OPTIMAL DEBT AND PROFITABILITY IN THE TRADEOFF THEORY

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ABSTRACT

I develop a dynamic model of leverage with tax deductible interest and an endogenous cost of default. The interest rate includes a premium to compensate lenders for expected losses in default. A borrowing constraint is generated by lenders' unwillingness to lend an amount that would trigger immediate default. When the borrowing constraint is not binding, the tradeoff theory of debt holds: optimal debt equates the marginal tax shield and the marginal expected cost of default. Contrary to conventional interpretation, increases in current or future profitability reduce the optimal leverage ratio when the tradeoff theory holds, consistent with empirical findings.

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The tradeoff theory of capital structure is the longest standing theory of capital structure[1] and underlies much of the large body of empirical work that studies capital structure. In the tradeoff theory, the optimal amount of debt equates the marginal benefit of a dollar of debt arising from the tax deductibility of interest payments with the marginal cost of a dollar of debt arising from increased exposure to default. This framework implies that changes in leverage over time, or variation in leverage across firms, can be attributed to differences in the interest tax shield and/or differences in the (marginal) cost of default. The tradeoff theory has been interpreted[2] to imply that more profitable firms should have higher leverage ratios—a prediction that is contrary to the empirical fact[3] that more profitable firms tend to have lower leverage ratios.

In this paper, I develop and analyze a model of capital structure that incorporates an interest tax shield as well as the possibility of default. In some situations, the optimal amount of debt will be determined by the equality of the marginal benefit of debt arising from the interest tax shield and the marginal cost of debt associated with increased probability of default—that is, optimal debt will be characterized by the tradeoff theory. However, in other situations within the model, optimal debt will not be characterized by the equality of marginal benefit and marginal cost that epitomizes the tradeoff theory. Because the model allows for situations in which the tradeoff theory holds and situations in which it does not hold, the model has the potential to guide empirical tests by including both a null hypothesis in terms of the tradeoff theory and an alternative hypothesis that offers an explanation of leverage other than the tradeoff theory. In particular, I show that the model developed here accommodates situations in which higher profitability (either current profitability or expected future profitability) is associated with lower leverage. Furthermore, if the probability of default is nonzero, these situations arise when and only when the tradeoff theory is operative. That is, the empirical finding that more profitable firms tend to have lower leverage ratios, which has been viewed by others as evidence against the tradeoff theory, is viewed as evidence in favor of the tradeoff theory when viewed through the lens of the model presented here.

As the tradeoff theory has developed over the past half century, it has become increasingly

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[1]Robichek and Myers (1966), Kraus and Litzenberger (1973), and Scott (1976).
[2]Scott (1976), Fama and French (1992), Frank and Goyal (2007), and Strebulaev (2007).
[3]Fama and French (1992).

complex, especially in empirical structural models of the firm that are designed to capture realistic features of a firm's environment.[4] The model I develop here will be stripped of these complexities so that I can focus on its new features and implications in a framework that admits analytic results without relying on numerical solution. The model's biggest departure from standard models of debt concerns the maturity of the debt. Many standard models of debt[5] assume that debt has infinite maturity and pays a fixed coupon over the infinite future, or until the firm defaults. Clearly, the assumption of infinite maturity is extreme, but it has been used productively over the years. I also make an extreme assumption about maturity, but in the opposite direction. I assume that debt must be repaid, with interest, right after it is issued. The standard specification with infinite maturity can be viewed as the limiting case of long-term debt and my specification with zero maturity can be viewed as the limiting case of short-term debt, such as commercial paper.

The specification of zero-maturity debt is motivated by two considerations. First, this specification makes salient the recurrent nature of the financing decision, in contrast to the once-and-for-all financing decision in many models of debt.[6] At each instant of time, the firm decides whether to repay its debt, with interest, or to default, and if it decides to repay its debt, it chooses the amount of debt to issue anew. Because I do not include any flotation, issuance, or adjustment costs, the amount of debt issued responds immediately and completely to changes in the firm's environment, and will not have the rich dynamics documented empirically and analyzed by Leary and Roberts (2005). The second reason for specifying debt to have zero maturity is that zero-maturity debt is always valued at par, which alleviates the need to calculate the value of debt that would arise with long-term debt. Therefore, the firm's decision about whether to default on debt, which depends on a comparison of the total value of the firm and the value of the firm's debt, becomes transparent.

Because firms have the opportunity to default on their debt, rational lenders need to take account of the probability of default, as well as their losses in the event of default, in order to determine the appropriate interest rate on their loans to the firm. In this paper, risk-neutral lenders require a premium above the riskless rate in order to compensate for the expected

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[4] Hennessy and Whited (2007).

[5] Modigliani and Miller (1958) and Leland (1994).

[6] Merton (1974) and Leland (1994).

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losses in the event of default. In addition, if the amount of debt were sufficiently large, it would trigger immediate default. In the current framework with zero-maturity debt, lenders avoid being subject to immediate default by refusing to lend an amount greater than the contemporaneous value of the firm, which itself depends on the amount of debt issued.

An important component of the firm's financing decision is the stochastic process for the firm's pre-tax pre-interest cash flow, that is, EBIT (earnings before interest and taxes). I specifiy a continuous-time continuous-state process for EBIT. Instead of a diffusion process, as in Leland (1994) for example, I specify a Markov process in which EBIT remains unchanged for a random length of time and then a new value of EBIT arrives at dates governed by a Poission process. For simplification, the new values of EBIT are independent across arrivals. With this stochastic specification, the firm's optimal capital structure depends on whether EBIT exceeds or falls short of an endogeneously determined critical value.

For values of EBIT that exceed the critical value, the tradeoff theory is operative, provided that the firm faces a positive probability of default. I show that, consistent with empirical findings, the optimal leverage ratio is a declining function of EBIT. In particular, when the tradeoff theory is operative, the optimal level of debt is invariant to EBIT when EBIT exceeds the critical value. Nevertheless, an increase in EBIT increases the total value of the firm so the optimal leverage ratio falls as profitability, measured by EBIT, increases. This negative relationship between profitability and the leverage ratio, which arises from the invariance of debt to EBIT, is reminiscent of Strebulaev's (2007) finding that when adjustment costs prevent the level of debt from changing, an increase in profitability reduces the leverage ratio. However, the mechanisms that make debt invariant to profitability are different in the current model and in Strebulaev's model. Specifically, the invariance of optimal debt arises in the current model in the complete absence of any flotation, issuance, or adjustment costs that prevent debt from adjusting in Stebulaev's model.

For values of EBIT below the critical value, the borrowing constraint that prevents immediate default binds. In this situation, the tradeoff theory is not operative at the margin and the optimal leverage ratio is invariant to EBIT. To summarize, the model predicts a negative relationship between the optimal leverage ratio and contemporaneous EBIT (which is consistent with empirical findings) when the tradeoff theory is operative but not when the constraint on the firm's borrowing is binding.

To examine the relationship between the optimal amount of debt and *expected future* prof-

itability, I analyze the impact of a rightward translation of the unconditional distribution of profitability. This favorable shift of the unconditional distribution of profitability increases the continuation value of the firm, which the firm's current shareholders stand to lose in the event of default; therefore, a favorable shift of the unconditional distribution increases the cost of default facing these shareholders.[7] In response to the increased cost of default, the firm adjusts its capital structure to reduce the probability of default. Whether this change in capital structure reduces the optimal amount of debt depends on the unconditional distribution of profitability. In the focal case of a uniform distribution, a rightward translation of the distribution reduces optimal debt when the tradeoff theory is operative. In fact, this finding holds more broadly within the class of truncated exponential distributions for density functions that do not slope upward too steeply. Only for sufficiently steeply upward-sloping density functions will optimal debt increase in response to a rightward translation of the distribution function when the tradeoff theory is operative. Thus, whether one analyzes the relationship between the optimal leverage ratio and *contemporaneous* EBIT (as described above) or between the optimal leverage ratio and *long-run average* EBIT, the model predicts a negative relationship between the optimal leverage ratio and EBIT if the tradeoff theory is operative, but not if the firm faces a binding constraint on its borrowing.

Section 1 presents the economic environment facing a firm, including the opportunity to borrow and the ability to default on outstanding debt. The availability and terms of loans to the firm depend on the valuation of the firm, which is presented in Section 2. Section 3 characterizes the optimal level of debt and the associated value of shareholders' equity, which together sum to the total valuation of the firm. An important component of this analysis is the critical value of current EBIT, which is the boundary between low values of EBIT for which the borrowing constraint binds and high values of EBIT for which the borrowing constraint does not bind. Depending on whether this critical value of EBIT is at the minimum value of the support of EBIT, the maximum value of the support of EBIT, or in between, the firm will find itself in one of three regimes, which are characterized in Section 4. Although optimal behavior in two of the three regimes can be derived very

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[7]To the extent that creditors recover a portion of their loans in the event of default, the social cost of default is smaller than the loss to current shareholders. In this paper, creditors receive a fraction $1 - \alpha$ of the continuation value of the firm in the event of default, so the social cost of default is a fraction α of the continuation value. Therefore, the social cost of default, as well as private cost to current shareholders, is increased by a favorable shift of the unconditional distribution of profitability.

easily, optimal behavior in one of the regimes (denoted as Regime II) entails more extensive analysis, which is presented in Section 5. Of particular interest is subsection 5.2, which interprets the analytic results in terms of the tradeoff theory. Section 6 demonstrates that Regime I, II, or III will prevail depending on whether the tax rate is low, intermediate, or high, and it provides explicit expressions for the values of the tax rate associated with the boundaries between adjacent regimes. Section 7 analyzes the impact of a rightward translation of the unconditional distribution of EBIT on optimal debt, shareholder equity, the critical value of EBIT, and the probability of default. Concluding remarks are presented in Section 8. To avoid disruption in the narrative flow of the paper, the proofs of all lemmas, propositions, and corollaries are in Appendix A. In addition, an online appendix presents closed-form solutions for optimal debt and the value of the firm in the special case in which the unconditional distribution of EBIT is uniform and creditors recover zero in the event of default.

1 The Firm's Economic Environment

Let $\phi(t)$ be EBIT, the pre-tax net cash flow from operations, before interest, at time t. The realizations of $\phi(t)$ are generated by an exogenous Markov process and thus are independent of any actions taken by the firm. If EBIT at time t_0 is $\phi(t_0)$, then $\phi(t)$ remains equal to $\phi(t_0)$ for all $t > t_0$ until some random date $t_1 > t_0$. At time t_1, a new value of EBIT, $\phi(t_1)$, is drawn from a distribution with c.d.f. $F(\phi)$, with $F(\Phi_L) = 0$, $F(\Phi_H) = 1$, and finite density $f(\phi) = F'(\phi) > 0$ everywhere on the support $[\Phi_L, \Phi_H]$, where $-\infty < \Phi_L < \Phi_H < \infty$. In addition, assume that $f'(\phi) \geq 0$ for all ϕ in the support of $F(\phi)$, which will ensure that the firm's objective function is concave in debt. The unconditional expected value of EBIT is assumed to be positive, that is,

$$E\{\phi\} = \int_{\Phi_L}^{\Phi_H} \phi \, dF(\phi) > 0, \tag{1}$$

which implies $\Phi_H > 0$ but does not place a restriction on the sign of Φ_L. The arrival date of a new value of EBIT, that is, the timing of t_1, is governed by a Poisson process with an instantaneous probability λ of an arrival of a new value of $\phi(t)$. The realizations of new values of EBIT are i.i.d. with c.d.f. $F(\phi)$.

At time t, the firm borrows an amount $D(t)$ from risk-neutral lenders and then pays interest of $(rD(t) + Q(t))\,dt$ over the next interval dt of time, where r is the instantaneous riskless interest rate and $Q(t)$ is a default premium paid to lenders to compensate for the probability that the firm will default on its debt during the next interval dt of time. If lenders receive no payment in the event of default, that is, if the recovery rate on defaulted debt is zero, then $Q(t) = P(t)D(t)$, where $P(t)$ is the probability that the firm will default during the next interval, dt, of time. However, to the extent that lenders are able to recover some portion of their loans when the firm defaults, the default premium, $Q(t)$, will be lower than $P(t)D(t)$. I will specify the default premium in this case more fully after I describe the valuation of the firm. Taxes are levied at rate $\tau > 0$ on pre-tax cash flow, which is EBIT less interest payments, $\phi(t) - (rD(t) + Q(t))$. When cash flow is negative, the firm pays negative taxes.

The maturity of the debt is vanishingly small and the firm can "rollover" its debt by issuing new debt immediately after paying interest and repaying the principal on its existing debt. The firm is managed on behalf of the current shareholders who, like lenders, are risk-neutral. Shareholders discount future expected flows at rate ρ and I will assume that shareholders and lenders have equal rates of time preference, so that $r = \rho$. Henceforth, I will use ρ for the common value of the discount rate of shareholders and the riskless interest rate, so the after-tax cash flow of the firm at time t, denoted as $C(t)$, is

$$C(t) \equiv (1 - \tau)\left[\phi(t) - (\rho D(t) + Q(t))\right]. \tag{2}$$

The support of the distribution for EBIT may include negative values of EBIT, since Φ_L may be negative. If a negative realization of EBIT is sufficiently large in absolute value or sufficiently persistent, current shareholders might choose to cease operation altogether and abandon ownership of the firm if doing so would allow them to avoid a stream of future cash flows with a negative expected present value. Thus, the shareholders of a firm might cease operation either as a response to unfavorable current and prospective future EBIT or as a means to avoid repaying the firm's debt. My focus in this paper is on leverage and the potential default associated with it, so I will restrict the stochastic process for EBIT to be such that in the complete absence of borrowing, the firm would never find it optimal to cease

operation.[8] To formalize this restriction, define $W\left(\phi\left(t\right)\right)$ to be the expected present value of current and future cash flows, if the firm does not ever issue any debt, and continues to operate forever, regardless of the realization of $\phi\left(t\right)$. Therefore,

$$W\left(\phi\left(t\right)\right) \equiv E_t\left\{\int_t^\infty \left(1-\tau\right)\phi\left(s\right)e^{-\rho(s-t)}ds\right\}, \tag{3}$$

where $E_t\left\{\right\}$ denotes the expectation conditional on information available at time t. As shown in the lemma below, $W\left(\phi\left(t\right)\right)$ is an increasing linear function with slope $\frac{1-\tau}{\rho+\lambda}$.

Lemma 1 $W\left(\phi\left(t\right)\right) = \frac{1-\tau}{\rho+\lambda}\left[\phi\left(t\right) + \frac{\lambda}{\rho}E\left\{\phi\right\}\right]$.

Corollary 2 If $\Phi_L > -\frac{\lambda}{\rho}E\left\{\phi\right\}$, then $W\left(\phi\left(t\right)\right) > 0$ for all $\phi\left(t\right) \in [\Phi_L, \Phi_H]$.

Henceforth, I assume that

$$\Phi_L > -\frac{\lambda}{\rho}E\left\{\phi\right\}, \tag{4}$$

so Corollary 2 implies that in the complete absence of borrowing, the firm would never find it optimal to cease operation.

2   Valuation of the Firm

Suppose that the firm arrives at time t without any outstanding debt. The shareholders in the firm choose an amount of zero-maturity debt $D\left(t\right)$ to issue at time t. This debt issuance is a discrete transfer of funds to the current shareholders, who could, in principle, simply take the amount $D\left(t\right)$ and then default. However, defaulting on the debt would cause shareholders to give up their claims on future cash flows from operations. To continue their ownership of the firm, the current shareholders would have to repay the debt, with interest, after an instant, and then they could immediately borrow again and continue operation. Because of the Markovian nature of the stochastic process for $\phi\left(t\right)$, an ongoing firm's optimal choice of debt and the instantaneous probability of default are simply functions of the contemporaneous value of $\phi\left(t\right)$. If EBIT remains continuously equal to $\phi\left(t\right)$ from time

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[8] Leland (1994, p. 1217) effectively makes the same assumption by specifying a stochastic process for the "asset value," which corresponds to $W\left(\phi\left(t\right)\right)$ in equation (3), that is bounded away from zero.

t to time $s > t$, the firm will choose to set $D(s)$ equal to $D(t)$. Let $t' > t$ be the first time after time t that ϕ changes. Thus, for all $s \in [t, t')$, the after-tax cash flow, $C(s)$, equals $(1 - \tau)[\phi(t) - (\rho D(t) + Q(t))]$. At time t', a new value of ϕ is drawn from the distribution $F(\phi)$ and a new value of the after-tax cash flow, $C(t') = (1 - \tau)[\phi(t') - (\rho D(t') + Q(t'))]$, is realized.

Define $V(\phi(t), D)$ to be the expected present value of current and prospective future funds available to current shareholders if they (1) arrive at time t without any outstanding debt; (2) issue an amount D of debt at time t and maintain a constant amount of debt equal to D until time $t' > t$, which is the first time after t that a new value of ϕ is drawn; and (3) issue optimal amounts of debt and behave optimally from time t' onward. Define

$$\widehat{V}(\phi(t)) \equiv \max_{D \leq \widehat{V}(\phi(t))} V(\phi(t), D) \tag{5}$$

as the maximized value of $V(\phi(t), D)$ attained when the firm chooses D subject to the constraint that D cannot exceed $\widehat{V}(\phi(t))$. I will refer to this constraint as a borrowing constraint. It reflects the fact that if D were to exceed $\widehat{V}(\phi(t))$ the firm would default because repaying the amount D would cost the firm more than the benefit of retaining ownership of the firm.[9]

The Markovian structure of the firm's decision problem implies that

$$
\begin{aligned}
V(\phi(t), D) \;=\; & D \\
& + E_t \left\{ \int_t^{t'} [(1 - \tau)[\phi(s) - (\rho D + Q(s))]] e^{-\rho(s-t)} ds \right\} \\
& + E_t \left\{ e^{-\rho(t'-t)} \int_{\widehat{V}(\phi(t')) \geq D} \left[\widehat{V}(\phi(t')) - D \right] dF(\phi(t')) \right\}.
\end{aligned}
\tag{6}
$$

The first term on the right hand side of equation (6) is the lump sum of funds that shareholders receive at time t when the firm issues debt D. The second term on the right hand side of equation (6) is the expected present value of cash flows, net of interest and taxes,

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[9]Corollary 2 and the maintained assumption $\Phi_L > -\frac{\lambda}{\rho} E\{\phi\}$ imply that the value of the firm would be positive if it always issues zero debt. Therefore, $V(\phi(t), 0) > 0$ so $\widehat{V}(\phi(t)) > 0$ and $D = 0$ is always feasible. Proposition 4 states that optimal D is positive. Therefore, I do not include include a non-negativity constraint on D since it would never bind.

that will flow to the firm until time t'. The third term on the right hand side of equation (6) is the expected present value of the "continuation value" of the firm at time t'. Since the firm has the option to default on its debt and to cease operation, this continuation value must be non-negative, in contrast to the second term, which could be positive, negative, or zero. At time t', the maximized expected present value of all the cash flows from that point onward would be $\widehat{V}(\phi(t'))$, if the firm were to arrive at time t' with zero debt. However, the firm arrives at time t' with debt equal to D and the current shareholders will choose to repay that debt and thereby retain their ownership of the firm if and only if $\widehat{V}(\phi(t')) \geq D$. Hence, the integral in the third term is limited to values of $\phi(t')$ for which $\widehat{V}(\phi(t')) \geq D$. For these values of $\phi(t')$, the firm chooses to repay its debt at time t', and $\widehat{V}(\phi(t')) - D \geq 0$ is the value of the current shareholders' stake in the firm at time t'.

Since the arrival of a new value of ϕ is governed by a Poisson process, the density of the arrival date t' is $\lambda e^{-\lambda(t'-t)}$, which implies that $E_t\left\{\int_t^{t'} e^{-\rho(s-t)}ds\right\} = \frac{1}{\rho+\lambda}$ and $E_t\left\{e^{-\rho(t'-t)}\right\} = \frac{\lambda}{\rho+\lambda}$. Therefore, equation (6) can be rewritten as

$$
\begin{aligned}
V(\phi(t), D) &= D + \frac{1-\tau}{\rho+\lambda}\left[\phi(t) - (\rho D + Q(t))\right] \\
&\quad + \frac{\lambda}{\rho+\lambda}\int_{\widehat{V}(\phi(t'))\geq D}\left[\widehat{V}(\phi(t')) - D\right]dF(\phi(t')).
\end{aligned}
\tag{7}
$$

Having defined the conditions that would lead the shareholders to default on their debt, I can specify the default premium $Q(t)$ more completely. A firm that arrives at time t with outstanding debt D will optimally choose to default on that debt if $\widehat{V}(\phi(t)) < D$. I will assume that in the event of default, the lenders take ownership of the firm. However, the event of default and the consequent transfer of ownership imposes a deadweight loss equal to a fraction α, $0 < \alpha \leq 1$, of the value of the firm. Therefore, the value of the firm to the new owners, i.e., the former lenders, is $(1-\alpha)\widehat{V}(\phi(t))$. Thus, the net loss to the lenders in the event of default is $D - (1-\alpha)\widehat{V}(\phi(t)) > 0$ for any $\phi(t)$ such that $\widehat{V}(\phi(t)) < D$. The default premium $Q(t)$ is the probability-weighted sum of these losses over the values of $\phi(t)$ that lead to default, and can be written as a function of D as

$$
Q(t) = \widetilde{Q}(D) \equiv \lambda\int_{\widehat{V}(\phi)<D}\left[D - (1-\alpha)\widehat{V}(\phi)\right]dF(\phi),
\tag{8}
$$

where λ is the instantaneous probability that a new value of ϕ is drawn and the integration is performed over new values of ϕ that lead the firm to default. Note that in the case in which $\alpha = 1$, so that all firm value is destroyed and lenders do not recover anything in the case of default, $Q(t)$ is simply $P(t)D$, where $P(t) = \widetilde{P}(D) \equiv \lambda \int_{\widehat{V}(\phi)<D} dF(\phi)$ is the instantaneous probability of default.

Substitute the expression for $Q(t)$ from equation (8) into equation (7) and rearrange to obtain

$$
\begin{aligned}
V(\phi(t), D) & = \frac{\tau}{\rho + \lambda} \left[\rho + \lambda \int_{\widehat{V}(\phi)<D} dF(\phi) \right] D + \frac{1-\tau}{\rho + \lambda} \phi(t) \\
& \quad + \frac{\lambda}{\rho + \lambda} \left[(1-\tau)(1-\alpha) \int_{\widehat{V}(\phi)<D} \widehat{V}(\phi) dF(\phi) + \int_{\widehat{V}(\phi)\geq D} \widehat{V}(\phi) dF(\phi) \right].
\end{aligned}
\tag{9}
$$

The expression for $V(\phi(t), D)$ in equation (9) is the sum of four terms:
(1) $\frac{\tau}{\rho+\lambda} \left[\rho + \lambda \int_{\widehat{V}(\phi)<D} dF(\phi) \right] D = \frac{\tau}{\rho+\lambda} \left[\rho + \widetilde{P}(D) \right] D$ is the expected present value of the tax shield until the next arrival of a new value of ϕ, if $\alpha = 1$ so that the interest rate is $\rho + \widetilde{P}(D)$; (2) $\frac{1-\tau}{\rho+\lambda} \phi(t)$ is the expected present value of EBIT, less taxes on EBIT, until the next arrival of a new value of ϕ; (3) $\frac{\lambda}{\rho+\lambda}(1-\tau)(1-\alpha) \int_{\widehat{V}(\phi)<D} \widehat{V}(\phi) dF(\phi)$ captures the fact that lenders recover a fraction $(1-\alpha)$ of the value of the firm at the time of default, which reduces (relative to the case with $\alpha = 1$) the total interest payments at each point of time by $(1-\alpha) \int_{\widehat{V}(\phi)<D} \widehat{V}(\phi) dF(\phi)$. The expected present value of the reduction in after-tax interest payments until the next arrival of a new value of ϕ is $\frac{\lambda}{\rho+\lambda}(1-\tau)(1-\alpha) \int_{\widehat{V}(\phi)<D} \widehat{V}(\phi) dF(\phi)$; and (4) $\frac{\lambda}{\rho+\lambda} \int_{\widehat{V}(\phi)\geq D} \widehat{V}(\phi) dF(\phi)$ is the expected present value of the continuation value of the firm at the time of the next arrival of a new value of ϕ. Inspection of equation (9) reveals that $V(\phi(t), D)$ is additively separable in D and $\phi(t)$. This additive separability has the important implication that the value of D that maximizes $V(\phi(t), D)$ is independent of $\phi(t)$, provided that the borrowing constraint does not bind.[10]

The firm will choose the amount of debt, D, to maximize $V(\phi(t), D)$. However, in the absence of any asymmetric or private information, lenders recognize that the firm would default immediately if it were able to borrow an amount $D > \widehat{V}(\phi(t))$. Therefore, the

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[10]Formally, this statement is $\arg\max_D V(\phi(t), D)$ is independent of $\phi(t)$, which ignores the constraint $D \leq \widehat{V}(\phi(t))$.

firm's debt must satisfy $D \leq \widehat{V}(\phi(t))$. Taking account of this constraint on the amount that the firm can borrow, the optimal amount of debt for shareholders to issue at time t is

$$\widehat{D}(\phi(t)) \equiv \arg \max_{D \leq \widehat{V}(\phi(t))} V(\phi(t), D). \tag{10}$$

Therefore, the optimal value of $V(\phi(t), D)$ is $\widehat{V}(\phi(t)) = V\left(\phi(t), \widehat{D}(\phi(t))\right)$.

Proposition 3 $\widehat{V}(\phi(t))$ *is strictly increasing in* $\phi(t)$. *Moreover, for any* $\phi_2 > \phi_1$ *in the support* $[\Phi_L, \Phi_H]$, $\frac{\widehat{V}(\phi_2) - \widehat{V}(\phi_1)}{\phi_2 - \phi_1} \geq \frac{1-\tau}{\rho + \lambda} > 0$.

The proof of Proposition 3 is in Appendix A, but the logic of the proof is straightforward. Since $V(\phi(t), D)$ is strictly increasing in $\phi(t)$, the firm would be able to maintain the same level of debt at ϕ_2 as at ϕ_1, without running up against the borrowing constraint. Therefore, if EBIT were to increase to ϕ_2 from ϕ_1, the value of $V(\phi(t), D)$ would increase by $\frac{1-\tau}{\rho+\lambda}(\phi_2 - \phi_1)$, with unchanged D. Allowing for the possibility that optimal debt changes when EBIT increases to ϕ_2 from ϕ_1 implies that $\widehat{V}(\phi(t))$ increases by at least $\frac{1-\tau}{\rho+\lambda}(\phi_2 - \phi_1)$.

Proposition 3 states that $\widehat{V}(\phi(t))$ is strictly increasing, which implies that $\widehat{V}(\phi(t))$ is invertible. The inverse $\widehat{V}^{-1}(x)$ is defined on the domain $\left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H)\right]$ and is strictly increasing on this domain. If the firm has an amount D of debt outstanding at time t, it will default at time t if and only if $\phi(t) < \widehat{V}^{-1}(D)$.[11] Hence, for $\widehat{V}(\Phi_L) \leq D \leq \widehat{V}(\Phi_H)$, equation (9) can be rewritten as

$$
\begin{aligned}
V(\phi(t), D) &= \frac{\tau}{\rho + \lambda}\left(\rho + \lambda F\left(\widehat{V}^{-1}(D)\right)\right) D + \frac{1-\tau}{\rho + \lambda}\phi(t) \\
&\quad + \frac{\lambda}{\rho + \lambda}\left[\begin{array}{c} (1-\tau)(1-\alpha)\int_{\Phi_L}^{\widehat{V}^{-1}(D)} \widehat{V}(\phi) f(\phi) d\phi \\ + \int_{\widehat{V}^{-1}(D)}^{\Phi_H} \widehat{V}(\phi) f(\phi) d\phi \end{array}\right].
\end{aligned} \tag{11}
$$

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[11] Formally, the set of ϕ that lead to default is $\left\{\phi : \widehat{V}(\phi) < D\right\} = \left\{\phi : \phi < \widehat{V}^{-1}(D)\right\}$ and the set of ϕ that does not lead to default is $\left\{\phi : \widehat{V}(\phi) \geq D\right\} = \left\{\phi : \phi \geq \widehat{V}^{-1}(D)\right\}$.

Differentiate equation (11) with respect to D for $D \in \left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H) \right]$ to obtain[12]

$$\frac{\partial V(\phi(t), D)}{\partial D} = \frac{\tau}{\rho + \lambda} \left(\rho + \lambda F\left(\widehat{V}^{-1}(D) \right) \right) \tag{12}$$
$$- (1 - \tau)\alpha \frac{\lambda}{\rho + \lambda} Df\left(\widehat{V}^{-1}(D) \right) \widehat{V}^{-1\prime}(D).$$

The two terms on the right-hand side of equation (12) illustrate the basic tension that underlies the tradeoff theory of debt. As I will show in greater detail in subsection 5.2, the first term, which is positive, reflects the value of the tax shield provided by the tax-deductibility of interest payments. The tax shield pushes shareholders to choose higher levels of debt. The second term, which is negative, reflects the possibility that shareholders will find it optimal to default on their debt, thereby giving up the continuation value of the firm. This possibility pushes shareholders to choose lower levels of debt.

The expression for $\frac{\partial V(\phi(t), D)}{\partial D}$ in equation (12) contains $\widehat{V}^{-1\prime}(D)$, which equals $\frac{1}{\widehat{V}'(\widehat{V}^{-1}(D))} > 0$. Proposition 3 implies that $\widehat{V}'(\phi) \geq \frac{1-\tau}{\rho+\lambda} > 0$, so $\widehat{V}^{-1\prime}(D) \leq \frac{\rho+\lambda}{1-\tau}$. Therefore, equation (12) implies that

$$\frac{\partial V(\phi(t), D)}{\partial D} \geq \frac{\tau}{\rho + \lambda} \left(\rho + \lambda F\left(\widehat{V}^{-1}(D) \right) \right) - \alpha\lambda Df\left(\widehat{V}^{-1}(D) \right), \tag{13}$$

with equality if $\widehat{V}'(\phi) = \frac{1-\tau}{\rho+\lambda}$.

Since shareholders owe a liability of D, the value of shareholders' equity in the firm, $S(\phi(t))$, is

$$S(\phi(t)) = \widehat{V}(\phi(t)) - \widehat{D}(\phi(t)) \geq 0, \tag{14}$$

where the inequality follows from the restriction $D \leq \widehat{V}(\phi(t))$. Because shareholders receive an amount D immediately when they borrow, they choose D to maximize $V(\phi(t), D)$ $= S(\phi(t)) + D$ rather than to maximize shareholder equity exclusive of D, $S(\phi(t)) = V(\phi(t), D) - D$.

Recall that I have restricted attention to situations in which, in the complete absence of debt, shareholders would never choose to shut down the firm. That is, I assume that

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[12]For $D < \widehat{V}(\Phi_L)$, equation (9) can be rewritten as $V(\phi(t), D) = \tau \frac{\rho}{\rho+\lambda} D + \frac{1-\tau}{\rho+\lambda}\phi(t) + \frac{\lambda}{\rho+\lambda} E\left\{ \widehat{V}(\phi) \right\}$ so $\frac{\partial V(\phi(t), D)}{\partial D} = \tau \frac{\rho}{\rho+\lambda}$.

$\Phi_L > -\frac{\lambda}{\rho} E\{\phi\}$, which implies (Corollary 2) that $W(\Phi_L) > 0$. Shareholders can issue any amount of debt less than or equal to $W(\Phi_L)$ without exposing themselving to the possibility of default. Therefore, provided that $\tau > 0$, the optimal amount of debt for the shareholders to issue is at least $W(\Phi_L)$. This result is formalized in Proposition 4.

Proposition 4 *If* $0 < \tau < 1$, *then* $\widehat{D}(\phi(t)) \geq \widehat{V}(\Phi_L) \geq \frac{1}{1-\tau} W(\Phi_L) = \frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho} E\{\phi\}\right] >$ 0.

Corollary 5 *If* $0 < \tau < 1$, *then* $\widehat{D}(\Phi_L) = \widehat{V}(\Phi_L)$, *and* $S(\Phi_L) = 0$.

Corollary 5 follows directly from Proposition 4, which implies that $\widehat{D}(\Phi_L) \geq \widehat{V}(\Phi_L)$, and from the constraint $\widehat{D}(\Phi_L) \leq \widehat{V}(\Phi_L)$. Therefore, $\widehat{D}(\Phi_L) = \widehat{V}(\Phi_L)$. Hence, when EBIT takes on its minimum value Φ_L, the optimal amount of debt equals $\widehat{V}(\Phi_L)$, which implies that shareholders' equity, $S(\Phi_L) = \widehat{V}(\Phi_L) - \widehat{D}(\Phi_L)$, is zero.

Proposition 6 *If (1)* $0 < \tau < \frac{\alpha}{1+\alpha}$ *and (2)* $f(\phi) > 0$ *is non-decreasing for all* $\phi \in [\Phi_L, \Phi_H]$, *then* $\widehat{V}(\phi(t))$ *is concave on the domain* $[\Phi_L, \Phi_H]$.

Proposition 6 is proved in Appendix A by showing that $\widehat{V}(\phi(t))$ is the fixed point of a contraction mapping that takes concave functions into concave functions.

Proposition 6 provides sufficient conditions for the value function, $\widehat{V}(\phi(t))$, to be concave. The concavity of the value function is to be contrasted with Proposition II in Kraus and Litzenberger (1973), hereafter KL, which provides sufficient conditions for the value function to be convex over at least part of the domain of $\phi(t)$. In contrast to the model I present here, the model in KL is a single-period model with a finite number of states and with an exogenous cost of being insolvent in each state. Interpretating the cost of being insolvent in the model presented here as the deadweight cost of default $\alpha\widehat{V}(\phi(t))$, the conditions for non-concavity in KL imply essentially that $\alpha\widehat{V}(\phi(t)) f(\phi(t))$ is decreasing in $\phi(t)$. Since $\widehat{V}(\phi(t))$ is strictly increasing in $\phi(t)$, the KL conditions require $f(\phi(t)) > 0$ to be decreasing over some interval of values of $\phi(t)$, a condition that is ruled out by assumption in Proposition 6. Hence there is no conflict between Proposition 6 above and Proposition II in KL.[13]

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[13]The model in KL has a finite number, n, of states indexed by j, where X_j is EBIT in state j, P_j is the price of a claim on a dollar in state j, C_j is the (exogenous) cost of being insolvent in state j, and T_j is the tax rate in state j. The states are ordered so that X_j is non-decreasing in j. For simplicity,

One of the steps in the proof of Proposition 6 involves proving that if $\widehat{V}\left(\phi\left(t\right)\right)$ is concave, then $V\left(\phi\left(t\right),D\right)$ is strictly concave in D for $\widehat{V}\left(\Phi_L\right) \leq D \leq \widehat{V}\left(\Phi_H\right)$, which is Corollary 7 below.

Corollary 7 *to Proposition 6.   If (1) $0 < \tau < \frac{\alpha}{1+\alpha}$ and (2) $f\left(\phi\right) > 0$ is non-decreasing for all $\phi \in \left[\Phi_L, \Phi_H\right]$, then $V\left(\phi\left(t\right),D\right)$ is strictly concave in D for $\widehat{V}\left(\Phi_L\right) \leq D \leq \widehat{V}\left(\Phi_H\right)$.*

Corollary 7 provides sufficient conditions for $V\left(\phi\left(t\right),D\right)$ to be strictly concave in D. It is straightforward to show that a necessary condition for this strict concavity is that default imposes a deadweight cost, that is, $\alpha > 0$.[14]

3   Optimal Debt, Firm Value, and Shareholder Equity

In this section, I characterize the optimal value of debt and the resulting values of the firm and shareholder equity. The first step is to recognize the lower and upper bounds on the optimal level of debt. The lower bound on optimal debt is provided by Proposition 4, which states that the optimal amount of debt, $\widehat{D}\left(\phi\left(t\right)\right)$, is greater than or equal to $\widehat{V}\left(\Phi_L\right)$. The upper bound on optimal debt is based on the constraint that prevents the firm from borrowing an amount that would lead to immediate default. This constraint is $\widehat{D}\left(\phi\left(t\right)\right) \leq \widehat{V}\left(\phi\left(t\right)\right) \leq \widehat{V}\left(\Phi_H\right)$, where the second inequality follows from the fact that $\widehat{V}\left(\phi\left(t\right)\right)$ is increasing in $\phi\left(t\right)$. Therefore, $\widehat{D}\left(\phi\left(t\right)\right) \in \left[\widehat{V}\left(\Phi_L\right), \widehat{V}\left(\Phi_H\right)\right]$.

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consider the case in which X_j is strictly increasing in j. Proposition II in KL states that if (a) $C_{k-1}P_{k-1} > (X_k - X_{k-1})\sum_{j=k}^{n} T_j P_j$ and (b) $(X_{k+1} - X_k)\sum_{j=k+1}^{n} T_j P_j > C_k P_k$, then $V\left(X_{k-1}\right) > V\left(X_k\right) < V\left(X_{k+1}\right)$, where $V\left(\right)$ is the market value of the firm. Since $T_k P_k \geq 0$, conditions (a) and (b) imply $C_{k-1}\frac{P_{k-1}}{X_k - X_{k-1}} > \sum_{j=k}^{n} T_j P_j \geq \sum_{j=k+1}^{n} T_j P_j > C_k \frac{P_k}{X_{k+1} - X_k}$. Under risk neutrality, as in the current paper, P_j is simply the probability of state j and $\frac{P_{j-1}}{X_j - X_{j-1}}$ can be interpretated as the probability density of state j, denoted here as f_j. Therefore, assumptions (a) and (b) imply that $C_j f_j$ is decreasing in j, equivalently decreasing in X_j. If the cost of being insolvent in the current model is $\alpha\widehat{V}\left(\phi\left(t\right)\right)$, conditions (a) and (b) imply that $\alpha\widehat{V}\left(\phi\left(t\right)\right)f\left(\phi\left(t\right)\right)$ is decreasing in $\phi\left(t\right)$, which is inconsistent with $\widehat{V}\left(\phi\left(t\right)\right)$ being strictly increasing (Proposition 3) and the assumption that $f\left(\phi\left(t\right)\right)$ is non-decreasing.

[14] To see the necessity of this condition, set $\alpha = 0$ in equation (12) and differentiate the resulting equation with respect to D to obtain $\frac{\partial^2 V(\phi(t),D)}{\partial D \partial D} = \frac{\tau}{\rho+\lambda}\lambda f\left(\widehat{V}^{-1}\left(D\right)\right)\widehat{V}^{-1\prime}\left(D\right) > 0$ because $f\left(\widehat{V}^{-1}\left(D\right)\right) > 0$ and $\widehat{V}^{-1\prime}\left(D\right) > 0$. Therefore, if $\alpha = 0$, then $V\left(\phi\left(t\right),D\right)$ is strictly convex in D. Thus, $\alpha > 0$ is a necessary condition for strict concavity of $V\left(\phi\left(t\right),D\right)$ in D.

Define D^* as the value of debt that maximizes $V(\phi(t), D)$ over the interval $\left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H)\right]$, ignoring the borrowing constraint $D \leq \widehat{V}(\phi(t))$. Formally,

$$D^* \equiv \arg \max_{\widehat{V}(\Phi_L) \leq D \leq \widehat{V}(\Phi_H)} V(\phi(t), D). \tag{15}$$

The strict concavity of $V(\phi(t), D)$ in D implies that D^* is unique. The value of D^* is invariant to $\phi(t)$ because, as noted earlier, $V(\phi(t), D)$ is additively separable in $\phi(t)$ and D. If $D^* \leq V(\phi(t), D^*)$, the constraint $D \leq \widehat{V}(\phi(t))$ does not bind and the firm will optimally issue D^* of debt. However, if $D^* > \widehat{V}(\phi(t))$, the firm will not be able to issue as much as D^* of debt. I will prove that there is a unique critical value of $\phi(t)$, which I denote ϕ^*, that is such that $\widehat{D}(\phi(t)) = D^*$ for $\phi(t) \geq \phi^*$ and $\widehat{D}(\phi(t)) < D^*$ for $\phi(t) < \phi^*$, where $\widehat{D}(\phi(t))$ is the optimal level of debt defined in equation (10).

The following lemma helps prove the existence and uniqueness of the critical value ϕ^*.

Lemma 8 $D^* \equiv \arg\max_{\widehat{V}(\Phi_L) \leq D \leq \widehat{V}(\Phi_H)} V(\phi(t), D)$ *satisfies* $V(\Phi_L, D^*) \leq D^* \leq V(\Phi_H, D^*)$.

Define ϕ^* implicitly by the following equation

$$V(\phi^*, D^*) = D^*. \tag{16}$$

Lemma 8 helps prove the following Proposition.

Proposition 9 *There exists a unique value* $\phi^* \in [\Phi_L, \Phi_H]$ *such that* $V(\phi^*, D^*) = D^*$.

Since $V(\phi(t), D^*)$ is strictly increasing in $\phi(t)$, ϕ^* is the lowest value of $\phi(t)$ for which the firm will be able to borrow as much as D^* at time t. For $\phi(t) \geq \phi^*$, the borrowing constraint is not strictly binding, while for $\phi(t) < \phi^*$, the borrowing constraint is strictly binding. The following corollary presents the relationship between D^* and ϕ^*, which will be useful in later derivations.

Corollary 10 $\widehat{V}(\phi^*) = D^*$ *and* $\widehat{V}^{-1}(D^*) = \phi^*$.

The following proposition characterizes optimal debt, $\widehat{D}(\phi(t))$, shareholder equity, $S(\phi(t))$, and total firm valuation, $\widehat{V}(\phi(t))$, first for $\phi(t) \geq \phi^*$ and then for $\phi(t) \leq \phi^*$.

Proposition 11 *Define ϕ^* so that $V(\phi^*, D^*) = D^*$. Then*

- *(1) for any $\phi(t) \geq \phi^*$,*

  (a) $\widehat{D}(\phi(t)) = D^*$,

  (b) $S(\phi(t)) = \frac{1-\tau}{\rho+\lambda}[\phi(t) - \phi^*]$, and

  (c) $\widehat{V}(\phi(t)) = \frac{1-\tau}{\rho+\lambda}[\phi(t) - \phi^*] + D^*$;

- *(2) for any $\phi(t) \leq \phi^*$,*

  (a) $\widehat{D}(\phi(t)) \equiv \widehat{V}(\phi(t))$ is concave and strictly increasing in $\phi(t)$, and

  (b) $S(\phi(t)) = 0$.

For high values of $\phi(t)$, specifically for $\phi(t) \geq \phi^*$, the borrowing constraint $D \leq \widehat{V}(\phi(t))$ does not bind. Therefore, the optimal value of debt is D^* for all $\phi(t) \geq \phi^*$. Shareholder equity is simply $S(\phi(t)) = \frac{1-\tau}{\rho+\lambda}[\phi(t) - \phi^*]$. Since $\widehat{D}(\phi(t))$ is invariant to $\phi(t)$ for these values of $\phi(t)$, the total value of the firm, $\widehat{V}(\phi(t))$, is a linear function of $\phi(t)$ with slope $\frac{1-\tau}{\rho+\lambda}$. For low values of $\phi(t)$, specifically for $\phi(t) \leq \phi^*$, the borrowing constraint $D \leq \widehat{V}(\phi(t))$ is binding: shareholders borrow as much as they can, so that $\widehat{D}(\phi(t)) \equiv \widehat{V}(\phi(t))$ and shareholder equity is driven to zero.

The following corollary states that the optimal value of debt never falls below previous optimal values of debt as long as the firm does not default. This result is consistent with the time path of debt in Goldstein, Ju, and Leland (2001), in which it is *assumed* that, outside of default, the firm does not decrease its debt.[15]

Corollary 12 *If the firm does not default at any $t \in [t_1, t_2]$, then $\widehat{D}(\phi(t_2)) \geq \widehat{D}(\phi(t_1))$.*

The following corollary, which follows immediately from Proposition 11, describes the optimal leverage ratio.

Corollary 13 *Define the optimal leverage ratio as $L(\phi(t)) \equiv \frac{\widehat{D}(\phi(t))}{\widehat{V}(\phi(t))}$. If $\phi(t) \leq \phi^*$, then $L(\phi(t)) \equiv 1$; if $\phi(t) \geq \phi^*$, then $L(\phi(t)) = \frac{1}{\frac{1-\tau}{\rho+\lambda}\frac{\phi(t)-\phi^*}{D^*}+1}$, which is strictly decreasing in $\phi(t)$.*

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[15] The second paragraph of Goldstein, Ju, and Leland (2001) begins "Below, we consider only the option to increase future debt levels. While in theory management can both increase and decrease future debt levels, Gilson (1997) finds that transactions costs discourage debt reductions outside of Chapter 11." (p. 483)

The tradeoff theory is operative, that is, optimal debt is determined by equating the marginal tax shield associated with interest deductibility and the marginal default cost, only when $\phi(t) \geq \phi^*$. Therefore, Corollary 13 implies that when the tradeoff theory is operative, the optimal leverage ratio is a decreasing function of contemporaneous profitability.

Corollary 12 states that if the firm does not default at any $t \in [t_1, t_2]$, then $\widehat{D}(\phi(t))$ is a (weakly) monotonic function of time for $t \in [t_1, t_2]$. However the optimal leverage ratio, $L(\phi(t))$, described in Corollary 13, is not a monotonic function of time for $t \in [t_1, t_2]$. For instance, suppose that for dates $t_L < t_M < t_H$ in $[t_1, t_2]$, $\phi^* < \phi(t_L) < \phi(t_M) > \phi(t_H) > \phi^*$. Then $\widehat{D}(\phi(t_L)) = \widehat{D}(\phi(t_M)) = \widehat{D}(\phi(t_H)) = D^*$ and $\widehat{V}(\phi(t_L)) < \widehat{V}(\phi(t_M)) > \widehat{V}(\phi(t_H))$, so $L(\phi(t_L)) > L(\phi(t_M)) < L(\phi(t_H))$. In this case, the optimal leverage ratio first decreases over time and then increases over time.

For a firm that follows the optimal debt policy $\widehat{D}(\phi(t))$, the instantaneous probability of default when EBIT equals $\phi(t)$ is

$$P(\phi(t)) \equiv \lambda \int_{\widehat{V}(\phi) < \widehat{D}(\phi(t))} dF(\phi). \tag{17}$$

Proposition 14 $P(\phi(t)) = \lambda \min[F(\phi(t)), F(\phi^*)]$. *Both* $P(\phi(t))$ *and* $\widetilde{Q}\left(\widehat{D}(\phi(t))\right) \equiv \lambda \int_{\widehat{V}(\phi) < \widehat{D}(\phi(t))} \left[\widehat{D}(\phi(t)) - (1-\alpha)\widehat{V}(\phi)\right] dF(\phi)$ *are strictly increasing in* $\phi(t)$ *for* $\phi(t) < \phi^*$ *and invariant to* $\phi(t)$ *for* $\phi(t) \geq \phi^*$.

Proposition 14 implies that within the set of low values of EBIT, specifically for $\phi(t) < \phi^*$, higher values of EBIT are associated with *higher* probabilities of default and *higher* default premia. The borrowing constraint is binding for low values of EBIT, that is, $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$, for $\phi(t) < \phi^*$. Therefore, within the set of low values of EBIT, the firm borrows greater amounts for higher values of $\phi(t)$, thereby increasing the probability that the next arrival of a new value of EBIT will lead the firm to choose to default.

The following proposition states that when the borrowing constraint is binding, the firm's net cash flow is negative.

Proposition 15 *If* $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$, *then after-tax cash flow* $\widehat{C}(\phi(t)) \equiv (1-\tau)\left[\phi(t) - \left(\rho\widehat{D}(\phi(t)) + Q(t)\right)\right] \leq 0$, *with strict inequality if* $\phi(t) < \Phi_H$.

Proposition 11 implies that when $\phi(t) < \phi^*$, the borrowing constraint is binding and optimal debt is strictly increasing in $\phi(t)$ so that $\rho\widehat{D}(\phi(t))$ is strictly increasing. Proposition

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14 states that when $\phi(t) < \phi^*$, the default premium $\widetilde{Q}\left(\widehat{D}(\phi(t))\right)$ is strictly increasing in $\phi(t)$. Therefore, interest payments $\rho\widehat{D}(\phi(t)) + \widetilde{Q}\left(\widehat{D}(\phi(t))\right)$ are strictly increasing in $\phi(t)$ when $\phi(t) < \phi^*$. Nevertheless, as indicated in Corollary 16 below, optimal after-tax cash flow, $\widehat{C}(\phi(t))$, is strictly increasing in $\phi(t)$ when $\phi(t) < \phi^*$. Optimal after-tax cash flow, $\widehat{C}(\phi(t))$, is also strictly increasing in $\phi(t)$ for $\phi(t) \geq \phi^*$ because optimal debt and interest payments are invariant to $\phi(t)$ for $\phi(t) \geq \phi^*$. Therefore, optimal after-tax cash flow, $\widehat{C}(\phi(t))$, is strictly increasing in $\phi(t)$, regardless of whether the borrowing constraint is binding.

Corollary 16 $\widehat{C}(\phi(t))$ *is strictly increasing in* $\phi(t)$.

4 Three Regimes

In this section, I define and analyze three regimes that are defined by whether $D^* \geq \widehat{V}(\Phi_L)$ strictly binds, $D^* \leq \widehat{V}(\Phi_H)$ strictly binds, or neither bound strictly binds. Formally, Regime I is the set of configurations of λ, ρ, α, τ, and $F(\phi)$ for which the lower bound on D^* strictly binds, Regime II is the set of configurations of λ, ρ, α, τ, and $F(\phi)$ for which neither bound strictly binds, and Regime III is the set of configurations of λ, ρ, α, τ, and $F(\phi)$ for which the upper bound on D^* strictly binds. The term "regime" does not imply that the configuration is temporary or can potentially change; the configuration of λ, ρ, α, τ, and $F(\phi)$ remains fixed forever.

To characterize the regimes, use the definition of D^* in equation (15), which states that D^* is the value of D that maximizes $V(\phi(t), D)$ subject to $\widehat{V}(\Phi_L) \leq D \leq \widehat{V}(\Phi_H)$. Therefore,

$$D^* = \arg\max_D V(\phi, D) + \omega_1\left(D - \widehat{V}(\Phi_L)\right) + \omega_2\left(\widehat{V}(\Phi_H) - D\right), \tag{18}$$

where $\omega_1 \geq 0$ is the multiplier on the constraint $\widehat{V}(\Phi_L) \leq D^*$ and $\omega_2 \geq 0$ is the multiplier on the constraint $D^* \leq \widehat{V}(\Phi_H)$. The first-order condition associated with the maximization in equation (18) is

$$\frac{\partial V(\phi, D^*)}{\partial D} = \omega_2 - \omega_1 \tag{19}$$

and the complementary slackness conditions are

$$\omega_1 \left(D^* - \widehat{V}\left(\Phi_L\right) \right) = 0 \tag{20}$$

and

$$\omega_2 \left(\widehat{V}\left(\Phi_H\right) - D^* \right) = 0. \tag{21}$$

Three distinct regimes are defined according to whether the multipliers ω_1 and ω_2 are positive or zero:[16]

- Regime I: $\omega_1 > 0$ and $\omega_2 = 0$

- Regime II: $\omega_1 = \omega_2 = 0$

- Regime III: $\omega_1 = 0$ and $\omega_2 > 0$

The following Lemma will help evaluate the partial derivative $\frac{\partial V(\phi, D)}{\partial D}$ in Regimes I and II.

Lemma 17 *If $\omega_2 = 0$, then $\widehat{V}'\left(\phi^*\right) = \frac{1-\tau}{\rho+\lambda}$ and $\widehat{V}^{-1\prime}\left(D^*\right) = \frac{\rho+\lambda}{1-\tau}$.*

The critical value ϕ^* and the value of D^* are straightforward to characterize in Regimes I and III, as in the following Proposition.

Proposition 18 *In Regime I, which prevails if and only if $\omega_1 > 0$, $D^* = \widehat{V}\left(\Phi_L\right)$ and $\phi^* = \Phi_L$. In Regime III, which prevails if and only if $\omega_2 > 0$, $D^* = \widehat{V}\left(\Phi_H\right)$ and $\phi^* = \Phi_H$.*

In Regime II, the critical value ϕ^* could potentially be any value of ϕ in the support $[\Phi_L, \Phi_H]$. I will analyze Regime II in more detail in Section 5. The remainder of this section will focus on Regimes I and III.

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[16] Since $\widehat{V}\left(\Phi_L\right) \neq \widehat{V}\left(\Phi_H\right)$, ω_1 and ω_2 cannot both be positive.

4.1 Regime I

In Regime I, $\omega_1 > 0$ and $\omega_2 = 0$, so the first-order condition in equation (19) implies $\frac{\partial V(\phi, D^*)}{\partial D} = -\omega_1 < 0$. Thus, the marginal tax shield, which provides an incentive to increase the amount of debt, is overwhelmed by the probability of losing the continuation value of the firm as a result of default. Therefore, the optimal value of debt is $\widehat{V}(\Phi_L)$, which is the highest value of debt that the firm can issue without facing a positive probability of default.

Proposition 19 *Define* $D_0 \equiv \frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$. *In Regime I (where $\omega_1 > 0$ and $\omega_2 = 0$), for all $\phi(t) \in [\Phi_L, \Phi_H]$,*

1. $\widehat{D}(\phi(t)) = D^* = D_0,$

2. $S(\phi(t)) = \frac{1-\tau}{\rho+\lambda}(\phi(t) - \Phi_L),$ *and*

3. $\widehat{V}(\phi(t)) = D_0 + \frac{1-\tau}{\rho+\lambda}(\phi(t) - \Phi_L).$

Proposition 19 states that in Regime I, the optimal value of debt equals D_0, which is invariant to the tax rate and invariant to the current value of EBIT. The invariance of optimal debt with respect to the tax rate reflects the fact that *at the margin* the tax shield is so weak that it is completely outweighed by the cost of default. Therefore, shareholders choose not to expose the firm to the probability of default. Nevertheless, because the tax rate τ is positive, the firm takes advantage of the tax shield on interest by choosing the maximal amount of debt that does not risk default. In fact, the firm uses the tax shield to completely insulate its value from taxes when $\phi(t) = \Phi_L$. That is, when $\phi(t) = \Phi_L$, the ability to borrow and deduct interest payments increases the total value of the firm to $\widehat{V}(\Phi_L) = D_0 = \frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$ from $\frac{1-\tau}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$ in the absence of debt. As is evident from Lemma 1, $\widehat{V}(\Phi_L) = D_0 = \frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$ is the value of the firm when $\phi(t) = \Phi_L$ if $\tau = 0$ and the firm never borrows nor ceases operation.

4.2 Regime III

In Regime III, $\omega_1 = 0$ and $\omega_2 > 0$, so equation (19) implies that $\frac{\partial V\left(\phi, \widehat{V}(\Phi_H)\right)}{\partial D} > 0$. Since $V(\phi, D)$ is strictly concave in D, $\frac{\partial V(\phi, D)}{\partial D} > 0$ for all $D \in \left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H)\right]$. The marginal tax shield associated with an increase in debt overwhelms the additional default probability

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associated with an increase in debt. Therefore, the firm will issue as much debt as it can, driving shareholders' equity to zero, so that $\widehat{D}(\phi(t)) \equiv \widehat{V}(\phi(t))$ for all $\phi(t) \in [\Phi_L, \Phi_H]$.

5   Regime II

In Regime II, D^* can be anywhere in $\left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H)\right]$ and the critical value ϕ^* can be anywhere in $[\Phi_L, \Phi_H]$.   In subsection 5.1, I derive the values of ϕ^* and D^* and thereby complete the calculation of the optimal values of debt, equity, and total firm value for $\phi(t) \geq \phi^*$, as shown in Proposition 11. In subsection 5.2, I interpret the expression for D^* in terms of the tradeoff theory, and in subsection 5.3, I calculate $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$ for $\phi(t) \leq \phi^*$.

5.1 ϕ^* and D^* in Regime II

In Regime II, $\omega_1 = \omega_1 = 0$, so equation (19) implies that $\frac{\partial V(\phi, D^*)}{\partial D} = 0$. Lemma 17 implies that

$$\widehat{V}^{-1\prime}(D^*) = \frac{\rho + \lambda}{1 - \tau}, \quad \text{in Regimes I and II.} \tag{22}$$

Now evaluate $\frac{\partial V(\phi(t), D)}{\partial D}$ in equation (12) at $D = D^*$, using Corollary 10 and equation (22), and set $\frac{\partial V(\phi(t), D^*)}{\partial D}$ equal to zero to obtain

$$\frac{\partial V(\phi(t), D^*)}{\partial D} = \frac{\tau}{\rho + \lambda}(\rho + \lambda F(\phi^*)) - \alpha \lambda D^* f(\phi^*) = 0. \tag{23}$$

The first-order condition in equation (23) involves both D^* and ϕ^*. A second expression involving D^* and ϕ^* follows from evaluating equation (11) at $\phi(t) = \phi^*$ and $D = D^* =$

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$V(\phi^*, D^*)$ to obtain[17]

$$D^* = \frac{1}{\rho + \lambda}\phi^* \tag{26}$$

$$+\frac{1}{\rho + \lambda F(\phi^*)}\left[(1-\alpha)\lambda\int_{\Phi_L}^{\phi^*}\widehat{V}(\phi)f(\phi)\,d\phi + \frac{\lambda}{\rho+\lambda}\int_{\phi^*}^{\Phi_H}\phi f(\phi)\,d\phi\right].$$

Use equation (26) to eliminate D^* from equation (23) to obtain

$$\frac{\partial V(\phi^*, D^*)}{\partial D} = \frac{1}{\rho+\lambda}\frac{1}{\rho+\lambda F(\phi^*)}H(\phi^*) = 0, \tag{27}$$

where

$$H(x) \equiv \tau\left[\rho+\lambda F(x)\right]^2 - \alpha\lambda f(x)\left(\begin{array}{c}\left[\rho+\lambda F(x)\right]x \\ +(1-\alpha)\lambda(\rho+\lambda)\int_{\Phi_L}^{x}\widehat{V}(\phi)f(\phi)\,d\phi \\ +\lambda\int_{x}^{\Phi_H}\phi f(\phi)\,d\phi\end{array}\right). \tag{28}$$

The first-order condition for the optimal value of ϕ^*, equivalently for the optimal value of debt, $D^* = \widehat{D}(\phi^*) = \widehat{V}(\phi^*)$, is $H(\phi^*) = 0$. Since $V(\phi^*, D)$ is strictly concave in D (Corollary 7), the first-order condition $H(\phi^*) = 0$ has at most one root.

Depending on the parameter values and the distribution function $F(\phi)$, the equation $H(\phi) = 0$ may or may not have a root in $[\Phi_L, \Phi_H]$. Evaluate $H(x)$ at $x = \Phi_L$ and $x = \Phi_H$,

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[17]Evaluate equation (11) at $\phi(t) = \phi^*$ and $D = D^*$ using Corollary 10 to set $\widehat{V}^{-1}(D^*) = \phi^*$ and equation (16) to set $V(\phi^*, D^*) = D^*$ to obtain

$$D^* = \tau\frac{\rho + \lambda F(\phi^*)}{\rho+\lambda}D^* + \frac{1-\tau}{\rho+\lambda}\phi^* \tag{24}$$

$$+\frac{\lambda}{\rho+\lambda}(1-\tau)(1-\alpha)\int_{\Phi_L}^{\phi^*}\widehat{V}(\phi)f(\phi)\,d\phi + \frac{\lambda}{\rho+\lambda}\int_{\phi^*}^{\Phi_H}\widehat{V}(\phi)f(\phi)\,d\phi.$$

Use Proposition 11 to substitute $\frac{1-\tau}{\rho+\lambda}[\phi-\phi^*]+D^*$ for $\widehat{V}(\phi)$ where $\phi \geq \phi^*$ in equation (24) to obtain

$$[\rho+\lambda F(\phi^*)]D^* = \frac{\rho + \lambda F(\phi^*)}{\rho+\lambda}\phi^* + (1-\alpha)\lambda\int_{\Phi_L}^{\phi^*}\widehat{V}(\phi)f(\phi)\,d\phi + \frac{\lambda}{\rho+\lambda}\int_{\phi^*}^{\Phi_H}\phi f(\phi)\,d\phi. \tag{25}$$

Divide both sides of equation (25) by $[\rho+\lambda F(\phi^*)]$ to obtain equation (26) in the text.

respectively, to obtain

$$H\left(\Phi_L\right) = \tau\rho^2 - \alpha\lambda f\left(\Phi_L\right)\left(\rho\Phi_L + \lambda E\left\{\phi\right\}\right) \tag{29}$$

and

$$H\left(\Phi_H\right) = \left[\tau\left(\rho + \lambda\right) - \alpha\lambda f\left(\Phi_H\right)\left(\Phi_H + \left(1 - \alpha\right)\lambda E\left\{\widehat{V}\left(\phi\right)\right\}\right)\right]\left(\rho + \lambda\right). \tag{30}$$

If $H\left(\Phi_L\right) \geq 0 \geq H\left(\Phi_H\right)$, then $H\left(x\right) = 0$ has a root in $\left[\Phi_L, \Phi_H\right]$. It follows from equations (29) and (30) that $H\left(\Phi_L\right) \geq 0 \geq H\left(\Phi_H\right)$ if and only if

$$\tau_L \leq \tau \leq \tau_H, \tag{31}$$

where

$$\tau_L \equiv \alpha\frac{\lambda}{\rho}f\left(\Phi_L\right)\left(\Phi_L + \frac{\lambda}{\rho}E\left\{\phi\right\}\right) > 0 \tag{32}$$

and

$$\tau_H \equiv \alpha\frac{\lambda}{\rho + \lambda}f\left(\Phi_H\right)\left(\Phi_H + \left(1 - \alpha\right)\lambda E\left\{\widehat{V}\left(\phi\right)\right\}\right) > 0. \tag{33}$$

In order for the interval of values of τ in equation (31) to be non-degenerate, τ_L must be less than τ_H. The following proposition provides sufficient conditions for $\tau_L < \tau_H$.

Proposition 20 *If* $\Phi_L + \frac{\lambda}{\rho}E\left\{\phi\right\} > 0$, $F\left(\Phi_L\right) = 0$, $F\left(\Phi_H\right) = 1$ *and* $f'\left(\phi\right) \geq 0$ *everywhere on the support* $\left[\Phi_L, \Phi_H\right]$, *then* $\tau_H < \frac{\alpha}{2}$ *is a sufficient condition for the interval* $\left[\tau_L, \tau_H\right]$ *to be non-degenerate.*

The following corollary presents sufficient conditions for $H\left(\phi\right) = 0$ to have a root in $\left[\Phi_L, \Phi_H\right]$.

Corollary 21 *If* $\Phi_L + \frac{\lambda}{\rho}E\left\{\phi\right\} > 0$, $F\left(\Phi_L\right) = 0$, $F\left(\Phi_H\right) = 1$, $f'\left(\phi\right) \geq 0$ *everywhere on the support* $\left[\Phi_L, \Phi_H\right]$, *and* $\tau_L \leq \tau \leq \tau_H < \frac{\alpha}{2}$, *then there is a unique value* $\phi^* \in \left[\Phi_L, \Phi_H\right]$ *for which* $H\left(\phi^*\right) = 0$.

The critical value ϕ^* is given by the root of $H\left(\phi^*\right) = 0$. For any $\phi\left(t\right) \geq \phi^*$, the optimal value of debt equals D^*, which is given by equation (26).

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5.2　The Tradeoff Theory

The first-order condition in equation (23) embodies the tradeoff theory. To interpret this condition, it is useful to multiply both sides by $\rho + \lambda$ to obtain

$$\tau \left(\rho + \lambda F \left(\phi^* \right) \right) = \left(\rho + \lambda \right) \alpha \lambda D^* f \left(\phi^* \right). \tag{34}$$

It is simplest to begin with the case in which $\alpha = 1$ so that default completely destroys the productive capability of the firm. In this case, the default premium $Q\left(t\right)$ in equation (8) is simply $\lambda F\left(\phi^*\right)D^*$ so that total interest payments are $\rho D^* + \lambda F\left(\phi^*\right)D^*$ and the interest rate on debt is $\rho + \lambda F\left(\phi^*\right)$. The left hand side of equation (34) is the marginal tax shield, over the next interval dt of time, associated with an additional dollar of debt. To see why, suppose that the firm issues an additional dollar of debt at time t_0, pays interest at rate $\rho + \lambda F\left(\phi^*\right)$ over the subsequent interval dt of time, then repays the dollar of debt at time $t_0 + dt$ with probability $e^{-\lambda F(\phi^*)dt}$, which is the probability that it is not optimal to default at $t_0 + dt$ or earlier. Therefore, *if the interest rate were to remain unchanged at $\rho + \lambda F\left(\phi^*\right)$*, the additional dollar of debt would increase the expected present value of the firm's after-tax cash flow over the next interval dt of time by $1 - \left(1 - \tau\right)\left(\rho + \lambda F\left(\phi^*\right)\right)dt - e^{-\rho dt}e^{-\lambda F(\phi^*)dt}$, which equals the left hand side of equation (34) for small dt. Thus, the left hand side of equation (34) is the marginal tax shield associated with an additional dollar of debt.

The right hand side of equation (34) is the marginal cost, over the next interval dt of time, associated with the increased probability of default resulting from an additional dollar of debt. By increasing the probability of default, a one-dollar increase in debt increases the default premium, and hence increases the interest rate, $\rho + \lambda F\left(\phi^*\right)$, paid by the firm. To measure the increase in the probability of default, define the function $\psi\left(D\right)$ implicitly by $D = V\left(\psi\left(D\right),D\right)$ and note from Proposition 9 that $\phi^* = \psi\left(D^*\right)$. Total differentiation of $D = V\left(\psi\left(D\right),D\right)$ with respect to D yields $1 = \frac{\partial V(\psi(D),D)}{\partial \phi}\psi'\left(D\right) + \frac{\partial V(\psi(D),D)}{\partial D}$. Evaluating this expression at D^* and using $\phi^* = \psi\left(D^*\right)$ and $\frac{\partial V(\psi(D^*),D^*)}{\partial D} = 0$ yields $\psi'\left(D^*\right) = \left[\frac{\partial V(\phi^*,D^*)}{\partial \phi}\right]^{-1} = \frac{\rho+\lambda}{1-\tau}$, where the second equality follows from the fact that $V\left(\phi,D\right)$ is linear in ϕ with slope $\frac{1-\tau}{\rho+\lambda}$. Therefore, a one-unit increase in D will increase the threshhold level of ϕ at which default occurs by $\psi'\left(D^*\right) = \frac{\rho+\lambda}{1-\tau}$ and thus increase the probability of default by $\lambda f\left(\phi^*\right)\psi'\left(D^*\right)$ $= \lambda f\left(\phi^*\right)\frac{\rho+\lambda}{1-\tau}$, which, in the case with $\alpha = 1$, increases the interest rate by $\lambda f\left(\phi^*\right)\frac{\rho+\lambda}{1-\tau}$, and increases the total flow of *after-tax* interest payments at time t_0 by $\lambda f\left(\phi^*\right)\left(\rho+\lambda\right)D^*$, which

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is the right hand side of equation (34) when $\alpha = 1$. Therefore, equation (34) represents the equality of the marginal tax shield and the marginal cost associated with the increased probability of default, which is the essence of the tradeoff theory.

In the more general case in which $\alpha \leq 1$, the interpretation of equation (34) in terms of the tradeoff theory is more nuanced. Define $R(D, \phi_0)$ as the flow of (pre-tax) interest payments at time t if the amount of outstanding debt is D and if the firm defaults if and only if $\phi < \phi_0$. Thus,

$$R(D, \phi_0) \equiv (\rho + \lambda F(\phi_0)) D - \lambda (1 - \alpha) \int_{\Phi_L}^{\phi_0} \widehat{V}(\phi) \, dF(\phi), \qquad (35)$$

which is the sum of ρD and the default premium $Q(t)$ in equation (8), using the fact that $\left\{ \phi : \widehat{V}(\phi) < D^* \right\} = \{ \phi : \phi < \phi^* \}$. *If the value of EBIT that triggers default, ϕ_0, were to remain unchanged*, an increase in debt would increase interest payments by $\frac{\partial R(D^*, \phi_0)}{\partial D} = \rho + \lambda F(\phi_0)$. Therefore, when $D = D^*$ and $\phi_0 = \phi^*$, a one-dollar increase in D would increase the tax shield associated with interest deductibility by

$$\tau \frac{\partial R(D^*, \phi^*)}{\partial D} = \tau (\rho + \lambda F(\phi^*)). \qquad (36)$$

Therefore, the left hand side of equation (34) is the marginal tax shield.

Now consider the impact on interest payments of an increase in ϕ_0, *holding the amount of debt unchanged*. Partially differentiating equation (35) with respect to ϕ_0, evaluating this derivative at $\phi_0 = \phi^*$, and using $\widehat{V}(\phi^*) = D^*$ (from Corollary 10) yields

$$\frac{\partial R(D^*, \phi^*)}{\partial \phi_0} = \alpha \lambda f(\phi^*) D^*. \qquad (37)$$

Therefore, equation (37) implies that if ϕ_0 were to increase by $\psi'(D^*)$, as would be the case if D were to increase by one dollar, after-tax interest payments would increase by

$$(1 - \tau) \frac{\partial R(D^*, \phi^*)}{\partial \phi_0} \psi'(D^*) = (\rho + \lambda) \alpha \lambda f(\phi^*) D^*. \qquad (38)$$

The left hand side of equation (38) is the marginal default cost, measured in flow terms, that reflects the increased probability of default when D increases by one dollar. Specif-

ically, a one-dollar increase in D increases the default threshhold ϕ_0 by $\psi'(D^*) = \frac{\rho + \lambda}{1-\tau}$, thereby increasing total interest costs by $\frac{\partial R(D^*, \phi^*)}{\partial \phi_0} \psi'(D^*)$ and after-tax interest costs by $(1-\tau) \frac{\partial R(D^*, \phi^*)}{\partial \phi_0} \psi'(D^*)$, which is the left hand side of equation (38). The right hand side of equation (38) is identical to the right hand side of equation (34), so the right hand side of equation (34) is the marginal expected cost of default associated with a one-dollar increase in D. Thus, equation (34) is a statement of the tradeoff theory, equating the marginal value of the tax shield associated with an additional dollar of debt, expressed per unit of time, and the marginal cost of an increased probability of default resulting from this increased debt, also expressed per unit of time.

5.3 Firm in Regime II with a Low Level of $\phi(t)$

Suppose that the firm is in Regime II and that EBIT, $\phi(t)$, has a low value. Specifically, $\phi(t) \le \phi^*$ so that $S(\phi(t)) = 0$. In this case, the firm issues as much debt as it can, driving the equity value to zero, so $\widehat{V}(\phi(t)) = \widehat{D}(\phi(t))$.

Evaluate equation (11) at $D = \widehat{D}(\phi(t))$, so that the left hand side becomes $V\left(\phi(t), \widehat{D}(\phi(t))\right) = \widehat{V}(\phi(t))$ and $\widehat{V}^{-1}(D)$ on the right hand side becomes $\widehat{V}^{-1}\left(\widehat{D}(\phi(t))\right) = \widehat{V}^{-1}\left(\widehat{V}(\phi(t))\right) = \phi(t)$. Therefore,

$$
\begin{aligned}
[(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]\,\widehat{D}(\phi(t)) \;=\;& (1-\tau)\phi(t) \tag{39}\\
&+ (1-\alpha)\lambda(1-\tau)\int_{\Phi_L}^{\phi(t)} \widehat{V}(\phi)\,dF(\phi)\\
&+ \lambda \int_{\phi(t)}^{\phi^*} \widehat{V}(\phi)\,dF(\phi) + \Gamma,
\end{aligned}
$$

where

$$
\Gamma \equiv \lambda \int_{\phi^*}^{\Phi_H} \widehat{V}(\phi)\,dF(\phi). \tag{40}
$$

Equation (39) is a functional equation in $\widehat{D}(\phi(t)) \equiv \widehat{V}(\phi(t))$ for $\phi(t) \le \phi^*$. A strategy for solving this functional equation is to differentiate both sides with respect to $\phi(t)$ and use the fact that $\widehat{V}(\phi(t)) \equiv \widehat{D}(\phi(t))$ for $\phi(t) \le \phi^*$ to obtain the following first-order linear

ordinary differential equation (ODE)

$$[(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]\,\widehat{D}'(\phi(t)) + (1-\tau)\alpha\lambda f(\phi(t))\,\widehat{D}(\phi(t)) = 1-\tau. \qquad (41)$$

This ODE is solved in Appendix B. It is straightforward to verify that the following equation is a solution to the ODE in equation (41) for $\phi(t) \leq \phi^*$, where C is a constant of integration,

$$\widehat{D}(\phi(t)) = [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{\frac{\alpha(1-\tau)}{\tau}} \qquad (42)$$
$$\times \left[C + (1-\tau)\int [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}-1}\,d\phi(t) \right].$$

The constant of integration, C, is determined by the boundary condition

$$\widehat{D}(\phi^*) = \frac{1}{(1-\tau)\rho + \lambda(1-\tau F(\phi^*))}\left[(1-\tau)\left(\phi^* + (1-\alpha)\lambda \int_{\Phi_L}^{\phi^*} \widehat{V}(\phi)\,dF(\phi) \right) + \Gamma \right], \qquad (43)$$

which follows from evaluating the functional equation (39) at $\phi(t) = \phi^*$.

An alternative, but equivalent, boundary condition, which takes the form of a value-matching condition, is

$$\widehat{D}(\phi^*) = D^*, \qquad (44)$$

where D^* is the optimal level of debt in equation (26) for high values of $\phi(t)$, i.e., $\phi(t) \geq \phi^*$.[18]

To evaluate the slope of $D(\phi(t))$ with respect to $\phi(t)$ at $\phi(t) = \phi^*$, evaluate the ODE in equation (41) at $\phi(t) = \phi^*$ and use the boundary condition $\widehat{D}(\phi^*) = D^*$ along with the

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[18]To see that the boundary conditions in equations (43) and (44) are equivalent, use Proposition 11 to substitute $\frac{1-\tau}{\rho+\lambda}[\phi - \phi^*] + D^*$ for $\widehat{V}(\phi)$ when $\phi \geq \phi^*$ to obtain

$$\Gamma = \lambda(1-F(\phi^*))\left[D^* - \frac{1-\tau}{\rho+\lambda}\phi^* \right] + \lambda\frac{1-\tau}{\rho+\lambda}\int_{\phi^*}^{\Phi_H} \phi\,dF(\phi). \qquad (45)$$

Then substitute this expression for Γ into equation (43), set $D^* = \widehat{D}(\phi^*)$, and rearrange to obtain

$$[\rho + \lambda F(\phi^*)]\,D^* = \frac{\rho + \lambda F(\phi^*)}{\rho+\lambda}\phi^* + (1-\alpha)\lambda\int_{\Phi_L}^{\phi^*} \widehat{V}(\phi)\,dF(\phi) + \frac{\lambda}{\rho+\lambda}\int_{\phi^*}^{\Phi_H} \phi(t)\,dF(\phi), \qquad (46)$$

which is equivalent to $\widehat{D}(\phi^*) = D^*$, where D^* is given by equation (26).
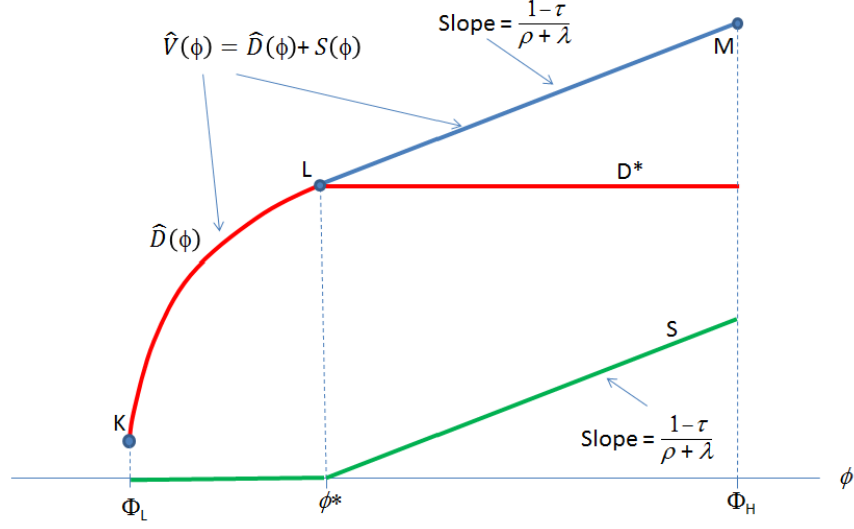
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Figure 1: Optimal Debt, Equity, and Total Value in Regime II

expression for $\alpha \lambda f(\phi^*) D^*$ implied by equation (23), and simplify, to obtain

$$\widehat{D}'(\phi^*) = \frac{1-\tau}{\rho+\lambda}. \tag{47}$$

Since equation (47) was derived from the ODE that holds for $\phi \leq \phi^*$, the derivative in equation (47) is actually the left-hand derivative. Since $\widehat{V}(\phi(t)) \equiv \widehat{D}(\phi(t))$ for $\Phi_L \leq \phi(t) \leq \phi^*$, equation (47) implies that the left-hand derivative of $\widehat{V}(\phi(t))$ at $\phi(t) = \phi^*$ is $\frac{1-\tau}{\rho+\lambda}$. Proposition 11 implies that the slope of $\widehat{V}(\phi(t))$ is $\frac{1-\tau}{\rho+\lambda}$ for $\phi(t) \geq \phi^*$ so the right-hand derivative of $\widehat{V}(\phi(t))$ at $\phi(t) = \phi^*$ is also $\frac{1-\tau}{\rho+\lambda}$. Therefore, the right-hand and left-hand derivatives of $\widehat{V}(\phi(t))$ at $\phi(t) = \phi^*$ are equal to each other.

Figure 1 illustrates the equity value, optimal debt, and the total value of the firm, each as a function of ϕ, for a firm in Regime II. The value of shareholders' equity, $S(\phi)$, is a convex piecewise linear function of ϕ. For $\phi \leq \phi^*$, shareholders' equity is identically zero, and for $\phi \geq \phi^*$, their equity is an increasing positive linear function of ϕ with slope equal to $\frac{1-\tau}{\rho+\lambda}$. For $\phi \geq \phi^*$, the optimal level of debt is constant and equal to D^*, as shown by the horizontal line segment starting at point L and extending to the right. For $\phi \leq \phi^*$, the

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optimal value of debt is shown by the increasing (concave[19]) curve labelled $\widehat{D}(\phi)$ connecting points K and L. The total value of the firm, $\widehat{V}(\phi)$, which is the sum of shareholders' equity and the optimal amount of debt at each ϕ, is shown by the curve through points K, L, and M. The value-matching condition discussed above is illustrated by the fact that the curve representing $\widehat{V}(\phi)$ for $\phi \leq \phi^*$ meets the curve representing $\widehat{V}(\phi)$ for $\phi \geq \phi^*$ at point L. Furthermore, as already discussed, the left-hand derivative and the right-hand derivative of $\widehat{V}(\phi)$ are equal to each other at $\phi = \phi^*$, so that the meeting of the curve through K and L and the line segment through L and M is smooth, that is, differentiable, at point L.[20]

6 Threshhold Tax Rates

The tax rate determines whether the firm is in Regime I, II, or III. Specifically, Corollary 21 states that if the tax rate τ is in $[\tau_L, \tau_H]$, the firm will be in Regime II. For lower values of the tax rate, the marginal tax shield is so small that it is overwhelmed by the marginal default cost, and hence the firm is in Regime I, where it does not expose itself to any chance of default. For sufficiently high values of the tax rate, the marginal tax shield overwhelms the marginal default cost and hence the firm is in Regime III, where it borrows as much as lenders are willing to lend. These findings are summarized in the following Proposition.

Proposition 22 *Suppose $\tau_L < \tau_H < \frac{\alpha}{2}$.*

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[19]Differentiate equation (41) with respect to $\phi(t)$ to obtain $[(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]\widehat{D}''(\phi(t)) = -[\alpha - (1+\alpha)\tau]\lambda f(\phi(t))\widehat{D}'(\phi(t)) - (1-\tau)\alpha\lambda f'(\phi(t))\widehat{D}(\phi(t))$. Since $(1-\tau)\rho + \lambda(1-\tau F(\phi(t))) > 0$, and for $\phi(t) \leq \phi^*$, $\widehat{D}'(\phi(t)) > 0$ and $\widehat{D}(\phi(t)) > 0$, it follows that $\widehat{D}''(\phi(t)) < 0$ if (a) $\tau \leq \frac{\alpha}{1+\alpha}$ and (b) $f'(\phi(t)) \geq 0$. Note that if $\alpha = 0$, so that there are zero deadweight costs of default, then $\widehat{D}''(\phi(t)) = \tau\lambda f(\phi(t))\widehat{D}'(\phi(t)) > 0$. Therefore, consistent with the discussion following Corollary 7, $\widehat{V}(\phi(t)) = \widehat{D}(\phi(t))$ is convex when $\alpha = 0$.

[20]This smooth meeting of the curve through K and L and the line segment through L and M has a superficial similarity to the smooth-pasting condition that arises in optimal stopping problems with an underlying diffusion process. In those problems, the value-matching and smooth-pasting conditions are two separate boundary conditions that help to pin down two parameters in the solution. However, in the current stochastic framework, the fundamental stochastic variable has finite variation whereas in optimal stopping problems with a diffusion process, the underlying stochastic variable has infinite variation. In the current framework, equality of the left-hand derivative and the right-hand derivative of $\widehat{V}(\phi(t))$ at $\phi = \phi^*$ arises as a consequence of the value-matching condition. That equality does not impose any additional structure or restriction on the solution.
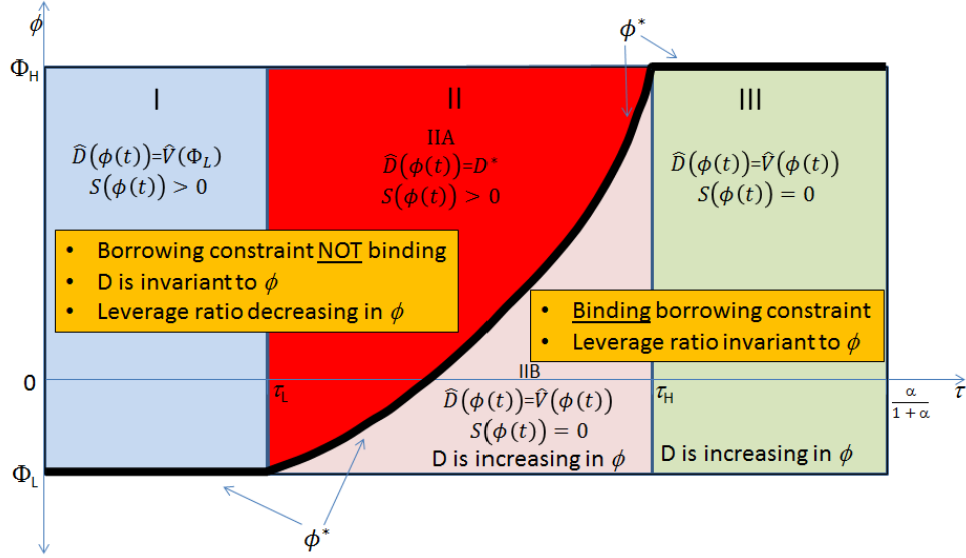
Figure 2: Three Regimes

- *(1) If $\tau < \tau_L$, then the firm is in Regime I and $\widehat{D}(\phi(t)) = \frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$ for all $\phi(t) \in \left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H)\right]$.*

- *(2) If $\tau_L \leq \tau \leq \tau_H$, then the firm is in Regime II and*
    - *(a) $\widehat{D}(\phi(t)) = D^*$ for $\phi(t) \geq \phi^*$*
    - *(b) $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$ for $\phi(t) \leq \phi^*$.*

- *(3) If $\tau_H < \tau < \frac{\alpha}{1+\alpha}$, then the firm is in Regime III and $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$ for all $\phi(t) \in \left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H)\right]$.*

Figure 2 illustrates the three regimes and displays the behavior of debt and the leverage ratio in each regime. The tax rate τ is measured along the horizontal axis and EBIT, ϕ, is measured along the vertical axis. Regime I prevails for $\tau < \tau_L$, Regime II prevails for $\tau_L \leq \tau \leq \tau_H$, and Regime III prevails for $\tau_H < \tau < \frac{\alpha}{1+\alpha}$. The thick line that is horizontal at $\phi = \Phi_L$ in Regime I, upward sloping in Regime II, and horizontal at $\phi = \Phi_H$ in Regime III shows the value of ϕ^* for each value of τ. Specifically, ϕ^* is the ordinate of each point on this line. Everywhere above this line, that is, throughout Regime I and in the upper portion

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of Regime II, which is labelled IIA, $\phi(t) > \phi^*$ so the borrowing constraint is not binding and $\widehat{D}(\phi(t)) = D^*$. Since $\widehat{D}(\phi(t))$ is invariant to ϕ and $\widehat{V}(\phi(t))$ is strictly increasing in ϕ in these regions, the optimal leverage ratio, $L(\phi(t)) \equiv \frac{\widehat{D}(\phi(t))}{\widehat{V}(\phi(t))}$, is strictly decreasing in ϕ throughout Regimes I and IIA. Everywhere below this line, that is, in the lower portion of Regime II, which is labelled IIB, and in Regime III, $\phi(t) < \phi^*$, so the borrowing constraint is binding and the leverage ratio is invariant to ϕ.

7 Effect of a Rightward Translation of $F(\phi)$

I have already examined the relationship between optimal debt—measured either by the optimal amount of debt issued, $\widehat{D}(\phi(t))$ or the optimal leverage ratio, $L(\phi(t))$—and *contemporaneous* profitability, $\phi(t)$. Now I turn attention to the relationship between optimal debt and the prospects for *future* profitability represented by the distribution function $F(\phi)$. In particular, I will look at an improvement in future prospects for profitability that is captured by a rightward translation of the distribution $F(\phi)$ to a new distribution, indexed by $m \in \mathcal{R}^+$, $G_m(\phi) = F(\phi - m)$ on the support $[\Phi_L(m), \Phi_H(m)]$, where $\Phi_L(m) = \Phi_L(0) + m$, $\Phi_H(m) = \Phi_H(0) + m$ and $[\Phi_L(0), \Phi_H(0)]$ is the support of the original distribution $F(\phi)$. For a given value of m, let $\widehat{V}(\phi(t); m)$ be the maximized firm value when $\phi = \phi(t)$, let $\widehat{D}(\phi(t); m)$ be the optimal value of debt when $\phi = \phi(t)$, let $P(\phi(t); m)$ be the instantaneous probability of default when $\phi = \phi(t)$, and let $\phi^*(m)$ and $D^*(m)$ be the values of ϕ^* and D^*, respectively.

The following proposition states that a rightward translation of the distribution $F(\phi)$ increases the value of the firm, $\widehat{V}(\phi(t); m)$, for any given $\phi(t)$ in the intersection of the supports of the original distribution and the new distribution.

Proposition 23 *For $0 < m < \Phi_H - \Phi_L$ and any $\phi(t) \in [\Phi_L + m, \Phi_H]$, $\widehat{V}(\phi(t); m) > \widehat{V}(\phi(t); 0)$.*

7.1 Effect of a Rightward Translation of $F(\phi)$ in Regime I

In Regime I, the firm issues the amount of debt shown in Proposition 19, which is the highest amount of debt that it can issue without exposing itself to a chance of default. Use

Proposition 19 along with $\frac{d\Phi_L(m)}{dm} = \frac{dE\{\phi\}}{dm} = 1$ to obtain

$$\frac{d\widehat{D}\left(\phi\left(t\right)\right)}{dm} = \frac{1}{\rho} > 0, \quad \text{in Regime I.} \tag{48}$$

Proposition 19 implies that in Regime I, $\widehat{D}\left(\phi\left(t\right)\right) = \widehat{V}\left(\Phi_L\right)$, which is the expected present value of EBIT over the infinite future, conditional on $\phi\left(t\right) = \Phi_L$. A rightward translation of the distribution $F\left(\phi\right)$ by one unit effectively increases EBIT by one unit in every state, which increases the expected present value of EBIT by $\frac{1}{\rho}$, since the firm will never default and hence will receive a stream of EBIT forever. Therefore, as shown in equation (48), the optimal amount of debt increases by $\frac{1}{\rho}$ units.

Differentiating the expression for $S\left(\phi\left(t\right)\right)$ in Proposition 19 with respect to m and using $\frac{d\Phi_L(m)}{dm} = 1$ yields

$$\frac{dS\left(\phi\left(t\right)\right)}{dm} = -\frac{1-\tau}{\rho+\lambda} < 0, \quad \text{in Regime I.} \tag{49}$$

Thus, a rightward translation of $F\left(\phi\right)$, which improves the prospects of the firm, reduces the value of equity, *for any given* $\phi\left(t\right)$. The value of equity falls because the increase in the firm's debt outweighs the increase in total firm value resulting from improved future prospects of the firm. However, the entire distribution shifts to the right so the *unconditional* expected value of $S\left(\phi\left(t\right)\right)$, which is $\frac{1-\tau}{\rho+\lambda}\left(E\left\{\phi\right\} - \Phi_L\right)$ in Regime I, is unchanged since $\frac{d\Phi_L(m)}{dm} = \frac{dE\{\phi\}}{dm} = 1$.

To examine the impact on the total value of the firm for any given value of $\phi\left(t\right)$, simply add equations (48) and (49) to obtain

$$\frac{d\widehat{V}\left(\phi\left(t\right)\right)}{dm} = \frac{d\widehat{D}\left(\phi\left(t\right)\right)}{dm} + \frac{dS\left(\phi\left(t\right)\right)}{dm} = \frac{\tau\rho+\lambda}{\rho\left(\rho+\lambda\right)} > 0, \quad \text{in Regime I.} \tag{50}$$

Thus, consistent with Proposition 23, a rightward translation of $F\left(\phi\right)$ increases the total value of the firm for any given $\phi\left(t\right)$.

To calculate the effect on the optimal leverage ratio, differentiate $L\left(\phi\left(t\right)\right) \equiv \frac{\widehat{D}(\phi(t))}{\widehat{V}(\phi(t))}$ with respect to m, and use equations (48) and (50) to obtain

$$\frac{dL\left(\phi\left(t\right)\right)}{dm} = \frac{1}{\widehat{V}\left(\phi\left(t\right)\right)} \frac{1}{\rho} \left[1 - L\left(\phi\left(t\right)\right)\frac{\tau\rho+\lambda}{\rho+\lambda}\right] > 0, \quad \text{in Regime I,} \tag{51}$$

where the inequality follows from $L\left(\phi\left(t\right)\right) \leq 1$ and $\tau < 1$. Thus, in Regime I, a rightward

translation of $F(\phi)$ increases the leverage ratio for any given $\phi(t)$.

7.2 Effect of a Rightward Translation of $F(\phi)$ in Regime II

In Regime II, $\omega_1 = \omega_2 = 0$ so that $\frac{\partial V(\phi(t), D^*(m); m)}{\partial D} = 0$. Corollary 10 implies $\widehat{V}(\phi^*(m); m) = D^*(m)$ so that for a given value of m, the first-order condition in equation (23) can be written as

$$\frac{\partial V\left(\phi(t), \widehat{V}(\phi^*(m), m); m\right)}{\partial D} \tag{52}$$
$$= \frac{\tau}{\rho + \lambda}(\rho + \lambda F(\phi^*(m) - m)) - \alpha\lambda\widehat{V}(\phi^*(m), m) f(\phi^*(m) - m) = 0.$$

Differentiate equation (52) with respect to m and rearrange to obtain

$$\left[\begin{array}{c}\frac{\tau}{\rho+\lambda}\lambda f(\phi^*(m) - m) \\ -\alpha\lambda\widehat{V}(\phi^*(m); m) f'(\phi^*(m) - m)\end{array}\right][\phi^{*\prime}(m) - 1] = \alpha\lambda f(\phi^*(m) - m)\frac{d\widehat{V}(\phi^*(m); m)}{dm}, \tag{53}$$

where $\frac{d\widehat{V}(\phi^*(m); m)}{dm} = \frac{\partial\widehat{V}(\phi^*(m); m)}{\partial\phi}\phi^{*\prime}(m) + \frac{\partial\widehat{V}(\phi^*(m); m)}{\partial m}$ is the effect on $\widehat{V}(\phi^*)$ of a small increase in m.[21] Use equation (52) to substitute $\frac{\tau}{\rho+\lambda}\frac{\rho+\lambda F(\phi^*(m)-m)}{f(\phi^*(m)-m)}$ for $\alpha\lambda\widehat{V}(\phi^*(m), m)$ in equation (53) to obtain

$$\frac{\tau}{\rho + \lambda}[\phi^{*\prime}(m) - 1]\chi(\phi^*(m) - m) = \alpha\lambda f(\phi^*(m) - m)\frac{d\widehat{V}(\phi^*(m); m)}{dm}, \tag{54}$$

where

$$\chi(\phi) \equiv \lambda f(\phi) - (\rho + \lambda F(\phi))\frac{f'(\phi)}{f(\phi)}. \tag{55}$$

I will use equation (54) to examine the impacts on ϕ^*, D^*, and $F(\phi^*)$ of a rightward translation of the distribution $F(\phi)$. As a preliminary step, I present the following lemma.

Lemma 24 If (1) $0 < \tau < \frac{\alpha}{1+\alpha}$ and (2) $f(\phi) > 0$ is non-decreasing for all ϕ, then $\phi^{*\prime}(m) < 1$.

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[21] Proposition 23 implies that $\frac{\partial\widehat{V}(\phi^*(m); m)}{\partial m} > 0$, but does not imply that $\frac{d\widehat{V}(\phi^*(m); m)}{dm}$ is positive. Indeed, $\frac{d\widehat{V}(\phi^*(m); m)}{dm}$ will be negative when $\chi(\phi^*)$, defined in equation (55), is positive.

Since an increase in m does not increase $\phi^*(m)$ by an amount greater than or equal to the increase in m, it reduces the probability of default if the firm optimally issues debt equal to $D^*(m)$. Alternatively, if $\phi(t) < \phi^*(m)$ so that $\widehat{D}(\phi(t);m) < D^*(m)$, an increase in m also reduces the probability of default, which is $\lambda F(\phi(t) - m)$. More formally, we have the following proposition.

Proposition 25 If (1) $0 < \tau < \frac{\alpha}{1+\alpha}$ and (2) $f(\phi) > 0$ is non-decreasing for all ϕ, then $\frac{dP(\phi(t);m)}{dm} < 0$.

A rightward translation of the distribution $F(\phi)$ can increase, decrease, or leave unchanged the value of D^*. The following proposition shows that the direction of the impact on D^* depends on the sign of $\chi(\phi^*)$, which is defined in equation (55).

Proposition 26 If (1) $0 < \tau < \frac{\alpha}{1+\alpha}$ and (2) $f(\phi) > 0$ is non-decreasing for all ϕ, then $sign\left(\frac{d\widehat{V}(\phi^*(m);m)}{dm}\right) = sign\left(D^{*\prime}(m)\right) = -sign\left(\chi\left(\phi^*(m) - m\right)\right)$.

Lemma 24 states that $\phi^{*\prime}(m)$ is less than one. The following proposition provides a sufficient condition for $\phi^{*\prime}(m)$ to be negative.

Proposition 27 If (1) $0 < \tau < \frac{\alpha}{1+\alpha}$, (2) $f(\phi) > 0$ is non-decreasing for all ϕ, and (3) $\chi(\phi^*(m) - m) \geq 0$, then $\phi^{*\prime}(m) < 0$.

If the distribution $F(\phi)$ is uniform, so that $f'(\phi) \equiv 0$, it satisfies the condition in Propositions 26 and 27 that $f(\phi)$ is non-decreasing. With $F(\phi)$ being uniform, $\chi(\phi) = \lambda f(\phi) = \frac{\lambda}{\Phi_H - \Phi_L} > 0$. Therefore, Propositions 26 and 27 imply that with a uniform distribution, a rightward translation reduces D^* and reduces ϕ^*.

Figure 3 illustrates the impact of a rightward translation of $F(\phi)$ for the case in which $\chi(\phi^*(m) - m) > 0$, which includes the uniform distribution. In response to a rightward translation of $F(\phi)$, the value of ϕ^* changes from its initial value of $\phi^*(0)$ to its new value of $\phi^*(m)$. In addition, the curves representing equity value, optimal debt, and the total value of the firm move from the solid curves to the dashed curves. As shown, the rightward translation of $F(\phi)$ has no effect on equity for $\phi \leq \phi^*(m)$ and increases the equity value by a constant amount, $\frac{1-\tau}{\rho+\lambda}(\phi^*(0) - \phi^*(m)) > 0$ for $\phi > \phi^*(m)$. The total value of the firm increases for all $\phi(t)$ (Proposition 23), as shown by the shift from the solid curve through
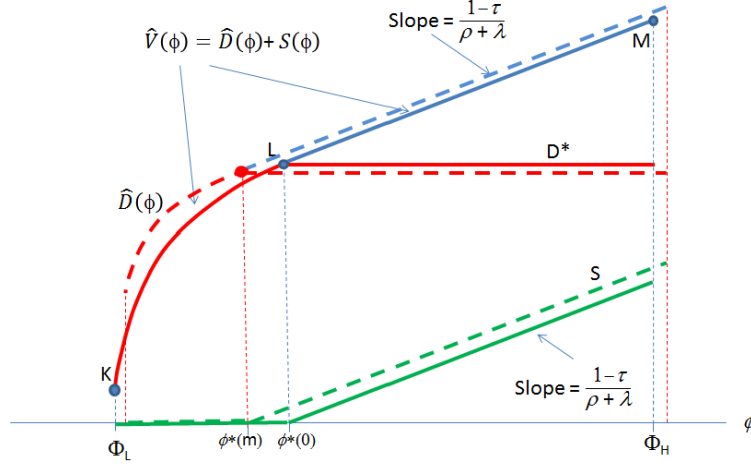
Figure 3: Effect of Rightward Translation of $F(\phi)$ in Regime II (when $\chi(\phi) > 0$)

K, L, and M, to the dashed curve that lies above it. The response of optimal debt to a rightward translation of $F(\phi)$ depends on whether ϕ is large or small. For $\phi(t) \geq \phi^*(0)$, the optimal value of debt, which is invariant to $\phi(t)$ in this range, falls in response to a rightward translation of $F(\phi)$. As discussed earlier, the continuation value of the firm increases, thereby increasing the cost of losing this future value by defaulting on debt. In response to this increased cost of default, the firm reduces its exposure to default by reducing the amount of debt it issues and reduces the critical value ϕ^*. For $\phi(t) \leq \phi^*(m)$, the optimal amount of debt increases in response to a rightward translation of $F(\phi)$. The reason for this increase in the optimal amount of debt is that the borrowing constraint is binding for low values of $\phi(t)$ and a rightward translation of $F(\phi)$ increases the total value of the firm, which allows the firm to borrow an increased amount. Finally, there is some $\widetilde{\phi} \in (\phi^*(m), \phi^*(0))$, not labelled in Figure 3, such that a rightward translation of $F(\phi)$ increases optimal debt for $\phi(t) < \widetilde{\phi}$ and decreases optimal debt for $\phi(t) > \widetilde{\phi}$.[22]

To understand the role of $\chi(\phi)$ in determining the impact of a rightward translation of

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[22]The text has shown that $D^*(m) < D^*(0) = \widehat{D}(\phi^*(0); 0)$. Note that $D^*(m) = \widehat{D}(\phi^*(m); m) = \widehat{V}(\phi^*(m); m) > \widehat{V}(\phi^*(m); 0) = \widehat{D}(\phi^*(m); 0)$, where the final equality follows from the fact that the borrowing constraint binds for $\phi(t) = \phi^*(m)$ under the original distribution $F(\phi)$. Therefore, $\widehat{D}(\phi^*(m); 0) < D^*(m) < \widehat{D}(\phi^*(0); 0)$. Since $\widehat{D}(\phi; m)$ is increasing in ϕ, there is a unique $\widetilde{\phi} \in (\phi^*(m), \phi^*(0))$ for which $\widehat{D}(\widetilde{\phi}; 0) = D^*(m)$.

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$F(\phi)$, it is helpful to recall that $\frac{\partial V\left(\phi(t),\widehat{V}(\phi^*(m),m);m\right)}{\partial D}$ in the first-order condition in equation (52) equals the marginal tax shield minus the marginal default cost. *Holding $\phi^*(m)$ fixed,* an increase in m *reduces* the marginal tax shield (by reducing $F(\phi^*(m))$) and, if $F(\phi)$ is uniform so that $f(\phi^*(m))$ is invariant to m, increases the marginal cost of default (by increasing $\widehat{V}(\phi^*(m),m)$), thereby making $\frac{\partial V\left(\phi(t),\widehat{V}(\phi^*(m),m);m\right)}{\partial D}$ negative. Because $V(\phi,D)$ is strictly concave in D, a reduction in D is needed to restore the first-order condition. In order to obtain the opposite effect, that is, in order for an increase in m to increase the optimal value of D^*, an increase in m must *reduce* the marginal default cost—and must do so by more than the reduction in the marginal tax shield. Such a reduction in the marginal default cost requires that an increase in m reduces $f(\phi^*(m) - m)$, for given $\phi^*(m)$, by a sufficiently large amount, which will occur if $f'(\phi^*)$ is large enough to make $\chi(\phi^*) < 0$.

To distinguish situations in which D^* falls in response to a rightward translation of $F(\phi)$ from situations in which D^* increases in response to a rightward translation of $F(\phi)$, consider a 3-parameter distribution that includes the uniform distribution as a special case. Specifically, consider the truncated exponential distribution $F(\phi) = \frac{1-e^{-\eta(\phi-\Phi_L)}}{1-e^{-\eta(\Phi_H-\Phi_L)}}$ on the support $[\Phi_L, \Phi_H]$.[23] The associated density function is $f(\phi) = \frac{\eta e^{-\eta(\phi-\Phi_L)}}{1-e^{-\eta(\Phi_H-\Phi_L)}} > 0$, which implies $f'(\phi) = -\eta f(\phi)$. To ensure that $\widehat{V}(\phi;m)$ is concave in ϕ and that $V(\phi,D;m)$ is strictly concave in D, I restrict attention to $\eta \le 0$ so that $f(\phi)$ is non-decreasing. It is convenient to re-parameterize the distribution in terms of η, $\delta \equiv \Phi_H - \Phi_L > 0$, and $\theta \equiv \frac{\eta}{1-e^{-\eta\delta}} > 0$, so that $F(\phi) = \frac{\theta}{\eta}\left(1 - e^{-\eta(\phi-\Phi_L)}\right)$, $f(\phi) = \theta e^{-\eta(\phi-\Phi_L)}$, $f(\Phi_L) = \theta$ and $f(\Phi_H) = \theta - \eta > 0$. Substitute $\frac{1}{\eta}(\theta - f(\phi))$ for $F(\phi)$ and $f'(\phi) = -\eta f(\phi)$ into the definition of $\chi(\phi)$ in equation (55) to obtain

$$\chi(\phi) = \rho\eta + \lambda\theta, \tag{56}$$

which is independent of ϕ. The following corollary provides a condition on $\eta\delta$ that determines the sign of $\chi(\phi)$ and hence determines the sign of $D^{*\prime}(\phi)$.

Corollary 28 *If (1) $0 < \tau < \frac{\alpha}{1+\alpha}$ and (2) the unconditional distribution of ϕ is the truncated exponential distribution $F(\phi) = \frac{1-e^{-\eta(\phi-\Phi_L)}}{1-e^{-\eta\delta}}$ for $\phi \in [\Phi_L, \Phi_H]$ where $\delta \equiv \Phi_H - \Phi_L > 0$ and*

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[23] The uniform distribution is the limiting case as η approaches zero, as may be seen by using L'Hopital's Rule to obtain $\lim_{\eta\to 0} F(\phi) = \lim_{\eta\to 0} \frac{(\phi-\Phi_L)e^{-\eta(\phi-\Phi_L)}}{(\Phi_H-\Phi_L)e^{-\eta(\Phi_H-\Phi_L)}} = \frac{\phi-\Phi_L}{\Phi_H-\Phi_L}$.

$\eta \leq 0$, then $D^{*\prime}(m) \lesseqgtr 0$ as $\eta\delta \gtreqless -\ln\left(1 + \frac{\lambda}{\rho}\right)$.

Since $-\ln\left(1 + \frac{\lambda}{\rho}\right) < 0$, Corollary 28 indicates that a rightward translation of the truncated exponential distribution will reduce D^* for negative values of η that are sufficiently small in absolute value, as well as for the uniform distribution ($\eta = 0$). In order to get the opposite result, that is, to get D^* to increase in response to a rightward translation of $F(\phi)$, η must be sufficiently negative, that is, the density function must have a sufficiently steep upward slope.

The tradeoff theory of capital structure is operative only in Regime II and, indeed, only for $\phi(t) \geq \phi^*$ in Regime II. When the tradeoff theory is operative, i.e., when $\phi(t) \geq \phi^*$, the optimal value of debt equals D^*. Thus Proposition 26 and Corollary 28 describe conditions under which optimal debt increases or decreases in response to a rightward translation of the distribution $F(\phi)$ when the tradeoff theory is operative. When $\phi(t) < \phi^*$, the tradeoff theory is not operative and $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$. Therefore, Proposition 23 implies that the optimal value of debt increases in response to a rightward translation of $F(\phi)$ in Regime II when $\phi(t) < \phi^*$. Of course, whenever $\phi(t) \leq \phi^*$, the optimal leverage ratio equals one and thus is invariant to a translation of $F(\phi)$.

7.3 Effect of a Rightward Translation of $F(\phi)$ in Regime III

In Regime III, the borrowing constraint binds for all $\phi(t)$, that is, $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$ for all $\phi(t)$. Therefore, Proposition 23 implies that in Regime III a rightward translation of $F(\phi)$ increases $\widehat{V}(\phi(t))$ and hence increases $\widehat{D}(\phi(t))$ for all $\phi(t)$. Of course, with $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$ for all $\phi(t)$, the optimal leverage ratio equals one for all $\phi(t)$ in Regime III.

7.4 Summary of Relationship between Profitability and Optimal Debt

Table 1 summarizes the findings about the impact of increased profitability on optimal debt when $\chi(\phi) > 0$, which includes the case in which $F(\phi)$ is uniform. The top third of the table summarizes the characteristics of the different regimes. The tradeoff theory will be operative if and only if (1) the borrowing constraint (B.C., in the table) is not binding

Characteristics of Regimes			
Regime I	Regime IIA	Regime IIB	Regime III
B.C.* not binding	B.C.* not binding	B.C.* binds	B.C.* binds
Pr{default}=0	Pr{default}>0	Pr{default}>0	Pr{default}>0
Not Tradeoff	Tradeoff Operative	Not Tradeoff	Not Tradeoff
Increase in $\phi(t)$			
Regime I	Regime IIA	Regime IIB	Regime III
$\widehat{D}$ unchanged	$\widehat{D}$ unchanged	$\widehat{D}\uparrow$	$\widehat{D}\uparrow$
$S\uparrow$	$S\uparrow$	$S\equiv 0$	$S\equiv 0$
$\widehat{V}\uparrow$	$\widehat{V}\uparrow$	$\widehat{V}\uparrow$	$\widehat{V}\uparrow$
$L\downarrow$	$L\downarrow$	$L\equiv 1$	$L\equiv 1$
Rightward Translation of $F(\phi)$			
Regime I	Regime IIA	Regime IIB	Regime III
$\widehat{D}\uparrow$	$\widehat{D}\downarrow$ if $\chi>0$	$\widehat{D}\uparrow$	$\widehat{D}\uparrow$
$S\downarrow$	$S\uparrow$	$S\equiv 0$	$S\equiv 0$
$\widehat{V}\uparrow$	$\widehat{V}\uparrow$	$\widehat{V}\uparrow$	$\widehat{V}\uparrow$
$L\uparrow$	$L\downarrow$ if $\chi>0$	$L\equiv 1$	$L\equiv 1$
*B.C. refers to the borrowing constraint $D(\phi)\leq\widehat{V}(\phi)$			

Table 1: Effects of Increased Profitability

and (2) the firm faces a positive probability of default. Only Regime IIA meets these two criteria, so the tradeoff theory is operative in Regime IIA, but not in the other regimes. Among the regimes in which the firm faces a positive probability of default (Regimes IIA, IIB, and III), an increase in profitability—either an increase in current profitability, $\phi(t)$, or a rightward translation of the unconditional distribution $F(\phi)$—can reduce the optimal leverage ratio only in Regime IIA, that is, only when the tradeoff theory is operative. Thus, in the context of the model presented here, in which the tradeoff theory can be operative or not, the empirical finding of a negative relationship between profitability and the leverage ratio is consistent only with the tradeoff theory being operative, if the probability of default is positive.

8 Concluding Remarks

This paper develops and analyzes a simple model of debt choice that is rich enough to include situations in which the tradeoff theory is operative as well as situations in which it is not operative. There are two payoffs to analyzing a model in which the tradeoff theory may or may not be operative. First, analysis of this model points to factors that determine whether or not the tradeoff theory is operative in particular situations. Second, the model allows us to compare the optimal behavior in situations in which the tradeoff theory is operative with optimal behavior in situations in which it is not operative. To the extent that optimal behavior differs depending on whether or not the tradeoff theory is operative, one could potentially use such differences in behavior as a basis for empirically testing the tradeoff theory.

The equality of the marginal tax shield resulting from interest deductibility and the marginal cost of increased default probability associated with increased debt is the defining feature of the tradeoff theory of debt. The mere presence of interest deductiblity and deadweight costs of default is not sufficient to ensure that the tradeoff theory is operative. I have demonstrated three situations in which the tradeoff theory is not operative despite the presence of interest deductibility and deadweight default costs. First, if the tax rate is very low (Regime I), the marginal benefit of the tax shield associated with an additional dollar of debt is completely overwhelmed by the marginal cost associated with increased default probability resulting from an additional dollar of debt. In this case, the firm will take

advantage of the tax shield offered by interest deductibility, but will only borrow as much as it can without exposing itself to any possibility of default. Thus, the tradeoff theory is not operative for tax rates that are sufficiently low. Second, if the tax rate is sufficiently high (but not so high as to violate conditions for concavity of the value of the firm), the marginal benefit of the tax shield associated with an additional dollar of debt completely overwhelms the marginal cost associated with increased default probability resulting from an additional dollar of debt. In this case, the firm borrows as much as lenders are willing to lend. The tradeoff theory is not operative because the borrowing constraint is strictly binding so the marginal benefit of the tax shield fails to equal the marginal cost of increased default probability. Third, even if the tax rate is neither too low nor too high, the tradeoff theory will fail to be operative if the current value of EBIT is lower than the critical value, denoted by ϕ^* in the model. In this situation, the low value of EBIT implies that the current value of firm is low, which implies that the constraint on how much the firm can borrow is strictly binding. In this situation, the marginal benefit of the tax shield exceeds the marginal cost of increased default probability, so, again, the tradeoff theory is not operative. The only situation in the model in which the tradeoff theory is operative is when the tax rate is neither too low nor too high and the current value of EBIT is higher than the critical value ϕ^*. In this case, the optimal value of debt equates the marginal benefit of the tax shield and the marginal cost of increased default probability. In the particular stochastic environment analyzed here, the optimal value of debt is invariant to the contemporaneous value of EBIT provided that the tradeoff theory is operative.

The relationship between profitability and optimal borrowing depends on whether the tradeoff theory is operative. Table 1 summarizes the impact on optimal borrowing of an increase in profitability. In this table are two measures of leverage—the amount of debt issued and the leverage ratio—and two measures of profitability—current EBIT and the unconditional distribution from which new values of EBIT are drawn. Two major lessons emerge from this table. First, when the tradeoff theory is operative, the leverage ratio is a decreasing function of current EBIT; and provided that the unconditional density function of EBIT does not slope upward too steeply, a rightward translation of the unconditional distribution of EBIT reduces both the level of optimal debt and the optimal leverage ratio. This finding that an increase in profitability leads to a reduction in the leverage ratio is contrary to standard interpretations of the tradeoff theory, but it is consistent with empirical

analyses of this relationship. The second lesson is that provided the firm faces a positive probability of default, the optimal leverage ratio is negatively related to profitability only if the tradeoff theory is operative. So the empirical finding of a negative relationship between borrowing and profitability is not consistent with the alternatives to the tradeoff theory in this model.

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A Appendix: Proofs

Proof. of Lemma 1. For any $s \geq t$, the probability that at least one new value of ϕ has been drawn by time s is $1-e^{-\lambda(s-t)}$. Therefore, $E_t \{\phi(s)\} = e^{-\lambda(s-t)}\phi(t)+\left(1 - e^{-\lambda(s-t)}\right) E\{\phi\}$, so $W(\phi(t)) = (1 - \tau)\left\{\int_t^\infty \left[e^{-\lambda(s-t)}\phi(t) + \left(1 - e^{-\lambda(s-t)}\right) E\{\phi\}\right] e^{-\rho(s-t)}ds\right\} = (1 - \tau)\left[\frac{1}{\rho+\lambda}\phi(t) + \left(\frac{1}{\rho} - \frac{1}{\rho+\lambda}\right) E\{\phi\}\right] = (1 - \tau)\frac{1}{\rho+\lambda}\left[\phi(t) + \frac{\lambda}{\rho}E\{\phi\}\right].$ ∎

Proof. of Corollary 2. Lemma 1 implies that $W(\phi(t)) \geq \frac{1-\tau}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right] > 0$ for all $\phi(t) \in [\Phi_L, \Phi_H]$. ∎

Proof. of Proposition 3. For any $\phi_2 > \phi_1$, equation (7) implies $V(\phi_2, D) > V(\phi_1, D)$. Therefore, $V\left(\phi_2, \widehat{D}(\phi_1)\right) > V\left(\phi_1, \widehat{D}(\phi_1)\right)$ and since $\widehat{D}(\phi_1) \leq V\left(\phi_1, \widehat{D}(\phi_1)\right)$, it follows that $\widehat{D}(\phi_1) \leq V\left(\phi_2, \widehat{D}(\phi_1)\right)$ so that $\widehat{D}(\phi_1)$ is a feasible level of debt when $\phi(t) = \phi_2$. Since $V(\phi, D)$ is additively separable in ϕ and D, and linear in ϕ with slope $\frac{1-\tau}{\rho+\lambda}$, it follows that $V(\phi_2, D) = V(\phi_1, D) + \frac{1-\tau}{\rho+\lambda}(\phi_2 - \phi_1)$. Hence, $\widehat{V}(\phi_2) = V\left(\phi_2, \widehat{D}(\phi_2)\right) \geq V\left(\phi_2, \widehat{D}(\phi_1)\right) = V\left(\phi_1, \widehat{D}(\phi_1)\right) + \frac{1-\tau}{\rho+\lambda}(\phi_2 - \phi_1) = \widehat{V}(\phi_1) + \frac{1-\tau}{\rho+\lambda}(\phi_2 - \phi_1)$. Therefore, $\widehat{V}(\phi_2) - \widehat{V}(\phi_2) \geq \frac{1-\tau}{\rho+\lambda}(\phi_2 - \phi_1) > 0$ and hence $\frac{\widehat{V}(\phi_2)-\widehat{V}(\phi_2)}{\phi_2-\phi_1} \geq \frac{1-\tau}{\rho+\lambda}$. ∎

Proof. of Proposition 4. The first step is to prove that $\widehat{D}(\phi) \geq \min_\phi \widehat{V}(\phi)$. Suppose the contrary, so that $\widehat{D}(\phi) < \min_\phi \widehat{V}(\phi)$. Then regardless of the new value of ϕ that is realized when the value of ϕ changes, the shareholders will choose not to default on the debt. In this case, equation (9) becomes $V\left(\phi(t), \widehat{D}(\phi)\right) = \frac{\tau}{\rho+\lambda}\rho\widehat{D}(\phi) + \frac{1-\tau}{\rho+\lambda}\phi(t) + \frac{\lambda}{\rho+\lambda}\int \widehat{V}(\phi) dF(\phi)$, which implies that $\frac{\partial V\left(\phi(t),\widehat{D}(\phi)\right)}{\partial D} = \frac{\tau}{\rho+\lambda}\rho > 0$. Since the constraint $D(\phi) < V(\phi, D)$ is not binding and $\frac{\partial V\left(\phi(t),\widehat{D}(\phi)\right)}{\partial D} > 0$, $\widehat{D}(\phi)$ cannot be the optimal value of debt. Therefore, $\widehat{D}(\phi) \geq \min_\phi \widehat{V}(\phi)$. Since $\widehat{V}(\phi(t))$ is strictly increasing (Proposition 3), $\min_\phi \widehat{V}(\phi) = \widehat{V}(\Phi_L) \geq V(\Phi_L, 0)$. Recall that $W(\Phi_L)$ is the expected present value of the flow of net cash flow for a firm that sets $D = 0$ and never ceases operation. Therefore, $V(\Phi_L, 0) \geq W(\Phi_L) = \frac{1-\tau}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$, where the equality follows from Lemma 1 and the final inequality follows from equation (4) ∎

Proof. of Corollary 5. $\widehat{V}(\Phi_L) \geq \widehat{D}(\phi) \geq \widehat{V}(\Phi_L)$, where the first inequality is the constraint that prevents the firm from borrowing an amount that leads to immediate default and the second inequality follows from Proposition 4. Since $S(\phi(t)) = \widehat{V}(\phi(t)) - \widehat{D}(\phi(t))$, $S(\Phi_L) = 0$. ∎

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Proof. of Proposition 6

Define the operator T by

$$\left(T\widehat{V}\right)(\phi(t)) = \max_{D \le \widehat{V}(\phi(t))} V(\phi(t), D)$$

$$= \max_{D \le \widehat{V}(\phi(t))} \left[\begin{array}{c} \frac{\tau}{\rho+\lambda}\left(\rho + \lambda F\left(\widehat{V}^{-1}(D)\right)\right) D + \frac{1-\tau}{\rho+\lambda}\phi(t) \\ + \frac{\lambda}{\rho+\lambda}(1-\tau)(1-\alpha)\int_{\Phi_L}^{\widehat{V}^{-1}(D)} \widehat{V}(\phi) f(\phi) d\phi \\ + \frac{\lambda}{\rho+\lambda}\int_{\widehat{V}^{-1}(D)}^{\Phi_H} \widehat{V}(\phi) f(\phi) d\phi \end{array} \right],$$

where the second equality follows from equation (11). The proof will proceed by showing that the operator T takes concave functions of ϕ into concave functions of ϕ and that T is a contraction, so that it has a unique fixed point, which is a concave function of ϕ.

Differentiate $\frac{\partial V(\phi(t),D)}{\partial D}$ in equation (12) with respect to D to obtain

$$\frac{\partial^2 V(\phi(t),D)}{(\partial D)^2} = [(1+\alpha)\tau - \alpha]\frac{\lambda}{\rho+\lambda}f\left(\widehat{V}^{-1}(D)\right)\widehat{V}^{-1\prime}(D) \tag{57}$$

$$- (1-\tau)\alpha\frac{\lambda}{\rho+\lambda}D\left[\begin{array}{c} f'\left(\widehat{V}^{-1}(D)\right)\left[\widehat{V}^{-1\prime}(D)\right]^2 \\ +f\left(\widehat{V}^{-1}(D)\right)\widehat{V}^{-1\prime\prime}(D) \end{array} \right].$$

Proposition 3 implies that $\widehat{V}^{-1}(D)$ is increasing so that $\widehat{V}^{-1\prime}(D) > 0$. Therefore, since $f\left(\widehat{V}^{-1}(D)\right) > 0$, the assumption $0 \le \tau < \frac{\alpha}{1+\alpha}$ implies that the first of the two terms in the expression for $\frac{\partial^2 V(\phi(t),D)}{(\partial D)^2}$ in equation (57), that is, $[(1+\alpha)\tau - \alpha]\frac{\lambda}{\rho+\lambda}f\left(\widehat{V}^{-1}(D)\right)\widehat{V}^{-1\prime}(D)$, is negative.

Suppose that $\widehat{V}(\phi(t))$ is concave, which implies that $\widehat{V}^{-1}(D)$ is convex, so that $\widehat{V}^{-1\prime\prime}(D) \ge 0$. Therefore, the assumption that $f(\phi)$ is non-decreasing implies that the second of the two terms in the expression for $\frac{\partial^2 V(\phi(t),D)}{(\partial D)^2}$ in equation (57), that is

$-(1-\tau)\alpha\frac{\lambda}{\rho+\lambda}D\left[f'\left(\widehat{V}^{-1}(D)\right)\left[\widehat{V}^{-1\prime}(D)\right]^2 + f\left(\widehat{V}^{-1}(D)\right)\widehat{V}^{-1\prime\prime}(D)\right]$, is non-positive. Therefore, $\frac{\partial^2 V(\phi(t),D)}{(\partial D)^2} < 0$ so $V(\phi(t), D)$ is strictly concave in D.

The next step is to prove that if $V(\phi(t), D)$ is strictly concave in D, then $\widehat{V}(\phi(t)) = \max_{0 \le D \le \widehat{V}(\phi(t))} V(\phi(t), D)$ is concave. Since $V(\phi(t), D)$ is additively separable in $\phi(t)$ and D, linear in $\phi(t)$ with slope $\frac{1-\tau}{\rho+\lambda}$, and strictly concave in D, $\widehat{V}(\phi(t))$ can be writ-

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ten as $\widehat{V}(\phi(t)) = \max_{0 \le D \le \widehat{V}(\phi(t))} \frac{1-\tau}{\rho+\lambda}\phi(t) + v(D)$ where $v(D)$ is strictly concave. Consider ϕ_1 and ϕ_2 and the respective corresponding optimal values of debt $\widehat{D}(\phi_1)$ and $\widehat{D}(\phi_2)$. Therefore, $\widehat{V}(\phi_i) = \frac{1-\tau}{\rho+\lambda}\phi_i + v\left(\widehat{D}(\phi_i)\right)$ and $\widehat{D}(\phi_i) \le \widehat{V}(\phi_i)$, $i = 1, 2$. Now consider $\phi(\gamma) = \gamma\phi_1 + (1-\gamma)\phi_2$ and $D(\gamma) = \gamma\widehat{D}(\phi_1) + (1-\gamma)\widehat{D}(\phi_2)$ for $0 \le \gamma \le 1$. Observe that $D(\gamma) = \gamma\widehat{D}(\phi_1) + (1-\gamma)\widehat{D}(\phi_2) \le \gamma\widehat{V}(\phi_1) + (1-\gamma)\widehat{V}(\phi_2) = \gamma\left[\frac{1-\tau}{\rho+\lambda}\phi_1 + v\left(\widehat{D}(\phi_1)\right)\right] + (1-\gamma)\left[\frac{1-\tau}{\rho+\lambda}\phi_2 + v\left(\widehat{D}(\phi_2)\right)\right] = \frac{1-\tau}{\rho+\lambda}\phi(\gamma) + \gamma v\left(\widehat{D}(\phi_1)\right) + (1-\gamma)v\left(\widehat{D}(\phi_2)\right) \le \frac{1-\tau}{\rho+\lambda}\phi(\gamma) + v\left(\widehat{D}(\gamma)\right) = T\widehat{V}(\phi(\gamma))$, where the final inequality follows from the strict concavity of $v(D)$ but allows for the possibility that $\widehat{D}(\phi_1) = \widehat{D}(\phi_2)$ so the inequality is a weak inequality. Therefore, since $\widehat{V}(\phi(\gamma)) \ge \frac{1-\tau}{\rho+\lambda}\phi(\gamma) + v\left(\widehat{D}(\gamma)\right)$, we have that $\widehat{V}(\phi(\gamma)) \ge \gamma\widehat{V}(\phi_1) + (1-\gamma)\widehat{V}(\phi_2)$ and $D(\gamma) \le \widehat{V}(\phi(\gamma))$ so $D(\gamma)$ is a feasible level of debt when $\phi = \phi(\gamma)$. Therefore, $T\widehat{V}(\phi(t))$ is concave if $V(\phi(t), D)$ is strictly concave in D.

We have shown that if $\widehat{V}(\phi)$ is concave, then $V(\phi(t), D)$ is strictly concave in D, and that if $V(\phi(t), D)$ is strictly concave in D, then $T\widehat{V}(\phi)$ is concave. Therefore, the operator T takes concave functions into concave functions. It remains to show that T is a contraction so that there is a unique concave function that is a fixed point of T.

Monotonicity: I will show that if $\widehat{V}_2(\phi(t)) \ge \widehat{V}_1(\phi(t))$, then $\left(T\widehat{V}_2\right)(\phi(t)) \ge \left(T\widehat{V}_1\right)(\phi(t))$. Suppose that $\widehat{V}_2(\phi(t)) \ge \widehat{V}_1(\phi(t))$.

$$\left(T\widehat{V}_i\right)(\phi(t)) = \max_{D \le \widehat{V}_i(\phi(t))}\left[\begin{array}{c} \frac{\tau}{\rho+\lambda}\rho D + \frac{1-\tau}{\rho+\lambda}\phi(t) \\ +\frac{\lambda}{\rho+\lambda}\int_{\Phi_L}^{\widehat{V}_i^{-1}(D)}\left[(1-\tau)(1-\alpha)\widehat{V}_i(\phi) + \tau D\right]dF(\phi) \\ +\frac{\lambda}{\rho+\lambda}\int_{\widehat{V}_i^{-1}(D)}^{\Phi_H}\widehat{V}_i(\phi)dF(\phi) \end{array}\right]. \quad (58)$$

Let $\widehat{D}_i(\phi(t)) = \arg\max_{D \le \widehat{V}_i(\phi(t))}\left[\begin{array}{c} \frac{\tau}{\rho+\lambda}\rho D + \frac{1-\tau}{\rho+\lambda}\phi(t) \\ +\frac{\lambda}{\rho+\lambda}\left[\begin{array}{c}\int_{\Phi_L}^{\widehat{V}_i^{-1}(D)}\left[(1-\tau)(1-\alpha)\widehat{V}_i(\phi) + \tau D\right]dF(\phi) \\ +\int_{\widehat{V}_i^{-1}(D)}^{\Phi_H}\widehat{V}_i(\phi)dF(\phi)\end{array}\right] \end{array}\right]$

and note that
$\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi)\right) \le \widehat{V}_1^{-1}\left(\widehat{D}_1(\phi)\right)$. Therefore, rewrite the expression for $\left(T\widehat{V}_1\right)(\phi(t))$ as
$\left(T\widehat{V}_1\right)(\phi(t)) = \frac{\tau}{\rho+\lambda}\rho\widehat{D}_1(\phi(t)) + \frac{1-\tau}{\rho+\lambda}\phi(t)$

$$+\frac{\lambda}{\rho+\lambda}\begin{bmatrix}\int_{\Phi_L}^{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}\left[(1-\tau)(1-\alpha)\widehat{V}_1(\phi)+\tau\widehat{D}_1(\phi(t))\right]dF(\phi)\\+\int_{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}\left[(1-\tau)(1-\alpha)\widehat{V}_1(\phi)+\tau\widehat{D}_1(\phi(t))\right]dF(\phi)\\\int_{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\Phi_H}\widehat{V}_1(\phi)\,dF(\phi)\end{bmatrix}$$

Now use the fact that for $\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)\le\phi\le\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)$, $\widehat{V}_1(\phi)\le\widehat{D}_1(\phi(t))$ so $(1-\tau)(1-\alpha)\widehat{V}_1(\phi)+\tau\widehat{D}_1(\phi(t))\le(1-\tau)(1-\alpha)\widehat{D}_1(\phi(t))+\tau\widehat{D}_1(\phi(t))\le D_1(\phi)$ to obtain

$$\left(T\widehat{V}_1\right)(\phi(t))\le\frac{\tau}{\rho+\lambda}\rho\widehat{D}_1(\phi(t))+\frac{1-\tau}{\rho+\lambda}\phi(t)$$

$$+\frac{\lambda}{\rho+\lambda}\begin{bmatrix}\int_{\Phi_L}^{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}\left[(1-\tau)(1-\alpha)\widehat{V}_1(\phi)+\tau\widehat{D}_1(\phi(t))\right]dF(\phi)\\+\int_{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}\widehat{D}_1(\phi(t))\,dF(\phi)\\\int_{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\Phi_H}\widehat{V}_1(\phi)\,dF(\phi)\end{bmatrix}$$

Since $\widehat{D}_1(\phi(t))\le\widehat{V}_1(\phi(t))<\widehat{V}_2(\phi(t))$ is a feasible choice of debt under $\widehat{V}_2(\phi(t))$, equation (58) implies

$$\left(T\widehat{V}_2\right)(\phi(t))\ge\frac{\tau}{\rho+\lambda}\rho\widehat{D}_1(\phi(t))+\frac{1-\tau}{\rho+\lambda}\phi(t)$$

$$+\frac{\lambda}{\rho+\lambda}\begin{bmatrix}\int_{\Phi_L}^{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}\left[(1-\tau)(1-\alpha)\widehat{V}_2(\phi)+\tau\widehat{D}_1(\phi(t))\right]dF(\phi)\\+\int_{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}\widehat{V}_2(\phi)\,dF(\phi)\\\int_{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\Phi_H}\widehat{V}_2(\phi)\,dF(\phi)\end{bmatrix}$$

and so

$$\left(T\widehat{V}_2\right)(\phi(t))-\left(T\widehat{V}_1\right)(\phi(t))\ge\frac{\lambda}{\rho+\lambda}\begin{bmatrix}(1-\tau)(1-\alpha)\int_{\Phi_L}^{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}\left[\widehat{V}_2(\phi)-\widehat{V}_1(\phi)\right]dF(\phi)\\+\int_{\widehat{V}_2^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}\left[\widehat{V}_2(\phi)-D_1(\phi)\right]dF(\phi)\\\int_{\widehat{V}_1^{-1}\left(\widehat{D}_1(\phi(t))\right)}^{\Phi_H}\left[\widehat{V}_2(\phi)-\widehat{V}_1(\phi)\right]dF(\phi)\end{bmatrix}\ge$$

0.

Therefore, the operator T satisfies the monotonicity property of the Blackwell conditions.

Discounting: Now suppose that $\widehat{V}_2(\phi(t))=\widehat{V}_1(\phi(t))+a$, where $a>0$ and note that $\min_{\Phi_L\le\phi(t)\le\Phi_H}\widehat{V}_2(\phi(t))\ge a$. Observe that $\widehat{V}_2^{-1}(D+a)=\widehat{V}_1^{-1}(D)$ so that

$$V_2(\phi(t),D+a)=\begin{bmatrix}\frac{\tau}{\rho+\lambda}\left(\rho+\lambda F\left(\widehat{V}_2^{-1}(D+a)\right)\right)(D+a)+\frac{1-\tau}{\rho+\lambda}\phi(t)\\+\frac{\lambda}{\rho+\lambda}(1-\tau)(1-\alpha)\int_{\Phi_L}^{\widehat{V}_2^{-1}(D+a)}\widehat{V}_2(\phi)\,dF(\phi)\\+\frac{\lambda}{\rho+\lambda}\int_{\widehat{V}_2^{-1}(D+a)}^{\Phi_H}\widehat{V}_2(\phi)\,dF(\phi)\end{bmatrix},$$

which implies that

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$$V_2\left(\phi\left(t\right),D+a\right)=\left[\begin{array}{c}\frac{\tau}{\rho+\lambda}\left(\rho+\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)\right)\left(D+a\right)+\frac{1-\tau}{\rho+\lambda}\phi\left(t\right)\\+\frac{\lambda}{\rho+\lambda}\left(1-\tau\right)\left(1-\alpha\right)\int_{\Phi_L}^{\widehat{V}_1^{-1}(D)}\left[\widehat{V}_1\left(\phi\right)+a\right]dF\left(\phi\right)\\+\frac{\lambda}{\rho+\lambda}\int_{\widehat{V}_1^{-1}(D)}^{\Phi_H}\left[\widehat{V}_1\left(\phi\right)+a\right]dF\left(\phi\right)\end{array}\right].$$

Therefore,

$$V_2\left(\phi\left(t\right),D+a\right)=\left[\begin{array}{c}V_1\left(\phi\left(t\right),D\right)+\frac{\tau}{\rho+\lambda}\left(\rho+\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)\right)a\\+\frac{\lambda}{\rho+\lambda}\left(1-\tau\right)\left(1-\alpha\right)\int_{\Phi_L}^{\widehat{V}_1^{-1}(D)}adF\left(\phi\right)\\+\frac{\lambda}{\rho+\lambda}\int_{\widehat{V}_1^{-1}(D)}^{\Phi_H}adF\left(\phi\right)\end{array}\right],$$

$$V_2\left(\phi\left(t\right),D+a\right)=\left[\begin{array}{c}V_1\left(\phi\left(t\right),D\right)\\+\frac{a}{\rho+\lambda}\left[\tau\rho+\tau\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)\right]\\+\frac{a}{\rho+\lambda}\left[\left(1-\tau\right)\left(1-\alpha\right)\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)+\lambda-\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)\right]\end{array}\right],$$

$$V_2\left(\phi\left(t\right),D+a\right)=\left[\begin{array}{c}V_1\left(\phi\left(t\right),D\right)\\\frac{a}{\rho+\lambda}\left[\begin{array}{c}\tau\rho+\lambda+\tau\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)-\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)\\+\left(1-\tau\right)\left(1-\alpha\right)\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)\end{array}\right]\end{array}\right],$$

$$V_2\left(\phi\left(t\right),D+a\right)=\left[\begin{array}{c}V_1\left(\phi\left(t\right),D\right)\\\frac{a}{\rho+\lambda}\left[\tau\rho+\lambda-\alpha\left(1-\tau\right)\lambda F\left(\widehat{V}_1^{-1}\left(D\right)\right)\right]\end{array}\right].$$

Hence

$$V_2\left(\phi\left(t\right),D+a\right)\le V_1\left(\phi\left(t\right),D\right)+\frac{\tau\rho+\lambda}{\rho+\lambda}a,$$

$$V_2\left(\phi\left(t\right),D+a\right)\le V_1\left(\phi\left(t\right),D\right)+\beta a,$$

where $\beta\equiv\frac{\tau\rho+\lambda}{\rho+\lambda}<1$.

Define $\widehat{D}_2\left(\phi\left(t\right)\right)\equiv\arg\max_{0\le D\le\widehat{V}_2(\phi(t))}V_2\left(\phi\left(t\right),D\right)$. Therefore, $\max_{0\le D\le\widehat{V}_2(\phi(t))}V_2\left(\phi\left(t\right),D\right)$ $=V_2\left(\phi\left(t\right),\widehat{D}_2\left(\phi\left(t\right)\right)\right)$ and $\widehat{D}_2\left(\phi\left(t\right)\right)\le V_2\left(\phi,\widehat{D}_2\left(\phi\left(t\right)\right)\right)$. Note that $V_2\left(\phi,\widehat{D}_2\left(\phi\left(t\right)\right)\right)\le$ $V_1\left(\phi,\widehat{D}_2\left(\phi\left(t\right)\right)-a\right)+\beta a$, (where $V_1\left(\phi,\widehat{D}_2\left(\phi\left(t\right)\right)-a\right)$ is well-defined because $\min_{\Phi_L\le\phi(t)\le\Phi_H}\widehat{V}_2\left(\phi\left(t\right)\right)\ge a$, so Proposition 4 implies that $\widehat{D}_2\left(\phi\left(t\right)\right)\ge a$, which implies

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that $\widehat{D}_2 (\phi(t)) - a \geq 0$) which implies $V_1 \left(\phi, \widehat{D}_2 (\phi(t)) - a \right) \geq V_2 \left(\phi(t), \widehat{D}_2 (\phi(t)) \right) - \beta a$
$\geq \widehat{D}_2 (\phi(t)) - \beta a = \widehat{D}_2 (\phi(t)) - a + (1 - \beta) a > \widehat{D}_2 (\phi(t)) - a$, so that $\widehat{D}_2 (\phi(t)) - a$
is a feasible level of debt under $V_1 (\phi(t), D)$. Therefore, $\max_{0 \leq D \leq \widehat{V}_1(\phi(t))} V_1 (\phi(t), D) \geq$
$\max_{0 \leq D \leq \widehat{V}_2(\phi(t))} V_2 (\phi(t), D) - \beta a$, so that $\left(T \left(\widehat{V}_1 + a \right) \right) (\phi(t)) \leq \left(T \widehat{V}_1 \right) (\phi(t)) + \beta a$. Hence,
the operator T satisfies the discounting property of the Blackwell conditions.

Therefore, the operator T takes concave functions on the domain $[\Phi_L, \Phi_H]$ into concave
functions on the domain $[\Phi_L, \Phi_H]$, and it has a unique fixed point $\widehat{V} (\phi(t))$. Therefore,
$\widehat{V} (\phi(t))$ is concave on the domain $[\Phi_L, \Phi_H]$. ■

Proof. of Corollary 7. The proof of Proposition 4 proves that if (1) $0 < \tau < \frac{\alpha}{1+\alpha}$; (2)
the positive-valued function $f (\phi)$ is non-decreasing for all $\phi \in [\Phi_L, \Phi_H]$; and (3) $\widehat{V} (\phi(t))$ is
concave, then $V (\phi(t), D)$ is strictly concave in D. ■

Proof. of Lemma 8. Since $D^* \leq \widehat{V} (\Phi_H)$ by definition, D^* is a feasible choice of debt when
$\phi = \Phi_H$. Since D^* maximizes $V (\phi, D)$, it is the optimal value of debt when $\phi = \Phi_H$. That is,
$\widehat{D} (\Phi_H) = D^*$ so $\widehat{V} (\Phi_H) = V (\Phi_H, D^*)$. Therefore, $D^* \leq \widehat{V} (\Phi_H) = V (\Phi_H, D^*)$. To prove
that $V (\Phi_L, D^*) \leq D^*$, suppose, contrary to what is to be proved, that $V (\Phi_L, D^*) > D^*$.
Therefore, D^* is a feasible choice of debt when $\phi = \Phi_L$ and it maximizes $V (\phi, D)$, so
$\widehat{D} (\Phi_L) = D^*$. Therefore, $\widehat{V} (\Phi_L) = V (\Phi_L, D^*) > D^* = \widehat{D} (\Phi_L)$, which contradicts Corollary
5. Therefore, $V (\Phi_L, D^*) \leq D^*$. ■

Proof. of Proposition 9. Since $V (\Phi_L, D^*) \leq D^* \leq V (\Phi_H, D^*)$ and $V (\phi(t), D^*)$ is strictly
increasing in $\phi(t)$, there is a unique value of $\phi \in [\Phi_L, \Phi_H]$ for which $V (\phi, D^*) = D^*$.
In fact, since $V (\phi(t), D)$ is linear in ϕ, we have $\frac{\Phi_H - \Phi_L}{V(\Phi_H, D^*) - V(\Phi_L, D^*)} = \frac{\phi^* - \Phi_L}{V(\phi^*, D^*) - V(\Phi_L, D^*)}$.
Use $V (\phi^*, D^*) = D^*$ and rearrange this equation to get the unique value of ϕ^*, which is
$\phi^* = \Phi_L + \frac{D^* - V(\Phi_L, D^*)}{V(\Phi_H, D^*) - V(\Phi_L, D^*)} [\Phi_H - \Phi_L]$. ■

Proof. of Corollary 10. By definition, D^* maximizes $V (\phi(t), D)$ so $\widehat{D} (\phi(t)) = D^*$ if
D^* is a feasible choice of debt, i.e., if $V (\phi(t), D^*) \geq D^*$. Since $V (\phi^*, D^*) = D^*$, D^* is
a feasible and hence the optimal level of debt when $\phi(t) = \phi^*$. Therefore, $\widehat{D} (\phi^*) = D^*$,
$\widehat{V} (\phi^*) = V \left(\phi^*, \widehat{D} (\phi^*) \right) = V (\phi^*, D^*) = D^*$. Since $\widehat{V} (\phi)$ is strictly increasing it is invertible.
Therefore, $\widehat{V} (\phi^*) = D^*$ implies that $\widehat{V}^{-1} (D^*) = \phi^*$. ■

Proof. of Proposition 11. If $\phi(t) \geq \phi^*$, $V (\phi(t), D^*) \geq V (\phi^*, D^*) = D^*$ so the constraint
$D \leq V (\phi(t), D)$ will not bind and hence the optimal value of debt is $\widehat{D} (\phi(t)) = D^*$.
Therefore, for any $\phi(t) \geq \phi^*$, the maximized value of $V (\phi(t), D)$ defined in equation (5) is

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$\widehat{V}(\phi(t)) = V\left(\phi(t), \widehat{D}(\phi(t))\right) = V(\phi(t), D^*)$ and equation (9) implies

$$\widehat{V}(\phi_2) - \widehat{V}(\phi_1) = \frac{1-\tau}{\rho+\lambda}(\phi_2 - \phi_1) \quad \text{for } \phi_1 \geq \phi^* \text{ and } \phi_2 \geq \phi^*. \tag{59}$$

Setting ϕ_1 in equation (59) equal to ϕ^*, and using Corollary 10 implies

$$\widehat{V}(\phi(t)) = \frac{1-\tau}{\rho+\lambda}[\phi(t) - \phi^*] + D^*, \quad \text{for } \phi(t) \geq \phi^*. \tag{60}$$

Since $\widehat{D}(\phi(t)) = D^*$ when $\phi(t) \geq \phi^*$, equations (60) and (14) imply

$$S(\phi(t)) = \frac{1-\tau}{\rho+\lambda}[\phi(t) - \phi^*], \quad \text{for } \phi(t) \geq \phi^*. \tag{61}$$

For values of $\phi(t) < \phi^*$, setting $D = D^*$ would violate the constraint $D \leq \widehat{V}(\phi(t))$ because $\widehat{V}(\phi(t)) \leq \widehat{V}(\phi^*) = D^*$, so the optimal value of D, $\widehat{D}(\phi(t))$, will be less than D^*. Since (Corollary 7) $V(\phi(t), D)$ is strictly concave in D, the constraint $V(\phi(t), D) \geq D$ will strictly bind for any $\phi(t) < \phi^*$ so that

$$\widehat{V}(\phi) \equiv \widehat{D}(\phi), \quad \text{for} \quad \phi(t) \leq \phi^*,$$

and

$$S(\phi(t)) = 0, \quad \text{for} \quad \phi(t) \leq \phi^*. \tag{62}$$

∎

Proof. of Corollary 12. (by contradiction) Suppose that the firm does not default at any time $t \in [t_1, t_2]$, and that $\widehat{D}(\phi(t_2)) < \widehat{D}(\phi(t_1))$. Therefore, there are consecutive arrival dates of new values of ϕ in $[t_1, t_2]$–say $t_1^0 < t_2^0$– such that $\widehat{D}(\phi(t_2^0)) < \widehat{D}(\phi(t_1^0))$. Since $\widehat{D}(\phi) \leq D^*$ for all ϕ in the support of $F(\phi)$, $\widehat{D}(\phi(t_2^0)) < \widehat{D}(\phi(t_1^0)) \leq D^*$, which implies that $\phi(t_2^0) < \phi^*$. Therefore, $\widehat{V}(\phi(t_2^0)) = \widehat{D}(\phi(t_2^0)) < \widehat{D}(\phi(t_1^0))$, where the equality follows from statement (2)(a) of Proposition 11. Since the firm arrives at time t_2^0 with debt equal to $\widehat{D}(\phi(t_1^0)) > \widehat{V}(\phi(t_2^0))$, the firm will default at time t_2^0, which is a contradiction. Therefore, $\widehat{D}(\phi(t_2)) \geq \widehat{D}(\phi(t_1))$ if the firm does not default in $[t_1, t_2]$. ∎

Proof. of Corollary 13. For $\phi(t) \geq \phi^*$, use Proposition 11 to substitute D^* for $\widehat{D}(\phi(t))$ and $\frac{1-\tau}{\rho+\lambda}[\phi(t) - \phi^*] + D^*$ for $\widehat{V}(\phi(t))$ in the definition of the leverage ratio, $L(\phi(t)) \equiv \frac{\widehat{D}(\phi(t))}{\widehat{V}(\phi(t))}$, and

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divide numerator and denominator by D^* to obtain $L\left(\phi\left(t\right)\right) = \frac{1}{\frac{1-\tau}{\rho+\lambda}\frac{\phi(t)-\phi^*}{D^*}+1}$. For $\phi\left(t\right) \leq \phi^*$,

Proposition 11 implies $\widehat{D}\left(\phi\left(t\right)\right) = \widehat{V}\left(\phi\left(t\right)\right)$ so the leverage ratio equals one. ∎

Proof. of Proposition 14. If $\phi\left(t\right) < \phi^*$, then $\widehat{D}\left(\phi\left(t\right)\right) = \widehat{V}\left(\phi\left(t\right)\right)$ so $\widehat{V}\left(\phi\right) < \widehat{D}\left(\phi\left(t\right)\right)$ if and only if $\widehat{V}\left(\phi\right) < \widehat{V}\left(\phi\left(t\right)\right)$, i.e., if and only if $\phi < \phi\left(t\right)$, since $\widehat{V}\left(\phi\right)$ is strictly increasing. Therefore, $P\left(\phi\left(t\right)\right) = \lambda F\left(\phi\left(t\right)\right) < \lambda F\left(\phi^*\right)$ and is strictly increasing in $\phi\left(t\right)$ for $\phi\left(t\right) < \phi^*$. Also, when $\phi\left(t\right) < \phi^*$, $Q\left(t\right) = \lambda \int_{\Phi_L}^{\phi(t)}\left[\widehat{D}\left(\phi\left(t\right)\right) - \left(1-\alpha\right)\widehat{V}\left(\phi\right)\right]dF\left(\phi\right)$, so $\frac{dQ(t)}{d\phi(t)} = \lambda\alpha f\left(\phi\left(t\right)\right)\widehat{D}\left(\phi\left(t\right)\right) + \lambda\widehat{D}'\left(\phi\left(t\right)\right)F\left(\phi\left(t\right)\right) > 0$, where the inequality follows from Proposition 11, (2)(a), which states that $\widehat{D}'\left(\phi\left(t\right)\right) > 0$ when $\phi\left(t\right) < \phi^*$. If $\phi\left(t\right) \geq \phi^*$, then $\widehat{D}\left(\phi\left(t\right)\right) = D^*$, so $\widehat{V}\left(\phi\right) < \widehat{D}\left(\phi\left(t\right)\right) = D^* = \widehat{V}\left(\phi^*\right)$ if and only if $\widehat{V}\left(\phi\right) < \widehat{V}\left(\phi\left(t\right)\right)$, i.e., if and only if $\phi < \phi^*$, since $\widehat{V}\left(\phi\right)$ is strictly increasing. Therefore, $P\left(\phi\left(t\right)\right) = \lambda F\left(\phi^*\right) \leq \lambda F\left(\phi\left(t\right)\right)$ for $\phi\left(t\right) \geq \phi^*$ and is invariant to $\phi\left(t\right)$. Therefore, $P\left(\phi\left(t\right)\right) = \lambda\min\left[F\left(\phi\left(t\right)\right), F\left(\phi^*\right)\right]$. When $\phi\left(t\right) \geq \phi^*$, $Q\left(t\right) = \lambda \int_{\Phi_L}^{\phi^*}\left[D^* - \left(1-\alpha\right)\widehat{V}\left(\phi\right)\right]dF\left(\phi\right)$ is invariant to $\phi\left(t\right)$. ∎

Proof. of Proposition 15. Suppose that $\widehat{D}\left(\phi\left(t\right)\right) = \widehat{V}\left(\phi\left(t\right)\right)$. Since $\widehat{V}\left(\phi\right)$ is strictly increasing, $\widehat{V}\left(\phi\right) \geq \widehat{V}\left(\phi\left(t\right)\right) = \widehat{D}\left(\phi\left(t\right)\right)$ for $\phi \geq \phi\left(t\right)$, with strict inequality if $\phi > \phi\left(t\right)$. Therefore,

$$\int_{\widehat{V}(\phi)\geq\widehat{D}(\phi(t))}\left[\widehat{V}\left(\phi\right) - \widehat{D}\left(\phi\left(t\right)\right)\right]dF\left(\phi\right) \geq 0, \tag{63}$$

with strict inequalty if $\phi\left(t\right) < \Phi_H$. Evaluate equation (7) at $D = \widehat{D}\left(\phi\left(t\right)\right) = \widehat{V}\left(\phi\left(t\right)\right) = V\left(\phi\left(t\right), \widehat{D}\left(\phi\left(t\right)\right)\right)$ to obtain

$$\widehat{V}\left(\phi\left(t\right)\right) = \widehat{D}\left(\phi\left(t\right)\right) + \frac{1-\tau}{\rho+\lambda}\left[\phi\left(t\right) - \left(\rho\widehat{D}\left(\phi\left(t\right)\right) + Q\left(t\right)\right)\right] \tag{64}$$
$$+ \frac{\lambda}{\rho+\lambda}\int_{\widehat{V}(\phi)\geq\widehat{D}(\phi(t))}\left[\widehat{V}\left(\phi\right) - \widehat{D}\left(\phi\left(t\right)\right)\right]dF\left(\phi\right).$$

Set $\widehat{D}\left(\phi\left(t\right)\right) = \widehat{V}\left(\phi\left(t\right)\right)$ and rearrange to obtain

$$\widehat{C}\left(\phi\left(t\right)\right) \equiv \left(1-\tau\right)\left[\phi\left(t\right) - \left(\rho\widehat{D}\left(\phi\left(t\right)\right) + Q\left(t\right)\right)\right] \tag{65}$$
$$= -\lambda\int_{\widehat{V}(\phi)\geq\widehat{D}(\phi(t))}\left[\widehat{V}\left(\phi\right) - \widehat{D}\left(\phi\left(t\right)\right)\right]dF\left(\phi\right) \leq 0,$$

where the inequality is implied by equation (63) and the inequality is strict if $\phi\left(t\right) < \Phi_H$. ∎

Proof. of Corollary 16. If $\phi(t) < \phi^*$, then $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$, and the proof of Proposition 15 implies that $\widehat{C}(\phi(t)) = -\lambda \int_{\phi(t)}^{\Phi_H} \left[\widehat{V}(\phi) - \widehat{D}(\phi(t)) \right] dF(\phi)$. Differentiating $\widehat{C}(\phi(t))$ with respect to $\phi(t)$ and using $\widehat{D}(\phi(t)) = \widehat{V}(\phi(t))$ yields $\widehat{C}'(\phi(t)) = \lambda \int_{\phi(t)}^{\Phi_H} \widehat{D}'(\phi(t)) dF(\phi) > 0$, where the inequality follows from part (2)(a) of Proposition 11. For $\phi(t) \geq \phi^*$, $\widehat{D}(\phi(t)) = D^*$ and the default premium $Q(t) = \widetilde{Q}(D^*)$ are invariant to $\phi(t)$. Therefore, $\widehat{C}(\phi(t)) = (1-\tau)\left[\phi(t) - \left(\rho D^* + \widetilde{Q}(D^*)\right)\right]$ so $\widehat{C}'(\phi(t)) = 1 - \tau > 0$. $\blacksquare$

Proof. of Lemma 17. Since $\widehat{V}(\phi^*) = V\left(\phi^*, \widehat{D}(\phi^*)\right)$, $\widehat{V}'(\phi^*) = \frac{\partial V\left(\phi^*, \widehat{D}(\phi^*)\right)}{\partial \phi} + \frac{\partial V\left(\phi^*, \widehat{D}(\phi^*)\right)}{\partial D} \widehat{D}'(\phi^*)$. Equation (19) and $\omega_2 = 0$ imply $\frac{\partial V(\phi^*, D^*)}{\partial D} = -\omega_1 \leq 0$ and Proposition 11 implies that $\widehat{D}'(\phi^*) \geq 0$. Therefore, $\widehat{V}'(\phi^*) \leq \frac{\partial V\left(\phi^*, \widehat{D}(\phi^*)\right)}{\partial \phi} = \frac{1-\tau}{\rho+\lambda}$. But Proposition 3 implies that $\widehat{V}'(\phi^*) \geq \frac{1-\tau}{\rho+\lambda}$. Therefore, $\frac{1-\tau}{\rho+\lambda} \leq \widehat{V}'(\phi^*) \leq \frac{1-\tau}{\rho+\lambda}$, which implies $\widehat{V}'(\phi^*) = \frac{1-\tau}{\rho+\lambda}$. Finally, the identity $\widehat{V}^{-1}\left(\widehat{V}(\phi)\right) = \phi$ implies $\widehat{V}^{-1\prime}\left(\widehat{V}(\phi^*)\right) \widehat{V}'(\phi^*) = 1$, so $\widehat{V}^{-1\prime}\left(\widehat{V}(\phi^*)\right) = \frac{1}{\widehat{V}'(\phi^*)} = \frac{\rho+\lambda}{1-\tau}$. Finally, Corollary 10 states that $\widehat{V}(\phi^*) = D^*$, so that $\widehat{V}^{-1\prime}(D^*) = \frac{\rho+\lambda}{1-\tau}$. $\blacksquare$

Proof. of Proposition 18. If $\omega_1 > 0$, the complementary slackness condition in equation (20) implies $D^* = \widehat{V}(\Phi_L)$ and then Proposition 3 and Corollary 10 imply $\phi^* = \Phi_L$. If $\omega_2 > 0$, the complementary slackness condition in equation (21) implies $D^* = \widehat{V}(\Phi_H)$ and then Proposition 3 and Corollary 10 imply $\phi^* = \Phi_H$. $\blacksquare$

Proof. of Proposition 19. Assume that Regime I prevails so that $\omega_1 > 0$. Proposition 18 states that $D^* = \widehat{V}(\Phi_L)$ and $\phi^* = \Phi_L$. Evaluate equation (9) at $\phi(t) = \phi^* = \Phi_L$ and substitute $\widehat{V}(\Phi_L)$ for $V(\phi, D)$ on the left hand side of equation (9) and for D on the right hand side of equation (9) to obtain

$$\widehat{V}(\Phi_L) = \frac{\tau}{\rho+\lambda}\rho \widehat{V}(\Phi_L) + \frac{1-\tau}{\rho+\lambda}\Phi_L + \frac{\lambda}{\rho+\lambda}E\left\{\widehat{V}(\phi)\right\} \quad \text{in Regime I.} \tag{66}$$

Since $\widehat{D}(\Phi_L) = D^* = \widehat{V}(\Phi_L)$ so $\phi^* = \Phi_L$, Proposition 11 implies

$$\widehat{V}(\phi(t)) = \frac{1-\tau}{\rho+\lambda}[\phi(t) - \Phi_L] + \widehat{V}(\Phi_L) \quad \text{in Regime I.} \tag{67}$$

Taking the unconditional expectation of both sides of equation (67) yields

$$E\left\{\widehat{V}(\phi)\right\} = \frac{1-\tau}{\rho+\lambda}[E\{\phi\} - \Phi_L] + \widehat{V}(\Phi_L) \quad \text{in Regime I.} \tag{68}$$

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Substitute equation (68) into equation (66) to obtain $\widehat{V}(\Phi_L) = \frac{\tau}{\rho+\lambda}\rho\widehat{V}(\Phi_L) + \frac{1-\tau}{\rho+\lambda}\Phi_L + \frac{\lambda}{\rho+\lambda}\left[\frac{1-\tau}{\rho+\lambda}[E\{\phi\} - \Phi_L] + \widehat{V}(\Phi_L)\right]$. Multiply both sides of this equation by $\rho + \lambda$ to obtain $\rho\widehat{V}(\Phi_L) = \tau\rho\widehat{V}(\Phi_L) + (1-\tau)\Phi_L + \lambda\frac{1-\tau}{\rho+\lambda}[E\{\phi\} - \Phi_L]$, which can be rewritten as $\rho\widehat{V}(\Phi_L) = \Phi_L + \lambda\frac{1}{\rho+\lambda}[E\{\phi\} - \Phi_L] = \frac{\rho}{\rho+\lambda}\Phi_L + \frac{\lambda}{\rho+\lambda}E\{\phi\}$. Therefore, $\widehat{V}(\Phi_L) = \frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$, so $\widehat{D}(\phi(t)) = D^* = \widehat{V}(\Phi_L) = \frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$. Proposition 11 implies that $S(\phi(t)) = \frac{1-\tau}{\rho+\lambda}[\phi(t) - \Phi_L]$ and $\widehat{V}(\phi(t)) = D^* + S(\phi(t))$. ∎

Lemma 29 below is not stated in the text. It is stated (and proved) here, immediately preceding the proof of Proposition 20, because its only role is to help prove that proposition.

Lemma 29 . *If $F(\Phi_L) = 0$, $F(\Phi_H) = 1$ and $f'(\phi) \geq 0$ everywhere on the support $[\Phi_L, \Phi_H]$, then $\Phi_L \leq \Phi_H - \frac{1}{f(\Phi_H)}$ and $E\{\phi\} \leq \Phi_H - \frac{1}{2f(\Phi_H)}$.*

Proof. of Lemma 29. Let $G(\phi)$ be the uniform distribution on $[\Phi_0, \Phi_H]$, where $\Phi_0 \equiv \Phi_H - \frac{1}{f(\Phi_H)}$ and $g(\phi) \equiv G'(\phi) = f(\Phi_H)$ for all $\phi \in [\Phi_0, \Phi_H]$. Since $g'(\phi) \equiv 0$, $f'(\phi) \geq g'(\phi)$, which, along with $f(\Phi_H) = g(\Phi_H)$, implies that $f(\phi) \leq g(\phi)$ for all ϕ in $[\Phi_0, \Phi_H]$. Hence, $F(\phi) = 1 - \int_\phi^{\Phi_H} f(x)\,dx \geq 1 - \int_\phi^{\Phi_H} g(x)\,dx = G(\phi)$ for all ϕ in $[\Phi_0, \Phi_H]$ and $F(\Phi_0) \geq G(\Phi_0) = 0$ so that $G(\phi)$ first-order stochastically dominates $F(\phi)$. To prove that $\Phi_L \leq \Phi_0$, suppose not, i.e., suppose that $\Phi_L > \Phi_0$. Then $F(\Phi_L) \geq G(\Phi_L) > 0$, which contradicts $F(\Phi_L) = 0$. Hence $\Phi_L \leq \Phi_0$. Let $E_F\{\phi\}$ be the expected value of ϕ under the distribution $F(\phi)$ and let $E_G\{\phi\}$ be the expected value of ϕ under the distribution $G(\phi)$. Then $E_F\{\phi\} \leq E_G\{\phi\}$ is an immediate consequence of the result that $G(\phi)$ first-order stochastically dominates $F(\phi)$. Since $E_G\{\phi\} = \frac{1}{2}(\Phi_0 + \Phi_H) = \frac{1}{2}\left(2\Phi_H - \frac{1}{f(\Phi_H)}\right) = \Phi_H - \frac{1}{2}\frac{1}{f(\Phi_H)}$, $E_F\{\phi\} \leq \Phi_H - \frac{1}{2}\frac{1}{f(\Phi_H)}$. ∎

Proof. of Proposition 20. Define $\gamma \equiv \frac{\lambda}{\rho}$ and use the definition of τ_L in equation (32) along with $f(\Phi_L) \leq f(\Phi_H)$, Lemma 29 (which is stated and proved immediately before this proof), and the assumption that $\Phi_L + \frac{\lambda}{\rho}E\{\phi\} > 0$ to obtain

$$\tau_L \leq \alpha\frac{\gamma}{1+\gamma}f(\Phi_H)\left((1+\gamma)^2\Phi_H - \frac{1+\gamma}{f(\Phi_H)}\left(1+\frac{\gamma}{2}\right)\right). \tag{69}$$

Use the definition of τ_H in equation (33), the definition $\gamma \equiv \frac{\lambda}{\rho}$, and the fact that

$\alpha \frac{\lambda}{\rho + \lambda} f(\Phi_H)(1 - \alpha) \lambda E\left\{\widehat{V}(\phi)\right\} \geq 0$ to obtain

$$\tau_H \geq \alpha \frac{\gamma}{1 + \gamma} f(\Phi_H) \Phi_H. \tag{70}$$

Subtract τ_L in equation (69) from τ_H in equation (70) to obtain

$$\tau_H - \tau_L \geq \alpha \gamma \frac{2 + \gamma}{1 + \gamma}\left[-\gamma \Phi_H f(\Phi_H) + \frac{1}{2}(1 + \gamma)\right]. \tag{71}$$

Multiply both sides of equation (70) by $-\frac{1+\gamma}{\alpha}$ to obtain

$$-\gamma f(\Phi_H) \Phi_H \geq -\frac{1 + \gamma}{\alpha} \tau_H. \tag{72}$$

Equations (71) and (72) imply

$$\tau_H - \tau_L \geq \gamma(2 + \gamma)\left(\frac{\alpha}{2} - \tau_H\right). \tag{73}$$

Therefore, if $\tau_H < \frac{\alpha}{2}$, then $\tau_H - \tau_L > 0$ and the interval (τ_L, τ_H) is not degenerate. ∎

Proof. of Corollary 21. From Proposition 20, $\tau_H < \frac{\alpha}{2}$ implies that $[\tau_L, \tau_H]$ is not degenerate, so that there exists a $\tau \in [\tau_L, \tau_H]$. Equations (29) and (32) imply that $H(\Phi_L) = \rho^2(\tau - \tau_L)$, so $\tau \geq \tau_L$ implies $H(\Phi_L) \geq 0$. Equations (30) and (33) imply that $H(\Phi_H) = (\rho + \lambda)^2(\tau - \tau_H)$, so $\tau \leq \tau_H$ implies $H(\Phi_H) \leq 0$. Therefore, $H(\phi) = 0$ has at least one root in $[\Phi_L, \Phi_H]$. To show that the root is unique differentiate $H(x)$ with respect to x and evaluate $H'(x)$ at $H(x) = 0$ to obtain $H'(\phi^*)|_{H(\phi^*)=0} = \left((2\tau - \alpha)[\rho + \lambda F(\phi^*)] - \alpha(1 - \alpha)\lambda(\rho + \lambda)\widehat{V}(\phi^*)f(\phi^*)\right)\lambda f(\phi^*) - \tau[\rho + \lambda F(\phi^*)]^2 \frac{f'(\phi^*)}{f(\phi^*)} < 0$, where the inequality follows from the assumptions that $\tau < \frac{\alpha}{2} < \frac{\alpha}{1+\alpha}$ and $f'(\phi^*) \geq 0$. ∎

Proof. of Proposition 22. The proof proceeds by presenting, for each regime, the value of D^* that satisfies the first-order condition in equation (19) and leads to satisfaction of the complementary slackness conditions in equations (20) and (21). Because the maximand in equation (18) is strictly concave in D, a value of D that satisfies equations (19), (20), and (21) is the unique value of D^*.

(I) Suppose that $\tau < \tau_L$. In this case, the proof proceeds by showing that equations (19), (20), and (21) are satisfied by $D^* = \widehat{V}(\Phi_L)$, $\omega_1 > 0$, and $\omega_2 = 0$. Suppose that

$D^* = \widehat{V}(\Phi_L) < \widehat{V}(\Phi_H)$. Since $D^* < \widehat{V}(\Phi_H)$, complementary slackness in equation (21) implies $\omega_2 = 0$. Therefore, $\widehat{V}^{-1\prime}(D^*) = \frac{\rho+\lambda}{1-\tau}$ (Lemma 17), so that equation (12) can be written as

$$\frac{\partial V\left(\phi(t),\widehat{V}(\Phi_L)\right)}{\partial D} = \frac{\tau}{\rho+\lambda}\left(\rho + \lambda F\left(\widehat{V}^{-1}\left(\widehat{V}(\Phi_L)\right)\right)\right) - \alpha\lambda\widehat{V}(\Phi_L)f\left(\widehat{V}^{-1}\left(\widehat{V}(\Phi_L)\right)\right).$$

(74)

Use $\widehat{V}^{-1}\left(\widehat{V}(\Phi_L)\right) = \Phi_L$, $F(\Phi_L) = 0$, and Proposition 19 to substitute $\frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]$ for $\widehat{V}(\Phi_L)$ to rewrite equation (74) as

$$\frac{\partial V\left(\phi(t),\widehat{V}(\Phi_L)\right)}{\partial D} = \frac{\tau}{\rho+\lambda}\rho - \alpha\lambda\frac{1}{\rho+\lambda}\left[\Phi_L + \frac{\lambda}{\rho}E\{\phi\}\right]f(\Phi_L).$$

(75)

Use the definition of τ_L in equation (32) to rewrite equation (75) as[24]

$$\frac{\partial V\left(\phi(t),\widehat{V}(\Phi_L)\right)}{\partial D} = \frac{\rho}{\rho+\lambda}(\tau - \tau_L) < 0.$$

(76)

Since $\frac{\partial V(\phi,D^*)}{\partial D} < 0$, equation (19) implies $\omega_1 > 0$. Therefore, $\omega_1 > 0$ and $\omega_2 = 0$, so the firm is in Regime I when $\tau < \tau_L$. In Regime I, $\widehat{D}(\phi(t)) = D^* = \widehat{V}(\Phi_L)$.

(II) Suppose that $\tau_L \leq \tau \leq \tau_H$. Recall from subsection 5.1 that if $\tau_L \leq \tau \leq \tau_H$, there is a unique $D^* \in \left[\widehat{V}(\Phi_L), \widehat{V}(\Phi_H)\right]$ for which $\frac{\partial V(\phi(t),D^*)}{\partial D} = 0$. Thus, the first-order condition in equation (19) implies that $\omega_1 = \omega_2$. Since $\widehat{V}(\Phi_L) \neq \widehat{V}(\Phi_H)$, the complementary slackness conditions in equations (20) and (21) imply that at least one of ω_1 and ω_2 must be zero. Therefore, $\omega_1 = \omega_2 = 0$, so the firm is in Regime II.

(III) Suppose that $\frac{\alpha}{1+\alpha} > \tau > \tau_H$ and $\tau_H < \frac{\alpha}{2}$. The proof proceeds by showing that $D^* = \widehat{V}(\Phi_H)$, $\omega_1 = 0$, and $\omega_2 > 0$ satisfy equations (19), (20), and (21). Suppose that $D^* = \widehat{V}(\Phi_H) > \widehat{V}(\Phi_L)$ so that the complementary slackness condition in equation (20) implies that $\omega_1 = 0$. Evaluate $\frac{\partial V(\phi(t),D)}{\partial D}$ in equation (13) at $D = \widehat{V}(\Phi_H)$ and use

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[24] The partial derivative in equation (76) is the right-hand derivative with respect to D at $D = \widehat{V}(\Phi_L)$. Inspection of footnote 2 reveals that the left-hand derivative at $\widehat{V}(\Phi_L)$ is $\tau\frac{\rho}{\rho+\lambda}$, which exceeds the right hand derivative because $\tau_L > 0$.

$\widehat{V}^{-1}\left(\widehat{V}\left(\Phi_H\right)\right) = \Phi_H$ and $F\left(\widehat{V}^{-1}\left(\Phi_H\right)\right) = 1$ to obtain

$$\frac{\partial V\left(\phi\left(t\right), \widehat{V}\left(\Phi_H\right)\right)}{\partial D} \geq \tau - \alpha\lambda\widehat{V}\left(\Phi_H\right)f\left(\Phi_H\right). \tag{77}$$

Use Lemma 30 to substitute $\frac{1}{\rho+\lambda}\left[\Phi_H + (1-\alpha)\lambda E\left\{\widehat{V}\left(\phi\right)\right\}\right]$ for $\widehat{V}\left(\Phi_H\right)$ in equation (77) to obtain

$$\frac{\partial V\left(\phi\left(t\right), \widehat{V}\left(\Phi_H\right)\right)}{\partial D} \geq \tau - \alpha\frac{\lambda}{\rho+\lambda}\left[\Phi_H + (1-\alpha)\lambda E\left\{\widehat{V}\left(\phi\right)\right\}\right]f\left(\Phi_H\right). \tag{78}$$

Use the definition of τ_H in equation (33) to rewrite equation (78) as

$$\frac{\partial V\left(\phi\left(t\right), \widehat{V}\left(\Phi_H\right)\right)}{\partial D} \geq \tau - \tau_H > 0, \tag{79}$$

which satisfies $\frac{\partial V(\phi, D^*)}{\partial D} = \omega_2 - \omega_1 > 0$ when $\omega_1 = 0$ and $\omega_2 > 0$. Therefore, the firm is in Regime III when $\tau > \tau_H$. ∎

Proof. of Proposition 23. Denote the current time as t_0 and the arrival times of new values of ϕ consecutively as $t_1 < t_2 < t_3, ...$, where t_1 is the first arrival time after t_0. Consider the following feasible financing plan under $G_m\left(\phi\right)$: (1) if $\phi\left(t\right) \in \left[\Phi_L\left(m\right), \Phi_H\left(0\right)\right]$, set $D\left(\phi\left(t\right);m\right) = \widehat{D}\left(\phi\left(t\right);0\right)$ and pay the same default premium that would be paid under $F\left(\phi\right)$, so that the firm's cash flow, say $\widetilde{C}\left(\phi\left(t\right)\right)$, is the same as under $F\left(\phi\right)$ when the firm follows the optimal policy, say $C_F\left(\phi\left(t\right)\right)$; (2) if $\phi\left(t\right) \in \left[\Phi_H\left(0\right), \Phi_H\left(m\right)\right]$, set $D\left(\phi\left(t\right);m\right) = \widehat{D}\left(\Phi_H\left(0\right);0\right)$ and pay the same default premium that would be paid under $F\left(\phi\right)$ when $\phi\left(t\right) = \Phi_H\left(0\right)$, so that the firm's cash flow under this policy exceeds the cash flow when $\phi\left(t\right) = \Phi_H\left(0\right)$; (3) at time t_j, $j = 1, 2, 3, ...$, (a) if $\phi\left(t_j\right) \in \left[\Phi_L\left(m\right), \Phi_H\left(0\right)\right]$, default if any only if $\phi\left(t_j\right) < \min\left[\phi\left(t_{j-1}\right), \phi^*\left(0\right)\right]$, that is, if and only if the firm would optimally default under $F\left(\phi\right)$ when $\phi = \phi\left(t_j\right)$ and (b) if $\phi\left(t_j\right) \in \left[\Phi_H\left(0\right), \Phi_H\left(m\right)\right]$, do not default. For $\phi\left(t_0\right) \in \left[\Phi_L\left(m\right), \Phi_H\left(0\right)\right]$, the firm will have the same cash flow during the interval $\left[t_0, t_1\right)$ as it would under the optimal policy under $F\left(\phi\right)$. For subsequent time intervals $\left[t_j, t_{j+1}\right)$, the expected present value of the cash flow over the interval is $\widetilde{v}$ $= \frac{1}{\rho+\lambda}\int_{\Phi_L(m)}^{\Phi_H(m)}\widetilde{C}\left(\phi\right)f\left(\phi - m\right)d\phi$ under $G_m\left(\phi\right)$ and is $v_F \equiv \frac{1}{\rho+\lambda}\int_{\Phi_L(0)}^{\Phi_H(0)}C_F\left(\phi\right)f\left(\phi\right)d\phi$ un-

der $F(\phi)$. For $\phi \in [\Phi_L(m), \Phi_H(0)]$, $\widetilde{C}(\phi) = C_F(\phi) > C_F(\phi - m)$ since $C_F(\phi)$ is strictly increasing and for $\phi \in (\Phi_H(0), \Phi_H(m)]$, $\widetilde{C}(\phi) \geq \widetilde{C}(\Phi_H(0)) = C_F(\Phi_H(0)) > C_F(\phi - m)$ since $\phi - m \leq \Phi_H(0)$ and $C_F(\phi)$ is strictly increasing. Therefore, $\widetilde{v} > \frac{1}{\rho+\lambda} \int_{\Phi_L(m)}^{\Phi_H(m)} C_F(\phi - m) f(\phi - m) d\phi = \frac{1}{\rho+\lambda} \int_{\Phi_L(0)}^{\Phi_H(0)} C_F(\phi) f(\phi) d\phi = v_F$. Therefore, it suffices to prove that there is a positive probability that the firm following the policy above under $G(\phi)$ does not default at time t_1. The firm will default at time t_1 if and only if $\phi(t_1) \in [\Phi_L(m), \Phi_H(0)]$ and $\phi(t_1) < \min[\phi(t_0), \phi^*(0)]$. Therefore, the firm will not default if $\phi(t_1) \geq \Phi_H(0) \geq \phi^*(0)$. Therefore, the probability that the firm does not at time t_1 is at least $1 - G_m(\Phi_H(0)) > 0$ since $G_m(\Phi_H(0)) = F(\Phi_H(0) - m) < 1$. ∎

Proof. of Lemma 24. Suppose that the distribution shifts to the right by $d > 0$ and consider whether $\widehat{\phi} \equiv \phi^*(m) + d$ satisfies the first-order condition in equation (52) under the new distribution. That is, consider whether $\Psi \equiv \frac{\tau}{\rho+\lambda}\left(\rho + \lambda F\left(\widehat{\phi} - (m+d)\right)\right) - \alpha\lambda\widehat{V}\left(\widehat{\phi}, m+d\right) f\left(\widehat{\phi} - (m+d)\right)$ equals zero. The definition of $\widehat{\phi}$ implies that $\widehat{\phi} - (m+d) = \phi^*(m) - m$, so $\Psi = \frac{\tau}{\rho+\lambda}(\rho + \lambda F(\phi^*(m) - m)) - \alpha\lambda\widehat{V}(\phi^*(m) + d, m+d) f(\phi^*(m) - m)$. Use equation (52) to replace $\frac{\tau}{\rho+\lambda}(\rho + \lambda F(\phi^*(m) - m))$ by $\alpha\lambda\widehat{V}(\phi^*(m), m) f(\phi^*(m) - m)$ in Ψ to obtain $\Psi = \alpha\lambda f(\phi^*(m) - m)\left[\widehat{V}(\phi^*(m), m) - \widehat{V}(\phi^*(m) + d, m+d)\right]$. Since $d > 0$, we have $\widehat{V}(\phi^*(m) + d, m+d) > \widehat{V}(\phi^*(m), m+d) > \widehat{V}(\phi^*(m), m)$ where the first inequality follows from Proposition 3 and the second inequality follows from Proposition 23. Since $\alpha\lambda f(\phi^*(m) - m) > 0$ and $\widehat{V}(\phi^*(m) + d, m+d) > \widehat{V}(\phi^*(m), m)$, we have $\Psi < 0$, that is, $\frac{\partial V\left(\phi(t), \widehat{V}(\widehat{\phi}, m+d); m+d\right)}{\partial D} < 0$ and since (from Corollary 7) $V(\phi(t), D; m)$ is strictly concave in D, the value of D^* is less than $\widehat{V}\left(\widehat{\phi}, m+d\right)$ so $\phi^*(m+d) < \widehat{\phi} \equiv \phi^*(m) + d$. Therefore, for any $d > 0$, $\frac{\phi^*(m+d) - \phi^*(m)}{d} < 1$, so $\phi^{*\prime}(m) < 1$. ∎

Proof. of Proposition 25. Proposition 14 implies that $P(\phi(t); m) = \lambda \min[F(\phi(t) - m), F(\phi^*(m) - m)]$. $\frac{dF(\phi^*(m) - m)}{dm} = f(\phi^*(m) - m)(\phi^{*\prime}(m) - 1) < 0$, where the strict inequality follows from $f(\phi^*(m) - m) > 0$ and Lemma 24. Also, $\frac{dF(\phi(t) - m)}{dm} = -f(\phi(t) - m) < 0$. Therefore, $\frac{dP(\phi(t); m)}{dm} < 0$. ∎

Proof. of Proposition 26. Corollary 10 implies that $sign\left(D^{*\prime}(m)\right) = sign\left(\frac{d\widehat{V}(\phi^*(m); m)}{dm}\right)$. Since $\frac{\tau}{\rho+\lambda} > 0$, $\alpha\lambda f(\phi^*(0)) > 0$, and $\phi^{*\prime}(m) < 1$ (Lemma 24), equation (54) implies $sign\left(\frac{d\widehat{V}(\phi^*(m); m)}{dm}\right) = -sign(\chi(\phi^*(m) - m))$. Therefore, $sign\left(\frac{d\widehat{V}(\phi^*(m); m)}{dm}\right) = sign\left(D^{*\prime}(m)\right) = -sign(\chi(\phi^*(m) - m))$. ∎

Proof. of Proposition 27. The proof of Proposition 26 implies that if $\chi(\phi^*(m) - m) \geq 0$,

then $\frac{d\widehat{V}(\phi^*(m);m)}{dm} \leq 0$. Since $\frac{d\widehat{V}(\phi^*(m);m)}{dm} = \frac{\partial\widehat{V}(\phi^*(m);m)}{\partial\phi}\phi^{*\prime}(m) + \frac{\partial\widehat{V}(\phi^*(m);m)}{\partial m}$, it follows that if $\chi(\phi^*(m) - m) \geq 0$, then $\frac{\partial\widehat{V}(\phi^*(m);m)}{\partial\phi}\phi^{*\prime}(m) = \frac{d\widehat{V}(\phi^*(m);m)}{dm} - \frac{\partial\widehat{V}(\phi^*(m);m)}{\partial m} \leq -\frac{\partial\widehat{V}(\phi^*(m);m)}{\partial m} < 0$, where the final inequality follows from Proposition 23. Finally, since $\frac{\partial\widehat{V}(\phi^*(m);m)}{\partial\phi} > 0$ (Proposition 3), $\phi^{*\prime}(m) \leq -\frac{\frac{\partial\widehat{V}(\phi^*(m);m)}{\partial m}}{\frac{\partial\widehat{V}(\phi^*(m);m)}{\partial\phi}} < 0$. $\blacksquare$

Proof. of Corollary 28. Substitute $\theta \equiv \frac{\eta}{1-e^{-\eta\delta}}$ into equation (56) and rearrange to obtain[25] $\chi(\phi) = \left(1 + \frac{\lambda}{\rho} - e^{-\eta\delta}\right)\frac{\rho\eta}{1-e^{-\eta\delta}}$. Since $\frac{\rho\eta}{1-e^{-\eta\delta}} > 0$ for any $\eta \neq 0$, $\chi(\phi) \gtreqless 0$ as $\eta\delta \gtreqless -ln\left(1 + \frac{\lambda}{\rho}\right)$. This inequality, together with Proposition 26, implies that $D^{*\prime}(m) \lesseqgtr 0$ as $\eta\delta \gtreqless -ln\left(1 + \frac{\lambda}{\rho}\right)$. $\blacksquare$

B   Appendix: Solution to ODE in equation (41)

The ODE in equation (41) holds for $\phi(t) \leq \phi^*$. To solve this ODE, rewrite this equation in canonical form as

$$\widehat{D}'(\phi(t)) + \frac{\alpha(1-\tau)\lambda f(\phi(t))}{(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))}\widehat{D}(\phi(t)) = \frac{1-\tau}{(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))}. \tag{B.1}$$

Next multiply both sides of equation (B.1) by the integrating factor $\exp\left(\int \frac{\alpha(1-\tau)\lambda f(\phi(t))}{(1-\tau)\rho+\lambda-\tau\lambda F(\phi(t))}d\phi(t)\right)$ $= [(1-\tau)\rho + \lambda - \tau\lambda F(\phi(t))]^{-\frac{\alpha(1-\tau)}{\tau}}$ to obtain

$$\begin{bmatrix} [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}}D'(\phi(t)) + \\ [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}-1} \\ \times \alpha(1-\tau)\lambda f(\phi(t))D(\phi(t)) \end{bmatrix} = (1-\tau)[(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}-1}. \tag{B.2}$$

Integrating both sides of equation (B.2) yields

$$\begin{pmatrix} [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}} \\ \times D(\phi(t)) \end{pmatrix} = \begin{bmatrix} C \\ + (1-\tau)\int [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}-1}d\phi(t) \end{bmatrix}, \tag{B.3}$$

---

[25]For $\eta = 0$, use L'Hopital's Rule to obtain $\lim_{\eta\to 0}\chi(\phi) = \lim_{\eta\to 0}\frac{\left(1+\frac{\lambda}{\rho}-e^{-\eta\delta}\right)\rho\eta}{1-e^{-\eta\delta}} = \lim_{\eta\to 0}\frac{\left(1+\frac{\lambda}{\rho}-e^{-\eta\delta}\right)\rho+\left(\delta e^{-\eta\delta}\right)\rho\eta}{\delta e^{-\eta\delta}} = \frac{\lambda}{\delta}$. Equivalently, $\lim_{\eta\to 0}\theta = \lim_{\eta\to 0}\frac{1}{\delta e^{-\eta\delta}} = \frac{1}{\delta}$, so $\lim_{\eta\to 0}\chi(\phi) = \lim_{\eta\to 0}(\rho\eta + \lambda\theta) = \frac{\lambda}{\delta} > 0$. Therefore, if $\eta = 0$, then $\chi(\phi) > 0$ and Proposition 26 implies $D^{*\prime}(m) < 0$.

where C is a constant of integration. Divide both sides of equation (B.3) by $[(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}}$ to obtain

$$
\begin{aligned}
D(\phi(t)) \;=\;& [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{\frac{\alpha(1-\tau)}{\tau}} \\
& \times \left[C + (1-\tau) \int [(1-\tau)\rho + \lambda(1-\tau F(\phi(t)))]^{-\frac{\alpha(1-\tau)}{\tau}-1} d\phi(t) \right].
\end{aligned}
\tag{B.4}
$$

C   Online Appendix: Closed-Form Solutions: $\alpha = 1$ and $F(\phi)$ is Uniform on $[\Phi_L, \Phi_H]$

In this online appendix, I derive closed-form solutions for optimal debt and shareholders' equity in the special case in which (1) $\alpha = 1$, so that default completely destroys the value of the firm; and (2) the unconditional distribution $F(\phi)$ is uniform. The assumption that $\alpha = 1$ simplifies the solution of the shareholders' problem in Regime II by permitting the calculation of ϕ^*, $\widehat{D}(\phi(t))$, and $\widehat{V}(\phi(t))$ for $\phi(t) \geq \phi^*$ without having to simultaneously calculate $\widehat{V}(\phi(t))$ for values of $\phi(t) < \phi^*$.[26]

It will be convenient to represent the uniform distribution on $[\Phi_L, \Phi_H]$ by the mean $\mu \equiv \frac{1}{2}(\Phi_L + \Phi_H)$ and by $\delta \equiv \Phi_H - \Phi_L$. Therefore, $f(x) = \frac{1}{\delta}$, $F(x) = \frac{x - \mu + \frac{1}{2}\delta}{\delta}$, and hence $x = \mu - \frac{1}{2}\delta + \delta F(x)$, for $x \in [\Phi_L, \Phi_H]$.

Proposition 19 contains closed-form solutions for $\widehat{D}(\phi(t))$, $S(\phi(t))$, and $\widehat{V}(\phi(t))$ in Regime I for the more general case in which $0 < \alpha \leq 1$ and $f'(\phi) \geq 0$, which includes the uniform distribution as a special case. This Appendix presents closed-form solutions that apply in Regime II and Regime III in the special case with $\alpha = 1$ and uniform $F(\phi)$.

C.1   Closed-Form Solutions for Regime II

Consider Regime II. It is straighforward to show that for $x \in [\Phi_L, \Phi_H]$

$$\int_x^{\Phi_H} \phi \, dF(\phi) = (1 - F(x))\left(\mu + \frac{1}{2}\delta F(x)\right) = \mu - \left(\mu - \frac{1}{2}\delta\right) F(x) - \frac{1}{2}\delta [F(x)]^2. \qquad (\text{C.1})$$

Therefore, $H(x)$ in equation (28), with $\alpha = 1$, can be written as

$$H(x) = \tau [\rho + \lambda F(x)]^2 - \frac{\lambda}{\delta}\left(\begin{array}{c} (\rho + \lambda F(x))\left(\mu - \frac{1}{2}\delta + \delta F(x)\right) \\ + \lambda (1 - F(x))\left(\mu + \frac{1}{2}\delta F(x)\right) \end{array}\right). \qquad (\text{C.2})$$

Notice that x enters the right hand side of equation (C.2) only through $F(x)$. Moreover, equation (C.2) is quadratic in $F(x)$ so it is convenient to write this equation as

$$\widetilde{H}(F) = \left(\begin{array}{c} \frac{1}{2}(2\tau - 1)\lambda^2 F^2 + (2\tau - 1)\rho\lambda F \\ + \tau\rho^2 - \frac{\lambda}{\delta}\left[(\rho + \lambda)\mu - \frac{1}{2}\rho\delta\right] \end{array}\right), \qquad (\text{C.3})$$

where $F = F(x)$.

---

[26]In the more general case with $\alpha \neq 0$, $\widehat{V}(\phi(t))$ for $\phi \geq \phi^*$ depends on $\alpha\widehat{V}(\phi(t))$ for $\phi(t) < \phi^*$.

Under the uniform distribution and with $\alpha = 1$, τ_L, which is defined in equation (32), is

$$\tau_L = \frac{\lambda}{\rho} \frac{1}{\delta} \left(\left(1 + \frac{\lambda}{\rho} \right) \mu - \frac{1}{2} \delta \right) = \frac{\lambda}{\rho} \frac{1}{\delta} \left(\Phi_L + \frac{\lambda}{\rho} \mu \right). \tag{C.4}$$

Evaluate $\widetilde{H}(F)$ at $F = 0$, which corresponds to evaluating $H(x)$ at $x = \Phi_L$, and use equation (C.4) to obtain

$$\widetilde{H}(0) = (\tau - \tau_L) \rho^2. \tag{C.5}$$

Under the uniform distribution and with $\alpha = 1$, τ_H, which is defined in equation (33), is

$$\tau_H = \frac{\lambda}{\rho + \lambda} \frac{1}{\delta} \Phi_H = \frac{\lambda}{\rho + \lambda} \frac{1}{\delta} \left(\mu + \frac{1}{2} \delta \right). \tag{C.6}$$

Evaluate $\widetilde{H}(F)$ at $F = 1$, which corresponds to evaluating $H(x)$ at $x = \Phi_H$, and use equation (C.6) to obtain

$$\widetilde{H}(1) = (\tau - \tau_H)(\lambda + \rho)^2. \tag{C.7}$$

Assume that

$$\tau_L \leq \tau \leq \tau_H \tag{C.8}$$

so that Regime II prevails. Equations (C.8), (C.5), and (C.7) together imply that $\widetilde{H}(0) \geq 0 \geq \widetilde{H}(1)$ and there is a unique root of $\widetilde{H}(F) = 0$ in the interval $[0, 1]$; that root corresponds to the unique root of $H(x) = 0$ in the interval $[\Phi_L, \Phi_H]$.

Use the expression for τ_L in equation (C.4) to rewrite $\widetilde{H}(F)$ in equation (C.3) as

$$\widetilde{H}(F) = [1 - 2\tau] \rho^2 \left[-\frac{1}{2} \left(\frac{\lambda}{\rho} \right)^2 F^2 - \frac{\lambda}{\rho} F + \frac{\tau - \tau_L}{1 - 2\tau} \right], \tag{C.9}$$

which is a quadratic function in $\frac{\lambda}{\rho} F$ so that $\widetilde{H}(F) = 0$ has a positive root

$$\frac{\lambda}{\rho} F^* = -1 + \sqrt{\frac{1 - 2\tau_L}{1 - 2\tau}}. \tag{C.10}$$

Note that the root depends on ρ and λ only through their ratio.

Use the fact that $\phi^* = \delta F(\phi^*) + \mu - \frac{1}{2}\delta = \delta F(\phi^*) + \Phi_L$ to obtain

$$\phi^* = \mu - \frac{1}{2}\delta - \delta \frac{\rho}{\lambda} + \delta \frac{\rho}{\lambda} \sqrt{\frac{1 - 2\tau_L}{1 - 2\tau}} = \Phi_L + \delta \frac{\rho}{\lambda} \left(-1 + \sqrt{\frac{1 - 2\tau_L}{1 - 2\tau}} \right). \tag{C.11}$$

To calculate the optimal level of debt, D^*, substitute $f(\phi^*) = \frac{1}{\delta}$ and equation (C.10) into equation (23) and set $\alpha = 1$ to obtain

$$D^* = \tau \frac{\delta}{\lambda} \frac{\rho + \lambda F(\phi^*)}{\rho + \lambda} = \frac{\tau\delta}{\lambda} \frac{\rho}{\rho + \lambda} \sqrt{\frac{1 - 2\tau_L}{1 - 2\tau}}. \tag{C.12}$$

Next calculate the optimal amount of debt when $\phi(t) \leq \phi^*$ by solving the functional equation in (39). For $\phi(t) \leq \phi^*$, the borrowing constraint is binding so $\widehat{V}(\phi(t)) \equiv \widehat{D}(\phi(t))$. Therefore, set $\widehat{V}(\phi(t)) = \widehat{D}(\phi(t))$, $dF(\phi) = \frac{1}{\delta}d\phi$, and $\alpha = 1$ in the functional equation (39) to obtain

$$(A - B\phi(t))\widehat{D}(\phi(t)) = (1 - \tau)\phi(t) + \frac{\lambda}{\delta}\int_{\phi(t)}^{\phi^*} \widehat{D}(x)\,dx + \Gamma, \tag{C.13}$$

where

$$A \equiv (1 - \tau)\rho + \lambda + \frac{\tau\lambda}{\delta}\left(\mu - \frac{1}{2}\delta\right) > 0 \tag{C.14}$$

and

$$B \equiv \frac{\tau\lambda}{\delta} > 0, \tag{C.15}$$

so that

$$A - B\phi(t) = (1 - \tau)\rho + \lambda[1 - \tau F(\phi(t))] > 0. \tag{C.16}$$

In Appendix C.3, I show that the functional equation in equation (C.13) is satisfied by

$$\widehat{D}(\phi(t)) = (A - B\phi(t))^{\frac{1-\tau}{\tau}}C + \tau\frac{1}{B}, \tag{C.17}$$

where C is a constant of integration that is determined by the boundary condition in equation (44) (or equivalently by equation (43)), so that

$$\widehat{D}(\phi(t)) = \frac{\delta}{\lambda}\left[1 - \left(\frac{A - B\phi^*}{\rho + \lambda}\right)\left(\frac{A - B\phi(t)}{A - B\phi^*}\right)^{\frac{1-\tau}{\tau}}\right], \text{ for } \phi(t) \leq \phi^*. \tag{C.18}$$

Differentiating equation (C.18) with respect to $\phi(t)$ and using $\frac{B}{\tau} = \frac{\lambda}{\delta}$ from equation (C.15) yields

$$\widehat{D}'(\phi(t)) = \frac{1 - \tau}{\rho + \lambda}\left(\frac{A - B\phi(t)}{A - B\phi^*}\right)^{\frac{1}{\tau} - 2} \geq \frac{1 - \tau}{\rho + \lambda}, \text{ for } \phi(t) \leq \phi^*, \tag{C.19}$$

where the inequality follows from $\tau < \frac{\alpha}{1+\alpha} \leq \frac{1}{2}$ and the inequality is strict for $\phi(t) < \phi^*$. Since $\frac{1-\tau}{\rho+\lambda} > 0$, $\widehat{D}'(\phi(t)) > 0$ for $\phi(t) < \phi^*$. Therefore, consistent with Proposition 11, for low values of $\phi(t)$, that is, for $\phi(t) \leq \phi^*$, the optimal amount of debt is strictly increasing in $\phi(t)$. Evaluating

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the derivative in equation (C.19) at $\phi = \phi^*$ yields $\widehat{D}'\left(\phi^*\right) = \frac{1-\tau}{\rho+\lambda}$, which is consistent with equation (47).[27]

C.2 Closed-Form Solutions for Regime III

In Regime III, the borrowing constraint binds for all values of EBIT so that $\widehat{D}\left(\phi\left(t\right)\right) = \widehat{V}\left(\phi\left(t\right)\right)$ for all $\phi\left(t\right) \in \left[\Phi_L, \Phi_H\right]$. Set $\widehat{D}\left(\phi\right) = \widehat{V}\left(\phi\right)$ in equation (39), so that optimal debt is given by equation (C.17). However, the constant of integration, C, is determined by a different boundary condition than in Regime II. To obtain an appropriate boundary condition, set $\alpha = 1$ in equation (9) and evaluate this equation at $\phi\left(t\right) = \Phi_H$ and $D = \widehat{V}\left(\Phi_H\right)$ to obtain

$$V\left(\Phi_H, \widehat{V}\left(\Phi_H\right)\right) = \tau\widehat{V}\left(\Phi_H\right) + \frac{1-\tau}{\rho+\lambda}\Phi_H. \tag{C.23}$$

Since $\widehat{D}\left(\Phi_H\right) = \widehat{V}\left(\Phi_H\right)$, the left hand side of equation (C.23) becomes $V\left(\Phi_H, \widehat{V}\left(\Phi_H\right)\right) = V\left(\Phi_H, \widehat{D}\left(\Phi_H\right)\right)$ $= \widehat{V}\left(\Phi_H\right)$ and equation (C.23) implies

$$\widehat{V}\left(\Phi_H\right) = \frac{1}{\rho+\lambda}\Phi_H, \tag{C.24}$$

so that the boundary condition in Regime III, when $\alpha = 1$, is

$$\widehat{D}\left(\Phi_H\right) = \frac{1}{\rho+\lambda}\Phi_H. \tag{C.25}$$

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[27] As a check, note that when $\tau = \tau_L$, so that $\phi^* = \Phi_L$ and $F\left(\phi^*\right) = 0$, equation (C.18) implies that

$$\widehat{D}\left(\Phi_L\right) = \frac{\delta}{\lambda}\left[1 - \left(\frac{A - B\Phi_L}{\rho+\lambda}\right)\right]. \tag{C.20}$$

Equation (C.16) implies that $A - B\Phi_L = \left(1 - \tau_L\right)\rho + \lambda$ when $\tau = \tau_L$ so equation (C.20) can be rewritten as

$$\widehat{D}\left(\Phi_L\right) = \frac{\delta}{\lambda}\frac{\rho}{\rho+\lambda}\tau_L. \tag{C.21}$$

Substitute τ_L from equation (C.4) into equation (C.21) recognizing that $\mu - \frac{1}{2}\delta = \Phi_L$ to obtain

$$\widehat{D}\left(\Phi_L\right) = \frac{1}{\rho+\lambda}\left(\frac{\lambda}{\rho}\mu + \Phi_L\right), \tag{C.22}$$

which is identical to the optimal value of debt in Regime I shown in Proposition 19.

As shown in Appendix C.3, the boundary condition in equation (C.25), along with equation (C.17), implies that[28]

$$\widehat{D}\left(\phi\left(t\right)\right) = \frac{\delta}{\lambda}\left[1 - \left(1 - \tau_H\right)\left(\frac{A - B\phi\left(t\right)}{A - B\Phi_H}\right)^{\frac{1-\tau}{\tau}}\right]. \tag{C.26}$$

C.3   Appendix: Solving the Functional Equation in (C.13)

The functional equation in equation (C.13) holds for $\phi\left(t\right) \leq \phi^*$. To solve the functional equation in equation (C.13), differentiate this equation with respect to $\phi\left(t\right)$ and use $\frac{B}{\tau} = \frac{\lambda}{\delta}$ from equation (C.15) to write the resulting differential equation in canonical form as

$$\widehat{D}'\left(\phi\left(t\right)\right) + \frac{1-\tau}{\tau}\frac{B}{A - B\phi\left(t\right)}\widehat{D}\left(\phi\left(t\right)\right) = \frac{1-\tau}{A - B\phi\left(t\right)}. \tag{C.27}$$

Multiply both sides of equation (C.27) by the integrating factor $\left(A - B\phi\left(t\right)\right)^{-\frac{1-\tau}{\tau}}$ to obtain

$$\left(A - B\phi\left(t\right)\right)^{-\frac{1-\tau}{\tau}}\widehat{D}'\left(\phi\left(t\right)\right) + \frac{1-\tau}{\tau}B\left(A - B\phi\left(t\right)\right)^{-\frac{1}{\tau}}\widehat{D}\left(\phi\left(t\right)\right) = \left(1 - \tau\right)\left(A - B\phi\left(t\right)\right)^{-\frac{1}{\tau}}. \tag{C.28}$$

Integrate both sides of equation (C.28) and then divide both sides of the resulting equation by the integrating factor $\left(A - B\phi\left(t\right)\right)^{-\frac{1-\tau}{\tau}}$ to obtain

$$\widehat{D}\left(\phi\left(t\right)\right) = \left(A - B\phi\left(t\right)\right)^{\frac{1-\tau}{\tau}}C + \tau\frac{1}{B}, \tag{C.29}$$

where C is a constant of integration. To determine the value of C, substitute the solution for $D\left(\phi\left(t\right)\right)$ from equation (C.29) into the functional equation (C.13) to obtain

$$\left(A - B\phi\left(t\right)\right)\left[\left(A - B\phi\left(t\right)\right)^{\frac{1-\tau}{\tau}}C + \tau\frac{1}{B}\right] = \left(1 - \tau\right)\phi\left(t\right) + \frac{\lambda}{\delta}\int_{\phi\left(t\right)}^{\phi^*}\left[\left(A - Bx\right)^{\frac{1-\tau}{\tau}}C + \tau\frac{1}{B}\right]dx + \Gamma. \tag{C.30}$$

Evaluating equation (C.30) at $\phi\left(t\right) = \phi^*$ and simplifying yields

$$C = \left[\phi^* + \Gamma - \tau\frac{A}{B}\right]\left(A - B\phi^*\right)^{-\frac{1}{\tau}}. \tag{C.31}$$

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[28] As a check, note that when $\tau = \tau_H$, so that $\phi^* = \Phi_H$ and $F\left(\phi^*\right) = 1$, equation (C.26) and the expression for τ_H in equation (C.6) imply that $\widehat{D}\left(\Phi_H\right) = \frac{\delta}{\lambda}\tau_H = \frac{1}{\rho+\lambda}\Phi_H$, which is identical to equation (C.25).

Finally, substitute equation (C.31) into equation (C.29) to get an expression for the optimal amount of debt

$$\widehat{D}\left(\phi\left(t\right)\right) = \tau\frac{1}{B} + J\left(\frac{A - B\phi\left(t\right)}{A - B\phi^*}\right)^{\frac{1-\tau}{\tau}}, \tag{C.32}$$

where

$$J \equiv \frac{(1 - \tau)\phi^* + \Gamma}{A - B\phi^*} - \tau\frac{1}{B}. \tag{C.33}$$

To calculate J in Regime II, evaluate equation (C.32) at $\phi\left(t\right) = \phi^*$ and use the value-matching condition in equation (44) to obtain

$$D^* = \tau\frac{1}{B} + J, \quad \text{in Regime II.} \tag{C.34}$$

Use the expression for D^* in equation (C.12) and the definition $B \equiv \frac{\tau\lambda}{\delta}$ to obtain

$$D^* = \tau\frac{\rho + \lambda F\left(\phi^*\right)}{\rho + \lambda}\frac{\tau}{B}, \quad \text{in Regime II.} \tag{C.35}$$

Equating the right hand sides of equations (C.34) and (C.35) yields

$$J = \left[\tau\frac{\rho + \lambda F\left(\phi^*\right)}{\rho + \lambda} - 1\right]\frac{\tau}{B} < 0, \quad \text{in Regime II,} \tag{C.36}$$

where the inequality follows from $\lambda > 0$, $0 < \tau < 1$, and $B > 0$. Use equation (C.16) to obtain

$$\tau\frac{\rho + \lambda F\left(\phi^*\right)}{\rho + \lambda} - 1 = -\frac{A - B\phi^*}{\rho + \lambda}, \quad \text{in Regime II.} \tag{C.37}$$

Substitute equation (C.37) into equation (C.36) to obtain

$$J = -\frac{A - B\phi^*}{\rho + \lambda}\frac{\tau}{B}, \quad \text{in Regime II.} \tag{C.38}$$

Substituting equation (C.38) into equation (C.32) yields

$$\widehat{D}\left(\phi\left(t\right)\right) = \frac{\tau}{B}\left[1 - \left(\frac{A - B\phi^*}{\rho + \lambda}\right)\left(\frac{A - B\phi\left(t\right)}{A - B\phi^*}\right)^{\frac{1-\tau}{\tau}}\right], \quad \text{in Regime II.} \tag{C.39}$$

which is the same as equation (C.18) in the text.

To calculate J in Regime III, use $\phi^* = \Phi_H$, so that equation (40) implies $\Gamma = 0$ and equation

(C.16) implies $A - B\phi^* = (1 - \tau)(\rho + \lambda)$, to rewrite equation (C.33) as

$$J = \frac{\Phi_H}{\rho + \lambda} - \frac{\tau}{B}, \quad \text{in Regime III.} \tag{C.40}$$

Substituting equation (C.40) into equation (C.32) and using $\frac{\tau}{B} = \frac{\delta}{\lambda}$ yields

$$\widehat{D}(\phi(t)) = \frac{\delta}{\lambda}\left[1 - \left(1 - \frac{\lambda}{\rho + \lambda}\frac{1}{\delta}\Phi_H\right)\left(\frac{A - B\phi(t)}{A - B\Phi_H}\right)^{\frac{1-\tau}{\tau}}\right], \quad \text{in Regime III.} \tag{C.41}$$

Now use the expression for τ_H in equation (C.6) to obtain

$$\widehat{D}(\phi(t)) = \frac{\delta}{\lambda}\left[1 - (1 - \tau_H)\left(\frac{A - B\phi(t)}{A - B\Phi_H}\right)^{\frac{1-\tau}{\tau}}\right], \quad \text{in Regime III,} \tag{C.42}$$

which is identical to equation (C.26) in the text.

C.4   Appendix: Verification that Equation (C.32) Satisfies Functional Equation (C.13)

Use equation (C.32) to express the right hand side of equation (C.13) as

$$(1 - \tau)\phi(t) + \frac{\lambda}{\delta}\int_{\phi(t)}^{\phi^*}\widehat{D}(x)\,dx + \Gamma = (1 - \tau)\phi(t) + \frac{\lambda}{\delta}\int_{\phi(t)}^{\phi^*}\left[\frac{\tau}{B} + J\left(\frac{A - Bx}{A - B\phi^*}\right)^{\frac{1-\tau}{\tau}}\right]dx + \Gamma. \tag{C.43}$$

Perform the integration on the right hand side of equation (C.43) to obtain

$$(1 - \tau)\phi(t) + \frac{\lambda}{\delta}\int_{\phi(t)}^{\phi^*}\widehat{D}(x)\,dx + \Gamma = \left[\begin{array}{c} (1 - \tau)\phi(t) + \frac{\lambda}{\delta}\frac{\tau}{B}(\phi^* - \phi(t)) \\ +\frac{\lambda}{\delta}J\tau\left(-\frac{A - B\phi^*}{B}\right)\left(\frac{A - Bx}{A - B\phi^*}\right)^{\frac{1}{\tau}}|_{\phi(t)}^{\phi^*} + \Gamma \end{array}\right], \tag{C.44}$$

which can be written as

$$(1 - \tau)\phi(t) + \frac{\lambda}{\delta}\int_{\phi(t)}^{\phi^*}\widehat{D}(x)\,dx + \Gamma = \left[\begin{array}{c} (1 - \tau)\phi(t) + \frac{\lambda}{\delta}\frac{\tau}{B}(\phi^* - \phi(t)) \\ +\frac{\lambda}{\delta}J\tau\left(-\frac{A - B\phi^*}{B}\right)\left[1 - \left(\frac{A - B\phi(t)}{A - B\phi^*}\right)^{\frac{1}{\tau}}\right] + \Gamma \end{array}\right]. \tag{C.45}$$

Use the definition $B \equiv \frac{\tau \lambda}{\delta}$ to obtain

$$(1-\tau) \phi(t) + \frac{\lambda}{\delta} \int_{\phi(t)}^{\phi^*} \widehat{D}(x) \, dx + \Gamma = -\tau \phi(t) + \phi^* - J\left(A - B\phi^*\right)\left[1 - \left(\frac{A - B\phi(t)}{A - B\phi^*}\right)^{\frac{1}{\tau}}\right] + \Gamma. \quad \text{(C.46)}$$

Use the definition $J \equiv \frac{(1-\tau)\phi^* + \Gamma}{A - B\phi^*} - \frac{\tau}{B}$ in equation (C.33), which implies $(A - B\phi^*) J = \phi^* + \Gamma - \tau\frac{A}{B}$, to obtain

$$(1-\tau) \phi(t) + \frac{\lambda}{\delta} \int_{\phi(t)}^{\phi^*} \widehat{D}(x) \, dx + \Gamma = -\tau \phi(t) + \frac{A}{B}\tau + J\left(A - B\phi^*\right)\left(\frac{A - B\phi(t)}{A - B\phi^*}\right)^{\frac{1}{\tau}}. \quad \text{(C.47)}$$

Multiply and divide the final term on the right hand side of equation (C.47) by $A - B\phi(t)$ and simplify to obtain

$$(1-\tau) \phi(t) + \frac{\lambda}{\delta} \int_{\phi(t)}^{\phi^*} \widehat{D}(x) \, dx + \Gamma = (A - B\phi(t))\left[\frac{\tau}{B} + J\left(\frac{A - B\phi(t)}{A - B\phi^*}\right)^{\frac{1}{\tau}-1}\right]. \quad \text{(C.48)}$$

Use the solution for the level of debt in equation (C.32) on the right hand side of equation (C.48) to obtain

$$(1-\tau) \phi(t) + \frac{\lambda}{\delta} \int_{\phi(t)}^{\phi^*} \widehat{D}(x) \, dx + \Gamma = (A - B\phi(t)) \widehat{D}(\phi(t)), \quad \text{(C.49)}$$

which is the same as equation (C.13). In Regime II, the solution for the optimal level of debt

in equation (C.32) becomes equation (C.18) as demonstrated in the derivation of equation (C.39) and in Regime III, the solution for the optimal level of debt in equation (C.32) becomes equation (C.26) as demonstrated in the derivation of equation (C.42).