NBER WORKING PAPER SERIES

GLOBAL SUNSPOTS AND ASSET PRICES IN A MONETARY ECONOMY

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Working Paper 20831
http://www.nber.org/papers/w20831

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
January 2015

I would like to thank Fernando Alvarez, Markus Brunnermeir, Emmanuel Farhi, Leland Farmer, Xavier Gabaix, Nicolae Garleanu, Valentin Haddad, Lars Hansen, Nobu Kiyotaki, Robert Lucas, Nancy Stokey, Harald Uhlig, Ivan Werning and Pawel Zabczyk for their comments on earlier versions of the ideas contained in this paper. I would also like to thank participants at the NBER Economic Fluctuations and Growth Meeting in February of 2014, the NBER 2014 summer workshop on Asset pricing, the 2014 summer meetings of the Society for Economic Dynamics in Toronto Canada, and the Brigham Young University Computational Public Economics Conference in Park City Utah, December 2014. Earlier versions of this work were presented at the Bank of England, the Board of Governors of the Federal Reserve, Harvard University, the International Monetary Fund, the London School of Economics, the London Business School, Penn State University, the University of Chicago, the Wharton School and Warwick University. I would especially like to thank C. Roxanne Farmer for her editorial assistance. The views expressed herein are those of the author and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 20831
January 2015
JEL No. E3,E43,G12

ABSTRACT

The representative agent (RA) model, widely used by macroeconomists, successfully explains the co-movements among consumption, investment, employment and GDP. It is much less successful at explaining asset price data. Here, I construct a simple heterogeneous agent model, driven by sunspots, that provides a bridge between macroeconomics and finance theory. Most existing sunspot models use local linear approximations: instead, I construct global sunspot equilibria. My agents are expected utility maximizers with logarithmic utility functions, there are no fundamental shocks and markets are sequentially complete. Despite the simplicity of these assumptions, I am able to go a considerable way towards explaining features of asset pricing data that have presented an obstacle to previous models that adopted similar assumptions. My model generates volatile persistent swings in asset prices, a substantial term premium for long bonds and bursts of conditional volatility in rates of return. If my explanation for asset price volatility is accepted, models that build on my framework have the potential to unify macroeconomics with finance theory in a simple and parsimonious way.

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1 Introduction

The representative agent (RA) model has been used by macroeconomists to understand business cycles for more than thirty years. This model, when supplemented by price rigidities and financial frictions, does a reasonable job of replicating the co-movements of consumption, investment, GDP and employment in past data (Smets and Wouters, 2003, 2007). But it fails badly when confronted with financial market facts (Cochrane, 2011).

The following three features of asset price data are anomalies from the perspective of the RA model. 1) Asset prices are persistent and volatile and price dividend ratios are mean reverting. 2) Aggregate consumption is smooth but the return to a riskless asset is five hundred basis points less than the return to the stock market. 3) Asset price volatility is non-constant and non-Gaussian, and models that assume that asset prices are log normally distributed with time-invariant volatility are rejected decisively by the data (Bollerslev, Engle, and Nelson, 1994).[1]

This paper constructs a heterogenous agent general equilibrium model that helps to explain all three anomalies. In this model, asset price fluctuations are caused by random shocks to the price level that reallocate consumption across two kinds of people. Asset prices are volatile and price dividend ratios are persistent even though there is no fundamental uncertainty and financial markets are sequentially complete.

My work differs in three ways from standard asset pricing models. First, I allow for birth and death by exploiting Blanchard's (1985) concept of perpetual youth. Second, there are two types of people that differ in the rate at which they discount the future.[2] Third, my model contains an asset, gov-

[1] An active body of scholars seek to explain these anomalies. Some of the approaches that have been tried include richer utility specifications (Abel, 1990; Constantinides, 1990; Campbell and Cochrane, 1999) adding technology shocks with exogenous time-varying volatility (Bansal and Yaron, 2004), and assuming that technology is occasionally hit by rare disasters (Reitz, 1988; Barro, 2005, 2006; Wachter, 2013; Gabaix, 2012).

[2] Different discount rates may arise either because agents have different time-preference

ernment debt, denominated in dollars. All three of these assumptions have appeared before in previous work.[3] My contribution is to combine them in a way that generates novel results.

Because I am interested in the effects of distributional shocks, my baseline model has no fundamental uncertainty of any kind. I characterize equilibria as a first order non-linear difference equation in two state variables. This equation has a unique feasible steady state that is a saddle point. I compute the full set of dynamic equilibria by solving for a one dimensional manifold in a two dimensional space. Trajectories that begin on this manifold converge to the steady state and those trajectories completely characterize the set of perfect foresight equilibria.

Because debt is denominated in dollars, there is a free initial condition and the initial price level is indeterminate. I exploit this indeterminacy to construct a set of stochastic rational expectations equilibria driven by purely non-fundamental uncertainty. Following David Cass and Karl Shell (1983), I refer to the random variables that drive these rational expectations equilibria as sunspots.

Most sunspot models add a shock to the perfect foresight equilibria of a model that has been linearized around an indeterminate steady state.[4] This method may be used to generate local sunspot equilibria but there is no guarantee that the sunspot solutions of a linear approximation remain valid

rates (some agents are more patient) or because they have different ages. In this second interpretation, explored in Farmer (2014a), the agents transit randomly from youth to middle age to death.

[3]Farmer (2002a) develops a version of Blanchard's (1985) perpetual youth model with capital and aggregate uncertainty and Farmer (2002b) adds nominal government debt to this framework to explain asset price volatility. Farmer, Nourry, and Venditti (2012), Gârleanu, Kogan, and Panageas (2012) and Gârleanu and Panageas (2014) develop versions of the Blanchard framework with two agents. The results in the current paper rely on all three of these pieces; perpetual youth, multiple types and nominal debt.

[4]Farmer and Woodford (1984, 1997) is the first example of this type in a one dimensional model and Woodford (1986) extends the technique to higher dimensions. See Benhabib and Farmer (1999) and Farmer (1999), for further applications of this method to business cycle models.

once the variance of the shocks becomes large. In this paper, I exploit the nonlinear nature of my solution to compute global sunspot equilibria. This feature enables the model to generate a substantial term premium for assets that exhibit duration risk.

Although I model an endowment economy, the framework I provide can easily be extended to allow for production by adding capital and a labor market. If my explanation for asset price volatility is accepted, models that build on this framework have the potential to unify macroeconomics with finance theory in a simple and parsimonious way.

2 Antecedents

This paper draws on ideas developed at the University of Pennsylvania in the 1980s (Farmer, 2014b). Using the term "sunspots" to refer to nonfundamental uncertainty, David Cass and Karl Shell (1983) showed that sunspots can have real effects on consumption, even in the presence of a complete set of financial markets. Using the term "self-fulfilling prophecies" to refer to nonfundamental uncertainty, Costas Azariadis (1981) showed that nonfundamental shocks could be added to a DSGE model to drive business cycles. Drawing on both of these ideas, Roger Farmer and Michael Woodford (1984; 1997) combined sunspots with indeterminacy to generate a model where sunspot shocks explain persistent fluctuations in GDP.

In this paper I move the sunspot research agenda forward by developing a sunspot model of asset pricing that represents an alternative to the widely used representative agent approach (Abel, 1990; Campbell and Cochrane, 1999; Bansal and Yaron, 2004).

My work is most closely related to four unpublished working papers, Farmer (2002a,b, 2014a) and Farmer, Nourry, and Venditti (2012). In Farmer (2002a) and Farmer (2002b) I constructed a perpetual youth model of the kind developed by Blanchard (1985). I added aggregate shocks, and I used the resulting framework to understand features of asset pricing data. The models developed in those papers exploited the existence of an indeterminate steady state, but they relied on the unrealistic feature that the equilibrium is dynamically inefficient.

In joint work with Carine Nourry and Alain Venditti (Farmer, Nourry, and Venditti, 2012) we thought we had solved the problem of dynamic inefficiency by constructing sunspot equilibria in a model with a unique perfect foresight equilibrium. Unfortunately, that turns out not to be the case as the putative equilibria we construct in that working paper fail to equate the marginal rates of substitution of each type of agent in every state. Consequently, the paper does not fulfil its claim to generate sunspot equilibria.

In this paper I combine ideas from all of these working papers in a novel way. First, I reintroduce nominally denominated government debt as in my (2002b) paper. Second, I exploit the idea that there are two types of agents, as in Farmer, Nourry, and Venditti (2012). And third, I introduce a technique to construct global sunspots that is a development of an idea first introduced in Farmer (2014a).

This is not the only paper to explore heterogenous agents models to understand asset pricing data. Gârleanu, Kogan, and Panageas (2012) build a two agent lifecycle model where the agents have recursive preferences but a common discount factor and they show that this model generates intergenerational shifts in consumption patterns that they call 'displacement risk'. In a related paper Gârleanu and Panageas (2014) study asset pricing in a continuous time stochastic overlapping generations model. In contrast to my work, these papers focus on fundamental equilibria and they adopt the, now common, assumption of Epstein Zin preferences (Epstein and Zin, 1991, 1989).

Challe (2004) generates return predictability in an overlapping generations model and Guvenen (2009) constructs a production economy that he solves computationally. Constantinides and Duffie (1996) exploit crosssection heterogeneity of the income process to show that uninsurable income risk across consumers can potentially explain *any* observed process for asset prices. Kubler and Schmedders (2011) construct a heterogenous agent overlapping generations model with sequentially complete markets. By dropping the rational expectations assumption, they are able to generate substantial asset price volatility. In a related paper, Feng and Hoelle (2014) generate large welfare distortions from sunspot fluctuations.

My work differs from these papers by providing a simple and analytically tractable model that provides a bridge between asset pricing models and business cycle models. In contrast to the now familiar assumption of Epstein Zin preferences, my agents are expected utility maximizers with logarithmic utility functions. I abstract from fundamental shocks and I assume that markets are sequentially complete. Despite the simplicity of these assumptions, I am able to go a considerable way towards explaining features of asset pricing models that have presented an obstacle to previous models that adopted similar assumptions.

3 The Structure of my Model

This section lays out the structure of my model, and it explains how fiscal and monetary policy interact. Sections 3.1 – 3.4 lay out the assumptions about the environment and Section 3.5 discusses an important implication of the absence of intergenerational transfers. This assumption implies that the model is non-Ricardian in the sense of Barro (1974).

3.1 Assumptions about people, apples and trees

There are two types of people. Each type is endowed with one unit of a unique perishable commodity in every period in which he is alive; I call this an apple. The wealth of a person in the year of his birth is equal to the discounted present value of his apples. I call this a tree.

People have logarithmic preferences and discount factors β_1 and β_2. Type 1 people are more patient than type 2 people. This assumption is represented by the inequalities,

$$0 < \beta_2 < \beta_1 < 1. \tag{1}$$

People of each type die with probability $1 - \pi$ and when a person dies he is replaced by a new person of the same type. The model contains μ_i type i people, where $\sum_i \mu_i = 1$, hence, there is a constant population of measure 1.

3.2 Assumptions about uncertainty

Uncertainty in period t is indexed by a random variable S_t with compact support $\boldsymbol{S}$

$$S_t \in \boldsymbol{S}.$$

I refer to a $\tau-$period sequence S_t^τ as a $\tau-$period *history* with root S_t,

$$S_t^\tau = \{S_t, S_{t+1}, ...S_\tau\}.$$

The root is the initial date-state pair and a history, S_t^τ is a $\tau - t$ dimensional random variable with support $\boldsymbol{S}_t^{\tau-t}$.

In the remainder of the paper, I will drop t subscripts to cut down on notation. Instead, I will use the notation x to refer to $x_t(S)$ and $x(S')$ to refer to $x_{t+1}(S')$. All real date t variables are functions of the current realization of S.

3.3 Assumptions about the asset markets

Asset markets are sequentially complete. Three assets are actively traded; Arrow securities, government debt, and trees.

An Arrow security costs $Q(S')$ apples at date t and pays one apple at date $t+1$ if and only if state S' occurs. In aggregate, people of type i hold

$a_i(S)$ type S securities at date t in state S. The quantities of each security demanded by each type may be positive or negative.[5]

Government debt costs $Q^N D'$ dollars at date t and is a claim to D' dollars at date $t+1$. Because the dollar price of apples is a random variable, the real return to government debt is also random.

A tree costs p_k apples at date t and delivers one apple every period in which the issuer of the asset remains alive. The price of a tree is computed recursively from the pricing equation,

$$p_k = 1 + \pi E\left[p_k(S') Q(S')\right].$$

The term π appears in this expression to reflect the fact that the tree will be worthless next period with probability $(1-\pi)$. This reflects the probability that the person issuing the claim has died.

Let $Q\left(S_t^T\right)$ be the price today of a claim to one apple in history S_t^T. I assume that

$$\lim_{T\to\infty}\inf Q\left(S_t^T\right) = 0, \qquad \text{for all } S_t^\tau.$$

This is the stochastic generalization, for this economy, of the assumption that the interest rate is greater than the growth rate and it rules out equilibria that are dynamically inefficient.

3.4 Assumptions about government

Government consists of a central bank and a treasury. The treasury issues dollar denominated one-period debt and faces the budget constraint

$$Q^N D' = D - \tau p, \tag{2}$$

[5]Because I make assumptions that allow me to aggregate the consumption decision of each type, I do not refer to the asset holdings of individual agents. But in fact these asset holdings display a rich pattern of heterogeneity. Asset holdings depend not just on type, but also on the state into which a person was born.

at every date t where τ is the proportional tax rate and p is the dollar price of an apple.

The central bank sets the gross interest rate equal to a constant, that I denote by R^N, where $R^N \equiv 1/Q^N$, in every period. A monetary policy rule of this kind is called *passive*. The treasury issues sufficient nominal debt to roll over its existing debt, net of tax revenues. A fiscal policy of this kind is called *active*.[6]

Michael Woodford (1995), has shown that, in representative agent economies, the combination of an active fiscal policy and a passive monetary policy leads to a unique equilibrium price level. This result is known as the fiscal theory of the price level and it *does not hold* in the model I develop in this paper.[7]

Dividing Equation (2) by p and multiplying and dividing the left-hand side by p', we can write the following expression for the evolution of government debt

$$Q'b' = b - \tau, \tag{3}$$

where

$$b \equiv \frac{D}{p}, \quad \text{and} \quad Q' = \frac{p'}{pR^N}.$$

In Section 5.2 I will combine Equation (3) with a difference equation in p_k and b that arises from the assumption that the marginal rates of substitution of each type are equal state by state. This leads to two difference equations in two variables, p_k and b. For any given initial condition, pinned down by the initial price of apples, these difference equations fully characterize the set of perfect foresight equilibria.

[6] This terminology is due to Leeper (1991).

[7] The fiscal theory of the price level treats the government budget constraint as a valuation equation. For a given net present value of tax revenues, there is a unique price level for which the budget is exactly balanced. That is not true in my model. Instead, variations in the price level redistribute the tax burden of the debt between the current generation and future generations.

3.5 Aggregate wealth and Ricardian equivalence

The aggregate wealth of the private sector consists of the after tax value of existing trees, plus the value of government debt.

I will use the symbol W to represent aggregate private wealth,

$$W = p_k - T + b, \tag{4}$$

and the symbols τ and T to represent the tax rate and the tax obligations of current generations. T and τ are related by the identity,

$$T \equiv p_k \tau.$$

Because the economy is closed, government debt is the liability of private agents. But some of the people who will repay that debt have not yet been born.[8] Using $\bar{T}$ to represent the tax liability of future generations, the net present value of the government's assets must equal the net present value of its liabilities,

$$b \equiv T + \bar{T}. \tag{5}$$

Note however that

$$b \neq T.$$

This model is non-Ricardian in the sense of Barro (1974) because future, as yet unborn generations, are partially liable for the debts incurred by the treasury on behalf of the current generation.

The fact that the model is non-Ricardian depends, not just on demographics, but also on the assumption that there are no active intergenerational

[8] Gârleanu, Kogan, and Panageas (2012) refer to the risk introduced by incomplete participation as 'displacement risk'. In their work, all uncertainty is fundamental. Farmer, Nourry, and Venditti (2012) also cite incomplete participation as a reason for the existence of sunspot equilibria. However, their paper does not allow for a nominal asset. As a consequence, the equilibrium in Farmer, Nourry, and Venditti (2012) is unique. I am indebted to Pawel Zabczyk, Markus Brunnermeir and Valentin Haddad for discussions which helped me to clarify this issue.

transfers. This is an important assumption because it allows me to construct sunspot equilibria in which people born into different sunspot states have different utilities. One might think that, if people cared for their children, they would make asset market trades on their behalf that would eliminate the effects of nonfundamental uncertainty. That argument is incorrect.

In order for asset market trades to eliminate sunspot uncertainty it must be possible for a person to leave his children with positive bequests in some states of nature and with negative bequests in others. Although these trades would never be observed on the equilibrium path, their conceptual existence is required in order to enforce uniqueness of the fundamental equilibrium. The fact that western legal codes prohibit debt bondage is sufficient to rule out trades of this kind.

4 Household choice

In this section I solve individual maximization problems and, in Sections 5 and 6, I put the solutions to these problems together with the market clearing conditions to characterize equilibria.

4.1 Utility maximization as a recursive problem

Agents have logarithmic preferences and an agent of type i solves the problem

$$J_i\left[W_i\right] = \max_{\{a_i(S')\}} \left\{\log C_i + \pi\beta_i E J\left[W_i'\left(S'\right)\right]\right\},$$

such that

$$\sum_{S'} \pi Q\left(S'\right) W_i'\left(S'\right) + C_i \leq a_i\left(S\right), \tag{6}$$

and

$$W_i \equiv a_i\left(S\right).$$

W_i is wealth at date t in state S, and a_i is the holding of a type i agent of security S. In the period of his birth, the wealth of a person of type i is equal to

$$W_{i,0} = p_k(S)(1-\tau). \tag{7}$$

4.2 Annuities, life insurance and lifecycle utility

Because this is a lifecycle economy I must keep track of peoples' assets when they die. I follow Blanchard (1985) by assuming that there exist complete annuities markets. The term π multiplies each security price in Equation (6) because a person who holds a positive amount of security S' simultaneously purchases an annuities contract. He earns a return greater than the market return in state S' in return for leaving his assets to the annuities company in the event of his death. Similarly, a person who borrows security S' is required to purchase a life insurance policy that discharges his debt in the event of his death.

4.3 Consumption demand functions

I have made three strong assumptions. First, every person has the same probability of death, independent of his current age. Second, preferences are logarithmic, and third, markets are sequentially complete. The first two of these assumptions are common to all models that use Blanchard's 1985 perpetual youth model. The third assumption, of sequentially complete markets, allows me to easily solve my model when there are two types of people.

I show, in Appendix A, that these assumptions imply that the aggregate consumption of the two types are linear functions of their wealth.

$$AC_1 = W_1 \equiv \alpha_1(S), \qquad BC_2 = W_2 \equiv \alpha_2(S), \tag{8}$$

where the parameters A and B are functions of the discount factors, β_i and of the survival probability, π,

$$A = \frac{1}{1 - \beta_1 \pi}, \quad B = \frac{1}{1 - \beta_2 \pi}.$$

The assumption that type 1 agents are more patient than type 2 agents implies that

$$A > B.$$

5 Perfect foresight equilibria

In this section I derive an expression for the equilibrium price of an Arrow security and I characterize perfect foresight equilibria as the solution to a pair of difference equations.

5.1 Marginal rates of substitution

Let m_i be the marginal rate of substitution of a type i person who is alive in two consecutive periods. When preferences are logarithmic and markets are complete, the marginal rates of substitution of each type are equal to the ratios of their consumptions, weighted by the discount rate and the probability that they will survive,

$$m_1 = \frac{\pi \beta_1 c_1}{c_1^O(m')}, \quad \text{and} \quad m_2 = \frac{\pi \beta_2 c_2}{c_2^O(m')}. \tag{9}$$

I am using lower-case c_i to represent the consumption of an individual person of type i, and upper case C_i to mean the aggregate consumption of all people of type i. The superscript O on the term c_i^O indexes a person who was alive in the previous period.

Following this convention, $c_i^O(m')$ is the consumption, next period, of a type i person who is still alive and $C_i^O(m')$ is the aggregate consumption of all of these people. I show in Appendix B, that Equation (9) can be aggregated across people and that the ratios of consumptions of each type in two consecutive periods obeys the same equation as individual marginal utilities,

$$m_1 = \frac{\pi \beta_1 C_1}{C_1^O(S')}, \quad \text{and} \quad m_2 = \frac{\pi \beta_2 C_2}{C_2^O(S')}. \tag{10}$$

Further, I show that the numerators and denominators of Equations (10) can be expressed as affine functions of the components of aggregate wealth,

$$C_i = \theta_{0,i} + \theta_{1,i}\left[p_k(1-\tau) + b\right], \tag{11}$$

$$C_i^O(S') = \eta_{0,i} + \eta_{1,i} p'_k(S')(1-\tau) + \eta_{2,i} b(S'), \tag{12}$$

where the coefficients of these equations are functions of the deep parameters, β_1, β_2 and π.

Although the coefficients of C_i depend only on the sum, $p_k + b$, the terms $p_k(S')$ and $b(S')$ appear in the expression for $C_i^O(S')$ with different coefficients. This important property follows from the fact that the newborns next period do not hold government debt. It is important because the fact that $\eta_{1,i}$ and $\eta_{2,i}$ are different implies that variations in the composition of wealth between trees and government debt will influence the pricing kernel.

5.2 Characterizing perfect foresight equilibria

To characterize equilibria I will derive two functions ψ and ϕ. The function ψ describes a relationship between the price of a tree and the value of debt. The function ϕ describes the pricing kernel.

In a competitive equilibrium with complete markets, the marginal utility of consumption of each type must be equal in every state. Combining equations (10), (11) and (12), these marginal utilities can be written as functions of p_k, b, p'_k and b',

$$m_1(p_k, b, p'_k, b') = m_2(p_k, b, p'_k, b'). \tag{13}$$

Solving (13) for p'_k leads to the definition of ψ,

$$p'_k = \psi\left(p_k, b, b'\right), \tag{14}$$

Replacing Equation (14) in the either of the functions $m_i\left(\cdot\right)$, for $i \in \{1, 2\}$ we obtain the following definition of the pricing kernel ϕ,

$$\phi\left(p_k, b, b'\right) \equiv m_i\left[p_k, b, \psi\left(p_k, b, b'\right), b'\right]. \tag{15}$$

Next, I derive a pair of difference equations that characterize competitive equilibria. Equation (14), defines one difference equation in two variables, p_k and b. Substituting from Equation (15) into the government budget constraint leads to a second difference equation in p_k and b

$$b = \tau + \phi\left(p_k, b, b'\right) b'. \tag{16}$$

To study the properties of a perfect foresight equilibrium, I define a variable m'

$$m' = \phi\left(p_k, b, b'\right), \tag{17}$$

and, in Appendix C, I derive a transformation of variables that rewrites equations (14) and (16) as an equivalent system in the variables $\{b, m\}$.[9] The transformed system has the form,

$$\left[\begin{array}{c} m' - F\left(m, b\right) \\ b' - G\left(m, b\right) \end{array}\right] = 0. \tag{18}$$

In a separate Appendix, available online, I publish the code used to solve the model and I show that, for the parameter values used in my calibration,

[9]This transformation is convenient because, as I will demonstrate in Section 6, there are equilibria of this model where m' is a random variable. Given an expression for the evolution of the sequence $\{m\}$, I can price any asset by computing the conditional expectation of its return with m'.

there exists a unique feasible steady state, $\{\bar{m}, \bar{b}\}$ that satisfies the equation

$$\left[\begin{array}{c} \bar{m} - F\left(\bar{m}, \bar{b}\right) \\ \bar{b} - G\left(\bar{m}, \bar{b}\right) \end{array}\right] = 0. \tag{19}$$

Further, this steady state is a saddle point

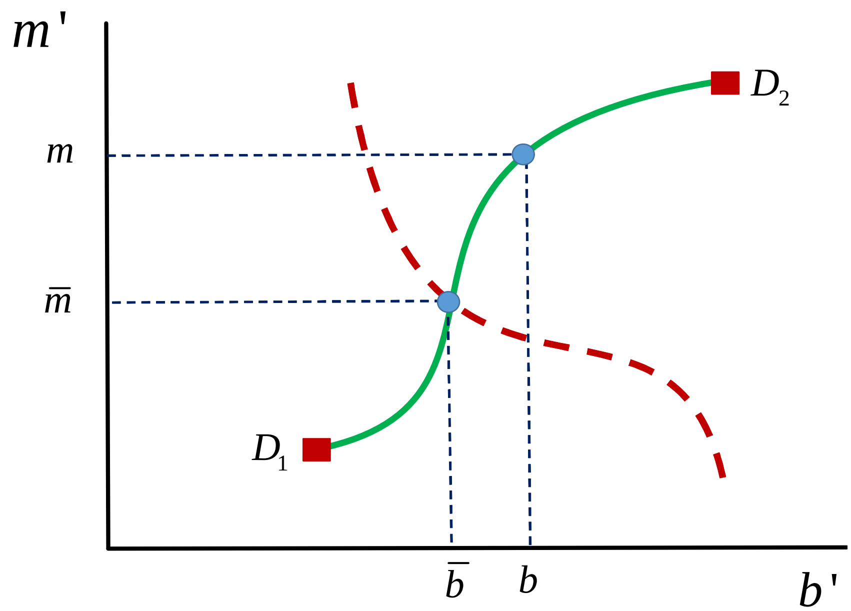

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Figure 1: The set of pefect foresight equilibria

The axes of Figure 1 represent values of $\{b, m\}$. The map defined in Equation (18) sends every point in this space to some other point. The upward sloping solid green curve is the stable manifold and the downward sloping dashed red curve is the unstable manifold. The stable manifold is a set $D = [D_1, D_2]$ and a function $g : D \to \mathbb{R}$,

$$b = g(m), \tag{20}$$

with the property that every point that begins on this manifold follows a first order difference equation $f : D \to D$,

$$m' = f(m), \tag{21}$$

that converges to the steady state $\{\bar{b}, \bar{m}\}$.

5.3 Why there are multiple perfect foresight equilibria

The stable manifold, $g(m)$ is one of two solutions to the functional equation,

$$\begin{aligned} m' &= F[m, g(m)] \equiv f(m), \\ g(m') &= G[m, g(m)] \equiv g[f(m)]. \end{aligned} \tag{22}$$

In the first period of the model, type 1 people enter the period with a net claim on type 2 people that I represent by $a_{1,0}$. This initial condition imposes a linear restriction on $p_{k,0}$ and b_0,

$$\delta_0 + \delta_1 b_0 + \delta_2 p_{k,0} = a_{1,0}, \tag{23}$$

where δ_0, δ_1 and δ_2 are functions of the deep parameters. After transforming the system to the new coordinates $\{m, b\}$, Equation (23) implicitly imposes a linear restriction on m_0 and b_0.

The trajectory that originates at $\{m_0, b_0\}$, calculated by iterating the equation,

$$m' = f(m), \quad m_0 = \bar{m}, \tag{24}$$

characterizes a perfect foresight equilibrium. If b_0 were fixed in units of apples, the point $\{m_0, b_0\}$ would be unique. But because debt is denominated in dollars, Equation (23) does not uniquely determine the values m_0 and b_0. As a consequence, there are multiple initial price levels, all associated with a different perfect foresight equilibrium and a different initial point on the stable manifold.

6 Rational expectations equilibria

In this section, I show how to construct a set of rational expectations equilibria by randomizing over the perfect foresight equilibria of the underlying model. In these equilibria, people form self-fulfilling beliefs about the distribution of future prices. These beliefs are functions of the realization of an extraneous random variable, a sunspot, and they are enforced by the existence of a complete set of Arrow securities that trigger payments between agents of different types in response to the realization of the sunspot.

6.1 Randomizations over perfect foresight equilibria

In the finite Arrow-Debreu model there is, generically, a finite odd number of equilibria. But one cannot construct new stochastic equilibria by randomizing across the existing perfect foresight equilibria. This is a direct implication of the first welfare theorem which asserts that every competitive equilibrium is Pareto optimal. Because people are assumed to be risk averse, they would always prefer the mean of a gamble to the gamble itself. And, in the case of sunspot fluctuations, that mean is available.

That result breaks down when there is incomplete participation in asset markets as a consequence of birth and death (Cass and Shell, 1983). In that case, one can construct randomizations across the perfect foresight equilibria of the model that are themselves equilibria.

To construct equilibria of this kind, I generate sequences of random variables $\left\{\tilde{m}, \tilde{b}\right\}$ that satisfy the equations,

$$E\left[\begin{array}{c}\tilde{m}' - F(m,b) \\ \tilde{b}' - G(m,b)\end{array}\right] = 0. \tag{25}$$

Because there is a complete set of Arrow securities, Equation (26) also holds in every state,

$$\begin{bmatrix} m'(S') - F(m,b) \\ b'(S') - G(m,b) \end{bmatrix} = 0. \tag{26}$$

For any pair of values $\{b, m\}$ there are many values of m' and b' that are consistent with (26). But not all of these continuation values are valid rational expectations equilibria since most of them eventually violate either a boundedness condition or a non-negativity constraint. There is, nevertheless, a large set of continuation values that are valid equilibria. These are the ones that begin on, and remain on, the stable manifold.

Conventional rational expectations models select the initial price level by choosing the unique belief that is consistent with the existence of a stationary equilibrium. The fact that this initial price is indeterminate allows me to construct rational expectations equilibria that are enforced by self-fulfilling beliefs, encoded into the prices of Arrow securities.

6.2 Beliefs and sunspots

What enforces a sunspot equilibrium? Suppose that Mr. A and Mr. B believe the writing of an influential financial journalist, Mr. W. Mr. W writes a weekly column for the, fictitious, *Lombard Street Journal* and his writing is known to be an uncannily accurate prediction of asset prices. Mr. W only ever writes two types of article; one of them, his optimistic piece, has historically been associated with a 10% increase in the price of trees. His second, pessimistic piece, is always associated with a 10% fall in the price of trees.

Mr. A and Mr. B are both aware that Mr. W makes accurate predictions and, wishing to insure against wealth fluctuations, they use the articles of Mr. W to write a contract. In the event that Mr. W writes an optimistic piece, Mr. A agrees, in advance, that he will transfer wealth to Mr. B. In the event that Mr. W writes a pessimistic piece, the transfer is in the other direction. These contracts have the effect of ensuring that Mr. W's predictions are self-fulfilling.[10] How can that be an equilibrium?

There are three groups of people involved in any potential trade. Patient agents alive today, impatient agents alive today, and agents of both types who will be born tomorrow. Fluctuations in the price of trees cause a wealth redistribution from the newly born, to the existing generations. This wealth redistribution operates by a transfer of tax obligations to or from the unborn. Because the existing agents have different propensities to consume out of wealth, they choose to change their net obligations to each other in different ways depending on whether the transfer from the unborn is positive or negative. In a rational expectations equilibrium, the different behaviors of Mr. A and Mr. B are self-fulfilling.

7 Global numerical approximations to equilibria

In this section I introduce a new method for computing sunspot-driven rational expectations equilibria. The usual method of computing sunspot equilibria proceeds by linearizing a dynamic stochastic general equilibrium model around an indeterminate steady state and adding random shocks to the resulting linear system (Farmer, 1999; Woodford, 1986). This method produces a valid approximation to the equilibria of a non-linear model but the accuracy of the approximation decreases as the variance of the shocks becomes larger. In this section, I show how to construct a higher order global approximation that remains valid for shocks that move the pricing kernel over the entire range of its support.

[10] I have shown in Farmer (2002c), that self-fulfilling beliefs can be enforced by what I call a 'belief function'; a new fundamental that has the same methodological status as preferences, technology and endowments. In Farmer (2012), I estimated a model in which the belief function is a primitive and I showed that it fits post-war US data better than a standard New-Keynesian model.

7.1 The method described

To construct global sunspot equilibria, I map the pricing kernel into the interval $[0, 1]$ and I assume that, for any value of m, the variable $\tilde{m}'$ has a Beta distribution with mean $f(m)$, where $m' = f(m)$ is the stable manifold of the map (26). That assumption implies that in any given period, people believe that $\tilde{m}'$ is a random variable with support D for every value of $m \in D$. In words, however well the economy is doing today, there is always positive probability that next period will be associated with an extreme value in which the discount factor is at its upper or lower bound.

The Beta distribution, (Johnson, Kotz, and Balakrishnan, 1995, Chapter 21), is characterized by two parameters, α and β and if $\tilde{m}'$ has a beta distribution, its conditional expectation is given by the expression,

$$E[\tilde{m}' \mid m] = \frac{\alpha}{\alpha + \beta}.$$

Alternatively, one may parameterize the Beta distribution by the mean μ and the 'sample size', V, where

$$\alpha = V\mu, \quad \text{and} \quad \beta = V(1 - \mu).$$

By modeling $\tilde{m}'$ as a Beta distributed random variable, I am able to capture in a parsimonious way, the idea that people believe that equilibria will be selected by the psychology of market participants.

One possible approach to modeling sunspots would be to fix the sample size, V. This leads to the following dependence of the parameters α and β on m,

$$\alpha(m) = Vf(m), \quad \beta(m) = V(1 - f(m)).$$

However, this approach is problematic since for m close to D_1 or D_2, probability mass piles up at the boundaries. It seems desirable to retain the property that the distribution has a single interior peak, a condition that requires that α and β are both greater than 1. For that reason, I chose to let V be state dependent.

In my simulations, I chose a parameter $k > 1$ and I picked V such that

$$V(m) = k \max\left[\frac{1}{f(m)}, \frac{1}{1 - f(m)}\right]. \tag{27}$$

This choice of V guarantees that a and β are both greater than 1 (and hence the distribution has a single interior mode), with strict equality only when $k = 1$.

Figure 2 depicts the distribution of $\tilde{m}'$ for three different values of m. The figure is drawn for the choice of $k = 2$, which corresponds to my baseline calibration. Higher values of k generate pictures with the same qualitative features but with a lower variance for each distribution.

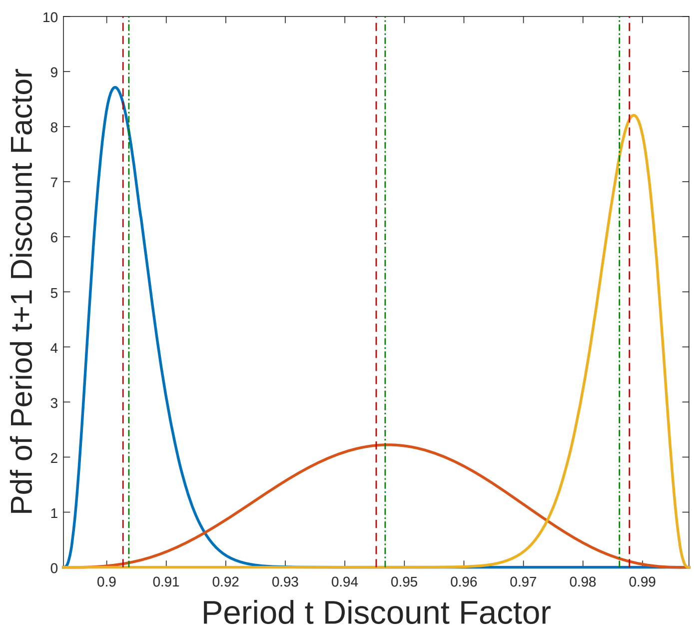

Figure 2: The Distribution of m' when $k = 2$

The three dashed vertical red lines on Figure 2 depict values of m. I chose values,

$$m = \left[\begin{array}{ccc} 0.903 & 0.945 & 0.988 \end{array} \right],$$

which correspond to the midpoint of the support of $\tilde{m}'$ and a distance of 0.01 from each end.

The dot-dashed vertical green lines depict the function $f(m)$. These correspond to the values,

$$f(m) = \left[\begin{array}{ccc} 0.904 & 0.947 & 0.987 \end{array} \right].$$

For each value of m the associated single-peaked curve is the Beta distribution associated with that realization of $\tilde{m}'$, with mean $f(m)$. Notice that the variance of $\tilde{m}'$ is greater when m is in the center of the set D than at either end. This property is dictated by three assumptions. First, $\tilde{m}'$ has a Beta distribution, second, $\tilde{m}'$ has full support for every m, and third, the distribution of $\tilde{m}'$ has a single interior peak.

7.2 Calibrating the model

We have many examples of sunspot models. The interesting question is whether a calibrated version of a sunspot model can help us understand the behavior of asset prices. To address this question, I calibrated the model to the parameter values reported in Table 1.

The parameter π is the probability that a person will survive into the subsequent period and, when $\pi = 0.98$, the typical person has an expected life of 50 years. I arrived at this number by splitting the age distribution of the US population into quintiles and weighting each quintile by life expectancy using mortality tables.

The choice of μ_1 to be 0.5 was arbitrary. I did, however, conduct robustness checks and the results I report below are not sensitive to alternative choices.

Table 1

Parameter Description	Parameter Name	Parameter Value
Survival probability	π	0.98
Fraction of type 1 in the population	μ_1	0.5
Gross nominal interest rate	R^N	1.05
Discount factor of type 1	β_1	0.98
Discount factor of type 2	β_2	0.90
Variance parameter	k	2
Primary surplus	τ	0.02

To describe monetary policy, I chose $R^N = 1.05$. That choice is frequently cited by central bankers as the 'normal value' for interest rates and it is consistent with a safe real rate of 3% and an inflation target of 2%. It would be interesting to study the properties of the model under the assumption that R^N reacts to realized inflation. I will leave that task for future work.

The parameters β_1 and β_2, affect the steady state discount factor and one can show that

$$\beta_2 < \bar{m} < \beta_1.$$

I chose values of 0.98 and 0.9 by experimenting with the model to find values that led to a mean safe rate of 3%. The gap between these two discount factors determines the possible range of sunspot fluctuations and it needs to be relatively large if the model is to have a hope of capturing observed asset price movements. For any value of the support of $\tilde{m}'$, the parameter k determines the variance of the sunspot distribution for any given m. I experimented with different values of k and chose $k = 2$ to match asset returns with an approximate range of plus or minus 20%. Higher values of k lead to lower asset return volatility and lower values lead to higher volatility. Finally, I chose a value of τ of 2% to match the mean post-war primary government budget surplus.

Table 2

Variable Name	Parameter Name	Parameter Value
Equilibrium discount factor	$\bar{m}$	0.97
Equilibrium government debt	$\bar{b}$	0.69
Equilibrium asset price	$\bar{p}_k$	20.6
Return to a tree	R^R	1.03
Return to debt	R^S	1.03

The calibration of Table 1 implies the steady state values for $\bar{m}$, $\bar{b}, \bar{p}_k, R^R$ and R^S, reported in Table 2. Here, R^R and R^S are the real gross returns to holding a tree, or to holding government debt, in the non-stochastic steady state. These are the same and both are equal to 1.03, corresponding to a real interest rate of 3%.

7.3 Approximate global solutions

A perfect foresight solution to the model is characterized by a set D and a pair of functions $f(m): D \to D$ and $g(m): D \to \mathbb{R}$ such that

$$\begin{aligned} m' &= F[m, g(m)] \equiv f(m), \\ g(m') &= G[m, g(m)] \equiv g[f(m)], \end{aligned} \tag{28}$$

for all $m \in D$. To solve these equations I used Chebyshev collocation as described in Judd (1998). That method converts the operator equation, (28), into a non-linear algebraic equation in the coefficients of two unknown polynomials $\hat{f}(m)$ and $\hat{g}(m)$. These polynomials approximate the functions $f(m)$ and $g(m)$ and by increasing the number of terms in the polynomial, one can achieve an arbitrary close approximation to f and g. In practice, I used polynomials of order 3.

To compute the boundaries of the set D, I solved Equation (29)

$$C_1[D_1, g(D_1)] = 0, \quad C_2[D_2, g(D_2)] = 0, \tag{29}$$

to find the two points where one or the other type consumes the entire endowment of the economy. For the calibration from Table 1, the lower boundary, D_1 is equal to 0.893 and the upper boundary, D_2, is equal to 0.998. When $m = D_1$, type 2 agents consume all of GDP. When $m = D_2$, type 1 agents consume everything.

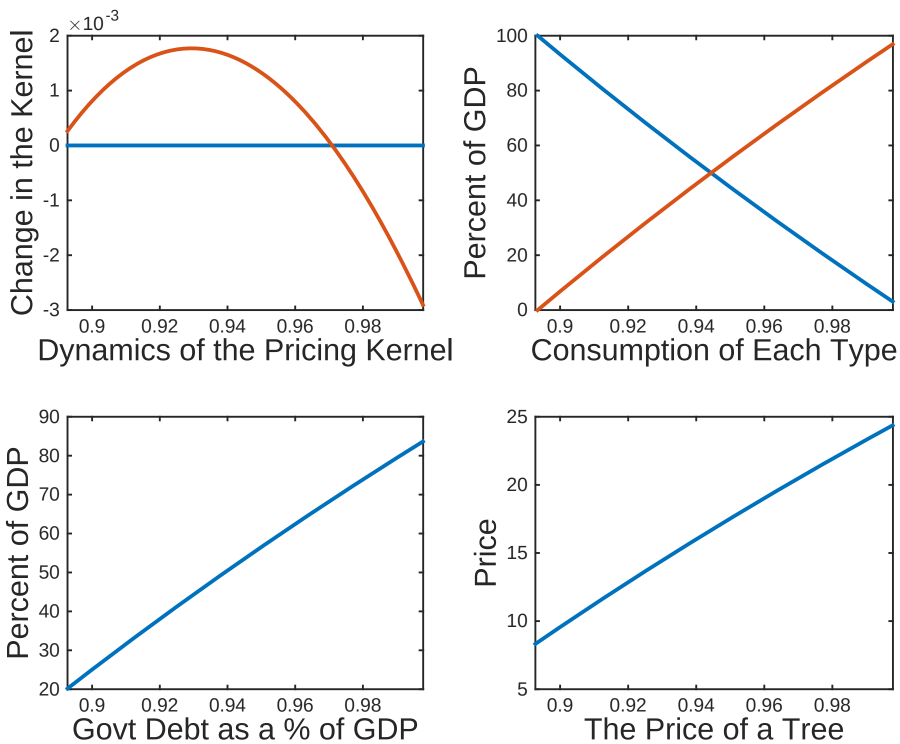

Figure 3: Some properties of the global solution

The top left panel of Figure 3 graphs the function $\hat{f}(m) - m$, on the vertical axis as a function of m on the horizontal axis. The point where the curve crosses the zero axis corresponds to the steady state $\bar{m} = 0.97$ and the range of m is defined by the set D. I have graphed the change in m as a function of m, rather than m' as a function of m, because in a plot of

m' against m, it is difficult to discern the difference between m' and the 45 degree line.

The top right panel of Figure 2 graphs the consumption of type 1 people, this is the upward sloping curve, and the consumption of type 2 people, this is the downward sloping curve. The lower left panel of Figure 3 is the function $b = \hat{g}(m)$. This panel shows that, when the primary surplus is 2% of GDP, government debt can attain values between 20% and 85% of GDP.

The lower right panel of Figure 3 is the price of a tree as a function of m. This panel demonstrates that, for the calibration in Table 1, the price of a tree can vary between 8 and 24. This fact is significant since p_k determines the lifetime wealth of a newborn. A person born into the world when $p_k = 24$ will be three times better off during his life than a person born into the world when $p_k = 8$.

8 Explaining the three puzzles

In the introduction, I identified three asset pricing puzzles. 1) Asset prices are persistent, volatile and mean reverting. 2) Aggregate consumption is smooth but the return to a riskless asset is five hundred basis points less than the return to the stock market, and 3) Asset price volatility is non-constant and non-Gaussian. This section describes the method I used to simulate data from the model and it presents a series of graphs that depict the characteristics of these simulated data. I use these simulations to ask: how far can my model go towards explaining the three asset pricing puzzles?

8.1 Excess volatility

To simulate data, I initialized $m_0 = \bar{m}$ and I generated 60 years of data by drawing a sequence of Beta distributed random variables that obey the recursion,

$$m' = B\left[V(m), \hat{f}(m)\right],$$

where $B(V, \mu)$ is the beta distribution parameterized by sample size V and mean μ. I chose V to be a function of m, using the method described in Section 7.1, Equation (27).

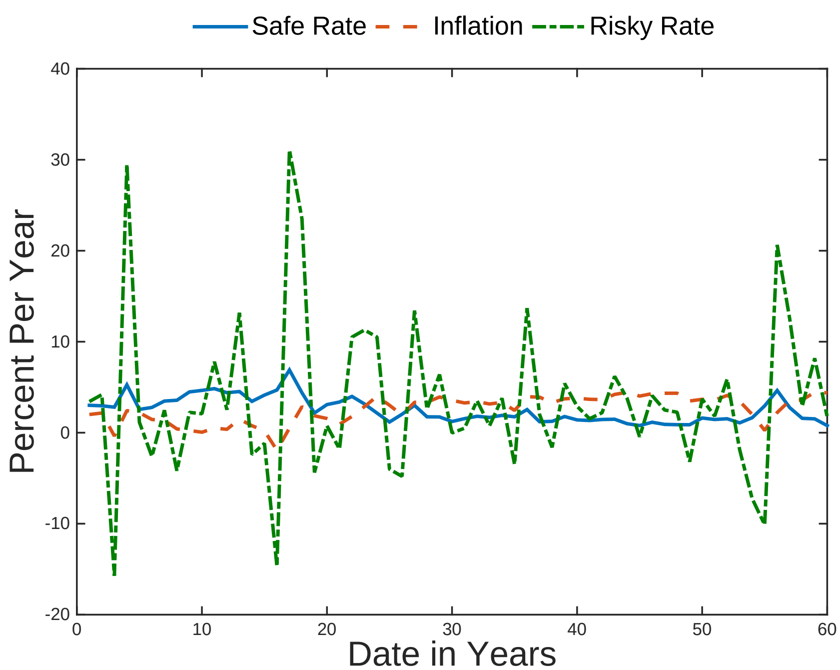

Figure 4: 60 Years of Simulated Data

Figure 4 plots the data generated from a single 60 year simulation. The green dashed series, is the real risky return, r^R. This is the payoff from buying a tree in period t and selling it again in period $t+1$.

$$r^R \equiv 100 \left(\frac{\pi p_k'}{p_k - 1} - 1 \right).$$

The solid blue line is the real safe return r^S. This is equal to

$$r^S \equiv 100 \left(\frac{1}{\int_{D_1}^{D_2} m \Pr \left[m'; V(m), \hat{f}(m) \right] dm'} - 1 \right),$$

where $\Pr\left[m'; V(m), \hat{f}(m)\right]$ is the density function of a Beta distributed random variable, defined over the set D, with mean $\hat{f}(m)$ and sample size $V(m)$. A person could earn this return by buying a bundle of Arrow securities, one for each realization of the future value of m'.

The dashed red line is the realized inflation rate predicted by the model. Notice that the movements in inflation are small, relative to asset price fluctuations. That is encouraging since it means that the model does not generate asset price volatility at the cost of counterfactually large fluctuations in goods prices and inflation.

Table 3 reports the means and standard deviations of r^S and r^R for this draw of sixty years of data, along with the Sharpe ratio, defined as

$$\text{Sharpe} = \frac{r^R - r^S}{\sigma^r},$$

where σ^r is the standard deviation of r^R.

Table 3:	Safe Rate	Risky Rate	Sharpe Ratio
Mean	2.59	3.46	
Std. Dev.	1.4	8.6	0.1

Two features stand out from this simulation. First; the risky return is highly volatile fluctuating in this sample between a high of 30% and a low of -16%. Second; the return from buying a long claim and holding it for a year has a return which is almost 1% higher than the riskless rate. The fact that asset prices are volatile and mean reverting, even when aggregate consumption is constant, is the first feature of the data that I set out to understand. The following section probes more deeply into the second feature, the ability of my model to understand the equity premium puzzle.

8.2 The Sharpe ratio, the equity premium and the term premium

In the US data, the mean return to equity has been, on average, 5% higher than the return to government bonds. Because it is possible to leverage returns through borrowing, finance economists focus instead on a different statistic; the Sharpe ratio. The Sharpe ratio, defined as the excess return on a risky asset divided by its standard deviation, has varied in US data between 0.25 and 0.5 depending on the time period and the frequency over which it is measured (Cochrane, 2001).

Figure 4 suggests my model can explain part, but not all, of the equity premium. In one simulated data series of 60 years, the excess return was approximately 1% and the Sharpe ratio was 0.1. This fact raises raises several questions. First; is the result a fluke?

The average Sharpe ratio in 60 years of simulated data is a random variable and because asset returns are so volatile, its standard deviation is high. To examine the ability of my model to produce a high Sharpe ratio, I simulated 500 draws of 60 years of data and I plotted the empirical frequency distributions of the riskless rate, the mean return to holding a tree and the Sharpe ratio. The results are graphed in Figure 5.

The lower panel of Figure 5 plots the distributions of safe and risky returns. The more dispersed distribution, plotted in red, is the safe return. The more concentrated blue curve is the return to holding a tree. This figure shows that a 1% equity premium is not a fluke; it is characteristic of the invariant distribution of returns.

The upper panel of Figure 5 plots the distribution of Sharpe ratios in these 500 simulations. This figure shows that the mean Sharpe ratio is around 0.1 and there is a non-trivial probability of observing a Sharpe ratio of 0.2 or higher.

Andrew Abel (1999) has pointed to the important distinction between the equity premium and the term premium. The equity premium is the excess return to holding a long dated claim to an uncertain income stream such as equity. The term premium is the excess return to holding a long-dated claim to a safe income stream such as a thirty year treasury bond. Abel finds that about 1/4 of the equity premium puzzle can be attributed solely to the term premium, a finding that is consistent with the data generated by my model.

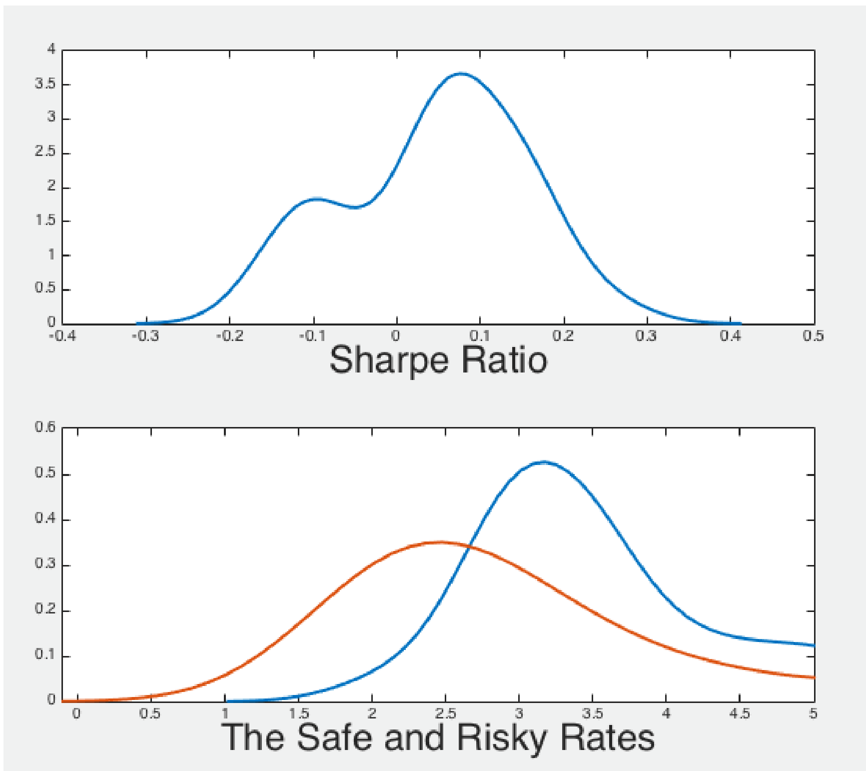

Figure 5: The Sharpe ratio and the equity premium

Is this a success? Partially. My model has logarithmic preferences, expected utility and no fundamental uncertainty and yet it is able to generate a substantial Sharpe ratio. It seems likely that a version of my model that allows for more risk aversion and aggregate fundamental uncertainty will be able to do much better in this dimension.

8.3 Conditional volatility

Traditional asset pricing models rely on time varying volatility to explain asset prices (Bansal and Yaron, 2004). The fact that asset prices display bursts of volatility was highlighted by the ARCH and GARCH models of Engle (1982) and Bollerslev (1986) and has since become a staple feature of asset pricing models.

In much of the finance literature, conditional volatility is introduced by assuming that shocks to dividend growth are driven by an exogenous stochastic process with a time-varying standard deviation. The model I develop in this paper generates endogenous conditional volatility.

The intuition for this result is contained in Figure 2. The assumption that expectations are rational requires that the pricing kernel should be mean reverting. The fact that the support of $\tilde{m}'$ is bounded implies that the variance of $\tilde{m}'$ is endogenously higher when m is in the middle of its support than when it is at either end.

If the discount factor strays towards the middle of its range following a large negative shock, there is an increased probability that it will be hit with an even larger negative shock that sends it towards the lower bound of its support. Once it reaches that region, the variance of future shocks falls and it takes a longer time to escape back towards the mean of the invariant distribution. This feature generates endogenous bursts of stochastic volatility.

One such burst is depicted in Figure 6. The top panel of this figure depicts the risky rate, the safe rate and the inflation rate for one draw of sixty years of data. The shaded region between observations 23 and 43 depicts an episode where the volatility of the return to a tree is higher than at other times. The lower panel blows up this picture to show more clearly the behavior of the safe return and the inflation rate. Notice that a period of high volatility is associated with a higher than average safe rate and a period of deflation.

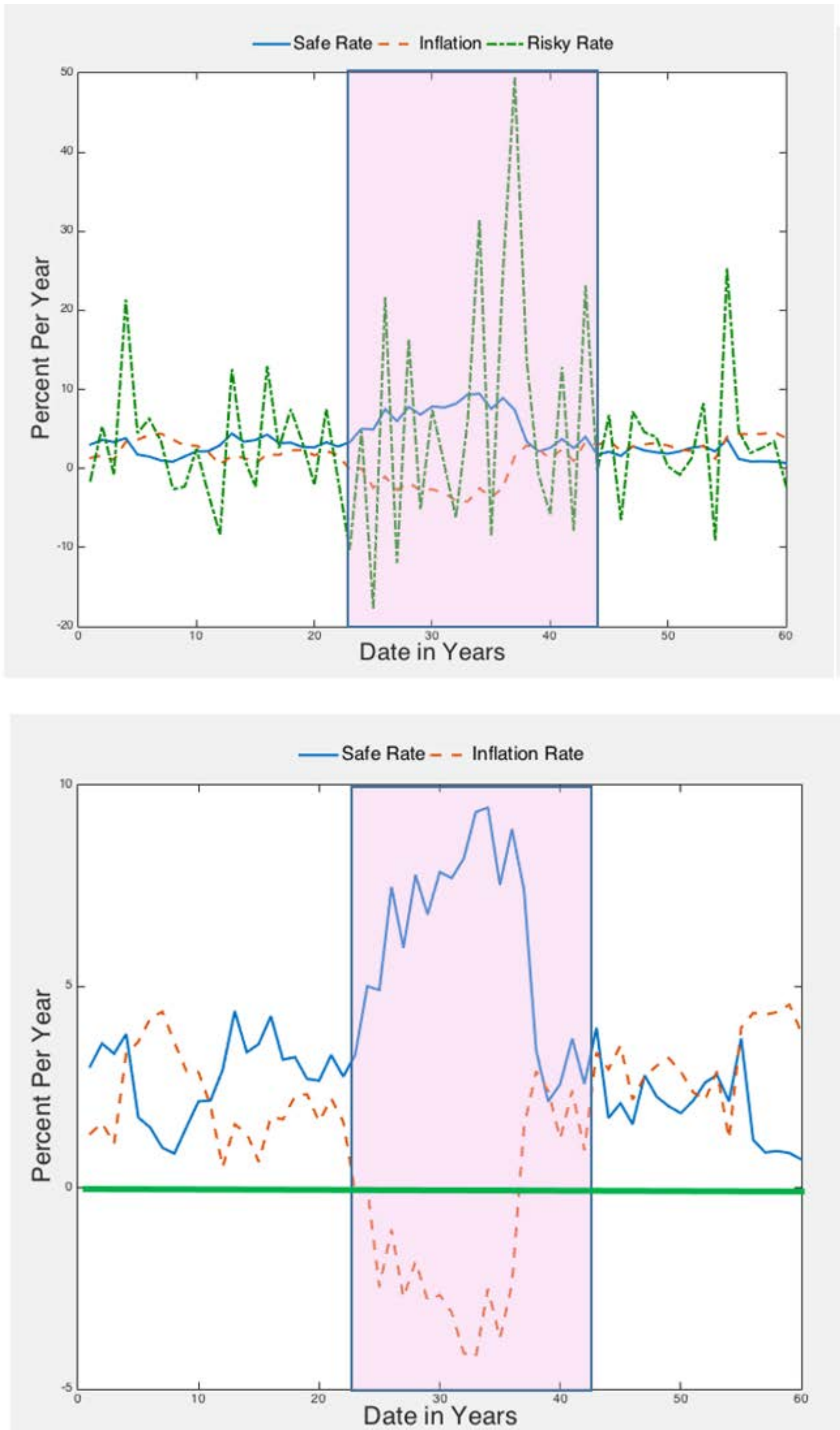

Figure 6: A Burst of Volatility

The safe rate of interest climbs to nearly 10% over the period of increased volatility and the inflation rate becomes negative for more than a decade. These results are suggestive of the Great Depression or the 2008 financial crisis.

9 Conclusion

In this paper, I have presented a theory that explains asset pricing data in a new way. In contrast to much of the existing literature in both macroeconomics and finance, my work is based on the idea that most asset price fluctuations are caused by non-fundamental shocks to beliefs. My model produces data that display volatile asset prices, a sizeable term premium and bursts of time varying volatility. If one accepts the argument that a simpler explanation is a better one, the fact that I am able to reproduce these empirical facts in a model with logarithmic preferences and no fundamental shocks suggests that the model is on the right track.

My model is rich in its implications. It provides a simple theory of the pricing kernel that can be used to price other assets. The model is open to more rigorous econometric testing and its parameters can be estimated, rather than calibrated, using non-linear methods. It provides a theory of the term structure of interest rates that can be tested against observed bond yields and by adding a richer theory, in which output fluctuates as a consequence of labor supply or because of movements in the unemployment rate, the theory can be expanded to distinguish between the term premium and the equity premium. I view all of these extensions as grist for the mill of future research. Conducting these extensions is an important task because my model is not just a positive theory of asset prices; it is ripe with normative implications.

In my baseline calibration, I chose parameters to match key features of the data and I generated simulated data series that closely mimic observed inflation, interest rates and asset prices in the real world. In these simulations, asset price fluctuations cause Pareto inefficient reallocations of wealth between current and future generations and these reallocations lead to substantial fluctuations in welfare. If my model is correct, and these fluctuations are the main reason why asset prices move in the real world, stabilizing asset prices through monetary and fiscal interventions will be unambiguously welfare improving.

Appendix A: Optimal decision rules

Let $J_i(W)$ represent the value function of a person of type i. This function obeys the Bellman equation,

$$J_i[W_i] = \max_{\{W_i(S')\}} \left\{ \log \left[W_i - \sum_{S'} \pi Q(S') W_i'(S') \right] \right. \\ \left. + \pi \beta_i E J[W_i'(S')] \right\}, \quad \text{(A1)}$$

where

$$W_i - \sum_{S'} \pi Q(S') W_i'(S') \equiv C_i. \quad \text{(A2)}$$

The unknown functions $J_i(W)$ must satisfy the following envelope condition,

$$J_{iW}[W_i] = \frac{1}{C_i}, \quad \text{(A3)}$$

and the Euler equations for each state

$$-\frac{\pi Q(S')}{C_1} + \beta_i \pi J_W[W'(S')] = 0. \quad \text{(A4)}$$

Since this is a logarithmic problem with complete markets I will guess that the value functions take the form

$$J_1(W) = A\log(W_1), \quad J_2(\alpha) = B\log(W_2), \tag{A5}$$

and verify this conjecture by finding values for the numbers A and B such that equations (A3) and (A4) hold. By replacing the unknown functions $J_i(\cdot)$ with their conjectured functional forms from Equation (A5) we arrive at Equations (A6) and (A7).

$$C_1 = \frac{W_1}{A}, \quad C_2 = \frac{W_2}{B}, \tag{A6}$$

$$C_1 = \frac{Q'(S')W_1'(S')}{\beta_i A}, \quad C_2 = \frac{Q'(S')W_2'(S')}{\beta_i B}. \tag{A7}$$

The two budget equations, for each type, (A2), together with the four first order conditions, (A6) – (A7), constitute six equations in the six unknowns, W_1', W_2', C_1, C_2, A and B. To solve these equations, substitute from (A7), state by state, into Equation (A2), and cancel C_i from each side to give the expressions

$$A = \frac{1}{1-\beta_1\pi}, \quad B = \frac{1}{1-\beta_2\pi}. \tag{A8}$$

Combining these solutions for A and B with (A6) gives the consumption rules that we seek.

Appendix B: Deriving an expression for the pricing kernel

In Appendix B we seek to establish that the first order condition

$$Q(S') = \frac{\pi\beta_i c_i}{c_i'^{O}(S')}, \tag{B1}$$

implies that

$$Q\left(S'\right)=\frac{\theta_{0,i}+\theta_{1,i}\left[p_k\left(1-\tau\right)+b\right]}{\eta_{0,i}+\eta_{1,i}p_k'\left(S'\right)\left(1-\tau\right)+\eta_{2,i}b'\left(S'\right)}. \tag{B2}$$

The following argument follows closely from the argument developed in Farmer, Nourry, and Venditti (2011). We begin with some definitions. Let c_i^O be the consumption of a type i person who was alive in the previous period and let c_i^Y denote the consumption of a newborn of type i. Further, let C_i^Y be the aggregate consumption of all newborns of type i. To prove that (B2) follows from (B1), we must find expressions for c_i and $c_i'^O$ as functions of p_k and b.

The following steps imply that Equation (B1) must also hold not only for individuals, but also in aggregate. Multiplying both sides of (B1) by $c_i'^O\left(S'\right)$ and adding up over all people of type i who are alive in two consecutive periods gives the expression,

$$Q\left(S'\right)C_i'^O=\pi\beta_iC_i. \tag{B3}$$

Rearranging, leads to the expression.

$$Q\left(S'\right)=\frac{\pi\beta_iC_i}{C_i'^O\left(S'\right)}. \tag{B4}$$

This establishes the claim following Equation (9) in Section 5.1.

Goods and asset market clearing imply

$$C_1+C_2=1, \tag{B5}$$

and

$$W_1+W_2=p_k\left(1-\tau\right)+b. \tag{B6}$$

Combing these equations with the solutions for consumption from Appendix

A, we have that,

$$C_1 = \frac{[p_k (1-\tau) + b - B]}{A - B}, \quad \text{and} \quad C_2 = \frac{[A - p_k (1-\tau) - b]}{A - B}. \tag{B7}$$

It follows that the coefficients of the numerators of (B2) are given by the following definitions,

$$\theta_{1,0} \equiv \frac{-B\beta_1\pi}{A - B}, \qquad \theta_{1,1} \equiv \frac{\beta_1\pi}{A - B}, \tag{B8}$$

$$\theta_{2,0} \equiv \frac{A\beta_2\pi}{A - B}, \qquad \theta_{2,1} \equiv \frac{-\beta_2\pi}{A - B}. \tag{B9}$$

Next we seek expressions for the denominator of Equation (B2).

The aggregate consumption of all type i people alive in period $t+1$ can be decomposed into the consumption of those who were alive in period t and the consumption of the newborns. Let $\mathcal{A}_t$ be the index set of all type i people alive at date t and let $\mathcal{N}_{t+1}$ be the index set of all type i newborns at date $t+1$. Using these definitions,

$$\sum_{\mathcal{A}_{t+1}} c_i' (S') = \pi \sum_{\mathcal{A}_t} c_i'^O (S') + \sum_{\mathcal{N}_{t+1}} c_i'^Y (S'), \tag{B10}$$

where π premultiplies the first term on the right-side of this expression to reflect the fact a fraction $1-\pi$ of the previous generations have died. We can rewrite Equation (B10), using the definitions of C_i', $C_i'^Y$ and $C_i'^O$, as follows,

$$C_i'^O (S') = \frac{C_i' (S') - C_i'^Y (S')}{\pi}. \tag{B11}$$

Now we seek an expression for $C_i'^Y (S)$ as a function of wealth. There are $1-\pi$ newborns of each type, each of whom consumes a fraction of his wealth. These facts lead to the equations,

$$C_1'^Y (S') = A^{-1} p_k (S') (1-\tau) (1-\pi), \tag{B12}$$

and

$$C_2^{\prime Y}\left(S^{\prime}\right)=B^{-1}p_k\left(S^{\prime}\right)\left(1-\tau\right)\left(1-\pi\right), \tag{B13}$$

which determine the aggregate consumptions of newborns of each type. Combining (B12) and (B13) with (B11), making use of (B7), leads to the expressions we seek,

$$C_1^{\prime O}\left(S^{\prime}\right)=\frac{\left[p_k\left(S^{\prime}\right)\left(1-\tau\right)+b\left(S^{\prime}\right)-B\right]}{\pi\left(A-B\right)}-\frac{p_k\left(S^{\prime}\right)\left(1-\tau\right)\left(1-\pi\right)}{\pi A}, \tag{B14}$$

and

$$C_2^{\prime O}\left(S^{\prime}\right)=\frac{\left[A-p_k\left(S^{\prime}\right)\left(1-\tau\right)-b\left(S^{\prime}\right)\right]}{\pi\left(A-B\right)}-\frac{p_k\left(S^{\prime}\right)\left(1-\tau\right)\left(1-\pi\right)}{\pi B}. \tag{B15}$$

These equations express the denominators of Equations (B2) as functions of the components of wealth. It follows that the coefficients $\eta_{i,0}, \eta_{i,1}$ and $\eta_{i,2}$ from Equation (12) in Section 5.1 are defined as,

$$\begin{aligned}
\eta_{1,0} &\equiv \frac{-B}{\pi\left(A-B\right)}, \quad \eta_{1,1}\equiv\left(1-\tau\right)\left[\frac{1}{\pi\left(A-B\right)}-\frac{\left(1-\pi\right)}{\pi A}\right], \\
\eta_{1,2} &\equiv \frac{1}{\pi\left(A-B\right)},
\end{aligned} \tag{B16}$$

and

$$\begin{aligned}
\eta_{2,0} &\equiv \frac{A}{\pi\left(A-B\right)}, \quad \eta_{2,1}\equiv-\left(1-\tau\right)\left[\frac{1}{\pi\left(A-B\right)}+\frac{\left(1-\pi\right)}{\pi B}\right], \\
\eta_{2,2} &\equiv \frac{-1}{\pi\left(A-B\right)}.
\end{aligned} \tag{B17}$$

Appendix C: Transforming variables

We seek to derive a map $\{b,m\}\rightarrow\{b^{\prime},m^{\prime}\}$ given the functions ψ and ϕ,

$$p_k^{\prime}=\psi\left(p_k,b,b^{\prime}\right), \tag{C1}$$

$$m' = \phi\left(p_k, b, b'\right), \tag{C2}$$

and the government budget equation,

$$b = m'b' + \tau. \tag{C3}$$

Equations (C1)–(C3) constitute three equations in the three unknowns b, p_k and p'_k which may be solved to find three functions

$$b = \theta_1\left(b', m'\right), \quad p'_k = \theta_2\left(b', m'\right) \quad \text{and} \quad p_k = \theta_3\left(b', m'\right). \tag{C4}$$

Substituting $\theta_2(\cdot)$ and $\theta_3(\cdot)$ from (C4) into (C1),

$$\theta_2\left(b', m'\right) = \psi\left[\theta_3\left(b', m'\right), b, b'\right]. \tag{C5}$$

Solving equations (C3) and (C5) for m' and b' as functions of m and b leads to the functions we seek,

$$m' = F(m, b), \tag{C6}$$

$$b' = G(m, b). \tag{C7}$$

The existence of the functions F and G is not guaranteed for all parameter values. The online Appendix provides code to compute F and G for my baseline calibration and to establish numerically that equations (C6) and (C7) have a unique steady state for which debt and the consumptions of each group are non-negative.

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