price q_t, as well as the drifts of aggregate consumption of experts C_t and households $\underline{C}_t$. We also have another asset-pricing condition (3.12), which does not contain any drift terms. In the end we have three functions but only two drift conditions. As a result, the time dimension of our computation involves only two functions - the value functions of experts and households - and the third function, the price, is found for each time point through a separate procedure.[13]

Our procedure is literally the analogue of value function iteration (but with multiple agents affecting the evolving stochastic state). It is convenient to derive directly the equations that value functions must satisfy. The value functions of experts and households can be presented in the form

$$v_t \frac{K_t^{1-\gamma}}{1-\gamma} = \frac{v_t}{(\eta_t q_t)^{1-\gamma}} \frac{N_t^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \underline{v}_t \frac{K_t^{1-\gamma}}{1-\gamma}.$$

Since the marginal utilities of consumption and wealth must be the same, we have

$$C_t^{-\gamma} = \frac{v_t}{(\eta_t q_t)^{1-\gamma}} N_t^{-\gamma} = \frac{v_t}{\eta_t q_t} K_t^{-\gamma} \quad \Rightarrow \quad C_t = N_t \frac{(\eta_t q_t)^{1/\gamma - 1}}{v_t^{1/\gamma}} = K_t \frac{(\eta_t q_t)^{1/\gamma}}{v_t^{1/\gamma}}. \tag{3.23}$$

[13] If we used the shooting method to find the equilibrium with CRRA utilities, we would have a system of second-order differential equations for the value functions, and a first-order differential equation for the price.

Hence, the risk premia of households and experts are given by

$$\varsigma_t = \gamma\sigma_t^C = -\sigma_t^v + \sigma_t^\eta + \sigma_t^q + \gamma\sigma \quad \text{and} \quad \underline{\varsigma}_t = \gamma\sigma_t^{\underline{C}} = -\sigma_t^{\underline{v}} - \frac{\eta\sigma_t^\eta}{1-\eta} + \sigma_t^q + \gamma\sigma. \tag{3.24}$$

Since

$$\underbrace{\int_0^t e^{-\rho s}\frac{C_s^{1-\gamma}}{1-\gamma}\,ds}_{\text{utility flow}} + e^{-\rho t}\underbrace{v_t\frac{K_t^{1-\gamma}}{1-\gamma}}_{\text{continuation utility}}$$

is by standard dynamic programming arguments a martingale and

$$\frac{d(K_t^{1-\gamma})}{K_t^{1-\gamma}} = \left((1-\gamma)(\Phi(\iota_t)-\delta) - \frac{\gamma(1-\gamma)}{2}\sigma^2\right)dt + (1-\gamma)\sigma\,dZ_t,$$

we have

$$\frac{C_t^{1-\gamma}}{1-\gamma} - \rho v_t\frac{K_t^{1-\gamma}}{1-\gamma} + v_t\frac{K_t^{1-\gamma}}{1-\gamma}\left(\mu_t^v + (1-\gamma)(\Phi(\iota_t)-\delta) - \frac{\gamma(1-\gamma)}{2}\sigma^2 + \sigma_t^v(1-\gamma)\sigma\right) = 0.$$

Using (3.23), we obtain

$$\mu_t^v = \rho - \frac{(\eta_t q_t)^{1/\gamma-1}}{v_t^{1/\gamma}} - (1-\gamma)(\Phi(\iota_t)-\delta) + \frac{\gamma(1-\gamma)}{2}\sigma^2 - \sigma_t^v(1-\gamma)\sigma. \tag{3.25}$$

Likewise,

$$\mu_t^{\underline{v}} = \underline{\rho} - \frac{((1-\eta_t)q_t)^{1/\gamma-1}}{\underline{v}_t^{1/\gamma}} - (1-\gamma)(\Phi(\iota_t)-\delta) + \frac{\gamma(1-\gamma)}{2}\sigma^2 - \sigma_t^{\underline{v}}(1-\gamma)\sigma. \tag{3.26}$$

Given μ_t^v and $\mu_t^{\underline{v}}$, we obtain partial differential equations for the functions $v(\eta,t)$ and $\underline{v}(\eta,t)$ using Ito's Lemma, and they are as follows:

$$\mu_t^v v(\eta,t) = \mu_t^\eta\eta\, v_\eta(\eta,t) + \frac{(\sigma_t^\eta\eta)^2}{2}v_{\eta\eta}(\eta,t) + v_t(\eta,t) \quad \text{and} \tag{3.27}$$

$$\mu_t^{\underline{v}}\,\underline{v}(\eta,t) = \mu_t^\eta\eta\,\underline{v}_\eta(\eta,t) + \frac{(\sigma_t^\eta\eta)^2}{2}\underline{v}_{\eta\eta}(\eta,t) + \underline{v}_t(\eta,t). \tag{3.28}$$

Description of the Procedure

Below we outline the procedure of how we solve for the equilibrium using equations (3.27) and (3.28). There are three parts.

- The terminal conditions $v(\eta, T)$ and $\underline{v}(\eta, T)$
- The static step: finding capital price $q(\eta)$, allocations $\psi(\eta)$ and $\chi(\eta)$, volatilities and drifts at a given time point t given the value functions $v(\eta, t)$ and $\underline{v}(\eta, t)$, and
- The time step: finding $v(\eta, t - \Delta t)$ and $\underline{v}(\eta, t - \Delta t)$ from prices, allocations, volatilities and drifts at time t.

The terminal conditions. Our terminal conditions specify the utilities of the representative expert and household, as functions of the experts' wealth share η_t. We have not performed a detailed theoretical study of acceptable terminal conditions, but in practice any reasonable guess works well for a wide range of parameters.

For example, if we set $q_T = 1$ and $C_T/K_T = a\eta_T$, then (3.23) implies that

$$v_T = \eta_T \left(a\eta_T\right)^{-\gamma} \quad \text{and} \quad \underline{v}_T = (1 - \eta_T) \left(a(1 - \eta_T)\right)^{-\gamma}. \tag{3.29}$$

The static step. Suppose we know value functions through $v(\eta, t)$ and $\underline{v}(\eta, t)$. Let us describe how we can compute the price q_t and characterize equilibrium dynamics at time t.

There are three regions. When η is close enough to 0, then the experts' risk premia are so much higher than those of households that $\psi_t < 1$, i.e. households hold capital, and equation (3.12) holds. In this region experts issue the maximal allowed equity share to households, so $\chi_t = \underline{\chi}$, since the households' risk premia are lower. In the middle region, $\psi_t = 1$, i.e. only experts hold capital, but the experts' risk premia are still higher than those of households so $\chi_t = \underline{\chi}$. Finally, when $\eta \geq \underline{\chi}$, the capital is allocated efficiently to experts (i.e. $\psi_t = 1$) and risk can shared perfectly between households and experts by setting $\chi_t = \eta_t$. In the last region, (3.15) implies that $\sigma^\eta = 0$, so there is no endogenous risk, and risk premia of experts and households are both equal to $\varsigma_t = \underline{\varsigma}_t = \gamma\sigma$ by (3.24).

In the region where $\psi_t < 1$ we solve for $q(\eta)$, $\psi(\eta)$ and $\sigma + \sigma_t^q$ from a system of the following three equations, which ultimately gives us a first-order ODE in $q(\eta)$. We obtain the first by combining (3.12) and (3.24) together with evolution of η equation (3.13), we

have

$$\frac{a-\underline{a}}{q_t} = \underline{\chi}\underbrace{\left(\frac{\underline{v}'(\eta)}{\underline{v}(\eta)} - \frac{v'(\eta)}{v(\eta)} + \frac{1}{\eta(1-\eta)}\right)(\underline{\chi}\psi_t - \eta)(\sigma+\sigma_t^q)^2}_{\left(\sigma_t^{\underline{v}} - \sigma_t^v + \frac{\sigma_t^\eta}{1-\eta}\right)(\sigma+\sigma_t^q)}. \tag{3.30}$$

The second we obtain from (3.14) and Ito's Lemma,

$$(\sigma+\sigma^q)\left(1 - (\underline{\chi}\psi - \eta)\frac{q'(\eta)}{q(\eta)}\right) = \sigma. \tag{3.31}$$

Finally, from (3.23) and an analogous condition for households, the market-clearing condition for output is

$$\underbrace{\frac{(\eta_t q_t)^{1/\gamma}}{v_t^{1/\gamma}} + \frac{((1-\eta_t)q_t)^{1/\gamma}}{\underline{v}_t^{1/\gamma}}}_{(C_t+\underline{C}_t)/K_t} = a\psi + \underline{a}(1-\psi) - \iota(q(\eta)). \tag{3.32}$$

Once ψ_t reaches 1, condition (3.30) is no longer relevant. From then on, we set $\psi_t = 1$, find $q(\eta)$ from (3.32) and $\sigma + \sigma_t^q$ from (3.31). Once η_t reaches $\underline{\chi}$, we enter the last region. There we set $\psi_t = 1$, $\chi_t = \eta_t$, compute $q(\eta)$ from (3.32) and set $\sigma_t^q = 0$.

Once we know function $q(\eta)$ in all three regions, we can find the volatility of η_t from (3.13) and the volatilities of v_t and $\underline{v}_t$ from Ito's Lemma, i.e.

$$\sigma_t^\eta = \frac{\chi_t\psi_t - \eta_t}{\eta_t}(\sigma+\sigma_t^q), \quad \sigma_t^v = \frac{v'(\eta)}{v(\eta)}\sigma_t^\eta\eta, \quad \text{and} \quad \sigma_t^{\underline{v}} = \frac{\underline{v}'(\eta)}{\underline{v}(\eta)}\sigma_t^\eta\eta. \tag{3.33}$$

We find the required risk premia ς_t and $\underline{\varsigma}_t$ from (3.24) and the drift of η_t from (3.13), i.e.

$$\mu_t^\eta = \left(\frac{a-\iota_t}{q_t} - \frac{(\eta_t q_t)^{1/\gamma-1}}{v_t^{1/\gamma}}\right) + \sigma_t^\eta(\varsigma_t - \sigma - \sigma_t^q) - (\sigma+\sigma_t^q)(1-\underline{\chi})\left(\varsigma_t - \underline{\varsigma}_t\right).$$

Finally, we solve for the drifts of v_t and $\underline{v}_t$ from (3.25).

The time step. Once we have all characteristics of the equilibrium at a given time point t, we can solve for the value functions at an earlier time step $t - \Delta t$ from equations (3.27) and (3.28). These are parabolic equations, which can be solved using either explicit or implicit methods.

Summary. *Set terminal conditions for value functions $v(\eta, T)$ and $\underline{v}(\eta, T)$ according to (3.29) on a grid over η. Divide the interval $[0, T]$ into small subintervals. Going backwards*

in time, for each subinterval $[t-\Delta t, t]$ perform the static step and then the time step. That is, from value functions $v(\eta,t)$ and $\underline{v}(\eta,t)$ find the drift and volatility of η as well as the drifts of v and $\underline{v}$ using the following procedure (static step). Start from an initial condition near $(\eta=0,\psi=0)$ (perturb the condition to avoid division by 0). Solve (3.30), (3.31) and (3.32) (as a first-order ordinary differential equation for $q(\eta)$) until ψ reaches 1. Then set $\psi=1$ and use (3.32) to find $q(\eta)$ and (3.31) to find σ^q. Throughout, use $\chi_t=\max(\underline{\chi},\eta)$. With functions (of η) q, σ^q, ψ and χ obtained in this way, compute volatilities from (3.33), ς_t and $\underline{\varsigma}_t$ from (3.24), μ_t^η from (3.13) and the drifts of v_t and $\underline{v}_t$ from (3.25). Then (this is the time step) solve the partial differential equations (3.27) and (3.28) for v and $\underline{v}$ backward in time over the interval $[t-\Delta t,t]$, using fixed functions μ_t^v, $\mu_t^{\underline{v}}$, μ_t^η and σ_t^η of η computed by the static step. Continue until time 0. We get convergence when T is sufficiently large.

Remark. The static step alone is sufficient to solve for the equilibrium prices, allocations and dynamics in a model with logarithmic utility (i.e. $\gamma=1$), since in this case we know that $(C_t+\underline{C}_t)/(q_tK_t)=\rho\eta+\underline{\rho}(1-\eta)$ and expert and household risk premia are $\varsigma_t=\sigma_t^N=\chi_t\psi_t/\eta_t(\sigma+\sigma_t^q)$ and $\underline{\varsigma}_t=(1-\chi_t\psi_t)/(1-\eta_t)(\sigma+\sigma_t^q)$. Hence, equations (3.30) and (3.32) become

$$\frac{a-\underline{a}}{q_t}=\underline{\chi}\frac{\underline{\chi}\psi_t-\eta}{\eta(1-\eta)}(\sigma+\sigma_t^q)^2 \quad \text{and} \quad (\rho\eta+\underline{\rho}(1-\eta))q_t=a\psi+\underline{a}(1-\psi)-\iota(q(\eta)). \tag{3.34}$$

Equation (3.31) remains the same.

For logarithmic utility, however, we do not immediately obtain the agents' value functions. Those can be found using an extra step.

3.6 Examples of Solutions: CRRA Utility

In this section, we illustrate solutions generated by our code, using the iterative method, and what we learn from them. We use baseline parameters $\rho=6\%$, $r=5\%$, $a=11\%$, $\underline{a}=3\%$, $\delta=5\%$, $\sigma=10\%$, $\underline{\chi}=0.5$, $\gamma=2$ and an investment function of the form $\Phi(\iota)=\log(\kappa\iota+1)/\kappa$ with $\kappa=10$. We then study how several parameters, specifically $\underline{a}$, σ, $\underline{\chi}$ and γ affect the equilibrium.

Figure 5 illustrates the equilibrium for the baseline set of parameters. Notice that capital price q_t has a kink - the kink separates the crisis region near $\eta=0$ where $\psi_t<1$, i.e. households hold some capital, and the normal region where experts hold all capital in the economy.

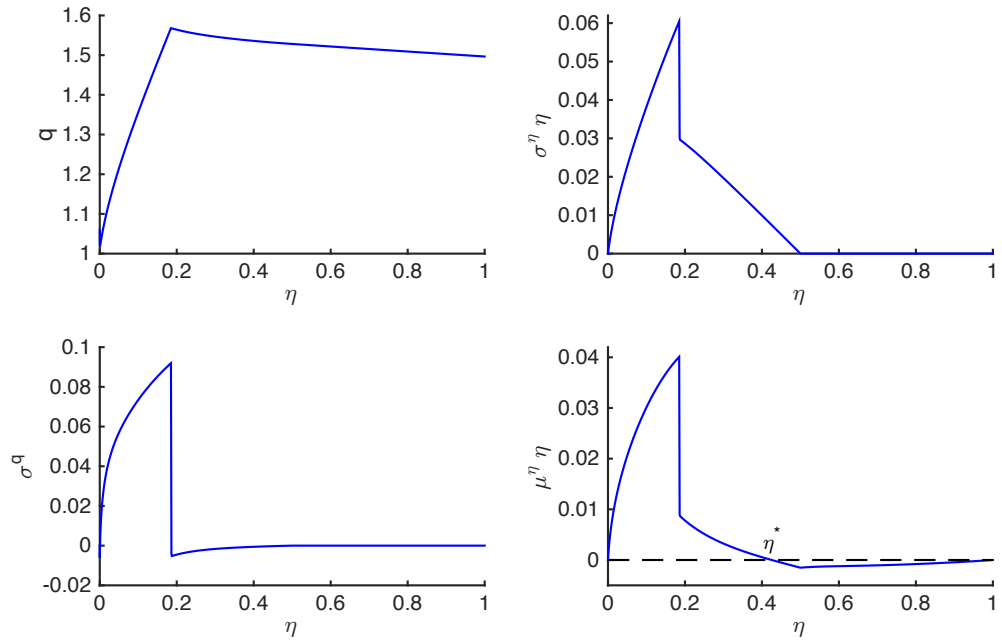

Figure 5: Equilibrium for the baseline set of parameters.

Here, point η^* where the drift of η_t becomes 0 plays the role of a steady state of the system. In the absence of shocks, the system stays still at the steady state and in response to small shocks, drift pushes the system back to the steady state. Moving away from the crisis regime, at η^* risk premia decline sufficiently so that the experts' earnings are exactly offset by their slightly higher consumption rates.

Above $\eta = \underline{\chi} = 0.5$ is the region of perfect risk sharing, where the volatility of η is zero. Since the drift in that region is negative, the system never ends up there (and if the initial condition is $\eta_0 > \underline{\chi}$, then η_t drifts deterministically down to $\underline{\chi}$).

Figure 6 shows the effect of σ on the equilibrium dynamics. We bound the horizontal axis at $\eta = \underline{\chi} = 0.5$, since the system never enters the region $\eta > \underline{\chi}$. The steady state η^* declines as σ falls, as risk premia decline in the normal regime, until η^* coincides with the boundary of the crisis region for low σ (this happens for $\sigma = 0.01$ in Figure 6). We also observe the volatility paradox: as σ declines, endogenous risk σ_t^q does not have to fall, and may even rise.

But what happens as $\sigma \to 0$? Does endogenous risk disappear altogether, and does the solution converge to first best? It turns out that no: in the limit as $\sigma \to 0$, the boundary of the crisis region η^ψ converges not to 0 but to a finite number.

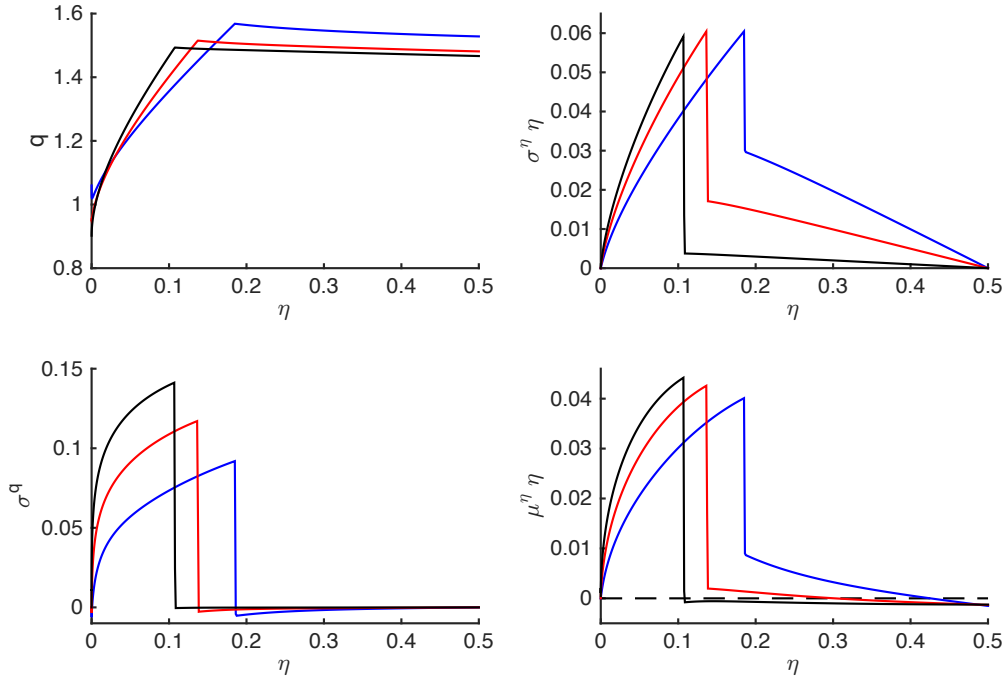

Figure 6: Equilibrium for $\sigma = .1$ (blue), .05 (red) and .01 (black).

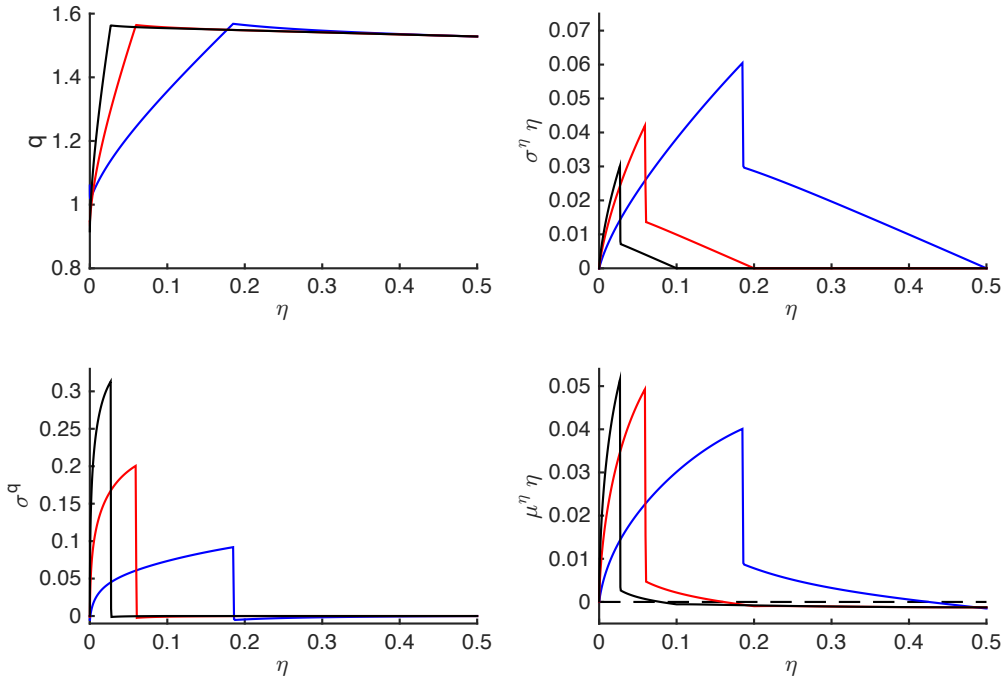

Figure 7: Equilibrium for $\underline{\chi} = .5$ (blue), .2 (red) and .1 (black).

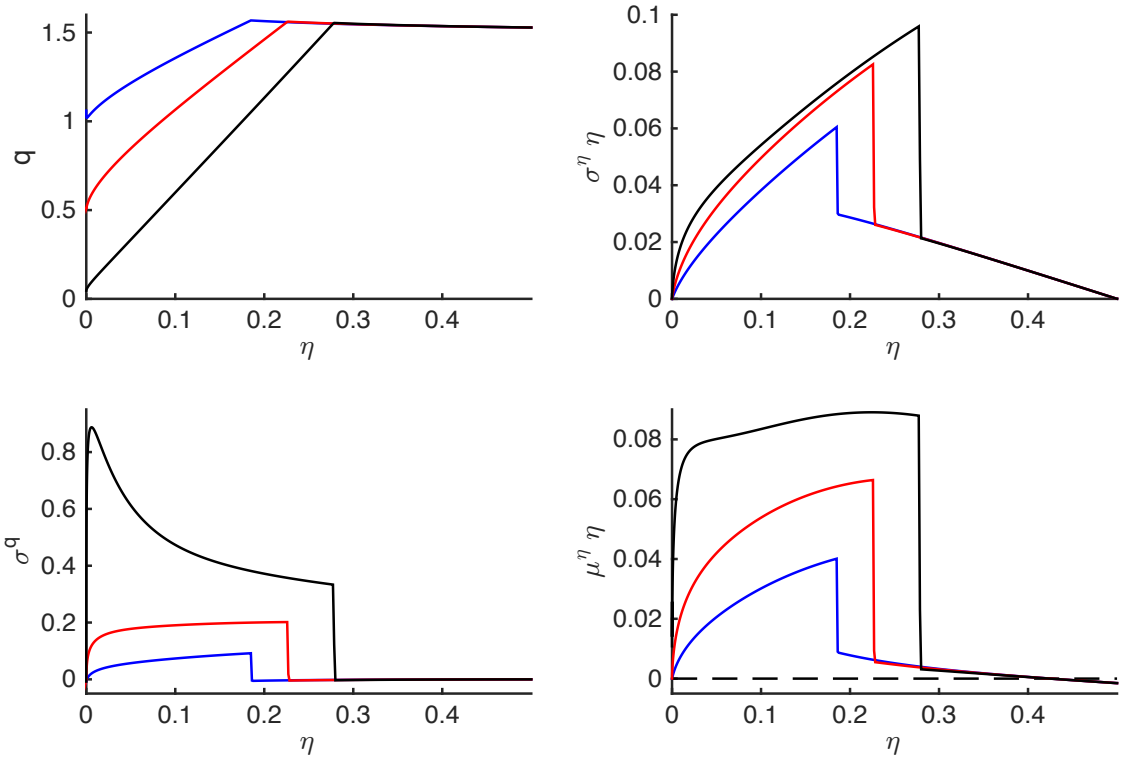

Figure 8: Equilibrium for $\underline{a} = .03$ (blue), -.03 (red) and -.09 (black).

Likewise, what happens if financial frictions become relaxed, and experts are able to hold capital while retaining a smaller portion of risk? It is tempting to conjecture that as financial frictions become relaxed, the system becomes more stable. Yet, as the bottom left panel of Figure 7 demonstrates, endogenous risk σ_t^q rises sharply as $\underline{\chi}$ declines.[14]

It turns out that a crucial parameter that affects system stability is the household productivity parameter $\underline{a}$. The level of endogenous risk in crises depends strongly on the illiquidity of capital - the difference between parameter a and $\underline{a}$ that determines how much less households value capital, in the event that they have to buy it, relative to experts. Figure 8 illustrates the equilibrium for several values of $\underline{a}$. Note that endogenous risk in crises rises sharply as $\underline{a}$ drops. However, the dynamics in the normal regime and the level of η^* have extremely low sensitivity to $\underline{a}$ - only dynamics in the crisis regime are extremely sensitive. This is a surprise. While expert leverage responds endogenously to fundamental risk σ in the normal regime it does not respond strongly to endogenous tail risk. In fact, for logarithmic

[14]Of course, there is a discontinuity at both $\sigma = 0$ and $\underline{\chi} = 0$. As financial frictions disappear altogether, the crisis region disappears.

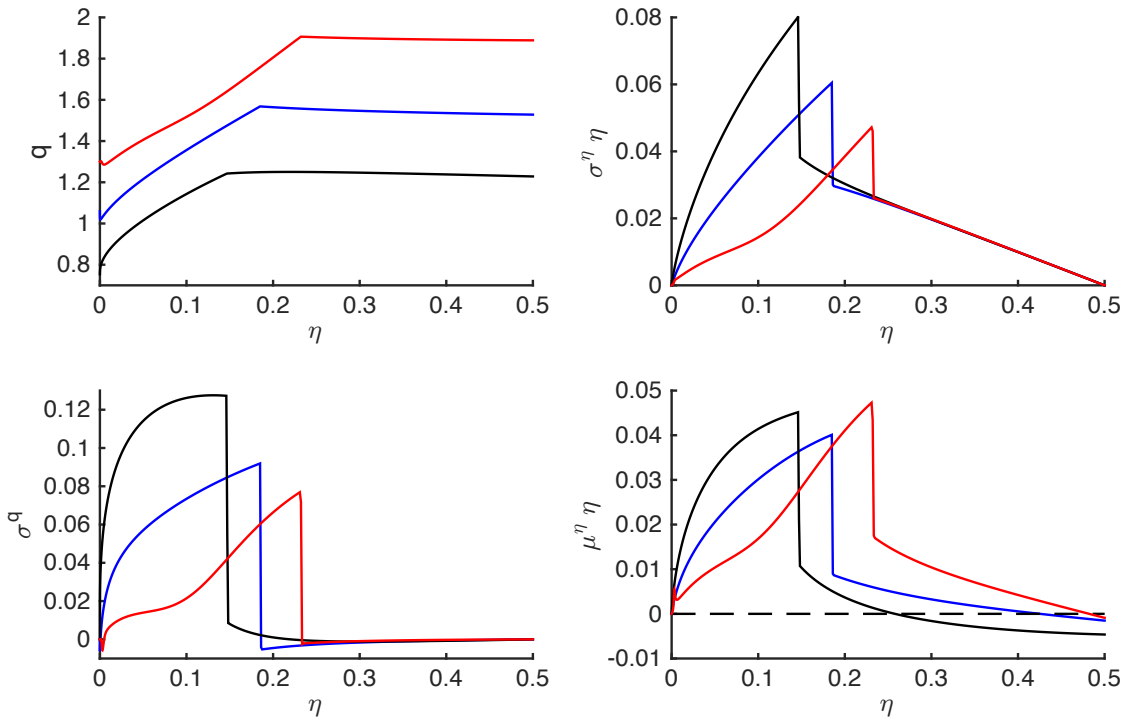

Figure 9: Equilibrium for $\gamma = 2$ (blue), 5 (red) and 0.5 (black).

utility it is possible to prove analytically that the dynamics in the normal regime do not depend on $\underline{a}$ at all (but here we illustrate the dynamics for $\gamma = 2$).

Finally, let us consider risk aversion γ. There are several effects. Lower risk aversion leads to a smaller crisis region (but with greater endogenous risk), and lower steady state η^* as the risk premia become lower. In this example, higher risk aversion leads to a higher price of capital, as risk creates a precautionary savings demand.

4 A Simple Monetary Model

So far we focused on a real model with a single risky asset, physical capital and a risk-free asset. Now, building on Brunnermeier and Sannikov's (2015a) "I Theory of Money" we introduce instead of the (real) risk-free asset, another asset, money. In general, money has three roles: it is a unit of account, it facilitates transactions, and it serves as a store of value (safe asset). Here, we focus on its role as a store of value, which arises in our setting due to incomplete markets frictions. Unlike in New Keynesian models, which focus on the role of

money as a unit of account and rely on price and wage rigidities as the key frictions, prices are fully flexible in our model.

In this section focuses on the following:

1. Money can have positive value despite the fact that it never pays any dividend. That is, money is a bubble.

2. Money helps agents to share risks in an economy that is plagued by financial frictions. Hence, having a nominal store of value instead of a real short-term risk-free bond alters the equilibrium risk dynamics.

3. The "*Paradox of Prudence*" coined in Brunnermeier and Sannikov (2015a) arises. Experts hold money to self-insure against idiosyncratic shocks, an action which is micro-prudent but macro-*im*prudent. By selling capital to achieve a greater portfolio weight on money, experts depress aggregate investment and growth, leading to lower returns on all assets (including money). The Paradox of Prudence is in the risk space what Keynes' Paradox of Thrift is for the consumption-savings decision. The Paradox of Thrift describes how each person's attempt to save more paradoxically lowers overall aggregate savings.

4.1 Model with Idiosyncratic Capital Risk and Money

Let us return to the Basak-Cuoco model of Section 2 with experts holding physical capital and households who cannot, i.e. $\underline{a} = -\infty$. We introduce the following two modifications: (i) Capital has in addition to aggregate risk also idiosyncratic risk. (ii) There is no risk-free asset, but there is money in fixed supply. Agents can long and short it and want to hold it to self-insure against idiosyncratic risk.

More formally, we assume as before that each expert operates a linear production technology, ak_t, with productivity a, but now they also face idiosyncratic risk $\tilde{\sigma} d\tilde{Z}_t$ in addition to aggregate risk σdZ_t. That is a single expert's capital k_t evolves according to

$$dk_t/k_t = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t.$$

The shock dZ_t is the same for the whole economy, while the shock $d\tilde{Z}_t$ is expert-specific and orthogonal to dZ_t. Idiosyncratic shocks cancel out in the aggregate.

Since idiosyncratic risk is uninsurable due to markets incompleteness, experts also want to hold money. Money is an infinitely divisible asset in fixed supply, which can be traded

without frictions. Since money does not pay off any dividends it has value in equilibrium only because agents want to self-insure against idiosyncratic shocks to their capital holdings. In other words, money is a bubble, like in Samuelson (1958) and Bewley (1980). Unlike in Bewley (1980), our idiosyncratic shocks are not endowment shocks, but investment shocks like in Angeletos (2007). We assume that idiosyncratic risk of the dividend-paying capital is large enough, $\tilde{\sigma} > \sqrt{\rho}$, so that money, which does not pay dividends, still has value in equilibrium. This is unlike Diamond (1965) who introduces physical capital in Samuelson's OLG model and Aiyagari (1994) who introduces capital in Bewley's incomplete markets setting. In those models, the presence of capital crowds out money as a store of value.[15]

Experts can invest in (outside) money and capital, while households like in Section 2 only hold money. We also assume for simplicity that all agents have logarithmic utility with time preference rate ρ.[16]

As before let us follow our four step approach to solve the model.

4.2 The 4-Step Approach

Step 1: Postulate Price and SDF Processes. In this monetary setting we now have to postulate not only a process for the price of capital, but also for the "real price" of money. We denote (without loss of generality) the value of the total money stock in terms of the numeraire (the consumption good) by $p_t K_t$. We normalize the total value of the money stock by K_t to emphasize that, everything else being equal, the value of money should be proportional to the size of the economy.

$$\begin{aligned} \frac{dq_t}{q_t} &= \mu_t^q dt + \sigma_t^q dZ_t, \\ \frac{dp_t}{p_t} &= \mu_t^p dt + \sigma_t^p dZ_t, \end{aligned}$$

In addition, like in Section 3 we postulate the processes for individual experts' and households' stochastic discount factors:

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \varsigma_t dZ_t - \tilde{\varsigma}_t d\tilde{Z}_t \quad \text{and} \quad \frac{d\underline{\xi}_t}{\underline{\xi}_t} = -\underline{r}_t dt - \underline{\varsigma}_t dZ_t,$$

[15] We assume that money is intrinsically worthless, and so along with the equilibrium in which money has value, there is also an equilibrium in which money has no value. However, in a perturbation of the model, in which agents get small utility from holding money (e.g. because money facilitates transactions), only the equilibrium with full value of money survives.

[16] Solving this model with CRRA models using the results on page 22 in Section 3.1 is a worthwhile exercise.

where r_t and $\underline{r}_t$ are the (real) shadow risk-free interest rates of experts and households, respectively. Note that shadow risk free rates need not be identical, since no real risk-free asset is traded. Note also that experts require a risk premium not only for the aggregate risk ς_t but also for the idiosyncratic risk they have to bear $\tilde{\varsigma}_t$.

We will show that there exists an equilibrium in which the wealth share η_t evolves deterministically and so do the prices q_t and p_t. Hence, for simplicity we set $\sigma_t^q = \sigma_t^p = 0$. Under this conjecture the return on physical capital accruing to experts is

$$dr_t^k = \frac{a - \iota_t}{q_t} dt + (\Phi(\iota_t) - \delta + \mu_t^q) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t$$

and world stock of money $p_t K_t$ earns the (real) return of

$$dr_t^M = (\Phi(\iota_t) - \delta + \mu_t^p)\, dt + \sigma dZ_t,$$

where ι_t is the investment rate in global capital.

Step 2: Equilibrium Conditions. First, note that the optimal investment rate is determined by q_t through $\Phi'(\iota_t) = 1/q_t$. Second, the optimal consumption rate of all agents is simply ρ times their net worth, since the utility of all agents is logarithmic with time preference rate ρ. Hence, aggregate demand for the consumption good is $\rho(q_t + p_t)K_t$. Given total supply of consumption goods after investing, we have the following goods market equilibrium condition:

$$\rho(q_t + p_t)K_t = (a - \iota)K_t.$$

Next, we solve the experts' and households' portfolio problems. Notice that, given the returns dr_t^M and dr_t^k on capital and money, the only two assets traded in this economy, all agents have exposure $\sigma\, dZ_t$ to aggregate risk. At the same time, experts also have exposure $x_t \tilde{\sigma}\, d\tilde{Z}_t$ to their individual idiosyncratic shocks, where x_t is the experts' portfolio weight on capital. Hence, the required risk premia of these log-utility agents are

$$\varsigma_t = \underline{\varsigma}_t = \sigma \quad \text{and} \quad \tilde{\varsigma}_t = x_t \tilde{\sigma}.$$

The experts' and households' asset pricing equations for money, respectively, are

$$\frac{E_t[dr_t^M]}{dt} - r_t = \frac{E_t[dr_t^M]}{dt} - \underline{r}_t = \underbrace{\sigma^2}_{=\varsigma_t \sigma = \underline{\varsigma}_t \sigma}.$$

Thus, $r_t = \underline{r}_t$: even though there is no risk-free real asset in this economy, both agent types would agree on a single real risk-free real interest rate.

The experts' asset pricing equation for physical capital is

$$\frac{E_t[dr_t^k]}{dt} - r_t = \varsigma_t \sigma + \tilde{\varsigma}_t \tilde{\sigma},$$

reflecting the fact that experts are also exposed to idiosyncratic risk for which they earn an extra risk premium. Hence,

$$\frac{E_t[dr_t^k]}{dt} - \frac{E_t[dr_t^M]}{dt} = x_t \tilde{\sigma}^2. \tag{4.1}$$

Capital market clearing implies that

$$x_t = \frac{q_t K_t}{\eta_t (p_t + q_t) K_t} = \frac{1}{\eta_t} \frac{q_t}{p_t + q_t}, \tag{4.2}$$

Step 3: Evolution of η. Experts' aggregate net worth N_t evolves according to

$$\frac{dN_t}{N_t} = r_t + \sigma(\underbrace{\sigma}_{\varsigma_t} dt + dZ_t) + x_t \tilde{\sigma} \tilde{\varsigma}_t \, dt - \rho \, dt,$$

given their exposures to aggregate and idiosyncratic risk, and since idiosyncratic risk cancels out in the aggregate. The law of motion of aggregate wealth is

$$\frac{d((q_t + p_t)K_t)}{(q_t + p_t)K_t} = r_t + \sigma(\sigma \, dt + dZ_t) + \eta_t x_t \tilde{\sigma} \tilde{\varsigma}_t \, dt - \rho \, dt,$$

where $\eta_t = \frac{N_t}{(q_t + p_t)K_t}$ is the experts' net worth share and $\eta_t x_t = q_t/(p_t + q_t)$ is the exposure to idiosyncratic risk in the world portfolio. Hence,

$$\frac{d\eta_t}{\eta_t} = x_t^2 (1 - \eta_t) \tilde{\sigma}^2 \, dt = \left(\frac{q_t}{p_t + q_t} \right)^2 \frac{1 - \eta_t}{\eta_t^2} \tilde{\sigma}^2 dt. \tag{4.3}$$

Step 4: Derive ODEs for the postulated price processes q and p as a function of the state variable η. We omit this step as it is similar to the previous section.

4.3 Observations and Limit Case

The increase in experts' wealth share η_t, or equivalently the decline of households' wealth share, $1 - \eta_t$, results in part from the fact that experts earn a risk premium from taking on

idiosyncratic risk. The higher the idiosyncratic risk $\tilde{\sigma}^2$, the faster experts' wealth share rises towards 100%. Interestingly, it is the fact that experts are unable to share idiosyncratic risk which makes them richer over time compared to households.

Money allows for some sharing of idiosyncratic risk, since the experts' exposure to idiosyncratic risk of $x_t\tilde{\sigma}$ is less than what it would have been without money, i.e. $\tilde{\sigma}/\eta_t$, as long as $x_t < 1/\eta_t$ or $p_t > 0$.

Comparison with Real Model. It is instructive to contrast the settings of this section with that of Section 2, where households hold the real risk-free asset instead of money. The evolution η follows now (4.3) instead of (2.11). Note that in both settings the experts' wealth share drifts towards 100%. However, there are crucial differences. In the setting with nominal money, aggregate risk is shared fully between experts and households. Hence, both groups receive a risk premium and therefore aggregate risk does not impact the wealth share in the model with money. In contrast, in the real model experts hold all the aggregate risk and hence only they earn a risk premium, leading to a positive drift in η. More importantly, aggregate risk sharing with money makes the evolution of experts' wealth share deterministic. In contrast, in the real model that experts' wealth share is necessarily stochastic, as revealed by (2.11).

The Only Experts Case. Finally, we are able to derive a closed form solution for the absorbing state $\eta = 1$ to which the system drifts. When the state $\eta = 1$ is reached $\mu^q(1) = \mu^p(1) = 0$ and thus experts' asset pricing equation (4.1) and capital market clearing (4.2) can be combined and simplified as follows

$$\frac{1}{\tilde{\sigma}^2}\frac{a-\iota}{q} = \frac{E[dr^k - dr^M]/dt}{\tilde{\sigma}^2} = x_t = \frac{q}{p+q} \tag{4.4}$$

Combining equation (4.4) with the goods market clearing condition

$$\rho(p+q)K_t = (a-\iota)K_t \tag{4.5}$$

and the optimal investment rate

$$\iota = \frac{q-1}{\kappa}, \tag{4.6}$$

for the functional form $\Phi(\iota) = \frac{1}{\kappa}\log(\kappa\iota + 1)$ one obtains the "money equilibrium," in which money is a bubble with

$$q = \frac{1+\kappa a}{1+\kappa\sqrt{\rho}\tilde{\sigma}} \text{ and } p = \frac{\tilde{\sigma}-\sqrt{\rho}}{\sqrt{\rho}}q.$$

The "money equilibrium" exists as long as $\tilde{\sigma} > \sqrt{\rho}$.

In addition, there exists a "moneyless equilibrium", obtained by setting $p = 0$ and solving (4.5) with (4.6) to obtain

$$q^0 = \frac{1+\kappa a}{1+\kappa\rho} \text{ and } p^0 = 0.$$

Equation (4.4) is no longer relevant because money is no longer an asset in which agents can put their wealth.

Note that the price of capital for the "moneyless" equilibrium is the same as in the real economy of Section 2. The growth rate of the economy in both equilibria is given by $g = \frac{1}{\kappa}\log q - \delta$. In the money equilibrium, q is lower and so is overall economic growth, but experts have to bear less risk.

Financial Deepening. Financial deepening or innovation that lower the amount of idiosyncratic risk households have to bear also lowers the value of money, p. However, it increases the price of capital q and with it, the investment rate, ι, and the overall economic growth rate g. Surprisingly, $q + p$ declines. That is, financial deepening lowers total wealth in the economy.

The Paradox of Prudence. The Paradox of Prudence arises when experts try to lower their risk by tilting their portfolio away from real investment and towards safe asset, money. Scaling back risky asset holding can be micro-prudent, but macro-*im*prudent. As experts try to lower their (idiosyncratic) risk exposure, the price of capital falls in Brunnermeier and Sannikov (2015a). This behavior lowers overall economic growth and with it the real return on money holdings. Since each individual expert takes prices and rates of return as given, they do not internalize this pecuniary externality. As shown in Brunnermeier and Sannikov (2015a), money holdings in this model are inefficiently high if $\tilde{\sigma}(1-\kappa\rho) > 2\sqrt{\rho}$. Our Paradox of Prudence is analogous to Keynes' Paradox of Thrift, but the former is about changes in portfolio choice and risk, while the latter refers to the consumption-savings decision.[17]

[17]Keynes' Paradox of Thrift states that an increase in the savings propensity can paradoxically lower aggregate savings. An increase in savings propensity lowers consumption demand. If the increased savings are "parked in (bubbly) money" instead of additional real investments, aggregate demand becomes depressed.

5 Critical Assessment and Outlook

The economy with two types of agents gives rise to a number of general ideas - we describe these broader ideas in this section. We would like to make the point that continuous time has the capacity to build upon many ideas present in the literature, with fuller and less stylized models, and to drive a deeper understanding of financial frictions in the macroeconomy in new ways. We comment on how the methodology we presented above can be extended, and used fruitfully, in higher-dimensional state spaces. We also comment on the issues of uniqueness of equilibria and the characterization of the full set of equilibrium possibilities when multiple equilibria exist.

One key idea is that the wealth distribution in the economy matters. In the models we solved in Sections 2 and 3, the wealth distribution is characterized by a single state variable, the wealth share of experts η_t. When η_t is low, experts become undercapitalized. More generally, other sectors can become undercapitalized. Mian and Sufi (2009) argue that a big drag on the economy in the recent financial crisis has been the fact that many households are undercapitalized. Caballero et al. (2008) discuss how during Japan's lost decade it was the corporate sector that became undercapitalized. The general message here is that the wealth distribution across sectors matters for the level of economic activity - asset allocation - as well as the rates of earnings and risk exposures of various sectors. These earnings and risk exposures in turn drive the stochastic evolution of the wealth distribution.

The idea that the wealth distribution drives economic cycles is not new in the literature. Kiyotaki and Moore (1997) and Bernanke et al. (1999) consider the fluctuations of the wealth of a class of agents near the steady state. Of course, continuous-time methods facilitate a full solution of this type of a model. He and Krishnamurthy (2013) consider a model similar to the ones we presented here, but without asset misallocation and with a somewhat different assumption of the earnings of the households' holdings of expert equity.[18]

More broadly, several papers introduce the idea of intergenerational wealth distribution. This idea exists already in Bernanke and Gertler (1999), where the wealth of old entrepreneurs affects wages in the labor market, which in turn impact the accumulation of wealth by young entrepreneurs. Myerson (2012) builds a model with T generations of bankers, in which the wealth distribution evolves in cycles, causing cycles in real activities.

This lowers aggregate income. Saving a fraction of now lower income can lower overall dollar savings.

[18] In that model, households earn more than their required return, and therefore there is rationing of experts' shares. Effectively, the alternative assumption gives households some market power, which intermediaries do not have. This leads to a lower intermediary earnings rate and a slower recovery from crisis.

When the wealth of old bankers is high, risk premia are low, and hence earnings of young bankers are low. Wealth distribution across sectors also matters. Brunnermeier and Sannikov (2015b) develop a rather symmetric model, in which there are two sectors that produce two essential goods, and either one of the sectors can become undercapitalized. Brunnermeier and Sannikov (2012) discuss the idea that multiple sectors can be undercapitalized, and that monetary policy can affect "bottlenecks" through its redistributive consequences. They envision an economy in which multiple assets are traded, and agents within various sectors hold specific portfolios, backed by a specific capital structure. Brunnermeier and Sannikov (2015a) provide formal backing of these ideas using a three-sector model, in which traded assets include capital, money and long-term bonds, and monetary policy can affect the prices of these assets (and hence affect the sectors that hold theses assets) in various ways.

This leads us to the obvious question about the capacity of continuous-time models to develop these complex ideas. Can continuous-time methods successfully handle models with multiple state variables, which describe e.g. the distribution of wealth across sectors together with the composition of productive capital? We believe that yes - we are highly optimistic about the potential of continuous-time models. Certainly, the curse of dimensionality still exists. However, models with as many as four state variables should be solvable through a system of partial differential equations in a matter of minutes, if not faster, through the use of efficient computational methods. The authors of this chapter have some experience with computation, and on a personal level many possibilities seem feasible now which appeared out of reach five years ago. To gauge computational speed, DeMarzo and Sannikov (2016) solve a model with three state variables, using a system of two partial differential equations. In addition the procedure involves an integration step somewhat reminiscent of the "static step" of the procedure in subsection 3.5. With $201 \times 51 \times 51$ grid points, the procedure using the explicit(!) method takes only a minute to compute the optimal contract. The implicit method of solving partial differential equations, which we use to compute the examples in subsection 3.6 is significantly faster. For example, when solving a partial differential equation of the parabolic type in two dimensions (all equations for computing the value function using the iterative method are parabolic), with N grid points in space, one needs $O(N^2)$ grid points in time to ensure that the computational procedure is stable, when using the explicit method. In contrast, when using the implicit method, stability does not depend on the length of the time step, i.e. the time step can be kept constant when greater resolution is required along the space dimension. Hence, we believe that by making a claim that models with four state

variables are feasible to solve, we are in fact quite conservative.

We think that the iterative method, based on value function iteration for each type of agent, should prove quite fruitful. This method is based on backward induction starting from a terminal condition on the state space. At each new time interval, we start with value functions computed for the end of the interval. These value functions determine the agents' incentives through their continuation values from various portfolio choices. As a result, we can determine at each time point the allocations of assets and risk consistent of equilibrium - this is the "static step" - and hence we can compute the value function one period earlier. We see this method as fairly general and suitable for multiple dimensions.

In contrast, the shooting method aims at solving for the fixed point - equilibrium value functions and allocations in an infinite-horizon economy - up front. The straightforward extension of this method to multidimensional state spaces may be difficult to implement, as one would have to guess functions that match boundary conditions on the entire periphery of the state space, instead of just two endpoints. Nevertheless, procedures that use variations of policy iteration may lead to an efficient way of solving for a fixed point.

What makes continuous-time models particularly tractable is that transitions are local (when shocks are Brownian) - hence it is possible to determine the agents' optimal decisions and solve for their value functions by evaluating only first and second derivatives. In discrete time, with discrete transitions, the agents' decisions at any point may depend on entire value functions.

What about environments with so many dimensions that the straightforward discretization of the state space makes computation infeasible, due to the curse of dimensionality? Here, we are curious about the idea of describing state variables through certain essential moments - following the suggestions of Krusell and Smith (1998). We have not processed this possibility sufficiently to comment on it in the chapter, but generally we are very eager to know about ways to choose moments that describe the state space in a meaningful way for a given model. We should say, however, that continuous time can be helpful here as well, for describing continuation values and prices as functions of moments.

We finish this section by discussing the question of equilibrium uniqueness in the model we presented and in more complex models we envision. First, consider a finite-horizon economy that we are solving for via an iterative procedure. The procedure has two steps - the time step of value function iteration and the static step that determines prices and allocation. The time step cannot be a source of nonuniqueness - given continuation values, transition probabilities and payoff flows, the value function one period earlier is fully determined. The

static step may or may not lead to nonuniqueness. In the model of Section 3 there are multiple nonstationary equilibria. For example, at any time point, the price of capital q_t can jump. If q_t jumps up by 10% then the risk-free asset must have an instantaneous return of 10% as well to ensure that the markets for capital and the risk-free asset clear. Of course, by the market-clearing condition for output (3.32), the price of capital q_t must correspond to the allocation of capital $\psi_t \in [0, 1]$. The allocation itself must be justified by the local volatility of capital, so that all agents have incentives to hold their portfolios. However, the possibility of jumps opens up room to many possibilities.

We compute the Markov equilibrium, in which prices and allocations are functions of η. If so, then the price of capital $q(\eta_t)$ must satisfy the differential equation that follows from (3.30) and (3.31). Notice that there are two values of $\sigma + \sigma_t^q$ consistent with the quadratic equation (3.30), positive and negative. We select the positive value, since otherwise amplification is negative, in the sense that a positive fundamental shock would result in a drop in the value of capital. Hence, the equilibrium we compute is the unique Markov equilibrium, in which the return on capital is always positively correlated with fundamental shocks to capital.

In more general models, we envision that some of the same forces are present. We also anticipate that, when there are multiple equilibria, it may be of interest to characterize the whole set of equilibria via an appropriate recursive structure. To answer this question, one may need to construct/compute a correspondence from the state space to the vector of equilibrium payoffs of all agent types. We envision that this correspondence can be found recursively by solving for the boundaries of attainable equilibrium payoff sets backwards in time, but the details of this procedure are certainly work in progress.

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