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LIQUIDITY AND RISK MANAGEMENT, EXECUTIVE COMPENSATION,  
AND INVESTMENT WITH INALIENABLE HUMAN CAPITAL:  
OPTIMAL CONTRACTING AND CORPORATE VALUATION

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Liquidity and Risk Management, Executive Compensation, and Investment with Inalienable Human Capital: Optimal Contracting and Corporate Valuation

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**ABSTRACT**

A risk-averse entrepreneur with access to a profitable venture needs to raise funds from investors. She cannot indefinitely commit her human capital to the venture, which limits the firm's debt capacity, distorts investment and compensation, and constrains the entrepreneur's risk-sharing. This puts dynamic liquidity and state-contingent risk allocation at the center of corporate financial management. The firm balances mean-variance investment efficiency and the preservation of financial slack. We show that in general the entrepreneur's net worth is over-exposed to idiosyncratic risk and under-exposed to systematic risk. These distortions are greater the closer the firm is to exhausting its debt capacity.

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Consider a risk-averse entrepreneur who has access to a profitable venture with an initial capital stock  $K_0$ . This entrepreneur needs to raise start-up funds and on occasion additional working capital from investors. In a first-best Modigliani-Miller environment the entrepreneur would be able to diversify away her idiosyncratic risk, fully pledge the market value of her venture, and raise the funds from investors against a promised competitive risk-adjusted return. However, if the entrepreneur is essential to the venture and cannot irrevocably dedicate her human capital to the firm, the promised return may not be credible. We show that this inalienability of the entrepreneur's human capital, or what is also commonly referred to as *key-man risk*, has critical implications not only for the firm's financing capacity, investment and compensation, but also for its liquidity and risk management policy. The more liquidity or the larger borrowing capacity the firm has, the greater is its ability to retain talent by making credible future compensation promises. In addition, by managing the firm's exposures to idiosyncratic and aggregate risk the firm can reduce the cost of retaining talent and lower the firm's cost of financing.

In sum, our paper offers a new theory of corporate liquidity and risk management based on the inalienability of risky human capital. Even when there are no capital market frictions, corporations add value by optimally managing risk and liquidity because doing so allows them to reduce the cost of key-man risk to investors. This rationale for corporate risk and liquidity management is particularly relevant for technology firms where key-man risk is acute.

The main building blocks of our model are as follows. The entrepreneur has constant relative risk-averse preferences and seeks to smooth consumption. The firm's operations are exposed to both idiosyncratic and aggregate risk. The firm's capital is illiquid and is exposed to stochastic depreciation. It can be accumulated through investments that are subject to adjustment costs. The entrepreneur faces risk with respect to both the firm's performance and her outside options. To best retain the entrepreneur, the firm optimally compensates her by smoothing her consumption and limiting her risk exposure. But to be able to do so the firm must manage its liquidity and risk allocation. The firm's optimized balance sheet is composed of illiquid capital,  $K$ , and cash or marketable securities,  $S$ , on the asset side. The liability side is composed of equity and a line of credit (when  $S$  is negative), with a limit that depends on the entrepreneur's outside option.

The solution of this problem has the following key elements. The entrepreneur manages the firm's risk by choosing optimal loadings on the idiosyncratic and market risk factors. The firm's liquidity is augmented through retained earnings from operations and through returns from its portfolio of marketable securities, including its hedging and insurance positions. The scaled state variable is the firm's liquidity-to-capital ratio  $s = S/K$ . When liquidity is abundant ( $s$  is large) the firm is essentially unconstrained and can choose its policies to maximize its market value (or equivalently the entrepreneur's net worth.) The firm's investment policy then approaches the Hayashi (1982) risk-adjusted first-best benchmark, and its consumption and asset allocations approach the generalized Merton (1971) consumption and mean-variance portfolio rules. In particular, the entrepreneur is completely insulated from idiosyncratic risk.

In contrast, when the firm exhausts its credit limit, its objective is exclusively to ensure that the

entrepreneur gets at least as much as her outside option, which is achieved by optimally preserving liquidity  $s$  and eliminating the volatility of  $s$  at the endogenously determined debt limit  $\underline{s}$ . As one would expect, preserving liquidity requires cutting investment and consumption, engaging in asset sales, and lowering the systematic risk exposure of the entrepreneur’s net worth. More surprisingly, preserving financial slack also involves retaining some exposure to idiosyncratic risk. That is, relative to the first-best, the entrepreneur’s net worth is over-exposed to idiosyncratic risk and under-exposed to systematic risk, as this helps reduce (or even eliminate) the volatility of  $s$ .

In short, the risk management problem for the firm boils down to a compromise between achieving mean-variance efficiency for the entrepreneur’s net worth and preserving the firm’s financial slack. The latter is the dominant consideration when liquidity  $s$  is low.

The first model to explore the corporate finance consequences of inalienable human capital is Hart and Moore (1994). They consider an optimal financial contract between an entrepreneur and outside investors to finance a single project, with a finite horizon, and no cash-flow uncertainty. Both the entrepreneur and investors are assumed to have linear utility functions. They argue that the inalienability of the entrepreneur’s human capital implies that debt is an optimal financial contract.

We generalize the Hart and Moore model in several important directions. Our first generalization is to consider an infinitely-lived firm, with ongoing investment subject to adjustment costs, and an entrepreneur with a strictly concave utility function. The firm’s financing constraint is always binding in Hart and Moore (1994), but in our model the financing constraint is generically non-binding. Because it is optimal to smooth investment and consumption, the firm does not want to run through its stock of liquidity in one go. This naturally gives rise to a theory of liquidity management even when there is no uncertainty. We describe this special case in Section VII. Our second generalization is to introduce both idiosyncratic and aggregate risk, which leads to a theory of corporate risk management that ties together classical intertemporal asset pricing and portfolio choice theory with corporate liquidity demand. Investors set the market price of risk, which the entrepreneur takes as given to determine the firm’s optimal risk exposures and how they should vary with the firm’s stock of liquidity. By generalizing the Hart and Moore model to include ongoing investment, consumption smoothing, uncertainty, and risk aversion for both entrepreneur and investors, we are able to show that inalienability of human capital not only gives rise to a theory of debt capacity, but also a dynamic theory of liquidity and risk management, which is fundamentally connected to the entrepreneur’s optimal compensation.

Corporate risk management in our analysis is not about achieving an optimal risk-return profile for investors, they can do that on their own, but about offering optimal risk-return profiles to risk-averse, under-diversified, key employees (the entrepreneur in our setting) with inalienable human capital constraint. In our setup the firm is, in effect, both the employer and the asset manager for its key employees. This perspective on corporate risk management is consistent with Duchin et al. (2016), who find that non-financial firms invest 40% of their liquid savings in risky financial assets. They find that the less constrained firms invest more in the market portfolio, which is

consistent with our predictions. In addition, when firms are severely financially constrained, we show that they cut compensation, reduce corporate investment, engage in asset sales, and reduce hedging positions, with the primary objective of surviving by honoring liabilities and retaining key employees. These latter predictions are in line with the findings of Rampini, Sufi, and Viswanathan (2014), Brown and Matsa (2016), and Donangelo (2016).

Corporate liquidity management in our model is not about avoiding costly external financing, but about compensation smoothing, which requires in particular maintaining liquidity buffers in low productivity states. This motive generally outweighs the countervailing investment financing motive of Froot, Scharfstein, and Stein (1993), which prescribes building liquidity buffers in high productivity states, where investment opportunities are good. If the firm finds itself in the low productivity state, we show that it is optimal for the entrepreneur to take a pay cut, consistent with the evidence on executive compensation and corporate cash holdings (e.g. Ganor, 2013). It is possible for the firm to impose a pay cut because in a low productivity state the entrepreneur's outside options are also worth less. It is also optimal to sell insurance in a low productivity state to generate valuable liquidity. The optimality of selling insurance when productivity is low is not driven by risk shifting incentives as in Jensen and Meckling (1976), but rather by the firm's need to replenish liquidity. Asset sales in response to a negative productivity shock (also optimal in our setting) are commonly emphasized (Campello, Giambona, Graham, and Harvey, 2011). But our analysis further explains why it is also optimal to sell insurance and cut pay in response to low productivity shocks.

Our theory is particularly relevant for human-capital intensive, high-tech, firms. These firms often hold substantial cash and employee stock-option pools. We explain why these pools may be necessary to make future compensation promises credible and thereby retain highly valued employees. When stock options vest and are exercised, companies generally engage in stock repurchases so as to avoid excessive stock dilution. But such repurchase programs require funding, which partly explains why these companies hold such large liquidity buffers.

We show that the firm's optimal liquidity and risk management problem can also be reformulated as a dual optimal contracting problem between a well diversified risk-averse investor and an entrepreneur with inalienable human capital. In the contracting formulation, the state variables are the certainty equivalent wealth  $W$  that the investor promises to the entrepreneur and the firm's capital stock  $K$ . Analogous to our primal formulation, the ratio,  $w = W/K$ , is the scaled state variable that describes how constrained the firm is.

As Table I below summarizes, this dual contracting problem is equivalent to the entrepreneur's liquidity and risk management problem:  $s = -p(w)$ , where  $p(w)$  is the investor's scaled value in the contracting problem, and  $w = m(s)$ , where  $m(s)$  is the entrepreneur's scaled certainty-equivalent wealth in the liquidity and risk management problem. A key observation here is that the credit constraint  $s \geq \underline{s}$  is the outcome of an optimal financial contracting problem under an inalienability constraint for the entrepreneur given by  $w \geq \underline{w}$ .

**[Place Table I about here]**

Ai and Li (2015) consider a closely related contracting problem. They characterize optimal CEO compensation and corporate investment under limited commitment, but they do not consider the implementation of the contract through corporate liquidity and risk management policies. Their formulation differs from ours in two other important respects. First, they assume that investors are risk neutral, so that they cannot make a meaningful distinction between idiosyncratic and aggregate risk. Second, their limited-commitment assumption does not take the form of an inalienability-of-human-capital constraint. In their setup, the entrepreneur is assumed to abscond with the firm’s capital, and when she does so she can only continue operating under autarky. In our setup the entrepreneur is free to leave and can therefore offer her human capital to another firm under an optimal contract. Ai and Li’s limited-commitment assumptions lead them to substantially different predictions. First, autarky is such a severe punishment (because the entrepreneur is then fully exposed to the firm’s operating risk) that the limited commitment constraint barely binds and may not result in any distortions in investment and consumption. Even for a relatively low risk aversion coefficient for the entrepreneur the first-best outcome is attainable. Second, for low risk aversion, the dynamics for the entrepreneur’s consumption are such that consumption is constant as long as the limited commitment constraint does not bind and adjusts up only when the constraint is binding. In contrast, in our model the inalienability-of-human-capital constraint distorts consumption, investment and risk-exposures even for high coefficients of risk aversion of the entrepreneur. Moreover, these policies respond smoothly to changes in the firm’s liquidity. We provide a detailed discussion of the difference between the autarky and recontracting assumptions in Section V.F.

Rampini and Viswanathan (2010, 2013) develop a limited-commitment-based theory of risk management that focuses on the tradeoff between exploiting current versus future investment opportunities. If the firm invests today it may exhaust its debt capacity and thereby forego future investment opportunities. If instead the firm foregoes investment and hoards its cash it is in a position to be able to exploit potentially more profitable investment opportunities in the future. The difference between our theory and theirs is mainly due to our assumptions of risk aversion for the entrepreneur and investors, our modeling of limited commitment in the form of risky inalienable human capital, and our assumption of physical capital illiquidity via  $q$  theory of investment. We focus on a different aspect of corporate liquidity and risk management, namely the management of risky human capital and key-man risk. In particular, we emphasize the benefits of risk management to help smooth consumption of the firm’s stakeholders (entrepreneur, managers, key employees).

Berk, Stanton and Zechner (2010) analyze a model where the firm trades off the tax benefits of debt with the cost of under-insuring risk-averse employees. Building on Merton’s intertemporal portfolio choice framework, Wang, Wang, and Yang (2012) study a risk-averse entrepreneur’s optimal consumption-savings, portfolio choice, and capital accumulation decisions when facing uninsurable capital and productivity risks. Unlike Wang, Wang, and Yang (2012), our model features optimal liquidity and risk management policies that arise endogenously from an underlying financial contracting problem.

Our theory has elements in common with the literature on contracting under limited commitment following Harris and Holmstrom (1982). They analyze a model of optimal insurance for a risk-averse worker, who is unable to commit to a long-term contract. Lustig, Syverson and Nieuwerburgh (2011) build on Harris and Holmstrom (1982) by incorporating organizational capital. Eisfeldt and Papanikolaou (2013) consider the asset-pricing implications of limited commitment and organizational capital risk.

In terms of methodology, our paper builds on the dynamic contracting in continuous time work of Holmstrom and Milgrom (1987), Schaettler and Sung (1993), and Sannikov (2008), among others. Our model is evidently related to the dynamic corporate security design literature in the vein of DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), and DeMarzo and Fishman (2007b).<sup>1</sup> As in DeMarzo and Sannikov (2006), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo, Fishman, He, and Wang (2012), our continuous-time formulation allows us to provide sharper closed-form solutions for consumption, investment, liquidity and risk management policies, up to an ordinary differential equation (ODE) for the investors' scaled value  $p(w)$ . These papers also focus on the implementation of the optimal contracting solution via corporate liquidity (cash and credit line) and (inside and outside) equity. Two key differences are: (1) risk aversion and (2) systematic and idiosyncratic risk, which together lead to a theory of the firm's off-balance-sheet (zero-NPV) futures and insurance positions in addition to the "marketable securities" entry on corporate balance sheets. A third difference is that these papers focus on dynamic moral hazard, which is different from our focus on inalienability of risky human capital. A fourth difference is our generalization of the  $q$ -theory of investment to settings with inalienable human capital.<sup>2</sup>

Our theory is also related to the liquidity asset pricing theory of Holmstrom and Tirole (2001). We significantly advance their agenda of developing an asset pricing/portfolio choice theory based on corporate liquidity. They consider a three-period model with risk neutral agents, where firms are financially constrained and therefore have higher value when they hold more liquidity. Their assumptions of risk neutrality and no consumption smoothing limit the integration of asset pricing and corporate finance theories.

There is also an extensive macroeconomics literature on limited commitment.<sup>3</sup> Green (1987), Thomas and Worrall (1990), Marcet and Marimon (1992), Kehoe and Levine (1993) and Kocherlakota (1996) are important early contributions on optimal contracting under limited commitment. Alvarez and Jermann (2000) extend welfare theorems to economies with limited commitment. Our entrepreneur's optimization problem is related to the agent's dynamic optimization problem in Alvarez and Jermann (2000) and Chien and Lustig (2010) to allow for recontracting after default. While their focus is on optimal consumption allocation, we focus on consumption, liquidity and risk allocation, as well as corporate investment.

Albuquerque and Hopenhayn (2004), Quadrini (2004), Clementi and Hopenhayn (2006), and Lorenzoni and Walentin (2007) characterize financing and investment decisions under limited commitment or asymmetric information. Kehoe and Perri (2002) and Albuquerque (2003) analyze the implications of limited commitment for international business cycles and foreign direct investments.

Miao and Zhang (2015) develop a duality-based solution method for limited-commitment problems.

Our analysis also contributes to the executive compensation literature, (see Frydman and Jenter, 2010, and Edmans and Gabaix, 2016, for recent surveys). Our model brings out an important positive link between (a.) executive compensation and (b.) corporate liquidity and risk management, and helps explain why companies typically cut compensation and investment, and reduce risk exposures when liquidity is tight. DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), and DeMarzo, Fishman, He, and Wang (2012) also provide financial implementation with cash and/or credit line and link to executive compensation.<sup>4</sup>

Finally, our paper is clearly related to the voluminous economics literature on human capital that builds on Ben-Porath (1967) and Becker (1975).

## I. The Model

We consider an intertemporal optimization problem faced by a risk-averse entrepreneur, who cannot irrevocably promise to operate the firm indefinitely under all circumstances. This inalienability problem for the entrepreneur results in *endogenous* financial constraints distorting her consumption, savings, investment, and exposures to systematic and idiosyncratic risks. To best highlight the central economic mechanism arising from the inalienability of human capital, we remove all other financial frictions from the model and assume that financial markets are otherwise fully competitive and all state-contingent claims can be traded frictionlessly.

### A. Production Technology and Preferences

**Production and Capital Accumulation.** The firm's capital stock  $K$  evolves according to a controlled Geometric Brownian Motion (GBM) process:

$$dK_t = (I_t - \delta_K K_t)dt + \sigma_K K_t \left( \sqrt{1 - \rho^2} dZ_{h,t} + \rho dZ_{m,t} \right), \quad (1)$$

where  $I$  is the firm's rate of gross investment,  $\delta_K \geq 0$  is the expected rate of depreciation, and  $\sigma_K$  is the volatility of the capital depreciation shock.<sup>5</sup> Without loss of generality, we decompose risk into two orthogonal components: an idiosyncratic shock represented by the standard Brownian motion  $Z_h$  and a systematic shock represented by the standard Brownian motion  $Z_m$ .<sup>6</sup> The parameter  $\rho$  measures the correlation between the firm's capital risk and systematic risk, so that the firm's systematic volatility is equal to  $\rho\sigma_K$  and its idiosyncratic volatility is given by

$$\nu_K = \sigma_K \sqrt{1 - \rho^2}. \quad (2)$$

The capital stock includes both physical capital and intangible capital (such as, patents, know-how, brand value, and organizational capital).

As in Hart and Moore (1994), production requires combining the entrepreneur's inalienable human capital and the firm's physical assets. When either the entrepreneur's human capital or



the firm's physical capital is missing, no output is produced and no value creation is possible. In other words, value is created by matching the entrepreneur's human capital and the firm's physical capital stock. The entrepreneur's human capital is captured by a parameter denoted by  $A$ . Human capital is more valuable when it is deployed on a larger capital stock  $K_t$ . Specifically we assume that the firm's output produced by the match is given by  $AK_t$ . This formulation encapsulates the idea that the added value of the entrepreneur's human capital is risky to the extent that the firm's capital  $K_t$  is risky. We generalize our model to introduce shocks to productivity  $A$  in Section VI. An important simplifying assumption throughout our analysis is that the entrepreneur's human capital is always best matched with the firm's physical capital stock, so that there is no separation under the optimal contract.<sup>7</sup>

Investment involves an adjustment cost as in the standard  $q$ -theory of investment, so that the firm's *free cash flow* (net of capital costs but before consumption) is given by:

$$Y_t = AK_t - I_t - G(I_t, K_t), \quad (3)$$

where the price of the investment good is normalized to one and  $G(I, K)$  is the standard adjustment cost function. Note that  $Y_t$  can take negative values, which simply means that additional financing may be needed to close the gap between contemporaneous revenue,  $AK_t$ , and total investment costs.

We further assume that the adjustment cost  $G(I, K)$  is homogeneous of degree one in  $I$  and  $K$  (a common assumption in the  $q$ -theory of investment) and express  $G(I, K)$  as follows:

$$G(I, K) = g(i)K, \quad (4)$$

where  $i = I/K$  denotes the investment-capital ratio and  $g(i)$  is increasing and convex in  $i$ . As Hayashi (1982) has shown, under this homogeneity property Tobin's average and marginal  $q$  are equal in the first-best benchmark.<sup>8</sup> However, under inalienability of human capital an endogenous wedge between Tobin's average and marginal  $q$  will emerge in our model.<sup>9</sup>

**Preferences.** The infinitely-lived entrepreneur has a standard concave utility function over positive consumption flows  $\{C_t; t \geq 0\}$  given by:

$$J_t = \mathbb{E}_t \left[ \int_t^\infty \zeta e^{-\zeta(v-t)} U(C_v) dv \right], \quad (5)$$

where  $\zeta > 0$  is the entrepreneur's subjective discount rate,  $\mathbb{E}_t[\cdot]$  is the time- $t$  conditional expectation, and  $U(C)$  takes the standard constant-relative-risk-averse utility (CRRA) form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \quad (6)$$

with  $\gamma > 0$  denoting the coefficient of relative risk aversion. We normalize the flow payoff with  $\zeta$  in (5), so that the utility flow is given by  $\zeta U(C)$ .<sup>10</sup>

### B. Complete Financial Markets

We assume that financial markets are perfectly competitive and complete. By using essentially the same argument as in the Black-Merton-Scholes option pricing framework, we can dynamically complete markets with three long-lived assets (Harrison and Kreps, 1979 and Duffie and Huang, 1985): Given that the firm's production is subject to two shocks,  $\mathcal{Z}_h$  and  $\mathcal{Z}_m$ , financial markets are dynamically complete if the following three non-redundant financial assets can be dynamically and frictionlessly traded:

- a. A risk-free asset that pays interest at a constant risk-free rate  $r$ ;
- b. A hedging contract that is perfectly correlated with the idiosyncratic shock  $\mathcal{Z}_h$ . There is no up-front cost to enter this hedging contract as the risk involved is purely idiosyncratic and thus the counter-party earns no risk premium. The transaction at inception is therefore off-the-balance sheet. The instantaneous payoff for each unit of the contract is  $\nu_K d\mathcal{Z}_{h,t}$ .
- c. A stock market portfolio. The incremental return  $dR_{m,t}$  of this asset is

$$dR_{m,t} = \mu_m dt + \sigma_m d\mathcal{Z}_{m,t}, \quad (7)$$

where  $\mu_m$  and  $\sigma_m$  are constant drift and volatility parameters. As this risky asset is only subject to the systematic shock we refer to it as the *market portfolio*.

Dynamic and frictionless trading with these three securities implies that the following unique stochastic discount factor (SDF) exists (e.g., Duffie, 2001):

$$\frac{d\mathbb{M}_t}{\mathbb{M}_t} = -r dt - \eta d\mathcal{Z}_{m,t}, \quad (8)$$

where  $\mathbb{M}_0 = 1$  and  $\eta$  is the Sharpe ratio of the market portfolio given by:

$$\eta = \frac{\mu_m - r}{\sigma_m}. \quad (9)$$

The SDF  $\mathbb{M}$  follows a geometric Brownian motion where the drift is equal to the negative risk-free rate, as required under no arbitrage. By definition the SDF is only exposed to the systematic shock  $\mathcal{Z}_m$ . Fully diversified investors will only demand a risk premium for their exposures to systematic shocks.

**Dynamic Trading.** Let  $\{S_t; t \geq 0\}$  denote the entrepreneur's liquid wealth process. When  $S_t > 0$ , the entrepreneur's savings are positive and when  $S_t < 0$ , she is a borrower. The entrepreneur continuously allocates  $S_t$  between the risk-free asset and the stock market portfolio  $\Phi_{m,t}$  whose

return is given by (7). Moreover, the entrepreneur chooses a pure idiosyncratic-risk hedging position  $\Phi_{h,t}$ . Her liquid wealth  $S_t$  then evolves as follows:

$$dS_t = (rS_t + Y_t - C_t)dt + \Phi_{h,t}\nu_K d\mathcal{Z}_{h,t} + \Phi_{m,t}[(\mu_m - r)dt + \sigma_m d\mathcal{Z}_{m,t}]. \quad (10)$$

The first term in (10),  $rS_t + Y_t - C_t$ , is simply the sum of the interest income  $rS_t$  and net operating cash flows,  $Y_t - C_t$ , the second term,  $\Phi_{h,t}\nu_K d\mathcal{Z}_{h,t}$ , is the exposure to the idiosyncratic shock  $\mathcal{Z}_h$ , which earns no risk premium, and the third term,  $\Phi_{m,t}[(\mu_m - r)dt + \sigma_m d\mathcal{Z}_{m,t}]$ , is the excess payoff from the market portfolio.

In the absence of any risk exposure  $rS_t + Y_t - C_t$  is simply the rate at which the entrepreneur saves as in standard permanent-income models. However, in general, saving all liquid wealth  $S$  at the risk-free rate is sub-optimal. By dynamically engaging in risk taking and risk management, through the risk exposures  $\Phi_h$  and  $\Phi_m$ , the entrepreneur will do better.

Next, we use dynamic programming to characterize the firm's liquidity and risk management policies.

### C. Dynamic Programming

Let  $J(K, S)$  denote the entrepreneur's value function. The entrepreneur's *liquid* wealth  $S$  and *illiquid* productive capital  $K$  play different roles and accordingly both serve as natural state variables. By the standard dynamic programming argument, the solution for  $J(K, S)$  in the interior region is characterized by the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} \zeta J(K, S) = & \max_{C, I, \Phi_h, \Phi_m} \zeta U(C) + (rS + \Phi_m(\mu_m - r) + AK - I - G(I, K) - C)J_S(K, S) \\ & + (I - \delta_K K)J_K(K, S) + \frac{\sigma_K^2 K^2}{2} J_{KK}(K, S) \\ & + (\nu_K^2 \Phi_h + \rho \sigma_K \sigma_m \Phi_m) K J_{KS}(K, S) + \frac{(\nu_K \Phi_h)^2 + (\sigma_m \Phi_m)^2}{2} J_{SS}(K, S). \end{aligned} \quad (11)$$

The first term on the right side of (11) represents the entrepreneur's utility over consumption; the second term is the product of the marginal value of liquidity,  $J_S(K, S)$ , and the savings rate for  $S$ ; the third term is the product of net investment,  $(I - \delta_K K)$ , and the marginal value of capital  $J_K(K, S)$ ; the last three terms (involving  $J_{KK}(K, S)$ ,  $J_{KS}(K, S)$  and  $J_{SS}(K, S)$ ) correspond to the quadratic-variation and co-variation effects of  $K$  and  $S$ .

The entrepreneur chooses consumption  $C$ , investment  $I$ , idiosyncratic-risk hedge  $\Phi_h$ , and market-portfolio allocation  $\Phi_m$ , to maximize her lifetime utility. With a concave utility function  $U(C)$ , optimal consumption is determined by the first-order condition (FOC):

$$\zeta U'(C) = J_S(K, S), \quad (12)$$

which equates the marginal utility of consumption  $\zeta U'(C)$  with  $J_S(K, S)$ , the marginal value of

liquid wealth. The FOC for investment  $I$ ,

$$(1 + G_I(I, K)) J_S(K, S) = J_K(K, S), \quad (13)$$

is somewhat less obvious. It equates: (a.) the marginal cost of investing in illiquid capital, given by the product of the marginal cost of investing  $(1 + G_I)$  and the marginal value of liquid savings  $J_S(K, S)$ , with (b.) the entrepreneur's marginal value of investing in illiquid capital  $J_K(K, S)$ .

The optimal stock-market portfolio allocation  $\Phi_m$  satisfies the following FOC:

$$\Phi_m = -\frac{\eta}{\sigma_m} \frac{J_S(K, S)}{J_{SS}(K, S)} - \frac{\rho\sigma_K}{\sigma_m} \frac{K J_{KS}(K, S)}{J_{SS}(K, S)}. \quad (14)$$

The first term in (14) is in the spirit of Merton's mean-variance demand and the second term is the hedging demand with respect to the firm's systematic risk exposure. Similarly, the optimal hedge against idiosyncratic risk  $\Phi_h$  is given by the following FOC:

$$\Phi_h = -\frac{K J_{KS}(K, S)}{J_{SS}(K, S)}. \quad (15)$$

Note that the numerators in both (14) and (15) involve the cross partial,  $J_{KS}(K, S)$ .

Equations (11), (12), (13), (14), and (15) jointly characterize the interior solution of the firm's optimization problem.

**The Entrepreneur's Certainty-Equivalent Wealth  $M(K, S)$ .** A key step in our derivation is to establish that the entrepreneur's value function  $J(K, S)$  takes the following form:<sup>11</sup>

$$J(K, S) = \frac{(bM(K, S))^{1-\gamma}}{1-\gamma}, \quad (16)$$

where  $M(K, S)$  is the entrepreneur's certainty-equivalent wealth and  $b$  is the constant:<sup>12</sup>

$$b = \zeta \left[ \frac{1}{\gamma} - \frac{1}{\zeta} \left( \frac{1-\gamma}{\gamma} \right) \left( r + \frac{\eta^2}{2\gamma} \right) \right]^{\frac{\gamma}{\gamma-1}}. \quad (17)$$

In words,  $M(K, S)$  is the dollar amount that the entrepreneur would demand to permanently give up her productive human capital and retire as a Merton-style consumer living under complete markets. By linking the entrepreneur's value function  $J(K, S)$  to her certainty-equivalent wealth  $M(K, S)$  we are able to transform the entrepreneur's payoff from the value function,  $J(K, S)$ , to the certainty-equivalent wealth,  $M(K, S)$ .

This transformation is conceptually important, as it allows us to measure payoffs in dollars and thereby to make the economics of the entrepreneur's problem more intuitive. In particular, it is only possible to determine the marginal value of liquidity,  $M_S(K, S)$ , after making the transformation from  $J(K, S)$  to  $M(K, S)$ . As we will show, the economics of the entrepreneur's problem and the solution of the entrepreneur's liquidity and risk management problem are closely tied to the

marginal value of liquidity  $M_S(K, S)$ .

**Reduction to One Dimension.** An additional simplifying step is to exploit the model's homogeneity property to reduce the entrepreneur's problem to one dimension. Scaling the variables expressed in dollar units by  $K_t$ , we use lower-case letters to denote the following variables: consumption  $c_t = C_t/K_t$ , investment  $i_t = I_t/K_t$ , liquidity  $s_t = S_t/K_t$ , idiosyncratic-risk hedge  $\phi_{h,t} = \Phi_{h,t}/K_t$ , and market-portfolio position  $\phi_{m,t} = \Phi_{m,t}/K_t$ . We also express the entrepreneur's certainty equivalent wealth  $M(K_t, S_t)$  as follows:

$$M(K_t, S_t) = m(s_t) \cdot K_t. \quad (18)$$

**Endogenous Risk Aversion  $\gamma_e$ .** To better interpret our solution it is helpful to introduce the following measure of endogenous relative risk aversion for the entrepreneur, denoted by  $\gamma_e$  and defined as follows:

$$\gamma_e \equiv -\frac{J_{SS}}{J_S} \times M(K, S) = \gamma m'(s) - \frac{m(s)m''(s)}{m'(s)}. \quad (19)$$

In (19) the first identity sign gives the definition of  $\gamma_e$  and the second equality follows from the homogeneity property. What economic insights does  $\gamma_e$  capture and why do we introduce  $\gamma_e$ ? First, inalienability of human capital results in a form of endogenous market incompleteness. Therefore, the entrepreneur's *endogenous* risk aversion is captured by the curvature of her value function  $J(K, S)$  rather than her utility function  $U(\cdot)$ . We can characterize the entrepreneur's coefficient of endogenous absolute risk aversion with the standard definition via her value function:  $-J_{SS}(K, S)/J_S(K, S)$ . But how do we link this absolute risk aversion measure to a relative risk aversion measure? We need to multiply absolute risk aversion,  $-J_{SS}(K, S)/J_S(K, S)$ , with an appropriate measure of the entrepreneur's wealth. There is no well-defined market measure of the entrepreneur's total wealth under inalienability. However, the entrepreneur's certainty equivalent wealth  $M(K, S)$  is a natural measure. This motivates our definition of  $\gamma_e$  in (19).<sup>13</sup> We will show that the inalienability of human capital causes the entrepreneur to be under-diversified and hence effectively more risk averse, so that  $\gamma_e(s) > \gamma$ .<sup>14</sup> The second equality in (19) confirms this result, as her certainty equivalent wealth  $m(s)$  is concave in  $s$  with  $m'(s) > 1$ , which we establish below.

Next, we characterize the evolution of  $s$  given the policy functions  $\phi_h(s)$ ,  $\phi_m(s)$ ,  $c(s)$ , and  $i(s)$ .

**Dynamics of the Liquidity Ratio  $\{s_t : t \geq 0\}$ .** Given policies  $c(s)$ ,  $i(s)$ ,  $\phi_h(s)$ , and  $\phi_m(s)$ , we can express the dynamics for the liquidity ratio  $s_t$  using Ito's formula as:

$$ds_t = d(S_t/K_t) = \mu^s(s_t)dt + \sigma_h^s(s_t)dZ_{h,t} + \sigma_m^s(s_t)dZ_{m,t}, \quad (20)$$

where the idiosyncratic volatility function for  $s_t$ ,  $\sigma_h^s(\cdot)$ , and the systematic volatility function for  $s_t$ ,  $\sigma_m^s(\cdot)$ , are respectively given by:

$$\sigma_h^s(s) = (\phi_h(s) - s)\nu_K, \quad (21)$$

$$\sigma_m^s(s) = \phi_m(s)\sigma_m - \rho\sigma_K s, \quad (22)$$

and the drift function for  $s_t$ ,  $\mu^s(\cdot)$ , is given by:

$$\mu^s(s_t) = y(s_t) + \phi_m(s_t)(\mu_m - r) - c(s_t) + (r + \delta_K - i(s_t))s_t - (\nu_K\sigma_h^s(s_t) + \rho\sigma_K\sigma_m^s(s_t)), \quad (23)$$

where  $y_t = Y_t/K_t$  is the scaled free-cash flow (before consumption):

$$y(s_t) = A - i(s_t) - g(i(s_t)). \quad (24)$$

Next, we discuss the first-best solution in Section II and the inalienability solution in Section III. A key difference between the two is the determination of the endogenous debt capacity, which corresponds to the left boundary conditions. Inalienability causes debt capacity to be much lower than the first-best level, which in turn causes policy functions to be non-linear, as we demonstrate in Section III.

## II. First-Best Solution

We report the first-best closed-form solution and provide a brief discussion of key economic insights. Appendix A provides the proof. Under the first-best, markets are (dynamically) complete and the entrepreneur's certainty-equivalent wealth coincides with the mark-to-market valuation of her net worth. Moreover, entrepreneur's consumption and production decisions can be separated.<sup>15</sup>

**Investment, Tobin's  $q$ , CAPM  $\beta$ , and Gordon Growth Formula.** The market value of the firm's capital stock is  $Q_t^{FB} = q^{FB}K_t$ , where  $q^{FB}$  is the endogenously determined Tobin average  $q$ . The FOC for investment implies:

$$q^{FB} = 1 + g'(i^{FB}), \quad (25)$$

which equates Tobin's  $q$  to the marginal cost of investing,  $1 + g'(i)$ . Adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that  $q^{FB} \neq 1$  in general.

Under the first best, financing policies are irrelevant. Therefore, consider a firm whose only asset is its capital stock. Then, this firm's value is  $Q_t^{FB}$ . Tobin's average  $q$ ,  $q^{FB}$ , also satisfies the following present value formula:

$$q^{FB} = \max_i \frac{A - i - g(i)}{r_K - (i - \delta_K)}, \quad (26)$$

where  $r_K$  is the expected rate of return for the firm whose only asset is its capital stock:

$$r_K = r + \rho\eta\sigma_K = r + \beta^{FB} \times (\mu_m - r) . \quad (27)$$

Because of the SDF given in (8) and our model's homogeneity property, the CAPM holds in our model for the firm whose only asset is its capital stock and  $\beta^{FB}$  in (27) is the CAPM  $\beta$  for this firm:

$$\beta^{FB} = \frac{\rho\sigma_K}{\sigma_m} . \quad (28)$$

Equation (26) is the Gordon growth formula with an endogenously determined growth rate. The numerator is the scaled free-cash flow  $y = A - i - g(i)$  and the denominator is given by the difference between the cost of capital  $r_K$  and the free-cash flow's expected growth rate ( $i^{FB} - \delta_K$ ). Equation (26) shows that the production side of our model generalizes Hayashi (1982) to situations where a firm faces both idiosyncratic and systematic risk, and where systematic risk commands a risk premium.

We can equivalently write the formula (26) as follows:

$$q^{FB} = \max_i \frac{A - i - g(i)}{r - (i - \delta)} , \quad (29)$$

where  $\delta$  is the risk-adjusted depreciation rate:

$$\delta = \delta_K + \rho\eta\sigma_K . \quad (30)$$

Note that (29) is the Gordon growth formula under the risk-neutral measure.<sup>16</sup>

Having characterized investment and the value of capital, we next turn to consumption and dynamic risk management. This part of our model is a generalized version of Merton (1969).

**Consumption, Hedging, and Portfolio Choices.** Because markets are dynamically complete, the entrepreneur's total wealth,  $M_t^{FB}$ , is equal to the sum of wealth  $S_t$  and the market value of capital  $Q_t^{FB}$ :

$$M_t^{FB} = S_t + Q_t^{FB} = (s_t + q^{FB})K_t = m^{FB}(s_t)K_t . \quad (31)$$

Scaled consumption is proportional to scaled net worth:

$$c^{FB}(s) = \chi m^{FB}(s) = \chi (s + q^{FB}) , \quad (32)$$

where  $\chi$  is Merton's marginal propensity to consume (MPC) given by:

$$\chi = r + \frac{\eta^2}{2\gamma} + \gamma^{-1} \left( \zeta - r - \frac{\eta^2}{2\gamma} \right) . \quad (33)$$

Because markets are complete and MM conditions hold, the entrepreneur's endogenous relative risk aversion, defined in equation (19), is equal to  $\gamma$ .

The FOC for  $\phi_h^{FB}(s_t)$  then yields

$$\phi_h^{FB}(s) = -q^{FB}. \quad (34)$$

The entrepreneur completely neutralizes her idiosyncratic risk exposure (due to her long position in the business venture) by going short and setting  $\phi_h^{FB}(s) = -q^{FB}$ , leaving her net worth  $M^{FB}$  with a zero net exposure to idiosyncratic risk  $\mathcal{Z}_h$ .

Similarly, the FOC for  $\phi_m^{FB}(s_t)$  yields

$$\phi_m^{FB}(s) = \frac{\eta}{\gamma\sigma_m} m^{FB}(s) - \beta^{FB} q^{FB}. \quad (35)$$

The first term achieves the target mean-variance aggregate risk exposure for her net worth  $M^{FB}$  and the second term in (35),  $-\beta^{FB} q^{FB}$ , fully offsets the entrepreneur's exposure to the aggregate shock through the firm's operations.

**Total Wealth and Debt Capacity.** Total wealth,  $M_t$ , evolves via the following GBM process:

$$dM_t^{FB} = M_t^{FB} \left[ \left( r - \chi + \frac{\eta^2}{\gamma} \right) dt + \frac{\eta}{\gamma} d\mathcal{Z}_{m,t} \right], \quad (36)$$

The entrepreneur's net worth has zero net exposure to the idiosyncratic shock  $\mathcal{Z}_{h,t}$  under the first-best. The debt capacity under the first-best is  $q^{FB}$  per unit of capital, so that  $s \geq -q^{FB}$  and  $m(s) \geq m(-q^{FB}) = 0$ . Because the entrepreneur has access to a credit line up to  $q^{FB}$  per unit of capital at the risk-free rate  $r$ , she can achieve first-best consumption smoothing and investment, attaining the maximal value of capital at  $q^{FB} K_t$  and the maximal net worth  $m^{FB}(s)$  given in (31).

Next we turn to the inalienability solution.

### III. Inalienable-Human-Capital Solution

In this section, we simplify the policy functions, derive the ODE for  $m(s)$ , and characterize the debt capacity under inalienable human capital.

#### A. Optimal Policy Functions and the ODE for $m(s)$

Substituting the value function given by (16) into the optimality conditions (12), (13), (15), and (14) and using (18), we obtain the following policy functions in terms of the liquidity ratio  $s$ . **Consumption  $C_t$  and Corporate Investment  $I_t$ .** The consumption policy is given by:

$$c(s) = \chi m'(s)^{-1/\gamma} m(s), \quad (37)$$

where  $\chi = \zeta^{\frac{1}{\gamma}} b^{\frac{\gamma-1}{\gamma}}$  is the marginal propensity to consume (MPC) under the first-best and given by (33). Under inalienability, consumption is nonlinear and depends on both the entrepreneur's



certainty equivalent wealth,  $m(s)$ , and the marginal value of wealth,  $m'(s)$ . Note that the entrepreneur's consumption is increasing in liquidity  $s$ . This can be seen by differentiating  $c(s)$  with respect to  $s$  and noting that  $m(s)$  is concave in  $s$ :

$$c'(s) = \chi \left[ m'(s)^{1-\frac{1}{\gamma}} - \frac{1}{\gamma} m''(s) m'(s)^{-\left(1+\frac{1}{\gamma}\right)} m(s) \right] > 0. \quad (38)$$

In Section V.D, we illustrate how the inalienability-of-human-capital constraint can generate very large marginal propensities to consume for the entrepreneur when the entrepreneur is close to exhausting her borrowing capacity.

Similarly, investment  $i(s)$  is given by

$$1 + g'(i(s)) = \frac{m(s)}{m'(s)} - s, \quad (39)$$

which also depends on  $m(s)$  and  $m'(s)$ . As one may expect,  $i(s)$  is increasing in  $s$ . To see this, differentiating  $i(s)$  with respect to  $s$  yields:

$$i'(s) = -\frac{1}{\theta} \frac{m(s)m''(s)}{m'(s)^2} > 0. \quad (40)$$

The positive investment-liquidity sensitivity again follows from the concavity of  $m(s)$ .

**Idiosyncratic Risk Hedge  $\Phi_{h,t}$  and Market Portfolio Allocation  $\Phi_{m,t}$ .** Simplifying (14) and (15) gives the following optimal idiosyncratic risk hedge  $\phi_h(s)$ :

$$\phi_h(s) = - \left( \frac{\gamma m(s)}{\gamma_e(s)} - s \right). \quad (41)$$

As we show in Section V,  $\phi_h(s) < 0$  for all  $s$ . Because the entrepreneur is exposed to idiosyncratic risk through the firm's operations, she optimally reduces this exposure by taking a short position in the hedging asset. However, with inalienability the hedging demand  $\phi_h(s)$  does not completely eliminate the entrepreneur's exposure to idiosyncratic risk. Indeed, note that since  $\gamma_e(s) > \gamma$  under inalienability, (41) implies that incomplete idiosyncratic risk hedging is optimal.

The optimal market portfolio allocation  $\phi_m(s)$  is given by

$$\phi_m(s) = \frac{\eta}{\sigma_m} \frac{m(s)}{\gamma_e(s)} - \frac{\rho\sigma_K}{\sigma_m} \left( \frac{\gamma m(s)}{\gamma_e(s)} - s \right) = \frac{\mu_m - r}{\sigma_m^2} \frac{m(s)}{\gamma_e(s)} - \beta^{FB} \phi_h(s), \quad (42)$$

where  $\beta^{FB}$  is the CAPM beta for the market value of capital under the first-best as given in (28), and  $\gamma_e(\cdot)$  is the entrepreneur's effective risk aversion given by (19). The first term in (42) reflects the mean-variance demand for the market portfolio, which differs from the standard Merton model in two ways: (1.) risk aversion  $\gamma$  is replaced by the effective risk aversion  $\gamma_e(s)$  and (2.) net worth is replaced by certainty equivalent wealth  $m(s)$ .

The second term in (42) gives the hedging demand with respect to systematic risk  $\mathcal{Z}_m$ . This

systematic risk hedging demand term is proportional to the idiosyncratic risk hedging demand,  $\phi_h(s)$ , where the proportionality coefficient is  $\beta^{FB}$ .

The optimal market portfolio allocation  $\phi_m(s)$ , balances achieving mean-variance efficiency for the entrepreneur's certainty equivalent wealth, as is reflected in the first term in (42), and maximizing the firm's financing capacity, as is reflected in the second term in (42). Overall, maximizing financing capacity amounts to both increasing the idiosyncratic risk exposure,  $|\phi_h(s)|$ , and reducing the systematic risk exposure,  $|\phi_m(s)|$ , away from the first best, as  $s$  moves closer to  $\underline{s}$ .

**ODE for  $m(s)$ .** Substituting the policy functions for  $c(s)$ ,  $i(s)$ ,  $\phi_h(s)$ , and  $\phi_m(s)$  and the value function (16) into the HJB equation (11) and using the homogeneity property, we obtain the following ODE for  $m(s)$ :

$$0 = \frac{m(s)}{1-\gamma} \left[ \gamma \chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] + [rs + A - i(s) - g(i(s))] m'(s) + (i(s) - \delta)(m(s) - sm'(s)) + \left( \frac{\gamma \sigma_K^2}{2} - \rho \eta \sigma_K \right) \frac{m(s)^2 m''(s)}{\gamma_e(s) m'(s)} + \frac{\eta^2 m'(s) m(s)}{2\gamma_e(s)}. \quad (43)$$

### B. Inalienable Human Capital and Endogenous Debt Capacity

The entrepreneur has the option to walk away at any time from her current firm of size  $K_t$ , thereby leaving behind all her liabilities. Her next-best alternative is to manage a firm of size  $\alpha K_t$ , where  $\alpha \in (0, 1)$  is a constant. That is, under this alternative, her talent creates less value, as  $\alpha < 1$ . Therefore, as long as the entrepreneur's liabilities are not too large the entrepreneur prefers to stay with the firm.<sup>17</sup>

The inalienability of her human capital gives rise to an endogenous debt capacity, denoted by  $\underline{S}_t$ , that satisfies:

$$J(K_t, \underline{S}_t) = J(\alpha K_t, 0). \quad (44)$$

That is,  $\underline{S}_t$  equates the value for the entrepreneur  $J(K_t, \underline{S}_t)$  of remaining with the firm and her outside option value  $J(\alpha K_t, 0)$  associated with managing a smaller firm of size  $\alpha K_t$  and no liabilities. Given that it is never efficient for the entrepreneur to quit on the equilibrium path,  $J(K, S)$  must satisfy the following condition:

$$J(K_t, S_t) \geq J(K_t, \underline{S}_t). \quad (45)$$

We can equivalently express the inalienability constraint given by (44) and (45) as:<sup>18</sup>

$$S_t \geq \underline{S}_t = \underline{S}(K_t), \quad (46)$$

where  $\underline{S}(K_t)$  defines the endogenous credit capacity as a function of the capital stock  $K_t$ . When  $S_t < 0$ , the entrepreneur draws on a line of credit (LOC) and services her debt at the risk-free rate  $r$  up to  $\underline{S}(K_t)$ . Note that debt is risk-free because (46) ensures that the entrepreneur does not walk away from the firm in an attempt to evade her debt obligations.

Substituting the value function,  $J(K, S)$ , given in (44) and simplifying the value-matching condition given in (44), we further obtain the following condition for  $m(s)$  at  $s = \underline{s}$ :

$$m(\underline{s}) = \alpha m(0). \quad (47)$$

Note that when  $\alpha = 0$  the entrepreneur has no outside option, so that  $m(\underline{s}) = 0$ , corresponding to the first-best case. By optimally setting  $\underline{s} = -q^{FB}$  we attain the first-best outcome where the entrepreneur can potentially pledge the entire market value of capital,  $q^{FB}$ , which is equal to Tobin's average  $q$  under the first-best. At the other extreme, when  $\alpha = 1$ , the entrepreneur's outside option is as good as her current employment. No long-term contract can then retain the entrepreneur, so that the model has no solution. Therefore, in order for the inalienability of human capital problem to have an interesting and non-degenerate solution, it is necessary to require that  $0 < \alpha < 1$ . For these values of  $\alpha$ , (47) implies that  $m(\underline{s}) > 0$ .<sup>19</sup>

We simplify the credit constraint given in (46) by expressing it in terms of scaled liquidity  $s$ ,

$$s_t \geq \underline{s}. \quad (48)$$

As in the household buffer-stock savings literature (e.g., Deaton (1991)), the *risk-averse* entrepreneur manages her liquid holdings  $s$  with the objective of smoothing her consumption. Setting  $s_t = \underline{s}$  for all  $t$  is too costly and suboptimal in terms of consumption smoothing. Although the credit constraint (48) rarely binds, it has to be satisfied with probability one. Only then can we ensure that the entrepreneur always stays with the firm.

Given that  $\{s_t : t \geq 0\}$  is a diffusion process and hence is continuous, in order to satisfy the inalienability constraint (48), it is necessary that both the idiosyncratic and systematic volatility at  $\underline{s}$  be equal to zero:

$$\sigma_h^s(\underline{s}) = 0 \quad \text{and} \quad \sigma_m^s(\underline{s}) = 0. \quad (49)$$

Otherwise, the probability of crossing a candidate debt limit of  $\underline{s}$  to its left is strictly positive, violating the credit constraint (48). By substituting  $\phi_h(s)$  given by (41) and  $\phi_m(s)$  given by (42) into the volatility functions, (21) and (22), we can equivalently express (49) as:

$$\frac{m(\underline{s})}{\gamma_e(\underline{s})} = 0. \quad (50)$$

In other words, at the endogenously determined  $\underline{s}$ , either the entrepreneur's scaled certainty equivalent wealth is zero,  $m(\underline{s}) = 0$ , or the entrepreneur is effectively infinitely risk averse,  $\gamma_e(\underline{s}) = \infty$ .<sup>20</sup> With inalienable human capital,  $m(\underline{s}) > 0$ , so that the volatility boundary conditions (50) can only be satisfied if  $\gamma_e(\underline{s}) = \infty$ , which is equivalent to

$$m''(\underline{s}) = -\infty. \quad (51)$$

That is, the inalienability condition (47) implies that the curvature of  $m(s)$  approaches infinity

when the entrepreneur runs out of liquidity at the endogenous boundary  $s = \underline{s}$ . Preserving her long-term relationship with the firm at  $\underline{s}$  is then so valuable that the entrepreneur does not want to take the chance that  $s$  crosses  $\underline{s}$ , implying that the entrepreneur is infinitely risk averse to the volatility in  $s$ .

Finally, when the entrepreneur is infinitely wealthy, she has no reason to quit and hence

$$\lim_{s \rightarrow \infty} m(s) = m^{FB}(s) = s + q^{FB}. \quad (52)$$

That is, the boundary condition at the right end of  $s$  under inalienability is the first-best solution.

**Summary.** We summarize the solution for the inalienability case in the theorem below.

**THEOREM 1:** *When  $0 < \alpha < 1$ , the solution to the inalienability problem is such that  $m(s)$  solves the ODE (43) subject to the FOCs (37) for consumption  $c(s)$ , (39) for investment  $i(s)$ , (41) for idiosyncratic risk hedge  $\phi_h(s)$ , (42) for market portfolio allocation  $\phi_m(s)$ , and the conditions (47) and (51) at the endogenous left boundary,  $\underline{s}$ , and (52) when  $s \rightarrow \infty$ .*

## IV. Equivalent Optimal Contract

We consider next the long-term contracting problem between an infinitely-lived, fully diversified, risk-averse investor (*the principal*) and an infinitely-lived, financially constrained, risk-averse, entrepreneur (*the agent*). The output process  $Y_t$  is publicly observable and verifiable. In addition, the entrepreneur cannot privately save.<sup>21</sup> The contract specifies an investment process  $\{I_t; t \geq 0\}$  and a compensation  $\{C_t; t \geq 0\}$  process, both of which depend on the entire history of idiosyncratic and aggregate shocks  $\{\mathcal{Z}_{h,t}, \mathcal{Z}_{m,t}; t \geq 0\}$ .

Because the risk-averse investor is fully diversified and markets are complete, the investor chooses investment  $\{I_t; t \geq 0\}$  and compensation  $\{C_t; t \geq 0\}$  to maximize the risk-adjusted discounted value of free cash flows:

$$F(K_0, V_0) = \max_{C, I} \mathbb{E}_0 \left[ \int_0^\infty \mathbb{M}_t (Y_t - C_t) dt \right], \quad (53)$$

where  $K_0$  is the initial capital stock and  $V_0$  is the entrepreneur's initial utility. Given that the investor is fully diversified, we use the same SDF,  $\mathbb{M}$  which is given in (8), to evaluate the present value of cash flows  $(Y_t - C_t)$ . Note that it is possible that  $Y_t < C_t$ . The contracting problem is subject to the entrepreneur's inalienability constraints at all future dates  $t \geq 0$  and the participation constraint at time 0. We denote by  $\underline{V}(K_t)$  the entrepreneur's endogenous outside utility payoff, so that the inalienability constraint at time  $t$  is given by:

$$V_t \geq \underline{V}(K_t), \quad t \geq 0, \quad (54)$$

where  $V_t$  is the entrepreneur's promised utility specified under the contract.

### A. Recursive Formulation

We proceed in three steps to transform the optimal contracting problem into a recursive form: (1) we define the entrepreneur's promised utility  $V$  and the principal's value  $F(K, V)$  in recursive form; (2) we map promised utility  $V$  into promised certainty-equivalent wealth  $W$ ; and (3) we simplify the contracting problem into a one-dimensional problem. While step (1) is standard in the recursive contracting literature, step (2) is less common but is essential to allow us to connect the contracting problem to the liquidity and risk management problem analyzed before. Derivations for results in this section are provided in Appendix B.

**The Investor's Value Function  $F(K_t, V_t)$ .** By using the martingale representation theorem, we show that the expected change of the entrepreneur's promised utility satisfies

$$\mathbb{E}_t[\zeta U(C_t)dt + dV_t] = \zeta V_t dt, \quad (55)$$

where  $\zeta U(C_t)dt$  is the utility of current compensation and  $dV_t$  is the change in promised utility. The realized change of the entrepreneur's promised utility,  $dV$ , implied by (55), can be written as the sum of: *i*) the expected change  $\mathbb{E}_t[dV_t]$  (the drift term); *ii*) a martingale term driven by the idiosyncratic shock,  $\mathcal{Z}_h$ ; and *iii*) a martingale term driven by the systematic shock,  $\mathcal{Z}_m$ :

$$dV_t = \zeta(V_t - U(C_t))dt + z_{h,t}V_t d\mathcal{Z}_{h,t} + z_{m,t}V_t d\mathcal{Z}_{m,t}, \quad (56)$$

where  $\{z_{h,t}; t \geq 0\}$  and  $\{z_{m,t}; t \geq 0\}$  respectively control the idiosyncratic and systematic volatilities of the entrepreneur's promised utility  $V$ .

We can then write the investor's value function  $F(K_t, V_t)$  in terms of: *i*) the entrepreneur's promised utility  $V_t$ ; and, *ii*) the venture's capital stock  $K_t$ . The contracting problem specifies investment  $I_t$ , compensation  $C_t$ , idiosyncratic risk exposure  $z_{h,t}$  and systematic risk exposure  $z_{m,t}$  to maximize the investor's risk-adjusted present discounted value of free cash flows. The following HJB equation for the investor's value  $F(K, V)$  holds:

$$\begin{aligned} rF(K, V) = \max_{C, I, z_h, z_m} & (Y - C) + (I - \delta K)F_K + [\zeta(V - U(C)) - z_m \eta V]F_V \\ & + \frac{\sigma_K^2 K^2 F_{KK}}{2} + \frac{(z_h^2 + z_m^2)V^2 F_{VV}}{2} + (z_h \nu_K + z_m \rho \sigma_K)KV F_{VK}. \end{aligned} \quad (57)$$

**From Promised Utility  $V_t$  To Promised Certainty-Equivalent Wealth  $W_t$ .** To link the optimal contract to the optimal liquidity and risk management policies derived in the preceding section, we need to express the entrepreneur's promised utility in dollars (units of consumption) rather than in utils. Let  $W$  denote the promised (certainty-equivalent) wealth, the amount that the entrepreneur would demand to permanently give up her productive human capital, walk away from the long-term contracting relationship, and retire as a Merton-style consumer living under complete markets.

We show that  $W_t$  can be linked to the promised utility,  $V_t$ , via  $V_t = V(W_t)$ , where

$$V(W_t) = U(bW_t), \quad (58)$$

where  $U(\cdot)$  is given in (6) and  $b$  is given in (17). Differentiating (58), we obtain  $V'(W_t) = bU'(bW_t)$  and  $V''(W_t) = b^2U''(bW_t)$ . In addition, the following SDE for  $W_t$  holds:

$$\begin{aligned} dW_t &= \frac{1}{V'(W_t)} [\zeta(V_t - U(C_t))dt + z_{h,t}V_t dZ_{h,t} + z_{m,t}V_t dZ_{m,t}] - \frac{(z_{h,t}^2 + z_{m,t}^2)V_t^2 V''(W_t)}{2(V'(W_t))^3} dt \\ &= \left[ \frac{\zeta(U(bW_t) - U(C_t))}{V'(W_t)} - \frac{(x_{h,t}^2 + x_{m,t}^2)K_t^2 V''(W_t)}{2V'(W_t)} \right] dt + x_{h,t}K_t dZ_{h,t} + x_{m,t}K_t dZ_{m,t}, \end{aligned} \quad (59)$$

where

$$x_{m,t} = \frac{z_{m,t}V(W_t)}{K_t V'(W_t)} \quad \text{and} \quad x_{h,t} = \frac{z_{h,t}V(W_t)}{K_t V'(W_t)}. \quad (60)$$

Note that  $x_{h,t}$  and  $x_{m,t}$  are the idiosyncratic and systematic volatilities of  $W_t$  (scaled by contemporaneous  $K_t$ ). As will become clear,  $x_{m,t}$  and  $x_{h,t}$  are closely tied to the firm's optimal risk management policies  $\phi_{h,t}$  and  $\phi_{m,t}$  analyzed earlier.

**Reduction to One Dimension.** We can reduce the contracting problem to one dimension, with the scaled wealth  $w_t = W_t/K_t$  as the unique state variable, by re-writing the investor's value  $F(K_t, V_t)$  as follows:

$$F(K_t, V_t) \equiv F(K_t, U(bW_t)) = P(K_t, W_t) = p(w_t) \cdot K_t. \quad (61)$$

It is then sufficient to solve for  $p(w)$  and characterize the scaled consumption, investment, idiosyncratic risk hedge, and stock market allocation rules as functions of  $w$ .

**The Principal's Endogenous Risk Aversion  $\gamma_p$ .** It is again helpful to introduce a measure of endogenous risk aversion for the principal. Let  $\gamma_p$  denote the principal's risk-aversion under the contract:

$$\gamma_{p,t} \equiv \frac{W_t P_{WW}(K_t, W_t)}{P_W(K_t, W_t)} = \frac{w_t p''(w_t)}{p'(w_t)} > 0. \quad (62)$$

The identity sign gives the definition of  $\gamma_p$ , and the equality sign follows from the homogeneity property.<sup>22</sup> As  $w$  is a liability for the investor we have  $p'(w) < 0$ . This is why, unlike in the standard definition of risk aversion, there is no minus sign in (62).

Under the first-best, the investor's value is linear in  $w$ , so that  $p''(w) = 0$  and the principal's effective risk aversion  $\gamma_p^{FB}(w)$  is zero for all  $w$ . Under inalienability, we can show that the investor's endogenous risk aversion  $\gamma_p(w) > 0$  since  $p(w)$  is decreasing and concave.

### B. Optimal Policy Functions

**Consumption  $C_t$  and Corporate Investment  $I_t$ .** Substituting (61) into (B5) and (B6), we obtain the following consumption and investment functions. Optimal consumption is  $C_t = \mathbf{c}(w_t)K_t$ , where  $\mathbf{c}(w)$  is given by:

$$\mathbf{c}(w) = \chi (-p'(w))^{1/\gamma} w, \quad (63)$$

and again  $\chi$  is the MPC under the first best given in (33). Under inalienability, consumption depends on both  $w$  and the investor's marginal value of liquidity  $p'(w)$ . Similarly, optimal investment is  $I_t = \mathbf{i}(w_t)K_t$ , where  $\mathbf{i}(w)$  is given by the following FOC:

$$1 + g'(\mathbf{i}(w)) = p(w) - wp'(w). \quad (64)$$

The left side of (64) is the marginal cost of investing and the right side of (64) is the marginal value of capital  $P_K(K, W) = p(w) - wp'(w)$ .

**Idiosyncratic Risk Exposure  $x_h(w)$  and Systematic Risk Exposure  $x_m(w)$ .** Substituting the principal's endogenous coefficient of risk aversion  $\gamma_p(w)$  given in (62) into the optimal risk exposures, given in (B7) and (B8), and simplifying, we obtain the following simple and economically transparent expressions for  $x_h(w_t)$  and  $x_m(w_t)$ . First, the idiosyncratic risk exposure is

$$x_h(w) = \frac{\gamma_p(w)}{\gamma_p(w) + \gamma} \nu_K w. \quad (65)$$

This equation is reminiscent of the classic co-insurance formula, which involves the ratio between the principal's endogenous risk aversion,  $\gamma_p(w)$ , and the sum of the two parties' risk-aversion coefficients.

Second, the systematic risk exposure is

$$x_m(w) = \frac{\eta w}{\gamma_p(w) + \gamma} + \rho \sigma_K w \frac{\gamma_p(w)}{\gamma_p(w) + \gamma}, \quad (66)$$

where the first term is the mean-variance demand and the second term corresponds to the systematic risk hedging demand.

Under the first best, we have  $x_h^{FB}(w) = 0$  and  $x_m^{FB}(w) = \eta w/\gamma$ , since  $\gamma_p^{FB}(w) = 0$ . The result that  $x_h^{FB}(w) = 0$  means that the entrepreneur's promised net worth  $W_t$  has no net exposure to idiosyncratic risk  $\mathcal{Z}_{h,t}$ . The result that  $x_m^{FB}(w) = \eta w/\gamma$  is the contracting version for the standard mean-variance demand for the entrepreneur's net worth  $W$ .

In contrast, under inalienability optimal co-insurance involves the agent taking on some idiosyncratic risk as well as reducing her market risk exposure a bit from the first-best level,<sup>23</sup> as can be seen in the expressions for  $x_h(w)$  in (65) and  $x_m(w)$  in (66).

### C. Dynamics of Scaled Promised Wealth $w$

Applying Ito's formula to  $w_t = W_t/K_t$ , we obtain the following dynamics for  $w$ :

$$dw_t = d(W_t/K_t) = \mu^w(w_t)dt + \sigma_h^w(w_t)dZ_{h,t} + \sigma_m^w(w_t)dZ_{m,t}, \quad (67)$$

where the idiosyncratic and systematic volatilities for  $w$ ,  $\sigma_h^w(\cdot)$  and  $\sigma_m^w(\cdot)$ , are given by

$$\sigma_h^w(w) = -\nu_K \frac{\gamma w}{\gamma_p(w) + \gamma} < 0, \quad (68)$$

$$\sigma_m^w(w) = \left( \frac{\eta}{\gamma} - \rho\sigma_K \right) \frac{\gamma w}{\gamma_p(w) + \gamma}. \quad (69)$$

Note that both  $\sigma_h^w(w)$  and  $\sigma_m^w(w)$  are proportional to  $w/(\gamma_p(w) + \gamma)$ . Finally, the drift function  $\mu^w(\cdot)$  of  $w_t$  is given by:

$$\mu^w(w) = \frac{\zeta}{1-\gamma} \left( w + \frac{\mathbf{c}(w)}{\zeta p'(w)} \right) - w(\mathbf{i}(w) - \delta_K) + \frac{\gamma(x_h^2(w) + x_m^2(w))}{2w} - (\nu_K \sigma_h^w(w) + \rho\sigma_K \sigma_m^w(w)). \quad (70)$$

### D. ODE for $p(w)$

Substituting  $F(K, V) = p(w) \cdot K$  into the HJB equation (57), solving for  $p(w)$ , and substituting for the policy functions  $\mathbf{c}(w)$ ,  $\mathbf{i}(w)$ ,  $x_h(w)$  and  $x_m(w)$ , we obtain the following ODE for the investor's value  $p(w)$ :

$$\begin{aligned} rp(w) = & A - \mathbf{i}(w) - g(\mathbf{i}(w)) + \frac{\chi\gamma}{1-\gamma} (-p'(w))^{1/\gamma} w + (\mathbf{i}(w) - \delta)(p(w) - wp'(w)) \\ & + \frac{\zeta}{1-\gamma} wp'(w) + \left( \frac{\gamma\sigma_K^2}{2} - \rho\eta\sigma_K \right) \frac{w^2 p''(w)}{\gamma_p(w) + \gamma} - \frac{\eta^2}{2} \frac{wp'(w)}{\gamma_p(w) + \gamma}, \end{aligned} \quad (71)$$

where  $\mathbf{i}(w)$  is given by (64) and  $\gamma_p(w)$  is given by (62). Again, this ODE for  $p(w)$  characterizes the interior solution for both the first-best and inalienability cases. The only difference between the two problems is reflected in the inalienability constraint to which we turn next.

### E. Inalienability Constraint

The entrepreneur's outside option at any time is to manage a new firm with effective size  $\alpha K_t$  but with no legacy liabilities. Other than the size of the capital stock  $K$ , the new firm's production technology is identical to the one that she has just abandoned. Let  $\tilde{V}(\cdot)$  and  $\tilde{W}(\cdot)$  be the entrepreneur's utility and the corresponding certainty-equivalent wealth in this new firm, and suppose as before that investors in the new firm make zero net profits under competitive markets. Then, from (61) we obtain the following condition:

$$F(\alpha K_t, \tilde{V}(\alpha K_t)) = P(\alpha K_t, \tilde{W}(\alpha K_t)) = 0. \quad (72)$$



When the entrepreneur is indifferent between leaving her employer or not we have

$$\underline{W}(K_t) = \widetilde{W}(\alpha K_t), \quad (73)$$

where  $\underline{W}(K_t)$  is the lowest possible value for the entrepreneur's promised wealth such that her inalienability constraint is satisfied. Equation (73) is equivalent to:

$$\underline{w}_t \equiv \underline{W}(K_t)/K_t = \widetilde{W}(\alpha K_t)/K_t = \alpha \widetilde{W}(\alpha K_t)/(\alpha K_t) = \alpha \widetilde{w}_t, \quad (74)$$

where the last equality follows from the assumption that the new firm's capital is a constant fraction,  $\alpha$ , of the original firm's contemporaneous capital stock. The homogeneity property and the condition given in (72) together imply that  $p(\widetilde{w}) = 0$ . Thus, substituting  $\underline{w}_t = \alpha \widetilde{w}_t$  into  $p(\widetilde{w}_t) = 0$  we obtain the following simple expression for the inalienability constraint (where  $0 < \alpha < 1$ ):

$$p(\underline{w}/\alpha) = 0. \quad (75)$$

Note that inalienability implies that the entrepreneur's minimum wealth must be strictly positive,  $\underline{w} > 0$ . For the first-best case, however,  $\underline{w} = 0$ .

In both the first-best and inalienability cases the volatility functions  $\sigma_h^w(w)$  and  $\sigma_m^w(w)$  are equal to zero at  $\underline{w}$  to ensure that  $w$  never crosses  $\underline{w}$  to the left ( $w \geq \underline{w}$ ):

$$\sigma_h^w(\underline{w}) = 0 \quad \text{and} \quad \sigma_m^w(\underline{w}) = 0. \quad (76)$$

Equations (68) and (69) imply that the boundary conditions given in (76) are equivalent to:

$$\frac{\gamma \underline{w}}{\gamma_p(\underline{w}) + \gamma} = 0. \quad (77)$$

Equation (77) holds when either  $\underline{w} = 0$  (for the first-best case) or  $\gamma_p(\underline{w}) = \infty$  (under inalienability), which is equivalent to

$$p''(\underline{w}) = -\infty. \quad (78)$$

That is, inalienability causes the principal to be infinitely risk averse with respect to  $w$  at  $\underline{w}$ ! Even though the principal is well diversified, he is endogenously infinitely risk averse at  $\underline{w}$  with respect to his investment with the entrepreneur. As  $w$  approaches  $\underline{w}$ ,  $p(w)$  is strictly positive and reaches its maximum value (recall that  $p(\underline{w}) = -\underline{s} > 0$ ). Preserving his long-term relationship and investment with the entrepreneur at  $\underline{w}$  is then so valuable that the investor does not want to take the chance that  $w$  crosses  $\underline{w}$ , implying that the principal is infinitely risk averse to the volatility in  $w$ .

As for the primal liquidity and risk management problem, our contracting analysis reveals that the boundary conditions under inalienability are fundamentally different from those for the first-best: under inalienability  $\gamma_p(\underline{w}) = \infty$ , while under the first-best  $\gamma_p(w) = 0$  for all  $w$ . The first-best solution confirms the conventional wisdom for hedging, which calls for the complete elimination

of idiosyncratic risk exposures for the risk-averse entrepreneur. This conventional wisdom applies only to a complete-markets, Arrow-Debreu, world. With inalienability, this conventional wisdom no longer holds.

We summarize the contracting solution under inalienability in the theorem below.

**THEOREM 2:** *When  $0 < \alpha < 1$ , the optimal contract under inalienability is such that  $p(w)$  solves the ODE (71) subject to the FOCs (63) for  $c(w)$ , (64) for  $i(w)$ , (65) for idiosyncratic risk exposure  $x_h(w)$ , (66) for systematic risk exposure  $x_m(w)$ , and the boundary conditions (75) and (78). In addition, the drift  $\mu^w(w)$  given in (70) must be non-negative at  $\underline{w}$ , so that  $w$  is weakly increasing at  $\underline{w}$  with probability one.*

Finally, to complete the characterization of the optimal contracting solution we set the entrepreneur's initial reservation utility  $V_0^*$  such that  $F(K_0, V_0^*) = 0$  to be consistent with the general assumption that capital markets are competitive.

#### F. Equivalence

By equivalence, we mean that the resource allocations  $\{C_t, I_t; t \geq 0\}$  under the two problem formulations are identical for any path  $\{\mathcal{Z}_h, \mathcal{Z}_m\}$ . We demonstrate this equivalence in Appendix (B.B), by verifying that the following holds:

$$s = -p(w) \quad \text{and} \quad w = m(s), \quad (79)$$

The preceding equation implies that  $-p \circ m(s) = s$ . In other words, the state variable  $s$  in the primal liquidity and risk management problem is shown to be equal to  $-p(w)$ , the negative of the value function in the dual contracting problem. Correspondingly the scaled wealth function  $m(s)$  in the primal problem is equal to  $w$ , the scaled promised wealth, the state variable in the contracting problem.

**[Place Table II about here]**

Table II provides a detailed side-by-side comparison of the two problem formulations along all three relevant dimensions of the model: (a.) the state variable, (b.) the policy rules, and (c.) the value functions for both the inalienability and first-best cases. Panels A, B, and C offer a side-by-side mapping for the state variable, value function, and policy rules under the two formulations. The differences between the inalienability and first-best cases are entirely driven by the conditions pinning down the firm's borrowing capacity, as we highlight in Panels D and E.

Panel D describes the conditions for the borrowing capacity in the inalienability case ( $0 < \alpha < 1$ ). The entrepreneur's inalienability of human capital implies that  $m(\underline{s}) = \alpha m(0)$  given in (47) and  $p(\underline{w}/\alpha) = 0$  given in (75) have to be satisfied at the respective free boundaries  $\underline{s}$  and  $\underline{w}$  in the two formulations. Given these inalienability constraints, the volatility conditions can only be satisfied if the curvatures of the value functions,  $m(s)$  and  $p(w)$ , approach  $-\infty$  at the left boundaries. We also verify that the drift conditions at the left boundaries hold.

Panel E summarizes the first-best case, where  $\alpha = 0$ . The investor's value is given by the difference between the market value of capital,  $q^{FB}$ , and the promised wealth to the entrepreneur,  $w_t$ :  $p^{FB}(w_t) = q^{FB} - w_t$ . Equivalently,  $w_t = m^{FB}(s_t) = s_t + q^{FB}$ . The first-best policy rules such as consumption, investment, under the two formulations are consistent. For consumption, we have  $c^{FB}(w_t) = \chi w_t = \chi m^{FB}(s_t) = c^{FB}(s_t)$ . For investment, both formulations yield the same constant investment-capital ratio,  $i^{FB}$ . The optimal idiosyncratic risk exposure  $x_h^{FB}(w) = 0$  shuts down the idiosyncratic risk exposure of  $W_t$ , which is equivalent to setting the idiosyncratic risk hedge  $\phi_h^{FB}(s) = -q^{FB}$  in the primal formulation, thus eliminating idiosyncratic risk for  $M_t$ . The optimal systematic risk exposure  $x_m^{FB}(w) = \eta w/\gamma$  yields the aggregate volatility of  $\eta/\gamma$  for  $W_t$ , which is consistent with the fact that  $\phi_m^{FB}(s)$  given in (35) implies an aggregate volatility of  $\eta/\gamma$  for  $M_t$ . Last but not least, the borrowing limits in the two formulations are also consistent, in that  $\underline{w}^{FB} = 0$  if and only if  $\underline{s}^{FB} = -q^{FB}$ : the condition that the lower boundary for  $w$  is zero is equivalent to the property that the entrepreneur can at any time  $t$  borrow up to the entire market value of capital  $q^{FB}K_t$ .

## V. Quantitative Analysis

In this section, we present our main qualitative and quantitative results. For simplicity, we choose the widely-used quadratic adjustment cost function,  $g(i) = \theta i^2/2$ , for which we have explicit formulae for Tobin's  $q$  and optimal  $i$  under the first-best:<sup>24</sup>

$$q^{FB} = 1 + \theta i^{FB} \quad \text{and} \quad i^{FB} = r + \delta - \sqrt{(r + \delta)^2 - 2 \frac{A - (r + \delta)}{\theta}}. \quad (80)$$

Our model is parsimonious with eleven parameters. We set the entrepreneur's coefficient of relative risk aversion to  $\gamma = 2$ , the equity risk premium  $(\mu_m - r)$  to 6%, and the annual volatility of the market portfolio return to  $\sigma_m = 20\%$ , implying a Sharpe ratio of  $\eta = (\mu_m - r)/\sigma_m = 30\%$ . We choose the annual risk-free rate to be  $r = 5\%$  and set the entrepreneur's discount rate at  $\zeta = r = 5\%$ . These are standard parameter values in the asset pricing literature.

For the production-side parameters, we take the estimates in Eberly, Rebelo, and Vincent (2009) and set the annual productivity  $A$  at 20% and the annual volatility of capital shocks at  $\sigma_K = 20\%$ . We set the correlation between the market portfolio return and the firm's depreciation shock at  $\rho = 0.2$ , which implies that the idiosyncratic volatility of the depreciation shock is  $\nu_K = 19.6\%$ . We fit the first-best values of  $q^{FB}$  and  $i^{FB}$  to the sample averages by setting the adjustment cost parameter at  $\theta = 2$  and the (expected) annual capital depreciation rate at  $\delta_K = 11\%$ , both of which are in line with estimates in Hall (2004) and Riddick and Whited (2009). These parameters imply that  $q^{FB} = 1.264$ ,  $i^{FB} = 0.132$ , and  $\beta^{FB} = 0.2$ . Finally, we set the inalienability parameter at  $\alpha = 0.8$ . The parameter values for our baseline calculation are summarized in Table III.

[Place Table III about here]

### A. Firm Value and Endogenous Debt Capacity

We begin by linking the value functions of the two optimization problems,  $p(w)$  and  $m(s)$ .

**Liquidity Ratio  $s$  and Certainty-Equivalent Wealth  $m(s)$ .** Panels A and C of Figure 1 plot  $m(s)$  and the marginal value of liquidity  $m'(s)$ , respectively. Under the first-best, the entrepreneur's scaled net worth is simply given by the sum of her financial wealth  $s$  and the market value of the capital stock:  $m^{FB}(s) = s + q^{FB} = s + 1.264$ . Note that  $m^{FB}(s) \geq 0$  implies that  $s \geq -q^{FB}$ , so that the debt limit under the first-best is  $\underline{s}^{FB} = -q^{FB}$ .

As one would expect,  $m(s) < m^{FB}(s) = q^{FB} + s$  due to inalienability. Moreover,  $m(s)$  is increasing and concave. The higher the liquidity  $s$  the less constrained is the entrepreneur, so that  $m'(s)$  decreases. In the limit, as  $s \rightarrow \infty$ ,  $m(s)$  approaches  $m^{FB}(s) = q^{FB} + s$  and  $m'(s) \rightarrow 1$ . The equilibrium credit limit under inalienability is  $\underline{s} = -0.208$ , meaning that the entrepreneur's maximal borrowing capacity is 20.8% of the contemporaneous capital stock  $K$ , which is as little as one-sixth of the first-best debt capacity. The corresponding scaled certainty-equivalent wealth is  $m(-0.208) = 0.959$ . When the endogenous financial constraint binds at  $\underline{s} = -0.208$ , the marginal certainty equivalent value of liquidity  $m'(s)$  is highest and is equal to  $m'(-0.208) = 1.394$ . Figure 1 clearly illustrates that the first-best and inalienability cases are fundamentally different.<sup>25</sup>

[Place Figure 1 about here]

**Promised Wealth  $w$  and Investors' Value  $p(w)$ .** Panels B and D of Figure 1 plot  $p(w)$  and  $p'(w)$ , respectively. Under the first-best, compensation to the entrepreneur is simply a one-to-one transfer from investors:  $p^{FB}(w) = q^{FB} - w = 1.264 - w$ . With inalienable human capital,  $p(w) < q^{FB} - w$ , and  $p(w)$  is decreasing and concave. As  $w$  increases the entrepreneur is less constrained. In the limit, as  $w \rightarrow \infty$ ,  $p(w)$  approaches  $q^{FB} - w$ , and  $p'(w) \rightarrow -1$ . The entrepreneur's inability to fully commit not to walk away *ex post* imposes a lower bound  $\underline{w}$  on  $w$ . For our parameter values,  $\underline{w} = 0.959$ . Note that  $\underline{w} = 0.959 = m(\underline{s}) = m(-0.208)$ . This result is no coincidence and is implied by our equivalence result between the two optimization problems. The entrepreneur receives at least 95.9% in promised certainty-equivalent wealth for every unit of capital stock, which is strictly greater than  $\alpha = 0.8$  since the capital stock generates strictly positive net present value under the entrepreneur's control.

Panels A and B of Figure 1 illustrate how  $(s, m(s))$  is the "mirror-image" of  $(-p(w), w)$ . To be precise, rotating Panel B counter-clock-wise by  $90^\circ$  (i.e., turning the original  $x$ -axis for  $w$  into the new  $y$ -axis  $m(s)$ ) and then adding a minus sign to the horizontal  $x$ -axis (i.e., setting  $-p(w) = s$ ), we produce Panel A. Panel C shows that the entrepreneur's marginal value of liquidity  $m'(s)$  is greater than 1, which means that the liquid asset is valued more than its face value by the financially constrained entrepreneur. Panel D illustrates the same idea viewed from the investor's perspective: the marginal cost of a monetary transfer to is less than one for the investor,  $-1 < p'(w) < 0$ , because compensating the entrepreneur relaxes her financial constraint which in turn generates

value. Despite being fully diversified the investor behaves in an under-diversified manner due to the entrepreneur's inalienability constraints. This is reflected in the concavity of the investor's value function  $p(w)$ .

### B. Idiosyncratic Risk Management

[Place Figure 2 about here]

Panels A and B of Figure 2 plot the idiosyncratic-risk hedging demand  $\phi_h(s)$  and  $x_h(w)$  in the two formulations. Note that  $\phi_h(s)$  and  $x_h(w)$  respectively control the idiosyncratic volatilities of *total* liquid wealth  $S$  and certainty equivalent wealth  $W$ , as seen in (10) and (59). In Panels C and D of Figure 2, we plot the idiosyncratic volatilities of respectively scaled liquidity  $s$ ,  $\sigma_h^s(s)$ , and scaled wealth  $w$ ,  $\sigma_h^w(w)$ , which are directly linked to the risk management policies  $\phi_h(s)$  and  $x_h(w)$ . A key observation is that the volatility of  $S$  is different from the volatility of scaled liquidity,  $s = S/K$ . Equation (21), which states  $\sigma_h^s(s_t) = (\phi_h(s_t) - s_t)\nu_K$ , makes clear that  $\sigma_h^s(s_t)$  is affected by the hedging position  $\phi_h(s_t)\nu_K$ , which drives changes in  $S$ , and by  $-s_t\nu_K$ , through the idiosyncratic risk exposure of  $K$ , which influences compensation through the inalienability constraint. Proceeding in the same way for the contracting formulation, we obtain the following expression linking  $x_h(w)$  and  $\sigma_h^w(w)$ :

$$\sigma_h^w(w_t) = -\frac{\gamma}{\gamma_p(w_t)}x_h(w_t). \quad (81)$$

Consider now the first-best solution given by the dotted lines in Figure 2. Panel A shows that the first-best idiosyncratic-risk hedging demand is constant:  $\phi_h(s_t) = -q^{FB} = -1.264$ . Panel B confirms the same result, as  $x_h^{FB}(w_t) = 0$  for all  $w_t$ , which establishes the classic first-best result that optimal hedging for a *risk-averse* entrepreneur involves a zero net exposure to idiosyncratic shocks. Stated equivalently, the first-best idiosyncratic risk hedging policy completely insulates the entrepreneur's net worth  $M_t^{FB} = S_t + q^{FB}K_t$  from the idiosyncratic shock  $\mathcal{Z}_h$ , as one can see from the dynamics of  $M$  given in (36).

Panels C and D reveal a less obvious but important insight for the first-best case, namely that complete idiosyncratic risk hedging of net worth implies neither zero volatility for  $s$  nor for  $w$  in general. It is only when the entrepreneur has fully exhausted her debt capacity, when  $s_t = -q^{FB}$  (and equivalently  $w_t = 0$ ), that the volatility of scaled  $s$  and of  $w$  are zero:  $\sigma_h^s(s_t) = \sigma_h^w(w_t) = 0$ . When  $s_t > -q^{FB}$  (and  $w_t > 0$ ), the first-best solution is such that  $|\sigma_h^s(s_t)|$  and  $|\sigma_h^w(w_t)|$  strictly increase with respectively  $s_t = S_t/K_t$  and  $w_t = W_t/K_t$ , because of the impact of the idiosyncratic shock  $\mathcal{Z}_h$  on the firm's capital stock.

Consider next the inalienability case. Panels A and B strikingly reveal how different the hedging policy under inalienability is from the first-best. Because the endogenous debt limit  $|\underline{s}| = 0.208$  (and  $\underline{w} = 0.959$ ) under inalienability is much tighter than the first-best limit,  $|\underline{s}^{FB}| = q^{FB} = 1.264$  (and  $\underline{w}^{FB} = 0$ ), the entrepreneur is severely constrained in her ability to hedge out the idiosyncratic risk exposure of her certainty equivalent wealth  $M$ .

A key optimality condition is that the entrepreneur has to honor her liabilities with probability one, which requires that  $\sigma_h^s(\underline{s}) = 0$  and  $\sigma_h^w(\underline{w}) = 0$ . This equilibrium condition of zero volatility together with the inalienability conditions  $m(\underline{s}) = \alpha m(0)$  and  $p(\underline{w}/\alpha) = 0$  imply endogenous infinite risk aversion at  $\underline{s}$  and  $\underline{w}$ , meaning that  $\gamma_e(\underline{s}) = \infty$  and  $\gamma_p(\underline{w}) = \infty$  as shown in Figure 3.<sup>26</sup>

[Place Figure 3 about here]

Zero idiosyncratic volatility for  $s$  at  $\underline{s}$  (and equivalently for  $w$  at  $\underline{w}$ ) is achieved by setting the hedging position to  $\phi_h(\underline{s}) = \underline{s}$  (and equivalently  $x_h(\underline{w}) = \nu_K \underline{w}$ ). These expressions encapsulate the following general insight about hedging key-man risk. Suppose that the entrepreneur's scaled liquidity is at its limit,  $s_t = \underline{s}$ , and consider the consequences of a positive idiosyncratic shock  $dZ_{h,t}$ . Among other effects, such a shock increases the outside value of the entrepreneur's human capital and increases the entrepreneur's incentives to leave the firm.<sup>27</sup> How can the entrepreneur hedge against this risk so as to continue honoring her outstanding debt liabilities? By setting  $\phi_h(s) = s$  at the credit limit  $\underline{s}$ , as we explain next. Let  $Z_{h,t+\Delta} = Z_{h,t} + \sqrt{\Delta}$  denote the outcome of a positive shock over a small time increment  $\Delta$ . We can calculate the resulting liquidity ratio  $s_{t+\Delta}$  as follows:<sup>28</sup>

$$s_{t+\Delta} \equiv \frac{S_{t+\Delta}}{K_{t+\Delta}} \approx \frac{S_t + \phi_{h,t} K_t \nu_K \sqrt{\Delta}}{(1 + \nu_K \sqrt{\Delta}) K_t} = \frac{s_t + \phi_{h,t} \nu_K \sqrt{\Delta}}{(1 + \nu_K \sqrt{\Delta})}, \quad (82)$$

where the numerator uses (10) for  $dS_t$  and the denominator uses (1) for  $dK_t$ . To ensure that the credit constraint is satisfied at  $t + \Delta$  we have to set  $s_{t+\Delta} = s_t = \underline{s}$  in (82), which means that  $\phi_h(\underline{s}) = \underline{s} < 0$ . Had the entrepreneur chosen a larger hedging position, say  $|\phi_h(\underline{s})| > |\underline{s}|$ , or in the extreme scenario  $|\phi_h(\underline{s})| = |\phi_h^{FB}| = q^{FB}$ , we would have  $s_{t+\Delta} < s_t = \underline{s} < 0$ , violating the equilibrium condition  $s \geq \underline{s}$ . Following essentially the same argument for  $w = W/K$ , we can verify that  $x_h(\underline{w}) = \nu_K \underline{w} > 0$ , which implies that the entrepreneur's net worth  $W$  is overexposed to idiosyncratic risk relative to the first-best.

In words, the hedging positions at  $\underline{s}$  and  $\underline{w}$  are set so as to exactly offset the impact of the idiosyncratic shock  $Z_h$  on  $K_t$  in  $s_t = S_t/K_t$  and  $w_t = W_t/K_t$  and thereby turn off the volatilities of  $s$  at  $\underline{s}$  and  $w$  at  $\underline{w}$ . These hedging positions, in turn, significantly expose the entrepreneur's net worth  $W$  to idiosyncratic risk.

Turning now to the right end of the support for  $s$  and  $w$ , we observe that as  $s \rightarrow \infty$  (and equivalently  $w \rightarrow \infty$ ), the inalienability constraint becomes irrelevant. As a result, the entrepreneur achieves perfect risk sharing:  $\lim_{s \rightarrow \infty} \phi_h(s) = \phi_h^{FB} = -q^{FB}$  and  $\lim_{w \rightarrow \infty} x_h(w) = x_h^{FB} = 0$ .

With inalienability, the idiosyncratic risk hedge  $|\phi_h(\underline{s})| = |\underline{s}|$  at the debt limit is much lower than when the entrepreneur is unconstrained. More generally, when  $s$  moves away from the debt limit  $\underline{s}$ ,  $|\phi_h(s)|$  in effect becomes a 'weighted average' of the first-best policy of maximizing net worth and the zero-volatility policy for  $s$  at the debt limit, with an increasing weight put on the first-best policy as  $s$  increases. Correspondingly, Panel B shows that as the entrepreneur's promised scaled certainty equivalent wealth  $w$  increases, the entrepreneur becomes less exposed to idiosyncratic

risk, i.e.,  $x_h(w)$  decreases with  $w$  and eventually approaches zero as  $w \rightarrow \infty$ .<sup>29</sup> To summarize, the ‘key-man’ risk management problem for the firm boils down to a compromise between the maximization of the entrepreneur’s net worth, which requires full insurance against idiosyncratic risk, and the maximization of the firm’s financing capacity, which involves reducing the volatility of scaled liquidity and hence exposing the entrepreneur to idiosyncratic risk. This compromise can be seen as a general principle of idiosyncratic risk management for financially constrained firms that emerges from our analysis.<sup>30</sup>

### C. Optimal Equity Market Exposure

Panels A and B of Figure 4 plot the entrepreneur’s market portfolio allocation  $\phi_m(s)$  and the entrepreneur’s systematic risk exposure  $x_m(w)$  in the two formulations. Recall that  $\phi_m$  and  $x_m$  respectively control the systematic volatilities of liquid wealth  $S$  and certainty equivalent wealth  $W$ , as seen in (10) and (59). Panels C and D of Figure 4 plot the systematic volatility of *scaled* liquidity  $s$ ,  $\sigma_m^s(s)$ , and of *scaled*  $w$ ,  $\sigma_m^w(w)$ , respectively.

Again, we show that the policies  $\phi_m(s)$  and  $x_m(w)$ , plotted in Panels A and B, are directly linked to the corresponding volatilities,  $\sigma_m^s(s)$  and  $\sigma_m^w(w)$ , plotted in Panels C and D. Equation (22), which states  $\sigma_m^s(s_t) = (\phi_m(s_t) - \beta^{FB} s_t) \sigma_m$ , makes clear that  $\sigma_m^s(s_t)$  is affected by the market allocation term,  $\phi_m(s_t) \sigma_m$ , and by  $-s_t \beta^{FB} \sigma_m = -s_t \rho \sigma_K$ , which comes from the systematic risk exposure of  $K$ . Proceeding in the same way for the contracting problem, we obtain the following expression linking  $x_m(w)$  and  $\sigma_m^w(w)$ :

$$\sigma_m^w(w_t) = x_m(w_t) - \rho \sigma_K w_t. \quad (83)$$

Again, the key observation is that the systematic volatility of  $W$ , which is equal to  $x_m(w_t) K_t$ , is different from  $\sigma_m^w(w_t)$ , the systematic volatility for  $w = W/K$ .

Consider now the first-best solution given by the dotted lines in Figure 4. Panels A and B plot the classic Merton (1969) portfolio allocation result, which is linear in  $s$  and  $w$ . Panels C and D reveal a less emphasized insight, which however is important for our risk management analysis, namely that the systematic volatilities for scaled  $s$  and  $w$ ,  $\sigma_m^s(s_t)$  and  $\sigma_m^w(w_t)$ , are also linear in  $s$  and  $w$ , respectively. It is only when the entrepreneur has fully exhausted her debt capacity at  $s_t = -q^{FB}$  (and  $w_t = 0$ ), that we have  $\sigma_m^s(s_t) = \sigma_m^w(w_t) = 0$ .

Consider next the inalienability case. Panels A and B again reveal how different the risk exposures are from the first-best. Recall that the debt limit under inalienability  $|\underline{s}| = 0.21$  (and  $\underline{w} = 0.96$ ) is much tighter than the first-best debt limit,  $|\underline{s}^{FB}| = q^{FB} = 1.264$  (and  $\underline{w}^{FB} = 0$ ). As a result, the entrepreneur is endogenously more risk averse,  $\gamma_e(s) > \gamma$ , as shown in Panel A of Figure 3 and  $m(s)$  is lower than the first-best level for all  $s$ . Equivalently, in the contracting problem the principal is also endogenously more risk averse,  $\gamma_p(w) > 0$ , as shown in Panel B of Figure 3 and  $p(w)$  is lower than the first-best level for all  $w$ . It follows that the entrepreneur allocates less of her net worth to the stock market for any  $s$ , and equivalently the principal exposes the agent to

less systematic risk for any  $w$ . At the debt limit, in particular, the endogenous risk aversion for both the entrepreneur and the principal approach infinity,  $\gamma_e(\underline{s}) = \infty$  and  $\gamma_p(\underline{w}) = \infty$ , so that the systematic volatilities for both  $s$  and  $w$  approach zero:  $\sigma_m^s(\underline{s}) = \sigma_m^w(\underline{w}) = 0$ .

It is important to note that zero systematic volatilities are achieved by setting  $\phi_m(\underline{s}) = \beta^{FB} \underline{s}$  and  $x_m(\underline{w}) = \rho \sigma_K \underline{w}$ , as can be seen from (22) and (83). Remarkably, while the mean-variance term vanishes at the debt limit, the hedging term does not, because the entrepreneur still needs to immunize the systematic risk exposures of  $s$  and  $w$  that come from  $K$ .<sup>31</sup>

At the other end of the support, as  $s \rightarrow \infty$  (and equivalently as  $w \rightarrow \infty$ ) and the inalienability constraint becomes irrelevant, we see that the entrepreneur achieves the first-best:  $\lim_{s \rightarrow \infty} \phi_m(s) = \phi_m^{FB}(s)$  and  $\lim_{w \rightarrow \infty} x_m(w) = x_m^{FB}(w) = \eta w / \gamma$ . In general, for any given  $s$ ,  $|\phi_m(s)|$  is a ‘weighted average’ of the first-best policy of maximizing net worth and the zero-volatility policy for  $s$  at the debt limit, with an increasing weight being put on the first-best policy as  $s$  increases (the same is true for  $x_m(w)$  as  $w$  increases.)

In sum, the risk management problem for the firm boils down to a compromise between achieving mean-variance efficiency for the entrepreneur’s net worth and maximizing the firm’s financing capacity. To expand its financing capacity the firm must reduce the volatility of  $s$  when  $s$  is low, which involves scaling back  $|\phi_h(s)|$  and  $|\phi_m(s)|$ . Overall, this strategy amounts to both reducing the systematic risk exposure and increasing the idiosyncratic risk exposure of the entrepreneur’s net worth. This last result can be seen more directly from the risk exposures of the agent’s net worth under the optimal contract. Indeed, the optimal contract requires that  $x_m(w) < x_m^{FB}(w) = \eta w / \gamma$  and  $x_h(w) > x_h^{FB}(w) = 0$ , as can be seen from Panels B in Figures 2 and 4.

[Place Figure 4 about here]

#### D. Investment and Compensation

[Place Figure 5 about here]

**Investment and its Sensitivity to Liquidity.** Figure 5 plots corporate investment and its sensitivity. Panels A and C plot  $i(s)$  and  $i'(s)$  for the primal problem, and Panels B and D plot  $i(w)$  and  $i'(w)$  for the contracting problem, respectively. The dotted lines describe the constant  $i^{FB} = 0.132$  under the first-best benchmark. Under inalienability, the investment-capital ratio is lower than  $i^{FB} = 0.132$  under all circumstances, and it increases from  $-0.043$  to  $i^{FB} = 0.132$  when  $s$  increases from  $\underline{s} = -0.208$  towards  $\infty$ , or equivalently when  $w$  increases from  $\underline{w} = 0.959$  towards  $\infty$ , as can be seen from Panels A and B, respectively. As the firm’s financial slack  $s$  (and equivalently  $w$ ) increases, under-investment distortions are reduced. Note also that a sufficiently constrained firm optimally sells assets,  $i_t < 0$ , in order to replenish valuable liquidity.

Finally, we note that in our model there is a debt over-hang effect even though debt is risk-free. The reason is that debt reduces valuable financial slack thus crowding out future investments.



[Place Figure 6 about here]

**Consumption and the Marginal Propensity to Consume (MPC).** The entrepreneur's FOC for consumption is the standard condition:  $\zeta U'(C) = J_S(K, S)$ . Panels A and C of Figure 6 plot  $c(s)$  and the MPC  $c'(s)$ . The dotted lines in Panels A and C describe Merton's linear consumption rule under the first-best:  $c^{FB}(s) = \chi(s + q^{FB})$ , where the constant MPC is  $\chi = 6.13\%$  and  $q^{FB} = 1.264$ . Under inalienability the entrepreneur under-consumes:  $c_t$  is lower than  $c^{FB}(s)$  under all circumstances. But, the higher the financial slack  $s$  the higher is the entrepreneur's consumption. It is striking that financially constrained entrepreneurs with  $s$  close to  $\underline{s} = -0.208$  have substantially larger MPCs than suggested by Friedman's permanent-income hypothesis. For example, when  $s = -0.2$ , the MPC is  $c'(-0.2) = 19.6\%$ , which is much higher than the MPC  $\chi = 6.13\%$  given by the standard permanent-income hypothesis. This prediction is consistent with empirical evidence in Parker (1999) and Souleles (1999).

The dual contracting problem conveys the same insights as the entrepreneur's liquidity and risk management problem. Panels B and D of Figure 6 show that  $c(w)$  is lower than the first-best consumption rule due to the inalienability constraint, and  $c(w)$  is increasing and concave in  $w$ .

#### E. Comparative Statics with respect to $\alpha$

[Place Figure 7 about here]

The value of  $\alpha$  measures the degree of the inalienability of the entrepreneur's human capital. The higher the value of  $\alpha$ , the more inalienable the entrepreneur's human capital. Figure 7 compares the solution for the baseline parameter values where  $\alpha = 0.8$  with the case where  $\alpha = 0.4$ . When  $\alpha$  decreases from 0.8 to 0.4, the debt capacity increases significantly from 21% to 69% of the capital stock, i.e.,  $(\underline{s})$  changes from  $-0.21$  to  $-0.69$ . As a result, with less inalienable human capital (lower  $\alpha$ ),  $m(s)$  increases, the marginal value of liquidity,  $m'(s)$ , decreases, both the idiosyncratic and systematic risk positions,  $|\phi_h(s)|$  and  $|\phi_m(s)|$ , increase, and both consumption and investment increase. Consistent with these predictions, Jeffers (2018) finds that stronger labor-contract enforcement through tighter non-compete clauses is associated with higher investment at human-capital-intensive firms.

#### F. Which Outside Option: Recontracting or Autarky?

When limited commitment is due to the inalienability of human capital it is natural to assume that the entrepreneur's outside option is employment at another firm, which involves *recontracting*.<sup>32</sup> At the new firm, the entrepreneur can combine her human capital with the new firm's capital stock under a new optimal contract. The point is that the mere decision to quit does not mean that the entrepreneur has to hide and can no longer engage in any contracts. In contrast, when limited commitment takes the form of absconsion it is more natural to assume that the entrepreneur has to continue in autarky<sup>33</sup>. The absconsion/autarky perspective<sup>33</sup> is more common in the literature.

Why does it matter whether the outside option is autarky or recontracting? We address this question in this section and show that even for reasonable coefficients of relative risk aversion, autarky is such an unappealing and costly option for the entrepreneur that the first-best allocation can be supported. That is, the autarky outside option loses its bite in generating plausible economic predictions.

**[Place Figure 8 about here]**

Autarky means that the entrepreneur is shut out of capital markets and therefore has to divide operating revenues  $AK_t$  into consumption and investment (including adjustment costs), so that  $AK_t = C_t + I_t + G_t$ . As we show, autarky is a severe punishment even for an entrepreneur with moderate risk aversion, as she is then fully exposed to the firm's operating shocks and cannot diversify them away. *Ex ante* limited commitment under these circumstances may not result in much or any distortion in investment and consumption. We illustrate this key insight in Panels A and B of Figure 8 by plotting  $m(s)$  and  $m'(s)$  for both  $\gamma = 2$  and  $\gamma = 5$ , when the outside option is autarky.

As risk aversion  $\gamma$  increases from 2 to 5,  $\underline{s}$  changes from  $-0.756$  to  $-q^{FB} = -1.264$ . Panel B shows furthermore that when  $\gamma = 2$  the marginal value of liquidity  $m'(\underline{s})$  decreases from 1.544 to unity as  $s$  increases from  $\underline{s} = -0.756$  to  $\infty$ . In contrast, when  $\gamma = 5$ , the marginal value of liquidity equals unity ( $m'(s) = 1$ ) for all  $s$  (see the dashed line in Panel B), achieving the first best. That is, the first best is attainable with  $\gamma = 5$  under autarky because the punishment is so severe. The limited commitment constraint never binds in equilibrium under autarky when  $\gamma = 5$ . This reduces the empirical relevance of the limited commitment model with autarky.

In contrast, under our recontracting formulation the first best is far from attainable. The reason is that the entrepreneur's risk aversion has comparable quantitative effects on her value function and her outside option value. Panels C and D of Figure 8 report  $m(s)$  and  $m'(s)$  with  $\gamma = 2$  and  $\gamma = 5$  for our recontracting formulation. We find that changes in risk aversion have almost no impact on debt capacity:  $\underline{s}$  barely changes from  $-0.208$  to  $-0.203$  as we increase  $\gamma$  from 2 to 5. Finally, observe that inalienability imposes a much tighter debt limit than under autarky. For example, even when  $\gamma = 2$ , the debt capacity under recontracting is 0.208, which is less than one-third of the debt capacity under autarky, 0.756.

**Comparisons with Ai and Li (2015).** The reformulation of our model with autarky as the outside option is closely related to the contracting problem analyzed by Ai and Li (2015). They consider a contracting problem between an infinitely-lived risk-neutral principal and a risk-averse agent with CRRA preferences, who is subject to a limited-commitment constraint with autarky as the outside option. The contracting formulation of our model differs from Ai and Li (2015) in several other respects. First, in our model both the principal and the entrepreneur are risk-averse and are exposed to both aggregate and idiosyncratic shocks. Given that the principal is risk-neutral in Ai and Li (2015), the distinction between aggregate and idiosyncratic shocks is not meaningful in their setup.<sup>34</sup> As we have shown, aggregate and idiosyncratic shocks have very different implications for

consumption, investment, portfolio choice, and risk management. Second, the *state variable* that we choose to work with in our contracting problem is the entrepreneur's promised certainty-equivalent wealth, while in Ai and Li (2015) it is the agent's promised utility. In other words, our units are *dollars* while Ai and Li's units are the agent's *utils*. It is only by expressing the entrepreneur's compensation in dollars that we can interpret the entrepreneur's future promised compensation as a liquidity buffer and measure how financially constrained the firm is with the investor's marginal value of liquidity  $p'(w)$ .

Third, the entrepreneur's consumption in our problem is stochastic, while in Ai and Li the agent's consumption is deterministic for a given time interval  $(t, t + s)$  where the constraints do not bind at any time  $v$  in this interval. This result follows from the following optimality condition:

$$e^{-\zeta s} \left( \frac{C_{t+s}}{C_t} \right)^{-\gamma} = \frac{M_{t+s}}{M_t} = \exp \left[ - \left( r + \frac{\eta^2}{2} \right) s - \eta (\mathcal{Z}_{m,t+s} - \mathcal{Z}_{m,t}) \right]. \quad (84)$$

which states that the entrepreneur's marginal rate of substitution (under full spanning) must equal to the investors' SDF. Simplifying (84) yields:

$$C_{t+s} = C_t e^{-(\zeta-r)s/\gamma} \exp \left[ \frac{1}{\gamma} \left( \frac{\eta^2 s}{2} + \eta (\mathcal{Z}_{m,t+s} - \mathcal{Z}_{m,t}) \right) \right]. \quad (85)$$

In Ai and Li (2015), consumption is deterministic,  $C_{t+\tau} = C_t e^{-(\zeta-r)\tau/\gamma}$ , as  $\eta = 0$  in their model.<sup>35</sup>

## VI. Persistent Productivity Shocks

We further extend the model by introducing persistent productivity shocks. The firm faces two conflicting forces in the presence of such shocks. First, as Froot, Scharfstein, and Stein (1993) have emphasized, the firm will want to have sufficient funding capacity to take maximal advantage of the investment opportunities that become available when productivity is high. To do so, the firm may want to take hedging positions that allow it to transfer funds from the low to the high productivity state. Second, the firm also wants to smooth the entrepreneur's compensation across productivity states, allowing the entrepreneur to consume a higher share of earnings in the low than in the high productivity state. To do so, the firm will need to ensure that it has sufficient liquidity and funding capacity in the low productivity state. This may require taking hedging positions such that funds are transferred from the high to the low productivity state.

Which of these two forces dominates? We show that even for extreme parameter values for the productivity shocks the consumption/compensation smoothing effect dominates. One reason is that, when productivity is high, the firm's endogenous credit limit is also high, so that transferring funds from the low to the high productivity state is less important. In contrast, the consumption smoothing benefits of transferring funds from the high to the low productivity state are significant.

We model persistent productivity shocks  $\{A_t; t \geq 0\}$  as a two-state Markov switching process,  $A_t \in \{A^L, A^H\}$  with  $0 < A^L < A^H$ . We denote by  $\lambda_t \in \{\lambda^L, \lambda^H\}$  the transition intensity from one

state to the other, with  $\lambda^L$  denoting the intensity from state  $L$  to  $H$ , and  $\lambda^H$  the intensity from state  $H$  to  $L$ . The counting process  $\{\mathcal{N}_t; t \geq 0\}$  (starting with  $\mathcal{N}_0 = 0$ ) keeps track of the number of times the firm has switched productivity  $\{A_s : s \leq t\}$  up to time  $t$ ; it increases by one whenever the state switches from either  $H$  to  $L$  or from  $L$  to  $H$ :  $d\mathcal{N}_t = \mathcal{N}_t - \mathcal{N}_{t-} = 1$  if and only if  $A_t \neq A_{t-}$ , and  $d\mathcal{N}_t = 0$  otherwise.

In the presence of such shocks the entrepreneur will want to purchase or sell insurance against stochastic changes in productivity. We characterize the optimal insurance policy against such shocks and how investment, compensation, risk management, and debt capacity vary with productivity. For brevity, we only consider the case where productivity shocks are purely idiosyncratic.<sup>36</sup>

**Productivity Insurance Contract.** Consider the following insurance contract offered at current time  $t-$ . Over the time interval  $dt = (t-, t)$ , the entrepreneur pays the unit insurance premium  $\xi_{t-}dt$  to the insurance counterparty in exchange for a unit payment at time  $t$  if and only if  $A_t \neq A_{t-}$  (i.e.,  $d\mathcal{N}_t = 1$ ). That is, the underlying event for this insurance contract is the change in productivity. Under our assumptions of perfectly competitive financial markets and idiosyncratic productivity shocks, the actuarially fair insurance premium is given by the intensity of the change in productivity state:  $\xi_{t-} = \lambda_{t-}$ .

Let  $\Pi_{t-}$  denote the number of units of insurance purchased by the entrepreneur at time  $t-$ . We refer to  $\Pi_{t-}$  as the insurance demand. If  $\Pi_{t-} < 0$ , the firm sells insurance and collects insurance premia at the rate of  $\lambda_{t-}\Pi_{t-}$ . Then,  $S_t$  evolves as follows:

$$dS_t = (rS_t + Y_t - C_t + \Phi_{m,t}(\mu_m - r) - \lambda_{t-}\Pi_{t-})dt + \Phi_{h,t}\nu_K d\mathcal{Z}_{h,t} + \Phi_{m,t}\sigma_m d\mathcal{Z}_{m,t} + \Pi_{t-}d\mathcal{N}_t. \quad (86)$$

Note that the only differences between (86) and (10) are the insurance premium payment  $\lambda_{t-}\Pi_{t-}$  and the contingent liability coverage  $\Pi_{t-}d\mathcal{N}_t$ .

The solution for the firm's value is a pair of state-contingent value functions  $J(K, S; A^L) \equiv J^L(K, S)$  and  $J(K, S; A^H) \equiv J^H(K, S)$ , which solve two inter-linked HJB equations, one for each state.<sup>37</sup> The HJB equation in state  $L$  is:<sup>38</sup>

$$\begin{aligned} \zeta J^L(K, S) = & \max_{C, I, \Phi_h, \Phi_m, \Pi^L} \zeta U(C) + (I - \delta_K K)J_K^L + \frac{\sigma_K^2 K^2}{2} J_{KK}^L \\ & + (rS + \Phi_m(\mu_m - r) + A^L K - I - G(I, K) - C - \lambda^L \Pi^L) J_S^L \\ & + (\nu_K^2 \Phi_h + \rho \sigma_K \sigma_m \Phi_m) K J_{KS}^L + \frac{(\nu_K \Phi_h)^2 + (\sigma_m \Phi_m)^2}{2} J_{SS}^L \\ & + \lambda^L [J^H(K, S + \Pi^L) - J^L(K, S)]. \end{aligned} \quad (87)$$

Two important features differentiate (87) from the HJB equation (11). First, the drift term involving the marginal utility of liquidity  $J_S^L$  now includes the insurance payment  $-\lambda^L \Pi^L$ . Second, the last term in (87) captures the adjustment of  $S$  by the amount  $\Pi^L$  and the corresponding change in the value function following a productivity change from  $A^L$  to  $A^H$ .

The inalienability constraint must hold at all  $t$  for both productivity states, so that

$$S_t \geq \underline{S}(K_t; A_t), \quad (88)$$

or equivalently,

$$s_t \geq \underline{s}(A_t). \quad (89)$$

Naturally, the firm's time- $t$  credit limit  $|\underline{s}(A_t)|$  depends on its productivity  $A_t$ . We use  $\underline{s}^H$  and  $\underline{s}^L$  to denote  $\underline{s}(A_t)$  when  $A_t = A^H$  and  $A_t = A^L$ , respectively.

The entrepreneur determines her optimal insurance demand  $\Pi^L$  in state  $L$  by differentiating (87) with respect to  $\Pi^L$  and setting  $\Pi^L$  to satisfy the FOC:

$$J_S^L(K, S) = J_S^H(K, S + \Pi^L), \quad (90)$$

provided that the solution  $\Pi^L$  to the above FOC satisfies the (state-contingent) condition:

$$S + \Pi^L \geq \underline{S}^H. \quad (91)$$

Otherwise, the entrepreneur sets the insurance demand so that  $\Pi^L = \underline{S}^H - S$ , in which case the firm will be at its maximum debt level  $\underline{S}^H$  when productivity switches from  $A^L$  to  $A^H$ .<sup>39</sup>

[Place Figure 9 about here]

**Quantitative Analysis.** We consider two sets of (annualized) parameter values. The first set is such that  $A^H = 0.25$ ,  $A^L = 0.14$ , and  $\lambda^L = \lambda^H = 0.2$ , with all other parameter values as in Table III. The transition intensities  $(\lambda^H, \lambda^L) = (0.2, 0.2)$  imply that the expected duration of each state is five years. The second set of parameter values is identical to the first, except that  $A^L = 0.05$ . That is, productivity in the low state,  $A^L$ , is much lower (0.05 instead of 0.14).

Figure 9 plots the entrepreneur's insurance demand  $\pi^H(s)$  as the solid line, and  $\pi^L(s)$  as the dashed line. Panel A plots the insurance demand in both states when productivity differences are  $(A^H - A^L)/A^H = (0.25 - 0.14)/0.25 = 44\%$ , while Panel B plots the insurance demand when productivity differences are very large,  $(A^H - A^L)/A^H = (0.25 - 0.05)/0.25 = 80\%$ . Remarkably, under both sets of parameter values the firm optimally buys insurance in state  $H$ ,  $\pi^H(s) > 0$ , and sells insurance in state  $L$ ,  $\pi^L(s) < 0$ . This result is not obvious *a priori*, for when productivity differences are large the benefit from transferring liquidity from state  $L$  to  $H$  and thereby taking better advantage of investment opportunities when they arise, could well be the dominant consideration for the firm's risk management. But that turns out not to be the case. Even when productivity differences are as large as 80%, the dominant consideration is still to smooth the entrepreneur's consumption. Moreover, a comparison of Panels A and B reveals that for the larger productivity differences, the insurance demand is also larger, with  $\pi^H(s)$  exceeding 0.2 everywhere in Panel B,

but remaining below 0.2 in Panel A, and  $\pi^L(s)$  attaining values lower than  $-0.25$  in Panel B (when  $s + \pi^L \geq \underline{s}^H$  is not binding), while  $\pi^L(s)$  is always larger than  $-0.2$  in Panel A.<sup>40</sup>

[Place Figure 10 about here]

Figure 10 shows that  $m(s)$ , consumption  $c(s)$ , investment  $i(s)$ , and debt capacity  $|s|$  are higher in state  $H$  than in state  $L$ , as one would expect. Similarly, both the size of the idiosyncratic risk hedging position and of stock market exposures,  $|\phi_h(s)|$  and  $|\phi_m(s)|$ , are higher in state  $H$  than in state  $L$ . However, a somewhat subtle result is that marginal value of liquidity schedules,  $m'(s)$ , for state  $H$  and  $L$ , cross.

## VII. Deterministic Formulation à la Hart and Moore (1994)

Our contracting problem is also closely related to Hart and Moore's contracting problem under inalienability. Hart and Moore (1994) consider a special case with a single deterministic project and linear preferences for both the investor and the entrepreneur. They emphasize the idea that debt financing is optimal when the entrepreneur's human capital is inalienable. Our more general framework reveals that the optimality of debt financing is not a robust result. Instead, the robust ideas are that inalienability gives rise to: i) an endogenous financing capacity; and ii) an optimal corporate liquidity and risk management problem.

To highlight the critical role of liquidity management, it is instructive to consider the special case of our model where there are no shocks, so that  $\sigma_K = 0$  and  $\eta = 0$ , as in Hart and Moore (1994). Although output and capital accumulation become deterministic, this special case of our model is still more general than Hart and Moore (1994) in two respects: 1) the entrepreneur has a strictly concave utility function and therefore a strict preference for smoothing consumption; 2) the firm's operations are not fixed by a one-time lump-sum investment, but can be adjusted over time through capital accumulation (or decumulation). That is, our model can be viewed as a convex version of Hart and Moore (1994), as the additional controls in our deterministic formulation are consumption and investment, both of which are convex and characterized by FOCs.

With  $\sigma_K = 0$  and  $\eta = 0$ , the liquidity ratio  $s_t$  evolves at the rate of:

$$\mu^s(s_t) \equiv ds_t/dt = (r + \delta - i_t)s_t + A - i_t - g(i_t) - c_t, \quad (92)$$

given a contract  $\{c_t, i_t; t \geq 0\}$ . To ensure that the entrepreneur stays with the firm and the financing capacity is maximized,  $\mu^s(\underline{s}) = 0$  has to hold. The ODE given in (43) can be simplified to:

$$0 = \frac{m(s)}{1-\gamma} \left[ \gamma \chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] + [rs + A - i(s) - g(i(s))] m'(s) + (i(s) - \delta)(m(s) - sm'(s)), \quad (93)$$

where  $\chi = r + \gamma^{-1}(\zeta - r)$  and  $\lim_{s \rightarrow \infty} m(s) = q^{FB} + s$ .

Under the first-best, with  $i_t = i^{FB}$  and  $c_t = c^{FB}$ , the drift of  $s$ ,  $\mu_{FB}^s(s_t)$ , is then:

$$\mu_{FB}^s(s_t) = (r + \delta - i^{FB})(s_t + q^{FB}) - c^{FB} = -(i^{FB} - \delta + \gamma^{-1}(\zeta - r))m^{FB}(s_t), \quad (94)$$

where the first equality uses (29) and the second uses (32) and (33). It immediately follows that the first-best drift is negative,  $\mu_{FB}^s(s_t) \leq 0$ , if and only if the following condition holds:

$$i^{FB} \geq \delta - \gamma^{-1}(\zeta - r). \quad (95)$$

When does condition (95) hold? Under the auxiliary assumption that the entrepreneur's discount rate  $\zeta$  equals the interest rate  $r$ , (95) holds if and only if the firm's first-best net investment policy is positive:  $i^{FB} \geq \delta$ . In other words, condition (95) requires the firm to grow under the first-best policy, which is the natural case to focus on. The alternative case is when (95) is not satisfied. Then the firm's size is decreasing over time even under the first-best policy. In this latter case, the inalienability of human capital constraint is irrelevant and the first-best outcome (optimal downsizing) is attained.<sup>41</sup> We summarize this discussion in the proposition below.

**PROPOSITION 1:** *When (95) is satisfied the drift of  $s$  equals zero at the endogenous debt limit  $\underline{s}$ :  $\mu^s(\underline{s}) = 0$ . When (95) is not satisfied, the first-best outcome is obtained.*

[Place Figure 11 about here]

Figure 11 plots the solution when  $A = 0.185$ . Note that  $i^{FB} = 0.136$ , which is greater than  $\delta = \delta_K = 0.11$ . Hence, (95) is satisfied and the first-best is unattainable. The firm underinvests and under-compensates the entrepreneur relative to the first best, since the marginal value of liquidity is greater than one,  $m'(s) > 1$ . Liquidity  $s_t$  decreases over time and reaches  $\underline{s}$ , the permanently absorbing state. In our example,  $\underline{s} = -0.249$ . Starting at  $s_0 = 0$ , it takes 25.77 years to reach the absorbing state where the borrowing constraint binds permanently at  $s_{25.77} = \underline{s} = -0.249$ . Similarly, due to the friction of limited commitment the marginal value of liquidity is greater than one,  $m'(\underline{s}) = m'(-0.249) = 1.038 > 1$ . Panels C and D show that the entrepreneur reduces her consumption and investment smoothly even with no risk. Since  $m'(s) > 1$ , the MPC is greater than that under first-best case.

## VIII. Two-Sided Limited Commitment

In our baseline model, the firm's optimal policy requires that investors incur losses with positive probability. As Figure 1 illustrates, investors make losses,  $p(w) < 0$ , when  $w > 1.18$ . But investors' *ex ante* commitment to continue compensating the entrepreneur *ex post* even when doing so incurs large losses for investors may not be credible. What if investors cannot commit to such loss-making promises to the entrepreneur *ex post*? We next explore this issue and characterize the solution when neither the entrepreneur nor investors are able to commit.

Suppose that investors can only commit to making losses *ex post* up to a fixed fraction  $\ell$  of the total capital stock, so that  $p(w_t) \geq -\ell$  at all  $t$ . For expositional simplicity we set  $\ell = 0$ . Then, the main change relative to the one-sided commitment problem analyzed so far is that there is also an upper boundary  $\bar{s} = -p(\bar{w}) = 0$ . Note that under two-sided limited commitment with  $\ell = 0$ , the firm will never be in the positive savings region. As a result, the following new conditions hold at  $\bar{s} = 0$ :

$$\sigma_h^s(0) = \sigma_m^s(0) = 0. \quad (96)$$

Using the same argument as for (49), we may express (96) as  $m''(0) = -\infty$ , and we verify that  $\mu^s(s)$  given in (23) is weakly negative at  $\bar{s} = 0$ , so that  $s \leq \bar{s} = 0$  with probability one.

**[Place Figure 12 about here]**

Panel A of Figure 12 shows that investors' lack of commitment significantly destroys value. For example, at  $s = 0$ , under full commitment by investors,  $m(0) = 1.198$ , which is 42% higher than  $m(0) = 0.843$ , the value under two-sided limited commitment. With two-sided limited commitment,  $s$  lies between  $\underline{s} = -0.25$  and  $\bar{s} = 0$ , so that the entrepreneur has a larger credit limit of  $|\underline{s}| = 0.25$  instead of  $|\underline{s}| = 0.208$ , the debt capacity under one-sided limited commitment. However, a firm with a larger debt capacity is not necessarily less financially constrained, since investors' limited-liability constraint limits the entrepreneur's self-insurance capacity.

Interesting, the marginal value of liquidity under two-side limited commitment is lower than unity:  $m'(s) < 1$ , which is quite different from the one-sided case where  $m'(s) > 1$ . While an increase of liquidity mitigates the entrepreneur's inalienability, it makes the investor's limited-commitment more likely to bind in the future, so the net effect of increasing  $s$  on  $m'(s)$  is ambiguous. Value destruction arises from the direct effect of the entrepreneur's inability to hold liquid savings ( $s$  cannot be strictly positive), and also the indirect effect of distorting consumption decisions and investment. Panel C shows that the entrepreneur is under-compensated relative to the first best. Panel D shows that  $i(s)$  under two-sided limited-commitment fundamentally differs from that under one-sided limited-commitment. For example, at  $s = 0$ ,  $i(0) = 0.331$  under one-sided commitment, which is six times higher than  $i(0) = 0.053$  under two-sided limited commitment.

Compared with the first-best, the firm under-invests when  $s < -0.13$ , but over-invests when  $-0.13 < s \leq 0$ . Whether the firm under-invests or over-invests depends on the net effects of the entrepreneur's and investors' limited-commitment constraints. For sufficiently low values of  $s$  (when the entrepreneur is deep in debt) the entrepreneur's constraint matters more and hence the firm under-invests. When  $s$  is sufficiently close to zero, investors' limited-liability constraint has a stronger influence on investment. To ensure that  $s \leq 0$  the entrepreneur needs to transform liquid assets into illiquid capital even though this may compromise investment efficiency. This mechanism causes the firm to over-invest relative to the first-best.

Phrased in terms of the equivalent contracting problem, the intuition is as follows. Given that the entrepreneur cares about her total compensation  $W = w \cdot K$  and given that investors are constrained by their ability to promise the entrepreneur  $w$  beyond an upper bound  $\bar{w}$ , (in this case,



$\bar{w} = m(0) = 0.843$ ), investors reward the entrepreneur along the extensive margin, firm size  $K$ , which allows the entrepreneur to accumulate more human capital and earn higher compensation payoffs through over-investment.

Panels E and F plot the idiosyncratic risk hedge  $\phi_h(s)$  and the market portfolio allocation  $\phi_m(s)$ . Neither  $\phi_h(s)$  nor  $\phi_m(s)$  is monotonic in  $s$  under two-sided limited-commitment. The reason is that the volatilities  $\sigma_h^s(s)$  and  $\sigma_m^s(s)$  for  $s$  must be turned off at both  $\underline{s} = -0.25$  and  $\bar{s} = 0$  in order to prevent separation by the entrepreneur and investors (see Panels G and H.) This is achieved by setting  $\phi_h(\underline{s}) = \underline{s} = -0.25$ ,  $\phi_m(\underline{s}) = \beta^{FB}\underline{s} = -0.05$ , and  $\phi_h(0) = \phi_m(0) = 0$ , as implied by the volatility boundary conditions for  $\sigma_h^s(s)$  and  $\sigma_m^s(s)$  at  $\underline{s}$  and  $\bar{s}$ .

## IX. Conclusion

Talent retention is a major challenge for many companies, especially so for technology companies. It is obviously a central issue for human resource management. But, less obviously, it also has implications for corporate financial management, as our analysis has underscored. We have shown how human capital flight risk not only affects firms' compensation policy, but also their investment, financing capacity, liquidity and risk management policies. More liquidity and spare borrowing capacity buttress the firm's future compensation promises and allow the firm to retain talent in a more cost-efficient way.

Human capital flight risk provides a novel rationale for corporate risk management policies. The firm's goal in our analysis is not so much to improve the risk exposure of investors, but to offer constrained efficient risk exposures to its employees who have all their human capital tied up with the firm. Our theory helps explain in particular why when retained earnings rise firms choose to invest an increasing fraction of these earnings in risky financial assets (Duchin et al., 2016).

In sum, the corporate risk management problem in our model boils down to a compromise between (a.) the maximization of key employees' or the entrepreneur's net worth, which requires full insurance against idiosyncratic risk as well as a mean-variance efficient risk exposure to the stock market, and (b.) the maximization/preservation of the firm's borrowing capacity, which involves reducing the volatility of retained earnings per unit of capital. When the firm is close to depleting its line of credit, the priority is to survive. From a liquidity and risk management perspective, this means that the firm cuts back on expenditures, reduces compensation, and sells insurance in order to generate liquidity for survival. In contrast, when liquidity is plentiful the firm adapts its corporate policies so as to optimize the (mean-variance) preferences of its key employees.

Although our framework is already quite rich, we have imposed a number of strong assumptions, which are worth relaxing in future work. For example, one interesting direction is to allow for equilibrium separation between the entrepreneur and investors. This could arise, when after a productivity shock the entrepreneur is no longer the best user of the firm's capital stock. Investors may then want to redeploy their capital to other more efficient uses and the entrepreneur may also find her human capital more productive elsewhere. By allowing for equilibrium separation

our model could then be applied to study questions such as the life-span of entrepreneurial firms, managerial turnover, and how the choice of investment in firm-specific versus general human capital is affected by the firm's financial flexibility.

## Appendix A. The Entrepreneur's Optimization Problem

We conjecture that the entrepreneur's value function  $J(K, S)$  takes the following form:

$$J(K, S) = \frac{(bM(K, S))^{1-\gamma}}{1-\gamma} = \frac{(bm(s)K)^{1-\gamma}}{1-\gamma}, \quad (\text{A1})$$

where  $b$  is a constant that will be determined later. We then have:

$$J_S = b^{1-\gamma}(m(s)K)^{-\gamma}m'(s), \quad (\text{A2})$$

$$J_K = b^{1-\gamma}(m(s)K)^{-\gamma}(m(s) - sm'(s)), \quad (\text{A3})$$

$$J_{SK} = b^{1-\gamma}(m(s)K)^{-1-\gamma}(-sm(s)m''(s) - \gamma m'(s)(m(s) - sm'(s))), \quad (\text{A4})$$

$$J_{SS} = b^{1-\gamma}(m(s)K)^{-1-\gamma}(m(s)m''(s) - \gamma m'(s)^2), \quad (\text{A5})$$

$$J_{KK} = b^{1-\gamma}(m(s)K)^{-1-\gamma}(s^2m(s)m''(s) - \gamma(m(s) - sm'(s))^2). \quad (\text{A6})$$

Substituting these terms into the HJB equation (11) and simplifying, we obtain:

$$\begin{aligned} 0 = & \max_{c, i, \phi_h, \phi_m} \zeta m(s) \frac{\left(\frac{c}{bm(s)}\right)^{1-\gamma} - 1}{1-\gamma} + (i - \delta_K)(m(s) - sm'(s)) \\ & + (rs + \phi_m(\mu_m - r) + A - i - g(i) - c)m'(s) + \frac{\sigma_K^2}{2} \left( s^2 m''(s) - \frac{\gamma(m(s) - sm'(s))^2}{m(s)} \right) \\ & + (\nu_K^2 \phi_h + \rho \sigma_K \sigma_m \phi_m) \left( -sm''(s) - \frac{\gamma m'(s)(m(s) - sm'(s))}{m(s)} \right) \\ & + \frac{(\nu_K \phi_h)^2 + (\sigma_m \phi_m)^2}{2} \left( m''(s) - \frac{\gamma m'(s)^2}{m(s)} \right). \end{aligned} \quad (\text{A7})$$

The first order conditions for consumption and investment in (12) and (13) then become:

$$\zeta U'(c) = b^{1-\gamma} m(s)^{-\gamma} m'(s), \quad (\text{A8})$$

$$1 + g'(i) = \frac{m(s)}{m'(s)} - s. \quad (\text{A9})$$

From the first order conditions (15) and (14), we obtain (41) and (42).

Finally, substituting these policy functions for  $c(s)$ ,  $\phi_h(s)$ , and  $\phi_m(s)$  into (A7), we obtain the ODE for  $m(s)$ :

$$\begin{aligned} 0 = & \frac{m(s)}{1-\gamma} \left[ \gamma \chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] + [rs + A - i(s) - g(i(s))] m'(s) + (i(s) - \delta)(m(s) - sm'(s)) \\ & - \left( \frac{\gamma \sigma_K^2}{2} - \rho \eta \sigma_K \right) \frac{m(s)^2 m''(s)}{m(s) m''(s) - \gamma m'(s)^2} + \frac{\eta^2 m'(s)^2 m(s)}{2(\gamma m'(s)^2 - m(s) m''(s))}, \end{aligned} \quad (\text{A10})$$

where  $\chi$  is defined by

$$\chi \equiv b^{\frac{\gamma-1}{\gamma}} \zeta^{\frac{1}{\gamma}}. \quad (\text{A11})$$

And then substituting  $\gamma_e$  given by (19) into (A10), we obtain the ODE given in (43).

### Appendix A. First-Best

Under the first best, the value function is given by  $m^{FB}(s) = s + q^{FB}$ . Substituting this expression for  $m^{FB}(s)$  into the ODE (A10) we obtain:

$$\begin{aligned} 0 &= \frac{s + q^{FB}}{1 - \gamma} [\gamma\chi - \zeta] + [rs + A - i^{FB} - g(i^{FB})] + (i^{FB} - \delta)q^{FB} + \frac{\eta^2(s + q^{FB})}{2\gamma} \\ &= \left( \frac{\gamma\chi - \zeta}{1 - \gamma} + \frac{\eta^2}{2\gamma} + r \right) (s + q^{FB}) + [A - i^{FB} - g(i^{FB}) - (r + \delta - i^{FB})q^{FB}]. \end{aligned} \quad (\text{A12})$$

As (A12) must hold for all  $m^{FB}(s) = s + q^{FB}$ , we must have

$$\chi = r + \frac{\eta^2}{2\gamma} + \gamma^{-1} \left( \zeta - r - \frac{\eta^2}{2\gamma} \right), \quad (\text{A13})$$

as given by (33), and

$$0 = A - i^{FB} - g(i^{FB}) - (r + \delta - i^{FB})q^{FB}, \quad (\text{A14})$$

so that (29) holds. In addition, using (A11), we obtain the expression (17) for the coefficient  $b$ . Next, substituting  $m(s) = m^{FB}(s) = s + q^{FB}$  into (A8) and (A9) gives the first-best consumption rule (32) and investment policy (25). To ensure that the optimization problem is well posed, we require positive consumption and a positive Tobin's  $q$ , i.e.  $\chi > 0$  and  $q^{FB} > 0$ , which imply:

$$\text{Condition 1: } r + \frac{\eta^2}{2\gamma} + \gamma^{-1} \left( \zeta - r - \frac{\eta^2}{2\gamma} \right) > 0, \quad (\text{A15})$$

$$\text{Condition 2: } i^{FB} < r + \delta, \quad (\text{A16})$$

where  $i^{FB}$  is the solution of (29). Substituting  $m(s) = m^{FB}(s) = s + q^{FB}$  into (41) and (42) respectively, we obtain the first-best idiosyncratic risk hedge  $\phi_h^{FB}(s)$  given in (34) and the market portfolio allocation  $\phi_m^{FB}(s)$  given in (35).

The expected return for  $Q_t^{FB}$ ,  $\mu^{FB}$ , satisfies the CAPM where

$$\mu^{FB} = \frac{A - i^{FB} - g(i^{FB})}{q^{FB}} + (i^{FB} - \delta_K) = r + \delta - i^{FB} + (i^{FB} - \delta_K) = r + \beta^{FB} (\mu_m - r), \quad (\text{A17})$$

and  $\beta^{FB}$  is given by (28). The value of capital  $Q_t^{FB}$  follows a GBM process given by:

$$dQ_t^{FB} = Q_t^{FB} [(i^{FB} - \delta_K) dt + (\nu_K dZ_{h,t} + \rho\sigma_K dZ_{m,t})], \quad (\text{A18})$$

with the drift  $(i^{FB} - \delta_K)$ , idiosyncratic volatility  $\nu_K$ , and systematic volatility  $\rho\sigma_K$ . These coefficients are identical to those for  $\{K_t : t \geq 0\}$ . Next, we apply Ito's formula to  $M_t^{FB} = S_t + Q_t^{FB} =$

$S_t + q^{FB}K_t$  and obtain the following dynamics:

$$dM_t^{FB} = M_t^{FB} \left[ \left( r + \frac{\eta^2}{\gamma} - \chi \right) dt + \frac{\eta}{\gamma} dZ_{m,t} \right]. \quad (\text{A19})$$

### Appendix B. Inalienable Human Capital

From the monotonicity property of  $J(K, S)$  in  $S$ , it follows that the condition given in (45) reduces to  $S_t \geq \underline{S}_t = \underline{S}(K_t)$  given in (46). Then substituting the value function (16) into (44), we obtain  $M(K, \underline{S}) = M(\alpha K, 0)$ , which implies (47). The boundary conditions given in (49) are necessary to ensure that the entrepreneur will stay with the firm, which implies that

$$\phi_h(\underline{s}) = \underline{s} \quad \text{and} \quad \phi_m(\underline{s}) = \underline{s} \beta^{FB}. \quad (\text{A20})$$

Applying (A20) to (41) and (42), we show that (49) is equivalent to  $\lim_{s \rightarrow \underline{s}} m''(s) = -\infty$  as given in (51).

## Appendix B. Equivalent Optimal Contract

### Appendix A. Solution of the Contracting Problem

**HJB Equation for  $F(K, V)$ .** Using Ito's formula, we have

$$d(\mathbb{M}_t F(K_t, V_t)) = \mathbb{M}_t dF(K_t, V_t) + F(K_t, V_t) d\mathbb{M}_t + \langle d\mathbb{M}_t, dF(K_t, V_t) \rangle, \quad (\text{B1})$$

where

$$\begin{aligned} dF(K_t, V_t) &= F_K dK_t + \frac{F_{KK}}{2} \langle dK_t, dK_t \rangle + F_V dV_t + \frac{F_{VV}}{2} \langle dV_t, dV_t \rangle + F_{VK} \langle dV_t, dK_t \rangle \\ &= \left[ (I - \delta_K K) F_K + \frac{\sigma_K^2 K^2 F_{KK}}{2} + \zeta (V - U(C)) F_V \right] dt \\ &\quad + \left[ \frac{(z_h^2 + z_m^2) V^2 F_{VV}}{2} + (z_h \nu_K + z_m \rho \sigma_K) K V F_{VK} \right] dt \\ &\quad + V F_V (z_h dZ_{h,t} + z_m dZ_{m,t}) + \sigma_K K F_K \left( \sqrt{1 - \rho^2} dZ_{h,t} + \rho dZ_{m,t} \right). \end{aligned} \quad (\text{B2})$$

Using the SDF  $\mathbb{M}$  given in (8) and the following martingale representation,

$$\mathbb{E}_t[d(\mathbb{M}_t F(K_t, V_t))] + \mathbb{M}_t (Y_t - C_t) dt = 0, \quad (\text{B3})$$

we obtain (57), which is the HJB equation for the optimal contracting problem.

**Optimal Policy Functions and ODE for  $p(w)$ .** Applying Ito's formula to (61) and transforming (57) for  $F(K, V)$  into the HJB equation for  $P(K, W)$ , we obtain:

$$rP(K, W) = \max_{C, I, x_h, x_m} \left\{ Y - C + \left( \frac{\zeta(U(bW) - U(C))}{bU'(bW)} - x_m \eta K \right) P_W + (I - \delta_K K - \rho \eta \sigma_K K) P_K \right. \\ \left. + \frac{\sigma_K^2 K^2}{2} P_{KK} + \frac{(x_h^2 + x_m^2) K^2}{2} \frac{P_{WW} bU'(bW) - P_W b^2 U''(bW)}{bU'(bW)} \right. \\ \left. + (x_h \nu_K + x_m \rho \sigma_K) K^2 P_{WK} \right\}. \quad (\text{B4})$$

The FOCs for  $C$ ,  $I$ ,  $x_h$  and  $x_m$  are given by

$$U'(bW) = -\frac{\zeta}{b} P_W(K, W) U'(C), \quad (\text{B5})$$

$$1 + G_I(I, K) = P_K(K, W), \quad (\text{B6})$$

$$x_h = -\frac{\nu_K P_{WK}}{P_{WW} - P_W bU''(bW)/U'(bW)}, \quad (\text{B7})$$

$$x_m = -\frac{\rho \sigma_K P_{WK}}{P_{WW} - P_W bU''(bW)/U'(bW)} + \frac{\eta P_W}{K [P_{WW} - P_W bU''(bW)/U'(bW)]}. \quad (\text{B8})$$

By substituting  $P(K, W) = p(w)K$  into (B5)-(B8), we obtain the optimal consumption, investment, and risk management policies given by (63)-(66), respectively. By substituting  $P(K, W) = p(w)K$  and the corresponding optimal policies (63)-(66) into the PDE (B4), we find that the investor's value  $p(w)$  satisfies ODE (71).

**Dynamics of the Entrepreneur's Promised Scaled Wealth  $w$ .** Using Ito's formula, we have the following dynamics for  $W$ :

$$dW_t = \frac{\partial W}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 W}{\partial V^2} \langle dV_t, dV_t \rangle, \quad (\text{B9})$$

where we use  $\langle dV_t, dV_t \rangle$  to denote the quadratic variation of  $V$ ,  $\partial W / \partial V = 1/V'(W)$ , and  $\frac{\partial^2 W}{\partial V^2} = -\frac{V''(W)}{(V'(W))^3}$ . Substituting the dynamics of  $V$  given by (56) into (B9) yields (59). Using the dynamics for  $W$  and  $K$ , and applying Ito's formula to  $w_t = W_t/K_t$ , we can write the dynamic evolution of the certainty equivalent wealth  $w$  as given by (67).

### Appendix B. Equivalence

The optimization problem for the entrepreneur is equivalent to the dynamic optimal contracting problem for the investor in (53) if and only if the borrowing limits,  $\underline{S}(K)$ , are such that:

$$\underline{S}(K) = -P(K, \underline{W}), \quad (\text{B10})$$

where  $P(K, \underline{W})$  is the investor's value when the entrepreneur's inalienability constraint binds. We

characterize the implementation solution by first solving the investor's problem in (57) and then imposing the constraint (B10).

The optimal contracting problem gives rise to the investor's value function  $F(K, V)$ , with the promised utility to the entrepreneur  $V$  as the key state variable. The investor's value  $F(K, V)$  can be expressed in terms of the entrepreneur's promised certainty-equivalent wealth  $W$ ,  $P(K, W)$ . The optimization problem for the entrepreneur gives rise to the entrepreneur's value function  $J(K, S)$ , with  $S = -P(K, W)$  as the key state variable. Equivalently, the entrepreneur's objective is her certainty equivalent wealth  $M(K, S)$  and the relevant state variable is her savings  $S = -P$ .

The following relations between  $s$  and  $w$  hold:

$$s = -p(w) \text{ and } m(s) = w. \quad (\text{B11})$$

The standard chain rule implies:

$$m'(s) = -\frac{1}{p'(w)} \text{ and } m''(s) = -\frac{p''(w)}{p'(w)^3}. \quad (\text{B12})$$

Next, we demonstrate the equivalence between the two problems by showing first that by substituting  $s = -p(w)$  into the ODE for  $m(s)$ , we obtain the ODE for  $p(w)$ , and vice versa. Substituting (B11) and (B12) into the ODE (43) for  $m(s)$ , we obtain the ODE (71) for  $p(w)$ . Substituting (B11) and (B12) into consumption and investment policies (37) and (39) in the liquidity and risk management problem, we obtain the optimal consumption and investment policies, (63) and (64), in the contracting problem. Substituting (B11) and (B12) into (47) and (51), the boundary conditions for  $m(s)$ , we obtain (75) and (78), the boundary conditions for  $p(w)$ .

### *Appendix C. Autarky as the Entrepreneur's Outside Option*

Let  $\widehat{J}(K_t)$  denote the entrepreneur's value function under autarky defined as follows,

$$\widehat{J}(K_t) = \max_I \mathbb{E}_t \left[ \int_t^\infty \zeta e^{-\zeta(v-t)} U(C_v) dv \right]. \quad (\text{B13})$$

Under autarky, the entrepreneur's consumption  $C_t$  satisfies: output  $Y_t$ , in that

$$C_t = Y_t = A_t K_t - I_t - G(I_t, K_t). \quad (\text{B14})$$

The following proposition summarizes the main results.

**PROPOSITION 2:** *Under autarky, the entrepreneur's value function  $\widehat{J}(K)$  is given by*

$$\widehat{J}(K) = \frac{(b\widehat{M}(K))^{1-\gamma}}{1-\gamma}, \quad (\text{B15})$$

where  $b$  is given by (17) and  $\widehat{M}(K)$  is the certainty equivalent wealth under autarky given by

$$\widehat{M}(K) = \widehat{m}K, \quad (\text{B16})$$

where

$$\widehat{m} = \frac{(\zeta(1 + g'(\widehat{i}))(A - \widehat{i} - g(\widehat{i}))^{-\gamma})^{\frac{1}{1-\gamma}}}{b}, \quad (\text{B17})$$

and  $\widehat{i}$  is the optimal investment-capital ratio solving the following implicit equation:

$$\zeta = \frac{A - \widehat{i} - g(\widehat{i})}{1 + g'(\widehat{i})} + (\widehat{i} - \delta_K)(1 - \gamma) - \frac{\sigma_K^2 \gamma (1 - \gamma)}{2}. \quad (\text{B18})$$

**Proof of Proposition 2.** The value function  $\widehat{J}(K)$ , satisfies the following HJB equation:

$$\zeta \widehat{J} = \max_I \zeta \frac{C^{1-\gamma}}{1-\gamma} + (I - \delta_K K) \widehat{J}_K + \frac{\sigma_K^2 K^2}{2} \widehat{J}_{KK}. \quad (\text{B19})$$

Using  $\widehat{J}(K) = \frac{(b\widehat{M}(K))^{1-\gamma}}{1-\gamma}$  and  $c = A - i - g(i)$ , we have

$$\zeta = \max_i \zeta \left( \frac{A - i - g(i)}{\widehat{m}b} \right)^{1-\gamma} + (i - \delta_K)(1 - \gamma) - \frac{\sigma_K^2 \gamma (1 - \gamma)}{2}. \quad (\text{B20})$$

Using the FOC for  $i$ , we obtain (B17). Substituting (B17) into (B20), we obtain  $\widehat{i}$  given by (B18). Note that the entrepreneur's value function  $J(K, S)$  satisfies the following condition:

$$J(K_t, S_t) \geq \widehat{J}(K_t), \quad (\text{B21})$$

which implies that  $M(K_t, S_t) \geq \widehat{M}(K_t)$  and  $M(K_t, \underline{S}_t) = \widehat{M}(K_t)$ . By using the homogeneity property in  $K$ , we can also establish that the lower boundary  $\underline{S}$  satisfies  $m(\underline{S}) = \widehat{m}$ .

## Appendix C. Persistent Productivity Shocks

By using the dynamics given in (86), we obtain the HJB equation for the value function  $J^L(K, S)$  in State  $L$ , which is given by (87), and the following HJB equation for  $J^H(K, S)$  in State  $H$ :

$$\begin{aligned} \zeta J^H(K, S) = & \max_{C, I, \Phi_h, \Phi_m, \Pi^H} \zeta U(C) + (I - \delta_K K) J_K^H + \frac{\sigma_K^2 K^2}{2} J_{KK}^H \\ & + (rS + \Phi_m(\mu_m - r) + A^H K - I - G(I, K) - C - \lambda^H \Pi^H) J_S^H \\ & + (\nu_K^2 \Phi_h + \rho \sigma_K \sigma_m \Phi_m) K J_{KS}^H + \frac{(\nu_K \Phi_h)^2 + (\sigma_m \Phi_m)^2}{2} J_{SS}^H \\ & + \lambda^H [J^L(K, S + \Pi^H) - J^H(K, S)]. \end{aligned} \quad (\text{C1})$$

We then obtain the following main results:



PROPOSITION 3: In the region  $s > \underline{s}^L$ ,  $m^L(s)$  satisfies the following ODE:

$$\begin{aligned}
0 = & \max_{i^L, \pi^L} \frac{m^L(s)}{1-\gamma} \left[ \gamma \chi m^{L'}(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] + [rs + A^L - i^L - g(i^L) - \lambda^L \pi^L(s)] m^{L'}(s) \\
& - \left( \frac{\gamma \sigma_K^2}{2} - \rho \eta \sigma_K \right) \frac{m^L(s)^2 m^{L''}(s)}{m^L(s) m^{L''}(s) - \gamma m^{L'}(s)^2} + \frac{\eta^2 m^{L'}(s)^2 m^L(s)}{2(\gamma m^{L'}(s)^2 - m^L(s) m^{L''}(s))} \\
& + (i^L - \delta)(m^L(s) - s m^{L'}(s)) + \frac{\lambda^L m^L(s)}{1-\gamma} \left( \left( \frac{m^H(s + \pi^L)}{m^L(s)} \right)^{1-\gamma} - 1 \right), \quad (C2)
\end{aligned}$$

subject to the following boundary conditions:

$$\lim_{s \rightarrow \infty} m^L(s) = q_L^{FB} + s, \quad m^L(\underline{s}^L) = \alpha m^L(0), \quad \text{and} \quad m^{L''}(\underline{s}^L) = -\infty, \quad (C3)$$

where  $q_L^{FB}$  is provided below in Proposition 4. The insurance demand  $\pi^L(s)$  solves:

$$\frac{dm^H(s + \pi^L)}{ds} = \frac{dm^L(s)}{ds} \left( \frac{m^L(s)}{m^H(s + \pi^L)} \right)^{-\gamma}, \quad (C4)$$

as long as  $\pi^L(s)$  satisfies  $\pi^L(s) \geq \underline{s}^H - s$ . Otherwise, the entrepreneur sets  $\pi^L(s) = \underline{s}^H - s$ . We have another set of analogous equations and boundary conditions for  $m^H(s)$  and  $\pi^H(s)$  in state  $H$ .

The following proposition summarizes the solutions for the first-best case.

PROPOSITION 4: Under the first best, the firm's value  $Q_n^{FB}(K)$  in state  $n = \{H, L\}$  is proportional to  $K$ :  $Q_n^{FB}(K) = q_n^{FB} K$ , where  $q_H^{FB}$  and  $q_L^{FB}$  jointly solve:

$$(r + \delta - i_L^{FB}) q_L^{FB} = A^L - i_L^{FB} - g(i_L^{FB}) + \lambda^L (q_H^{FB} - q_L^{FB}), \quad (C5)$$

$$(r + \delta - i_H^{FB}) q_H^{FB} = A^H - i_H^{FB} - g(i_H^{FB}) + \lambda^H (q_L^{FB} - q_H^{FB}), \quad (C6)$$

and where  $i_L^{FB}$  and  $i_H^{FB}$  satisfy:  $q_L^{FB} = 1 + g'(i_L^{FB})$  and  $q_H^{FB} = 1 + g'(i_H^{FB})$ . The insurance demands in state  $L$  and  $H$  are respectively given by:  $\pi^L = q_H^{FB} - q_L^{FB}$  and  $\pi^H = q_L^{FB} - q_H^{FB}$ .

## Appendix D. Monotonicity and Concavity of the Value Function

**Lemma 1.** The value function  $J(K, S)$  is strictly increasing in  $S$ .

**Proof:** To see that  $J(K, S)$  is strictly increasing in  $S$ , consider  $S_{1,0} < S_{2,0}$  where  $S_{j,0} \geq \underline{s}K_{j,0}$  for  $j = 1, 2$ . We set  $K_{1,0} = K_{2,0} = K_0$ . Let  $\{C_{1,t}, I_{1,t}, \Phi_{1,h,t}, \Phi_{1,m,t}\}_{t=0}^{\infty}$  be the optimal policy with the given initial condition  $(K_0, S_{1,0})$ . Let  $\widehat{J}(K_0, S_{2,0})$  be the value function associated with an alternative policy  $\{\widehat{C}_{2,t}, \widehat{I}_{2,t}, \widehat{\Phi}_{2,h,t}, \widehat{\Phi}_{2,m,t}\}_{t=0}^{\infty}$  to be described below subject to the initial condition  $(K_0, S_{2,0})$ . Let  $\{K_{j,t}, S_{j,t}\}_{t=0}^{\infty}$  for  $j = 1, 2$  denote the implied liquidity and physical capital processes subject to the initial conditions  $(K_0, S_{j,0})$  for any admissible policy including both the optimal and candidate policies. We establish the following properties for  $\{\widehat{C}_{2,t}, \widehat{I}_{2,t}, \widehat{\Phi}_{2,h,t}, \widehat{\Phi}_{2,m,t}\}_{t=0}^{\infty}$ :

1.  $S_{2,t} \geq \underline{s}K_{2,t}$  for all  $t \geq 0$ ;

2. The value function,  $\widehat{J}(K_0, S_{2,0})$ , implied by this alternative policy is larger than  $J(K_0, S_{1,0})$ .

To construct  $\{\widehat{C}_{2,t}, \widehat{I}_{2,t}, \widehat{\Phi}_{2,h,t}, \widehat{\Phi}_{2,m,t}\}_{t=0}^{\infty}$ , we first define another policy,  $\{C'_t, I'_t, \Phi'_{h,t}, \Phi'_{m,t}\}_{t=0}^{\infty}$ , as follows:

$$C'_t = \lambda C_{1,t}, \text{ and } I'_t = I_{1,t}, \Phi'_{h,t} = \Phi_{1,h,t}, \Phi'_{m,t} = \Phi_{1,m,t}, \quad \text{for all } t \geq 0, \quad (\text{D1})$$

where  $\lambda > 1$  and  $\{C_{1,t}, I_{1,t}, \Phi_{1,h,t}, \Phi_{1,m,t}\}_{t=0}^{\infty}$  is the optimal policy defined earlier. Let  $\widehat{\tau}$  be the stopping time such that  $S_{2,t} = S_{1,t}$  for the first time under the policy  $\{C'_t, I'_t, \Phi'_{h,t}, \Phi'_{m,t}\}_{t=0}^{\infty}$  with the initial condition  $(K_0, S_{2,0})$ . We now define  $\{\widehat{C}_{2,t}, \widehat{I}_{2,t}, \widehat{\Phi}_{2,h,t}, \widehat{\Phi}_{2,m,t}\}_{t=0}^{\infty}$  as follows:

$$\widehat{C}_{2,t} = \begin{cases} C'_t = \lambda C_{1,t}, & \text{for } t \leq \widehat{\tau}, \\ C_{1,t}, & \text{for } t > \widehat{\tau}, \end{cases} \quad (\text{D2})$$

and

$$\widehat{I}_{2,t} = I_{1,t}, \widehat{\Phi}_{2,h,t} = \Phi_{1,h,t}, \widehat{\Phi}_{2,m,t} = \Phi_{1,m,t}, \quad \text{for all } t \geq 0. \quad (\text{D3})$$

With this constructed policy  $\{\widehat{C}_{2,t}, \widehat{I}_{2,t}, \widehat{\Phi}_{2,h,t}, \widehat{\Phi}_{2,m,t}\}_{t=0}^{\infty}$ , the dynamics for  $S_{1,t}$  and  $S_{2,t}$  when  $t \leq \widehat{\tau}$  are given by

$$dS_{1,t} = (rS_{1,t} - C_{1,t})dt + Y_t dt + \Phi_{1,h,t} \nu_K dZ_{h,t} + \Phi_{1,m,t} [(\mu_m - r)dt + \sigma_m dZ_{m,t}], \quad (\text{D4})$$

$$dS_{2,t} = (rS_{2,t} - \lambda C_{1,t})dt + Y_t dt + \Phi_{1,h,t} \nu_K dZ_{h,t} + \Phi_{1,m,t} [(\mu_m - r)dt + \sigma_m dZ_{m,t}]. \quad (\text{D5})$$

Since  $S_{1,\widehat{\tau}} = S_{2,\widehat{\tau}}$  and  $S_{1,t}$  and  $S_{2,t}$  have the same dynamics when  $t \geq \widehat{\tau}$ , we have  $S_{2,t} = S_{1,t}$  for all  $t \geq \widehat{\tau}$ . In addition,  $K_{1,t} = K_{2,t}$  for all  $t \geq 0$  since  $K_{1,0} = K_{2,0}$  and  $\widehat{I}_{2,t} \equiv I_{1,t}$  for all  $t \geq 0$ .

Condition 1 is satisfied under  $\{\widehat{C}_{2,t}, \widehat{I}_{2,t}, \widehat{\Phi}_{2,h,t}, \widehat{\Phi}_{2,m,t}\}_{t=0}^{\infty}$  because  $S_{2,t} \geq S_{1,t} \geq \underline{s}K_{1,t} = \underline{s}K_{2,t}$ . Condition 2 is also satisfied under  $\{\widehat{C}_{2,t}, \widehat{I}_{2,t}, \widehat{\Phi}_{2,h,t}, \widehat{\Phi}_{2,m,t}\}_{t=0}^{\infty}$  because  $\lambda > 1$  implies

$$J(K_0, S_{1,0}) < \mathbb{E}_0 \left[ \int_0^{\widehat{\tau}} \zeta e^{-\zeta t} U(\lambda C_{1,t}) dt + \int_{\widehat{\tau}}^{\infty} \zeta e^{-\zeta t} U(C_{1,t}) dt \right] = \widehat{J}(K_0, S_{2,0}). \quad (\text{D6})$$

By definition,  $J(K_0, S_{2,0})$  is the value function under the optimal policy with the initial condition  $(K_0, S_{2,0})$ ,  $\widehat{J}(K_0, S_{2,0}) \leq J(K_0, S_{2,0})$  thus holds. We have proven that  $J(K_0, S_{1,0}) < J(K_0, S_{2,0})$ .

**Lemma 2.** The value function  $J(K, S)$  is concave in  $S$ .

**Proof:** We use the same notations as in the proof for Lemma 1 whenever feasible. Let

$$S_0^\lambda = \lambda S_{1,0} + (1 - \lambda) S_{2,0}, \quad (\text{D7})$$

where  $0 \leq \lambda \leq 1$ . Let  $\widehat{J}(K_0, S_0^\lambda)$  be the value function associated with an alternative policy  $\{C_t^\lambda, I_t^\lambda, \Phi_{h,t}^\lambda, \Phi_{m,t}^\lambda\}_{t=0}^{\infty}$  to be described below subject to the initial conditions for  $(K_0^\lambda, S_0^\lambda)$  where  $K_0^\lambda = K_0$  and  $S_0^\lambda$  is given by (D7). Let  $\{K_t^\lambda, S_t^\lambda\}_{t=0}^{\infty}$  denote the implied liquidity and physical

capital processes subject to the initial conditions  $(K_0^\lambda, S_0^\lambda)$  for any admissible policy. We establish the following properties implied by the policy  $\{C_t^\lambda, I_t^\lambda, \Phi_{h,t}^\lambda, \Phi_{m,t}^\lambda\}_{t=0}^\infty$ :

1.  $S_t^\lambda \geq \underline{s}K_t^\lambda$  for all  $t \geq 0$ ;
2. The value function  $\widehat{J}(K_0, S_0^\lambda)$  is weakly larger than  $\lambda J(K_0, S_{1,0}) + (1 - \lambda)J(K_0, S_{2,0})$ .

We construct the policy  $\{C_t^\lambda, I_t^\lambda, \Phi_{h,t}^\lambda, \Phi_{m,t}^\lambda\}_{t=0}^\infty$  as follows:

$$\begin{aligned} C_t^\lambda &= \lambda C_{1,t} + (1 - \lambda)C_{2,t} + [\lambda G(I_{1,t}, K_{1,t}) + (1 - \lambda)G(I_{2,t}, K_{2,t}) - G(I_t^\lambda, K_t^\lambda)] \\ &\geq \lambda C_{1,t} + (1 - \lambda)C_{2,t} = \widehat{C}_t^\lambda, \end{aligned} \quad (\text{D8})$$

$$I_t^\lambda = \lambda I_{1,t} + (1 - \lambda)I_{2,t}, \quad (\text{D9})$$

$$\Phi_{h,t}^\lambda = \lambda \Phi_{1,h,t} + (1 - \lambda)\Phi_{2,h,t}, \quad (\text{D10})$$

$$\Phi_{m,t}^\lambda = \lambda \Phi_{1,m,t} + (1 - \lambda)\Phi_{2,m,t}. \quad (\text{D11})$$

Note that the convexity of the capital adjustment cost function  $G(I, K)$  gives rise to the inequality in (D8). First, we show that Condition 1 is satisfied. Equation (D9) implies that

$$dK_t^\lambda = (I_t^\lambda - \delta_K K_t^\lambda)dt + \sigma_K K_t^\lambda \left( \sqrt{1 - \rho^2} dZ_{h,t} + \rho dZ_{m,t} \right), \quad (\text{D12})$$

where  $K_t^\lambda = \lambda K_{1,t} + (1 - \lambda)K_{2,t}$ . Similarly, (D8), (D10), and (D11) imply that

$$dS_t^\lambda = (rS_t^\lambda + Y_t^\lambda - C_t^\lambda)dt + \Phi_{h,t}^\lambda \nu_K dZ_{h,t} + \Phi_{m,t}^\lambda [(\mu_m - r)dt + \sigma_m dZ_{m,t}], \quad (\text{D13})$$

where  $Y_t^\lambda = AK_t^\lambda - I_t^\lambda - G(I_t^\lambda, K_t^\lambda)$ . Therefore, we have  $S_t^\lambda = \lambda S_{1,t} + (1 - \lambda)S_{2,t}$  for all  $t \geq 0$ . The constraints  $S_{j,t} \geq \underline{s}K_{j,t}$  and additivity imply  $S_t^\lambda = \lambda S_{1,t} + (1 - \lambda)S_{2,t} \geq \lambda \underline{s}K_{1,t} + (1 - \lambda)\underline{s}K_{2,t} = \underline{s}K_t^\lambda$ , which is Condition 1.

Next, we use the monotonicity and concavity of the utility function  $U(\cdot)$  to prove Condition 2. The value function under the candidate policy satisfies

$$\begin{aligned} \widehat{J}(K_0, S_0^\lambda) &= \mathbb{E} \left[ \int_0^\infty \zeta e^{-\zeta t} U(C_t^\lambda) dt \right] \geq \lambda \mathbb{E} \left[ \int_0^\infty \zeta e^{-\zeta t} U(C_{1,t}) dt \right] + (1 - \lambda) \mathbb{E} \left[ \int_0^\infty \zeta e^{-\zeta t} U(C_{2,t}) dt \right] \\ &= \lambda J(K_0, S_{1,0}) + (1 - \lambda)J(K_0, S_{2,0}), \end{aligned} \quad (\text{D14})$$

where  $J(K, S)$  is the value function under the optimal policy and the inequality follows from  $U(C_t^\lambda) \geq U(\widehat{C}_t^\lambda) = U(\lambda C_{1,t} + (1 - \lambda)C_{2,t}) \geq \lambda U(C_{1,t}) + (1 - \lambda)U(C_{2,t})$ . We have thus proved the concavity of the value function.

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## Notes

<sup>1</sup>See also Biais, Mariotti, Rochet, and Villeneuve (2010), and Piskorski and Tchisty (2010), among others. Biais, Mariotti, and Rochet (2013) and Sannikov (2013) provide recent surveys of this literature. For static security design models, see Townsend (1979) and Gale and Hellwig (1985), Innes (1990), and Holmstrom and Tirole (1997).

<sup>2</sup>DeMarzo and Fishman (2007a), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo, Fishman, He and Wang (2012) incorporate investment into dynamic agency models.

<sup>3</sup>See Ljungqvist and Sargent (2004) Part V for a textbook treatment of limited-commitment models.

<sup>4</sup>See Stulz (1984, 1996), Smith and Stulz (1985), and Tufano (1996) for early work on the link between corporate hedging and executive compensation.

<sup>5</sup>Stochastic capital accumulation processes have been widely used in Corporate Finance, Asset Pricing, and Macro. Cox, Ingersoll, and Ross (1985), Jones and Manuelli (2005), Albuquerque and Wang (2008), and Brunnermeier and Sannikov (2014) are examples in general equilibrium with agency and financial constraints.

<sup>6</sup>The subscripts  $h$  and  $m$  for the two standard Brownian motions refer to idiosyncratic hedgeable risk and systematic market risk.

<sup>7</sup>Note that since there is no separation in equilibrium, we do not have to specify the firm's second best use of its physical capital.

<sup>8</sup>Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average  $q$ . Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

<sup>9</sup>An endogenous wedge between Tobin's average and marginal  $q$  also arises in cash-based models such as Bolton, Chen, and Wang (2011) and optimal contracting models such as DeMarzo, Fishman, He, and Wang (2012).

<sup>10</sup>This normalization is convenient in contracting models (see Sannikov, 2008). We can generalize these preferences to allow for a coefficient of relative risk aversion that is different from the inverse of the elasticity of intertemporal substitution, à la Epstein and Zin (1989). Indeed, as Epstein-



Zin preferences are homothetic, allowing for such preferences in our model will not increase the dimensionality of the optimization problem. Details are available upon request.

<sup>11</sup>Our conjecture is guided by the twin observations that: i) the value function for the standard Merton portfolio-choice problem (without illiquid assets) inherits the CRRA form of the agent's utility function  $U(\cdot)$  and, ii) the entrepreneur's problem is homogeneous in  $K$  and  $S$ .

<sup>12</sup>We infer the value of  $b$  from the solution of Merton (1971)'s closely related consumption and portfolio choice problem under complete markets. Note also that for the special case where  $\gamma = 1$  we have  $b = \zeta \exp \left[ \frac{1}{\zeta} \left( r + \frac{\eta^2}{2} - \zeta \right) \right]$ .

<sup>13</sup>See Wang, Wang, and Yang (2012) for a similar definition in a different setting where markets are exogenously incomplete.

<sup>14</sup>We will establish that under the first-best we have  $\gamma_e(s) = \gamma$ .

<sup>15</sup>The first-best case can be solved either via dynamic programming as we do here or via the Arrow-Debreu complete-markets/Cox-Huang martingale approach. The intuition that consumption and production decisions are independent are more transparent via the latter formulation. For brevity, we omit in this paper. See Duffie (2001) for a textbook treatment.

<sup>16</sup>By that we mean that  $\delta$  is the capital depreciation rate under the risk-neutral measure: The gap  $\delta - \delta_K$  is equal to the risk premium  $\rho\eta\sigma_K$  for capital shocks. The two Gordon growth formulae (26) and (29) are equivalent: The CAPM, implied by no arbitrage and the unique SDF given in (8), connects the two formulae under the physical and the risk-neutral measures.

<sup>17</sup>In practice entrepreneurs can sometimes partially commit themselves and lower their outside options by signing non-compete clauses. This possibility can be captured in our model by lowering the parameter  $\alpha$ , which relaxes the entrepreneur's inalienability-of-human-capital constraints.

<sup>18</sup> See Appendix A.B for technical details.

<sup>19</sup>Otherwise  $m(0) = m(\underline{s}) = 0$ , which does not make economic sense.

<sup>20</sup>We verify that the drift  $\mu^s(s)$  given in (23) is non-negative at  $\underline{s}$ , so that  $s$  is weakly increasing at  $\underline{s}$ .

<sup>21</sup>This is a standard assumption in the dynamic moral hazard literature (Ch. 10 in Bolton and Dewatripont, 2005). Di Tella and Sannikov (2016) develop a contracting model with hidden savings for asset management.

<sup>22</sup>Here, the subscript  $p$  refers to the principal, while the subscript  $e$  in  $\gamma_e$  refers to the en-

trepreneur's endogenous effective risk aversion in the liquidity and risk management problem analyzed earlier.

<sup>23</sup>Note that the coinsurance weight  $\frac{\gamma_p(w)}{\gamma_p(w)+\gamma}$  appears in (65) and (66).

<sup>24</sup>The necessary convergence condition is  $(r + \delta)^2 - 2\frac{A-(r+\delta)}{\theta} \geq 0$ .

<sup>25</sup>The first-best case is degenerate because the entrepreneur's indifference condition  $m(-q^{FB}) = 0$  implies zero-volatility of  $s$  at  $s = -q^{FB}$ . But this is not true for the inalienability case. Besides the indifference condition  $m(\underline{s}) = \alpha m(0)$ , we also need to provide incentives for the entrepreneur to choose zero volatility for  $s$  at the credit limit  $\underline{s}$ , which requires the entrepreneur to be endogenously infinitely risk averse at  $\underline{s}$ ,  $\gamma_e(\underline{s}) = \infty$ , meaning that  $m''(\underline{s}) = -\infty$ .

<sup>26</sup>This result can be seen from Panels B and D in Figure 1 where the slopes of  $m'(s)$  and  $p'(w)$  approach  $-\infty$  at  $\underline{s}$  and  $\underline{w}$ . Mathematically, this follows from the definition of  $\gamma_e$  given in (19),  $\sigma_h^s(s)$  given in (21), and  $m(\underline{s}) = 0.207$ . Similar mathematical reasoning applies for  $\gamma_p = \frac{wp''(w)}{p'(w)}$  in (62).

<sup>27</sup>A negative shock has the opposite effect on the entrepreneur's human capital and relaxes the inalienability constraint. Hence, we focus on the positive shock.

<sup>28</sup>The (diffusion) risk term for any stochastic process locally dominates its drift effect as the former is of order  $\sqrt{\Delta}$  and the latter is of order  $\Delta$ . We thus can drop the drift term in the limit for this calculation.

<sup>29</sup>There is a natural analogy here with the general principle in moral hazard theory that the agent's compensation trades off incentive and risk sharing considerations. Following Holmstrom (1979), this literature assumes that the agent's utility function is separable in effort and wealth (or consumption.) In our framework, exerting effort is analogous to staying with the firm. With this analogy, we note that our model does not assume the standard separability as that the severity of the agency problem depends on the distance of  $w$  to the debt limit  $\underline{w}$ . We therefore obtain a sharper result, namely that the more severe the agency problem is the less the agent is insured against idiosyncratic risk. See Sannikov (2008) for a continuous-time version of the classical moral hazard problem.

<sup>30</sup>Rampini, Sufi, and Viswanathan (2014) provide empirical evidence showing that more financially constrained firms hedge less. However, our analysis implies that more constrained firms have less volatile  $s$ .

<sup>31</sup>Note that the zero systematic volatility condition for  $s$  (and equivalently for  $w$ ) turns out to

be identical to the zero idiosyncratic volatility condition for  $s$  (and equivalently for  $w$ ).

<sup>32</sup>Unless, of course, the entrepreneur is prevented from working by a non-compete clause, which we have ruled out. However, in general, non-compete clauses are of finite duration and hence in theory, the employee still has options to re-contract in the future.

<sup>33</sup> Absconcion means “to hide away” or “to conceal” according to the Merriam-Webster Dictionary. If the entrepreneur were openly seen to use the pilfered capital elsewhere, she would be at risk of legal recovery and enforcement actions. To avoid these actions she has to hide and therefore cannot engage in any new contracts.

<sup>34</sup>In our model, the principal uses the SDF  $\mathbb{M}_t = e^{-rt} \exp\left(-\frac{\eta^2}{2}t - \eta \mathcal{Z}_{m,t}\right)$ , while in Ai and Li (2015), the principal uses  $\mathbb{M}_t = e^{-rt}$ . That is, the market price of risk is  $\eta > 0$  in ours and  $\eta = 0$  in theirs.

<sup>35</sup>With the additional assumption that  $\zeta = r$ , consumption between  $t$  and  $t+\tau$  is a sub-martingale in our model, while it is constant in Ai and Li (2015).

<sup>36</sup>We have analyzed more general situations that incorporate systematic productivity shocks. Generalizing our model to allow for a systematic risk premium requires an application of the standard change of measure technique by choosing different transition intensities under the physical measure and the risk-neutral measure. See for example, Bolton, Chen, and Wang (2013). As one may expect, the generalized liquidity and risk management problem in this section also has an equivalent optimal contracting formulation.

<sup>37</sup>For contracting models involving jumps and/or regime switching, see Biais, Mariotti, Rochet, and Villeneuve (2010), Piskorski and Tchisty (2010), and DeMarzo, Fishman, He and Wang (2012), among others.

<sup>38</sup>In the Appendix C, we provide the coupled equivalent HJB equation for  $J(K, S; A^H) \equiv J^H(K, S)$  in state  $H$ .

<sup>39</sup>There is an equivalent set of conditions characterizing  $\Pi^H$ , which we refer readers to the Appendix C.

<sup>40</sup>These results are robust and hold for other more extreme parameter values, which for brevity we do not report.

<sup>41</sup>For example, when productivity  $A = 0.18$  (together with  $\sigma_K = 0$  &  $\eta = 0$ ),  $q^{FB} = 1.17$  and  $i^{FB} = 0.0852$ . Because  $\delta = 11\%$  and  $r = \zeta = 5\%$ , it is immediate to see that (95) is violated

and hence  $\mu_{FB}^s(s_t) > 0$ . That is,  $s_t$  increases over time even under first-best and thus her limited-commitment constraint never binds. Of course, the net worth  $s + q^{FB}$  is positive which implies  $s \geq \underline{s}$  where  $\underline{s} = -q^{FB} = -1.17$  in this case.

**Table I**  
**Equivalence: Primal Optimization and Dual Contracting Problems**

	Primal Optimization	Dual Contracting
State Variable	$s$	$w$
Value Function	$m(s)$	$p(w)$

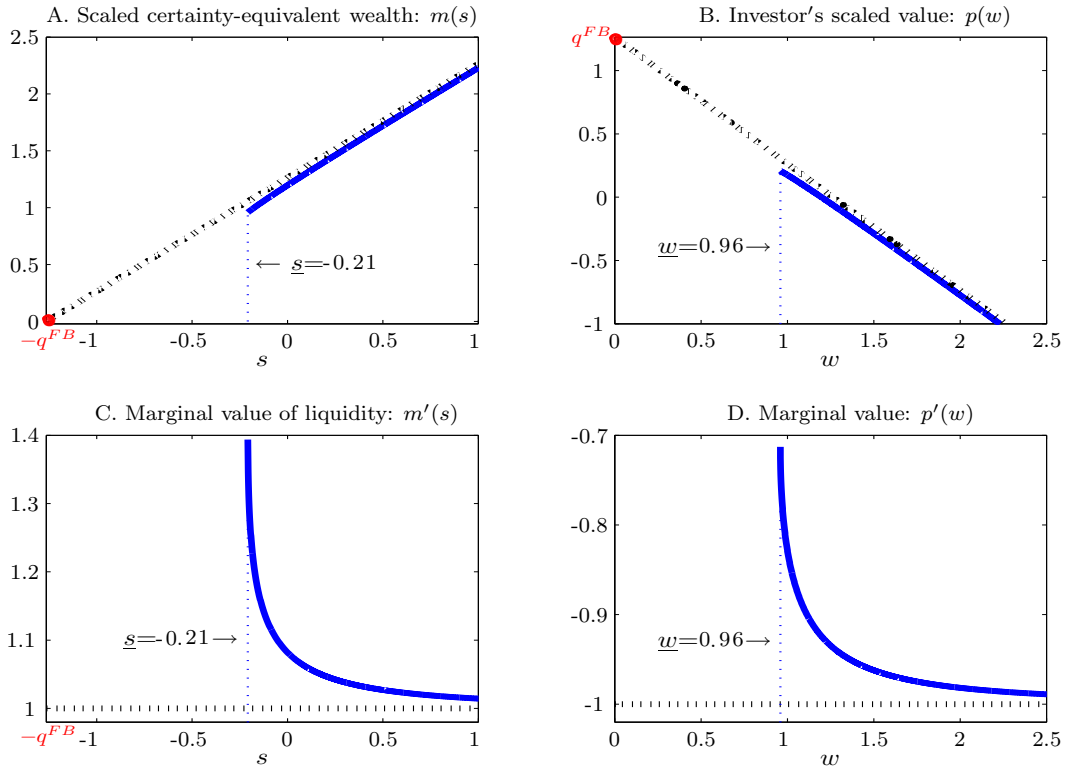
**Table II**  
**Comparison of Primal and Dual Optimization Problems.**

	<b>Primal Optimization</b>	<b>Dual Contracting</b>
<b>A. State Variable</b>	$s$	$w$
Drift	$\mu^s(s)$ given in (23)	$\mu^w(w)$ given in (70)
Idiosyncratic Volatility	$\sigma_h^s(s)$ given in (21)	$\sigma_h^w(w)$ given in (68)
Systematic Volatility	$\sigma_m^s(s)$ given in (22)	$\sigma_m^w(w)$ given in (69)
Admissible Range	$s \geq \underline{s}$	$w \geq \underline{w}$
<b>B. Value Function</b>	$m(s)$	$p(w)$
Interior Region	ODE given in (43)	ODE given in (71)
Right Limit	$\lim_{s \rightarrow \infty} m(s) = s + q^{FB}$	$\lim_{w \rightarrow \infty} p(w) = q^{FB} - w$
<b>C. Policy Rules</b>		
Compensation	$c(s)$ given in (37)	$\mathfrak{c}(w)$ given in (63)
Corporate Investment	$i(s)$ given in (39)	$i(w)$ given in (64)
Idiosyncratic Risk Hedge	$\phi_h(s)$ given in (41)	$x_h(w)$ given in (65)
Systematic Risk Exposure	$\phi_m(s)$ given in (42)	$x_m(w)$ given in (66)
<b>D. Inalienability Case: <math>0 &lt; \alpha &lt; 1</math></b>		
Inalienability Constraint	$m(\underline{s}) = \alpha m(0)$	$p(\underline{w}/\alpha) = 0$
Curvature Condition	$m''(\underline{s}) = -\infty$	$p''(\underline{w}) = -\infty$
<b>E. First-Best case: <math>\alpha = 0</math></b>		
Borrowing Limit	$\underline{s} = -q^{FB}$	$\underline{w} = 0$

**Table III**  
**Parameter Values**

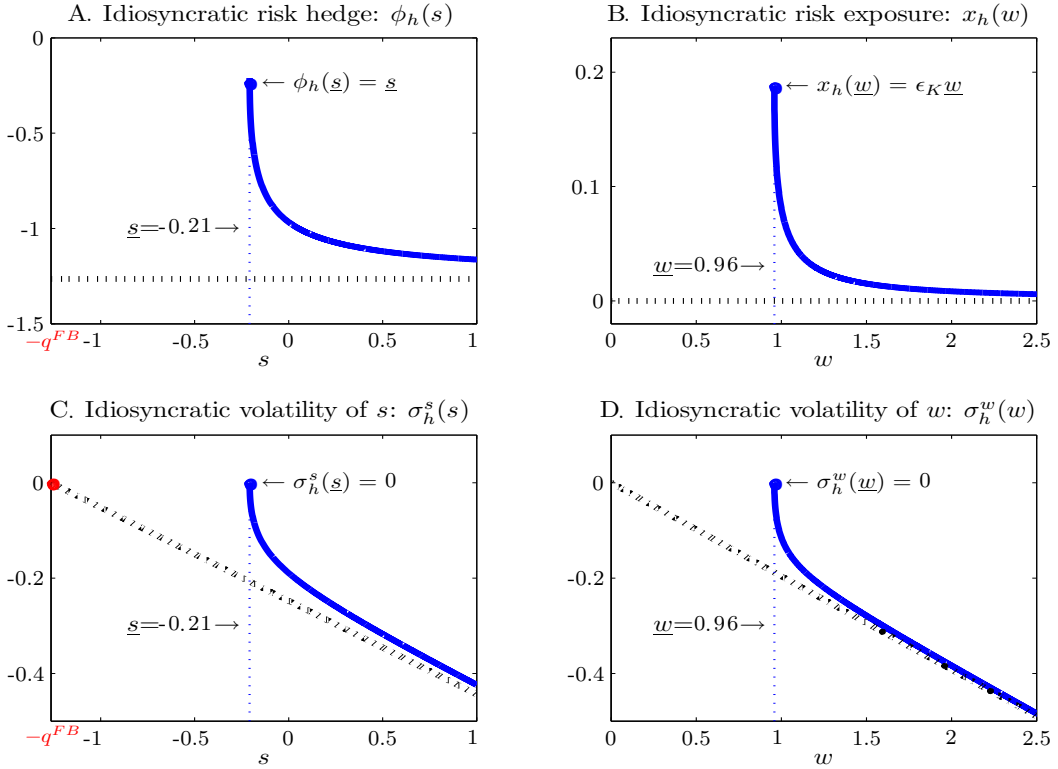
Parameters	Symbol	Value
Risk-free rate	$r$	5%
The entrepreneur's discount rate	$\zeta$	5%
Correlation	$\rho$	20%
Excess market portfolio return	$\mu_m - r$	6%
Volatility of market portfolio	$\sigma_m$	20%
The entrepreneur's relative risk aversion	$\gamma$	2
Capital depreciation rate	$\delta_K$	11%
Volatility of capital depreciation shock	$\sigma_K$	20%
Quadratic adjustment cost parameter	$\theta$	2
Productivity parameter	$A$	20%
Inalienability parameter	$\alpha$	80%

This table summarizes the parameter values for our baseline analysis in Section V. Whenever applicable, parameter values are annualized.

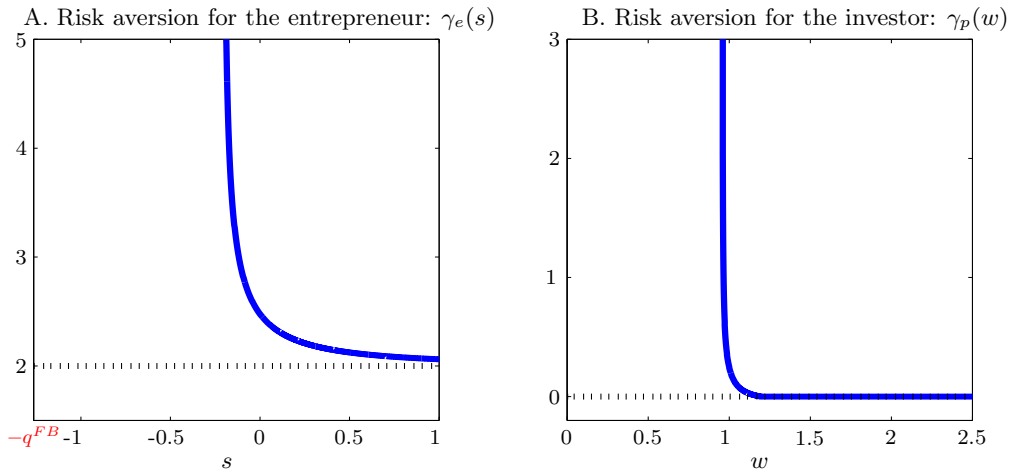


**Figure 1. Certainty equivalent wealth  $m(s)$  and investor's value  $p(w)$ .** The dotted lines depict the first-best results:  $m(s) = q^{FB} + s$  and  $m'(s) = 1$  for  $s \geq -q^{FB} = -1.264$ ,  $p(w) = q^{FB} - w$  and  $p'(w) = -1$  for  $w \geq \underline{w}^{FB} = 0$ . The solid lines depict the inalienability case:  $m(s)$  is increasing and concave where  $s \geq \underline{s} = -0.21$ , and  $p(w)$  is decreasing and concave where  $w \geq \underline{w} = 0.96$ . The debt limit  $\underline{s}$  is determined by  $m(\underline{s}) = \alpha m(0)$  and  $m''(\underline{s}) = -\infty$ , and  $\underline{w}$  is determined by  $p(\underline{w}/\alpha) = 0$  and  $p''(\underline{w}) = -\infty$ .

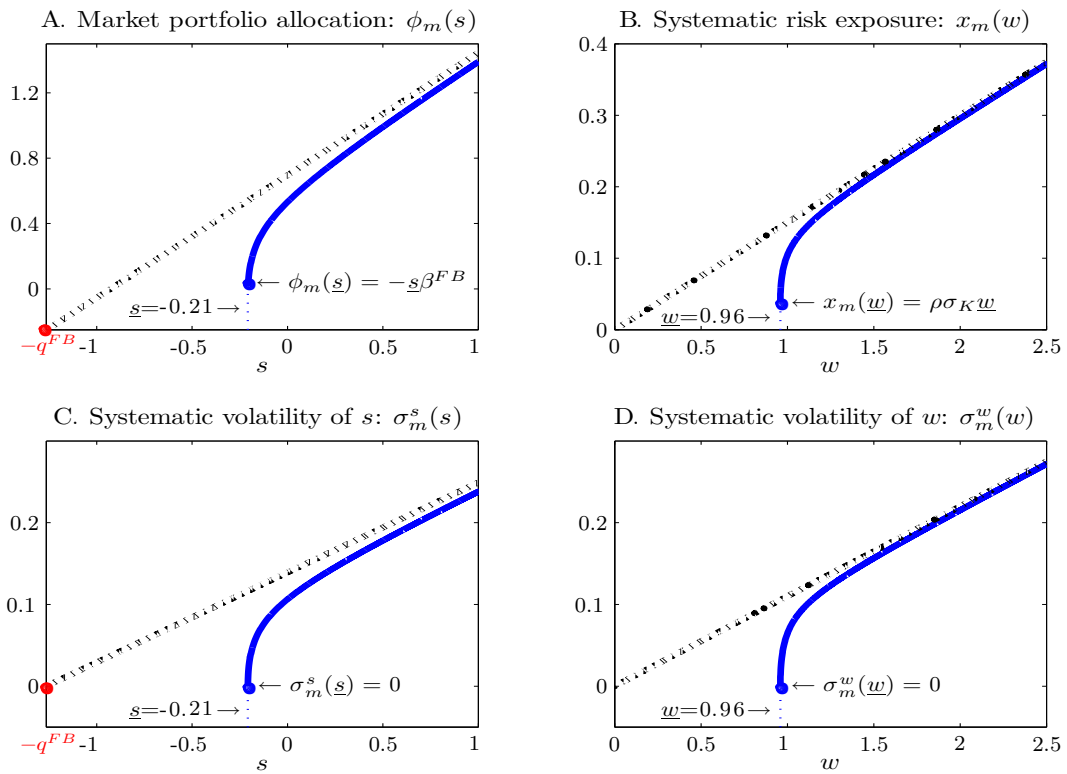




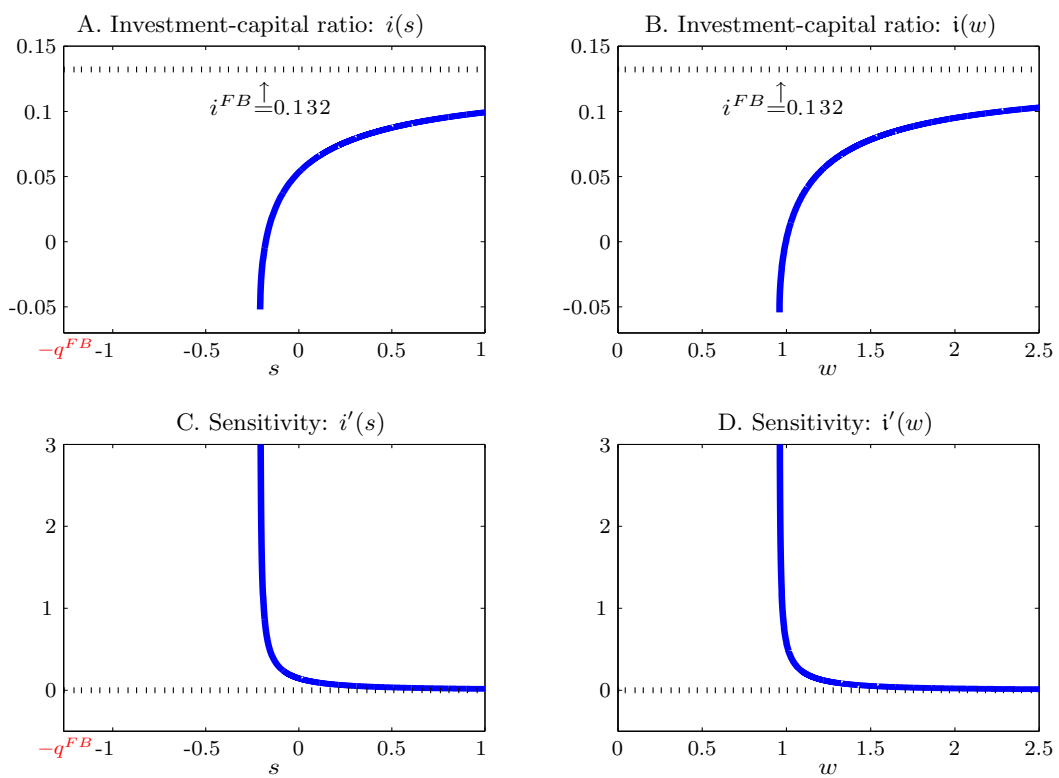
**Figure 2. Idiosyncratic risk management policies,  $\phi_h(s)$  and  $x_h(w)$ , and idiosyncratic volatilities for  $s$  and  $w$ ,  $\sigma_h^s(s)$  and  $\sigma_h^w(w)$ .** The dotted lines depict the first-best results:  $\phi_h^{FB}(s) = -q^{FB} = -1.264$  and  $x_h^{FB}(w) = 0$ . The solid lines depict the inalienability case: the entrepreneur hedges less than under the first best,  $|\phi_h(s)| < |\phi_h^{FB}(s)| = q^{FB}$ , and her idiosyncratic risk exposure is thus positive,  $x_h(w) > 0$ .



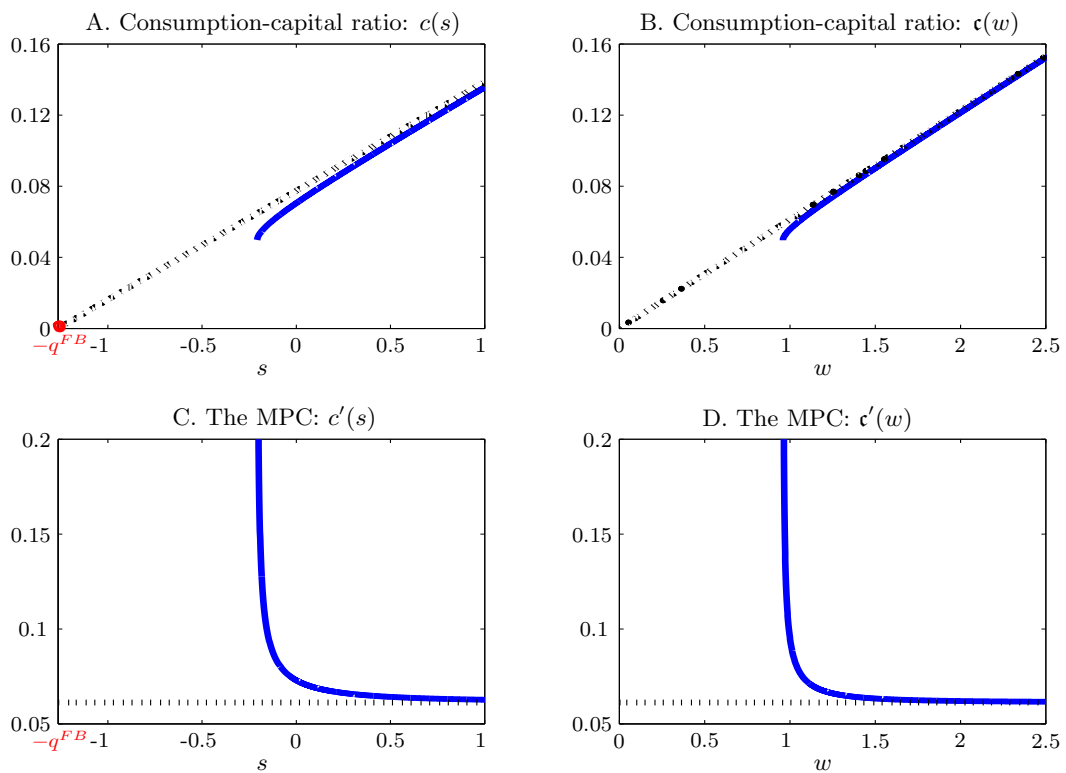
**Figure 3. Endogenous relative risk aversion for the entrepreneur and the investor,  $\gamma_e(s)$  and  $\gamma_p(w)$ .** The dotted lines depict the first-best results:  $\gamma_e^{FB}(s) = \gamma = 2$  and  $\gamma_p^{FB}(w) = 0$ .



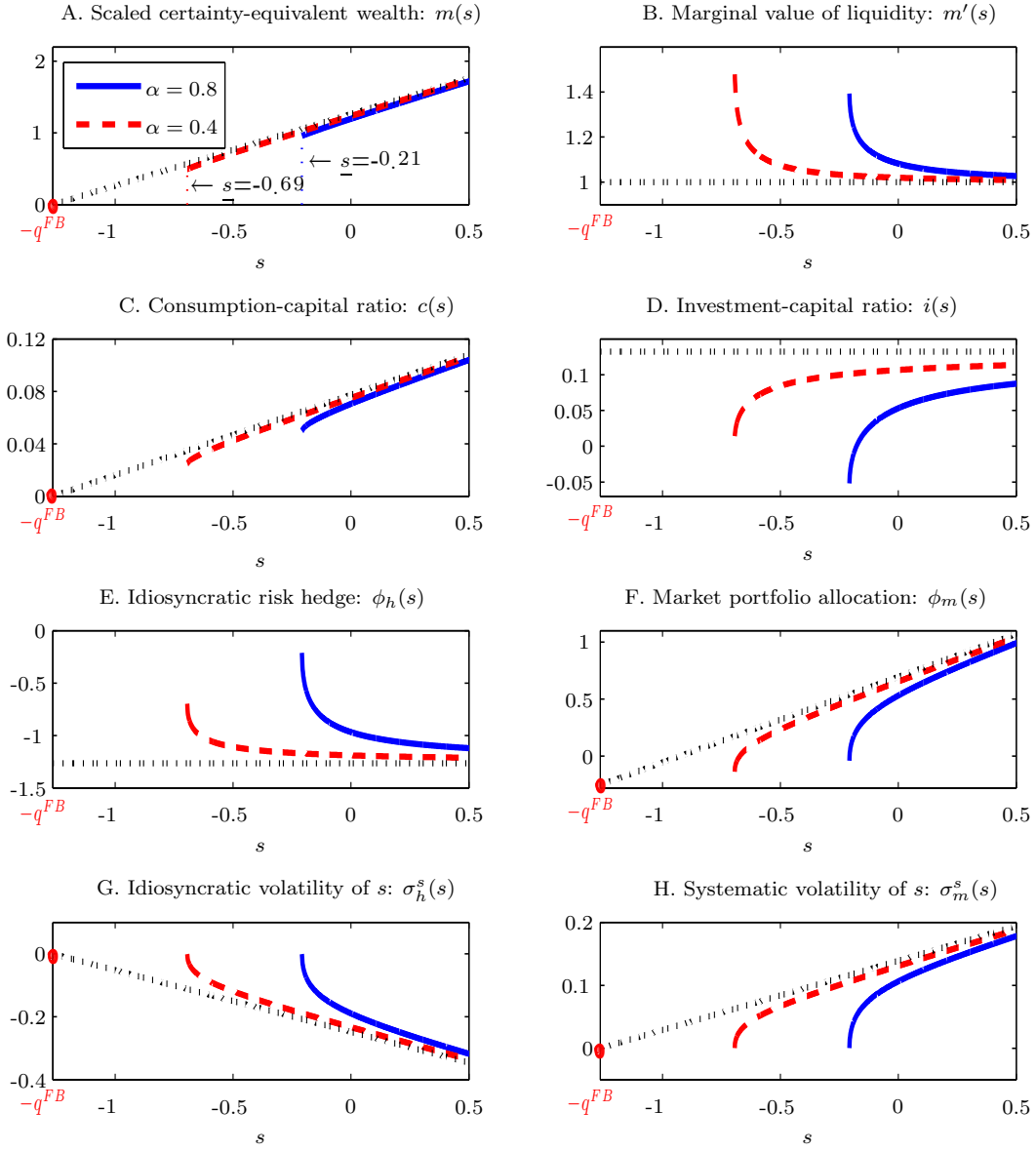
**Figure 4. Systematic risk exposures,  $\phi_m(s)$  and  $x_m(w)$ , and systematic volatilities for  $s$  and  $w$ ,  $\sigma_m^s(s)$  and  $\sigma_m^w(w)$ .** The dotted lines depict the first-best results. The solid lines depict the inalienability case: the entrepreneur's systematic risk exposures are lower than under the first best,  $\phi_m(s) < \phi_m^{FB}(s)$  and  $x_m(w) < x_m^{FB}(w)$ .



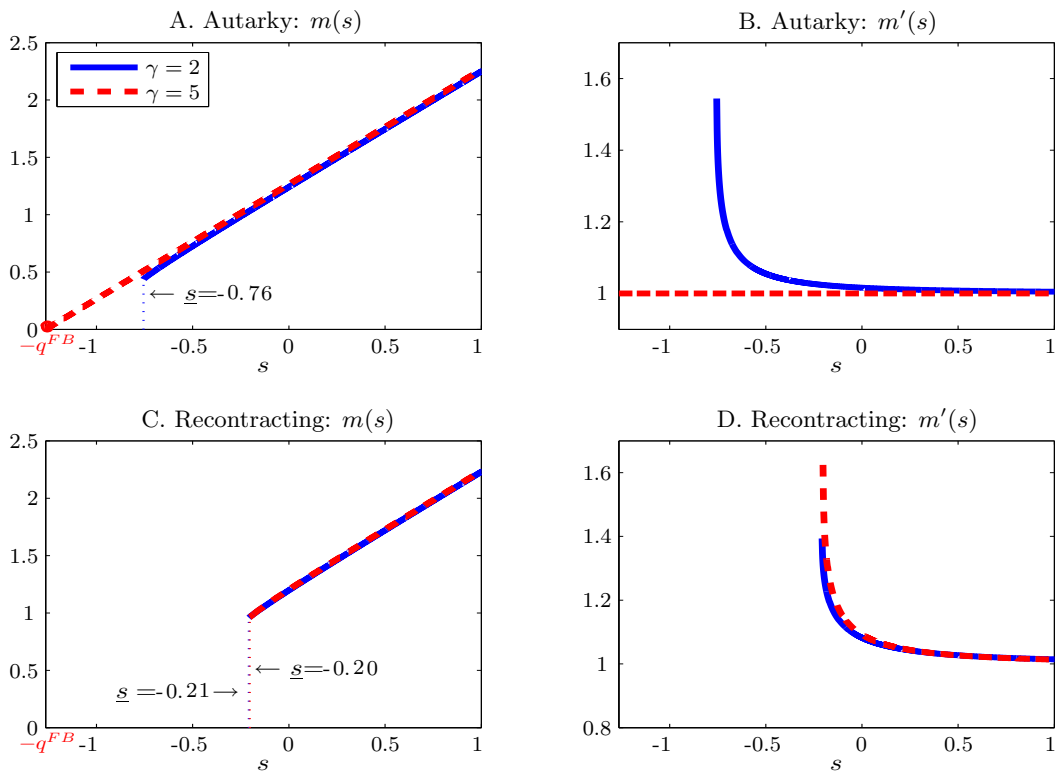
**Figure 5. Investment-capital ratio and its sensitivity.** The dotted lines depict the first-best results:  $q^{FB} = 1.264$  and  $i^{FB} = 0.132$ . The solid lines depict the inalienability case: the firm always under-invests and  $i(s)$  increases with  $s$  (equivalently  $i(w)$  increases with  $w$ .)



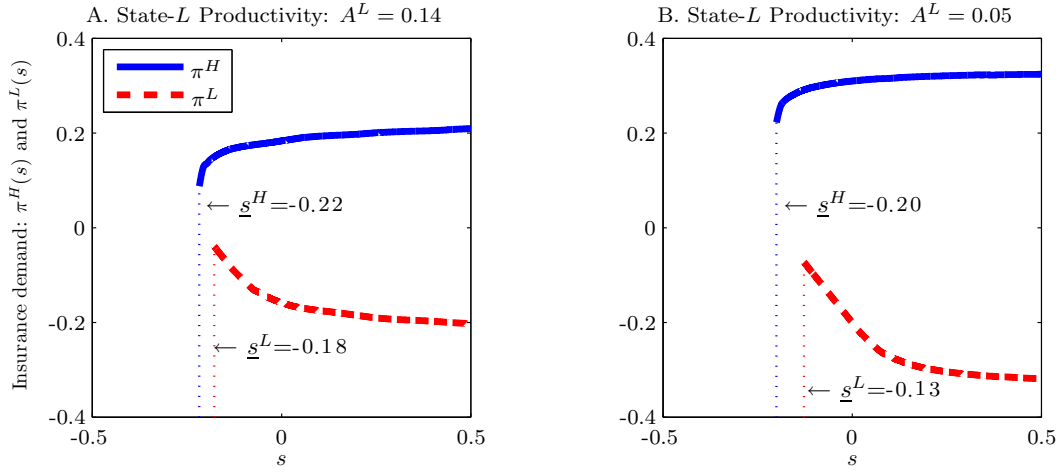
**Figure 6. Consumption-capital ratio and the MPC.** The dotted lines depict the first-best results:  $c^{FB}(s) = \chi m(s) = \chi w = \mathfrak{c}^{FB}(w)$ . The solid lines depict the inalienability case: the entrepreneur always under-consumes and  $c(s)$  is increasing and concave in  $s$  (equivalently  $\mathfrak{c}(w)$  is increasing and concave with  $w$ .)



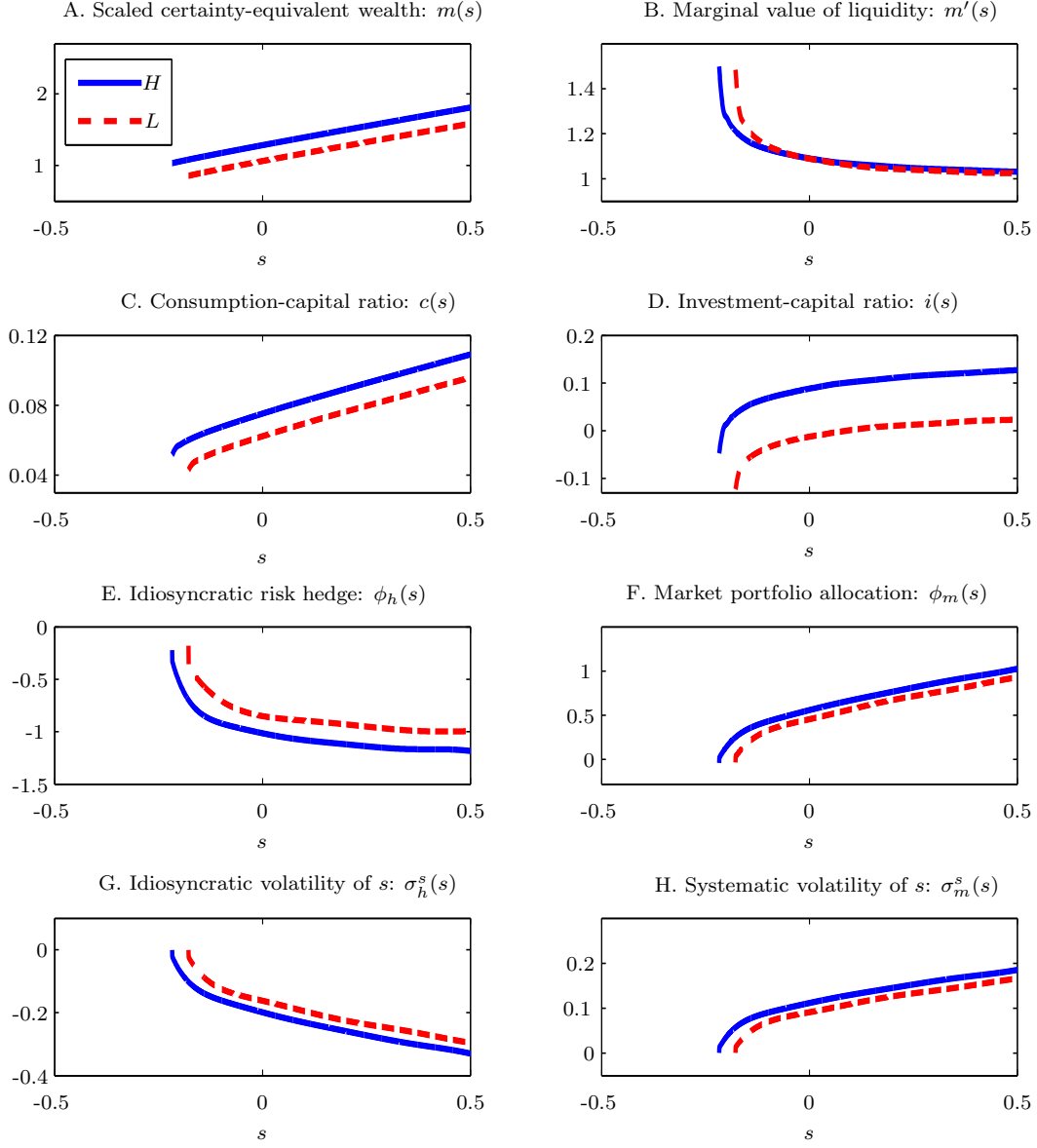
**Figure 7. Comparative Statics with respect to  $\alpha$ .** The lower the value of  $\alpha$ , the less inalienable the entrepreneur's human capital, the higher the debt capacity  $|s|$ , the less the firm under-invests and under-compensates the entrepreneur, the higher the idiosyncratic risk hedging demand, and the higher the entrepreneur's exposure to the stock market.



**Figure 8. Recontracting versus autarky.** Panels A and B plot the autarky case for  $\gamma = 2$  and  $\gamma = 5$ . Panels C and D plot the recontracting case for  $\gamma = 2$  and  $\gamma = 5$ . Under recontracting, the solutions for  $\gamma = 2$  and  $\gamma = 5$  are similar. For example,  $\underline{s} = -0.21$  for  $\gamma = 2$  and  $\underline{s} = -0.20$  for  $\gamma = 5$ . However, under autarky, when  $\gamma = 5$ , the solution features the first best and hence  $\underline{s} = -q^{FB}$ , but when  $\gamma = 2$ ,  $\underline{s} = -0.76$ .

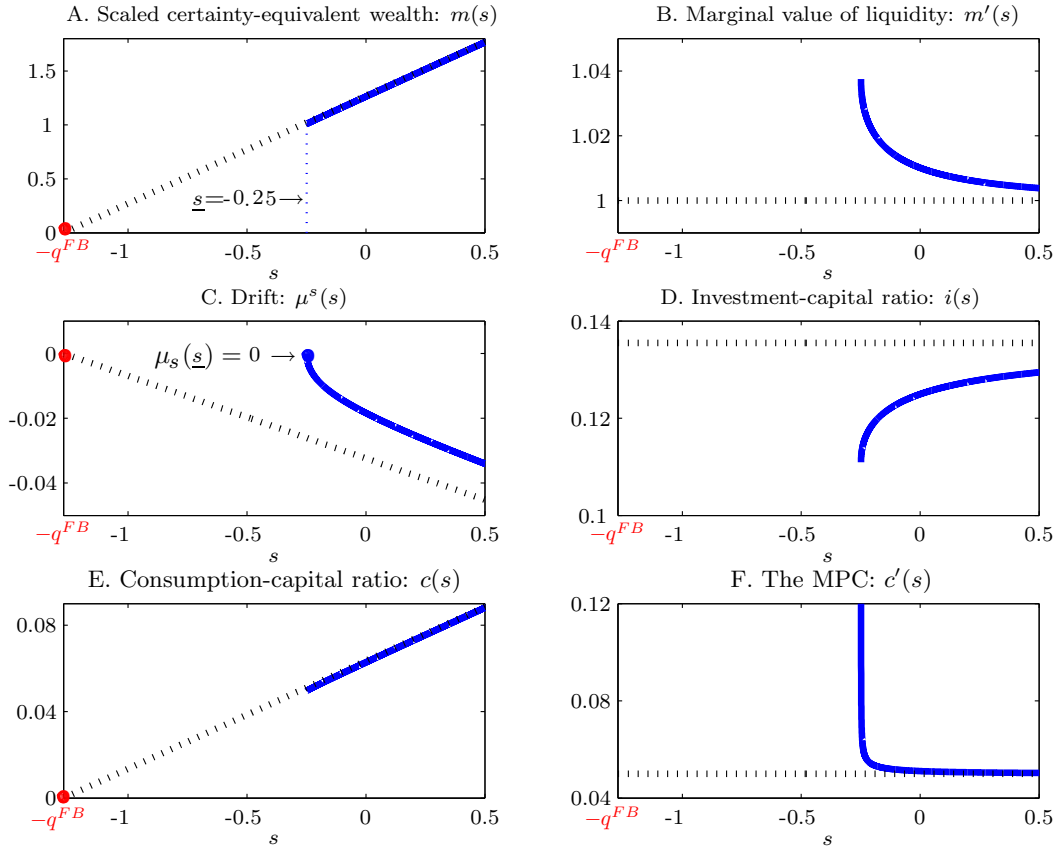


**Figure 9. Insurance demand under persistent productivity shocks:  $\pi^H(s)$  and  $\pi^L(s)$ .** State- $H$  productivity is  $A^H = 0.25$  in both panels. In Panel A, State- $L$  productivity is  $A^L = 0.14$ , and  $\underline{s}^H = -0.217$ , and  $\underline{s}^L = -0.178$ , and  $\pi^L(s) = \underline{s}^H - s$  when  $-0.178 < s < -0.129$ . In Panel B, State- $L$  productivity is  $A^L = 0.05$ , and  $\underline{s}^H = -0.200$ ,  $\underline{s}^L = -0.128$ , and  $\pi^L(s) = \underline{s}^H - s$  when  $-0.128 < s < 0.023$ .

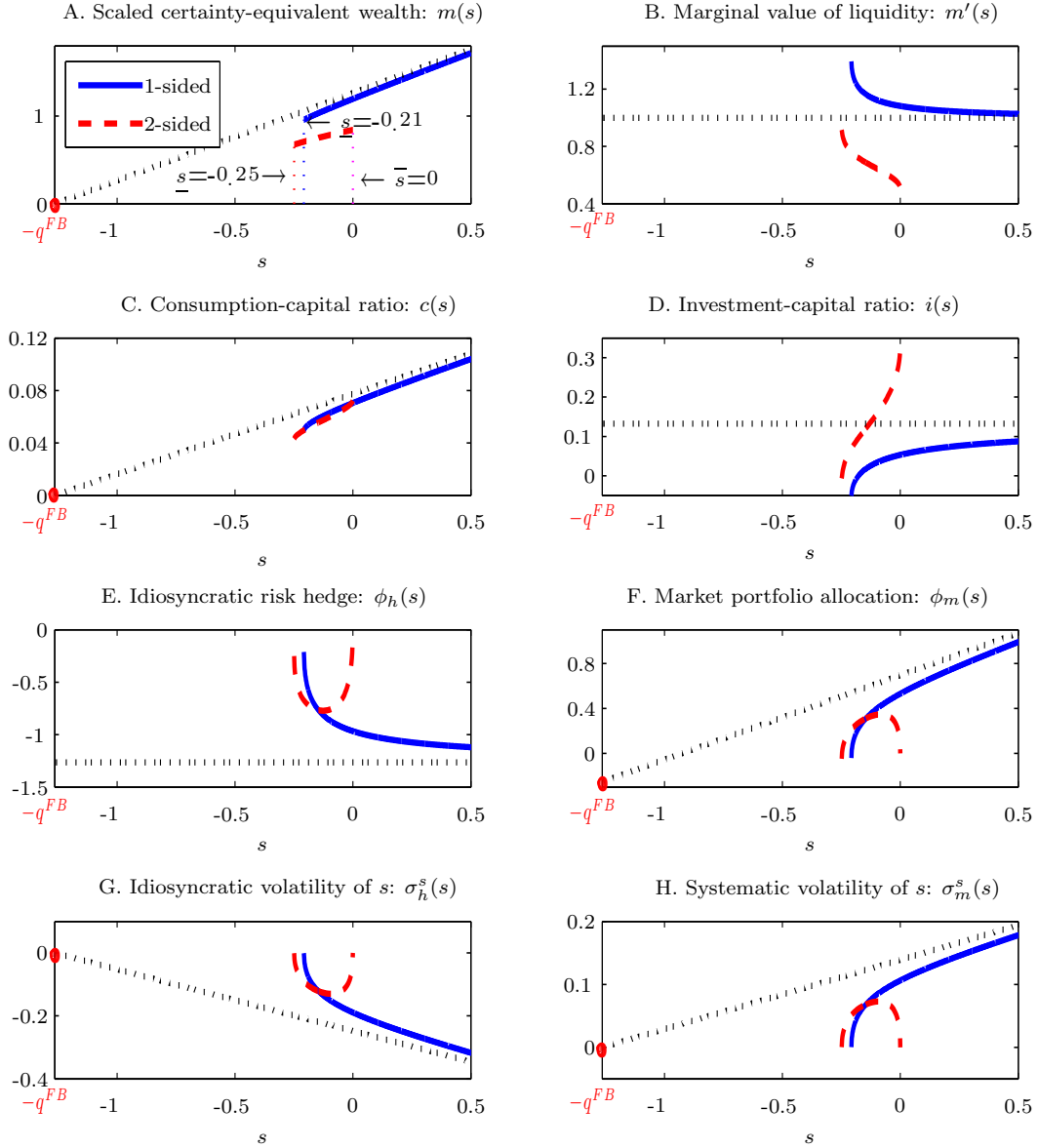


**Figure 10. The case with persistent productivity shocks.** Table III contains parameter values unless otherwise stated here:  $A^H = 0.25$ ,  $A^L = 0.14$ , and  $\lambda^L = \lambda^H = 0.2$ . Under inalienability,  $\underline{s}^H = -0.217$ , and  $\underline{s}^L = -0.178$ . Under the first best,  $q_H^{FB} = 1.357$ ,  $q_L^{FB} = 1.115$ ,  $i_H^{FB} = 0.179$ ,  $i_L^{FB} = 0.057$ .





**Figure 11. The deterministic case ( $\sigma_K = 0$  and  $\eta = 0$ ) where the firm is financially constrained.** Productivity  $A = 0.185$  and other parameter values are given in Table III. Under the first best, the firm's debt capacity is  $-\underline{s} = 0.25$ . The dotted lines depict the first-best results with  $q^{FB} = 1.271$  and  $i^{FB} = 0.136$ .



**Figure 12. Two-sided limited commitment.** The endogenous upper boundary  $\bar{s} = 0$ . Compared with the first best,  $s$  lies in the range  $(\underline{s}, \bar{s}) = (-0.25, 0)$  under two-sided limited commitment. The firm under-invests when  $s$  is close to  $\underline{s} = -0.25$  and over-invests when  $s$  is close to  $\bar{s} = 0$ . The credit limit under the two-sided limited commitment  $|\underline{s}| = 0.25$  is larger than the credit limit  $|\underline{s}| = 0.21$  in our baseline case with only inalienable human capital.