DYNAMIC SAVINGS CHOICES WITH DISAGREEMENTS

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ABSTRACT

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We study a dynamic savings game in continuous time, where decision makers rotate in and out of power. Agents value consumption more highly while in power. Our setup applies to individuals under a behavioral interpretation, or to governments under a political-economy interpretation. We prove existence of Markov equilibria by construction and provide tight characterizations. Our analysis isolates the importance of a local disagreement index $\beta(c)$ which we define as the ratio of marginal utilities for those in and out of power. If disagreement is constant our setup specializes to hyperbolic discounting and we provide novel results even in this context, but we also allow disagreement to vary with spending. When disagreements are sufficiently high we show that an equilibrium with dissavings exists; conversely, when disagreement are sufficiently low, an equilibrium with savings emerges. When disagreements vary sufficiently with spending rich dynamics are possible. In particular, an equilibrium with poverty traps—dissaving at low levels of wealth and savings at high levels of wealth—exists when disagreements decrease with spending. In contrast, when disagreements increase with spending, wealth may convergence to a unique interior steady state. We also investigate conditions for continuous, discontinuous and multiple equilibria. Finally, we show how the model can be solved in reverse, inverting to find primitives that support an equilibrium.

1 Introduction

Temptations or time-inconsistency problems help explain a number of phenomena and have received ample attention from the economics literature. However, the extent of these problems likely varies quite a bit according to the situation. In particular, there is no reason to expect the self-control problems to save faced by the rich to be the same

as those faced by the poor. Similarly, it has been observed that for political economy reasons, governments may behave in a present-biased manner, yet these problems may be quite different for advanced countries than for developing countries. The general point is that strength of the time-inconsistency problem may depend on the level of wealth or spending. This possibility has received relatively little attention from the literature.

This paper studies an infinite-horizon continuous-time savings game that accommodates flexible forms of time inconsistency. A sequence of decision makers rotate in and out of power.[1] An agent currently in power controls consumption and savings, choosing how much to spend subject to a borrowing constraint and a constant flow of income. Agents in power retain power for a stochastic interval of time and lose it at a Poisson rate to a successor. Once removed from power, an agent continues to care about the future spending path chosen by other agents. However, spending is enjoyed more while in power. This disagreement, captured by differences in the utility functions for those in and out of power, leads to a time-inconsistency problem in savings choices. As a result, one must approach the problem as a dynamic game.[2] We focus on Markov equilibria of this dynamic game, a widely used refinement in this literature to focus attention on situations without any commitment, including implicit forms obtained by trigger strategies.

Our model admits both a behavioral and political-economy interpretation. For the behavioral one, following Phelps and Pollak (1968) and Laibson (1997), the model may describe the problem of a single consumer playing an intertemporal game against future 'selves'. The disagreement on the utility function that we allow generates a time inconsistency problem that is similar, but strictly generalizes, hyperbolic discounting. For the political economy interpretation, the model describes a situation where the ruling party controls the budget and obtains private benefits from spending while in power, due to pork spending or outright transfers to ruling party members. This relates our work to political economy models of government debt, such as Alesina and Tabellini (1990), Amador (2002), Battaglini and Coate (2008), Azzimonti (2011) and others.

With few exceptions, the existing time-inconsistency literature has focused on saving games that are effectively variants of the hyperbolic discounting setup. In our model this amounts to the assumption of a uniform disagreement, with utility out of power proportional to utility in power. Our analysis also applies to this special case and delivers novel results of interest. At the same time one of our main goals is to explore forms of

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[1]The model also admits an reinterpretation with only two agents that rotate in and out of power (see Section 3.1).

[2]When preferences are the same in and out of power, the rotation of agents is of no consequence and the equilibrium coincides with the optimum for a dictator that always remains power—an application of the Principle of Optimality from dynamic programming.

disagreements that are not uniform. To this end, we consider general differences in the utility functions for those in and out of power. This gives rise to a time-inconsistency problem that is not uniform and, as a result, the incentives to save vary with wealth. We are especially interested in how the long-run dynamics of wealth play out depending on the form disagreements take.

Why would disagreements and temptations vary with spending? One straightforward answer is that there is no real reason to expect them to be uniform and that the possibility that they are not must be contemplated. For example, in the behavioral context, it seem plausible to assume that present-biased impulses and behaviors decrease with spending. A more sophisticated answer is offered by Banerjee and Mullainathan (2010), who provide a foundation for disagreements based on the notion that spending takes place over many goods, and disagreements focus on some goods.[3] The importance of disagreements then naturally vary with the level of spending, except in special cases. For example, in a behavioral setting agents may feel great pleasure from current consumption of tempting goods such as unhealthy foods, alcohol or drugs, but may not value future consumption of these goods by future 'selves'. If the marginal propensity to consume such goods falls with spending, then their relative importance diminishes at higher spending levels, giving rise to a situation with *decreasing* disagreements (we review this argument formally in Section 2). A similar argument applies in a political economy context as long as the marginal propensity to spend on pork is not constant. Indeed, the voting model in Battaglini and Coate (2008) provides a motive for *increasing* disagreements. One of the goals of this paper is to provide a framework that can encompass a wide class of assumptions on the form of disagreements, nesting the increasing and decreasing cases in particular.

Our first contribution is to provide general results on the existence and characterization of Markov equilibria. As is well known, dynamic saving games may be quite ill behaved. For example, in discrete-time settings Krusell and Smith (2003) proved that the hyperbolic discounting model has a continuum of local Markov equilibria with discontinuous policy functions; more recently Chatterjee and Eyigungor (2015) show that in discrete time all Markov equilibrium must be discontinuous (see also Morris and Postlewaite, 1997 and Morris, 2002). Properties such as these render these models relatively

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[3]In a behavioral context, Banerjee and Mullainathan (2010) focus on two- or three-period model with many goods and additively separable utilities, with disagreements over which goods should be valued. In a political economy context, Alesina and Tabellini (1990) consider an infinite-horizon model with a relatively general form of disagreement in the composition of spending across different goods (see their equations 1). However, for their analysis they specialize to corner cases and a more extreme and uniform disagreement (see their equations 4 and 5).

intractable and contribute towards making general existence and characterization results elusive. The literature has responded to these challenges in a number of ways. Harris and Laibson (2013) introduce a continuous-time model, focusing on a limit with 'instant gratification' and small noise in asset returns to apply the theory of viscosity solutions. Chatterjee and Eyigungor (2015) work in a discrete-time setting but introduce lotteries to smooth out the solution.

Our model is cast in continuous time and this turns out to be crucial to our approach, techniques and results. Our continuous-time formulation builds on Harris and Laibson (2013), but extends it to allow for more general disagreements. In addition, since we do not focus on the 'instant gratification' limit our solution strategy is different.[4] Our approach works with the differential equations characterizing a Markov equilibrium—the Hamilton-Jacobi-Bellman equation for the agent in power and the law of motion for welfare for those out of power. Since no general existence results are immediately available for such equations, our method is to attack these equations head on, to construct and characterize equilibria. Intuitively, since wealth evolves continuously over time, we can build up our characterization of behavior locally towards a global solution. Our direct attack on the differential system, as mentioned above, differs from prior approaches and allows us to prove the existence of relatively well-behaved Markov equilibria without lotteries, away from the instantaneous-gratification limit, and with relatively general forms of disagreement. Since our proofs proceed by construction, we are also able to characterize equilibria relatively sharply and provide a straightforward procedure for computation.

Our second contribution is to isolate the forces determining saving and dissaving and characterize the resulting dynamics for wealth. To do so, we introduce a local disagreement index $\beta(c)$, defined as the ratio of marginal utilities for agents in and out of power. The shape of this function summarizes how disagreements depend on spending. In the special case where $\beta(c)$ is constant we recover hyperbolic discounting.

Our first set of results involve cases where the disagreement index does not vary too much and is either sufficiently high or low. Under these conditions, we show that an equilibrium exists that features either saving or dissaving at all wealth levels. Specifically, we define a threshold $\hat{\beta}$ which depends on the interest rate and other parameters and show that when the disagreement index β lies above $\hat{\beta}$ there is an equilibrium with positive

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[4]Harris and Laibson (2013) show that under some conditions on the hyperbolic discounting parameter and the utility function there is a connection between Markov equilibria in the instantaneous gratification limit and the non-strategic choice problem for a time-consistent consumer with a modified utility function. When available, this connection is very useful and tractable. In the Online Appendix we extend their approach to allow for non-uniform disagreements. Unfortunately, this approach is not available away from the instant gratification limit and does require a few additional assumptions on disagreements and utilities (one of our goals is to place as few restrictions on disagreements as possible).

savings; when instead β lies below $\hat{\beta}$ an equilibrium with dissaving exists. A special case of interest is the hyperbolic discounting model where β is constant, but our results apply more generally.

Our second set of results consider cases where the disagreement index β does not lie on one side of $\hat{\beta}$, but instead crosses this threshold. We find that rich dynamics emerge, with saving and dissaving coexisting at different wealth levels. We focus on two polar opposite cases, when β is decreasing and when β is increasing.

In particular, if disagreements fall with spending then poverty traps—dissavings at low wealth levels, savings at high wealth levels—can arise. Intuitively, at low wealth levels the time-inconsistency problem is relatively severe because spending takes place in the range where disagreements are high. The incentive to consume is high because agents in power do not want to leave resources that may be spent when they are out of power. There is a feedback loop: the incentive to dissave is reinforced by the anticipation that successors overspend from the point of view of those in power. Indeed, under our assumptions poverty traps cannot arise without this feedback loop.[5] At high wealth levels the time-inconsistency problem is moderated by the fact that spending takes place in regions with lower disagreement. Positive savings may then emerge if the interest rate is high enough. Again, a feedback loop reinforces these incentives: the incentive to save is enhances if successors are not expected to overspend much.

Poverty traps may also emerge with uniform disagreements if one considers non-Markov subgame-perfect equilibria, as shown recently by Bernheim et al. (2015) in a hyperbolic discounting discrete-time setup. Although we obtain poverty traps the results are different, the one resting on the use of trigger strategies, the other on non-uniform disagreements. Interestingly, some comparative statics are also different. Higher labor income makes getting caught in a poverty trap more likely in their setup, but less likely in ours. Looser borrowing limits make poverty traps less likely in their setup, but more likely in ours. These differences highlight underlying differences in the mechanisms at work.

Turning to the opposite case, if disagreements rise with spending then the time-inconsistency problem worsens at higher wealth levels. As a result, we show that an equilibrium exists where the wealthy dissave, while the poor save, at least if the interest rate is high enough. In our equilibrium, from any initial wealth level, wealth converges to an interior steady state. This kind of stability is related to the mean-reverting forces in Battaglini and Coate (2008). Indeed, one might view their voting model as providing a foundation for disagreements to rise with spending.

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[5]In particular, they do not arise in a two-period adaptation of our model.

In addition to solving for an equilibrium given primitives using our differential approach, we show how our model can be fruitfully studied in reverse: inverting to find primitives that support any given postulated equilibrium. In particular, given a baseline utility for those in power and a discount rate, we construct a disagreement $\beta(c)$ function for any given smooth consumption function. This "inverse" perspective is not only technically tractable, but it may also be the more appropriate point of view in some circumstances, especially for the economist that observes behavior (the policy function), but has no direct measurement of disagreements.

Finally, we investigate the potential for continuous, discontinuous and multiple equilibria. We first provide sufficient conditions for a continuous equilibrium to exist. We then provide conditions that guarantee the existence of a discontinuous equilibrium. Since we find that both conditions are compatible, these results imply that multiple global equilibria arise in some cases. As mentioned earlier, Krusell and Smith (2003) already showed that local Markov equilibria are indeterminate in a discrete-time hyperbolic model (we also provide an analogous result for our setup). Importantly, this indeterminacy result is local in nature, applying only when constrained to an interval of wealth. In contrast, our multiplicity result constructs equilibria globally, over all wealth levels.

An Online Appendix collects a few additional results and extensions. We extend the equivalence result for the instantaneous gratification limit obtained by Harris and Laibson (2013) to our setting with nonuniform disagreements. We also introduce uncertainty in asset returns and income and develop some illustrative numerical examples. Lastly, we show how the model can be extended to replace the constant interest rate with a decreasing returns technology for saving.

The next section presents the model. Section 3 defines Markov equilibria and discusses our solution approach. Sections 4–6 contain our main results.

2 A Dynamic Savings Game

We first introduce the model environment, then offer a few interpretations and special cases.

2.1 Environment

Time is continuous with an infinite horizon, denoted by $t \in [0, \infty)$. We next describe the preferences and the constraints agents face.

Preferences.   At any moment t, the flow utility obtained from consumption by an agent in power is

$$U_1(c_t),$$

while that for an agent out of power is

$$U_0(c_t).$$

We assume the utility functions $U_1 : \mathbb{R}_+ \to \mathbb{R}$ and $U_0 : \mathbb{R}_+ \to \mathbb{R}$ are concave, increasing, continuous and differentiable. In addition, we assume U_1 satisfies the INADA conditions $U_1'(0) = \infty$ and $\lim_{c \to \infty} U_1'(c) = 0$.[6]

Agents in power are removed at a constant Poisson arrival rate $\lambda \geq 0$. Thus, tenure is stochastic with average length λ^{-1}. To simplify we assume power can never be regained; however, we later show that this is without loss of generality in the sense that if power can be regained at some Poisson rate the model can be transformed to a setting where power cannot be regained. Our Poisson alternation of power helps smooth out the time inconsistency problem and is borrowed from Harris and Laibson (2013), extended to allow for more general differences in preferences. We will also not focus on the 'instant gratification' limit $\lambda \to 0$, so our approach is different; we do derive results for this limit in the Online Appendix.

The continuation lifetime utility at time t for an agent in power is

$$V_t \equiv \mathbb{E}_t \left[\int_0^\tau e^{-\rho s} U_1(c_{t+s}) ds + e^{-\rho \tau} W_{t+\tau} \right] \tag{1}$$

where $\rho > 0$ is a discount rate and τ is the random time at which the agent currently in power is removed, distributed according to the c.d.f. $1 - e^{-\lambda \tau}$. Here W_t represents the continuation lifetime utility once out of power,

$$W_t \equiv \int_0^\infty e^{-\rho s} U_0(c_{t+s}) ds. \tag{2}$$

The expectation operator in these expressions averages over the only underlying shock in the economy, the alternation of power. In principle, consumption c_t could also be stochastic. However, given the symmetry of preferences and our focus on Markov equilibria, the path for consumption will be deterministic.

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[6]Concavity and differentiability of U_0 are not crucial for the analysis but serve to simplify the exposition for most of the results. Indeed, an earlier version of the paper focused on a case that had a convex kink in $U_0(c)$. Theorem 6 below actually assumes a concave kink in U_0.

Disagreement between those in and out of power captured by the difference in the utility functions U_0 and U_1, is crucial to our model. Crucial to our analysis is the introduction of a *local disagreement index*, defined as the ratio of marginal utilities

$$\beta(c) \equiv \frac{U_0'(c)}{U_1'(c)}.$$

The function $\beta(c)$ summarizes the difference between the utility functions U_1 and U_0. In particular, when $\beta(c) = 1$ for all c the utility functions coincide (up to a constant). As we shall see our local disagreement index is crucial in determining equilibrium behavior. Throughout the paper we assume that the marginal utility from consumption is higher while in power.

Assumption 1 (Present Bias). *The utility functions U_1 and U_0 are such that for all $c > 0$*

$$\beta(c) \in [\underline{\beta}, 1]$$

for some $\underline{\beta} > 0$.

When $\beta(c) < 1$, agents prefer to consume relatively more while in power, this leads to a present-bias time-inconsistency problem. Those out of power would like those in power to exercise restraint, lowering consumption and increasing savings. Those in power would like to commit their successors somehow, but have no way to do so.

We allow for general $\beta(c)$ to capture various possible patterns of disagreement. In particular, in some cases it may be natural to assume that disagreements are stronger at lower consumption levels, so that $\beta(c)$ is increasing. Another possibility is for disagreements to be strongest at higher spending levels, so that $\beta(c)$ is decreasing. Finally, the case with constant disagreements, is an important benchmark in the literature on hyperbolic discounting.

Budget Constraints and Borrowing Limits. While in power, the agent chooses $c_t \geq 0$ and wealth evolves according to the budget constraint

$$\dot{a}_t = ra_t + y - c_t, \tag{3}$$

where a_t denotes total asset wealth. The interest rate $r > 0$ and income $y \geq 0$ are exogenous and constant.

The agent is also subject to a borrowing constraint

$$\underline{a} \leq a_t, \tag{4}$$

The so-called natural borrowing limit assumes that $\underline{a} = -\frac{y}{r} \leq 0$, allowing the agent to borrow against all future income.[7] We assume the borrowing constraint is tighter so that $\underline{a} > -\frac{y}{r}$. Note that whenever $a_t = \underline{a}$ we require $c_t \leq y + r\underline{a}$ to ensure that $\dot{a} \geq 0$.

Cutoff Disagreement $\hat{\beta}$. Intuitively, when $r < \rho$ even a time consistent agent (i.e. $\lambda = 0$ or $\beta(c) = 1$) would dissave and the time inconsistency problem (i.e. $\lambda > 0$ and $\beta(c) < 1$) only reinforces this conclusion. Our formal results confirm this intuition. On the other hand, when $r > \rho$ a time consistent agent has an incentive to save, but this incentive may be counteracted by the time inconsistency problem. It is then unclear whether to expect savings or dissavings.

What turns out to be crucial is the value of our local disagreement index $\beta(c)$ relative to a cutoff defined by

$$\hat{\beta}(r, \rho, \lambda) \equiv \frac{\rho}{r} \left(1 - \frac{r - \rho}{\lambda} \right). \tag{5}$$

Note that $\hat{\beta}(r, \rho, \lambda) < 1$ if and only if $r > \rho$. As we will show, when $\beta(c) < \hat{\beta}(r, \rho, \lambda)$ temptations are strong enough to provide a force for dissaving; this is necessarily the case when $r < \rho$, given Assumption 1. In contrast, we will find that when $\beta(c) > \hat{\beta}(r, \rho, \lambda)$ temptations are sufficiently weak to allow for positive savings; this requires $r > \rho$, given Assumption 1. Note that this is consistent with the time consistent criterion for savings because when $\lambda = 0$ the sign of $\beta(c) - \hat{\beta}(r, \rho, \lambda)$ is dominated by the sign of $r - \rho$ determining whether $\hat{\beta}(r, \rho, 0)$ is $-\infty$ or ∞.

Normalization of Income and Asset Limit. Without loss of generality to the analysis that follows we normalize income to zero, $y = 0$, and work with a positive asset limit $\underline{a} > 0$. This is simply a normalization since, by a change of variables, one can transform a problem with positive income $y > 0$ to one without. To see this, define $\tilde{a}_t = a_t + \frac{y}{r}$. Then $\dot{\tilde{a}}_t = r\tilde{a}_t - c_t$ and $\tilde{a} \geq \underline{\tilde{a}} \equiv \underline{a} + \frac{y}{r}$. As a result of this transformation, the borrowing limit becomes a positive lower bound on assets, $\underline{\tilde{a}} > 0$.

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[7]The borrowing constraint can also be interpreted as a commitment device. In the case of consumers this may be forced savings and social security or illiquid assets. In the political-economy context, this could capture wealth funds, which limit discretionary spending from natural resources.

2.2 Interpretation and Special Cases

In this subsection we first discuss the important case where disagreement is uniform, so that $\beta(c)$ is constant. We then discuss a motivation for considering nonuniform patterns of disagreement. Finally, we provide a Generalized Euler equation that previews the role of our local index of disagreement $\beta(c)$ in affecting savings decisions.

Hyperbolic Discounting and Instantaneous Gratification. A special and interesting case occurs when

$$\beta(c) = \bar{\beta} \tag{6}$$

for some $\bar{\beta} < 1$, so that $U_0(c) = \bar{\beta} U_1(c)$. This corresponds to the continuous-time hyperbolic discounting model introduced by Harris and Laibson (2013), which in turn builds on discrete-time treatments in Harris and Laibson (2001), Laibson (1997) and Phelps and Pollak (1968). Harris and Laibson focus on the limit as $\lambda \to \infty$, the so-called 'Instantaneous Gratification' limit. They show that tractability is gained from the fact that $V_t = W_t$ in the limit, so that a single continuation value function suffices, and they relate the Markov equilibrium to an optimization problem with modified objectives.

Many Goods, Disagreement and Engel Curves. One interesting motivation for disagreements and their variability is to interpret c as overall spending on various goods. Differences in U_0 and U_1 then arise from disagreement on how to spend across goods. This notion is popular in the political economy literature on government spending and debt, as in Persson and Svensson (1989), Alesina and Tabellini (1990), Amador (2002) and Azzimonti (2011), although this literature often assumes simple specifications that imply uniform disagreements. Banerjee and Mullainathan (2010) work in a behavioral setting and consider richer disagreements across goods, examining the relationship between the shape of Engel curves and disagreements.

Here we develop these ideas within our formulation, linking U_1 and U_0 to the disagreement over different goods, the shape of Engel curves and our disagreement index $\beta(c)$. Suppose there are two goods x and z. Normalize both prices to unity. Assume additively separable utilities between x and z, given by $h(x)$ and $g(z)$, respectively. Now suppose x is equally valued by those in and out of power, while z is not valued at all by agents out of power (less extreme assumptions work similarly). Thus, utility for those in power equals $h(x) + g(z)$, while utility for those out of power equals $h(x)$.

At any moment, agents in power solve a static subproblem: splitting spending be-

tween x and z, given total spending c. This defines indirect utility functions

$$U_1 (c) = \max_{x+z=c} \{h(x) + g(z)\} \tag{7a}$$

$$U_0 (c) = h(\hat{x}(c)), \tag{7b}$$

where $(\hat{x}(c), \hat{z}(c))$ denotes the solution to the maximization. The next result shows that we can generate any desired U_1 and U_0 in this way, by appropriate choices of h and g.

Proposition 1. *Given U_1 and U_0 satisfying Assumption 1, there exists strictly concave functions h and g so that (7) holds.*

Proof. Appendix A.1. □

Note that $U_1'(c) = h'(\hat{x}(c))$ and $U_0'(c) = h'(\hat{x}(c))\hat{x}'(c)$, implying

$$\beta(c) = \hat{x}'(c) = 1 - \hat{z}'(c) \leq 1.$$

A high marginal propensity to spend on z lowers the marginal utility of spending for those out of power, since they do not value z. Thus, the shape of the Engel curve dictates the shape of the ratio $\beta(c)$. For example, when $\hat{z}(c)$ is concave, so that the marginal propensity to spend on z is decreasing, the ratio $\beta(c)$ is increasing. Intuitively, time inconsistency problems are then greater for relatively low levels of spending, where marginal increases in spending are allocated to the good that only those in power value.[8]

Generalized Euler Equation. To illustrate the central role played by the disagreement index $\beta(c)$ we show in the Online Appendix that

$$\frac{d}{dt}\left(U_1'(c_t)\right) = (\rho - r)U_1'(c_t)$$

$$+ \lambda \int_0^{T^*} e^{-\int_0^s (\rho + \lambda - r + \hat{c}'(a_{t+s_1}))ds_1} U_1'(c_{t+s})\left(1 - \beta(c_{t+s})\right)\hat{c}'(a_{t+s})ds,$$

$$+ \lambda e^{-\int_0^{T^*} (\rho + \lambda - r + \hat{c}'(a_{t+s_1}))ds_1} U_1'(ra_{T^*})\left(1 - \beta(ra_{T^*})\right),$$

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[8]Under hyperbolic discounting, $U_0(c) = \bar{\beta}U_1(c)$, the functions h, g in Proposition 1 are

$$h(x) = \bar{\beta}U_1\left(\frac{x}{\bar{\beta}}\right) \quad \text{and} \quad g(z) = (1 - \bar{\beta})U_1\left(\frac{z}{1 - \bar{\beta}}\right),$$

which implies that $\hat{x}(c) = \bar{\beta}c$ and $\hat{z}(c) = (1 - \bar{\beta})c$.

where $\hat{c}(a)$ is the consumption decision of the decision maker as a function of wealth in a Markov equilibrium, and T^* the time wealth reaches a steady state, i.e. $\hat{c}(a_{T^*}) = ra_{T^*}$; if $T^* = \infty$ the last term is zero. This is a continuous-time version of what Harris and Laibson (2001) call a Generalized Euler equation. When $\lambda = 0$ or $\beta(c) = 1$ it reduces to the standard Euler equation, stating that the marginal utility $U_1'(c_t)$ grows at rate $\rho - r$. However, when $\lambda > 0$, $\beta(c) < 1$ and $\hat{c}'(a) > 0$ the second and third terms on the right hand side are nonzero and positive. Intuitively, if an agent currently in power agent saves today the additional resources may be used consumed once this agent is out of power. The additional consumption is positive as long as $\hat{c}'(a) > 0$ but it is valued less since $\beta(c) < 1$. These positive terms contributes to a form of impatience pushing for a declining path for consumption and wealth, as does a higher value of ρ.

2.3 Full Commitment

An important benchmark is provided by the following problem with commitment, where we imagine an agent currently in power that can choose a deterministic path for consumption $\{c_t\}$ that is applied whether or not this agent remains in power.[9] Lifetime utility with such a commitment is given by

$$V_{sp}(a) \equiv \max_{\{c_t\}} \int_0^\infty e^{-\rho t} \left(e^{-\lambda t} U_1(c_t) + (1 - e^{-\lambda t}) U_0(c_t) \right) dt \qquad (8)$$

where the maximization is subject to $a_0 = a$, $\dot{a}_t = ra_t - c_t$ and $a_t \geq \underline{a}$. We assume that V_{sp} is finite.[10] Here $e^{-\lambda t}$ represents the probability that the agent is still in power after time t.[11] Clearly, any equilibrium must be worse for the agent in power than the commitment value, so that $V(a) \leq V_{sp}(a)$.

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[9]A better commitment technology would allow consumption to be contingent on whether the agent is currently in power or not. However, deterministic consumption is comparable to outcomes in our Markov equilibria.

[10]A sufficient condition that guarantees this property is:

$$\limsup_{t \to \infty} e^{-\rho t} U_1(e^{-rt}) \leq 0,$$

a standard restriction for deterministic optimization problems. This is immediately satisfied if U_1 is bounded from above. If U_1 is a power function, i.e. $U_1(c) = \frac{c^{1-\sigma}}{1-\sigma}$, and $\sigma < 1$ this is equivalent to $r(1 - \sigma) < \rho$.

[11]We derive (8) by integrating out, from the expected value

$$V_{sp} = \int_\tau \int_0^\infty e^{-\rho t} \left(\mathbf{1}_{\{t \leq \tau\}} U_1(c_t) + \mathbf{1}_{\{t > \tau\}} U_0(c_t) \right) dt dF(\tau),$$

the Poisson uncertainty in the stopping time τ at which the current decision maker is out of power. $F(\tau) = 1 - e^{-\lambda \tau}$ is the CDF for τ.

When $r < \rho$ then the commitment solution features dissaving. When $r > \rho$ the commitment solution eventually features positive savings.

Proposition 2. *Suppose Assumption 1. When $r < \rho$ the commitment solution features dissaving with $\dot{a} < 0$ whenever $a_t > \underline{a}$ and $a_t \to \underline{a}$ as $t \to \infty$. When $r > \rho$ the commitment solution eventually features positive savings, with wealth and consumption increasing without bound: there exists $T > 0$ such that $\dot{a}_t, \dot{c}_t > 0$ for all $t > T$ and $a_t, c_t \to \infty$ as $t \to \infty$.*

Proof. Appendix A.3. □

The first part of this result is immediate and follows from the fact that $r < \rho$ provides no incentives to save even if $\lambda = 0$; the fact that $\lambda > 0$ and $\beta(c) < 1$ only reinforces the incentive to dissave. The result for $r > \rho$ is only slightly more subtle and is driven by the fact that the probability of remaining in power eventually converges to zero. As a result, the commitment problem approaches a standard optimization problem with utility function U_0. When $r > \rho$, the solution requires consumption to increase without bound. As is well understood, without commitment this is no longer the case and dissaving may emerge even when $r > \rho$.

3 Markov Equilibria

We focus on Markov equilibria with wealth a_t as the state variable, consisting of a policy function $\hat{c}(a)$ for consumption that maximizes the right hand side of (1), taking as given the value function $W(a)$ defined by (2).

3.1 Differential Equations

As is well understood, the value and policy functions associated with Markov equilibria must satisfy differential equations which in our case are given by the following:

$$\rho V(a) = \max_{c \geq 0}\{U_1(c) + V'(a)(ra - c) + \lambda(W(a) - V(a))\}, \tag{9a}$$

$$\rho W(a) = U_0(\hat{c}(a)) + W'(a)(ra - \hat{c}(a)), \tag{9b}$$

where $\hat{c}(a)$ denotes the solution to the maximization in (9a), which is subject additionally to $c \leq r\underline{a}$ whenever $a = \underline{a}$. Equation (9a) is a recursive representation of the problem of maximizing (1) taking the value function $W(a)$ as given. The last term takes into account the probability of transitioning out of power with probability λ, at which point the

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continuation value jumps from $V(a)$ to $W(a)$. Equation (9b) is a recursive representation of condition (2), which defines $W(a)$ given the policy function $\hat{c}(a)$. Finally, the implied dynamics for wealth is

$$\dot{a}_t = ra_t - \hat{c}(a_t). \tag{10}$$

A Markov equilibrium is a pair of value functions $(V(a), W(a))$ and a consumption function $\hat{c}(a)$ satisfying (9) and the following properties: (a) V is continuous and piecewise differentiable; (b) W is piecewise continuous and piecewise differentiable; (c) W is continuous and differentiable at any point where $\hat{c}(a) \neq ra$; (d) W is continuous from the left at any point where $ra < \hat{c}(a)$ in a neighborhood to the left of a; (e) conversely, W is continuous from the right at any point where $ra > \hat{c}(a)$ in a neighborhood to the right of a; and (f) for any $a \geq \underline{a}$ the ODE $\dot{a}_t = ra_t - \hat{c}(a_t)$ with initial condition a_0 admits a solution $\{a_t\}_{t=0}^{\infty}$ with $a_t \geq \underline{a}$ for all $t \geq 0$ satisfying $\lim_{t \to \infty} e^{-\rho t} V(a_t) = 0$ and $\lim_{t \to \infty} e^{-\rho t} W(a_t) = 0$.

The conditions for a Markov equilibrium are relatively straightforward. The only subtle issue worth discussing here are the smoothness requirements for V and W. The function V must be continuous, as stated in condition (a), because it represents the value from a continuous-time optimal control problem with a controllable state with continuous payoffs in the control. That is, discontinuities in W cannot induce discontinuities in V as long as $\dot{a}$ is unrestricted. Although V is continuous, by (9a) it inherits kinks at points where W is discontinuous.

In contrast, the function W may be discontinuous, because it is not the value from a smooth optimization. However, since $W(a_t) = \int_t^{\infty} e^{-\rho(s-t)} U_0(\hat{c}(a_s)) \, ds$, the function W is continuous and differentiable (differentiating, one recovers (9b)) along interval of wealth that are connected by a path $\{a(t)\}$. These considerations lead to conditions (b)–(e). Note that when the policy function $\hat{c}$ is such that there are multiple disconnected ergodic sets for wealth, then W potentially jumps at the boundaries of these sets.[12]

3.2 Solution Method

We now describe our method for constructing Markov equilibria, which underlies our formal analysis and results, as well as numerical examples. Our approach constructs solutions to the differential system (9) by attacking these equations directly and constructing

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[12]In the Online Appendix, we prove a Verification Theorem showing that if (V, W) satisfies the above requirements, then V is the value function of a decision maker that maximizes (1) subject to the budget constraint (3) and the borrowing constraint (4). The difference relative to standard verification theorems (Fleming and Soner, 2005) is that in our case W may be discontinuous and V may not be differentiable at points of discontinuity of W.

a solution. That is, we do not appeal to general existence results for the system (9).

Indeed, we are unaware of the availability of any general results that would immediately establish the existence of a solution to these equations. This may seem surprising at first. After all, (9a) which is just a Hamilton-Jacobi-Bellman for V given W, and for which various existence results may apply (at least for a regular enough class of W functions). However, the main difficulty is not with solving (9a) for V given W. The problem lies in solving the system (9) jointly for both V and W. In particular, (9b) may seem reminiscent of a Hamilton-Jacobi-Bellman equation, but it is not, because $\hat{c}(a)$ does not maximize the right hand side (9b).

In contrast, Harris and Laibson (2013) do apply general existence results based on viscosity solutions in a related hyperbolic discounting model (constant β). They study the 'instantaneous gratification' limit as $\lambda \to \infty$ and show that under some conditions (9) is then equivalent to the HJB system for a time consistent consumer with a suitably modified utility function, that depends on wealth in addition to consumption. Unfortunately, this line of attack is not available in our case with finite λ, and it does require some additional assumptions on disagreements and utilities (one of our goals is to place as few assumptions on disagreements as needed).

Thus, we approach the problem differently and treat these two equations as ordinary differential equations (ODEs) in (V, W). We do not know of any off-the-shelf ODE existence results that immediately apply to these equations. There are two issues, both involving steady state points where $\hat{c}(a) = ra$. First, solving for $V'(a)$ given (9a) yields a law of motion that is not Lipschitz continuous around steady states. Second, more seriously, $W'(a)$ is not even determined at steady state points. Fortunately, we are able to resolve these two issues, as we explain below. Once these two issues at steady states are surmounted, our method is relatively straightforward and simple to implement.

Constant Wealth and Consumption. It is useful to define the value of holding wealth and consumption constant,

$$\bar{V}(a) = \frac{\rho}{\rho + \lambda} \frac{U_1(ra)}{\rho} + \frac{\lambda}{\rho + \lambda} \frac{U_0(ra)}{\rho}, \tag{11a}$$

$$\bar{W}(a) = \frac{U_0(ra)}{\rho}. \tag{11b}$$

The value for agents outside of power is simply the present value of consuming ra forever according to the utility function U_1. The value for those in power is a weighted average of the present value according to U_1 and the present value according to U_0.

These value functions will play an important role in our constructions and proofs. The equilibrium value functions (V, W) must coincide with $(\bar{V}, \bar{W})$ at steady states points, where $\hat{c}(a) = ra$ so that $\dot{a} = 0$.

Roots for V'(a).   Equation (9a) is an implicit differential equation, to be solved for $V'(a)$ at each point a, given values for $V(a)$ and $W(a)$. In the appendix we show that the right hand side of (9a) is a strictly convex function with a unique minimum, so that there are zero, one or two solutions (roots) for $V'(a)$. The first-order condition for the maximization in (9a) gives

$$U_1' \left(\hat{c} \left(a \right) \right) = V' \left(a \right), \tag{12}$$

which gives consumption as a decreasing function of $V'(a)$. Indeed, when two roots for $V'(a)$ exist, the lower one corresponds to dissaving $\hat{c}(a) > ra$ and $\dot{a} < 0$, while the higher one corresponds to saving, $\hat{c}(a) < ra$ and $\dot{a} > 0$. When a unique root for $V'(a)$ exists it is associated with $\hat{c}(a) = ra$ and $\dot{a} = 0$.

Solving the ODEs.   The equilibrium analysis in the next sections involves solving the ODE equations (9) for the value functions (V, W) and associated policy function $\hat{c}(a)$ to satisfy the equilibrium requirements (a)–(f) mentioned above.

We construct equilibria by solving the ODEs starting at the bottom and working up; or by starting at the top and working down; or by combining both procedures. In more detail, the construction involves the following parts: (i) solving the ODEs with the appropriate root over an interval of wealth; (ii) decide if and where to engineer a jump in W and the appropriate jump in W; and (iii) imposing boundary conditions, either at the borrowing constraint or for high enough wealth. Each one of the components is relatively straightforward. The great advantage of this technique is that part (i) is local in nature. We work up when we pick the root that implies dissaving; we work down when picking the root that implies saving. Part (ii) is aided by the fact that V must be continuous and that W can only jump to a self-generating value, i.e. consistent with (2). Finally, the boundary conditions required for part (iii) are supplied at the extremes by known solutions: either getting absorbed by the borrowing constraint at the bottom or reaching a high enough level of wealth where a known solution is available, such as the commitment solution.[13]

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[13]The equilibrium coincides with the commitment solution at a high enough level of wealth if disagreement disappears at a high enough level of consumption: $\frac{U_1'(c)}{U_0'(c)} = 1$ for $c \geq \bar{c}$. Alternatively, a known solution can be found if $U_0(c) = \beta U_1(c)$ for $c \geq \bar{c}$ and $U_1(c)$ is a power function for $c \geq \bar{c}$. Another possibility is imposing an upper bound on wealth. Basically, what we require is that there be a known solution for high enough wealth, to serve as a boundary condition.

Dealing with Singularities at Steady States. One technical challenge is that the differential system (9) is singular at steady states.[14] At any point a_0 with $V(a_0) = \bar{V}(a_0)$ and $W(a_0) = \bar{W}(a_0)$ we have $\hat{c}(a_0) = ra_0$, so condition (9b) at a_0 cannot determine $W'(a_0)$. As a result, we cannot apply standard existence theorems for regular ODEs.

The following lemma shows the existence of the solution (V, W) locally around such points of singularity. Away from these points the system (9) is non-singular, so we can apply standard ODE methods to extend the solution.

Lemma 1. *Suppose $\beta(ra_0) < \hat{\beta}(r, \rho, \lambda)$. Then the differential system (9) with initial condition $(V(a_0), W(a_0)) = (\bar{V}(a_0), \bar{W}(a_0))$ admits a solution over the interval $[a_0, a_0 + \omega]$ for some $\omega > 0$, with*

1. *$V(a) > \bar{V}(a)$ for $a > a_0$;*

2. *$\hat{c}(a) > ra$ for $a > a_0$, $\lim_{a \downarrow a_0} \hat{c}(a) = ra_0$ and $\lim_{a \downarrow a_0} \hat{c}'(a) = \infty$.*

Proof. The details of the proof are in Appendix C, here we provide a sketch. Consider initial values

$$(V_\epsilon(a_0), W_\epsilon(a_0)) = (\bar{V}(a_0), \bar{W}(a_0) - \epsilon),$$

where $\epsilon > 0$. With this boundary condition, the differential system is nonsingular at a_0, so we can find a solution over some interval $[a_0, a_0 + \omega]$ for some $\omega > 0$ that is independent of ϵ. We then take the limit $\epsilon \to 0$ and show that the sequence of solutions converges to a well-defined limit that constitutes a solution to the original system with the desired properties. $\square$

Note that under this construction wealth falls, $\dot{a} < 0$ for $a > a_0$, towards the steady state at a_0. This implies that the values constructed are "self generating" in the sense that the interval $[\underline{a}, \underline{a} + \omega]$ is ergodic, forming a closed self-referential system.

3.3 Recovery of Power

We assumed for simplicity that an agent ousted from power never recovers power. We now show that we can relax this and assume power can be recovered at some Poisson

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[14]When we rewrite system (9) as a differential algebraic equation (DAE), the steady states correspond to critical singular points. However, the DAE at this point does not satisfy the sufficient conditions provided in the literature, for example in Rabier and Rheinboldt (2002), for the existence and uniqueness of solutions around singular points of DAEs, except for the case $\lambda = 0$.

rate $\lambda_r > 0$. The differential system describing a Markov equilibrium is then

$$\rho V(a) = \max_c U_1(c) + V'(a)(ra - c) + \lambda(W(a) - V(a)),$$

$$\rho W(a) = U_0(\hat{c}(a)) + W'(a)(ra - \hat{c}(a)) + \lambda_r(V(a) - W(a)).$$

This system is different from (9a)–(9b) because the second equation includes the possibility of returning to power in the last term. However, our next result establishes that a setting with recovery of power is observationally equivalent to one without.

Proposition 3. *Consider an economy with utilities and Poisson rates $(U_1, U_0, \lambda, \lambda_r)$ with positive recovery probability $\lambda_r > 0$. Markov equilibria for this economy coincide with Markov equilibria for an economy with utilities and Poisson rates $(U_1, \tilde{U}_0, \tilde{\lambda}, 0)$ with no possible recovery of power, where $\tilde{\lambda} \equiv \lambda + \lambda_r$ and $\tilde{U}_0(c) \equiv \frac{\lambda}{\lambda + \lambda_r} U_0(c) + \frac{\lambda_r}{\lambda + \lambda_r} U_1(c)$.*

Proof. One can verify that the pair (V, W) satisfies the differential system with power recovery above for $(U_1, U_0, \lambda, \lambda_r)$ if and only if the pair $(V, \tilde{W})$ with $\tilde{W} \equiv \frac{\lambda}{\lambda + \lambda_r} W + \frac{\lambda_r}{\lambda + \lambda_r} V$ satisfies the differential system without power recovery (9a)–(9b) for $(U_1, \tilde{U}_0, \tilde{\lambda}, 0)$ with $\tilde{\lambda} \equiv \lambda + \lambda_r$ and $\tilde{U}_0(c) \equiv \frac{\lambda}{\lambda + \lambda_r} U_0(c) + \frac{\lambda_r}{\lambda + \lambda_r} U_1(c)$. $\square$

Intuitively, the possibility of recovering power makes an agent more invested in the future consumption possibilities, even after being ousted from power. However, this is similar to placing a higher value on consumption while out of power; this is why $\tilde{U}_0$ is a weighted average of U_0 than U_1 in our result. In a political economy setting, Amador (2002) and Azzimonti (2011) effectively assume that there are no benefits from consuming out of power, $U_0 = 0$.[15] With $\lambda_r = 0$ there would be no time inconsistency problem, only greater impatience with utility U_1 discounted at geometric rate $\rho + \lambda > \rho$. Thus, they focus on cases where the agent returns to power with positive probability, $\lambda_r > 0$. Proposition 3 shows that this is equivalent to a model without recovery of power but with a positive utility for those out of power: $\tilde{U}_0 = \bar{\beta} U_1$ for $\bar{\beta} = \lambda / (\lambda + \lambda_r) \in (0,1)$, a hyperbolic discounting setting.

As we have shown, studying the case where power is never recovered also captures situations where power can be recovered, with an appropriate adjustment to utility functions and the Poisson rate of losing power. This allows us to focus, without loss of generality, on cases where power cannot be recovered, $\lambda_r = 0$.

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[15]They motivate this by assuming there are many goods and agents in power consume different goods, with polar disagreement on what goods are valued, or, equivalently, that consumption is a private good enjoyed only by the agent in power.

4 Dissaving and Saving

In this section we begin to construct and characterize Markov equilibria, first focusing on situations where the ratio $\beta(c)$ is relatively stable and lies on one side of the critical cutoff $\hat{\beta}(r, \rho, \lambda)$ (defined by equation (5)). Our main results in this section establishes the existence of Markov equilibria with dissaving when $\beta < \hat{\beta}$ and positive savings when $\beta > \hat{\beta}$.

To anticipate the key role played by β relative to $\hat{\beta}$ it is useful to cover some special cases. The first such result is a simple adaptation of a well-known result by Laibson (1996) for the hyperbolic discounting case, where $\beta(c)$ is constant, also assuming no asset limit (i.e. the natural borrowing limit) and power utility functions.

Theorem 1 (Linear Markov Equilibria). *Suppose*

$$\beta(c) = \bar{\beta} \leq 1, \qquad U_1(c) = \frac{1}{1-\sigma}c^{1-\sigma} \qquad and \qquad \underline{a} = 0.$$

Then if $(1-\sigma)r < \rho$ there exists a unique linear Markov equilibrium $\hat{c}(a) = \psi a$ with saving $\psi < r$ when $\bar{\beta} > \hat{\beta}(r, \rho, \lambda)$ and dissaving $\psi > r$ when $\bar{\beta} < \hat{\beta}(r, \rho, \lambda)$. When $\bar{\beta} > \hat{\beta}(r, \rho, \lambda)$ the result holds even if $\underline{a} > 0$.

Proof. Appendix D. □

This result establishes the existence and uniqueness of linear equilibria, characterized by a constant savings rate $\psi - r$.[16] Crucially, the sign of this savings rate depends on β versus $\hat{\beta}$. When the interest rate is low enough or disagreements are high enough, $\beta < \hat{\beta}$, the temptation to overconsume is strong and the agent dissaves in equilibrium. When the interest rate is high enough or disagreements are low enough, so that $\beta > \hat{\beta}$, savings are positive.

The linear equilibrium breaks down if the disagreement ratio $\beta(c)$ is not constant, if utility functions are not power functions, or in the presence of a borrowing constraint $\underline{a} > 0$ if $\beta < \hat{\beta}$. Although the conditions for linear equilibria are very special in this sense, our analysis below will establish that the key conclusion is much more general: savings are positive or negative depending on β versus $\hat{\beta}$.

Before turning to these results, we state a very simple result showing that in the borderline case $\beta(c) = \hat{\beta}(r, \rho, \lambda)$ an equilibrium exists with zero savings. Since consumption

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[16]In discrete time Phelps and Pollak (1968) and Young (2007) show that multiple linear equilibria might exist. But in continuous time, we obtain the uniqueness of linear equilibria. However, we will show in Section 6 that non-linear multiple equilibria may exist.

and wealth remain constant, this result does not require power utility functions nor the absence of borrowing constraints (i.e. it holds even if $\underline{a} > 0$).

Theorem 2 (Zero Savings). *Assume that $\beta(c) = \hat{\beta}(r, \rho, \lambda)$ for all $c > r\underline{a}$. Then, $(V, W) = (\bar{V}, \bar{W})$ and $\hat{c}(a) = ra$ is a Markov equilibrium.*

Proof. Appendix E. □

4.1 Dissaving

Our first result constructs an equilibrium with dissaving, when $\beta(c) < \hat{\beta}(r, \rho, \lambda)$. Since $\beta(c) < 1$ by Assumption 1, this condition is ensured if $r \leq \rho$ or if $r > \rho$ and is small enough.

Theorem 3 (Dissaving). *Suppose Assumption 1 and that $\beta(c) < \hat{\beta}(r, \rho, \lambda)$ for all $c \geq r\underline{a}$. Then there exists a Markov equilibrium with $\hat{c}(\underline{a}) = r\underline{a}$ and $\hat{c}(a) \geq ra$ for $a \geq \underline{a}$; indeed, the latter inequality is strict, $\hat{c}(a) > ra$, except possibly on a countable set of steady state points where $\hat{c}(a) = ra$.*

Proof. Appendix E. □

Recall from Proposition 2 that when $r > \rho$ the commitment solution eventually involved positive savings, with consumption and wealth growing without bound. Thus, dissaving in such a case is due entirely to the lack of commitment and the time-inconsistency problem generated by the disagreement regarding utility functions. Dissaving is more likely the lower is β, the higher is λ or ρ, and the lower is r. Intuitively, a higher turnover rate λ amplifies the time inconsistency problem.

The proof of this theorem implies that there is always an equilibrium with $V \geq \bar{V}$ even though there may be equilibria with $V < \bar{V}$. Thus, there are always equilibria where a commitment device that imposes a borrowing constraint at the current wealth level would not be chosen by the agent in power. However, when $r \geq \rho$ then $W < \bar{W}$ away from steady states (where $W = \bar{W}$), so the agent out of power would always want to bind the agent in power to keep wealth constant at the current level of wealth.[17]

As we anticipated in Section 3.2, our proof is constructive and involves solving for an equilibrium with negative savings by solving the differential system (9). We start at $\underline{a}$ setting $V(\underline{a}) = \bar{V}(\underline{a})$ and $W(\underline{a}) = \bar{W}(\underline{a})$ as the boundary condition, then solve moving up a using the root for $V'(a)$ that implies dissaving. To guarantee that a global solution exists we need to ensure that we do not run out of a root for $V'(a)$, which may occur for

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[17]When $r < \rho$, it is possible that $W > \bar{W}$.

a region of the space $(a, V(a), W(a))$; to do so, we show that we can always engineer an appropriate jump in W to its steady state value $\bar{W}(a)$ to reset the differential equations before the differential system reaches this region.

Theorem 3 provides a general proof of existence of Markov equilibria. Recall that a special case is the standard model with hyperbolic discounting in the presence of a borrowing constraint.[18] Section 6 complements this existence result by providing further characterizations. We explore sufficient conditions for the equilibrium to be continuous or discontinuous. We also show that there may be multiple Markov equilibria.

4.2 Saving

We now consider the opposite case where disagreement is low enough and show that $\beta > \hat{\beta}$ provides conditions for positive savings to emerge in equilibrium. Given Assumption 1, $\beta > \hat{\beta}$ requires $r > \rho$.

Since our approach is constructive, it is useful to first pin down values and behavior above some threshold for wealth, allowing us to concentrate our construction on the remaining compact interval of wealth. The next assumptions are helpful in this regard.

Assumption 2. *Either (a) there is no disagreement for high enough consumption levels: $\beta(c) = 1$ for all $c \geq \bar{c}$ for some $\bar{c} > 0$; or (b) disagreement is constant and the utility functions are power functions for high enough consumption levels: $\beta(c) = \bar{\beta} < 1$ and $U_1(c) = \frac{1}{1-\sigma}c^{1-\sigma}$ for all $c \geq \bar{c}$ for some $\bar{c} > 0$.*

Importantly, this assumption only applies above possibly very high level of consumption and places no restrictions on the disagreement ratio $\beta(c)$ or utility functions below this threshold. We use Assumption 2 to provide a boundary condition at some high enough level of wealth $\bar{a}$. When Assumption 2a holds, the equilibrium coincides with the commitment solution above the threshold. When Assumption 2b holds, the equilibrium coincides with the linear equilibrium described in Theorem 1 above a threshold $\bar{a}$.[19] Using either of these two boundary conditions, the next result constructs an equilibrium with positive savings by solving the differential system (9).

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[18] Current existence proofs in the literature require additional assumptions or ingredients. For example, in discrete time, Harris and Laibson (2001) introduce income shocks . Chatterjee and Eyigungor (2015) introduce lotteries among the choices of the decision maker. Bernheim, Ray and Yeltekin (2015) show the the existence of Markov equilibrium for the special case of power utility and interest rate strictly greater than the discount factor, i.e. $r > \rho$ in our notation.

[19] Another simple, but perhaps less natural, way to provide an upper boundary condition is to assume a maximum wealth level $\bar{a}$ and restrict $a \in [\underline{a}, \bar{a}]$. When $\beta(r\bar{a}) > \hat{\beta}$, we can then construct an equilibrium with savings by using the boundary $\hat{c}(\bar{a}) = r\bar{a}$, $V(\bar{a}) = \bar{V}(\bar{a})$ and $W(\bar{a}) = \bar{W}(\bar{a})$. Theorem 6 involves a similar construction with positive savings to the left of an interior steady-state.

Theorem 4 (Saving). *Suppose that $\beta(\bar{c}) > \hat{\beta}(r, \rho, \lambda)$ and that Assumption 1 and 2 hold. Then there exists $\hat{a} < \frac{\bar{c}}{r}$ such that if $\underline{a} > \hat{a}$ there exists a Markov equilibrium with $\hat{c}(a) < ra$ and $\hat{c}'(a) > 0$ for $a \geq \underline{a}$. Moreover, if $\beta(c)$ is increasing then either $\hat{a}$ is such that $\beta(r\hat{a}) \leq \hat{\beta}(r, \rho, \lambda)$ or $\hat{a} = 0$.*

Proof. Appendix F. □

Just as for Theorem 3, the proof of this result is constructive and works by solving the differential system (9). We start at $\bar{a}$ using initial conditions for $(V(\bar{a}), W(\bar{a}))$ provided by the commitment solution (under Assumption 2a) or the linear solution (under Assumption 2b). We then solve downwards using the higher root for $V'(a)$, which is associated with positive savings. Positive savings implies that $\bar{a}$ will be reached from below, justifying its use as a boundary condition for the differential system. This construction stops either at $\hat{a} = 0$ or at a value of $\hat{a}$ where we run out of a root for $V'(a)$. In the latter case we actually show that $(V(\hat{a}), W(\hat{a})) = (\bar{V}(\hat{a}), \bar{W}(\hat{a}))$ and thus $\hat{c}(\hat{a}) = r\hat{a}$.

The last statement in the theorem implies that if $\beta(c)$ is increasing but always above $\hat{\beta}$ then $\hat{a} = 0$, which implies that savings occur for all wealth levels regardless of the asset limit $\underline{a}$. This leaves open the possibility that if $\beta(c)$ is decreasing but always above $\hat{\beta}$ that $\hat{a} > 0$, although we have been unable to provide an example. Speculating a little, although the level of disagreement is conducive to savings, the fact that disagreement rises with spending may discourage saving.

4.3 An Example

The following example illustrates the previous two theorems.

Example 1. Let the utility for the agent in power be given by

$$U_1(c) = \frac{c^{1-\bar{\sigma}}}{1 - \bar{\sigma}}$$

for $\sigma > 0$ and let disagreement be given by

$$\beta(c) = \begin{cases} \bar{\beta}\left(\alpha\left(\frac{c}{\bar{c}}\right)^{\gamma} + 1 - \alpha\right) & \text{if } c \leq \bar{c}, \\ \bar{\beta} & \text{if } c \geq \bar{c}. \end{cases}$$

with $\alpha, \bar{\beta} \leq 1$ and $\gamma > 0$. Under this specification $\beta(c)$ is an increasing and continuous function of c, reaching a plateau of $\bar{\beta}$ at $c = \bar{c}$. The implied U_0, which defined by U_1 and
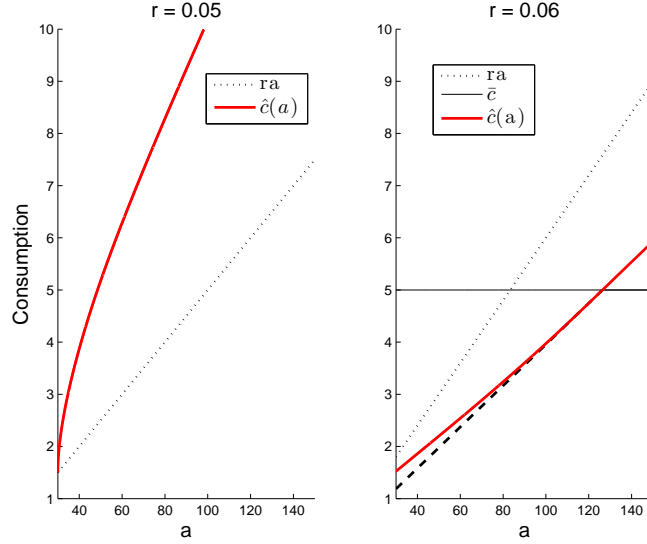
Figure 1: Consumption Functions under Different Interest Rates

$\beta(c)$ up to a constant, is concave as long as $\gamma\alpha \leq \bar{\sigma}$.[20] For our numerical examples we use the following parameters: $\rho = 0.05$, $\bar{\sigma} = \frac{1}{5}$, $\bar{\beta} = \frac{3}{4}$, $\alpha = \frac{3}{4}$, $\gamma = \frac{\bar{\sigma}}{\alpha}$, $\lambda = 0.05$, $\bar{c} = 5$ and $\underline{a} = 30$.

Figure 1 depicts the policy function $\hat{c}(a)$ (solid line) against ra (dotted line). The left panel sets $r = 0.05$ which ensures that $\beta(c) < \hat{\beta}(r, \rho, \lambda)$ so that Theorem 3 applies. Since $\hat{c}(a) > ra$ the agent is dissaving and wealth declines, $\dot{a} < 0$. The consumption function is concave. Indeed, it has infinite slope at the asset limit $\underline{a}$ and this limit is reached in finite time.

The right panel sets $r = 0.06$ ensuring that $\beta(c) > \hat{\beta}(r, \rho, \lambda)$ so that Theorem 4 applies. Since $\hat{c}(a) < ra$ the agent is saving and wealth rises, $\dot{a} > 0$, without bound, $a_t \to \infty$. For a high enough level of wealth the consumption function becomes linear, coinciding with the linear equilibrium, provided in Theorem 1 for the hyperbolic discounting model with $\beta(c) = \bar{\beta}$ (dashed line). This provides the boundary condition needed for our construction. For lower levels of wealth the equilibrium consumption function is nonlinear and slightly convex.

5 Non-Uniform Disagreement

We now turn our attention to situations where disagreement varies sufficiently so that $\beta(c)$ lies on both sides of $\hat{\beta}(r, \rho, \lambda)$, preventing the conditions for global dissaving or sav-

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[20]For $c < \bar{c}$, $U_0''(c) = \bar{\beta}\left(-\bar{\sigma}(1-\alpha) + \alpha\left(\frac{c}{\bar{c}}\right)^{\gamma}(\gamma - \bar{\sigma})\right)c^{-\bar{\sigma}-1} < 0$ if $\gamma\alpha \leq \bar{\sigma}$.

ing equilibria. We focus on two polar cases. In the first, disagreement is decreasing with consumption, so that temptations are stronger at low levels of consumption. We show how this creates the conditions for a poverty trap: dissaving at lower wealth levels coexisting with positive savings at high wealth levels. The second case we consider involves increasing disagreements and leads to convergence to an interior steady state.

5.1 Decreasing Disagreement: Poverty Traps

We start with the case where the disagreement index falls with consumption and lies on both sides of $\hat{\beta}(r, \rho, \lambda)$. By Assumption 1 this requires $r > \rho$.

Assumption 3. *The ratio $\beta(c)$ is weakly increasing in c and crosses $\hat{\beta}(r, \rho, \lambda)$.*

Since $\beta(c) > \hat{\beta}(r, \rho, \lambda)$ for high consumption levels, Theorem 4 provides an equilibrium with savings as long as the asset limit is high enough, so that $\underline{a} \geq \hat{a}$. However, if the asset limit is low enough so that $\underline{a} < \hat{a}$ the construction is incomplete and cannot be completed to yield an equilibrium with positive saving for all wealth levels. Indeed, since $\beta(r\hat{a}) < \hat{\beta}(r, \rho, \lambda)$ dissaving seems like a natural outcome for lower wealth levels, following Theorem 3. This opens the door to poverty traps: positive saving above a cutoff and dissaving below this cutoff. Intuitively, for lower wealth levels consumption lies in a region where disagreement is severe, lowering the incentive to save and perpetuating the time-inconsistency problem. At higher wealth levels disagreements are lower and the agent is able to save and overcome the time-inconsistency problem. Indeed, reaching regions with lower disagreements and time inconsistency provide an additional incentive to save, since the equilibrium allocation may get closer to the commitment solution.[21]

The next result is obtained by combining the constructions underlying Theorems 3 and 4, using Theorem 3 for low wealth levels and Theorem 4 for high wealth levels. The cutoff must be set at a point where the agent in power is indifferent between following the saving path versus the dissaving path.

Theorem 5 (Poverty Trap). *Suppose Assumptions 1, 2, and 3 hold. Then there exists a cutoff a^* and a Markov equilibrium with saving for $a > a^*$ and dissaving for $a < a^*$, i.e. $\hat{c}(a) < ra$ for $a > a^*$ and $\hat{c}(a) \geq ra$ for $a < a^*$.*

Proof. Appendix G ☐

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[21]For example, under Assumption 2a, since then the commitment solution is obtained at sufficiently high wealth levels.
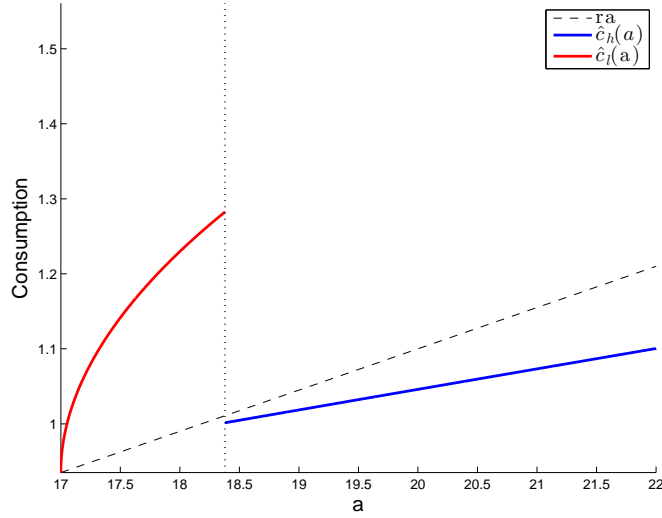
Figure 2: Consumption Function in an Equilibrium with Poverty Trap

This result still leaves open the possibility that $a^* = \underline{a}$, so that there is no region with dissaving, or that $a^* = \infty$, so that there is no region with positive saving. Indeed, these outcome may occur in some cases. Intuitively, dissaving may not occur if the difference between $\beta(c)$ and $\hat{\beta}(r, \rho, \lambda)$ are small. Likewise, positive savings may not occur if β is too low at the top. Ensuring an interior $\hat{a}$ requires additional assumptions.

The next result provides sufficient conditions for a^* to be interior.

Proposition 4. *Suppose Assumptions 1, 2a and 3 and $\underline{a} < \frac{\bar{c}}{\rho}$ (where $\bar{c}$ is defined in Assumption 2a). Then there exists $r^* > \rho$ such that for all $r \in (\rho, r^*]$ the cutoff a^* defined in Theorem 5 is interior: $\underline{a} < a^* < \infty$.*

Proof. Appendix G. □

This result requires a low enough interest rate. Intuitively, when the interest rate r is close to the discount rate ρ the benefit from saving is relatively small. Dissaving becomes relatively more attractive and an equilibrium where the latter occurs at low wealth levels becomes possible. Of course poverty traps with interior a^* also arise away from these sufficient conditions.[22]

Example 2. The following example illustrates a poverty trap under the conditions of Theorem 5. We use the parameters in Example 1 except that $\bar{\beta} = 1$ and $r = 0.055$. We let

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[22]In an earlier version of this paper, we showed the existence of a Markov equilibrium with a poverty trap for $U_1(c) = \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}}$ and $\beta(c) = \underline{\beta}$ for $c < \bar{c}$ and $\beta(c) = 1$ for $c \geq \bar{c}$.

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the asset limit at $\underline{a} = 17$. As depicted in Figure 2 ($\hat{c}_h$ and $\hat{c}_l$ are two pieces of the policy function $\hat{c}$, as constructed in the proof of Theorem 5), $\hat{c}(a) \geq ra$ for $a < a^*$ and $\hat{c}(a) < ra$ for $a > a^*$, where $a^* \approx 18.37$. Thus, there is poverty trap for all $a < a^*$.

We now discuss a few interesting comparative statics. First, consider the asset limit. We argue that loosening the asset limit due to a loosening of the borrowing constraint (recall the connection laid out in Section 2) can actually worsen the incentives for saving, prompting the agent to dissave instead. To see this, suppose initially that $\underline{a} \geq \hat{a}$ where $\hat{a}$ is defined by Theorem 4. An equilibrium with positive savings then exists, where wealth and consumption rising over time starting from any initial wealth level $a \geq \underline{a}$. Next, suppose $\underline{a}$ is lowered and that at its new level $\underline{a} < \hat{a}$ ($\hat{a}$ is unaffected by $\underline{a}$). Then, according to Theorem 5, an equilibrium exists where the agent dissaves below a cutoff $a^* > \hat{a}$, where wealth and consumption eventually fall to a lower level. It follows that for any wealth level in the intermediate region $[\hat{a}, a^*)$ the equilibrium switches from saving to dissaving. In this sense, loosening the asset limit leads an agent to switch from saving to dissaving.

Next, consider an increase in labor income y and suppose for the sake of the present discussion the borrowing limit is proportional to income. Recall that for convenience, we have been working with the transformations outlined in Section 2 that allows us to set income to zero, without loss of generality. Using these transformations, an increase in y amounts to an upward adjustment in the asset limit together with an upward adjustment in the initial (transformed) asset level. As we just argued, the asset limit and the cutoff a^* can move in opposite directions: falling when the asset limit rises. This is necessarily the case when the asset limit is high enough (near $\hat{a}$), which occurs for high enough income. When this is the case, both the increase in assets and the decrease in the cutoff decrease the chances of finding oneself below the cutoff and, thus, trapped in a poverty trap. Moreover, a large enough increase in y increases $\underline{a}$ enough and ensures $\underline{a} > \hat{a}$, making the poverty trap disappear altogether. We are then left with an equilibrium featuring global savings.

Our construction of a Markov equilibrium with a poverty trap relies on nonuniform disagreements. Interestingly, poverty traps are also possible with uniform disagreements, if one relaxes the equilibrium requirement from a Markov equilibrium to that of a sub-game perfect equilibrium. This has been shown by Bernheim et al. (2015). Their model is cast in discrete-time and assumes hyperbolic discounting and power utility. The best equilibrium is sustained by a trigger strategy that punishes deviations by reverting to the worst equilibrium, with dissaving. Intuitively, such punishments are not very effective at low wealth levels, near the asset limit, since there is not much wealth to be dissaved. As a result, good behavior and positive savings can only be sustained at high wealth lev-

els. Clearly, this result is driven by a different and complementary mechanism to ours. If we look at comparative statics, there are also interesting differences in predictions. Bernheim et al. (2015) show that when the borrow constraint is loosened, i.e $\underline{a}$ is lowered, the likelihood of a poverty trap decreases in their setting (punishments are more severe). It can also be shown that an increase in labor income, for given positive level of wealth, always increases the chance of being in a poverty trap in their setting.[23] These contrasting predictions serve to highlight the differences in mechanisms.

As discussed in the Introduction, Banerjee and Mullainathan (2010) study models with disagreements over many goods that imply nonuniform disagreements with respect to spending. In particular, they derive a result for the case of decreasing disagreements that relates to our poverty trap result. However, there are two important differences with our framework and result. First, they work with a two-period setting so they cannot study the long-run dynamics for wealth, making it somewhat difficult to define a poverty trap in this sense. Thus, what they prove is the possibility of a downward discontinuity in the consumption policy function, which may be related to the discontinuity we obtain at the cutoff a^*. The discontinuity in their setting is due to a non-convexity in the optimization problem, which brings us to the second difference: such a discontinuity would never arise in a two-period version of our model. The relevant optimization would be to maximize $U_1(a_0 - a_1) + e^{-\rho}U_0(e^r a_1)$ with respect to a_1. Since we assumed U_1 and U_0 to be strictly concave, the solution is unique and continuous in initial wealth a_0. Poverty traps arise in our model from the strategic interactions across different selves over longer horizons.

5.2 Increasing Disagreement: Convergence

We now consider the reverse situation, when disagreement rises with spending. The time-inconsistency problem is aggravated at higher wealth levels in this case. Intuitively, this may provide a force for dissaving at high wealth levels together with saving at low wealth levels. If these forces are strong enough they may generate convergence to a unique steady state wealth level. We now provide one such result. Our global convergence proof requires a downward jump in $\beta(c)$ at the steady state.[24] We conjecture that a similar result

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[23]This follows because with power utility their equilibria are homogeneous. The borrowing limit expands in proportion to the increase in income. If assets were also changed in the same proportion then the equilibrium would simply scale up. However, since assets are left unchanged and only income is increased, then when assets are positive this is effectively like being less rich (relative to income). As a result, the agent either remains or falls into the poverty trap region.

[24]This requires U_0 to feature a concave kink, which was ruled out up to now by our assumption that U_0 is everywhere differentiable, but we relax this assumption for the next result.

obtains for continuous $\beta(c)$ as long as the function is sufficiently decreasing.[25]

Assumption 4. *Suppose $\beta(c)$ is decreasing. Suppose further that $\beta(c)$ is continuous except at $c^* > 0$ where*

$$\lim_{c \uparrow c^*} \beta(c) > \hat{\beta}(r, \rho, \lambda) > \lim_{c \downarrow c^*} \beta(c).$$

Under this assumption, we show that there is a Markov equilibrium defined over some interval $[\underline{a}, \infty)$ with an interior steady state at $a^* \equiv \frac{c^*}{r}$ that is globally stable: the agent saves when wealth falls below a^* and dissaves when wealth is above a^*, so that $a_t \to a^*$ monotonically.

Theorem 6 (Convergence). *Suppose Assumption 1 and 4 hold. Then there exists $a_{\min} \geq 0$ such that when $\underline{a} > a_{\min}$ then a Markov equilibrium exists with a unique stable stationary state $a^* = \frac{c^*}{r} > \underline{a}$, so that $\hat{c}(a^*) = ra^*$, $\hat{c}(a) < ra$ for $a < a^*$ and $\hat{c}(a) \geq ra$ for $a > a^*$. If, in addition $\beta(c) = \bar{\beta}$ for all $c < c^*$, then $a_{\min} = 0$.*

Proof. Appendix H. □

Intuitively, at high wealth levels the time inconsistency problem leads to dissaving, while at low wealth levels the fact that $r > \rho$ leads to positive savings. Thus, the variable time-inconsistency problem provides a force for convergence, despite a constant interest rate. In Battaglini and Coate (2008), the authors obtain a similar convergence in a model with time-inconsistency arises from political economy frictions. In their paper, time-inconsistency is stronger when debt is low, i.e. wealth is high, and group-specific transfers are strictly positive ("business- as-usual" regime) and there is no time-inconsistency when debt is high, i.e. wealth is low, and group-specific transfers are absent ("responsible policymaking" regime).

Observational Non-Equivalence. Both the existence of equilibria with a poverty trap and the existence of equilibria featuring convergence to an interior steady state illustrate behavior that is patently *not* observationally equivalent to any time consistent consumer with additive utility and constant exponential discounting (with concave or convex utilities). A time-consistent agent would either save or dissave at all wealth levels, depending on the sign of $r - \rho$. Morris and Postlewaite (1997) constructs an example where observational equivalence fails, based on the presence of discontinuities in the equilibrium policy function. Discontinuities are never optimal for time-consistent agents, assuming concave utility functions. Interestingly, our convergence result instead provides a refutation of

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[25]In this case c^* is uniquely determined by $\beta(c^*) = \hat{\beta}$.

observational equivalence without the need to observe discontinuities in the policy function.

Local Indeterminacy. We now provide a local result of a different nature. For any wealth level $\tilde{a}$, suppose we restrict asset choices to a local neighborhood of $\tilde{a}$. We can then construct an equilibrium with $\tilde{a}$ as a stable steady state, as long as β is above $\hat{\beta}$ but not too high.

Theorem 7 (Local Indeterminacy). *Suppose $r > \rho$ and*

$$\beta(r\tilde{a}) \in (\hat{\beta}, 1 - \frac{r - \rho}{\lambda}) \tag{13}$$

for some $\tilde{a} > 0$. There exists a continuous local Markov equilibrium over an interval $[\underline{a}, \bar{a}]$ containing $\tilde{a}$ as the unique stable stationary state i.e. $\hat{c}(\tilde{a}) = r\tilde{a}$, $\hat{c}(a) < ra$ for $a < \tilde{a}$ and $\hat{c}(a) > ra$ for $a > \tilde{a}$.

Proof. Appendix H. □

This result is the continuous-time version of the indeterminacy result emphasized by Krusell and Smith (2003). One small but noteworthy difference is that in our continuous-time setting no discontinuities are present in the construction.

What does this indeterminacy result say and not say? For any $\tilde{a}$ where (13) holds, there is a local Markov equilibrium with $\tilde{a}$ as the unique steady-state. However, it must be emphasized that this result is local in nature in that wealth levels must be restricted to a neighborhood $[\underline{a}, \bar{a}]$ of $\tilde{a}$. Indeed, the interval $[\underline{a}, \bar{a}]$ generally depends on $\tilde{a}$.[26] Unfortunately, this local result cannot be immediately extended to obtain a global result. We have verified numerically that the equilibrium value and policy functions cannot be extended indefinitely to the right for high wealth levels a.[27,28] Indeed, one difficulty relative to our previous results is that $V(a) < \bar{V}(a)$ for all $a \geq \tilde{a}$ so one cannot apply the construction procedure behind Theorem 3, which required $V \geq \bar{V}$.

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[26]Consequently, this local indeterminacy result is not a multiple equilibrium result as it stands, because for each different $\tilde{a}$ the different intervals $[\underline{a}, \bar{a}]$ define different games. See Theorem 12 for a multiple equilibrium result in which different equilibria are defined in the same wealth space $[\underline{a}, \infty)$.

[27]For example, when $U_1(c) = \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}}$ with $\bar{\sigma} = \frac{3}{4}$, $\lambda = 1, \rho = 0.05, r = 0.06$ and $\beta \equiv \bar{\beta} = \frac{9}{10} > \hat{\beta}$, the solution with steady state at $\tilde{a} = \frac{1}{r} \approx 16.66$ cannot be extended past $a = 17.7$.

[28]When $\beta(c)$ is constant to the left of $\bar{c}$ then Theorem 6 shows that the equilibrium can be extended globally to the left.

5.3 Inverting: Consumption Functions to Disagreements

We now explore the model from a different angle, inverting it to solve for $\beta(c)$ given an equilibrium policy function for consumption $\hat{c}(a)$. It turns out that, at least for sufficiently smooth consumption functions this mapping is closed form and, thus, very tractable. To state the next result, define the local curvature of the utility function

$$\sigma\left(U_1, c\right) \equiv \frac{-U_1''(c)c}{U_1'(c)}, \tag{14}$$

which is strictly positive, since U_1 is assumed strictly increasing and strictly concave.

Theorem 8. *Suppose $\hat{c}(a)$ is the consumption function of a Markov equilibrium and $\hat{c}$ is strictly monotone (increasing or decreasing) and twice differentiable within an interval (a_1, a_2). Let $\zeta(c)$ denote the inverse of $\hat{c}(a)$ defined over the associated interval $(\hat{c}(a_1), \hat{c}(a_2))$. Then $\beta(c)$ must equal*

$$\beta(c) = \frac{1}{\lambda\zeta'}\left\{\alpha_1\zeta' + \alpha_2\left(\zeta'\right)^2 + \sigma c\zeta''\left(\frac{r\zeta - c}{c}\right)^2\right.$$

$$\left. + \sigma\left(2 + (2\rho + \lambda - 3r)\zeta'\right)\frac{r\zeta - c}{c} + (\sigma^2 + \sigma - c\sigma')\left(\frac{r\zeta - c}{c}\right)^2\right\} \tag{15}$$

where $\alpha_1 = (\rho + \lambda - r)$ and $\alpha_2 = (\rho - r)(\rho + \lambda - r)$.

Proof. Appendix I. □

    This characterization can be used to derive necessary conditions on the disagreement index $\beta(c)$ for certain properties to emerge in equilibrium. We next state a few examples.

    First, suppose the equilibrium features dissaving for high enough wealth levels, so that $ra - \hat{c}(a) < 0$. Suppose further that $\hat{c}'(a)$ and and $\hat{c}''(a)a$ converge as $a \to \infty$ with $\lim_{a\to\infty} c'(a) > r$. In addition, suppose U_1 is such that σ and $\sigma'c$ converge as $c \to \infty$. Then it follows that $\beta(c) < \hat{\beta}$ for all high consumption c.

    Next, suppose the equilibrium features an interior locally stable steady state $\tilde{a}$ with $\hat{c}(\tilde{a}) = r\tilde{a}$ and $\hat{c}'(\tilde{a}) > r$. Then we must have

$$\beta(r\tilde{a}) > \hat{\beta}.$$

If, in addition, $\hat{c}$ is weakly concave at $\tilde{a}$ so that $\hat{c}''(\tilde{a}) \le 0$ (i.e. marginal propensity to consume that falls with wealth) then we also obtain

$$\beta'(r\tilde{a}) < 0,$$

so that disagreements are increasing. This is consistent with our previous convergence result, which featured increasing disagreements. Moreover, whether or not $\hat{c}''(\tilde{a}) \leq 0$, if the interior steady state is globally stable, then we must have dissaving above $\tilde{a}$. Using the results from the previous paragraph, assuming the other conditions discussed there are met, this requires $\beta < \hat{\beta}$ for high consumption levels. Putting this together with $\beta > \hat{\beta}$ at the steady state we see that $\beta(c)$ must decrease on average and cross $\hat{\beta}$ from above. In this sense, increasing disagreements are necessary for an equilibrium to feature convergence of wealth to an interior steady state.

Equation (15) can also be used to construct equilibria globally, simply by postulating any monotone and twice differentiable consumption function over a range $(\underline{c}, \infty)$ and backing out the required disagreement index $\beta(c)$, obtained in closed form. We only need to ensure that the disagreement index obtained in this way satisfies $\beta(c) \in (0, 1]$. The next example illustrates this procedure.

Example 3. Assume that $U_1(c) \equiv \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}}$ and $\rho + \lambda > r$. Consider a linear consumption function $\hat{c}(a) = r\tilde{a} + \Psi(a - \tilde{a})$ with $\Psi > r$ and $\tilde{a} > 0$. This implies $\dot{a} = -\Psi(a - \tilde{a})$ so that $\tilde{a}$ is a stable steady state. Equation (15) then implies that β is quadratic in $\frac{1}{c}$,

$$\beta(c) = \bar{\beta}_0 + \bar{\beta}_1 \left(\frac{r\tilde{a}}{c} - 1 \right) + \bar{\beta}_2 \left(\frac{r\tilde{a}}{c} - 1 \right)^2,$$

where the coefficients are $\bar{\beta}_0 = \frac{(\rho - r + \Psi)(\rho + \lambda - r)}{\lambda \Psi}$, $\bar{\beta}_1 = \frac{\bar{\sigma}}{\lambda \Psi}(\lambda + 2\rho + 2\Psi - 3r)(\Psi - r)$ and $\bar{\beta}_2 = \bar{\sigma}(\bar{\sigma} + 1)\frac{1}{\lambda \Psi}(\Psi - r)^2$. One confirms that

$$\beta(r\tilde{a}) = \bar{\beta}_0 > \hat{\beta} \qquad \text{and} \qquad \beta'(r\tilde{a}) = -\bar{\beta}_1 \frac{1}{r\tilde{a}} < 0.$$

Figure 3 depicts an example using $\rho = 0.05$, $\bar{\sigma} = 0.8$, $\lambda = 0.05$, $r = 0.07$ and a slope of $\Psi = 0.08$ and $\tilde{a} = 10$. The upper panel depicts the linear consumption function, while the lower panel depicts the disagreement function $\beta(c)$ that sustains this linear policy function as an equilibrium. As it turns out, $\beta(c)$ is everywhere decreasing and $\lim_{c \to \infty} \beta(c) > 0$.

6 Continuity and Multiplicity

In this section, we focus attention on scenarios with dissaving and characterize them in greater detail. In particular, we investigate whether equilibria are continuous or discontinuous and explore the possibility of multiple equilibria.
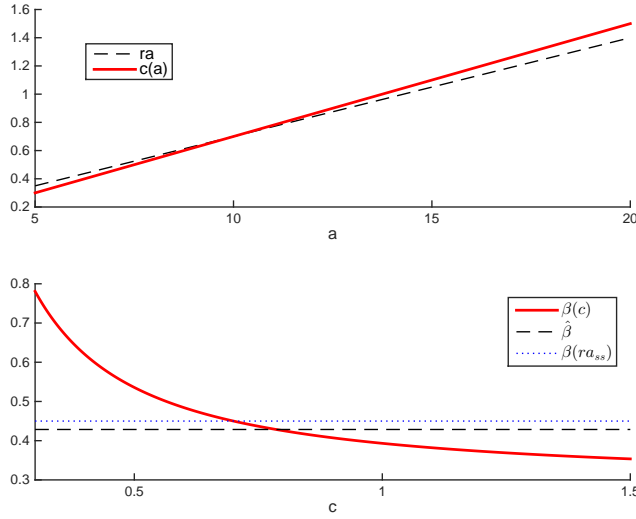
Figure 3: A linear consumption function and its implied disagreement index $\beta(c)$.

Theorem 3 showed that a Markov equilibria with dissaving exists whenever $\beta(c) < \hat{\beta}(r, \rho, \lambda)$. However, this still leaves open a few possibilities. There may exist a continuous equilibrium, one with a single stable steady state at the lower limit $\underline{a}$ and strict dissaving above $\underline{a}$. Alternatively, a discontinuous equilibrium may exist, with other steady states in addition to $\underline{a}$. Finally, both equilibria may exist. The following theorems provide some sufficient conditions for each of these possibilities.

Our first result shows that a continuous equilibrium always exists for interest rates below the discount rate.

Theorem 9 (Continuity I). *Suppose Assumption 1 and $r < \rho$. Then there exists a continuous Markov equilibrium with dissaving, i.e. $V, W, \hat{c}$ continuously differentiable and $\hat{c}(a) > ra$ for $a > \underline{a}$ and $\hat{c}(\underline{a}) = r\underline{a}$. Moreover, the consumption function is strictly increasing, i.e. $\hat{c}'(a) > 0$.*

Proof. Appendix J. □

Recall that the form of discontinuities we encounter are related to the possibility of alternative steady states above $\underline{a}$. When $r < \rho$ the forces for dissaving are present even without disagreements ($\beta = 1$ or $\lambda = 0$). Disagreement only reinforce this tendency for dissaving, preventing other steady states and implying that continuous equilibria exist.

In contrast, when $r \geq \rho$ dissaving arises from the time-inconsistency problem due to disagreements. To ensure the existence of a continuous equilibrium we must be sure that the time-inconsistency problem is strong enough. Our next result provides sufficient conditions that ensure this is the case.

Theorem 10 (Continuity II). *Suppose $r \geq \rho$, $\sup_{c \geq r\underline{a}} \beta(c) < \frac{\rho}{r}$ and*

$$\inf_{c \geq r\underline{a}} \frac{1 - \sigma(U_1, c)}{\beta(c)} > 1, \tag{16}$$

where the local curvature σ is defined in (14).Then a continuous Markov equilibrium exists with dissaving for all large enough λ or low enough $r \geq \rho$, i.e. $V, W, \hat{c}$ are continuously differentiable, with $\hat{c}(a) > ra$ for all $a > \underline{a}$, $\hat{c}(\underline{a}) = r\underline{a}$, and $\hat{c}'(a) > 0$. Finally, when $r = \rho$, a continuous Markov equilibrium exists for any $\lambda > 0$.

Proof. Appendix J. □

Theorem 10 requires the local curvature of the utility function to be low relative to the degree of time-inconsistency. In particular, it requires $\sigma < 1$ since $\hat{\beta} \leq 1$. Figure 4, based on Example 4 described below, illustrates a continuous Markov equilibrium in the left panels. The top panel shows the value functions, while the bottom panels presents the consumption policy function $\hat{c}(a)$ as well as ra for reference. Since $\hat{c}(a) \geq ra$ there is dissaving and $\dot{a} \leq 0$. Indeed, the equilibrium features $\hat{c}(a) > ra$ for all $a > \underline{a}$, so that $\dot{a} < 0$ for all $\underline{a}$ and, thus, $\underline{a}$ is a unique globally stable steady state.

Theorems 9 and 10 stand in contrast with the discrete-time results in Chatterjee and Eyigungor (2015, Theorems 3 and 4) that show that, in the hyperbolic discounting case with $r \leq \rho$ and power utility functions, a continuous Markov equilibrium does not exist. As our two theorems make clear, continuous Markov equilibria do exist under these conditions in our continuous time setting. Indeed, our results apply away from the hyperbolic discounting case and allow for $r > \rho$ and utilities that are not power functions.[29]

In the hyperbolic discounting case with power utility functions condition (16) is the opposite of the condition imposed by Harris and Laibson (2013) to guarantees the existence of an equilibrium in the instantaneous gratification limit $\lambda \to \infty$ (see our Online Appendix for details).[30] Consistent with this observation, numerically we find that under condition (16) the continuous Markov equilibrium identified by our theorem does not converge to a the solution of the limiting system, i.e. the instantaneous gratification

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[29]To understand the difference between continuous and discrete time in this regard note that Chatterjee and Eyigungor (2015) rule out continuous Markov equilibria by observing that there exists an interval of assets near the debt limit (the flat spot) where the decision maker chooses to go to the debt limit. A discontinuity in the policy function must appear close to the end of this interval. However, this is no longer true in continuous time because it always takes a positive amount of time to reach the debt limit, since wealth must move continuously.

To reconcile these results, it is natural to conjecture that in the discrete time approximation of our continuous time model, as periods are shortened by adjusting interest rates and discount rates, these discontinuities shrink in size.

[30]When $U_1(c) = \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}}$ and $U_0(c) = \bar{\beta} U_1(c)$, Harris and Laibson (2013) require that $1 - \bar{\beta} < \bar{\sigma}$.

limit. Instead, overconsumption becomes extreme, with the sequence of policy functions diverging with $\hat{c}(a) \to \infty$ as $\lambda \to \infty$ for any $a > \underline{a}$.

Our next result provides sufficient conditions to guarantee the existence of discontinuous Markov equilibria.

Theorem 11 (Discontinuity). *Suppose $r > \rho$, $\sup_{c \geq r\underline{a}} \beta(c) < \frac{\rho}{r}$ and either U_1 is bounded above or*

$$\sigma(U_1, c) > \underline{\sigma} \qquad \text{for all } c > \bar{c}, \tag{17}$$

for some $\bar{c} > 0$ and $\underline{\sigma} > 0$. Then there exists a Markov equilibrium with dissaving and a discontinuous policy function $\hat{c}(a)$ for all large enough λ.

Proof. Appendix J. ☐

The proof of this theorem involves showing that for large enough λ the solution to the differential system (9) delivers a value function V_λ that crosses $\bar{V}_\lambda$ from above at some finite $a_1(\lambda) > \underline{a}$. We can then engineer a jump in W_λ to $\bar{W}_\lambda$ at $a_1(\lambda)$ and apply Theorem 3 to construct the rest of the equilibrium, for $a \geq a_1(\lambda)$, by letting $a_1(\lambda)$ stand in for $\underline{a}$. This induces an equilibrium with a jump in $\hat{c}(a)$ at $a_1(\lambda)$.

Figure 4, based on Example 4 described below, presents a discontinuous Markov equilibrium with many steady-states in the right panels. The top panel shows the value functions, while the bottom panels presents the consumption policy function $\hat{c}(a)$ as well as ra for reference. Since $\hat{c}(a) \geq ra$ there is dissaving and $\dot{a} \leq 0$. However, the equilibrium features many steady states where $\hat{c}(a) = ra$. At these points $V(a) = \bar{V}(a)$ and $W(a) = \bar{W}(a)$ as required. Note that $V(a)$ is continuous, whereas $W(a)$ and $\hat{c}(a)$ feature jumps to the left of the steady state points.

When $r > \rho$, the conditions required by Theorems 10 and 11 are actually compatible, as can be easily seen. This then guarantees situations where multiple Markov equilibria coexist, both a continuous and a discontinuous equilibrium. Our next theorem summarizes this observation.

Theorem 12 (Multiplicity). *Suppose $r > \rho$, $\sup_{c \geq r\underline{a}} \beta(c) < \frac{\rho}{r}$ and both (16) and (17) hold. Then there for all λ large enough there exist at least two Markov equilibria.*

Unlike the local indeterminacy result offered by Krusell and Smith (2003) and echoed in our Theorem 7, the present result provides conditions for the global existence of multiple equilibria, over the entire set of wealth levels.

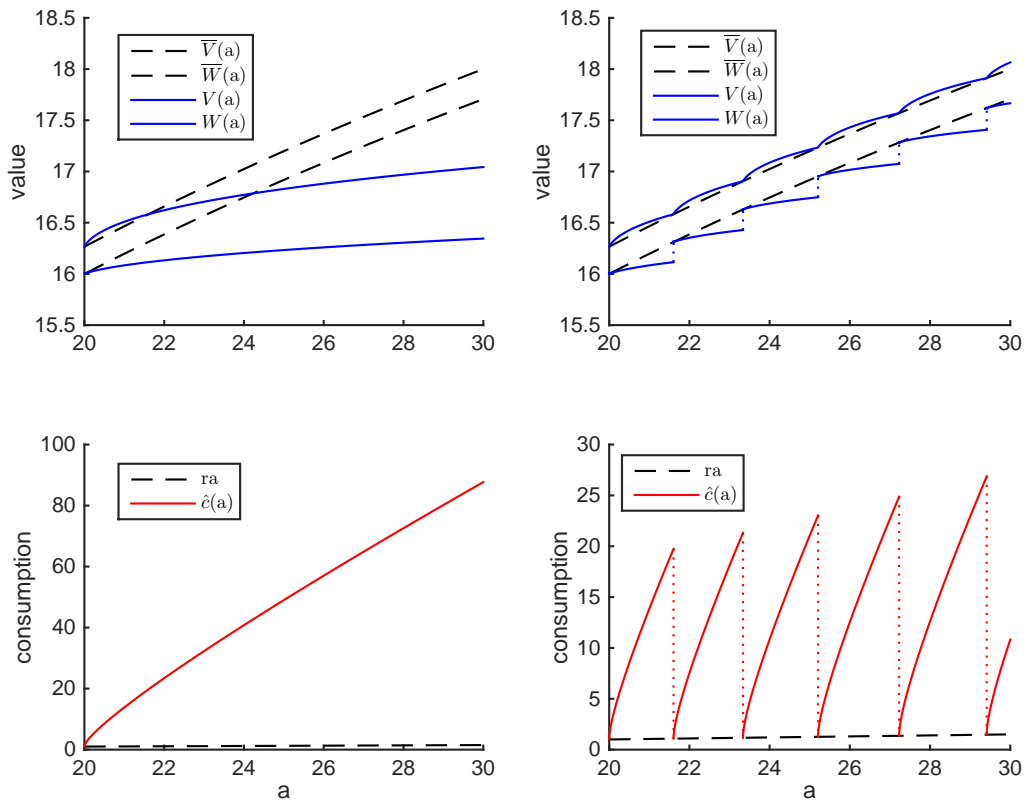The following numerical example illustrates these results.

Figure 4: Multiple Equilibria under hyperbolic discounting (in the figure, $V > W$ and $\overline{V} > \overline{W}$)

Example 4. Suppose hyperbolic discounting and power utilities

$$\beta(c) = \bar{\beta} \qquad \text{and} \qquad U_1(c) = \frac{c^{1-\bar{\sigma}}}{1-\bar{\sigma}},$$

with the following parameters similar to Harris and Laibson (2013)

$$\rho = 0.05 \qquad r = 0.05 \qquad \bar{\sigma} = \frac{3}{4}.$$

When $\bar{\beta}$ is sufficiently low, so that $\bar{\sigma} < 1 - \beta$ and λ is sufficiently high Theorems 10 and 11 can be applied.[31] Figure 4 describes two equilibria when $\beta = \frac{1}{5}$ and $\lambda = 12$. The equilibrium on the left panels (the upper panel depicts value functions and the lower panel depicts the policy function), has continuous value and policy functions as constructed in Theorem 10. Given that λ is sufficiently high, another equilibrium with multiple steady states exists and is depicted in the right panels.

Which equilibrium is more reasonable? We have explored introducing uncertainty using diffusion processes to either income or the rate of return to wealth. For parameters where Theorem 12 applies, we find that the noisy perturbations select the continuous equilibrium: as the variance of the shocks is taken to zero the equilibrium policy and value functions converge to the continuous equilibrium of the game without uncertainty. As Example 4 illustrates, the continuous equilibrium features greater dissaving. In this sense, uncertainty seems to exacerbate the time inconsistency problem.[32]

7 Conclusion

The study of time-inconsistency problems has produced many insights and potential explanations for behavior. We believe this fruitful exploration should continue into new territories. Here we have offered an effort to expand the range of disagreements under study in the context of a standard dynamic stationary savings game. In the process, we have offered an approach and results that are of interest, even for the case of uniform disagreements, i.e. hyperbolic discounting. We also characterized the rich dynamics that are possible with increasing or decreasing disagreements. Much more remains to be done in future research.

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[31]$\lambda = 12$ corresponds to monthly switching frequency.

[32]When a continuous equilibrium does not exist, for example when $r > \rho$ and condition (16) does not hold (the case considered by Harris and Laibson (2013)), we find that, for fixed λ, a solution fails to exist numerically when uncertainty is small enough. In addition, for larger noise variances, the solutions do not seem to converge as we reduce the variance. More details are provided in the Online Appendix.

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Appendix

A General Properties

A.1 Proof of Proposition 1

We show existence and uniqueness at the same time. By the Envelope Theorem

$$U_1'(c) = h'(\hat{x}(c)) = g'(\hat{z}(c)).$$

Moreover,

$$U_0'(c) = h'(\hat{x}(c))\hat{x}'(c).$$

Therefore, $\hat{x}'(c) = \frac{U_0'(c)}{U_1'(c)}$. Thus

$$\hat{x}(c) = \int_0^c \frac{U_0'(\tilde{c})}{U_1'(\tilde{c})} d\tilde{c},$$

which is strictly increasing in c since $U_0', U_1' > 0$. Furthermore,

$$\hat{z}(c) = c - \hat{x}(c) = \int_0^c \left(1 - \frac{U_0'(\tilde{c})}{U_1'(\tilde{c})}\right) d\tilde{c},$$

which is also increasing in c because $1 - \frac{U_0'(\tilde{c})}{U_1'(\tilde{c})} \geq 0$ by Assumption 1. So $h(x)$ and $g(z)$ are uniquely determined (up to constants) by

$$h'(\hat{x}(c)) = U_1'(c)$$

and

$$g'(\hat{z}(c)) = U_1'(c).$$

h and g are increasing because $h', g' > 0$ and are concave because $\hat{x}$ and $\hat{z}$ are increasing in c and U_1' is decreasing in c.

A.2 Roots of Hamilton-Jacobi-Bellman Equations

Write the HJB equation (9a) as

$$(\rho + \lambda) V(a) - \lambda W(a) = H(V'(a), a),$$ (18)

where

$$H(p,a) \equiv \sup_c \{U_1(c) + p(ra - c)\}. \tag{19}$$

The next lemma characterizes the function H.

Lemma 2. *For any a, the function $H(\cdot, a)$ defined by (19) is continuous, strictly convex and continuously differentiable for $p > 0$; has a unique interior minimum at $p = U'(ra)$; satisfies $\lim_{p \to \infty} H(p, a) = \infty$ and $H(0, a) = \lim_{p \to 0} H(p, a) = U_1(\infty)$.*

Proof. For any $p > 0$ a maximum is attained on the right hand side of (19) uniquely by the first-order condition $U_1'(c) = p$. This implies that $H(a, \cdot)$ differentiable with derivative $H_p(p, a) = ra - (U_1')^{-1}(p)$. This derivative is continuous and strictly increasing. Thus, $H(p, a)$ is strictly convex in p. Since $H_p(U_1'(ra), a) = 0$ then $p = U_1'(ra)$ is the unique minimum. Since $H(a, \cdot)$ is strictly convex it follows that $\lim_{p \to \infty} H(p, a) = \infty$.

Finally, by definition $H(0, a) \equiv \sup_c U_1(c) = \lim_{c \to \infty} U_1(c)$. This also coincides with $\lim_{p \to 0} H(p, a)$ since

$$\lim_{p \to 0} H(p, a) = \lim_{c \to \infty} \left(U_1(c) + U_1'(c)(ra - c) \right) \leq \lim_{c \to \infty} U_1(c),$$

$$\lim_{p \to 0} H(p, a) \geq \lim_{p \to 0} (U_1(p^{-\frac{1}{2}}) + p(ra - p^{-\frac{1}{2}})) = \lim_{c \to \infty} U_1(c). \qquad \square$$

This has immediate implications for the possible solutions to equation (18).

Lemma 3. *Consider solutions $p = V'(a)$ to equation (18), if*

Case 1. $(\rho + \lambda) V(a) - \lambda W(a) < U_1(ra)$, *then no solution exists;*

Case 2. $(\rho + \lambda) V(a) - \lambda W(a) = U_1(ra)$, *then the unique solution is given by $p = U_1'(ra)$;*

Case 3. $U_1(ra) < (\rho + \lambda) V(a) - \lambda W(a) \leq U_1(\infty)$, *then exactly two solutions p_1 and p_2 exist and $0 \leq p_1 < U_1'(ra) < p_2$;*

Case 4. $U_1(\infty) < (\rho + \lambda) V(a) - \lambda W(a)$, *then a unique solution exists and $U_1'(ra) < p$.*

Given Lemma 3, we define the following subsets of $\mathbb{R}^3$:

$$E \equiv \{(a, V, W) \,|\, a > 0 \text{ and } (\rho + \lambda) V - \lambda W > U_1(ra) \},$$
$$E_0 \equiv \{(a, V, W) \,|\, a > 0 \text{ and } U_1(\infty) > (\rho + \lambda) V - \lambda W > U_1(ra) \},$$
$$E_s \equiv \{(a, V, W) \,|\, a > 0 \text{ and } (\rho + \lambda) V - \lambda W = U_1(ra) \}$$

Lastly $\bar{E} = E \cup E_s$, and $\bar{E}_0 = E_0 \cup E_s$. Notice that E_s corresponds to the set of singular points of the differential (9) as an implicit ODE.

Using Lemma 3 we now rewrite system (9) as explicit ODEs. There are two systems to consider, depending on whether we consider the high or lower root.

Definition 1. Let $R_l\,(a, V, W)$ denote the lower root $p = V'(a)$ of equation (18). By Lemma 3, R_l is well-defined over $\bar{E}_0$ and is continuous in a, V, W. Let $S_l\,(a, V, W)$ denote the associated solution to W' in equation (9b), so that

$$S_l\,(a, V, W) = \frac{U_0(\hat{c}(a)) - \rho W}{\hat{c}(a) - ra}$$

with $\hat{c}(a) = (U_1')^{-1}\,(V'(a)) = (U_1')^{-1}\,(R_l(a, V, W)) > ra$, defined over E_0. By the Implicit Function Theorem, R_l and S_l are continuously differentiable in a, V, W over E_0.

Using R_l and S_l, system (9) can be represented as an explicit ODE

$$\begin{pmatrix} V'(a) \\ W'(a) \end{pmatrix} = \begin{pmatrix} R_l\,(a, V, W) \\ S_l\,(a, V, W) \end{pmatrix}. \tag{20}$$

This ODE is regular around $(a, V, W) \in E_0$. Around any regular point we can apply standard extension results (for example, Picard–Lindelöf theorem or Cauchy–Lipschitz theorem; see Hartman (2002) for a comprehensive exposition) to show that, the ODE (20) admits a unique solution (V, W) defined over a neighborhood of a, $(a - \epsilon, a + \epsilon)$, and is twice continuously differentiable (because R_l and S_l are continuously differentiable), such that $(V(a), W(a)) = (v, w)$.[33]

The next definition is analogous, but using the higher root of equation (18).

Definition 2. Let $R_h\,(a, V, W)$ be the higher root for $p = V'(a)$ of equation (18). By Lemma 3, R_h is well-defined over $\bar{E}$ and is continuous in a, V, W. Let $S_h\,(a, V, W)$ be the associated value W' in equation (9b), so that

$$S_h\,(a, V, W) = \frac{\rho W - U_0(\hat{c}(a))}{ra - \hat{c}(a)},$$

where $\hat{c}(a) = (U_1')^{-1}\,(V'(a)) = (U_1')^{-1}\,(R_h(a, V, W)) < ra$, defined over E. By the Implicit Function Theorem, R_h and S_h are continuously differentiable in (a, V, W) over E.

Using R_h and S_h, system (9) can be represented as an explicit ODE

$$\begin{pmatrix} V'(a) \\ W'(a) \end{pmatrix} = \begin{pmatrix} R_h\,(a, V, W) \\ S_h\,(a, V, W) \end{pmatrix}. \tag{21}$$

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[33]For any solution $x(a)$ to an ODE $x'(a) = F(x(a))$. If F is continuously differentiable then x is twice continuously differentiable, and $x''(a) = \nabla F(x) \cdot x' = \nabla F(x) \cdot F(x)$.

This ODE is regular around any $(a, V, W) \in E$. Just as with (20), standard extension results apply whenever (a, V, W) is regular.

A.3 Full Commitment Solution

A.3.1 Proof of Proposition 2

When $r < \rho$, the result comes directly from the first order condition in c_t and the observation that whenever a_t reaches $\underline{a}$ the optimal solution features $a_s = \underline{a}$ and $c_s = r\underline{a}$ for all $s \geq t$. In this Appendix, we show the result for when $r > \rho$.

From the evolution of wealth (3),

$$a_t = \int_0^\infty e^{-rs} c_{t+s} ds \geq \underline{a}. \tag{22}$$

Let $\overline{U}(c, t) = e^{-\lambda t} U_1(c) + (1 - e^{-\lambda t}) U_0(c)$. The objective function of the commitment problem can be rewritten as $\int_0^\infty e^{-\rho t} \overline{U}(c_t) dt$.

Consider a variation where we decrease c_t by $\varepsilon > 0$ and increase c_{t+s} by $e^{rs}\varepsilon$ then this increases the objective locally if and only if

$$\overline{U}_c(c_t, t) < e^{-(\rho - r)s} \overline{U}_c(c_{t+s}, t + s).$$

This variation satisfies the budget constraint and weakly increases wealth at any time so it is feasible. Thus a necessary condition for an optimum is that

$$\overline{U}_c(c_{t+s}, t + s) \leq e^{(\rho - r)s} \overline{U}_c(c_t, t).$$

In particular
$$\overline{U}_c(c_t, t) \leq e^{(\rho - r)t} \overline{U}_c(c_0, 0) \to 0,$$

as $t \to \infty$.

Moreover, by Assumption 1,

$$\overline{U}_c(c_t, t) = e^{-\lambda t} U_1'(c_t) + (1 - e^{-\lambda t}) U_0'(c_t) \geq \underline{\beta} U_1'(c_t). \tag{23}$$

Therefore $U_1'(c_t) \to 0$ as $t \to \infty$. By the INADA condition, this implies $c_t \to \infty$ which then requires $a_t \to \infty$, from (22).

Thus, there is a date T for which all $t \geq T$, the borrowing constraint, $a_t \geq \underline{a}$, is not binding. For any dates $t \geq T$, we can perform the same variation as above but with $\varepsilon < 0$,

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thus we must have

$$\overline{U}_c(c_{t+s}, t+s) = e^{(\rho-r)s}\overline{U}_c(c_t, t). \tag{24}$$

at an optimum. This then implies that c_t is monotone for all $t \geq \hat{T}$ for some $\hat{T} > T$.

Indeed, differentiating both sides of (24) with respect to s, together with letting $\mu = \overline{U}_c(c_T, T) > 0$ and using t standing for $T + s$ and T standing for t, we obtain

$$\left(e^{-\lambda t}U_1''(c_t) + (1 - e^{-\lambda t})U_0''(c_t)\right)\dot{c}_t = -(r-\rho)\mu e^{-(r-\rho)(t-T)} + \lambda e^{-\lambda t}(U_1'(c_t) - U_0'(c_t))$$

$$\leq -(r-\rho)\mu e^{-(r-\rho)(t-T)} + \lambda e^{-\lambda t}U_1'(c_t)(1 - \underline{\beta})$$

$$\leq -(r-\rho)\mu e^{-(r-\rho)(t-T)} + \lambda e^{-\lambda t}\mu e^{-(r-\rho)(t-T)}\frac{1 - \underline{\beta}}{\underline{\beta}},$$

where the first inequality comes from Assumption 1 that $U_0'(c_t) > \underline{\beta}U_0'(c_t)$ and the second inequality comes from (23) and (24). There exists $\hat{T} > T$ such that $(r-\rho) > \lambda e^{-\lambda t}\frac{1-\underline{\beta}}{\underline{\beta}}$ for all $t \geq \hat{T}$. Moreover, $e^{-\lambda t}U_1''(c_t) + (1 - e^{-\lambda t})U_0''(c_t) < 0$, therefore $\dot{c}_t > 0$ for all $t \geq \hat{T}$.

From (22), $\dot{a}_t > 0$ for all $t \geq \hat{T}$.

A.3.2 Properties of the Commitment Solution

In this subsection, we characterize the commitment solution under Assumption 2a, i.e. there exists $\bar{c}$ such that $U_1'(c) = U_0'(c)$ for all $c \geq \bar{c}$. The commitment solution will be used as a boundary condition for the equilibrium value functions in Theorems 4 and 5.

In this case

$$U_1(c) = U_0(c) + \bar{u}$$

for all $c \geq \bar{c}$ for some $\bar{u} \in \mathbb{R}$. Without loss of generality in what follows we normalize $\bar{u} = 0$.

Because $\beta(c) \leq 1$, from Assumption 1, $U_1(c) \leq U_0(c)$ for all $c \leq \bar{c}$. It is then immediate that

$$V_{sp}(a_0) \leq V_0^*(a_0) \equiv \max \int e^{-\rho t}U_0(c(t)) \qquad \text{s.t. } a_0 = \int_0^\infty e^{-rt}c_t dt.$$

Consider the variations as in the proof of Proposition 2, for an optimum to the maximization problem in the right-hand side,

$$U_0'(c_t) = e^{(\rho-r)t}U_0'(c_0)$$

which gives c_t as a strictly increasing function of c_0 and is strictly increasing in t because

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$\rho < r$. We then pin down c_0 as a function of a_0 from

$$a_0 = \int_0^\infty e^{-rt} c_t dt.$$

The right-hand side is strictly increasing in c_0, so $c_0 = \hat{c}_0(a_0)$ is uniquely pinned down from this equation and is strictly increasing in a_0.

Let $a_u = \max \left\{ \int_0^\infty e^{-rt} \left(U_0' \right)^{-1} \left(e^{(\rho-r)t} U_0'(\bar{c}) \right) dt, \underline{a} \right\}$. Then, for $a_0 \geq a_u$, $\hat{c}_0(a_0) \geq \bar{c}$. Because c_t is strictly increasing in t, $c(t) \geq \bar{c}$ for all $t \geq 0$, and $a(t) = \int_0^\infty e^{-rt} c_{t+s} ds \geq a_0 \geq \underline{a}$ for all $t \geq 0$. Then it follows that $V_{sp}(a_0) = V_0^*(a_0)$ for all $a_0 \geq a_u$.

It is standard to show that the value function $V_0^*(a)$ is concave and differentiable and satisfies an HJB equation,

$$\rho V_0^*(a) = \max_c U_0(c) + V_0^{*\prime}(a)(ra - c),$$

and the policy function $\hat{c}(a) = \arg\max_c U_0(c) + V_0^{*\prime}(a)(ra - c)$ gives the optimal solution $\hat{c}_0(a_0)$ defined above. Thus $\hat{c}$ satisfies $\hat{c}(a_0) \geq \bar{c}$ for all $a_0 \geq a_u$. It then follows that system (9) holds for $V = V_0^*$ and $W = V_0^*$ for all $a_0 \geq a_u$.

A.4 Useful Observations

The following general properties of the solutions to system (9) is also important for their characterization.

Lemma 4. *Assume that V, W and $\hat{c}$ constitutes a solution to the system (9). If V and W are continuously differentiable and V is twice differentiable at a, then*

$$(\rho + \lambda - r) V'(a) - \lambda W'(a) = V''(a)(ra - \hat{c}(a)) \tag{25}$$

and if $\hat{c}(a) \neq ra$:

$$\hat{c}'(a) = \frac{V''(a)}{U_1''(\hat{c}(a))} = \frac{1}{U_1''(\hat{c}(a))} \frac{(\rho + \lambda - r) V'(a) - \lambda W'(a)}{ra - \hat{c}(a)} \tag{26}$$

Proof. Differentiating (9a) with respect to a, we obtain

$$(\rho + \lambda) V'(a) - \lambda W'(a) = U_1'(\hat{c}(a))\hat{c}'(a) + V'(a)(r - \hat{c}'(a)) + V''(a)(ra - \hat{c}(a)).$$

Combining this with (12) and rearranging yield (25). Now differentiating (12) with respect

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to a,
$$U_1''(\hat{c}(a))\hat{c}'(a) = V''(a),$$

or equivalently $\hat{c}'(a) = \frac{V''(a)}{U_1''(\hat{c}(a))}$, which together with (25) yields (26). □

Lastly, we will also use the follow result to connect the comparison between $\beta(.)$ and $\hat{\beta}$ to the comparison between the slopes of $\overline{V}$ and U_1'.

Lemma 5. *For $a > 0$, $\beta(ra) < \hat{\beta}(r, \rho, \lambda)$ if and only if*

$$\overline{V}'(a) < U_1'(ra). \tag{27}$$

And $\beta(ra) = \hat{\beta}(r, \rho, \lambda)$ if and only if $\overline{V}'(a) = U_1'(ra)$.

Proof. Using the definition (11) for $\overline{V}$, we have

$$\overline{V}'(a) = \frac{r}{\rho + \lambda}U_1'(ra) + \frac{\lambda r}{(\rho + \lambda)\rho}U_0'(ra).$$

The condition that $\hat{\beta}(ra) < \hat{\beta}(r, \rho, \lambda)$ is equivalent to

$$\frac{r}{\rho + \lambda}U_1'(ra) + \frac{\lambda r}{(\rho + \lambda)\rho}U_0'(ra) < U_1'(ra).$$

The result then follows. Likewise, for the case with $\beta(ra) = \hat{\beta}(r, \rho, \lambda)$. □

B A Single-Crossing Property

The following simple result on the comparison between two functions plays a crucial role in the characterization of the solutions to system (9). Although this result is very simple, we do not know of any reference, so include it here for completeness.[34]

Lemma 6. *Let f and g be two continuously differentiable functions defined over $[\underline{a}, \overline{a}]$. Consider the subset satisfying the requirements that (1) $f(\underline{a}) \geq g(\underline{a})$; and (2) if $f(a) = g(a)$ for some $a \in [\underline{a}, \overline{a}]$ then $f'(a) > g'(a)$. Then $f(a) > g(a)$ for all $a \in (\underline{a}, \overline{a}]$.*

Proof. First, observe that, if $f(\underline{a}) = g(\underline{a})$, by property 2. $f'(\underline{a}) > g'(\underline{a})$, therefore $f(a) > g(a)$ in a neighborhood to the right of $\underline{a}$. If $f(\underline{a}) > g(\underline{a})$, we obtain the same result by continuity. Now, we prove the lemma by contradiction. Assume that, there exists $\tilde{a} \in [\underline{a}, \overline{a}]$ such that $f(\tilde{a}) \leq g(\tilde{a})$. By the Intermediate Value Theorem, we can assume that

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[34]See Cao (2014) for an earlier application of this result in the context of two-agent dynamic games.

$f(\tilde{a}) = g(\tilde{a})$, without loss of generality. Now let $a^* = \inf \{a \in [\underline{a}, \bar{a}] : f(a) = g(a)\}$. By continuity $f(a^*) = g(a^*)$. Moreover, $a^* > \underline{a}$ because $f(a) > g(a)$ in the right neighborhood of $\underline{a}$. By property 2), $f'(a^*) > g'(a^*)$. Together with $f(a^*) = g(a^*)$, this implies, $f(a) < g(a)$ in a neighborhood to the left of a^*. Therefore by the Intermediate Value Theorem, there exists $a^{**} \in (\underline{a}, a^*)$ such that $f(a^{**}) = g(a^{**})$. This contradicts the definition of a^* which is the infimum. $\qquad\square$

We also use a few variations of this lemma.

Variation 1. if 1)$f(\bar{a}) \geq g(\bar{a})$, and 2) if $f(a) = g(a)$, for some $a < \bar{a}$, then $f'(a) < g'(a)$, we have $f(a) > g(a)$ for all $a \in [\underline{a}, \bar{a}]$.

Variation 2. We can also relax condition 2, by the condition that if $f(a) = g(a)$ then $f'(\tilde{a}) > g'(\tilde{a})$ in a neighborhood to the left of a. Indeed, in the proof above, if $f(a^*) = g(a^*)$ and $f'(\tilde{a}) > g'(\tilde{a})$ in the left neighborhood of a^*, then for a in the left neighborhood of a^*,

$$
\begin{aligned}
f(a) &= f(a^*) - \int_a^{a^*} f'(\tilde{a})d\tilde{a} \\
&= g(a^*) - \int_a^{a^*} f'(\tilde{a})d\tilde{a} \\
&< g(a^*) - \int_a^{a^*} g'(\tilde{a})d\tilde{a} = g(a).
\end{aligned}
$$

We can then proceed as in the remaining of the proof. This variation is useful when f' or g' are not well-defined at some a.

C   Proofs for Lemma 1

C.1   Proof of Lemma 1

For any $\epsilon > 0$ sufficiently small, indeed satisfying

$$
\epsilon < \frac{\lim_{c \to +\infty} U_1(c) - U_1(ra_0)}{\lambda},
$$

consider the solution (V_ϵ, W_ϵ) to the ODE (20) satisfying the initial condition

$$
(V_\epsilon(a_0), W_\epsilon(a_0)) = \left(\overline{V}(a_0), \overline{W}(a_0) - \epsilon\right). \tag{28}
$$

Given that (20) is regular around a_0, we can apply standard ODE existence results to show that (V_ϵ, W_ϵ) exists and is unique over some interval $[a_0, a_0 + \omega_\epsilon]$ that depends on ϵ. We will use (V_ϵ, W_ϵ), together with the supporting results, Lemmas 7-10 in Subsection C.2, to construct the equilibrium described in Lemma 1 as follows:

First, Lemma 7 shows that there exists an $\omega > 0$ and $\bar{\epsilon} > 0$ such that for $0 < \epsilon < \bar{\epsilon}$ such that (V_ϵ, W_ϵ) are defined over $[a_0, a_0 + \omega]$. Second, Lemma 9 shows that for $0 < \epsilon < \bar{\epsilon}$, the slopes of V_ϵ and W_ϵ are uniformly bounded over $[a_0, a_0 + \omega]$. Finally, using these two results and applying the Dominated Convergence Theorem, we show that (V_ϵ, W_ϵ) converges to (V, W) for a subsequence $\epsilon_N \to 0$ and (V, W) is a solution to system (9).

We now describe this last step in detail. Lemma 7 shows that there exist $\omega > 0$ and $\bar{\epsilon} > 0$ such that: for any $\epsilon < \bar{\epsilon}$ the solution $(V_\epsilon(a), W_\epsilon(a))$ are defined over $[a_0, a_0 + \omega]$ and that $V_\epsilon(a) > \overline{V}(a)$ for all $a \in (a_0, a_0 + \omega]$. Lemma 9 implies that for all $a \in [a_0, a_0 + \omega]$,

$$0 \leq W'_\epsilon(a) \leq U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra),$$

$$0 \leq V'_\epsilon(a) \leq U'_1(ra).$$

Because the derivatives V'_ϵ and W'_ϵ are uniformly bounded, the families of functions $\{V_\epsilon\}$ and $\{W_\epsilon\}$ defined over $[a_0, a_0 + \omega]$ are uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem, there exists a sequence ϵ_N such that $(V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ converges uniformly to continuous functions (V, W). We now show that this candidate (V, W) is a solution to (9).

Because (V_ϵ, W_ϵ) is a solution to the ODE (20), for any two points $a_1 < a_2$ in the interval $[a_0, a_0 + \omega]$,

$$V_{\epsilon_N}(a_1) - V_{\epsilon_N}(a_2) = \int_{a_1}^{a_2} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) \, da.$$

Since R_l is continuous

$$\lim_{N \to \infty} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = R_l(a, V(a), W(a)).$$

Moreover, by Lemma 9, $R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ is uniformly bounded over $[a_1, a_2]$: $0 \leq R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = V'_{\epsilon_N}(a) \leq U'_1(ra) < U'_1(ra_0)$. Therefore, by the Dominated

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Convergence Theorem,

$$\lim_{N \to \infty} \int_{a_1}^{a_2} R_l\left(a, V_{\epsilon_N}\left(a\right), W_{\epsilon_N}\left(a\right)\right) da = \int_{a_1}^{a_2} \lim_{N \to \infty} R_l\left(a, V_{\epsilon_N}\left(a\right), W_{\epsilon_N}\left(a\right)\right) da$$

$$= \int_{a_1}^{a_2} R_l\left(a, V(a), W(a)\right) da.$$

Thus,

$$V\left(a_1\right) - V\left(a_2\right) = \lim_{N \to \infty} \left(V_{\epsilon_N}\left(a_1\right) - V_{\epsilon_N}\left(a_2\right)\right)$$

$$= \lim_{N \to \infty} \int_{a_1}^{a_2} R_l\left(a, V_{\epsilon_N}\left(a\right), W_{\epsilon_N}\left(a\right)\right) da$$

$$= \int_{a_1}^{a_2} R_l\left(a, V\left(a\right), W\left(a\right)\right) da. \tag{29}$$

Because R_l is continuous in a, V, W and V, W are continuous in a, the last equality implies that $V'(a) = R_l(a, V(a), W(a))$ for all $a \in [a_0, a_0 + \omega]$ (with V' standing for the right derivative of V at $a = a_0$).

Similarly, for any two points $a_1 < a_2$ in the interval $[a_0, a_0 + \omega]$,

$$W_{\epsilon_N}\left(a_1\right) - W_{\epsilon_N}\left(a_2\right) = \int_{a_1}^{a_2} S_l\left(a, V_{\epsilon_N}\left(a\right), W_{\epsilon_N}\left(a\right)\right) da.$$

By choosing ω sufficiently small, the last property in Lemma 7 applies for each $a \in (a_0, a_0 + \omega]$. We show that, $(a, V(a), W(a)) \in E_0$ for each $a \in (a_0, a_0 + \omega]$ and

$$\lim_{N \to \infty} S_l\left(a, V_{\epsilon_N}\left(a\right), W_{\epsilon_N}\left(a\right)\right) = S_l\left(a, V(a), W(a)\right). \tag{30}$$

Indeed, from the definition of $V_{\epsilon_N}, W_{\epsilon_N}$, $(\rho + \lambda)V_{\epsilon_N}(a) - \lambda W_{\epsilon_N}(a) > U_1(ra)$. Therefore, by pointwise convergence, $(\rho + \lambda)V(a) - \lambda W(a) \geq U_1(ra)$. We show by contradiction that $(\rho + \lambda)V(a) - \lambda W(a) > U_1(ra)$. Assume to the contrary that $(\rho + \lambda)V(a) - \lambda W(a) = U_1(ra)$. From the last property of Lemma 7, $V_{\epsilon_N}(a) \geq \overline{V}(a) + \gamma_a$ for $\epsilon_N < \epsilon_a$. Therefore, by pointwise convergence, $V(a) \geq \overline{V}(a) + \gamma_a$. This, together with the contradiction

assumption, implies that

$$W(a) < \overline{W}(a) - \frac{\rho + \lambda}{\lambda}\gamma_a.$$

In addition, by the continuity of R_l and by pointwise convergence,

$$\hat{c}_{\epsilon_N}(a) = \left(U_1'\right)^{-1}\left(R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a))\right) \to \left(U_1'\right)^{-1}\left(R_l(a, V(a), W(a))\right) = ra$$

as $N \to \infty$. Consequently, there exists $\delta \in (0,1)$ such that for N sufficiently high,

$$S_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = \frac{\rho W_{\epsilon_N}(a) - U_0(\hat{c}_{\epsilon_N}(a))}{ra - \hat{c}_{\epsilon_N}(a)} > \frac{\rho \overline{W}(a) - \frac{\rho(\rho+\lambda)}{\lambda}(1-\delta)\gamma_a - U_0(ra)}{ra - \hat{c}_{\epsilon_N}(a)}$$

$$\to \frac{\rho\overline{W}(a) - \frac{\rho(\rho+\lambda)}{\lambda}(1-\delta)\gamma_a - U_0(ra)}{ra - ra} = +\infty,$$

as $N \to \infty$, which contradicts the boundedness of $S_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ shown in Lemma 9. Therefore, we have shown by contradiction that $(\rho + \lambda)V(a) - \lambda W(a) > U_1(ra)$. By the continuity of S_l in E_0, we obtain the limit (30).

Since $0 < S_l\left(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)\right) < U_0'(ra) + \frac{\rho}{\lambda}U_1'(ra) < U_0'(ra_1) + \frac{\rho}{\lambda}U_1'(ra_1)$, by the Dominated Convergence Theorem, we can take the limit and conclude that

$$W(a_1) - W(a_2) = \int_{a_1}^{a_2} S_l\left(a, V(a), W(a)\right)da. \tag{31}$$

In addition, R_l, S_l are continuous over E_0, therefore (29) and (31) imply that (V, W) is a solution to ODE (20) over $(a_0, a_0 + \omega_0]$; this immediately implies that (9) holds for all $a \in (a_0, a_0 + \omega_0]$.

Next we show that (9) holds at $a = a_0$. We showed that $V'(a_0) = R_l\left(a_0, V\left(a_0\right), W\left(a_0\right)\right)$, so equation (9a) holds at $a = a_0$. Since $(V(a_0), W(a_0)) = \lim_{N \to \infty}(V_{\epsilon_N}(a_0), W_{\epsilon_N}(a_0)) = \left(\overline{V}(a_0), \overline{W}(a_0)\right)$ this implies that $V'(a_0) = U_1'(ra_0)$. Since $V'(a_0) = U_1'(ra_0)$, this gives $\hat{c}(a_0) = ra_0$, and so equation (9b) holds.

Having established the existence of (V, W), we turn to showing Properties 1) and 2).

Property 1: Notice that the right derivative of V at a_0, $V'(a_0) = R_l\left(a_0, \overline{V}(a_0), \overline{W}(a_0)\right) = U_1'(ra_0) > \overline{V}'(a_0)$, by Lemma 5. Together with $V(a_0) = \overline{V}(a_0)$, we have $V(a) > \overline{V}(a)$ in a neighborhood to the right of a_0. Restricting ω so that $a_0 + \omega$ lies in this neighborhood, we obtain the first property in Lemma 1.

Property 2: Because $\hat{c}(a) = \left(U_1'\right)^{-1}\left(V'(a)\right)$ and $\lim_{a \downarrow a_0} V'(a) = \lim_{a \downarrow a_0} R_l(a, V(a), W(a)) = U_1'(ra_0)$, $\lim_{a \downarrow a_0} \hat{c}(a) = ra_0$.

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By (26) in Lemma 4,

$$\hat{c}'(a) = \frac{1}{U_1''(\hat{c}(a))} \frac{(\rho + \lambda - r) V'(a) - \lambda W'(a)}{ra - \hat{c}(a)}.$$

From the derivation (34) in Lemma 9,

$$W'(a) = \lim_{\epsilon \to 0} W_\epsilon'(a) \leq \lim_{\epsilon \to 0} \left(U_0(ra) + \frac{\rho}{\lambda} U_1'(ra) - \frac{\rho}{\lambda} V_\epsilon'(a) \right),$$

and $V'(a) = \lim_{\epsilon \to 0} V_\epsilon'(a)$. Therefore

$$\hat{c}'(a) \geq \frac{1}{U_1''(\hat{c}(a))} \frac{\lim_{\epsilon \to 0} \left((2\rho + \lambda - r) V_\epsilon'(a) - \lambda U_0(ra) - \rho U_1'(ra) \right)}{ra - \hat{c}(a)}.$$

Because $\hat{c}(a) \to ra_0$ as $a \to a_0$, $\lim_{a \to a_0} (ra - \hat{c}(a)) = 0$. Moreover,

$$\lim_{(a,\epsilon) \to (a_0, 0)} \left((2\rho + \lambda - r) V_\epsilon'(a) - \lambda U_0(ra) - \rho U_1'(ra) \right) = (\rho + \lambda - r) U_1'(ra_0) - \lambda U_0'(ra_0) > 0,$$

where the last inequality comes from the fact that

$$\frac{U_0'(ra_0)}{U_1'(ra_0)} = \beta(ra_0) < \frac{\rho}{r} \left(\frac{\lambda + \rho - r}{\lambda} \right) \leq \frac{\lambda + \rho - r}{\lambda},$$

if $r \geq \rho$ and

$$\frac{U_0'(ra_0)}{U_1'(ra_0)} = \beta(ra_0) \leq 1 < \frac{\rho + \lambda - r}{\lambda},$$

if $r < \rho$. As a result, $\lim_{a \to a_0} \hat{c}'(a) = +\infty$. We have established the second property in Lemma 1.

C.2 Supporting Results for Lemma 1

The proof of Lemma 1 given above draws on the following results.

The first lemma below shows that there exists $\omega > 0$ and $\bar{\epsilon}$ such that for each $\epsilon \in (0, \bar{\epsilon})$, the solution (V_ϵ, W_ϵ) to ODE (20) are defined over $[a_0, a_0 + \omega]$ and $V_\epsilon(a) > \overline{V}(a)$. The proof of this lemma uses Lemma 8 that follow.

Lemma 7. *There exist $\omega > 0$ and $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, $(V_\epsilon(a), W_\epsilon(a))$ constructed in the proof of Lemma 1 is defined on $[a_0, a_0 + \omega]$. Moreover, $V_\epsilon(a) > \overline{V}(a)$ for all $a \in (a_0, a_0 + \omega]$. Lastly, there exists $\omega_0 < \omega$ such that for each $a \in (a_0, a_0 + \omega_0]$, there exist $\epsilon_a, \gamma_a > 0$ such that $V_\epsilon(a) > \overline{V}(a) + \gamma_a$ for all $0 < \epsilon < \epsilon_a$.*

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Proof. Let $\bar{\epsilon}_1 = \frac{1}{\lambda}(U_1(\infty) - U_1(ra_0)) > 0$. For $0 < \epsilon < \bar{\epsilon}_1$, let $[a_0, \tilde{a}_\epsilon)$ denote the (right) maximal interval of existence for (V_ϵ, W_ϵ).[35] Lemma 8 shows that if $\tilde{a}_\epsilon < \infty$ then

$$(\tilde{a}_\epsilon, V_\epsilon(\tilde{a}_\epsilon), W_\epsilon(\tilde{a}_\epsilon)) \in E_s.$$

In addition, $V_\epsilon(\tilde{a}_\epsilon) \leq \overline{V}(\tilde{a}_\epsilon)$.

Because R_l is continuous,

$$\lim_{\epsilon \to 0} R_l(a_0, V_\epsilon(a_0), W_\epsilon(a_0)) = R_l(a_0, \overline{V}(a_0), \overline{W}(a_0)) = U_1'(ra_0) > \overline{V}'(a_0),$$

where the last inequality is an application of Lemma 5 at a_0. Therefore, there exists $\bar{\epsilon}_2 > 0$, such that $V_\epsilon'(a_0) = R_l(a_0, V_\epsilon(a_0), W_\epsilon(a_0)) > \overline{V}'(a_0)$ for $0 < \epsilon < \bar{\epsilon}_2$. In this case, $V_\epsilon(a) > \overline{V}(a)$ in some neighborhood to the right of a_0.

For $0 < \epsilon < \min(\bar{\epsilon}_1, \bar{\epsilon}_2)$, let

$$\bar{a}_\epsilon = \sup\left\{a \in (a_0, \tilde{a}_\epsilon) : V_\epsilon(a') > \bar{V}(a') \text{ for all } a' \in (a_0, a)\right\}.$$

Because $V_\epsilon(a) > \overline{V}(a)$ in some neighborhood to the right of a_0, as shown above, $\bar{a}_\epsilon > a_0$.

We show by contradiction that there exist $\omega > 0$ and $0 < \bar{\epsilon} < \min(\bar{\epsilon}_1, \bar{\epsilon}_2)$, such that $\bar{a}_\epsilon > a_0 + \omega$ for all $\epsilon < \bar{\epsilon}$. Assume that this is not true, then there exists a sequence $\epsilon_N \to 0$ such that $\lim_{N \to \infty} \bar{a}_{\epsilon_N} = a_0$.

Because V_{ϵ_N} is continuous, $V_{\epsilon_N}(\bar{a}_{\epsilon_N}) \geq \overline{V}(\bar{a}_{\epsilon_N})$ (otherwise, $V_{\epsilon_N}(a) < \overline{V}(a)$ in the some neighborhood to the left of $\bar{a}_{\epsilon_N}$, which contradicts the definition of $\bar{a}_\epsilon$). If $V_{\epsilon_N}(\bar{a}_{\epsilon_N}) > \overline{V}(\bar{a}_{\epsilon_N})$, then $\bar{a}_{\epsilon_N} < \tilde{a}_{\epsilon_N}$, because if $\tilde{a}_{\epsilon_N} < \infty$ then $V_{\epsilon_N}(\tilde{a}_{\epsilon_N}) \leq \overline{V}(\tilde{a}_{\epsilon_N})$ as shown in Lemma 8. This also contradicts the definition of $\bar{a}_{\epsilon_N}$, because $V_{\epsilon_N}(a)$ is defined and is strictly greater than $\overline{V}(a)$ in a neighborhood of $\bar{a}_{\epsilon_N}$. Therefore $V_{\epsilon_N}(\bar{a}_{\epsilon_N}) = \overline{V}(\bar{a}_{\epsilon_N})$.

By the Mean Value Theorem, there exists $a^*_{\epsilon_N} \in [a_0, \bar{a}_{\epsilon_N}]$ such that

$$\frac{V_{\epsilon_N}(\bar{a}_{\epsilon_N}) - V_{\epsilon_N}(a_0)}{\bar{a}_{\epsilon_N} - a_0} = V_{\epsilon_N}'(a^*_{\epsilon_N}) = \frac{\overline{V}(\bar{a}_{\epsilon_N}) - \overline{V}(a_0)}{\bar{a}_{\epsilon_N} - a_0} \tag{32}$$

and by the definition of V_ϵ, W_ϵ:

$$V_{\epsilon_N}'(a^*_{\epsilon_N}) = R_l(a^*_{\epsilon_N}, V_{\epsilon_N}(a^*_{\epsilon_N}), W_{\epsilon_N}(a^*_{\epsilon_N})).$$

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[35]The definition of the maximal interval of existence is standard in the ODE literature. See, for example, Hartman (2002).

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By the monotonicity of V_ϵ and W_ϵ shown in Lemma 9,

$$\overline{V}(a_0) < V_{\epsilon_N}\left(a^*_{\epsilon_N}\right) < \overline{V}\left(\bar{a}_{\epsilon_N}\right),$$

and

$$W_{\epsilon_N}(a_0) = \overline{W}(a_0) - \epsilon_N < W_{\epsilon_N}\left(a^*_{\epsilon_N}\right).$$

Moreover, from the upper bound on W'_ϵ shown in Lemma 9 (using $V_{\epsilon_N}(a) \geq \overline{V}(a)$ for $a \in (a_0, \bar{a}_{\epsilon_N})$):

$$W_{\epsilon_N}\left(a^*_{\epsilon_N}\right) \leq W_{\epsilon_N}(a_0) + \left(U'_0(ra_0) + \frac{\rho}{\lambda}U'_1(ra_0)\right)\left(a^*_{\epsilon_N} - a_0\right).$$

Besides, by the contradiction assumption, $\lim_{N\to\infty} a^*_{\epsilon_N} = \lim_{N\to\infty} \bar{a}_{\epsilon_N} = a_0$. Therefore, by the Squeeze Principle, using the four inequalities above, we obtain

$$\lim_{N\to\infty} V_{\epsilon_N}\left(a^*_{\epsilon_N}\right) = \overline{V}(a_0)$$
$$\lim_{N\to\infty} W_{\epsilon_N}\left(a^*_{\epsilon_N}\right) = \overline{W}(a_0).$$

Thus, together with the continuity of R_l and (32), we obtain

$$\lim_{N\to\infty} R_l\left(a^*_\epsilon, V_\epsilon\left(a^*_\epsilon\right), W_\epsilon\left(a^*_\epsilon\right)\right) = R_l\left(a_0, \overline{V}(a_0), \overline{W}(a_0)\right) = U'_1(ra_0)$$
$$= \lim_{N\to\infty} \frac{\overline{V}\left(\bar{a}_{\epsilon_N}\right) - \overline{V}(a_0)}{\bar{a}_{\epsilon_N} - a_0} = \overline{V}'(a_0).$$

This leads to the desired contradiction because Lemma 5 for $a = a_0$ implies that $\overline{V}'(a_0) < U'_1(ra_0)$.

Finally, we show the last property by contradiction. Assume that it does not hold. Then there exists a sequence $a_N \to a_0$ such that for each N, there exists a sequence $\epsilon_{N,M} \to 0$ such that $V_{\epsilon_{N,M}}(a_N) \to \overline{V}(a_N)$. By choosing M sufficiently high, we have $0 < \epsilon_{N,M} < \frac{1}{N}$ and

$$\left|\frac{V_{\epsilon_{N,M}}(a_N) - \overline{V}(a_N)}{a_N - a_0}\right| < \frac{1}{N}.$$

By the Mean Value Theorem, there exists $\tilde{a}_N \in [a_0, a_N]$ such that

$$\frac{V_{\epsilon_{N,M}}(a_N) - \overline{V}(a_N)}{a_N - a_0} = \frac{V_{\epsilon_{N,M}}(a_N) - V_{\epsilon_{N,M}}(a_0) + \overline{V}_{\epsilon_{N,M}}(a_0) - \overline{V}(a_N)}{a_N - a_0}$$
$$= V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) - \overline{V}'(\tilde{a}_{N,M}).$$

Therefore

$$\left| V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) - \overline{V}'(\tilde{a}_{N,M}) \right| < \frac{1}{N}. \tag{33}$$

However,

$$V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) = R_l\left(\tilde{a}_{N,M}, V_{\epsilon_{N,M}}(\tilde{a}_{N,M}), W_{\epsilon_{N,M}}(\tilde{a}_{N,M})\right),$$

and by Lemma 9, as $N, M \to \infty$ $V_{\epsilon_{N,M}}(\tilde{a}_{N,M}) \to \overline{V}(a_0)$ and $W_{\epsilon_{N,M}}(\tilde{a}_{N,M}) \to \overline{W}(a_0)$. Therefore by the continuity of R_l,

$$V'_{\epsilon_{N,M}}(\tilde{a}_{N,M}) \to R_l\left(a_0, \overline{V}(a_0), \overline{W}(a_0)\right) = U'_1(ra_0).$$

Because $\tilde{a}_{N,M} \to a_0$

$$\overline{V}'(\tilde{a}_{N,M}) \to \overline{V}'(a_0).$$

Combining the last two limits with (33), we have $U'_1(ra_0) = \overline{V}'(a_0)$, which contradicts condition (27) for $a = a_0$ that $U'_1(ra_0) > \overline{V}'(a_0)$. Therefore by contradiction, the last property holds. $\qquad\square$

Lemma 8. *Consider the (right) maximal interval of existence, $[a_0, \tilde{a})$ for the solution (V_ϵ, W_ϵ) to the ODE (20) with the initial condition (28) and $0 \le \epsilon < \frac{1}{\lambda}(U_1(\infty) - U_1(ra_0))$. If $\tilde{a} < \infty$, then $\lim_{a\uparrow\tilde{a}} V_\epsilon(a) = V(\tilde{a})$ and $\lim_{a\uparrow\tilde{a}} W_\epsilon(a) = W_\epsilon(\tilde{a})$ and $(\tilde{a}, V_\epsilon(\tilde{a}), W_\epsilon(\tilde{a})) \in E_s$. In addition, $V_\epsilon(\tilde{a}) \le \overline{V}(\tilde{a})$.*

Proof. By Lemma 9, $V'_\epsilon(a), W'_\epsilon(a) > 0$. Therefore, the limits $\lim_{a\uparrow\tilde{a}} V_\epsilon(a) = V_\epsilon(\tilde{a})$ and $\lim_{a\uparrow\tilde{a}} W_\epsilon(a) = W_\epsilon(\tilde{a})$ exist. In addition, since $V'_\epsilon(a) = U'_1(\hat{c}_\epsilon(a)) < U'_1(ra)$, $V_\epsilon(\tilde{a}) < V_\epsilon(a_0) + \int_{a_0}^{\tilde{a}} U'_1(ra)da < \infty$ and $W_\epsilon(\tilde{a}) \le \frac{(\rho+\lambda)V_\epsilon(\tilde{a}) - U_1(r\tilde{a})}{\lambda} < \infty$. By Hartman (2002, Theorem 3.1), $(\tilde{a}, V_\epsilon(\tilde{a}), W_\epsilon(\tilde{a}))$ must lie in the boundary of E_0, i.e. Case 1: $(\rho + \lambda) V_\epsilon(\tilde{a}) - \lambda W_\epsilon(\tilde{a}) = U_1(\infty)$ or Case 2: $(\rho + \lambda) V_\epsilon(\tilde{a}) - \lambda W_\epsilon(\tilde{a}) = U_1(r\tilde{a})$. We first rule out Case 1 by showing that $(\rho + \lambda) V_\epsilon(\tilde{a}) - \lambda W_\epsilon(\tilde{a}) < U_1(\infty)$.

If $U_1(\infty) = \infty$, this is obvious. Now if $U_1(\infty) < \infty$. Let $a(t)$ denote the solution to the ODE, $a(0) = \tilde{a}$ and $\frac{da(t)}{dt} = ra(t) - \hat{c}_\epsilon(a(t))$ where $\hat{c}_\epsilon(a) = (U'_1)^{-1}(R_l(a, V_\epsilon(a), W_\epsilon(a)) > ra$. Consider the derivative:

$$\frac{d\left(e^{-(\rho+\lambda)t}V_\epsilon(a(t))\right)}{dt} = e^{-(\rho+\lambda)t}\left(-(\rho + \lambda)V_\epsilon(a(t)) + V'_\epsilon(a(t))(ra(t) - \hat{c}_\epsilon(a(t)))\right)$$

$$= e^{-(\rho+\lambda)t}\left(U_1(\hat{c}_\epsilon(a(t))) + \lambda W_\epsilon(a(t))\right),$$

where the second equality comes from the fact that V_ϵ is the solution of ODE (20). Let T

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denote the time at which $a(t)$ reaches a_0 (T can be $+\infty$), then

$$V_\epsilon(\tilde{a}) = \int_0^T e^{-(\rho+\lambda)t}(U_1(\hat{c}_\epsilon(a(t))) + \lambda W_\epsilon(a(t)))dt + e^{-(\rho+\lambda)T}\overline{V}(a_0).$$

Notice that, by Lemma 9, $W_\epsilon(a)$ is strictly increasing in a and a_t is strictly decreasing in t because $\hat{c}_\epsilon(a(t)) > ra(t)$. Therefore $W_\epsilon(a(t)) < W_\epsilon(a(0)) = W_\epsilon(\tilde{a})$. This implies

$$V_\epsilon(\tilde{a}) =< \int_0^T e^{-(\rho+\lambda)t}(U_1(\infty) + \lambda W_\epsilon(\tilde{a}))dt + e^{-(\rho+\lambda)T}\overline{V}(a_0)$$

$$= (1 - e^{-(\rho+\lambda)T})\frac{1}{\rho+\lambda}U_1(\infty) + \frac{\lambda}{\rho+\lambda}(1 - e^{-(\rho+\lambda)T})W_\epsilon(\tilde{a}) + e^{-(\rho+\lambda)T}\overline{V}(a_0).$$

By the definition of $\overline{V}(a)$,

$$\overline{V}(a_0) = \frac{1}{\rho+\lambda}U_1(ra_0) + \frac{\lambda}{\rho+\lambda}\overline{W}(a_0) = \frac{1}{\rho+\lambda}U_1(ra_0) + \frac{\lambda}{\rho+\lambda}(W_\epsilon(a_0) + \epsilon)$$

$$< \frac{1}{\rho+\lambda}U_1(\infty) + \frac{\lambda}{\rho+\lambda}W_\epsilon(a_0),$$

since $\epsilon < \frac{1}{\lambda}(U_1(\infty) - U_1(ra_0))$. Thus

$$V_\epsilon(\tilde{a}) < (1 - e^{-(\rho+\lambda)T})\frac{1}{\rho+\lambda}U_1(\infty) + \frac{\lambda}{\rho+\lambda}(1 - e^{-(\rho+\lambda)T})W_\epsilon(\tilde{a}) + e^{-(\rho+\lambda)T}\overline{V}(a_0)$$

$$< (1 - e^{-(\rho+\lambda)T})\frac{1}{\rho+\lambda}U_1(\infty) + \frac{\lambda}{\rho+\lambda}(1 - e^{-(\rho+\lambda)T})W_\epsilon(\tilde{a})$$

$$+ e^{-(\rho+\lambda)T}\left(\frac{1}{\rho+\lambda}U_1(\infty) + \frac{\lambda}{\rho+\lambda}W_\epsilon(\tilde{a})\right) = \frac{1}{\rho+\lambda}U_1(\infty) + \frac{\lambda}{\rho+\lambda}W_\epsilon(\tilde{a}).$$

Therefore $(\rho+\lambda)V_\epsilon(\tilde{a}) < U_1(\infty) + \lambda W_\epsilon(\tilde{a})$ which is equivalent to the desired inequality.

As we have ruled out Case 1, we must be in Case 2, i.e. $(\tilde{a}, V(\tilde{a}), W(\tilde{a})) \in E_s$.

Next, we show by contradiction that $V_\epsilon(\tilde{a}) \leq \overline{V}(\tilde{a})$. Assume to the contrary that $V_\epsilon(\tilde{a}) > \overline{V}(\tilde{a})$. Then $W_\epsilon(\tilde{a}) > \overline{W}(\tilde{a})$ because $(\rho+\lambda)\overline{V}(\tilde{a}) - \lambda\overline{W}(\tilde{a}) = U_1(r\tilde{a})$.

Since R_l is continuous over $\bar{E}_0$, $\lim_{a\uparrow\tilde{a}} V'_\epsilon(a) = U'_1(r\tilde{a})$ and $\lim_{a\uparrow\tilde{a}} \hat{c}_\epsilon(a) = r\tilde{a}$. Therefore

$$\lim_{a\uparrow\tilde{a}} W'_\epsilon(a) = \lim_{a\uparrow\tilde{a}} \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} = \frac{U_0(r\tilde{a}) - \rho W_\epsilon(\tilde{a})}{r\tilde{a} - r\tilde{a}} = -\infty,$$

which contradicts the property that $W'_\epsilon > 0$ established in Lemma 9. So by contradiction, $W_\epsilon(\tilde{a}) \leq \overline{W}(\tilde{a})$, and $V_\epsilon(\tilde{a}) \leq \overline{V}(\tilde{a})$. $\qquad\square$

The following lemma establishes the bounds on the derivative of V_ϵ and W_ϵ that are

important to apply the Dominated Convergence Theorem in Lemma 1. To prove this result, we use Lemma 10.

Lemma 9. *Consider the solution (V_ϵ, W_ϵ) to ODE (20) with the initial condition (28) defined over some interval $[a_0, \bar{a}]$. We have $0 < V_\epsilon'(a) \le U_1'(ra)$ and $0 < W_\epsilon'(a)$ for all $a \in [a_0, \bar{a}]$. Moreover, if $V_\epsilon(a) \ge \overline{V}(a)$, $W_\epsilon'(a) < U_0'(ra) + \frac{\rho}{\lambda} U_1'(ra)$.*

Proof. Since $\widehat{c}_\epsilon(a) > ra$ and $V_\epsilon'(a) = U_1'(\widehat{c}_\epsilon(a))$, we have $0 < V_\epsilon'(a) \le U_1'(ra)$ due to the concavity of U_1. If $r \ge \rho$, from Lemma 10, $W_\epsilon(a) < \overline{W}(a)$. Therefore,

$$W_\epsilon'(a) = \frac{U_0(\widehat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\widehat{c}_\epsilon(a) - ra} > \frac{U_0(ra) - \rho \overline{W}(a)}{\widehat{c}_\epsilon(a) - ra} = 0.$$

If $r < \rho$, Lemma 10 immediately implies that $W_\epsilon'(a) > 0$.

To show the upper bound on $W_\epsilon'(a)$ when $V_\epsilon(a) \ge \overline{V}(a)$, we use the facts that

$$(\rho + \lambda) V_\epsilon(a) - \lambda W_\epsilon(a) = U_1(\widehat{c}_\epsilon(a)) + V_\epsilon'(a)(ra - \widehat{c}_\epsilon(a))$$

and

$$(\rho + \lambda) \overline{V}(a) - \lambda \overline{W}(a) = U_1(ra).$$

By subtracting the two equalities side by side and rearranging,

$$
\begin{aligned}
\lambda \left(\overline{W}(a) - W_\epsilon(a) \right) = & -(\rho + \lambda) \left(V_\epsilon(a) - \overline{V}(a) \right) \\
& + U_1(\widehat{c}_\epsilon(a)) - U_1(ra) + V_\epsilon'(a)(ra - \widehat{c}_\epsilon(a)) \\
& \le U_1(\widehat{c}_\epsilon(a)) - U_1(ra) + V_\epsilon'(a)(ra - \widehat{c}_\epsilon(a)).
\end{aligned}
$$

where the last inequality comes from $V_\epsilon(a) \ge \overline{V}(a)$. It follows that

$$
\begin{aligned}
W_\epsilon'(a) = & \frac{U_0(\widehat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\widehat{c}_\epsilon(a) - ra} = \frac{U_0(\widehat{c}_\epsilon(a)) - U_0(ra) + \rho \left(\overline{W}(a) - W_\epsilon(a) \right)}{\widehat{c}_\epsilon(a) - ra} \\
\le & \frac{U_0(\widehat{c}_\epsilon(a)) - U_0(ra) + \frac{\rho}{\lambda} \left(U_1(\widehat{c}_\epsilon(a)) - U_1(ra) + V_\epsilon'(a)(ra - \widehat{c}_\epsilon(a)) \right)}{\widehat{c}_\epsilon(a) - ra} \\
= & \frac{U_0(\widehat{c}_\epsilon(a)) - U_0(ra) + \frac{\rho}{\lambda} \left(U_1(\widehat{c}_\epsilon(a)) - U_1(ra) \right)}{\widehat{c}_\epsilon(a) - ra} - \frac{\rho}{\lambda} V_\epsilon'(a) \\
< & \frac{U_0(\widehat{c}_\epsilon(a)) - U_0(ra)}{\widehat{c}_\epsilon(a) - ra} + \frac{\rho}{\lambda} \frac{U_1(\widehat{c}_\epsilon(a)) - U_1(ra)}{\widehat{c}_\epsilon(a) - ra} < U_0'(ra) + \frac{\rho}{\lambda} U_1'(ra), \quad (34)
\end{aligned}
$$

where the last inequality comes from the concavity of U_1 and U_0 and $\widehat{c}_\epsilon(a) > ra$. $\square$

Lemma 10. *Consider the solution (V_ϵ, W_ϵ) to ODE (20) with the initial condition (28) defined over some interval $[a_0, \bar{a}]$. We have*

1) If $r \geq \rho$, $W_\epsilon(a) < \overline{W}(a) \; \forall a > a_0$.

2) If $r < \rho$, $W_\epsilon(a) < U_0(\hat{c}_\epsilon(a)) \; \forall a > a_0$.

Proof. 1) $r \geq \rho$: We use Lemma 6 to show property 1). We just need to verify conditions 1) and 2) in Lemma 6. First by definition, $W_\epsilon(a_0) < \overline{W}(a_0)$, so condition 1) Lemma 6 is satisfied. For condition 2) in Lemma 6, we show that if $W_\epsilon(a) = \overline{W}(a)$, for some $\forall a > a_0$ then $W'_\epsilon(a) < \overline{W}'(a)$. Indeed,

$$
\begin{aligned}
W'_\epsilon(a) &= \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} = \frac{U_0(\hat{c}_\epsilon(a)) - \rho \overline{W}(a)}{\hat{c}_\epsilon(a) - ra} \\
&= \frac{U_0(\hat{c}_\epsilon(a)) - U_0(ra)}{\hat{c}_\epsilon(a) - ra} < U'_0(ra),
\end{aligned}
$$

where the last inequality comes from the fact that U_0 is strictly concave and $\hat{c}_\epsilon(a) > ra$. On the other hand, we also have

$$
\overline{W}'(a) = \frac{r}{\rho} U'_0(ra) \geq U'_0(ra),
$$

because $r \geq \rho$. Therefore, $\overline{W}'(a) > W'_\epsilon(a)$.

2) $r < \rho$: We also use Lemma 6 to show property 2). By the definition of V_ϵ, W_ϵ:

$$
W_\epsilon(a_0) = U_0(ra_0) - \epsilon < U_0(ra_0) < U_0(\hat{c}_\epsilon(a_0)).
$$

So condition 1) in Lemma 6 is satisfied. Now we show that condition 2) in Lemma 6 is also satisfied, i.e. if at some $a > a_0$, $W_\epsilon(a) = U_0(\hat{c}_\epsilon(a))$, we show that $W'_\epsilon(a) < \frac{d}{da}(U'_0(\hat{c}_\epsilon(a)))$. Indeed,

$$
W'_\epsilon(a) = \frac{U_0(\hat{c}_\epsilon(a)) - \rho W_\epsilon(a)}{\hat{c}_\epsilon(a) - ra} = 0.
$$

Moreover,

$$
\frac{d}{da}(U_0(\hat{c}_\epsilon(a))) = U'_0(\hat{c}_\epsilon(a))\hat{c}'_\epsilon(a).
$$

By (26), in addition to $W'_\epsilon(a) = 0$,

$$
\begin{aligned}
\hat{c}'_\epsilon(a) &= \frac{1}{U''_1(\hat{c}_\epsilon(a))} \frac{(\rho + \lambda - r)V'_\epsilon(a) - \lambda W'_\epsilon(a)}{ra - \hat{c}_\epsilon(a)} \\
&= \frac{1}{-U''_1(\hat{c}_\epsilon(a))} \frac{(\rho + \lambda - r)U'_1(\hat{c}_\epsilon(a))}{\hat{c}_\epsilon(a) - ra} > 0.
\end{aligned}
$$

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Therefore $W'_\epsilon(a) = 0 < \frac{d}{da}\left(U'_0(\hat{c}_\epsilon(a))\right)$. □

D   Proof of Theorem 1

Assume $\sigma \neq 1$; the case with $\sigma = 1$ is similar. To proceed, we guess

$$V(a) = \bar{v}\frac{a^{1-\sigma}}{1-\sigma} \qquad W(a) = \bar{w}\frac{a^{1-\sigma}}{1-\sigma},$$

and find $\bar{v}, \bar{w}$ to verify that V, W satisfy (9).[36] To show the uniqueness of the linear equilibrium, notice that in any linear equilibrium, V, W must have the functional form above.

Given the conjectured functional form, the first-order condition (9a) implies

$$\hat{c}(a) = \psi a,$$

where $\psi = \bar{v}^{-\frac{1}{\sigma}}$. Plugging this back into (9a) gives

$$(\rho + \lambda)\,\bar{v}\frac{a^{1-\sigma}}{1-\sigma} = \frac{1}{1-\sigma}\left(\left(\bar{v}a^{-\sigma}\right)^{-\frac{1}{\sigma}}\right)^{1-\sigma} + \left(\bar{v}a^{-\sigma}\right)\left(ra - \left(\bar{v}a^{-\sigma}\right)^{-\frac{1}{\sigma}}\right) + \lambda\bar{w}\frac{a^{1-\sigma}}{1-\sigma}$$

$$= \frac{\sigma}{1-\sigma}\left(\bar{v}a^{-\sigma}\right)^{-\frac{1-\sigma}{\sigma}} + \left(\bar{v}a^{-\sigma}\right)ra + \lambda\bar{w}\frac{a^{1-\sigma}}{1-\sigma}.$$

Canceling the $\frac{a^{1-\sigma}}{1-\sigma}$ terms and rearranging we obtain

$$(\lambda + \Delta)\,\bar{v} = \sigma\bar{v}^{1-\frac{1}{\sigma}} + \lambda\bar{w}, \tag{35}$$

where Δ is defined by

$$\Delta \equiv \rho - r(1-\sigma) > 0, \tag{36}$$

where the inequality comes from the restriction that value functions V, W are finite.

From the second equation in system (9) we have

$$\rho\bar{w}\frac{a^{1-\sigma}}{1-\sigma} = \bar{\beta}\frac{1}{1-\sigma}\left(\left(\bar{v}a^{-\sigma}\right)^{-\frac{1}{\sigma}}\right)^{1-\sigma} + \left(\bar{w}a^{-\sigma}\right)\left(ra - \left(\bar{v}a^{-\sigma}\right)^{-\frac{1}{\sigma}}\right).$$

---

[36]If $\sigma = 1$

$$V(a) = A_v + \frac{1}{\rho+\lambda}\left(1 + \lambda\frac{\bar{\beta}}{\rho}\right)\log a$$

$$W(a) = A_w + \frac{\bar{\beta}}{\rho}\log a.$$

Canceling the $\frac{a^{1-\sigma}}{1-\sigma}$ terms gives

$$\bar{w} = \frac{\bar{\beta}\bar{v}^{1-\frac{1}{\sigma}}}{\Delta + (1-\sigma)\,\bar{v}^{-\frac{1}{\sigma}}}. \tag{37}$$

Combining equations (35) and (37), we obtain

$$\lambda + \Delta = \sigma\bar{v}^{-\frac{1}{\sigma}} + \lambda\bar{\beta}\frac{\bar{\beta}\bar{v}^{-\frac{1}{\sigma}}}{\Delta + (1-\sigma)\,\bar{v}^{-\frac{1}{\sigma}}},$$

a single equation in $\bar{v}$. Define $\psi \equiv \bar{v}^{-\frac{1}{\sigma}}$. We then have a quadratic equation in ψ:

$$P(\psi) \equiv Q_2\psi^2 + Q_1\psi + Q_0 = 0, \tag{38}$$

with

$$Q_2 \equiv (1-\sigma)\,\sigma$$
$$Q_1 \equiv (\sigma + \bar{\beta} - 1)\,\lambda + \Delta\,(2\sigma - 1)$$
$$Q_0 \equiv -(\lambda + \Delta)\,\Delta.$$

If $\sigma < 1$ then $Q_2 > 0$ and $Q_0 < 0$. This implies that there exists a unique strictly positive ψ that is the solution to (38). The implied consumption rule yields finite utility.

If $\sigma > 1$ we have that $Q_2 < 0$. This implies that

$$P\left(\frac{\Delta}{\sigma - 1}\right) = -\frac{\sigma}{\sigma - 1}\Delta^2 + \left((\sigma + \bar{\beta} - 1)\,\lambda + \Delta\,(2\sigma - 1)\right)\frac{\Delta}{\sigma - 1} - (\lambda + \Delta)\,\Delta$$
$$= \frac{\bar{\beta}}{\sigma - 1}\lambda\frac{\Delta}{\sigma - 1} > 0.$$

Therefore, there exists two solutions $0 < \psi_1 < \frac{\Delta}{\sigma - 1} < \psi_2$ such that $P(\psi) = 0$. To know which root corresponds to a solution to (9), we observe that $c_t = \psi a_t$ so $\dot{a}_t = (r - \psi)a_t$ or $a_t = e^{(r-\psi)t}a_0$. Thus $V \propto \int e^{-\rho t}e^{(1-\sigma)(r-\psi)t}dt = \int_0^\infty e^{(-\Delta + \psi(\sigma - 1))t}dt$. For V to finite, we require $\psi < \frac{\Delta}{\sigma - 1}$. So only the smaller root to (38), ψ_1, yields finite value functions, and corresponds to a solution to (9).

Lastly, the derivations above directly imply the uniqueness of the linear equilibrium.

Now, we turn to the second part of the theorem. Given that $\hat{c}(a) = \psi a$, $\hat{c}(a) < ra$ if and only if $\psi < r$. For $\sigma < 1$ we have $r > 0$ so that $0 < \psi < r$ if and only if $P(r) > P(\psi) = 0$. For $\sigma > 1$, because $\rho > 0$, $r < \frac{\rho - r(1-\sigma)}{\sigma - 1} = \frac{\Delta}{\sigma - 1}$. Given that $P(\frac{\Delta}{\sigma - 1}) > 0$ and $P(\psi_1) = 0$,

$r > \psi$ if and only if $P(r) > P(\psi) = 0$. Thus, we need to establish that $P(r) > 0$. This is equivalent to

$$\beta > \frac{\rho}{r}\left(1 - \frac{r - \rho}{\lambda}\right) = \hat{\beta}.$$

Similarly, $\hat{c}(a) > ra$, i.e. $\psi > r$ if and only if $\bar{\beta} < \hat{\beta}$.

E   Proofs for Dissaving Equilibria

E.1   Proof of Theorem 2

Once we verify the differential system (9) all the equilibrium conditions in Subsection 3.1 are met. By Lemma 5, we have $\overline{V}'(a) = U_1'(ra)$, therefore $\hat{c}(a) = ra$ and equations (9) are satisfied by the definitions of $\overline{V}$ and $\overline{W}$.

E.2   Proof of Theorem 3

We prove this theorem by construction.

Lemma 1 shows that starting from $a_0 = \underline{a}$, ODE (20) with the boundary condition

$$(V(a_0), W(a_0)) = (\overline{V}(a_0), \overline{W}(a_0))$$

admits a solution defined over $[\underline{a}, \underline{a} + \omega)$ for some $\omega > 0$. Let (V_0, W_0) denote this solution. In addition, let $[\underline{a}, a^*)$ be the right maximal interval of existence for this solution. It is immediate that $a^* \geq \underline{a} + \omega$. If $a^* = \infty$, we have found a (continuous) Markov equilibrium, with $(V, W) = (V_0, W_0)$.

If $a^* < \infty$, following the steps in the proof of Lemma 8, we can show that

$$\lim_{a \uparrow a^*} V_0(a) \leq \overline{V}(a^*).$$

Moreover, as shown in Lemma 1, $V_0(a) > \bar{V}(a)$ in a neighborhood to the right of $\underline{a}$. Thus, by the Intermediate Value Theorem, there exists $a_1 \in (\underline{a}, a^*]$ such that $V_0(a_1) = \overline{V}(a_1)$.

Starting from a_1, we apply Lemma 1 again with a_1 standing for a_0 and construct the a solution (V_1, W_1) to ODE (20) with the boundary condition

$$(V_1(a_1), W_1(a_1)) = (\overline{V}(a_1), \overline{W}(a_1)).$$

Following this procedure, we obtain a sequence $a_0 = \underline{a} < a_1 < \dots$ with $\lim_{n \to \infty} a_n =$

$+\infty$ and a sequence of value functions (V_n, W_n) defined over $[a_n, a_{n+1}]$ with the boundary condition

$$(V_n(a_n), W_n(a_n)) = (\overline{V}(a_n), \overline{W}(a_n)).$$

The divergence of $\{a_n\}$ is shown in Lemma 11 below.

We define the value and consumption functions $(V, W, \hat{c})$ over the whole interval $[\underline{a}, \infty)$ as

$$(V(a), W(a), \hat{c}) = (V_n(a), W_n(a), \hat{c}_n(a)) \text{ for } a \in [a_n, a_{n+1}).$$

We verify that this construction satisfies all the conditions in Subsection 3.1 for a Markov equilibrium. Conditions (a)-(e) are satisfied by the construction of (V, W). Condition (f) on the existence of a_t is satisfied because by construction $\hat{c}(a)$ is differentiable and $\hat{c}(a) > ra$ outside steady-states $\{a_n\}$. Indeed, if $a(0) = a_n$ then $a(t) \equiv a_n$ for all $t \geq 0$ satisfies ODE (10) for all $t \geq 0$. If $a(0) \in (a_n, a_{n+1})$, the solution $a(t)$ to ODE (10) with the initial condition $a(t) = a(0)$ exists and is unique over a small interval $[0, \epsilon]$ because $\hat{c}(a)$ is continuously differentiable over (a_n, a_{n+1}). In addition, because $\hat{c}(a) > ra$, $a(t)$ is strictly decreasing in t. Let $[0, T)$ denote the right maximal interval of existence for $a(t)$ to ODE (10). If $T = \infty$, we obtain the existence of $a(t)$ to ODE (10) over the whole time interval $[0, \infty)$. If $T < \infty$ (in the Online Appendix we show that this is always the case), by Hartman (2002, Theorem 3.1), $a(T) = a_n$. Defining $a(t) = a_n$ for all $t \geq T$, we also obtain the existence of $a(t)$ to ODE (10) over the whole time interval $[0, \infty)$. Finally, the limits $\lim_{t\to\infty} e^{-\rho t} V(a_t)$ and $\lim_{t\to t} e^{-\rho t} W(a_t)$ are both equal to 0 because $\dot{a}_t \leq 0$, V and W are increasing over $[a_n, a_{n+1})$ and $a_t \geq a_n$.

Lemma 11. *If the sequence $\{a_n\}$ constructed in Theorem 3 has an infinite number of elements then*

$$\lim_{n\to\infty} a_n = +\infty.$$

Proof. The result is shown by contradiction. Assume that the sequence is infinite and is bounded above. By construction $\{a_n\}_{=0}^{\infty}$ is strictly increasing, thus the sequence converges to some a^∞. We assume by contradiction that a^∞ is finite. By construction, $V_n(a_n) = \overline{V}(a_n)$, $W(a_n) = \overline{W}(a_n)$ and $V_n(a_{n+1}) = \overline{V}(a_{n+1})$ and $V_n(a) > \overline{V}(a)$ for $a \in (a_n, a_{n+1})$. We can then apply Lemma 9 to show that $0 \leq V_n'(a) \leq U_1'(ra)$ and $0 \leq W_n'(a) \leq U_0'(ra) + \frac{\rho}{\lambda} U_1'(ra)$. By the Mean Value Theorem, there exists $a_n^* \in [a_n, a_{n+1}]$ such that

$$V_n'(a_n^*) = \frac{V_n(a_{n+1}) - V_n(a_n)}{a_{n+1} - a_n} = \frac{\overline{V}(a_{n+1}) - \overline{V}(a_n)}{a_{n+1} - a_n}.$$

Since $\{a_n\}$ converges to a^∞,

$$\lim_{n\to\infty} V'_n(a^*_n) = \overline{V}'(a^\infty). \tag{39}$$

On the other hand $V'_n(a^*_n) = R_l(a^*_n, V_n(a^*_n), W_n(a^*_n))$. Since $\overline{V}(a_n) \leq V_n(a^*_n) \leq \overline{V}(a_{n+1})$ and $\overline{W}(a_n) \leq W_n(a^*_n) \leq \overline{W}(a^*_n) + (U'_0(ra_n) + \frac{\rho}{\lambda}U'_1(ra_n))(a_{n+1} - a_n)$ and R_l is continuous, by the Squeeze Principle,

$$\begin{aligned}
\lim_{n\to\infty} V'_n(a^*_n) &= \lim_{i\to\infty} R_l(a^*_n, V_n(a^*_n), W_n(a^*_n)) \\
&= \lim_{i\to\infty} R_l(a^\infty, \overline{V}(a^\infty), \overline{W}(a^\infty)) \\
&= U'_1(ra^\infty). \tag{40}
\end{aligned}$$

The desired contradiction follows from the fact that (39) and (40) cannot happen at the same time given that $\overline{V}'(a^\infty) < U'_1(ra^\infty)$ by condition (27) at $a = a^\infty$. $\qquad\square$

F Proofs for Saving Equilibria

F.1 Proof of Theorem 4

We prove this theorem by construction.

Depending on condition (a) or (b) in Assumption 2, we define an wealth level a_u, and the value functions (V_u, W_u) over $[a_u, \infty)$ satisfying the differential equations (9) as following.

Case 1: Assumption 2a holds.

Without loss of generality we assume that $\bar{c}$ is the minimum consumption level such that Assumption 2a is satisfied, i.e. $\beta(c) = \frac{U'_0(c)}{U'_1(c)} < 1$ for all $c < \bar{c}$ and $\beta(c) = 1$ for all $c \geq \bar{c}$. Therefore there exists $\bar{u}$ such that $U_1(c) = U_0(c) + \bar{u}$ for all $c \geq \bar{c}$. In Subsection A.3.2, we show that there exists a_u such that $\hat{c}_0(a) > \bar{c}$ for $a > a_u$ and $\hat{c}_0(a_u) = \bar{c}$. For $a \geq a_u$, let

$$(V_u(a), W_u(a)) = \left(V_{sp}(a), V_{sp}(a) - \frac{\bar{u}}{\rho + \lambda}\right).$$

As shown in Subsection A.3.2, (V_u, W_u) satisfies the differential equations (9), with $\hat{c}_u = \hat{c}_0$. Moreover, $\hat{c}'_u(a) > 0$ and, because $r > \rho$, $\hat{c}_u(a) < ra$.

Case 2: Assumption 2b holds.

In Theorem 1, we show that for $a \geq a_u$, $(V_u(a), W_u(a)) = \left(\bar{v}\frac{a^{1-\sigma}}{1-\sigma}, \bar{w}\frac{a^{1-\sigma}}{1-\sigma}\right)$ and $\hat{c}_u(a) = \phi a$ satisfy the differential equations (9) over $[a_u, \infty)$ where and $a_u = \frac{\bar{c}}{\phi}$. It is immediate that $\hat{c}'_u(a) = \phi > 0$. Moreover, because $\bar{\beta} > \hat{\beta}$, $\phi < r$, so $\hat{c}_u(a) < ra$.

Having determined the value and policy functions at and beyond a_u, we construct the value and policy functions below a_u. Noticing that the initial values $(a_u, V_u(a_u), W_u(a_u)) \in E$, a solution (V_d, W_d) to the ODE (21) with the initial condition

$$(V_d(a_u), W_d(a_u)) = (V_u(a_u), W_u(a_u))$$

exists and is unique locally over an interval $(a_u - \epsilon, a_u + \epsilon)$.[37] Let $(\hat{a}, a_u + \epsilon)$ denote the (left) maximal interval of existence for this solution. We will show that $\hat{a} < \frac{\bar{c}}{r}$.

If $\hat{a} = 0$, this is immediate. If $\hat{a} > 0$, by Lemma 12, $V_d'(a), W_d'(a) > 0$ for all $a > \hat{a}$. So the limits $\lim_{a \downarrow \hat{a}} V_d(a) = V_d(\hat{a})$ and $\lim_{a \downarrow \hat{a}} W_d(a) = W_d(\hat{a})$ exist.[38] By Hartman (2002, Theorem 3.1), $(\hat{a}, V_d(\hat{a}), W_d(\hat{a})) \in E_s$. Therefore $\hat{c}_d(\hat{a}) = r\hat{a}$. As shown in Lemma 12, $V_d''(a) < 0$. This implies $\hat{c}_d'(a) = \frac{V_d''(a)}{U_1''(\hat{c}_d(a))} > 0$. Thus,

$$r\hat{a} = \hat{c}_d(\hat{a}) < \hat{c}_d(a_u) = \bar{c}.$$

So $\hat{a} < \frac{\bar{c}}{r}$.

Given the value and policy functions $(V_u, W_u, \hat{c}_u)$ and $(V_d, W_d, \hat{c}_d)$, for $\underline{a} > \hat{a}$, we define the value and policy functions $(V, W, \hat{c})$ over $[\underline{a}, \infty)$ as follows

$$(V, W, \hat{c}) = \begin{cases} (V_d, W_d, \hat{c}_d) & \text{if } \underline{a} \leq a < a_u \\ (V_u, W_u, \hat{c}_u) & \text{if } a \geq a_u. \end{cases}$$

As in the proof of Theorem 3, we can verify that this construction satisfies all the conditions in Subsection 3.1 for a Markov equilibrium. In addition, $\hat{c}(a) < ra$ and $\hat{c}'(a) > 0$.

If $\hat{a} > 0$, Lemma 14 below shows that $(V(\hat{a}), W(\hat{a})) = (\bar{V}(\hat{a}), \bar{W}(\hat{a}))$ and $\hat{c}(\hat{a}) = r\hat{a}$.

---

[37]Because of the uniqueness of the solution, $(V_d(a), W_d(a)) = (V_u(a), W_u(a))$ for all $a \in [a_u, a_u + \epsilon)$.

[38]We also show that $\lim_{a \downarrow \hat{a}} W_d(a) > -\infty$ and $\lim_{a \downarrow \hat{a}} V_d(a) > -\infty$. This is immediate if U_1 is bounded from below, and consequently U_0 is bounded from below by some $\underline{u}$, because $\beta(c) \leq 1$. Because $W_d'(a) > 0$ and $\hat{c}_d(a) < ra$ for $a > \hat{a}$, (9b) implies that $W_d(a) > \frac{1}{\rho}\underline{u}$ for $a > \hat{a}$. If U_1 is unbounded from below, we make the additional technical assumption that $\underline{\sigma} = \inf_c \sigma(U_1, c) > 1 - \frac{\rho}{r}$. We have

$$\hat{c}_d'(a) = \frac{(\rho + \lambda - r)V_d'(a) - \lambda W_d'(a)}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))} < \frac{(\rho - r)V_d'(a)}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))}$$

$$= \frac{(\rho - r)U_1'(\hat{c}_d(a))}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))} < \frac{r - \rho}{\underline{\sigma}} \frac{\hat{c}_d(a)}{ra - \hat{c}_d(a)},$$

because $W_d'(a) < V_d'(a)$ as shown in Lemma 12. Therefore $\hat{c}_d(a) > c^*(a) > 0$ where $c^*(a)$ is the solution to the ODE $c^{*\prime}(a) = \frac{r - \rho}{\underline{\sigma}} \frac{c^*(a)}{ra - c^*(a)}$ and $c^*(a_u) = \hat{c}_d(a_u) < ra_u$ (the closed form solution for $c^*(a)$ is a special case of the solution to the IG limit provided in the Online Appendix). Therefore, $\lim_{a \downarrow \hat{a}} W_d(a) > \frac{1}{\rho}U_0(c^*(\hat{a})) > -\infty$ for all $a \geq \hat{a}$. Finally, $\lim_{a \downarrow \hat{a}} V_d(a) > \frac{\frac{\lambda}{\rho}U_0(c^*(\hat{a})) + U_1(\hat{a})}{\rho + \lambda} > -\infty$.

In addition, when $\beta(c)$ is increasing, Lemma 15 shows that $V(a) > \bar{V}(a)$ for all $a > \hat{a}$. Therefore, $V'(\hat{a}) = U'_1(r\hat{a}) \geq \bar{V}'(\hat{a})$. By Lemma 5, $\beta(r\hat{a}) \leq \hat{\beta}(r, \rho, \lambda)$.

F.2   Supporting Results for Proof of Theorem 4

Lemma 12. *Assume $\rho < r$. Consider a solution (V, W) to ODE (21) defined over $(\hat{a}, a_u]$ with the initial condition $(V(a_u), W(a_u)) = (V_u(a_u), W_u(a_u))$ with a_u, V_u, W_u defined in Subsection F.1 (depending on Assumption 2a or 2b). Then for all $a < a_u$*

1) $(\rho + \lambda - r) V'(a) - \lambda W'(a) < 0$ and $W'(a) > 0$
2) $V''(a) < 0$
3) $V'(a) > W'(a)$.

Proof. We prove this lemma in two steps. Step 1: If properties 1),2) and 3) hold in a neighborhood to the left of a_u then they hold for all $a < \hat{a}$. Step 2: Verify that properties 1),2) and 3) hold in a neighborhood to the left of a_u separately under Assumption 2a or 2b.

Step 1: Assume that properties 1),2) and 3) hold in a neighborhood to the left of a_u, We show that 1),2), and 3) hold for all $a < a_u$.

We prove 1) separately for two cases: Case 1: $\rho + \lambda - r > 0$ and Case 2: $\rho + \lambda - r \leq 0$.

Case 1: By construction, $V'(a) = U'_1(\hat{c}(a)) > 0$. Therefore if $(\rho + \lambda - r) V'(a) - \lambda W'(a) < 0$, $W'(a) > 0$. We just need to show the first inequality.

We prove this inequality using Lemma 6 (Variation 1). Condition 1) of Lemma 6 (at a_u) is satisfied by assumption. We just need to verify Condition 2) of Lemma 6, i.e. if there exists $\breve{a} < a_u$ such that

$$(\rho + \lambda - r) V'(\breve{a}) - \lambda W'(\breve{a}) = 0. \tag{41}$$

then

$$(\rho + \lambda - r) V''(\breve{a}) > \lambda W''(\breve{a}).$$

By Lemma 4,

$$(\rho + \lambda - r) V'(a) - \lambda W'(a) = V''(a)(ra - \hat{c}(a)). \tag{42}$$

At $a = \breve{a}$, because of (41), and $r\breve{a} > \hat{c}(\breve{a})$, $V''(\breve{a}) = 0$.

Differentiating the second equation, (9b), in system (9), and using (26), we have

$$\rho W'(a) = \frac{U'_0(\hat{c}(a))}{U''_1(\hat{c}(a))} V''(a) \tag{43}$$

$$+ W''(a)(ra - \hat{c}(a)) + W'(a)\left(r - \frac{1}{U''_1(\hat{c}(a))} V''(a)\right).$$

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At $a = \breve{a}$, using the previous result that $V''(\breve{a}) = 0$, and rearranging, we arrive at

$$W''(\breve{a})(r\breve{a} - \widehat{c}(\breve{a})) = (\rho - r)W'(\breve{a}).$$

Because, $W'(\breve{a}) = \frac{(\rho + \lambda - r)V'(\breve{a})}{\lambda} > 0$ and $\rho - r < 0$, the right hand side is strictly negative. Moreover $ra - \widehat{c}(a) > 0$, therefore $W''(\breve{a}) < 0$. Thus,

$$W''(\breve{a}) < 0 = V''(\breve{a}),$$

i.e. we have verified the second condition in Lemma 6. Given that the two conditions of Lemma 6 are satisfied, this lemma implies the first property.

Case 2: Because $\rho + \lambda \leq r$ and $V'(a) > 0$, $(\rho + \lambda - r)V'(a) - \lambda W'(a) < 0$ if $W'(a) > 0$. Therefore we just need to show the last inequality. Again we prove this inequality using Lemma 6 (Variation 1). Condition 1) of Lemma 6 (at a_u) is shown in the proof of Theorem 4. We now verify Condition 2). If there exists $\breve{a} < a_u$ such that $W'(\breve{a}) = 0$ we show that $W''(\breve{a}) < 0$. From equation (42) at $\breve{a}$, $(\rho + \lambda - r)V'(\breve{a}) = V''(\widehat{a})(r\breve{a} - \widehat{c}(\breve{a}))$. This implies $V''(\breve{a}) < 0$. From (43), since $W'(\breve{a}) = 0$,

$$0 = \frac{U_0'(\widehat{c}(\breve{a}))}{U_1''(\widehat{c}(\breve{a}))}V''(\breve{a}) + W''(\breve{a})(r\breve{a} - \widehat{c}(\breve{a})).$$

Therefore $W''(\breve{a}) < 0$. Given that the two conditions of Lemma 6 are satisfied, this lemma implies $W'(a) > 0$ for all a.

The second property immediately follows using (42) and $ra - \widehat{c}(a) > 0$.

We also prove the third property similarly by using Lemma 6. Condition 1) in Lemma 6 is satisfied. We now verify that condition 2) is also satisfied. Indeed, if there exists $\breve{a} < a_u$ such that $W'(\breve{a}) = V'(\breve{a})$. By (42), at $a = \breve{a}$,

$$V''(\breve{a}) = \frac{(\rho - r)V'(\breve{a})}{r\breve{a} - \widehat{c}(\breve{a})} < 0.$$

Again by equation (43),

$$(r\breve{a} - \widehat{c}(\breve{a}))W''(\breve{a}) = (\rho - r)W'(\breve{a}) - \underbrace{(U_0'(\widehat{c}(\breve{a})) - W'(\breve{a}))\frac{1}{U_1''(\widehat{c}(\breve{a}))}V''(\breve{a})}_{<0}$$

$$> (\rho - r)W'(\breve{a}) = (\rho - r)V'(\breve{a}),$$

The second line comes from the assumption that $W'(\breve{a}) = V'(\breve{a}) = U_1'(\widehat{c}(\breve{a})) > U_0'(\widehat{c}(\breve{a}))$

(by properties 1) and 2), $\hat{c}'(a) > 0$ therefore $\hat{c}(\breve{a}) < \hat{c}(a_u) = \bar{c}$ and that $U_1'(c) > U_0'(c)$ for all $c < \bar{c}$ by Assumption 1). So

$$W''(\breve{a}) > \frac{(\rho - r) V'(\breve{a})}{r\breve{a} - \hat{c}(\breve{a})} = V''(\breve{a}).$$

So by Lemma 6, we obtain the third property.

Step 2: We show that properties 1),2) and 3) hold in a neighborhood to the left of a_u. We treat the two cases associated with Assumption 2a or 2b separately.

Under Assumption 2b, given the closed form solution given in Appendix D, we show that properties 1), 2), 3) are satisfied at a_u in Lemma 13 below. By continuity, properties 1), 2), 3) hold in a neighborhood to the left of a_u.

Under Assumption 2a. It is easy to verify that properties 1) and 2) are satisfied because $V_{sp}(a)$ is strictly concave and $W_u'(a) = V_u'(a) = V_{sp}'(a)$ for $a \geq a_u$, and thus $V'(a_u) = W'(a_u) = V_{sp}'(a_u) > 0$ and $V''(a_u) = V_{sp}''(a_u) < 0$. By continuity, properties 1) and 2) hold in a neighborhood to the left of a_u. Properties 3) is not satisfied at a_u because $V_u'(a_u) = W_u'(a_u)$ so $V'(a_u) = W'(a_u)$ but we show that $V'(a) < W'(a)$ in some neighborhood to the left of a_u.

Indeed, consider the solution (V_ϵ, W_ϵ) to ODE (21) with the initial condition

$$(V_\epsilon(a_u), W_\epsilon(a_u)) = \left(V_u(a_u) - \frac{\epsilon}{\rho + \lambda}, W_u(a_u) - \frac{\epsilon}{\lambda} \right).$$

Because $(a_u, V_u(a_u), W_u(a_u)) \in E$, by Hartman (2002, Theorem 3.2), there exists $\omega > 0$ such that (V_ϵ, W_ϵ) are defined over, $[a_u - \omega, a_u]$ and $(V_\epsilon, W_\epsilon) \to (V, W)$ uniformly over $[a_u - \omega, a_u]$ as $\epsilon \to 0$.

It easy to verify that $V_\epsilon'(a_u) > W_\epsilon'(a_u)$. Indeed, from the initial conditions, we have $\hat{c}_\epsilon(a_u) = \hat{c}_u(a_u)$, and $V_u'(a_u) = U_1'(\hat{c}_u(a_u)) = U_1'(\hat{c}_\epsilon(a_u)) = V_\epsilon'(a_u)$,

$$W_\epsilon'(a_u) = \frac{\rho \left(W_u(a_u) - \frac{\epsilon}{\lambda} \right) - U_0(\hat{c}_\epsilon(a_u))}{ra_u - \hat{c}_\epsilon(a_u)} < \frac{\rho W_u(a_u) - U_0(\hat{c}_u(a_u))}{ra_u - \hat{c}_u(a_u)}$$
$$= W_u'(a_u) = V_u'(a_u) = V_\epsilon'(a_u).$$

In addition, when ϵ sufficiently small, we also have properties 1) and 2) holds for V_ϵ, W_ϵ at a_u. Therefore, following the proofs in Step 1 above, we can show that properties 1),2),3) hold for all $a \in [a_u - \omega, a_u]$ for (V_ϵ, W_ϵ). In particular, $V_\epsilon'(a) > W_\epsilon'(a)$ for all $a \in [a_u - \omega, a_u]$.

Now as $\epsilon \to 0$, $(V_\epsilon, W_\epsilon) \to (V, W)$. So $V'(a) \geq W'(a)$ for all $a \in [a_u - \omega, a_u]$. We show

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by contradiction that $V'(a) > W'(a)$ for all $a \in (a_u - \omega, a_u)$. Assume to the contrary that $V'(\check{a}) = W'(\check{a})$ for some $\check{a} < a_u$. As in Step 1, this implies that $V''(\check{a}) < W''(\check{a})$ strictly, because $\beta(\hat{c}(\check{a})) < 1$ given that $\hat{c}(\check{a}) < \hat{c}(a_u) = \bar{c}$. Therefore in the right neighborhood of $\check{a}$, $V'(a) < W'(a)$, which contradicts the earlier result that $V'(a) \geq W'(a)$. Thus, $V'(a) > W'(a)$ for all $a \in (a_u - \omega, a_u)$. $\square$

Lemma 13. *The linear equilibria in Theorem 1 with $\hat{\beta} < \bar{\beta} < 1$ satisfies, for all $a > 0$*

1) $(\lambda + \rho - r)V'(a) < \lambda W'(a)$
2) $V''(a) > 0$
3) $W'(a) < V'(a)$

Proof. As shown in Theorem 1, because $\bar{\beta} > \hat{\beta}$, $\hat{c}(a) < ra$. By Lemma 4,

$$(\rho + \lambda - r)\, V'\,(a) - \lambda W'\,(a) = V''\,(a)\,(ra - \hat{c}\,(a))\,.$$

Because $V''(a) = -\bar{v}\sigma a^{-\sigma-1} < 0$, and $\hat{c}(a) < ra$,

$$(\lambda + \rho - r)V'(a) - \lambda W'(a) < 0.$$

The second inequality 2) is immediate because $V''(a) = -\bar{v}\sigma a^{-\sigma-1} < 0$.

The last inequality 3) is equivalent to $\frac{\bar{\beta}\psi}{\Delta + (1-\sigma)\psi} < 1$. After algebra manipulation, this inequality holds if and only if $(\sigma + \bar{\beta} - 1)\psi < \Delta$. This obviously holds if $\sigma + \bar{\beta} \leq 1$, because $\Delta, \psi > 0$. If $\sigma + \bar{\beta} > 1$, we show that $\psi < \frac{\Delta}{\sigma+\bar{\beta}-1}$. Indeed, $P\left(\frac{\Delta}{\sigma+\bar{\beta}-1}\right) = \frac{\bar{\beta}(1-\bar{\beta})}{(\sigma+\bar{\beta}-1)^2} > 0$. Therefore, $\psi < \frac{\Delta}{\sigma+\bar{\beta}-1}$. $\square$

G   Proofs for Poverty Trap Equilibria

G.1   Proof of Theorem 5

We prove this theorem by construction.

As shown in the proof of Theorem 4, there exists a unique solution (V_d, W_d) to ODE (21) that satisfies $(V_d(a_u), W_d(a_u)) = (V_u(a_u), W_u(a_u))$ (with a_u, V_u, W_u defined differently under Assumption 2a or 2b) defined over a maximal interval of existence $(\hat{a}, a_u]$, where $\hat{a} < \frac{\bar{c}}{r}$. We also show in the proof of Theorem 4 that V_d, W_d is defined and is continuous at $\hat{a}$, i.e. the limits $\lim_{a\downarrow\hat{a}} V_d(a) = V_d(\hat{a})$ and $\lim_{a\downarrow\hat{a}} W_d(a) = W_d(\hat{a})$ exist, and $\hat{c}_d(\hat{a}) = r\hat{a}$.

If $\hat{a} \leq \underline{a}$, let

$$(V, W, \hat{c}) = \begin{cases} (V_d, W_d, \hat{c}_d) & \text{if } \underline{a} \leq a < \hat{a} \\ (V_u, W_u, \hat{c}_u) & \text{if } a \geq \hat{a}. \end{cases}$$

If $\underline{a} < \hat{a}$, Lemma 14 below shows that $V_d(\hat{a}) = \overline{V}(\hat{a})$. In addition, Lemma 15 shows that $V_d(a) > \overline{V}(a)$ for all $a > \hat{a}$. Therefore,

$$V_d'(\hat{a}) = U_1'(r\hat{a}) \geq \overline{V}'(\hat{a}).$$

By Lemma 5, $\beta(r\hat{a}) \leq \hat{\beta}$.

If $\beta(r\hat{a}) = \hat{\beta}$. Let $\tilde{a} = \min\{a \geq \underline{a} : \beta(ra) = \hat{\beta}\}$. Because β is weakly increasing, $\beta(ra) = \hat{\beta}$ for all $a \in [\tilde{a}, \hat{a}]$ (Assumption 3). We define V_h, W_h over $[\tilde{a}, \infty)$ such that

$$(V_h(a), W_h(a), \hat{c}_h) = \begin{cases} (V_u(a), W_u(a), \hat{c}_u(a)) & \text{if } a_u \leq a \\ (V_d(a), W_d(a), \hat{c}_d(a)) & \text{if } \hat{a} \leq a < a_u \\ (\overline{V}(a), \overline{W}(a), ra) & \text{if } \tilde{a} \leq a < \hat{a}. \end{cases}$$

By Theorems 2 and 4, V_u, W_u satisfy (9) over $[\tilde{a}, \infty)$. Replacing $\hat{a}$ by $\tilde{a}$ if $\beta(r\hat{a}) = \hat{\beta}$, without loss of generality, we can assume that $\beta(ra) < \hat{\beta}$ for $a < \hat{a}$.

Iteratively, we construct a sequence $\{a_i\}$ starting with $a_0 = \underline{a}$, and for each $i \geq 0$, $a_i < \hat{a}$ and $\beta(ra_i) < \hat{\beta}$ and the value and policy functions $(V_i, W_i, \hat{c}_i)$ are determined as following:

Iteration i: Starting from $a_i < \hat{a}$, because $\beta(ra_i) < \hat{\beta}$, using Lemma 1, we show that ODE (20) admits a solution (V_i, W_i), with the initial condition $(V_i(a_i), W_i(a_i)) = (\overline{V}(a_i), \overline{W}(a_i))$, defined over a (right) maximal interval of existence $[a_i, a_i^*)$. Moreover $V_i(a) > \overline{V}(a)$ in a neighborhood to right of a_i. There are three possibilities:

i-1) $a_i^* < \hat{a}$. Then following the steps in Lemma 8, we can shows that $V_i(a_i^*) \leq \overline{V}(a_i^*)$. By the Intermediate Value Theorem, there exists $a_i < a_{i+1} < \hat{a}$, such that $V_i(a_{i+1}) = \overline{V}(a_{i+1})$ and $V_i(a) > \overline{V}(a)$ for $a \in (a_i, a_{i+1})$. Because $a_{i+1} < \hat{a}$, $\beta(ra_{i+1}) < \hat{\beta}$. Go to iteration $i + 1$ with a_{i+1} standing for a_i.

i-2) $a_i^* \geq \hat{a}$ and $V_i(a) \leq \overline{V}(a)$ for some $a < \hat{a}$. By the Intermediate Value Theorem, there exists $a_i \leq a_{i+1} < \hat{a}$, such that $V_i(a_{i+1}) = \overline{V}(a_{i+1})$ and $V_i(a) > \overline{V}(a)$ for $a \in (a_i, a_{i+1})$. Go to iteration $i + 1$ with a_{i+1} standing for a_i.

i-3) $a_i^* \geq \hat{a}$ and $V_i(a) > \overline{V}(a)$ for all $a < \hat{a}$. We stop the construction.

Following this procedure, we produce a strictly increasing sequence $\{a_i\}$ such that for each $i \geq 0$, $a_i < \hat{a}$ and $\beta(ra_i) < \hat{\beta}$ and the value functions (V_i, W_i) satisfies $(V_i(a_i), W_i(a_i)) = (\overline{V}(a_i), \overline{V}(a_i))$ and $V_i(a_{i+1}) = \overline{V}(a_{i+1})$ and $V_i(a) > \overline{V}(a)$ for all $a \in (a_i, a_{i+1})$. Let

$$(V_l(a), W_l(a), \hat{c}_l) = (V_i(a), W_i(a), \hat{c}_i(a)) \text{ for } a \in [a_i, a_{i+1}),$$

with $a_{n+1} = a_n^*$ if possibility n-3) is reached at some iteration n.

There are two possible cases:

Case 1: The sequence $\{a_i\}$ is finite, i.e. possibility n-3) is reached at some iteration n: We obtain a sequence $a_0 < a_1 < ... < a_n < \hat{a}$.

If $a_n^* = \infty$, we define the value and consumption functions $(V, W, \hat{c})$ over the whole interval $[\underline{a}, \infty)$ as

$$(V(a), W(a), \hat{c}(a)) = (V_l(a), W_l(a), \hat{c}_l(a)).$$

If $a_n^* < \infty$, following the steps in the proof of Lemma 8, we can show that $V_n(a_n^*) \leq \overline{V}(a_n^*)$.

Therefore, both V_n and V_h are defined over $[\hat{a}, a_n^*]$ and

$$V_h(\hat{a}) = \overline{V}(\hat{a}) \leq V_n(\hat{a})$$
$$V_h(a_n^*) \geq \overline{V}(a_{n+1}^*) \geq V_n(a_{n+1}^*).$$

By the Intermediate Value Theorem, there exists $a^* \in [\hat{a}, a_n^*]$ such that

$$V_n(a^*) = V_h(a^*).$$

We define $(V, W, \hat{c})$ as

$$(V, W, \hat{c}) = \begin{cases} (V_l, W_l, \hat{c}_l) & \text{if } a < a^* \\ (V_h, W_h, \hat{c}_h) & \text{if } a \geq a^*. \end{cases}$$

Case 2: The sequence $\{a_i\}$ is infinite (possibility i-3 is never reached). Then $\lim_{i \to \infty} a_i = a_\infty \leq \hat{a}$ and $\beta(r a_\infty) = \hat{\beta}$. Because $\beta(ra) < \hat{\beta}$ for $a < \hat{a}$. We have $a_\infty = \hat{a}$. In this case

$$(V, W, \hat{c}) = \begin{cases} (V_l, W_l, \hat{c}_l) & \text{if } a < a_\infty = \hat{a} \\ (V_h, W_h, \hat{c}_h) & \text{if } a \geq \hat{a}. \end{cases}$$

In all cases we can construct the value and policy functions $(V, W, \hat{c})$ over $[\underline{a}, \infty)$. As in the proof of Theorem 3, we can verify that this construction satisfies all the conditions in Subsection 3.1 for a Markov equilibrium. In addition, $\hat{c}(a) < ra$ for all $a \geq a^*$ and $\hat{c}(a) \geq ra$ for all $a < a^*$.

G.2   Supporting Results for Theorem 5

Lemma 14. *Given the definition of V_d and $\hat{a}$ in the of Theorem 5, if $\hat{a} > 0$ then $(V_d(\hat{a}), W_d(\hat{a})) = (\overline{V}(\hat{a}), \overline{W}(\hat{a}))$.*

Proof. As shown in the proof of Theorem 4, if $\hat{a} > 0$, $(\hat{a}, V_d(\hat{a}), W_d(\hat{a})) \in E_s$. Therefore $\hat{c}_d(\hat{a}) = r\hat{a}$.

First, we show that $W_d(\hat{a}) \leq \overline{W}(\hat{a})$. Assume by contradiction that, $W_d(\hat{a}) > \overline{W}(\hat{a})$, then

$$W'_d(a) = \frac{\rho W_d(a) - U_0(\hat{c}_d(a))}{ra - \hat{c}_d(a).} \longrightarrow \frac{\rho W_d(\hat{a}) - \rho U_0(r\hat{a})}{r\hat{a} - r\hat{a}} = +\infty$$

as a approaches $\hat{a}$ from the right because $\hat{c}_d(a) \to r\hat{a}$. Moreover, by the continuity of R_l, $\lim_{a \downarrow \hat{a}} V'_d(a) = U'_1(r\hat{a})$. This contradicts the property 3) in Lemma 12 that $W'_d(a) < V'_d(a)$ for all $a_u > a > \hat{a}$. Therefore, $W_d(\hat{a}) \leq \overline{W}(\hat{a})$.

We also show that $W_d(\hat{a}) \geq \overline{W}(\hat{a})$. Assume by contradiction that, $W_d(\hat{a}) < \overline{W}(\hat{a})$, then, similarly to the previous case,

$$W'_d(a) = \frac{\rho W_d(a) - U_0(\hat{c}_d(a))}{ra - \hat{c}_d(a).} \longrightarrow \frac{\rho W_d(\hat{a}) - \rho U_0(r\hat{a})}{r\hat{a} - r\hat{a}} = -\infty$$

as a approaches $\hat{a}$ from the right. This contradicts the property 1) in Lemma 12 that $W'_d(a) > 0$ for all $a > \hat{a}$. Therefore, $W_d(\hat{a}) \geq \overline{W}(\hat{a})$.

The two results imply that $W_d(\hat{a}) = \overline{W}(\hat{a})$. Combining this equality with the fact that $(\hat{a}, V_d(\hat{a}), W_d(\hat{a})) \in E_s$ yields $V_d(\hat{a}) = \overline{V}(\hat{a})$. □

Lemma 15. *Given the definition of V_d and $\hat{a}$ in the proof of Theorem 5, $V_d(a) > \overline{V}(a)$ for all $a > \hat{a}$.*

Proof. Let $\tilde{U}(c) \equiv U_1(c) + \frac{\lambda}{\rho} U_0(c)$. By the concavity of U_1 and U_0, $\tilde{U}$ is also strictly concave. We first show that $ra > \hat{c}_d(a) > c^*(a)$ where $c^*(a)$ is defined by

$$\tilde{U}'(c^*(a)) = V'(a) + \frac{\lambda}{\rho} W'(a).$$

Indeed, because $\tilde{U}$ is strictly concave, this is equivalent to $\tilde{U}'(c^*(a)) > \tilde{U}'(\hat{c}_d(a))$ or

$$V'_d(a) + \frac{\lambda}{\rho} W'_d(a) > U'_1(\hat{c}_d(a)) + \frac{\lambda}{\rho} U'_0(\hat{c}_d(a)).$$

Because $V'_d(a) = U'_1(\hat{c}_d(a))$ and $W'_d(a) > U'_0(\hat{c}_d(a))$ by Lemma 16, we obtain the desired inequality.

Now using system (9), substituting W_d by the right hand side of the second equation into the first equation, we obtain

$$(\rho + \lambda) V_d(a) = \tilde{U}(\hat{c}(a)) + (V'_d(a) + \frac{\lambda}{\rho} W'_d(a))(ra - \hat{c}(a)).$$

Let

$$F(a,c) \equiv \widetilde{U}(c) + (V_d'(a) + \frac{\lambda}{\rho}W_d'(a))(ra - c).$$

Because $\widetilde{U}$ is strictly concave, F is strictly concave in c. By the definition of $\widetilde{U}$ and $c^*(a)$, $\frac{\partial F(a,c)}{\partial c} = 0$ at $c = c^*(a)$ and $\frac{\partial F(a,c)}{\partial c} < 0$ for $c > c^*(a)$. Therefore

$$F(a,c^*(a)) > F(a,\hat{c}_d(a)) > F(a,ra) = (\rho + \lambda)\overline{V}(a).$$

Moreover, $F(a,\hat{c}_d(a)) = (\rho + \lambda)V_d(a)$, so $V_d(a) > \overline{V}(a)$. $\qquad\square$

Lemma 16. *Given the definition of W_d and $\hat{a}$ in the proof of Theorem 5, $W_d'(a) > U_0'(\hat{c}(a))$ for all $a \in (\hat{a}, a_u)$.*

Proof. Assumption 3 is equivalent to

$$\frac{-U_0''(c)}{U_0'(c)} \leq \frac{-U_1''(c)}{U_1'(c)} \tag{44}$$

for all $c \leq \bar{c}$. We use Lemma 6 to prove this lemma. Indeed, we first show that condition 2) in Lemma 6 is satisfied, i.e. if $W_d'(a) = U_0'(\hat{c}_d(a))$ then

$$\frac{d}{da}(W_d'(a)) < \frac{d}{da}(U_0'(\hat{c}_d(a))). \tag{45}$$

Indeed, differentiating equation (9b) with respect to a implies

$$\rho W_d'(a) = U_0'(\hat{c}_d(a))\hat{c}_d'(a) + W_d''(a)(ra - \hat{c}_d(a)) + W_d'(a)(r - \hat{c}_d'(a)).$$

Because $W_d'(a) = U_0'(\hat{c}_d(a))$, this equation simplifies to

$$W_d''(a) = \frac{(\rho - r)W_d'(a)}{ra - \hat{c}_d(a)} = \frac{(\rho - r)U_0'(\hat{c}_d(a))}{ra - \hat{c}_d(a)}.$$

On the other hand,

$$\begin{aligned}
\frac{d}{da}(U_0'(\hat{c}_d(a))) &= U_0''(\hat{c}_d(a))\hat{c}_d'(a) = U_0''(\hat{c}_d(a))\frac{V_d''(a)}{U_1''(\hat{c}_d(a))} \\
&= U_0''(\hat{c}_d(a))\frac{(\rho - r)U_1'(\hat{c}_d(a)) + \lambda(U_1'(\hat{c}_d(a)) - U_0'(\hat{c}_d(a)))}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))} \\
&\geq U_0''(\hat{c}_d(a))\frac{(\rho - r)U_1'(\hat{c}_d(a))}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))}, \tag{46}
\end{aligned}$$

where the last inequality comes from $U_1'(\hat{c}_d(a)) \geq U_0'(\hat{c}_d(a))$. Combining this with $\rho < r$ and condition (44), we have (45), but with weak inequality. Now we show that it holds with strict inequality. For $a < a_u$, because $\hat{c}_d' > 0$, $\hat{c}_d(a) < \hat{c}_d(a_u) = \bar{c}$, therefore $U_1'(\hat{c}_d(a)) > U_0'(\hat{c}_d(a))$ (this also holds for $a = a_u$ under Assumption 2b). Thus (46) holds with strict inequality. If $a = a_u$ and under Assumption 2a, $\hat{c}_d(a) = \bar{c}$, (44) holds with strict inequality (we assume that $U_1'(c) > U_0'(c)$ for $c < \bar{c}$). Hence, in either case, (45) holds with strict inequality.

Now, we show that condition 1) in Lemma 6 is also satisfied. Under Assumption 2b of Theorem 4 with power utility, it is shown in Lemma 17 that at a_u that $W_d'(a_u) > U_0'(\hat{c}_d(a_u))$. Under Assumption 2a of Theorem 4, $U_1'(c) = U_0'(c)$ for $c \geq \bar{c}$, given how a_u is defined in Subsection A.3.2, we have $W_d'(a_u) = V_d'(a_u) = U_1'(\bar{c}) = U_0'(\bar{c})$, so $W_d'(a) = U_0'(\hat{c}(a))$ at $a = a_u$. Therefore, by (45), $W_d'(a) > U_0'(\hat{c}_d(a))$ in the left neighborhood of a_u.[39]

Given that both conditions in Lemma 6 are satisfied, it implies that $W'(a) > U_0'(\hat{c}(a))$ for all $a \in (\hat{a}, a_u)$. $\qquad\square$

Lemma 17. *The linear equilibria described in Theorem 1 with $\bar{\beta} < 1$ satisfies $W'(a) > U_0'(\hat{c}(a))$ for all $a > 0$.*

Proof. From Theorem 1, $W'(a) > U_0'(\hat{c}(a))$ is equivalent to

$$\frac{\bar{\beta}\bar{v}^{1-\frac{1}{\sigma}}}{\Delta + (1-\sigma)\,\bar{v}^{-\frac{1}{\sigma}}} > \bar{\beta}\bar{v},$$

or equivalently $\psi > \frac{\Delta}{\sigma}$. This inequality holds because $P\left(\frac{\Delta}{\sigma}\right) = (\bar{\beta} - 1)\frac{\lambda}{\sigma} < 0$. $\qquad\square$

G.3   Proof of Proposition 4

Consider the construction in Theorem 5. Fixing $\tilde{a} \in (0, \frac{\bar{c}}{\rho})$. First we show that there exists r_1 such that for $r < r_1$ such that

$$V_d(a_u) < \int_{\tilde{a}}^{a_u} U_1'(ra)\,da + \overline{V}(\tilde{a}). \tag{47}$$

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[39] Another way to show this is to proceed as in the proof of Theorem 3 by considering the solution (V_ϵ, W_ϵ) to the ODE (21) with the initial condition $(V_\epsilon(a_u), W_\epsilon(a_u)) = \left(V_u(a_u) + \frac{\epsilon}{\rho+\lambda}, W_u(a_u) + \frac{\epsilon}{\lambda}\right)$. It easy to verify that $W_\epsilon'(a_u) > U_0'(\hat{c}_\epsilon(a_u))$ because $\hat{c}_\epsilon(a_u) = \hat{c}(a_u)$. Therefore by Lemma 16, $V_\epsilon'(a) > W_\epsilon'(a)$ for all $a < a_u$. As $\epsilon \to 0$, $(V_\epsilon, W_\epsilon) \to (V_d, W_d)$. As a result, $W_d'(a) \geq U_0'(\hat{c}_d(a))$ for all $a < a_u$. We can then apply Lemma 16 to show that $W_d'(a) > U_0'(\hat{c}_d(a))$ for all $a < a_u$ because $\beta(c) < 1$ for all $c < \bar{c}$, which implies that (46), and consequently (45), holds with strict inequality.

Indeed, $\int_{\tilde{a}}^{a_u} U_1'(ra)da = \frac{1}{r}U_1(ra_u) - \frac{1}{r}U_1(r\tilde{a})$. When $r \to \rho$, $V_d(a_u) \to \frac{1}{\rho+\lambda}\left(U_1(\bar{c}) + \frac{\lambda}{\rho}U_0(\bar{c})\right)$ and $\frac{1}{r}U_1(ra_u) \to \frac{1}{\rho}U_1(\bar{c})$. The right hand side of (47) converges to

$$\frac{1}{\rho}U_1(\bar{c}) - \frac{1}{\rho}U_1(\rho\tilde{a}) + \frac{1}{\rho+\lambda}(U_1(\rho\tilde{a}) + \frac{\lambda}{\rho}U_0(\rho\tilde{a}))$$
$$= \frac{1}{\rho}U_1(\bar{c}) + \frac{\lambda}{\rho(\rho+\lambda)}(U_0(\rho\tilde{a}) - U_1(\rho\tilde{a})).$$

Therefore, at the limit $r \to \rho$, (47) is equivalent to

$$U_0(\bar{c}) - U_1(\bar{c}) < U_0(\rho\tilde{a}) - U_1(\rho\tilde{a}) \tag{48}$$

By Mean Value Theorem, there exists $\check{c} \in (\rho\tilde{a}, \bar{c})$ such that

$$\frac{U_0(\bar{c}) - U_0(\rho\tilde{a})}{U_1(\bar{c}) - U_1(\rho\tilde{a})} = \frac{U_0'(\check{c})}{U_1'(\check{c})} = \beta(\check{c}) < \beta(\bar{c}) = 1,$$

which is equivalent to (48). So, by continuity, (47) holds when r belongs to a neighborhood to the right of ρ, $[\rho, r_1]$.

Under (47), we show by contradiction that $\hat{a} > \tilde{a}$. Assume $\tilde{a} \geq \hat{a}$. By Lemma 15, $V_d(\tilde{a}) \geq \overline{V}(\tilde{a})$. Because $\hat{c}(a) < ra$ for $\tilde{a} < a < a_u$, $V_d'(a) > U_1'(ra)$. So

$$V_d(a_u) - V_d(\tilde{a}) > \int_{\tilde{a}}^{a_u} U_1'(ra)da,$$

which contradicts (47). So $\hat{a} > \tilde{a} > 0$. Now pick any $\underline{a}$ such that $0 < \underline{a} < \tilde{a}$. We have $\hat{a} > \underline{a}$.

The construction in Theorem 5 implies that $a^* > \underline{a}$. It remains to show that $a^* < \infty$. More strongly, we show by contradiction that $a^* < a_u$. Assume $a^* \geq a_u$. By the definition of a_u, $V_u(a) = V_{sp}(a)$ for all $a \geq a_u$. Therefore,

$$V_u(a^*) = V_{sp}(a^*) > V_l(a^*),$$

which contradicts the definition of a^* that $V_u(a^*) = V_l(a^*)$.

H   Proofs for Convergence Equilibria

Proof of Theorem 6. First of all let $\alpha_1 = \lim_{c \uparrow c^*} \beta(c)$ and $\alpha_2 = \lim_{c \downarrow c^*} \beta(c)$. Assumption 4 implies that

$$\underline{\beta} \leq \alpha_2 < \alpha_1 \leq 1.$$

Let $\underline{r} = \rho\frac{\rho+\lambda}{\rho+\lambda(1-\alpha_2)}$ and $\bar{r} = \rho\frac{\rho+\lambda}{\rho+\lambda(1-\alpha_1)}$. Also by Assumption 4, we have

$$\rho < \underline{r} < r < \bar{r}.$$

Consider the initial condition at $a^* = \frac{c^*}{r}$, $(V(a^*), W(a^*)) = (\overline{V}(a^*), \overline{W}(a^*))$. We show that the ODE (20) admits a solution over $[a^*, \infty)$ with the initial condition at a^*. Similarly, we show that the ODE (21) admits a solution over $[\underline{a}, a^*]$, with the initial condition at a^*. Combining the two solutions, we obtain a Markov equilibrium defined over $[\underline{a}, \infty)$.

Indeed, starting at $a^* = \frac{c^*}{r}$, and the initial condition $(\overline{V}(a^*), \overline{W}(a^*))$, because $r < \rho\frac{\rho+\lambda}{\rho+\lambda(1-\alpha_2)}$, $\beta(ra_+^*) < \hat{\beta}$, we can use Lemma 1, to show the existence of a solution (V, W) to ODE (20), given the initial condition. The solution has a (right) maximal interval of existence $[a^*, \tilde{a})$. If $1 - \alpha_2 > \sigma(U_1, c)$ for all $c \geq c^*$, Theorem 10 shows that $\tilde{a} = +\infty$, for λ sufficiently high. Otherwise, we follows the steps in the proof of Theorem 3 to restart the procedure each time V crosses $\overline{V}$. In doing so, we obtain a Markov equilibrium over $[a^*, \infty)$ with $\hat{c}(a) > ra$ except for a countable set of steady states at which $\hat{c}(a) = ra$.

Starting at $a^* = \frac{c^*}{r}$, and the initial condition $(\overline{V}(a^*), \overline{W}(a^*))$, we also show that the ODE (21) admits a solution defined over $(\tilde{a}, a^*]$, where $\tilde{a} < a^*$. First consider the case $\alpha_1 < 1$ (the case $\alpha_1 = 1$ will be considered below). The proof follows closely the steps of Lemma 1, i.e. we start with the initial condition

$$\left(\overline{V}(a^*) + \epsilon, \overline{W}(a^*) + \delta(\epsilon)\epsilon\right), \tag{49}$$

where $\delta(\epsilon) \in \left[1, \frac{\rho+\lambda}{\lambda}\right)$ is chosen appropriately. In Lemma 18 below, we show that there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$, $\delta(\epsilon)$ can be chosen such that

$$\max\left\{(\rho + \lambda - r)\, V_\epsilon'(a^*), \lambda U_0'(\hat{c}_\epsilon(a^*))\right\} < \lambda W_\epsilon'(a^*) < \lambda V_\epsilon'(a^*).$$

Therefore, following the steps in Lemma 12, we can show that

$$\max\left\{(\rho + \lambda - r)\, V_\epsilon'(a), 0\right\} < \lambda W_\epsilon'(a) < \lambda V_\epsilon'(a), \tag{50}$$

for all a in the (left) maximal interval of existence for V_ϵ, W_ϵ.

As in the proof of Lemma 1, we show that there exists $\bar{\epsilon} > 0$ and $\omega > 0$ such that the ODE (21) with the initial condition (49) admits a unique solution (V_ϵ, W_ϵ) defined

over $[a^* - \omega, a^*]$. Moreover, $V_\epsilon(a) > \overline{V}(a)$ for $a < a^*$.[40] Therefore, we follow the steps in Lemma 12 to show that $V_\epsilon''(a) < 0$, for all $0 < \epsilon < \bar{\epsilon}$ and $a^* - \omega \leq a \leq a^*$.

Now let $\underline{a} = a^* - \frac{\omega}{2}$, we have

$$\overline{V}(a^*) + \bar{\epsilon} - \overline{V}(a^* - \omega) \geq V_\epsilon(a^*) - V_\epsilon(a^* - \omega) \geq V_\epsilon(\underline{a}) - V_\epsilon(a^* - \omega) > \frac{\omega}{2} V_\epsilon'(\underline{a}),$$

where the last inequality comes from the concavity of V_ϵ. So $V_\epsilon'(\underline{a}) < \frac{2}{\omega}\left(\overline{V}(a^*) + \bar{\epsilon} - \overline{V}(a^* - \omega)\right)$. Also by the concavity of V_ϵ

$$V_\epsilon'(a) \leq V_\epsilon'(\underline{a}) < \frac{2}{\omega}\left(\overline{V}(a^*) + \bar{\epsilon} - \overline{V}(a^* - \omega)\right),$$

for all $a \in [\underline{a}, a^*]$.

Together with (50), we have

$$0 < V_\epsilon'(a), W_\epsilon'(a) < \frac{2}{\omega}\left(\overline{V}(a^*) + \bar{\epsilon} - \overline{V}(a^* - \omega)\right)$$

for all $a \in [\underline{a}, a^*]$ and $\epsilon \in (0, \bar{\epsilon})$. Therefore, as in Lemma 1, we can apply Dominated Convergence Theorem to show that $(V_\epsilon, W_\epsilon) \to (V, W)$ over $[\underline{a}, a^*]$ for some subsequence of ϵ and (V, W) is a solution to the ODE (21) over $[\underline{a}, a^*]$. Furthermore, for all $a \in (\underline{a}, a^*]$, $(a, V(a), W(a))$ is a regular point.

When $\beta(c) = 1 - \alpha_1$ for $c \leq c^*$. Consider left maximal interval of existence, $(\hat{a}, a^*]$ of (V_ϵ, W_ϵ) as a solution to the ODE (21) from a^* with the initial conditions $(V_\epsilon(a^*), W_\epsilon(a^*)) = \left(\overline{V}(a^*) + \epsilon, \overline{W}(a^*) + \epsilon\right)$. As shown above, for ϵ sufficiently small, at a^*,

$$\max\left\{\frac{\rho + \lambda - r}{\lambda} V_\epsilon'(a^*), 0\right\} < W_\epsilon'(a^*) \leq V_\epsilon'(a^*)$$

$$V_\epsilon''(a^*) < 0$$

$$W_\epsilon'(a^*) \geq U_0'(\hat{c}_\epsilon(a^*))$$

$$V_\epsilon(a^*) > \overline{V}(a^*).$$

By Lemmas 12, 15, and 16 (when $\lim_{c \uparrow c^*} \beta(c) = 1$, $W_\epsilon'(a) = V_\epsilon'(a) = U_1'(\hat{c}_\epsilon(a)) =$

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[40]We also prove by contradiction: if $V_{\epsilon_N}(a_N) = \overline{V}(a_N)$ and $a_N \to a^*$ as $N \to \infty$,

$$\frac{V_{\epsilon_N}(a^*) - \overline{V}(a_N)}{a^* - a_N} > \frac{\overline{V}(a^*) - \overline{V}(a_N)}{a^* - a_N}$$
$$= R_h(\tilde{a}_N, V_{\epsilon_N}(\tilde{a}_N), W_{\epsilon_N}(\tilde{a}_N))$$

which at the limit contradicts the condition that $\overline{V}'(a^*_-) > U_1'(ra^*)$ since $\beta(ra^*_-) > \hat{\beta}$.

$U_0'(\hat{c}_\epsilon(a))$ for $a < a^*$, and $W_\epsilon'(a) > U_0'(\hat{c}_\epsilon(a))$ for $a < a^*$), these properties hold for all $a \in (\hat{a}, a^*)$. We show by contradiction that $\hat{a} = 0$. Assume by contradiction that this is not the case, i.e. $\hat{a} > 0$. Then the ODE (21) reaches a singular point at $\hat{a}$. By Lemma 14, $V_\epsilon(\hat{a}) = \overline{V}(\hat{a})$. Because $V_\epsilon(a) > \overline{V}(a)$ for all $a > \hat{a}$, $V_\epsilon'(\hat{a}) \geq \overline{V}'(\hat{a})$. In addition, because $\hat{a}$ is a singular point, $V_\epsilon'(\hat{a}) = U_1'(r\hat{a})$. So $U_1'(r\hat{a}) \geq \overline{V}'(\hat{a})$, or equivalently $\beta(r\hat{a}) = 1 - \alpha_1 \leq \hat{\beta}$, by Lemma 5. This contradicts the assumption that $r < \bar{r}$. We obtain the desired contradiction. Thus, (V_ϵ, W_ϵ) is defined over $(0, a^*]$.

Given any $\underline{a} > 0$, let $\omega \in (0, \underline{a})$. We have

$$\overline{V}(a^*) + \epsilon - \overline{V}(\frac{\omega}{2}) \geq V_\epsilon(a^*) - V_\epsilon(\frac{\omega}{2}) \geq V_\epsilon(\omega) - V_\epsilon(\frac{\omega}{2}) > \frac{\omega}{2} V_\epsilon'(\omega),$$

where the last inequality comes from the concavity of V_ϵ. So $V_\epsilon'(\omega) < \frac{2}{\omega} \left(\overline{V}(a^*) + \epsilon - \overline{V}(\frac{\omega}{2})\right)$. Also by the concavity of V_ϵ

$$0 < V_\epsilon'(a) \leq V_\epsilon'(\omega) < \frac{2}{\omega} \left(\overline{V}(a^*) + \bar{\epsilon} - \overline{V}(\frac{\omega}{2})\right),$$

for all $a \in [\omega, a^*]$. W_ϵ' are bounded by the same bounds. Similar to the proof of Lemma 1, there exists a subsequence $\epsilon \to 0$ such that (V_ϵ, W_ϵ) converges to some limit (V, W) which is a solution to ODE (21) over $[\omega, a^*]$, thus over $[\underline{a}, a^*]$, with the initial condition $\left(a^*, \overline{V}(a^*), \overline{W}(a^*)\right)$. $\qquad \square$

Lemma 18. *Assume that $\alpha_1 < 1$. There exists $\bar{\epsilon} > 0$ such that for $\epsilon \in (0, \bar{\epsilon})$, there exists $\delta(\epsilon) \in \left[1, \frac{\rho+\lambda}{\lambda}\right)$, such that*

$$\max\left\{(\rho + \lambda - r) V_\epsilon'(a^*), \lambda U_0'(\hat{c}_\epsilon(a^*))\right\} < \lambda W_\epsilon'(a^*) < \lambda V_\epsilon'(a^*),$$

where $(V_\epsilon'(a^*), W_\epsilon'(a^*)) = (R_h, S_h)\left(a^*, \overline{V}(a^*) + \epsilon, \overline{W}(a^*) + \delta(\epsilon)\epsilon\right)$ and

$$\hat{c}_\epsilon(a^*) = \left(U_1'\right)^{-1}\left(V_\epsilon'(a^*)\right) < ra^*.$$

Proof. Because

$$r > \underline{r} = \rho\frac{\rho + \lambda}{\rho + \lambda(1 - \alpha_2)} > \rho,$$

we have $(\rho + \lambda - r) < \lambda$.

First, we show that there exists, $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and $\delta \in \left[1, \frac{\rho+\lambda}{\lambda}\right)$,

$$U_0'(\hat{c}_{\epsilon,\delta}(a^*)) < V_{\epsilon,\delta}'(a^*) \tag{51}$$

where $\left(V'_{\epsilon,\delta}(a^*), W'_{\epsilon,\delta}(a^*) \right) = (R_h, S_h)\left(a^*, \overline{V}(a^*) + \epsilon, \overline{W}(a^*) + \delta\epsilon \right)$ and

$$\hat{c}_{\epsilon,\delta}(a^*) = \left(U'_1 \right)^{-1} \left(V'_{\epsilon,\delta}(a^*) \right) < ra^*.$$

Indeed, because R_h is continuous, for each $\delta \in \left[1, \frac{\rho+\lambda}{\lambda} \right)$,

$$\lim_{\epsilon \to 0} V'_{\epsilon,\delta}(a^*) = U'_1(ra^*)$$

$$\lim_{\epsilon \to 0} \hat{c}_{\epsilon,\delta}(a^*) = ra^*.$$

In addition, $\beta(ra^*_-) = \alpha_1 < 1$ which is equivalent to $U'_0(ra^*_-) < U'_1(ra^*)$. Therefore there exists, $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and $\delta \in \left[1, \frac{\rho+\lambda}{\lambda} \right)$ such that (51) holds. Because $\frac{\rho+\lambda-r}{\lambda} < 1$, this also implies

$$\max \left\{ \frac{\rho+\lambda-r}{\lambda} V'_{\epsilon,\delta}(a^*), U'_0(\hat{c}_{\epsilon,\delta}(a^*)) \right\} < V'_{\epsilon,\delta}(a^*).$$

By Lemma 19, we can choose $\bar{\epsilon}$ such that for all $\epsilon \in (0, \bar{\epsilon})$, $W'_{\epsilon,1}(a^*) < V'_{\epsilon,1}(a^*)$. It is easy to see that

$$\lim_{\delta \uparrow \frac{\rho+\lambda}{\lambda}} W'_{\epsilon,\delta}(a^*) = +\infty > \max \left\{ \frac{\rho+\lambda-r}{\lambda} V'_{\epsilon,\delta}(a^*), U'_0(\hat{c}_{\epsilon,\delta}(a^*)) \right\}.$$

So by the Intermediate Value Theorem, there exists $\delta(\epsilon)$ such that

$$\max \left\{ (\rho+\lambda-r) V'_\epsilon(a^*), \lambda U'_0(\hat{c}_\epsilon(a^*)) \right\} < \lambda W'_\epsilon(a^*) < \lambda V'_\epsilon(a^*).$$

$\square$

Lemma 19. *For $\epsilon > 0$, let $V_\epsilon(a^*) = \overline{V}(a^*) + \epsilon$ and $W_\epsilon(a^*) = \overline{W}(a^*) + \epsilon$. We have*

$$\lim_{\epsilon \to 0} S_h(a^*, V_\epsilon(a^*), W_\epsilon(a^*)) = U'_0(ra^*_-).$$

Proof. From the definition of R_h, S_h,

$$V'_\epsilon(a^*) = R_h(a^*, V_\epsilon(a^*), W_\epsilon(a^*))$$

$$W'_\epsilon(a^*) = S_h(a^*, V_\epsilon(a^*), W_\epsilon(a^*)).$$

Also by the definition of $V_\epsilon(a*), W_\epsilon(a^*)$, $(\lambda + \rho) V_\epsilon(a^*) - \lambda W_\epsilon(a^*) = U_1(ra^*) + \rho\epsilon$. Using

the Taylor expansion for $H(p,a)$ around $p^* = U_1'(ra^*)$, we obtain

$$\lambda\epsilon + U_1(ra^*) = H(V_\epsilon'(a^*), a^*)$$
$$= H(p^*, a^*) + \frac{\partial H(p^*, a^*)}{\partial p}(V_\epsilon'(a^*) - p^*)$$
$$+ \frac{1}{2}\frac{\partial^2 H(p^*, a^*)}{\partial p^2}(V_\epsilon'(a^*) - p^*)^2 + o((V_\epsilon'(a^*) - p^*)^2).$$

From the proof of Lemma 3, $H(p^*, a^*) = U_1(ra^*)$ and $\frac{\partial H(p^*,a^*)}{\partial p} = 0$. In addition,

$$\frac{\partial^2 H(p^*, a^*)}{\partial p^2} = -\frac{1}{U_1''\left((U_1')^{-1}(p^*)\right)} = -\frac{1}{U_1''(ra^*)} > 0.$$

Therefore

$$\rho\epsilon = -\frac{1}{2U_1''(ra^*)}(V_\epsilon'(a^*) - U_1'(ra^*))^2 + o((V_\epsilon'(a^*) - p^*)^2).$$

Consequently

$$V_\epsilon'(a^*) - U_1'(ra^*) = \sqrt{(-2U_1''(ra^*))\rho\epsilon} + o(\sqrt{\epsilon}).$$

By the definition of $\hat{c}_\epsilon$,

$$\hat{c}_\epsilon(a^*) - ra^* = (U_1')^{-1}(V_\epsilon'(a^*)) - (U_1')^{-1}(U_1'(ra^*))$$
$$= \frac{1}{U_1''(ra^*)}(V_\epsilon'(a^*) - U_1'(ra^*)) + o(V_\epsilon'(a^*) - U_1'(ra^*))$$
$$= \frac{1}{U_1''(ra^*)}\sqrt{(-2U_1''(ra^*))\rho\epsilon} + o(\sqrt{\epsilon}).$$

Therefore,

$$W_\epsilon'(r\underline{a}) = \frac{\rho W_\epsilon(a^*) - U_0(\hat{c}_\epsilon(a^*))}{ra^* - \hat{c}_\epsilon(\underline{a})} = \frac{U_0(ra^*) - U_0(\hat{c}_\epsilon(a^*)) - \rho\epsilon}{ra^* - \hat{c}_\epsilon(a^*)}$$
$$= \frac{U_0(ra^*) - U_0(\hat{c}_\epsilon(a^*))}{ra^* - \hat{c}_\epsilon(a^*)} - \frac{\rho\epsilon}{\frac{1}{-U_1''(ra^*)}\sqrt{(-2U_1''(ra^*))\rho\epsilon} + o(\sqrt{\epsilon})}$$
$$\longrightarrow U_0'(ra_-^*),$$

as $\epsilon \to 0$. $\qquad\square$

Proof of Theorem 7. We can show the existence of Markov equilibrium following the steps of the proofs of Lemma 1 and Theorem 6. In particular, we show that ODEs (21) (and

(20)) admit a solutions over some neighborhood to the left (and right) of $\tilde{a}$, with the initial condition $(V(\tilde{a}), W(\tilde{a})) = (\overline{V}(\tilde{a}), \overline{W}(\tilde{a}))$ by taking the limit of a sequence of solutions to ODEs (21) (and (20)) starting at non-singular initial conditions at $\tilde{a}$. However, in this (more informal) proof, we present an intuitive argument.

We look for an equilibrium defined in a local neighborhood of $\tilde{a}$ with $\hat{c}(a) = r\tilde{a} + \Psi(a - \tilde{a}) + o(a - \tilde{a})$. For stability, we require that $\Psi > r$ (and we verify this property below). This implies that around $\tilde{a}$, $\dot{a}_t \approx (r - \Psi)a_t$, therefore, $a_t \approx (a_0 - \tilde{a})e^{(r-\Psi)t} + \tilde{a}$. Now

$$
\begin{aligned}
W(a_t) &= \int_0^\infty e^{-\rho s} U_0(c_{t+s}) ds \\
&\approx \frac{1}{\rho} U_0(r\tilde{a}) + \int_0^\infty e^{-\rho s} U_0'(r\tilde{a})(c_{t+s} - r\tilde{a}) ds \\
&\approx \frac{1}{\rho} U_0(r\tilde{a}) + \int_0^\infty e^{-\rho s} U_0'(r\tilde{a}) \Psi(a_{t+s} - \tilde{a}) ds \\
&\approx \frac{1}{\rho} U_0(r\tilde{a}) + \int_0^\infty e^{-\rho s} U_0'(r\tilde{a}) \Psi(a_t - \tilde{a}) e^{(r-\Psi)s} ds \\
&= \frac{1}{\rho} U_0(r\tilde{a}) + U_0'(r\tilde{a}) \frac{\Psi}{\rho + \Psi - r}(a_t - \tilde{a}).
\end{aligned}
$$

Therefore, $W'(\tilde{a}) = U_0'(r\tilde{a}) \frac{\Psi}{\rho + \Psi - r}$.

By differentiating (9a), we obtain, at a^*, $(\rho + \lambda - r)V'(\tilde{a}) = \lambda W'(\tilde{a})$. In addition $V'(\tilde{a}) = U_1'(\hat{c}(\tilde{a})) = U_1'(r\tilde{a})$. These equalities imply

$$
(\rho + \lambda - r)U_1'(r\tilde{a}) = \lambda \frac{\Psi}{\rho + \Psi - r} U_0'(r\tilde{a})
$$

or equivalently

$$
\Psi = \frac{(r - \rho)(\rho + \lambda - r)}{\rho + \lambda - r - \lambda(\beta(r\tilde{a}))} > 0,
$$

since $\rho + \lambda - r > \lambda\beta(r\tilde{a}) > 0$. Lastly, because $\beta(r\tilde{a}) > \hat{\beta}$, $\Psi > r$. □

I   Derivations for Inverting Results in Section 5.3

Proof of Theorem 8. Differentiating (9a), and noticing that $V'(a) = U_1'(\hat{c}(a))$, we obtain

$$
(\rho + \lambda - r)U_1'(\hat{c}(a)) = U_1''(\hat{c}(a))\hat{c}'(a)(ra - \hat{c}(a)) + \lambda W'(a).
$$

From this equation, we can solve for $W'(a)$ as a function of $\hat{c}(a), \hat{c}'(a), a$:

$$\lambda W'(a) = (\rho + \lambda - r)U_1'(\hat{c}(a)) - U_1''(\hat{c}(a))\hat{c}'(a)\,(ra - \hat{c}(a)).$$

Differentiating the last equation, we can also write $W''(a)$ as a function of $\hat{c}, \hat{c}', \hat{c}'', a$

$$\begin{aligned}
\lambda W''(a) &= (\rho + \lambda - r)U_1''(\hat{c}(a))\hat{c}'(a) - U_1'''(\hat{c}(a))\left(\hat{c}'(a)\right)^2(ra - \hat{c}(a)) \\
&\quad - U_1''(\hat{c}(a))\hat{c}''(a)\,(ra - \hat{c}(a)) - U_1''(\hat{c}(a))\hat{c}'(a)\,(r - \hat{c}'(a)) \\
&= U_1''(\hat{c}(a))\left((\rho + \lambda - 2r)\hat{c}'(a) + \left(\hat{c}'(a)\right)^2 - \hat{c}''(a)\,(ra - \hat{c}(a))\right) \\
&\quad - U_1'''(\hat{c}(a))\left(\hat{c}'(a)\right)^2(ra - \hat{c}(a)).
\end{aligned}$$

Now differentiating (9b) and rearranging, we obtain

$$\left(\rho - r + \hat{c}'(a)\right)W'(a) = U_0'(\hat{c}(a))\,\hat{c}'(a) + W''(a)\,(ra - \widehat{c}(a)).$$

Substituting in the expressions of W' and W'' above, and using the fact that $U_0'(c) = \beta(c)U_1'(c)$, we arrive at

$$\begin{aligned}
&\left(\rho - r + \hat{c}'(a)\right)\left((\rho + \lambda - r)U_1'(\hat{c}(a)) - U_1''(\hat{c}(a))\hat{c}'(a)\,(ra - \hat{c}(a))\right) \\
&= \lambda\beta(\hat{c}(a))U_1'(\hat{c}(a))\,\hat{c}'(a) \\
&\quad + U_1''(\hat{c}(a))\left((\rho + \lambda - 2r)\hat{c}'(a) + \left(\hat{c}'(a)\right)^2 - \hat{c}''(a)\,(ra - \hat{c}(a))\right)(ra - \hat{c}(a)) \\
&\quad - U_1'''(\hat{c}(a))\left(\hat{c}'(a)\right)^2(ra - \hat{c}(a))^2.
\end{aligned}$$

Finally, dividing both sides by $U_1'(\hat{c}(a))$ and simplifying, we get

$$\begin{aligned}
&\left(\rho - r + \hat{c}'(a)\right)(\rho + \lambda - r) \\
&= \lambda\beta(\hat{c}(a))\hat{c}'(a) - \frac{U_1'''(\hat{c}(a))\hat{c}(a)^2}{U_1'(\hat{c}(a))}\left(\hat{c}'(a)\right)^2\left(\frac{ra - \hat{c}(a)}{\hat{c}(a)}\right)^2 \\
&\quad - \sigma(U_1, \hat{c}(a))\left((2\rho + \lambda - 3r)\hat{c}'(a) + 2\left(\hat{c}'(a)\right)^2 - \hat{c}''(a)\,(ra - \hat{c}(a))\right)\frac{(ra - \hat{c}(a))}{\hat{c}(a)}.
\end{aligned}$$

Since ζ is the inverse of $\hat{c}$, $a = \zeta(\hat{c}(a))$. Therefore, $\zeta' = \frac{1}{\hat{c}'}$ and $\zeta'' = \frac{-\hat{c}''}{\hat{c}'^2}$. In addition, from the definition of σ, (14), $\sigma' = -\frac{U_1'''}{U_1'}c + \frac{U_1''^2}{U_1'^2}c - \frac{U_1''}{U_1'} = \frac{1}{c}\left(-\frac{U_1'''}{U_1'}c^2 + \sigma^2 + \sigma\right)$, which implies $-\frac{U_1'''}{U_1'}c^2 = c\sigma' - \sigma^2 - \sigma$. Plugging these identities into the last equation, we arrive at (15). $\qquad\square$

Now, we apply (15) to the parametric Example 3. Noticing that $\sigma \equiv \bar{\sigma}$, then $\sigma' \equiv 0$, and $\zeta(c) \equiv \frac{c - r\tilde{a}}{\Psi} + \tilde{a}$, $\zeta' \equiv \frac{1}{\Psi}$, (15) becomes

$$\lambda \beta \frac{1}{\Psi} = \alpha_1 \frac{1}{\Psi} + \alpha_2 \left(\frac{1}{\Psi}\right)^2$$
$$+ \bar{\sigma}(2 + (2\rho + \lambda - 3r)\frac{1}{\Psi}) \left(1 - \frac{r}{\Psi}\right) \left(\frac{r\tilde{a}}{c} - 1\right) + (\bar{\sigma}^2 + \bar{\sigma}) \left(\left(1 - \frac{r}{\Psi}\right) \left(\frac{r\tilde{a}}{c} - 1\right)\right)^2,$$

since $\frac{r\zeta - c}{c} = \frac{r\left(\frac{c - r\tilde{a}}{\Psi} + \tilde{a}\right) - c}{c} = \left(1 - \frac{r}{\Psi}\right)\left(\frac{r\tilde{a}}{c} - 1\right)$. Dividing both sides by $\lambda \frac{1}{\Psi}$ and simplifying, we obtain the expression for $\beta(c)$ given in Example 3.

J Proofs for Further Characterizations

Proof of Theorem 9. Consider the construction of equilibrium in Theorem 3 and let $(V, W) = (V_0, W_0)$ and $[\underline{a}, a^*)$ denote its maximal interval of existence. First of all, we show that there exists $\epsilon \in (0, 1)$ such that $\lambda W'(a) < (1 - \epsilon)(\lambda + \rho - r) V'(a)$, for all $a \in [\underline{a}, a^*)$. Then we show that $a^* = \infty$.

Because $r < \rho$, there exists $\epsilon \in (0, 1)$ (sufficiently small) such that

$$\frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} > 1$$

and

$$(\rho - r)\frac{1}{\lambda}\frac{1 - \epsilon}{\epsilon} > \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda}.$$

In the proof of Lemma 1, we show that

$$\lim_{a \downarrow \underline{a}} W'(a) \leq U_0'(r\underline{a}) \leq U_1'(r\underline{a}) = \lim_{a \downarrow \underline{a}} V'(a).$$

Therefore, in the right neighborhood of $\underline{a}$,

$$\lambda W'(a) < (1 - \epsilon)(\lambda + \rho - r) V'(a),$$

because $(1 - \epsilon)(\lambda + \rho - r) > \lambda$.

We use Lemma 6 (Variation 2) to show that $\lambda W'(a) < (1 - \epsilon)(\rho + \lambda - r)V'(a)$ for all $a^* > a > \underline{a}$. We just showed that this is true in the right neighborhood of $\underline{a}$, so the first condition in Lemma 6 is satisfied. Now, we show that the second (relaxed) condition in Lemma 6 is also satisfied, i.e. if there exists $\tilde{a} > \underline{a}$ such that $\lambda W'(\tilde{a}) = (1 - \epsilon)(\rho + \lambda -$

$r)V'(\tilde{a})$, then $\lambda W''(a) < (1-\epsilon)(\rho + \lambda - r)V''(a)$ in the left neighborhood of $\tilde{a}$.[41]

Indeed, in the left neighborhood of $\tilde{a}$, $\lambda W'(a) \approx (1-\epsilon)(\rho + \lambda - r)V'(a)$, therefore

$$V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r)V'(a)}{\hat{c}(a) - ra} \approx -\frac{\epsilon(\rho + \lambda - r)V'(a)}{\hat{c}(a) - ra} < 0,$$

Differentiating equation (9b), we obtain

$$W''(a) = \frac{(U_0'(\hat{c}(a))\hat{c}'(a) - \rho W'(a))(\hat{c}(a) - ra) - (U_0(\hat{c}(a)) - \rho W(a))(\hat{c}'(a) - r)}{(\hat{c}(a) - ra)^2}$$

$$= \frac{(U_0'(\hat{c}(a))\hat{c}'(a) - \rho W'(a))}{\hat{c}(a) - ra} - \frac{W'(a)(\hat{c}'(a) - r)}{\hat{c}(a) - ra}.$$

Therefore,

$$W''(a) = \frac{(U_0'(\hat{c}(a)) - W'(a))\,\hat{c}'(a)}{\hat{c}(a) - ra} + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra}$$

$$= \frac{(U_0'(\hat{c}(a)) - W'(a))}{\hat{c}(a) - ra}\frac{V''(a)}{U_1''(\hat{c}(a))} + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra}$$

$$= \frac{(W'(a) - U_0'(\hat{c}(a)))}{(\hat{c}(a) - ra)(-U_1''(\hat{c}(a)))}V''(a) + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra}.$$

When a close to $\tilde{a}$, we also have:

$$V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r)V'(a)}{\hat{c}(a) - ra}$$

$$\approx \frac{-\lambda\frac{\epsilon}{1-\epsilon}W'(a)}{\hat{c}(a) - ra},$$

because, by continuity, when a close to $\tilde{a}$, $\lambda W'(a) \approx (1-\epsilon)(\rho + \lambda - r)V'(a)$. Therefore, $W'(a) \approx -\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}V''(a)(\hat{c}(a) - ra)$. Plugging this back to the expression for W'' above, we have

$$W''(a) \approx \frac{\left(\frac{(1-\epsilon)(\rho + \lambda - r)}{\lambda}U_1'(\hat{c}(a)) - U_0'(\hat{c}(a))\right)}{(\hat{c}(a) - ra)(-U_1''(\hat{c}(a)))}V''(a) - \frac{(r - \rho)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}V''(a)(\hat{c}(a) - ra)}{\hat{c}(a) - ra}$$

$$= \left(\left(\frac{(1-\epsilon)(\rho + \lambda - r)}{\lambda} - \frac{U_0'(\hat{c}(a))}{U_1'(\hat{c}(a))}\right)\frac{1}{\sigma(U_1, \hat{c}(a))}\frac{\hat{c}(a)}{\hat{c}(a) - ra} + (\rho - r)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}\right)V''(a)$$

$$< \frac{(1-\epsilon)(\rho + \lambda - r)}{\lambda}V''(a),$$

---

[41]We use Variation 2 of Lemma 6 because if $\tilde{a} = a^*$, W' and W'' might not exist at a^*.

where the last inequality comes from

$$\left(\frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda} - \frac{U_0'(\hat{c}(a))}{U_1'(\hat{c}(a))}\right) \frac{1}{\sigma(U_1,\hat{c}(a))} \frac{\hat{c}(a)}{\hat{c}(a)-ra}$$

$$\geq \left(\frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda} - 1\right) \frac{1}{\sigma(U_1,\hat{c}(a))} \frac{\hat{c}(a)}{\hat{c}(a)-ra} > 0,$$

and

$$(\rho-r)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon} > \frac{(1-\epsilon)(\rho+\lambda-r)}{\lambda}$$

and $V''(a) < 0$. Therefore both conditions 1) and 2) in Lemma 6 are satisfied, and by that lemma, $\lambda W'(a) \leq (1-\epsilon)(\rho+\lambda-r)V'(a)$ for all $a < a^*$.

We prove by contradiction that a^* is infinite. Assume by contradiction that a^* is finite. Let $F(a) = (\rho+\lambda)V(a) - \lambda W(a) - U_1(ra)$. At $a = a^*$, $F(a) = 0$ and

$$\begin{aligned} F'(a) &= (\rho+\lambda)V'(a) - \lambda W'(a) - rU_1'(ra) \\ &> (\rho+\lambda)V'(a) - (1-\epsilon)(\rho+\lambda-r)V'(a) - rU_1'(ra) \\ &= (\rho+\lambda-(1-\epsilon)(\rho+\lambda-r)-r)U_1'(ra) \\ &= \epsilon(\rho+\lambda-r)U_1'(ra) > 0. \end{aligned}$$

So $F(a) < 0$ in the left neighborhood of a^*. This is a contradiction. Thus $a^* = +\infty$, i.e. (V, W) is defined over $[\underline{a}, \infty)$.

By Lemma 4,

$$\begin{aligned} \hat{c}'(a) &= \frac{V''(a)}{U_1''(\hat{c}(a))} = \frac{(\lambda+\rho-r)V'(a) - \lambda W'(a)}{U_1''(\hat{c}(a))(\hat{c}(a)-ra)} \\ &> \frac{\epsilon(\lambda+\rho-r)V'(a)}{U_1''(\hat{c}(a))(\hat{c}(a)-ra)} > \frac{\epsilon\lambda V'(a)}{U_1''(\hat{c}(a))(\hat{c}(a)-ra)} > 0, \end{aligned}$$

where the last inequality comes from $r < \rho$. $\square$

Proof of Theorem 10. As in the proof of Theorem 9 (using the same definition of V, W and a^*), first, we show that there exists $\epsilon \in (0,1)$ such that $\lambda W'(a) < (1-\epsilon)(\lambda+\rho-r)V'(a)$, for all $a \in [\underline{a}, a^*)$. Then we show that $a^* = \infty$.

Condition (16) implies that $\sup_{c>r\underline{a}} \beta(c) < 1$. Therefore, there exists $\epsilon \in (0,1)$ such that

$$\beta(c) < 1 - \epsilon$$

and

$$(1-\epsilon) - \beta(c) > (1-\epsilon)\sigma(U_1,c)$$

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for all $c > r\underline{a}$.[42] Therefore, given $\rho \leq r$, there exists $\bar{\lambda} \geq 0$ such that for all $\lambda > \bar{\lambda}$, we have $\beta(r\underline{a}) < \hat{\beta}(r, \rho, \lambda)$ (since $\beta(r\underline{a}) < \frac{\rho}{r}$) and

$$\lambda U_0'(r\underline{a}) < (1 - \epsilon)(\lambda + \rho - r) U_1'(r\underline{a}) \tag{52}$$

and for all $c > r\underline{a}$,

$$\frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} - \frac{U_0'(c)}{U_1'(c)} > \left(\frac{r - \rho}{\lambda} \frac{1 - \epsilon}{\epsilon} + \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} \right) \sigma(U_1, c). \tag{53}$$

Moreover $\bar{\lambda}$ can be chosen to be increasing in r and $\bar{\lambda}(\rho) = 0$.[43,44]

By (52),

$$\rho + \lambda - r > 0. \tag{54}$$

Since $\beta(r\underline{a}) < \hat{\beta}$ we can apply Lemma 1. Besides, in the proof of Lemma 1, we show that

$$\lim_{a \downarrow \underline{a}} W'(a) \leq U_0'(r\underline{a}),$$

$$\lim_{a \downarrow \underline{a}} V'(a) = U_1'(r\underline{a}).$$

Therefore, by (52),

$$\lambda W'(a) < (1 - \epsilon)(\lambda + \rho - r) V'(a)$$

in the right neighborhood of $\underline{a}$.

Given these three conditions, as in the proof of Theorem 9, we use Lemma 6 (Variation 2) to show that $\lambda W'(a) < (1 - \epsilon)(\rho + \lambda - r) V'(a)$ for all $a > \underline{a}$. As shown above, this is true in the right neighborhood of $\underline{a}$ so the first condition in Lemma 6 is satisfied. Now we show that the second (relaxed) condition in Lemma 6 is also satisfied, i.e. if there exists $\tilde{a} > \underline{a}$ such that $\lambda W'(\tilde{a}) = (1 - \epsilon)(\rho + \lambda - r) V'(\tilde{a})$, we show that $\lambda W''(a) < (1 - \epsilon)(\rho + \lambda - r) V''(a)$ in the left neighborhood of $\tilde{a}$. Indeed, in the left neighborhood of $\tilde{a}$, $\lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r) V'(a)$, therefore

$$V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r) V'(a)}{\hat{c}(a) - ra} \approx -\frac{\epsilon(\rho + \lambda - r) V'(a)}{\hat{c}(a) - ra} < 0,$$

---

[42]This is equivalent to $1 - \sigma(U_1, c) > \frac{1}{1-\epsilon} \beta(c)$, for some $\epsilon \in (0, 1)$ which is true given (16).

[43]When $r = \rho$, (53) becomes $(1 - \epsilon) - \beta(c) > (1 - \epsilon)\sigma(U_1, c)$.

[44]Similarly, given $\lambda > 0$ there exists $r_1 > \rho$ such that for $r \in [\rho, r_1]$, (52) and (53) hold. The proof for existence of continuous Markov equilibrium then proceeds in exactly the same way.

Differentiating equation (9b) and simplifying as done in the proof of Theorem 9:

$$W''(a) = \frac{(W'(a) - U_0'(\hat{c}(a)))}{(\hat{c}(a) - ra)\left(-U_1''(\hat{c}(a))\right)}V''(a) + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra}.$$

When a close to $\tilde{a}$, we also have:

$$V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r)V'(a)}{\hat{c}(a) - ra} \approx \frac{-\lambda\frac{\epsilon}{1-\epsilon}W'(a)}{\hat{c}(a) - ra},$$

because, by continuity, when a close to $\tilde{a}$, $\lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r)V'(a)$. Therefore, $W'(a) \approx -\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}V''(a)(\hat{c}(a) - ra)$. Plugging this back to the expression for W'' above, we have

$$W''(a) \approx \frac{\left(\frac{(1-\epsilon)(\rho + \lambda - r)}{\lambda}U_1'(\hat{c}(a)) - U_0'(\hat{c}(a))\right)}{(\hat{c}(a) - ra)\left(-U_1''(\hat{c}(a))\right)}V''(a) - \frac{(r - \rho)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}V''(a)(\hat{c}(a) - ra)}{\hat{c}(a) - ra}$$

$$= \left(\left(\frac{(1-\epsilon)(\rho + \lambda - r)}{\lambda} - \frac{U_0'(\hat{c}(a))}{U_1'(\hat{c}(a))}\right)\frac{1}{\sigma(U_1, \hat{c}(a))}\frac{\hat{c}(a)}{\hat{c}(a) - ra} - (r - \rho)\frac{1}{\lambda}\frac{1-\epsilon}{\epsilon}\right)V''(a)$$

$$< \frac{(1-\epsilon)(\rho + \lambda - r)}{\lambda}V''(a),$$

where the last inequality comes from (53) and $V''(a) < 0$. Therefore both conditions 1) and 2) in Lemma 6 are satisfied, and by that lemma, $\lambda W'(a) \le (1 - \epsilon)(\rho + \lambda - r)V'(a)$ for all $a \le a^*$.

As in the proof of Theorem 9, we prove by contradiction that a^* is infinite. Assume by contradiction that a^* is finite. Let $F(a) = (\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)$. At $a = a^*$, $F(a) = 0$ and

$$
\begin{aligned}
F'(a) &= (\rho + \lambda)V'(a) - \lambda W'(a) - rU_1'(ra) \\
&> (\rho + \lambda)V'(a) - (1 - \epsilon)(\rho + \lambda - r)V'(a) - rU_1'(ra) \\
&= (\rho + \lambda - (1 - \epsilon)(\rho + \lambda - r) - r)U_1'(ra) \\
&= \epsilon(\rho + \lambda - r)U_1'(ra) > 0,
\end{aligned}
$$

where the last inequality comes from (54). So $F(a) < 0$ in the left neighborhood of a^*. This is a contradiction. Thus $a^* = +\infty$.

Similar to the proof of Theorem 9,

$$\hat{c}'(a) = \frac{(\lambda + \rho - r)V'(a) - \lambda W'(a)}{U_1''(\hat{c}(a))(\hat{c}(a) - ra)} > \frac{\epsilon(\lambda + \rho - r)V'(a)}{U_1''(\hat{c}(a))(\hat{c}(a) - ra)} > 0,$$

where the last inequality also comes from (54). □

Proof of Theorem 11. We use the notation $\overline{V}_\lambda, \overline{W}_\lambda$ for the functions defined in (11). Notice that

$$\overline{W}_\lambda(a) = \frac{1}{\rho} U_0(ra),$$

independent of λ, so we can drop the subscript λ.

First, we notice that since $\sup_{c \geq r\underline{a}} \beta(c) < \frac{\rho}{r}$, there exists $\lambda^* > 0$ such that $\beta(c) < \hat{\beta}(r, \rho, \lambda)$ for all $\lambda \geq \lambda^*$. Therefore, we can apply Lemma 1 and Theorem 3 to construct Markov equilibria with dissaving. Let V_λ, W_λ denote the equilibrium value functions constructed in the proof of Theorem 3. We show that there exists $\bar{\lambda} > \lambda^*$, such that for all $\lambda \geq \lambda^*$, V_λ crosses $\overline{V}_\lambda$ at some $a_1(\lambda) > \underline{a}$ and $\lim_{\lambda \to \infty} a_1(\lambda) = \underline{a}$. In addition, $W_\lambda(a_1(\lambda)) < \overline{W}_\lambda(a_1(\lambda))$. This result immediate implies the existence of a Markov equilibrium with dissaving and discontinuous policy function because starting from $a_1(\lambda)$ we can apply Theorem 3 to obtain a Markov equilibrium defined over $[a_1(\lambda), \infty)$ with $(V, W) = (\overline{V}_\lambda, \overline{W}_\lambda)$ at $a_1(\lambda)$. Combining this equilibrium with (V_λ, W_λ) defined over $[\underline{a}, a_1(\lambda))$, we obtain a discontinuous equilibrium over $[\underline{a}, \infty)$.

We prove the result by contradiction. Assume that the result does not hold, then there exists $\bar{a} > \underline{a}$ and a sequence of $\{\lambda_n\}_{n=1}^\infty$ with $\lim_{n \to \infty} \lambda_n = \infty$, such that

$$V_{\lambda_n}(a) > \overline{V}_{\lambda_n}(a)$$

for all $a \in (\underline{a}, \bar{a})$.[45] Because $\lim_{a \downarrow \underline{a}} \frac{U_0(ra) - U_0(r\underline{a})}{a - \underline{a}} = rU_0'(r\underline{a})$, and $r > \rho$, there exists $a_1 \in (\underline{a}, \bar{a})$ and $0 < \gamma$ such that

$$\frac{1}{\rho} \frac{U_0(ra_1) - U_0(r\underline{a})}{a_1 - \underline{a}} > (\gamma + 1) U_0'(r\underline{a}).$$

First, using Lemma 20 below, we show that

$$\lim_{n \to \infty} W_{\lambda_n}(a_1) = \overline{W}(a_1).$$

Indeed, by Lemma 10, $W_{\lambda_n}(a_1) - \overline{W}(a_1) \leq 0$. Therefore

$$\limsup_{n \to \infty} \left(W_{\lambda_n}(a_1) - \overline{W}(a_1) \right) \leq 0. \tag{55}$$

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[45]By Lemma 10, $W_{\lambda_n}(a) < \overline{W}_\lambda(a)$. Therefore $(\rho + \lambda_n) V_{\lambda_n}(a) - \lambda_n W_{\lambda_n}(a) > (\rho + \lambda_n) \overline{V}_{\lambda_n}(a) - \lambda_n \overline{W}_{\lambda_n}(a) = U_1(ra)$. Thus $V_{\lambda_n}, W_{\lambda_n}$ are defined and continuous over $[\underline{a}, \bar{a}]$.

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Now,

$$W_{\lambda_n}(a_1) - \overline{W}(a_1) = W_{\lambda_n}(a_1) - V_{\lambda_n}(a_1) + V_{\lambda_n}(a_1) - \overline{V}_{\lambda_n}(a_1)$$
$$+ \overline{V}_{\lambda_n}(a_1) - \overline{W}_{\lambda_n}(a_1).$$

By Lemma 20,

$$\lim_{n \to \infty} \left(V_{\lambda_n}(a_1) - W_{\lambda_n}(a_1) \right) = 0.$$

By the definition of $\overline{V}_\lambda, \overline{W}_\lambda$ in (11)

$$\lim_{n \to \infty} \left(\overline{V}_{\lambda_n}(a_1) - \overline{W}_{\lambda_n}(a_1) \right) = 0,$$

and by the contradiction assumption

$$V_{\lambda_n}(a_1) - \overline{V}_{\lambda_n}(a_1) \geq 0.$$

Thus

$$\liminf_{n \to \infty} \left(W_{\lambda_n}(a_1) - \overline{W}(a_1) \right) \geq 0. \tag{56}$$

Therefore by (55) and (56)

$$\lim_{n \to \infty} \left(W_{\lambda_n}(a_1) - \overline{W}(a_1) \right) = 0.$$

Given this limit, for $\epsilon > 0$, sufficiently small, there exists N such that $W_{\lambda_n}(a_1) - \overline{W}(a_1) > -\epsilon$ for all $n \geq N$. Now,

$$\frac{W_{\lambda_n}(a_1) - W_{\lambda_n}(\underline{a})}{a_1 - \underline{a}} > \frac{\overline{W}(a_1) - \overline{W}(\underline{a}) - \epsilon}{a_1 - \underline{a}} > (\gamma + 1) U_0'(r\underline{a}) - \frac{\epsilon}{a_1 - \underline{a}}. \tag{57}$$

By the Mean Value Theorem, there exists $a_n \in (\underline{a}, a_1)$ such that,

$$W_{\lambda_n}'(a_n) = \frac{W_{\lambda_n}(a_1) - W_{\lambda_n}(\underline{a})}{a_1 - \underline{a}}$$
$$\leq U_0'(ra_n) + \frac{\rho}{\lambda_n} U_1'(ra_n)$$
$$\leq U_0'(r\underline{a}) + \frac{\rho}{\lambda_n} U_1'(r\underline{a}), \tag{58}$$

where the first inequality comes from the proof of Lemma 9 (especially inequality (34)).

By choosing ϵ sufficiently small and n sufficiently large such that

$$\frac{\epsilon}{a_1 - a} + \frac{\rho}{\lambda_n} U_1'(r\underline{a}) < \gamma U_0'(r\underline{a}),$$

which contradicts (57) and (58). We obtain the desired contradiction. □

Lemma 20. *Assume that there exists $\bar{a} > \underline{a}$ and a diverging sequence $\{\lambda_n\}$ such that $V_{\lambda_n}(a) > \overline{V}_{\lambda_n}(a)$ for all $a \in (\underline{a}, \bar{a})$. Then*

$$\lim_{n \to \infty} (V_{\lambda_n}(a) - W_{\lambda_n}(a)) = 0, \tag{59}$$

for all $a \in (\underline{a}, \bar{a})$.

Proof. By Lemma 10, $W_{\lambda_n} \leq \overline{W}$ therefore

$$V_{\lambda_n}(a) - W_{\lambda_n}(a) \geq \overline{V}_{\lambda_m}(a) - \overline{W}_{\lambda_n}(a)$$

for all $a \in (\underline{a}, \bar{a})$.

To find an upper bound on $V_{\lambda_n} - W_{\lambda_n}$. We rewrite equation (9a) as

$$\lambda \left(V_\lambda(a) - W_\lambda(a) \right) = U_1(\hat{c}_\lambda(a)) + V_\lambda'(a)(ra - \hat{c}_\lambda(a)) - \rho V_\lambda(a).$$

Therefore

$$\lambda \left(V_\lambda(a) - W_\lambda(a) \right) \leq U_1(\hat{c}_\lambda(a)) - \rho \overline{V}_\lambda(\underline{a}), \tag{60}$$

because $V_\lambda(a) > V_\lambda(\underline{a}) = \overline{V}_\lambda(\underline{a})$, and $V_\lambda' \geq 0$, and $ra - \hat{c}_\lambda(a) < 0$.

Now if U_1 is bounded above

$$\lambda(V_\lambda(a) - W_\lambda(a)) \leq \sup_c U_1(c) - \rho \overline{V}_\lambda(\underline{a}).$$

Thus $\lambda |V_\lambda(a) - W_\lambda(a)|$ is bounded when $\lambda \to \infty$. Therefore (59) holds.

If U_1 is not bounded, but condition (17) is satisfied, we show, using Lemma 6, that there exists $\bar{\lambda}$ such that, when $\lambda > \bar{\lambda}$, $\hat{c}_\lambda(a) < \frac{2\lambda}{\underline{\sigma}} a$, for all $a \in (\underline{a}, \bar{a})$. Let $f(a) = \frac{2\lambda}{\underline{\sigma}} a$ and $g(a) = \hat{c}_\lambda(a)$. With $\lambda > \underline{\sigma} r$, $f(\underline{a}) = \frac{2\lambda}{\underline{\sigma}} > r\underline{a}$. We just need to verify that if $f(a) = g(a)$ then $f'(a) = \frac{2\lambda}{\underline{\sigma}} > g'(a) = \hat{c}_\lambda'(a)$. Indeed, by differentiating, the first order condition (12) with respect to a,

$$\hat{c}_\lambda'(a) = \frac{V_\lambda''(a)}{U_1''(\hat{c}_\lambda(a))}.$$

To get $V_\lambda''(a)$, differentiating (9a) with respect to a and use the first order condition for c,

we obtain

$$V_\lambda''(a)(ra - \hat{c}_\lambda(a)) = (\rho + \lambda - r)\, V_\lambda'(a) - \lambda W_\lambda'(a)$$
$$= (\rho + \lambda - r)\, U_1'(\hat{c}_\lambda(a)) - \lambda W_\lambda'(a).$$

Therefore, because $W_\lambda' \geq 0$ as shown in Lemma 9,

$$\hat{c}_\lambda'(a) = \frac{(\rho + \lambda - r)\, U_1'(\hat{c}_\lambda(a)) - \lambda W_\lambda'(a)}{-U_1''(\hat{c}_\lambda(a))\,(\hat{c}_\lambda(a) - ra)} \leq \frac{(\rho + \lambda - r)\, U_1'(\hat{c}_\lambda(a))}{-U_1''(\hat{c}_\lambda(a))\,(\hat{c}_\lambda(a) - ra)}$$
$$= (\rho + \lambda - r)\, \frac{1}{\sigma(U_1, \hat{c}_\lambda(a))} \frac{\hat{c}_\lambda(a)}{(\hat{c}_\lambda(a) - ra)} < \frac{\lambda}{\underline{\sigma}} \frac{\hat{c}_\lambda(a)}{(\hat{c}_\lambda(a) - ra)}$$
$$= \frac{\lambda}{\underline{\sigma}} \frac{\frac{2\lambda}{\underline{\sigma}}}{\frac{2\lambda}{\underline{\sigma}} - ra}.$$

By choosing $\bar{\lambda}$ sufficiently large, for all $\lambda > \bar{\lambda}$, $\frac{\frac{2\lambda}{\underline{\sigma}}}{\frac{2\lambda}{\underline{\sigma}} - ra} < 2$ for all $a \in (\underline{a}, \bar{a})$. Therefore, by Lemma 6, $\hat{c}_\lambda(a) < \frac{2\lambda}{\underline{\sigma}} a$.

Now, going back to inequality (60),

$$\lambda\,(V_\lambda(a) - W_\lambda(a)) \leq U_1\,(\hat{c}_\lambda(a)) - \rho \overline{V}_\lambda(\underline{a}) < U_1\left(\frac{2\lambda}{\underline{\sigma}} \bar{a}\right) - \rho \overline{V}_\lambda(\underline{a}).$$

By the INADA conditions

$$\lim_{\lambda \to \infty} \frac{U_1\left(\frac{2\lambda}{\underline{\sigma}} \bar{a}\right)}{\lambda} = 0.$$

It is easy to show that $\lim_{\lambda \to \infty} \frac{\overline{V}_\lambda(\underline{a})}{\lambda} = 0$. Thus we obtain the desired convergence (59). □