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A MICROFOUNDATION FOR APPLIED WORK

Allan Collard-Wexler
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“Nash-in-Nash” Bargaining: A Microfoundation for Applied Work
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ABSTRACT

A “Nash equilibrium in Nash bargains” has become a workhorse bargaining model in applied analyses of bilateral oligopoly. This paper proposes a non-cooperative foundation for “Nash-in-Nash” bargaining that extends the Rubinstein (1982) alternating offers model to multiple upstream and downstream firms. We provide conditions on firms' marginal contributions under which there exists, for sufficiently short time between offers, an equilibrium with agreement among all firms at prices arbitrarily close to “Nash-in-Nash prices” — i.e., each pair's Nash bargaining solution given agreement by all other pairs. Conditioning on equilibria without delayed agreement, limiting prices are unique. Unconditionally, they are unique under stronger assumptions.

Allan Collard-Wexler
Department of Economics
Duke University
233 Social Sciences
Durham, NC 27708
and NBER
allan.collard-wexler@duke.edu

Gautam Gowrisankaran
Department of Economics
University of Arizona
P.O. Box 210108
Tucson, AZ 85721-0108
and HEC Montreal
and also NBER
gowrisankaran@eller.arizona.edu

Robin S. Lee
Department of Economics
Harvard University
Littauer Center 120
Cambridge, MA 02138
and NBER
robinlee@fas.harvard.edu

1 Introduction

Bilateral bargaining between pairs of agents is pervasive in many economic environments. Manufacturers bargain with retailers over wholesale prices, and firms negotiate with unions over wages paid to workers. As an example, in 2013, private insurers in the United States paid hospitals \$348 billion and physicians and clinics \$267 billion for their services.[1] Private prices for medical services are determined neither by perfect competition, nor by take-it-or-leave-it offers (as is assumed in Bertrand competition). Instead, they are predominantly determined by bilateral negotiations between medical providers and insurers. Furthermore, these negotiations are typically interdependent: for instance, an insurer's value from having one hospital in its network depends on which other hospitals are already in its network.

A substantial theoretical literature has sought to understand the equilibrium outcomes of bilateral bargaining models in a variety of settings, including negotiations over wages (e.g., Jun, 1989; Stole and Zwiebel, 1996) and in networks (e.g., Corominas-Bosch, 2004; Manea, 2011). To derive meaningful predictions, many of these papers have focused on environments where a single agent is involved in all bargains, or where a transaction between two agents does not affect the value of trade for others. Concurrently, an applied literature—both empirical and theoretical—has focused on surplus division within bilateral oligopoly environments with the goal of evaluating a range of industrial organization questions, including: the welfare impact of bundling (Crawford and Yurukoglu, 2012), horizontal mergers (Chipty and Snyder, 1999), and vertical integration (Crawford, Lee, Whinston, and Yurukoglu, 2015) in cable television; the effects of price discrimination for medical devices (Grennan, 2013); and the price impact of hospital mergers (Gowrisankaran, Nevo, and Town, 2015) and health insurance competition (Ho and Lee, 2015). Increasingly, this applied literature is influencing antitrust and regulatory policy.[2] The applied literature has emphasized interdependencies and externalities across firms and agreements, because they are often fundamental to bilateral oligopoly environments.[3]

[1] See Exhibit 1 on p. 4 of "National Health Expenditure Accounts: Methodology Paper, 2013" at `https://www.cms.gov/Research-Statistics-Data-and-Systems/Statistics-Trends-and-Reports/NationalHealthExpendData/Downloads/DSM-13.pdf` accessed on August 2, 2015.

[2] The Federal Communication Commission used a bargaining model similar to that analyzed in this paper in its analysis of the Comcast-NBC merger (Rogerson, 2013) and in recent hospital merger cases (Farrell, Balan, Brand, and Wendling, 2011). Also, in a recent ruling in a restraint of trade case in sports broadcasting, Judge Shira Scheindlin's opinion heavily referenced the Crawford and Yurukoglu (2012) bargaining framework as an appropriate way to consider competition in this sector (c.f. Thomas Laumann v National Hockey League (J. Scheindlin, S.D.N.Y. 2015 12-cv-1817 Doc. 431)).

[3] For instance, hospital mergers may raise prices in a bargaining context because the loss to an insurance company from removing multiple hospitals is worse than the sum of the losses from removing individual hospitals (Capps, Dranove, and Satterthwaite, 2003); however, a bargaining model without interdependencies would typically rule out a price increase following a merger.

To tractably and feasibly analyze the division of surplus in bilateral oligopoly settings with interdependent payoffs, the applied literature has leveraged the relatively simple solution concept proposed by Horn and Wolinsky (1988) (which studied horizontal merger incentives in the presence of exclusive vertical relationships). This bargaining solution is a set of transfer prices between "upstream" and "downstream" firms where the price negotiated between any pair of firms is the Nash bargaining solution (Nash, 1950) for that pair *given that all other pairs reach agreement.* Because this solution can be cast as nesting separate bilateral Nash bargaining problems within a Nash equilibrium to a game played among all pairs of firms, we refer to it as the "Nash-in-Nash" solution.[4] The Nash-in-Nash solution provides easily computable payments for complicated environments with interdependencies. It is also based on marginal valuations, which fits well with classical price theory. Yet, the Nash-in-Nash solution has been criticized by some as an ad hoc solution that nests a cooperative game theory concept of Nash bargaining within a non-cooperative Nash equilibrium. Non-cooperative microfoundations for the Nash-in-Nash solution that have been previously developed generally have assumed that firms do not use all the information that may be at their disposal at any point in time: i.e., most use "delegated agent" models where firms involved in multiple bilateral bargains rely on separate agents for each negotiation, and agents (even those from the same firm) cannot communicate with one another during the course of bargaining.[5]

The purpose of this paper is to provide support for the Nash-in-Nash solution as a viable surplus division rule in the applied analysis of bilateral oligopoly by specifying a non-cooperative microfoundation that does not require firms to behave independently (or "schizophrenically") across bargains. We contribute to the "Nash program" of pairing non-cooperative and axiomatic approaches to strategic bargaining problems (c.f. Binmore, 1987; Serrano, 2005), and share the same motivation as Binmore, Rubinstein, and Wolinsky (1986) who sought to "provide a more solid grounding for applications of the Nash bargaining solution in economic modeling." We develop a simple extensive form bargaining game that extends their analysis and the classic Rubinstein (1982) model of alternating offers between two parties to the bilateral oligopoly case with multiple upstream and downstream firms. Focusing on environments where payments are lump-sum and where there are gains from

[4]Crucially, this solution assumes that each bilateral pair bargains as if the negotiated prices (or contracts) of all other pairs of firms do not adjust in response to a bargaining disagreement or breakdown. It is a type of *contract equilibrium* as defined in Cremer and Riordan (1987).

[5]For instance, Crawford and Yurukoglu (2012) sketch a non-cooperative extensive form game generating this solution, writing: "Each distributor and each conglomerate sends separate representatives to each meeting. Once negotiations start, representatives of the same firm do not coordinate with each other. We view this absence of informational asymmetries as a weakness of the bargaining model." See also Chipty and Snyder (1999) and Inderst and Montez (2014).

trade between every pair of firms that are allowed to contract, we prove two main sets of results.

The first set of results provides sufficient conditions for the existence of a *Nash-in-Nash limit equilibrium*—i.e., an equilibrium where all agreements are formed, with formation at prices that are arbitrarily close to the Nash-in-Nash solution when the time between offers is sufficiently short. Our conditions place limits on the extent to which the sum of a firm's marginal gains from individual agreements within a set can exceed its marginal gains from the entire set of agreements. These conditions are satisfied in environments where firms on the same side of the market are substitutes for one another, and also in environments with limited complementarities.

Our second set of results concerns the uniqueness of equilibrium prices. We prove that any *no-delay equilibrium*—i.e., an equilibrium where all agreements (that have not yet been formed) form immediately following every history of play (as is the case for the equilibria that we construct for our existence results)—must have agreements that form at prices that are arbitrarily close to the Nash-in-Nash solution when the time between offers is sufficiently short. We also provide sufficient conditions for all equilibria to have this property without conditioning on no-delay equilibria. Our results do not restrict attention to stationary strategies, as is the case with refinements such as Markov perfect equilibrium.

We believe that our work has three general takeaways. First, by extending the Binmore, Rubinstein, and Wolinsky (1986) non-cooperative foundation for Nash bargaining to environments with multiple upstream and downstream firms, we provide a microfoundation for applied work using the the Nash-in-Nash solution as a surplus division rule in bilateral oligopoly. Second, our equilibrium existence results clarify when Nash-in-Nash may be an appropriate solution concept. Finally, our uniqueness results suggest that the Nash-in-Nash solution may be a relatively robust outcome across a variety of settings.

Overview. We now briefly discuss our model, results, and proofs. Our extensive form game conditions on the set of bilateral agreements that can be formed between upstream and downstream firms, and adopts the Rubinstein (1982) bargaining protocol to a setting with multiple agents. In odd periods, each downstream firm makes simultaneous private offers to each upstream firm with which it has not yet formed an agreement; each upstream firm then accepts or rejects any subset of its offers. In even periods, roles are reversed, with upstream firms making private offers and downstream firms accepting or rejecting them. If an offer is accepted, a fixed payment (or "price") is made and an agreement forms between the two firms. At the end of each period, the set (or "network") of agreements that have been formed is observed by all firms, and upstream and downstream firms earn flow profits. These profits, assumed to be a primitive of the analysis, are a function of the entire set of agreements formed up to that point, and allow for flexible interdependencies across agreements.[6] We assume that there are gains from trade from each agreement within that set given that all other agreements form. Crucially, our model admits the possibility that a firm can engage in deviations across multiple negotiations, and also optimally respond to information acquired from one of its negotiations in others.

Our game has imperfect information since, within a period, firms do not see offers for agreements that do not involve them. We employ perfect Bayesian equilibrium with passive beliefs as our solution concept. Passive beliefs restricts firms to believe, upon receiving an off-equilibrium offer, that all unobserved actions remain equilibrium actions. This solution concept has been widely used and employed in the vertical contracting literature to analyze similar types of problems (c.f. McAfee and Schwartz, 1994).

We provide two sets of conditions that ensure the existence of a Nash-in-Nash limit equilibrium. The first set comprises a single condition which we label *weak conditional decreasing marginal contributions* (A.WCDMC). It requires that the marginal contribution from any set of agreements be no less than the sum of the marginal contributions from each individual agreement within that set for all firms when all other agreements have been formed. A.WCDMC is implied if firms on the same side of the market were viewed as at least weak substitutes by the other side. We show that A.WCDMC is necessary and sufficient for there to exist an equilibrium of our bargaining game where agreements form immediately at prices that correspond to the pairwise Rubinstein (1982) prices for each pair of firms (given that all other firms reach agreement). These prices converge to the Nash-in-Nash solution as the time period between offers goes to zero (Binmore, Rubinstein, and Wolinsky, 1986).

Our second set of conditions simultaneously weakens A.WCDMC for one side of the market (i.e., either upstream or downstream firms) and strengthens it for the other side, thereby extending the settings under which our results apply. The weaker condition—*feasibility* (A.FEAS)—states that the marginal profits to each firm from any set of its agreements are weakly greater than the sum of the Nash-in-Nash prices that are made (or received) for those agreements.[7] A.WCDMC implies A.FEAS. The stronger condition—*strong conditional decreasing marginal contributions* (A.SCDMC)—states that the marginal contribution to each firm from an agreement are lower when all other agreements have been formed than when certain subsets of agreements have been formed. A.SCDMC implies A.WCDMC. We prove

[6] We restrict our analysis to the case where the prices are lump-sum payments. Thus, in a setting where downstream firms engage in subsequent price competition for consumers after all bargains have concluded, this would imply that the negotiated prices with upstream firms represent fixed fees.

[7] Since downstream firms pay upstream firms, these payments are the prices for downstream firms and the negative of the prices for upstream firms.

that A.FEAS is a necessary condition for there to exist a Nash-in-Nash limit equilibrium where all agreements form immediately, and is sufficient when combined with A.SCDMC holding for one side of the market. We view our existence results to be the most important from the point of view of applied work, as they provide conditions under which the Nash-in-Nash solution is a reasonable surplus division rule.

We prove two results concerning the uniqueness of equilibrium outcomes. The first result is that, in any no-delay equilibrium, all agreements are formed at prices that are arbitrarily close to the Nash-in-Nash solution when the time between offers is sufficiently short. The bargaining literature has sometimes restricted attention to no-delay equilibria (e.g., Ray and Vohra, 2015), and we believe that our first uniqueness result highlights the robustness of the Nash-in-Nash solution in environments where delay is unlikely to occur. The second result—proven without conditioning on immediate agreement—is that all equilibria involve all agreements (that have not yet been formed) forming immediately at the pairwise Rubinstein (1982) prices, given a restriction on how firms break ties when indifferent over best responses and two conditions on profits that are stronger than those used to prove existence. In particular, we assume that both upstream and downstream firm profits satisfy our stronger decreasing returns assumption used to establish existence (A.SCDMC), and also satisfy a limited negative externalities condition (A.LNEXT). We prove our second uniqueness result via induction on the set of agreements formed at any point in time, and leverage: (i) our timing assumptions that allow for multiple offers to be made and multiple agreements formed at any period; (ii) the fact that our candidate equilibrium prices make a firm indifferent between accepting an offer and rejecting it (and forming the agreement in the next period); and (iii) our stronger assumptions on profits to rule out the possibility that some agreements do not immediately form following any history of play.

Related Literature. Although our results complement a broader theoretical literature that examines vertical contracting in industrial organization settings (e.g., Hart and Tirole (1990); O'Brien and Shaffer (1992); McAfee and Schwartz (1994); Segal (1999); Rey and Vergé (2004); c.f., Whinston (2006)), our modeling approach is most similar to that adopted by the literature on wage bargaining (e.g., Davidson, 1988; Jun, 1989; Stole and Zwiebel, 1996; Westermark, 2003; Brügemann, Gautier, and Menzio, 2015).[8] The wage bargaining literature typically examines extensive form bargaining games between a single firm and

[8]Another related literature examines multilateral or coalitional bargaining with more than two players (e.g., Chatterjee, Dutta, Ray, and Sengupta (1993); Chae and Yang (1994); Merlo and Wilson (1995); Krishna and Serrano (1996); c.f. Osborne and Rubinstein (1994); Muthoo (1999)). Our analysis focuses only on bilateral surplus division, as side payments among firms on the same side of the market (or between firms without a contractual relationship) would generally violate antitrust laws.

multiple workers (or a single union and multiple firms), and uses a payoff structure that is a special case of ours (i.e., multiple upstream firms, or workers, bargaining with a single downstream firm that accrues all profits). Part of this literature has analyzed games under which workers are paid according to the Nash-in-Nash solution as the period length between offers goes to zero.[9]

Other papers in the wage bargaining literature, including Stole and Zwiebel (1996) and Brügemann, Gautier, and Menzio (2015), provide alternative games under which Shapley values emerge as the division of surplus; both of these papers adopt the assumption that following a breakdown in negotiation between a firm and any given worker, all other negotiations restart and begin anew with any worker previously involved in a breakdown no longer involved in bargaining. Similar to papers that have provided extensive form representations of the generalization of the Shapley value to networked or bilateral oligopoly settings—by relying on renegotiation following disagreement (Navarro and Perea, 2013), or contracts that are contingent on the set of realized agreements (Inderst and Wey, 2003; de Fontenay and Gans, 2014)—our paper extends the settings under which the Nash-in-Nash solution emerges as an equilibrium outcome.[10] We further discuss the relationship between extensive form representations and surplus division rules in Section 5.

Our focus on environments where contracts are bilaterally negotiated and the *network* of agreements matter in determining profits distinguishes our analysis from a broader literature on multilateral and coalitional bargaining and contracting, both with and without externalities (e.g., Chatterjee, Dutta, Ray, and Sengupta, 1993; Krishna and Serrano, 1996; Gomes, 2005; Ray and Vohra, 2015).

Finally, many of the assumptions that we use have analogs in the literatures previously mentioned as well as in the network formation literature. This literature examines conditions under which efficient or pairwise stable networks form but is not primarily concerned with the division of surplus (e.g., Bloch and Jackson, 2007; Hellmann, 2013). For example, our A.WCDMC assumption is analogous to Bloch and Jackson's "superadditive in own-links" property and our A.LNEXT is strictly weaker than their "nonnegative externalities" assumption.

The remainder of our paper is organized as follows. Section 2 describes our extensive

[9]For example, Jun (1989) analyzes bargaining between a single firm and two workers in a model that coincides with ours. Westermark (2003)'s game form also uses an alternating offers framework, but assumes that agents act sequentially with proposers being randomly determined. Both of these papers provide results for cases with a single firm negotiating with two workers, with Westermark restricting attention to equilibria with stationary strategies.

[10]Also related to our paper is a literature that examines the trade of goods in fixed networks (e.g., Corominas-Bosch, 2004; Polanski, 2007; Manea, 2011; Elliott, 2015); most of this literature rules out externalities from trades that do not involve a given buyer or seller.

form bargaining model and our equilibrium concept. Section 3 provides our assumptions and results for the existence of Nash-in-Nash limit equilibria. Section 4 provides our assumptions and results for the uniqueness of equilibrium outcomes. Section 5 discusses connections between our results and the applied literature, as well as directions for future work. Section 6 concludes.

2 Model

Consider the bipartite negotiations between N upstream firms, $U_1, U_2, \ldots, U_N$, and M downstream firms, $D_1, D_2, \ldots, D_M$. We only permit agreements to be formed between upstream and downstream firms and not between firms on the same "side" of the market.[11] Let $\mathcal{G}$ represent the set of potential or feasible agreements; $\mathcal{G}$ can be represented by a bipartite network between upstream and downstream firms.[12] Denote a potential agreement between U_i and D_j as ij; the set of agreements that U_i can form as $\mathcal{G}_{i,U}$; and the set of agreements that D_j can form as $\mathcal{G}_{j,D}$. For any subset of agreements $\mathcal{A} \subseteq \mathcal{G}$, let $\mathcal{A}_{j,D} \equiv \mathcal{A} \cap \mathcal{G}_{j,D}$ denote the set of agreements in $\mathcal{A}$ that involve firm D_j, and let $\mathcal{A}_{-j,D} \equiv \mathcal{A} \setminus \mathcal{A}_{j,D}$ denote the set of all agreements in $\mathcal{A}$ that do not involve D_j. Define $\mathcal{A}_{i,U}$ and $\mathcal{A}_{-i,U}$ analogously.

We take as primitives profit functions $\{\pi_{i,U}(\mathcal{A})\}_{i=1,\ldots,N;\mathcal{A}\subseteq\mathcal{G}}$ and $\{\pi_{j,D}(\mathcal{A})\}_{j=1,\ldots,M;\mathcal{A}\subseteq\mathcal{G}}$, which represent the surpluses realized by upstream and downstream firms for a set or "network" of agreements that have been formed at any point in time. Importantly, profits from an agreement may depend on the set of other agreements formed; this allows for profit interdependencies and externalities across agreements. We assume that each upstream firm U_i and downstream firm D_j negotiate over a lump-sum payment (or "price") p_{ij} made from D_j to U_i in exchange for forming an agreement; this implies that profits (not including transfers) depend on the set of agreements formed but not on the negotiated prices.[13]

We model a dynamic game with infinitely many discrete periods. Periods are indexed $t = 1, 2, 3, \ldots$, and the time between periods is $\Lambda > 0$. Total payoffs (profits and prices)

[11] In many market settings, contractual agreements between two firms on the same side of the market can be interpreted as collusion and hence may constitute *per se* antitrust violations. Alternatively, agreements between two firms on the same side of the market can be viewed as a horizontal merger, in which case our analysis would treat those merged firms as one entity. We do not explicitly model the determination of such mergers in this paper. As well, vertical integration and price formation, as modeled in de Fontenay and Gans (2005), is also outside the scope of the paper.

[12] Note that $\mathcal{G}$ need not contain all agreements between all upstream and downstream firm pairs; some agreements may be infeasible or impossible to form. As in Lee and Fong (2013), there may be a prior network formation game that leads to a set of agreements $\mathcal{G}$ being feasible before bargaining commences. A model that determines the realized set of agreements is outside the scope of this paper, as we focus on the determination of transfers *given* the set of agreements $\mathcal{G}$.

[13] See Section 5 for further discussion on this point.

for each firm are discounted. The discount factors between periods for an upstream and a downstream firm are represented by $\delta_{i,U} \equiv \exp(-r_{i,U}\Lambda)$ and $\delta_{j,D} \equiv \exp(-r_{j,D}\Lambda)$ respectively.

The game begins in period $t_0 \geq 1$ with no agreements in $\mathcal{G}$ formed: i.e., all agreements in $\mathcal{G}$ are "open." In odd periods, each downstream firm D_j simultaneously makes private offers $\{p_{ij}\}_{ij\in\mathcal{G}_{j,D}}$ to each U_i with an offer in $\mathcal{G}_{j,D}$ with which it does not yet have an agreement; each upstream firm U_i then simultaneously accepts or rejects any offers it receives. In even periods, each U_i simultaneously makes private offers $\{p_{ij}\}_{ij\in\mathcal{G}_{i,U}}$ to downstream firms with which it does not yet have an agreement; each D_j then simultaneously accepts or rejects any offers that it receives. If D_j accepts an offer from U_i, or U_i accepts an offer from D_j, then an agreement is formed between two firms, and that agreement remains "formed" for the rest of the game. Each U_i receives its payment of p_{ij} from D_j immediately in the period in which an agreement is formed.

We assume that within a period, a firm only observes the set of contracts that it offers, or that are offered to it. However, at the end of any period, all information—including the terms of every contract, accepted or not—is observed by every firm; this implies that at the beginning of each period, every firm observes and can condition their strategies on a common *history of play*.[14] This history, denoted h^t, is the set of all actions (price offers and acceptances or rejections) that have been made by every firm in all periods prior to t.

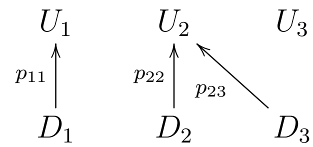
At the end of each period (after lump-sum payments from new agreements have been made), each upstream firm U_i and downstream firm D_j receives a flow payment equal to $(1-\delta_{i,U})\pi_{i,U}(\mathcal{A})$ or $(1-\delta_{j,D})\pi_{j,D}(\mathcal{A})$ (respectively), where $\mathcal{A}$ is the set of agreements that has been formed up to that point in time.[15]

Example. Figure 1 depicts a potential market with 3 upstream and 3 downstream firms. Assume that in period 1, the set of agreements $\mathcal{A}^1 \equiv \{11, 22, 23\}$ form. This implies that in period 1, U_1 receives a payment p_{11} from D_1, and U_2 receives p_{22} from D_2 and p_{23} from D_3; in the same period, each downstream firm D_j receives flow profits $(1-\delta_{j,D})\pi_{j,D}(\mathcal{A}^1)$ and each upstream firm U_i receives flow profits $(1-\delta_{i,U})\pi_{i,U}(\mathcal{A}^1)$. If, in period 2, D_1 forms an agreement with U_2 at some price p_{21} (and that is the only agreement that is formed), D_1

[14] We make this assumption for tractability. Without this assumption, we cannot condition on a common history of play, which would make our approach difficult. Institutionally, the contracted price between firms may not be observable to others (e.g., for competitive or antitrust concerns). Using a delegated agent model, de Fontenay and Gans (2005) examines the impact of vertical integration on bargaining assuming that agents do not observe the terms of contracts that do not involve them.

[15] Our model can also be recast without discounting but with an exogenous probability each period that negotiations end and with a lump-sum payment made based on the set of agreements that has been formed when negotiations end (Binmore, Rubinstein, and Wolinsky, 1986). The reason for this is that, in our model, externalities across firms generate endogenous "inside options" rather than "outside options" (Muthoo, 1999) since flow payments depend on the set of agreements that has been formed.

Figure 1: Example: Market with Three Agreements Already Formed



would pay U_2 a payment p_{21} in period 2, and all firms would earn period 2 flow profits as a function of the new realized set of agreements, $\mathcal{A}^2 \equiv \mathcal{A}^1 \cup \{12\}$.

Two points about our model are worth noting. First, while the flow profits continue to accrue to all firms forever, no additional actions are made after the last agreement is formed. Thus, the game can also be formulated to end in the period of last agreement, with each upstream firm U_i realizing a one-time payoff of $(1-\delta_{i,U})\pi_{i,U}(\mathcal{G})/(1-\delta_{i,U}) = \pi_{i,U}(\mathcal{G})$ (and analogously for each downstream firm). Second, if $M = N = 1$, our game is equivalent to the Rubinstein (1982) alternating offers model.

2.1 Equilibrium Concept

Rubinstein (1982) considers subgame perfect equilibria of his model. Because our model has imperfect information within a period (a firm only observes offers that it makes or receives), we use perfect Bayesian equilibrium as our solution concept. Perfect Bayesian equilibrium does not place restrictions on beliefs for information sets that are not reached in equilibrium, and thus may admit many equilibrium outcomes. To refine our predictions, we follow the literature on vertical contracting (e.g., Hart and Tirole, 1990; McAfee and Schwartz, 1994; Segal, 1999; de Fontenay and Gans, 2014) and assume that firms have *passive beliefs*:

Passive Beliefs. When a firm receives an out-of-equilibrium price offer following history of play h^t, the firm does not update its beliefs over any unobserved actions in the current period.

The assumption of passive beliefs is a key ingredient in many of the results of the literature on vertical contracting and opportunism, and in our setting restricts how beliefs can change following a deviation.[16] For example, this assumption implies that an upstream firm U_i, upon receiving an out-of-equilibrium price offer from some D_j in period t following history h^t,

[16] The vertical contracting literature has recognized an implicit relationship between passive beliefs and the Nash-in-Nash solution; e.g., Rey and Vergé (2004) state that "Horn and Wolinsky (1988) use a bilateral Nash bargaining approach that also relates somewhat to passive beliefs." It is possible that other belief restrictions may lead to different results (c.f. McAfee and Schwartz, 1994; Rey and Vergé, 2004).

believes that all other offers made by D_j and actions taken by other firms remain equilibrium actions. Henceforth, we use equilibrium to refer to a perfect Bayesian equilibrium with passive beliefs.

2.2 Nash-in-Nash and Rubinstein Prices

For exposition, it will be useful to define $\Delta\pi_{i,U}(\mathcal{A},\mathcal{B}) \equiv \pi_{i,U}(\mathcal{A})-\pi_{i,U}(\mathcal{A}\setminus\mathcal{B})$, for $\mathcal{B}\subseteq\mathcal{A}\subseteq\mathcal{G}$. This term is the increase in profits to U_i of adding agreements in $\mathcal{B}$ to the set of agreements $\mathcal{A}\setminus\mathcal{B}$. We refer to $\Delta\pi_{i,U}(\mathcal{A},\mathcal{B})$ as the *marginal contribution* to U_i of agreements $\mathcal{B}$ at $\mathcal{A}$. Correspondingly, let $\Delta\pi_{j,D}(\mathcal{A},\mathcal{B}) \equiv \pi_{j,D}(\mathcal{A}) - \pi_{j,D}(\mathcal{A}\setminus\mathcal{B})$. Importantly, as in Bloch and Jackson (2007), we define the marginal contribution of a set of agreements $\mathcal{B}$ as their value relative to removing them from the larger set $\mathcal{A}$ that includes $\mathcal{B}$.

For analysis, we assume that for any $ij\in\mathcal{G}$, the joint surplus created by U_i and D_j coming to an agreement (given that all other agreements in $\mathcal{G}$ have been formed) is positive:

Assumption 2.1 (A.GFT: Gains From Trade)

$$\Delta\pi_{i,U}(\mathcal{G},\{ij\})+\Delta\pi_{j,D}(\mathcal{G},\{ij\})>0 \qquad \forall ij\in\mathcal{G}\ .$$

This assumption implies that each pair of firms that can form an agreement in $\mathcal{G}$ has an incentive to keep that agreement given that all other agreements in $\mathcal{G}$ form. We believe that it is natural in many settings of interest as without A.GFT, firms may prefer to drop any agreements in which there are losses from trade. Rubinstein (1982) implies A.GFT as his first assumption, naming it "[the] 'pie' is desirable."

We now define "Nash-in-Nash" and "Rubinstein" prices for our game. Assume that A.GFT holds. For a given set of agreements $\mathcal{G}$ and set of positive bargaining weights $\{b_{i,U}\}_{\forall i}$ and $\{b_{j,D}\}_{\forall j}$, *Nash-in-Nash prices* are a vector of prices $\{p_{ij}^{\text{Nash}}\}_{\forall ij\in\mathcal{G}}$ such that, $\forall ij\in\mathcal{G}$:

$$\begin{aligned}
p_{ij}^{\text{Nash}} &\equiv \arg\max_p [\Delta\pi_{j,D}(\mathcal{G},\{ij\})-p]^{b_{j,D}}\times[\Delta\pi_{i,U}(\mathcal{G},\{ij\})+p]^{b_{i,U}}\\
&= \frac{b_{i,U}\Delta\pi_{j,D}(\mathcal{G},\{ij\})-b_{j,D}\Delta\pi_{i,U}(\mathcal{G},\{ij\})}{b_{i,U}+b_{j,D}}\ .
\end{aligned}$$

Each price p_{ij}^{Nash} corresponds to the Nash bargaining solution between D_j and U_i given that all other agreements in $\mathcal{G}$ are formed.

For a given set of agreements $\mathcal{G}$, *Rubinstein prices* are a vector of prices $\{p_{ij,D}^{\text{R}},p_{ij,U}^{\text{R}}\}_{\forall ij\in\mathcal{G}}$ such that, $\forall ij\in\mathcal{G}$:

$$p_{ij,D}^{R} = \frac{\delta_{i,U}(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\})-(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\})}{1-\delta_{i,U}\delta_{j,D}}\ ,$$

$$p_{ij,U}^{R} = \frac{(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - \delta_{j,D}(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\})}{1-\delta_{i,U}\delta_{j,D}} .$$

Each pair of prices $\{p_{ij,D}^{R}, p_{ij,U}^{R}\}$ correspond to the offers made in odd or even periods when D_j and U_i engage in a Rubinstein (1982) alternating offers bargaining game given that all other agreements in $\mathcal{G}$ are formed.

Let the Nash bargaining weights be parameterized so that $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$ and $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$. Then, as noted in Binmore, Rubinstein, and Wolinsky (1986), Rubinstein prices converge to the Nash-in-Nash prices as the time between offers becomes arbitrarily short:

Lemma 2.2 $\lim_{\Lambda\to 0} p_{ij,D}^{R} = \lim_{\Lambda\to 0} p_{ij,U}^{R} = p_{ij}^{Nash}$.

(All proofs are in the appendices.)

There are properties of Rubinstein and Nash-in-Nash prices that will prove crucial in our proofs. First, Rubinstein prices make the receiving agent indifferent between accepting its offer or waiting until the next period and having its counteroffer accepted given that all other agreements form. In our case, in an even (upstream-proposing) period, this implies that downstream firms are indifferent between accepting an offer and waiting until the next period (given that it believes that all agreements $\mathcal{G} \setminus \{ij\}$ have been or will be formed). Equivalently:

$$\underbrace{(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\})}_{\text{Loss in profit from waiting}} = \underbrace{p_{ij,U}^{R} - \delta_{j,D}p_{ij,D}^{R}}_{\text{Decrease in transfer payment from waiting}} \qquad \forall ij \in \mathcal{G}. \tag{1}$$

Correspondingly, for upstream firms in odd periods,

$$(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) = \delta_{i,U}p_{ij,U}^{R} - p_{ij,D}^{R} \qquad \forall ij \in \mathcal{G}. \tag{2}$$

Second, the value from agreement to each party is higher than prices paid or received, and Nash-in-Nash prices lie between the upstream and downstream proposing Rubinstein prices:

Lemma 2.3 *Assume A.GFT. Then $\forall ij \in \mathcal{G}$:*

$$\begin{aligned} \Delta\pi_{j,D}(\mathcal{G},\{ij\}) > p_{ij,U}^{R} > p_{ij,D}^{R} \ , \\ \Delta\pi_{i,U}(\mathcal{G},\{ij\}) > -p_{ij,D}^{R} > -p_{ij,U}^{R} \ , \\ p_{ij,U}^{R} > p_{ij}^{Nash} > p_{ij,D}^{R} \ . \end{aligned} \tag{3}$$

2.3 A Delegated Agent Model for Nash-in-Nash Prices

This subsection proposes an alternative “delegated agents” extensive form that generates Nash-in-Nash prices as $\Lambda \to 0$ (see Chipty and Snyder, 1999; de Fontenay and Gans, 2014; Inderst and Montez, 2014). In this model, firms appoint separate agents to conduct each bilateral bargain. Each bargain follows the alternating offers protocol based on Binmore, Rubinstein, and Wolinsky (1986): there is no discounting, but each bargain breaks down with a fixed exogenous probability—that is independent across bargains—at the end of every period, if an agreement has not yet been formed. Agents do not know the outcome of other bilateral bargains until their own bargain has concluded (by either breaking down or forming). Payments are made based on the set of agreements that have been formed after all bargains have concluded.

In Appendix A, we prove that A.GFT is sufficient for there to exist an equilibrium where all agreements in $\mathcal{G}$ immediately form at prices that are arbitrarily close to Nash-in-Nash prices as the probability of a bilateral bargain breaking down in each period goes to 0. However, this alternative model may be viewed as unsatisfying: firms do not leverage information learned in one negotiation in another, and cannot coordinate actions across different concurrent negotiations.

3 Equilibrium Existence

Our paper is concerned with equilibrium outcomes of our bargaining game as the time between offers becomes arbitrarily short. For this purpose, we define the concept of a *Nash-in-Nash limit equilibrium.* We say that a Nash-in-Nash limit equilibrium exists if, for any $\varepsilon > 0$ and t_0, there exists a $\bar{\Lambda} > 0$ such that for all $\Lambda \in (0, \bar{\Lambda}]$, there is an equilibrium with complete agreement (i.e., where all agreements in $\mathcal{G}$ are formed), and where all agreements are formed at prices that are within ε of Nash-in-Nash prices.

3.1 Equilibrium Existence at Rubinstein Prices

We first present a necessary and sufficient condition for there to exist an equilibrium of the bargaining game where, for any $\Lambda > 0$, all open agreements at any history of play immediately form at Rubinstein prices. Since Rubinstein prices converge to Nash-in-Nash prices as $\Lambda \to 0$ (by Lemma 2.2), this guarantees that a Nash-in-Nash limit equilibrium exists.

Assumption 3.1 (A.WCDMC: Weak Conditional Decreasing Marginal Contribution) *For upstream firms: for all $i = 1, \ldots, N$ and all $\mathcal{A} \subseteq \mathcal{G}_{i,U}$,*

$$\Delta\pi_{i,U}(\mathcal{G}, \mathcal{A}) \geq \sum_{ik \in \mathcal{A}} \Delta\pi_{i,U}(\mathcal{G}, \{ik\}).$$

For downstream firms: for all $j = 1, \ldots, M$ and all $\mathcal{A} \subseteq \mathcal{G}_{j,D}$,

$$\Delta\pi_{j,D}(\mathcal{G}, \mathcal{A}) \geq \sum_{hj \in \mathcal{A}} \Delta\pi_{j,D}(\mathcal{G}, \{hj\}).$$

This condition states that, for both upstream and downstream firms, the marginal contribution of any set of agreements to a firm is weakly greater than the sum of the marginal contributions of each individual agreement within the set when all other agreements in $\mathcal{G}$ have formed. This condition will generally be satisfied if firms on the same side of the market are substitutes, rather than complements, for firms on the other side of the market.

Similar assumptions have been used in the network formation and wage bargaining literature (e.g., Stole and Zwiebel, 1996; Westermark, 2003; Bloch and Jackson, 2007; Hellmann, 2013). Using Bloch and Jackson (2007)'s terminology, A.WCDMC is equivalent to assuming that profit functions for all firms are *superadditive in own-links* at $\mathcal{G}$.[17]

We now state our first existence result:

Theorem 3.2 (Existence at Rubinstein Prices) *Assume A.GFT. For any $\Lambda > 0$, there exists an equilibrium of the bargaining game where at every period t and history h^t:*

(a) all open agreements, denoted $\mathcal{C}(h^t)$, immediately form; and

(b) all agreements $ij \in \mathcal{C}(h^t)$ are formed at prices $p^R_{ij,D}$ ($p^R_{ij,U}$) if t is odd (even);

if and only if A.WCDMC holds.

The proof of Theorem 3.2, contained in Appendix C, constructs candidate equilibrium strategies based on the above statement and then confirms that these strategies are robust to one-shot deviations by any firm. Furthermore, it demonstrates that if A.WCDMC does not hold for some firm and a subset of its agreements, then that firm—when deciding to accept or reject that set of offers at Rubinstein prices—would have a profitable deviation of rejecting those offers.

[17]Bloch and Jackson (2007) define superadditivity in own-links as a condition analogous to A.WCDMC holding at *any* network, and not just at $\mathcal{G}$. Hellmann (2013) uses the same assumption as Bloch and Jackson (2007), but refers to it as "convex in own current links."

A.WCDMC rules out the possibility that any agent would wish to not form or delay forming any subset of its open agreements at Rubinstein prices. To see this, without loss of generality, consider any upstream firm U_i in an odd (downstream-proposing) period and any subset of its agreements $\mathcal{A} \subseteq \mathcal{G}_{i,U}$. A.WCDMC implies that the gain to U_i from accepting the agreements in $\mathcal{A}$ (given that all other agreements are formed) is weakly greater than the gain from rejecting the offers in $\mathcal{A}$ and forming the agreements in $\mathcal{A}$ in the subsequent period at candidate equilibrium prices; i.e.,:

$$\begin{aligned} &\overbrace{(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\mathcal{A}) + \sum_{ik\in\mathcal{A}} \left[p^R_{ik,D} - \delta_{i,U} p^R_{ik,U}\right]}^{\text{Change in } U_i\text{'s profits by forming agreements } \mathcal{A} \text{ in period } t \text{ as opposed to } t+1} \\ &\quad \geq \sum_{ik\in\mathcal{A}} \left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ik\}) + p^R_{ik,D} - \delta_{i,U} p^R_{ik,U}\right] = 0. \end{aligned} \tag{4}$$

where the inequality follows from A.WCDMC and the equality from (2). A similar inequality shows that each downstream firm D_j in even (upstream-proposing) periods would not wish to reject any subset of offers that it receives at even-period Rubinstein prices.

We illustrate the necessity of A.WCDMC for the existence of an equilibrium with immediate agreement at Rubinstein prices with two counterexamples. Consider first a setting where there are large complementarities across agreements. Assume that: there are three upstream parts "suppliers" that each provide a necessary component to a downstream "manufacturer" for production of an automobile, which is then resold for some fixed surplus; there are zero marginal costs for all firms; the manufacturer can sell the product for a surplus of 1 if all agreements are reached, and 0 otherwise; and firms share a common discount factor δ.

A.WCMDC does not hold here because the marginal contribution of a supplier to the manufacturer when not all agreements have been formed, at zero, is less than its marginal contribution of 1 when all agreements have been formed. The Nash-in-Nash prices here are a half of the marginal contributions when all agreements have been formed, or 0.5. At Nash-in-Nash prices, the downstream firm would realize a loss of 0.5 (with gross profits of 1 and supplier costs of 1.5), implying that the downstream firm would not wish to reach agreement at such prices with its suppliers. In this setting, it is implausible that transfers will be based on marginal contributions: either no agreements will be reached, or surplus division would be based on some other bargaining protocol or solution concept.[18]

Now consider a similar counterexample, but with two instead of three suppliers. We generalize the above example to assume that the manufacturer with one supplier earns profits

[18] This example is mathematically equivalent to example 1 in Westermark (2003), who makes a similar point with an example of a firm bargaining over wages with three workers.

a, $0 \leq a < 0.5$, so that the suppliers still exhibit complementarities from the perspective of the downstream firm. Note that the Rubinstein prices paid to each supplier U_i are $p^R_{i1,D} = \delta(1-a)/(1+\delta)$ and $p^R_{i1,U} = (1-a)/(1+\delta)$. Because profits to the manufacturer with one supplier are less than 0.5, A.WCDMC still does not hold here. Hence, by Theorem 3.2, there still is no equilibrium with immediate agreement at Rubinstein prices.

To understand why an equilibrium with immediate agreement at Rubinstein prices does not exist, consider any even (upstream-proposing) period where no agreements have yet been formed. By equation (1), the downstream firm will be exactly indifferent between (a) accepting both candidate equilibrium offers and (b) accepting one offer and rejecting the other (which will then be formed in the following period). In other words, the reduction in the present value of the payment from delaying an offer to the following (odd) period, $p^R_{i1,U} - \delta p^R_{i1,D}$, is equal to the loss in profits from this delay, $(1-\delta)(1-a)$. But, the loss in profits from rejecting the *second* offer is only $(1-\delta)a < (1-\delta)(1-a)$ (since $a < 0.5$), while the reduction in the present value of this payment remains $(1-\delta)(1-a)$. Thus, the downstream firm would strictly prefer rejecting both offers, implying that the strategy profile from Theorem 3.2 is not an equilibrium.

However, unlike in the three supplier example, the downstream firm here would strictly prefer agreement with both suppliers at any price below Nash-in-Nash to no agreement, as it would then obtain positive surplus instead of none (even in the limit case of perfect complements where $a = 0$). This suggests that a Nash-in-Nash limit equilibrium might exist here—though at different prices from Rubinstein. Our next results verify that this is the case.

3.2 Equilibrium Existence with Complementarities

We now provide an alternative set of conditions for the existence of a Nash-in-Nash limit equilibrium. Importantly, our conditions allow for limited complementarities and are not nested by A.WCDMC.

Our first condition ensures that all firms obtain value at the margin from all sets of its own agreements being formed at Nash-in-Nash prices:

Assumption 3.3 (A.FEAS: Feasibility)
For upstream firms: for all $i = 1, \ldots, N$ and all $\mathcal{A} \subseteq \mathcal{G}_{i,U}$,

$$\Delta\pi_{i,U}(\mathcal{G}, \mathcal{A}) \geq -\sum_{ij \in \mathcal{A}} p^{Nash}_{ij}.$$

For downstream firms: for all $j = 1, \ldots, M$ and all $\mathcal{A} \subseteq \mathcal{G}_{j,D}$,

$$\Delta\pi_{j,D}(\mathcal{G}, \mathcal{A}) \geq \sum_{ij \in \mathcal{A}} p_{ij}^{Nash}.$$

In words, A.FEAS states that each firm would prefer maintaining all of its agreements at Nash-in-Nash prices to dropping any set of its agreements (holding all other agreements fixed). This is a strictly weaker condition than A.WCDMC when A.GFT holds, as Nash-in-Nash prices are then strictly less (greater) than the marginal contributions of each agreement at $\mathcal{G}$ for downstream (upstream) firms (see Lemma 2.3). Similar conditions to A.FEAS have been used in other settings: e.g., in Stole and Zwiebel (1996), a feasibility condition ensures that each worker receives an equilibrium wage higher than her outside wage offer. In our setting, A.GFT alone is sufficient to ensure that this condition holds for all subsets involving a single agreement (see Lemma 2.3), but not for subsets that involve more than one agreement.

The following demonstrates the importance of A.FEAS.

Theorem 3.4 *A.FEAS is necessary for there to exist a Nash-in-Nash limit equilibrium where all agreements in $\mathcal{G}$ immediately form.*

The theorem states that if A.FEAS does not hold, then there is a $\varepsilon > 0$ and t_0 such that no $\bar{\Lambda} > 0$ exists where, for all $\Lambda \in (0, \bar{\Lambda}]$, there is an equilibrium where all agreements in $\mathcal{G}$ immediately form at t_0 at prices within ε of Nash-in-Nash prices. Its proof leverages the insight that if A.FEAS is violated but all agreements immediately form, then there exists a firm that would wish to reject (and never form) some set of its own agreements at prices that are arbitrarily close to Nash-in-Nash prices. For example, in our upstream supplier example with three suppliers, one can verify that A.FEAS does not hold for the downstream manufacturer when $\mathcal{A}$ includes all three suppliers; thus, Theorem 3.4 confirms the intuition from Section 3.1 that there is no Nash-in-Nash limit equilibrium for this three supplier game where all agreements immediately form. We discuss the role of equilibria with immediate agreement further in Section 4.

The second sufficient condition that we impose to establish existence is stronger than A.WCDMC (and hence A.FEAS) and, for existence, is only required to hold for *either* downstream *or* upstream firms:

Assumption 3.5 (A.SCDMC: Strong Conditional Decreasing Marginal Contribution)

(a) For upstream firms: for all $ij \in \mathcal{G}$, $\mathcal{B} \subseteq \mathcal{G}_{-j,D}$, and $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{G}_{j,D} \setminus \{ij\}$,

$$\pi_{i,U}(\mathcal{A} \cup \mathcal{B} \cup \{ij\}) - \pi_{i,U}(\mathcal{A}' \cup \mathcal{B}) \geq \Delta\pi_{i,U}(\mathcal{G}, \{ij\}).$$

(b) For downstream firms: for all $ij \in \mathcal{G}$, $\mathcal{B} \subseteq \mathcal{G}_{-i,U}$, and $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{G}_{i,U} \setminus \{ij\}$,

$$\pi_{j,D}(\mathcal{A} \cup \mathcal{B} \cup \{ij\}) - \pi_{j,D}(\mathcal{A}' \cup \mathcal{B}) \geq \Delta\pi_{j,D}(\mathcal{G}, \{ij\}).$$

A.SCDMC(a) states that the marginal contribution to any upstream firm U_i of agreement ij at $\mathcal{G}$—which is $\Delta\pi_{i,U}(\mathcal{G}, \{ij\})$—is no greater than the marginal contribution to U_i of that agreement at any subset of agreements $\mathcal{A} \cup \mathcal{B}$, even if D_j were to change its agreements (from $\mathcal{A}$ to $\mathcal{A}'$) when making such a comparison. A.SCDMC(b) states a similar condition for downstream firms. Unless explicitly noted, when we assume A.SCDMC, we assume that both parts (a) and (b) hold.

We rely on A.SCDMC to ensure that, whenever there are open agreements, a proposing firm would wish to make an acceptable offer and form an agreement, even if the receiving firm were to change its actions with respect to its other open agreements in that period. This rules out the possibility that an out-of-equilibrium offer from a proposing firm leads the receiving firm to change its set of other accepted offers, which in turn harms the proposing firm. A.SCDMC strictly implies A.WCDMC, as it places more restrictions on the marginal values of individual agreements when a subset of other agreements in $\mathcal{G}$ have formed than does A.WCDMC.

We are not aware of other papers that have used assumptions exactly analogous to A.SCDMC. While the network formation literature has obtained results on the efficiency and pairwise stability of networks by restricting the marginal contribution of a link ij to a firm i to be higher than the marginal contribution of that link when other firms also add links (Hellmann, 2013), it has not, to our knowledge, allowed the other firm j to adjust its links when computing the marginal contribution to i of a link ij (as in A.SCDMC).[19]

Before proceeding to our existence proof, we formalize the relationship of our three assumptions on marginal contributions:

Lemma 3.6 *Assume that A.GFT holds. Then:*

(a) A.SCDMC $\Rightarrow$ A.WCDMC $\Rightarrow$ A.FEAS.

(b) A.FEAS $\not\Rightarrow$ A.WCDMC $\not\Rightarrow$ A.SCDMC.

We now state our final existence result:

[19]Note also that a pairwise stability condition rules out the possibility that a link would not be formed if the two agents involved in the link value its formation holding fixed the actions of other players. In contrast, in our setting, even if two agents involved in a link would prefer that the link be formed (holding fixed all other actions), such an agreement may not be formed in a given period since a proposing firm cannot guarantee that other agreements will not change: i.e., a different offer made by the proposer may induce a change in the receiver's other acceptances, thereby affecting the proposer's profits.

Theorem 3.7 (Existence of a Nash-in-Nash Limit Equilibrium) *Assume A.GFT, A.FEAS, and either A.SCDMC(a) or A.SCDMC(b). Then, there exists a Nash-in-Nash limit equilibrium where all agreements in $\mathcal{G}$ immediately form.*

Our proof of existence in Theorem 3.7 is constructive.[20] In the equilibrium that we construct, if A.SCDMC(a) holds, then downstream firms always propose Rubinstein prices in odd periods.[21] In even periods, upstream firms propose offers that ensure that downstream firms do not want to reject a single offer or multiple offers, and also ensure that every downstream firm is indifferent between complete acceptance and rejecting some subset of its offers.

If A.WCMDC holds, then by Theorem 3.2 there exists an equilibrium with immediate agreement at Rubinstein prices. However, if A.SCDMC(a) holds (for upstream firms) but A.WCDMC does not hold (for downstream firms), then some downstream firm in an even period would prefer to reject of some set of offers as opposed to accepting all offers at Rubinstein prices. To eliminate the incentive to reject multiple offers, the even-period equilibrium prices are then lower than even-period Rubinstein prices (though still higher than odd-period Rubinstein prices).

By not imposing A.WCDMC on one side of the market (but still requiring that A.FEAS hold for that side), Theorem 3.7 admits certain forms of complementarities, thereby extending the settings under which there exists a Nash-in-Nash limit equilibrium where all agreements in $\mathcal{G}$ immediately form, relative to Theorem 3.2. For example, consider again the two upstream supplier counterexample from Section 3.1. Although this example does not satisfy A.WCDMC since the suppliers produce complementary inputs, A.FEAS will be satisfied on the downstream side so long as the profits with one supplier are not negative ($a \geq 0$). Moreover, A.SCDMC is trivially satisfied for the upstream suppliers (as they realize no flow profits). Thus, by Theorem 3.7, a Nash-in-Nash limit equilibrium exists (even in the limiting case of $a = 0$). To ensure that the manufacturer does not want to reject both offers, our constructed equilibrium for this example has even-period prices that are lower than Rubinstein prices, and in some cases, lower than Nash-in-Nash prices. [22]

[20] Our constructed strategies build on insights from Jun (1989), who analyzed a setting with a single firm negotiating with two workers.

[21] While we focus our discussion here on the case where A.SCMDC(a) holds, the case where A.SCDMC(b) holds is analogous.

[22] Because the manufacturer strictly prefers accepting both even-period offers to rejecting one, we can construct multiple equilibria which differ in the payment made to each upstream firm in even periods. These equilibria all make the manufacturer indifferent between rejecting and accepting both offers in even periods, and all have prices that are arbitrarily close to Nash-in-Nash prices for sufficiently short time periods.

4 Uniqueness of Equilibrium Outcomes

Having established results on existence, we now turn to the uniqueness of equilibrium outcomes. We prove two results. Our first result is that all *no-delay equilibria*—defined to be equilibria in which at any history of play h^t, all open agreements $\mathcal{C}(h^t)$ immediately form—have agreements formed at prices that are arbitrarily close to Nash-in-Nash prices for sufficiently short time periods. Note that the equilibria constructed to establish existence in Section 3 are no-delay equilibria. This result does not require any assumptions on profits, highlighting the generality of the Nash-in-Nash solution when agreement is immediate.

Our second result is complementary to the first, and provides sufficient conditions for all equilibria to have agreements formed at Rubinstein prices (and hence at prices that are arbitrarily close to Nash-in-Nash prices for sufficiently short time periods). This second result uses the same A.SCDMC condition used to establish existence, a further assumption on profits that limits how severe negative externalities can be across agreements (A.LNEXT), and a restriction on strategies that governs how ties are broken when firms are indifferent over actions. It does not require any assumptions on equilibrium behavior (such as no-delay).

4.1 Uniqueness for No-Delay Equilibria

Our first theorem states that for sufficiently short time periods, any no-delay equilibrium has prices that are arbitrarily close to Nash-in-Nash prices:

Theorem 4.1 (Uniqueness for No-Delay Equilibria) *For any $\varepsilon > 0$, there exists $\bar{\Lambda} > 0$ such that for any $\Lambda \in (0, \bar{\Lambda}]$, any no-delay equilibrium has prices at every period t and history h^t that are within ε of Nash-in-Nash prices.*

Theorem 4.1 does not use any assumptions on underlying profits, but instead conditions on the immediate formation of all open agreements following all histories of play. Restricting attention to equilibria without delay is an assumption that has been used in the bargaining literature (e.g., Brügemann, Gautier, and Menzio, 2015; Ray and Vohra, 2015).[23]

The intuition of the result is as follows. Given complete and immediate agreement at every period, we first show that a receiving firm cannot receive a worse offer than its Rubinstein price in that period. If it were to receive a worse offer, a deviation for this firm would

[23] Ray and Vohra (2015), in motivating such a restriction, note that delays in complete information bargaining models are "more artificial [than delays in bargaining with incomplete information] and stem from two possible sources. The first is a typical folk theorem-like reason in which history-dependent strategies are bootstrapped to generate inefficient outcomes... [The second] will only happen for protocols that are sensitive to the identity of previous rejectors," which is not the case for the model in Rubinstein (1982) or in our model.

be for it to reject this offer and, in the following period when only one agreement would remain open, form the agreement at the corresponding Rubinstein price (Rubinstein, 1982). Equations (1) and (2) ensure that such a deviation is profitable.

Next, we show that a proposing firm cannot make an offer that is accepted and significantly worse than its Nash-in-Nash price. Here, a no-delay equilibrium implies that a proposing firm anticipates that all open agreements will form in the next period, regardless of actions that are taken in the current period. If a proposing firm were to "withdraw" an offer that it is supposed to make (i.e., by making a sufficiently worse offer that would be rejected), that agreement—and potentially other agreements formed by the receiving firm—would be rejected and instead formed in the following period. Using the fact that prices in the subsequent period must be no worse for the proposing firm than Rubinstein prices (as noted above), we show that for a sufficiently short time period, a proposing firm would have a profitable deviation from withdrawing this agreement. Thus, the proposing firm would never form an agreement at a price significantly worse than the Nash-in-Nash price.

Consequently, in any no-delay equilibrium, prices must be arbitrary close to Nash-in-Nash prices as $\Lambda \to 0$. However, we have not addressed whether there exist equilibria with delay at some histories of play. We next provide sufficient conditions on primitives and strategies to rule out this possibility.

4.2 Uniqueness Without Conditioning on Immediate Agreement

Under stronger conditions than used to establish existence, we now prove that every equilibrium is a no-delay equilibrium where, for any history of play, all open agreements are formed at Rubinstein prices. Under these conditions, equilibrium outcomes will be unique. If there are multiple equilibria, they will only differ in their prescribed off-equilibrium-path play.

The first condition requires that A.SCDMC, introduced in Section 3.2 and used in establishing our second existence result, holds for both upstream and downstream firms. This condition, again, states that the marginal contribution to any firm forming an agreement when all other agreements have been formed can be no greater than that agreement's contribution when certain subsets of agreements have been formed.

Our second condition is new:

Assumption 4.2 (A.LNEXT: Limited Negative Externalities) *For all non-empty $\mathcal{A} \subseteq \mathcal{G}$, there exists $ij \in \mathcal{A}$ s.t.:*

$$\Delta\pi_{i,U}(\mathcal{G}, \mathcal{A}) \geq \sum_{ik \in \mathcal{A}_{i,U}} \Delta\pi_{i,U}(\mathcal{G}, \{ik\}) \text{ and}$$

$$\Delta\pi_{j,D}(\mathcal{G},\mathcal{A}) \geq \sum_{hj\in\mathcal{A}_{j,D}} \Delta\pi_{j,D}(\mathcal{G},\{hj\}).$$

A.LNEXT states that for any nonempty subset of agreements $\mathcal{A}$, there exists some agreement $ij \in \mathcal{A}$ such that the marginal contribution to U_i (and D_j) of agreements $\mathcal{A}$ at $\mathcal{G}$ is weakly greater than the sum of the individual marginal contributions of all agreements in $\mathcal{A}$ that involve U_i (and D_j) at $\mathcal{G}$. This is also equivalent to imposing a lower bound on the value to U_i of agreements in $\mathcal{A}_{-i,U}$ (with a similar condition holding for some D_j and agreements in $\mathcal{A}_{-j,D}$). We refer to the assumption as "limited negative externalities" since, when paired with A.SCDMC, this lower bound is weakly negative.[24] Importantly, for each different subset of agreements $\mathcal{A}$, there can be a different pair U_i and D_j that satisfy this condition. We use A.LNEXT to help rule out equilibria with delay as it ensures that at any history of play, there is some pair of firms with an open agreement that would prefer all remaining open agreements to form. Given A.SCDMC, A.LNEXT is implied by Bloch and Jackson (2007)'s nonnegative externalities condition.

Finally, we also restrict attention to equilibria that satisfy *common tie-breaking*: at any history of play h^t, if there are two information sets in which any receiving firm has the same set of best responses, the firm chooses the same best response across both information sets.[25] Similar types of restrictions—both informally and formally—have been used in the bargaining literature.[26]

We now state our second uniqueness result:

Theorem 4.3 (Uniqueness) *Assume A.GFT, A.SCDMC, and A.LNEXT. For any $\Lambda > 0$,* every *common tie-breaking equilibrium has the same properties as the equilibrium described*

[24]To see why A.LNEXT and A.SCDMC admit weakly negative externalities, focus on the A.LNEXT condition for some $\mathcal{A}$ and firm U_i, $ij \in \mathcal{A}$: $\Delta\pi_{i,U}(\mathcal{G},\mathcal{A}) \geq \sum_{ik\in\mathcal{A}_{i,U}} \Delta\pi_{i,U}(\mathcal{G},\{ik\})$. The left-hand side of this inequality can be expressed as $\Delta\pi_{i,U}(\mathcal{G},\mathcal{A}) = \Delta\pi_{i,U}(\mathcal{G},\mathcal{A}_{-i,U}) + \Delta\pi_{i,U}(\mathcal{G}\setminus\mathcal{A}_{-i,U},\mathcal{A}_{i,U})$. Substituting this expression into the A.LNEXT condition and re-arranging terms yields $\Delta\pi_{i,U}(\mathcal{G},\mathcal{A}_{-i,U}) \geq \sum_{ik\in\mathcal{A}_{i,U}} \Delta\pi_{i,U}(\mathcal{G},\{ik\}) - \Delta\pi_{i,U}(\mathcal{G}\setminus\mathcal{A}_{-i,U},\mathcal{A}_{i,U})$. Applying A.SCDMC, the right side of the inequality is less than or equal to 0. Thus, the marginal value of $\mathcal{A}_{-i,U}$ to U_i at $\mathcal{G}$ needs to be greater than a value that is weakly negative for U_i to satisfy the conditions of A.LNEXT.

[25]Common tie-breaking is substantively different than restricting strategies to be Markovian or stationary. Markov strategies require that firms follow the same actions across *different even or odd histories of play h^t* that share the same set of open agreements $\mathcal{C}(h^t)$. Common tie-breaking does not require this, but rather only restricts actions for receiving firms to be the same for a *given history of play h^t* across different sets of price offers that induce the same set of best responses.

[26]Ray and Vohra (2015) impose an equilibrium restriction that they call "compliance," which requires that a receiver of a bargaining proposal, when indifferent over a set of actions, chooses the action that is most preferred by the proposer (given equilibrium play). Brügemann, Gautier, and Menzio (2015) note that a related tie-breaking assumption, which they describe as "reasonable," is necessary to obtain uniqueness results under the Stole and Zwiebel (1996) union wage-bargaining model.

in Theorem 3.2: i.e., at every period t and history h^t, all open agreements $ij \in \mathcal{C}(h^t)$ immediately form at prices $p^R_{ij,D}$ ($p^R_{ij,U}$) if t is odd (even).

As discussed in the previous section, when A.SCDMC is violated, there may exist no-delay equilibria where prices are not Rubinstein but still arbitrarily close to Nash-in-Nash prices as $\Lambda \to 0$ (as in Theorem 3.7). It is an open research question whether, with weaker assumptions, all equilibria with complete (but not necessarily immediate) agreement have prices that are close to Nash-in-Nash prices as the time between periods becomes short.

We note that Theorem 4.3 also holds under an alternative set of assumptions: instead of assuming A.LNEXT and restricting attention to common tie-breaking equilibria, it is sufficient to impose a "no externality" assumption (formally defined in Appendix E) alongside A.SCDMC. This assumption still allows for interdependencies across agreements for a given firm, but rules out externalities. We prove the statement of Theorem 4.3 for both our main and alternative set of assumptions.

Overview of Proof and Role of Assumptions. The proof for Theorem 4.3 proceeds by induction on the set of open agreements at any history of the game, $\mathcal{C}$. The proof establishes that if any subgame that begins with a strict subset of $\mathcal{C}$ agreements being open results in all open agreements forming immediately at Rubinstein prices (i.e., the inductive hypothesis), then any subgame beginning with $\mathcal{C}$ open agreements also must result in all open agreements forming immediately at Rubinstein prices (i.e., the inductive step). The base case follows from Rubinstein (1982), who establishes this result for any subgame with a single open agreement.

We start by proving the simultaneity of agreements—i.e., in any equilibrium, if any open agreements are formed at period t, all open agreements are formed at t—when $\mathcal{C}$ contains agreements that involve multiple "receiving" firms (e.g., if t is odd, there are multiple upstream firms with open agreements). Here, A.SCDMC allows us to rule out equilibria where only a strict subset of open agreements are formed in a given period. To illustrate, consider a subgame where there are $\mathcal{C}$ open agreements, and suppose that in an equilibrium, at least one agreement in $\mathcal{C}$ forms at period t, but not all agreements in $\mathcal{C}$ do. Consider some agreement $ij \in \mathcal{C}$ that does not form at t and for which there exists another agreement that does form at t where the receiving firm at t differs from ij. We establish a contradiction by showing that the proposing firm involved in agreement ij will find it profitable to make an deviant offer at t that is slightly more generous than the Rubinstein price for this agreement. A.SCDMC ensures that the marginal value of forming this agreement to the receiving firm is weakly higher with open agreements than at $\mathcal{G}$, and hence the receiving firm will find it optimal to accept this offer. A.SCDMC further ensures that the proposing firm finds this deviation profitable even if the receiving firm were to change its set of acceptances upon receiving this deviant offer.

We next prove that when all open agreements are formed in a period where there are multiple receiving firms, they are all formed at Rubinstein prices. We show that a receiving firm will reject an offer that is worse than the Rubinstein price by leveraging the fact that offers are simultaneous, and that Rubinstein prices make a receiving firm indifferent between accepting and waiting (where, upon waiting, the agreement will then be formed at the Rubinstein price in the next period when only one open agreement remains). We show that a proposing firm will choose to adjust an offer that is worse for it than the Rubinstein price by showing that, if it were to adjust its offer slightly towards the Rubinstein price, this deviant offer would still be accepted by the receiving firm. Our restriction to common tie-breaking equilibria ensures that the receiving firm will not change its set of other acceptances upon accepting this deviant offer, thus ensuring that the deviation is profitable for the proposing firm.

We then prove the simultaneity of agreements at Rubinstein prices when open agreements are formed in a period where there is only a single receiving firm with open agreements. With multiple receiving firms, we are able to leverage our inductive hypothesis to prove the previous results; with a single receiving firm, we can no longer always rely on induction: if the receiving firm rejects all offers in a given period, the set of open agreements will remain the same in the following period. To proceed, we employ arguments similar to those used in Shaked and Sutton (1984) to prove the uniqueness result in Rubinstein (1982). While the proof with a single receiving firm is more involved than with multiple receiving firms, the role of our assumptions is similar.

Finally, we prove that any equilibrium results in all open agreements being formed immediately. We leverage A.LNEXT to rule out delay, as it implies that, at any history, there is a pair of firms with an open agreement that benefits from all open agreements forming.[27] Consequently, if there was not immediate agreement in a given period, there would be a profitable deviation for the proposer of this pair, whereby it would propose an offer to the receiver of this pair that would be accepted, resulting in the formation of all remaining open agreements by the next period at latest (by the inductive hypothesis). As with our proof of simultaneity, A.SCDMC ensures that if the proposer in this pair makes the offer at a price that is slightly more generous than Rubinstein, then the receiver would accept this offer and the deviation would be profitable for the proposer. This rules out equilibria with delayed agreement, and establishes our result.

[27] Relatedly, Jehiel and Moldovanu (1995) show that the presence of negative externalities can potentially lead to delay in negotiations in a finite horizon sequential bargaining game.

To illustrate the possibility of an equilibrium with delay when A.LNEXT is violated, consider the following counterexample. Let there be two upstream firms, U_1 and U_2 and two downstream firms D_1 and D_2, and let firms share a common discount factor δ. Assume that $\mathcal{G} = \{11, 22\}$, so that "1" firms cannot form agreements with "2" firms. Suppose that the marginal value of agreement $\{11\}$ is 1 to both U_1 and D_1. However, suppose that the establishment of agreement $\{22\}$ imposes a negative externality of -10 on both D_1 and U_1. Let the "2" firms' payoffs be symmetric to these. Note that profits in this example satisfy A.GFT and A.SCDMC but not A.LNEXT (due to the large negative externalities imposed across pairs). There are (at least) two equilibria of this game. First, since this example satisfies the assumptions of Theorem 3.4, one equilibrium involves the immediate formation of all agreements in $\mathcal{G}$ at Rubinstein prices. Second, another equilibrium has the four firms always proposing unattractive offers so that no agreements ever form in equilibrium. No pair has an incentive to "break" this second equilibrium because even though it knows that the marginal gain from its own agreement would be positive, the formation of this agreement would result in the other agreement forming in the following period and thereby impose a (present value) negative externality of -10δ.[28]

5 Relation of our Model to the Applied Literature

In this section, we discuss the connections between our model and applied papers that have, either directly or indirectly, appealed to Nash-in-Nash or related solutions.

Relation of Assumptions on Profits to the Applied Literature. Most of our results employ A.GFT (along with other assumptions). We believe that A.GFT will generally be satisfied for environments where, for some network $\mathcal{G}$, all agreements in $\mathcal{G}$ have either been observed to have formed in reality, or are expected to form, perhaps because they were announced in a prior network formation stage (as in Lee and Fong, 2013). Indeed, A.GFT is required for the bilateral Nash bargaining solution to be defined.

A number of papers focus on a single downstream firm which negotiates with multiple upstream entities. This has often been the case in the (applied theory) wage bargaining literature (e.g., Jun, 1989; Stole and Zwiebel, 1996; Westermark, 2003; Brügemann, Gautier, and Menzio, 2015). In these papers, a single firm bargains with multiple workers, profits are assumed to accrue only to the firm, and payments (i.e., wages) are lump sum. By Theorem 3.2, so long as A.FEAS holds for the firm's profits, a Nash-in-Nash limit equilibrium exists for

[28] Note that the equilibria in this counterexample are similar to defection and cooperative equilibria of a repeated prisoner's dilemma game.

these models (since A.SCDMC(a) trivially holds for workers' profits, which are zero). The fact that profits only accrue to the firm also implies that workers do not exert externalities on one another, and hence that profits satisfy a "no externality" condition (discussed briefly in Section 4.2 and formally defined in Appendix E). If workers are also substitutable (e.g., as implied by Westermark's "decreasing returns" assumption), then A.SCDMC and A.LNEXT also hold (see Appendix E), and any equilibrium will have complete and immediate agreement at Rubinstein prices, by Theorem 4.3. Substitutability among workers is commonly assumed in this literature.

Similarly, many papers consider environments where a single downstream firm negotiates with multiple upstream suppliers. These include applied theory papers which examine the impact of downstream "buyer power" on negotiated prices, upstream supplier incentives, and welfare (e.g., Inderst and Wey, 2007; Chipty and Snyder, 1999; Inderst and Valletti, 2011; O'Brien, 2014; Chen, 2015); and empirical work in health care markets (e.g., Capps, Dranove, and Satterthwaite, 2003; Grennan, 2013; Gowrisankaran, Nevo, and Town, 2015). Many of these papers assume that upstream firms earn a lump-sum payment (and no other profits), implying again that there are no externalities. For example, Chipty and Snyder (1999) study a monopolist content supplier negotiating with multiple downstream cable distributors. They assume that negotiations are for lump-sum payments at the efficient quantity, that downstream distributors are local monopolists which do not compete with one another, and that A.SCDMC holds, implying that A.LNEXT also holds. Capps, Dranove, and Satterthwaite (2003), in a bargaining model enriched and formalized by Gowrisankaran, Nevo, and Town (2015), study the negotiations between many hospitals and one insurer in an environment where A.WCDMC is satisfied.[29] Capps, Dranove, and Satterthwaite's payoffs are consistent with a model where the insurer reimburses hospitals both a lump sum upon the formation of an agreement and the marginal costs of serving patients (see Gowrisankaran, Nevo, and Town, 2015), again implying that A.LNEXT holds. Grennan (2013) models a single downstream firm (hospital) negotiating with multiple substitutable upstream firms (medical device manufacturers) in a setting where, again, A.WCDMC is satisfied. [30]

[29] Capps, Dranove, and Satterthwaite (2003) assume that the profit for an insurer is related to the ex ante surplus received by enrollees from the insurer's network of hospitals. In their framework, the total surplus of the insurer's network $\mathcal{H}$ can be expressed as $\sum_i \log\left(\sum_{j\in\mathcal{H}} u_{ij}\right)$ where u_{ij} is the exponentiated utility (net of an i.i.d. Type I extreme value error) that patient i receives from visiting hospital j and the 'i' sum is over the patients of the insurer. The marginal contribution of some hospital $k \notin \mathcal{H}$ to the insurer's network—denoted willingness-to-pay—is thus $WTP = \sum_i \log\left(u_{ik} + \sum_{j\in\mathcal{H}} u_{ij}\right) - \sum_i \log\left(\sum_{j\in\mathcal{H}} u_{ij}\right)$, which can be shown to be decreasing as we add elements to $\mathcal{H}$. The diminishing returns property also holds more generally, e.g. with random coefficients logit models (Berry, Levinsohn, and Pakes, 1995).

[30] In both Grennan (2013) and Gowrisankaran, Nevo, and Town (2015), contracts are not for lump sum payments but instead for linear fees; we discuss richer contract spaces below.

Applied papers that have used the Nash-in-Nash solution to model negotiations between multiple upstream and multiple downstream firms include Crawford and Yurukoglu (2012) and Crawford, Lee, Whinston, and Yurukoglu (2015) in cable television, and Lee and Fong (2013), Gowrisankaran, Nevo, and Town (2015), and Ho and Lee (2015) for health care markets. Given their underlying demand systems (based on similar ones to those discussed above), firms will view contracting partners as partial substitutes for one another (at the observed sets of agreements), implying that A.WCDMC will likely be satisfied in their settings.

Finally, Möllers, Normann, and Snyder (2016), following the approach of Martin, Normann, and Snyder (2001), conduct experiments involving a game played between three subjects; one subject plays the role of an upstream firm making offers to the other two subjects, representing downstream firms. In their setting, payments are lump-sum and the authors confirm that their payoffs satisfy our A.SCDMC and A.LNEXT assumptions. They argue that their experimental evidence is consistent with Nash-in-Nash surplus division when agents are allowed to engage in private communications while bargaining.

Other Bargaining Protocols. Our paper focuses on environments with lump-sum payments that do not depend on the set of other agreements that have formed. In other settings, realized transfers between firms may depend on the current set of agreements. This can occur if contracts: are renegotiated or bargaining restarts upon any disagreement (e.g., Stole and Zwiebel, 1996; Navarro and Perea, 2013; Brügemann, Gautier, and Menzio, 2015), are written to be contingent on the set of realized agreements (e.g., Inderst and Wey, 2003; de Fontenay and Gans, 2014), or have exogenously determined contingencies due to, for example, bankruptcy (e.g., Raskovich, 2003). These alternative bargaining protocols yield "Myerson-Shapley" values (Myerson, 1977; Jackson and Wolinsky, 1996; Navarro, 2007) under certain conditions. Intuitively, the ability of firms to void contracts or write contingent contracts changes their disagreement values, and can result in a firm being paid the average of its marginal contributions across different coalitions (as in Myerson-Shapley values) instead of its marginal contribution at $\mathcal{G}$.

Other Contract Spaces. The applied literature has often defined a Nash-in-Nash bargaining solution to be a set of bilateral *contracts* such that each contract is the solution to a bilateral Nash bargaining problem holding fixed the contracts of all other bilateral pairs. In our analysis, we have assumed that contracts are restricted to be lump-sum payments that do not depend on other agreements that have been formed; thus, firms' profits are not affected by the levels of realized payments. In other settings with larger contract spaces, negotiated contracts may directly affect firms' profits: e.g., if upstream and downstream firms negotiate over linear fees, and downstream firms engage in price competition with one another for consumers given these contracts, then the terms of all firms' contracts will affect downstream prices and hence realized firm surplus.

It is worth noting that employing Nash-in-Nash bargaining over other types of contracts as opposed to lump-sum payments (and allowing for other agents such as downstream firms and consumers to subsequently take actions that affect realized payments and profits) is feasible and has been employed in applied work (e.g., Crawford and Yurukoglu, 2012). As discussed, such use has been typically justified with a delegated agent model, similar to Section 2.3.

Extending the contract space may also change the set of environments in which the Nash-in-Nash solution concept can be applied. For example, consider again the "automobile supplier" example from Section 3 where agreements with all three upstream suppliers are required for a downstream firm to realize any surplus; recall that this example does not satisfy our feasibility assumption, A.FEAS. Now assume that the manufacturer bargains with each upstream supplier over a linear input fee. Following the bargaining stage, assume that the manufacturer chooses whether or not to produce and sell the car to a single consumer, who has unit demand with valuation 1. If the manufacturer has not secured all three contracts, it will not produce or sell any cars and the suppliers will not receive any payment.[31] Thus, the Nash-in-Nash solution in this setting results in each supplier being paid 1/4 of the total surplus. Notably, this corresponds to the Myerson-Shapley value, which is also realized in some of the alternative extensive forms discussed above.

Generally, when firm profits may depend on not only the set of agreements formed, but also upon the negotiated prices and contract terms themselves, analyses along the lines of that in the current paper—where delegated agents are not employed—become significantly more complicated. In particular, strategies may no longer condition only on the set of open agreements at any history of play, but also on the terms of contracts that have already been signed. Characterizing the properties of bargaining in bilateral oligopoly environments with more general contracts will require further assumptions on the underlying nature of both subsequent price competition among downstream firms and the nature of the derived demand functions for downstream and upstream firms by end consumers. This remains an open research agenda, and ultimately we view our current analysis as a step towards understanding the applicability of Nash-in-Nash to environments with more general contract spaces.

[31]See also Iozzi and Valletti (2014) who explore the importance of specifying whether or not disagreement is observable to rivals before a price competition stage between downstream firms.

6 Concluding Remarks

In this paper, we have provided a bargaining model that extends the Rubinstein (1982) alternating offers game to bilateral oligopoly. We establish two sets of results. Our first set of results proves that a Nash-in-Nash limit equilibrium exists if (1) there are gains from trade, (2) a feasibility condition holds, and (3) either a weak declining contribution condition holds, or a stronger declining contribution holds for one side of the market only. We believe that these conditions are satisfied in many bilateral bargaining environments considered in applied settings. We also show that the feasibility condition is necessary for a Nash-in-Nash limit equilibrium to exist. Our second set of results proves that the Nash-in-Nash outcome is unique for any no-delay equilibrium. Under stronger conditions on profits, it is also unique for any common tie-breaking equilibrium.

Instead of requiring that firms do not coordinate across multiple negotiations, our extensive form allows for firms to engage in deviations across multiple negotiations. We believe that our extensive form and profit assumptions reasonably capture aspects of bargaining protocols and firm competition in real-world industry settings. We further believe that our results provide justification for the use of the Nash-in-Nash solution as a credible bargaining framework for use in applied work.

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A Formal Results for the Delegated Agent Model

In this section, we present formal results for the *delegated agent model*, discussed in Section 2.3. The extensive form for this representation involves separate bilateral negotiations between delegated agents, or representatives, of each firm.[32] We develop this model and show that this representation also admits the Nash-in-Nash bargaining solution as an equilibrium outcome if A.GFT holds.

Our model is as follows. For every negotiation $ij \in \mathcal{G}$, U_i and D_j send individual representatives, denoted as $\mathcal{U}^{ij}$ and $\mathcal{D}^{ij}$, who engage in the alternating-offers bargaining protocol of Binmore, Rubinstein, and Wolinsky (1986), where negotiation breakdowns are independent across negotiations and profits are realized once all negotiations have concluded or broken down. Each representative seeks to maximize her firm's total expected profits across all bargains. However, she does not know the outcome of any other bilateral bargain until her own bargain has concluded or broken down. One interpretation is that each pair of agents for a negotiation are sequestered in separate bargaining rooms, and no one outside the room knows the status of the bargain until it has concluded or broken down.

Under these conditions, we show that a Nash-in-Nash limit equilibrium exists:

Theorem A.1 *Assume A.GFT and that every negotiation $ij \in \mathcal{G}$ is conducted by delegated agents from $\mathcal{U}_i$ and $\mathcal{D}_j$. Then there exists an equilibrium of the* delegated agent model *where all agreements $ij \in \mathcal{G}$ are immediately formed at prices $\hat{p}_{ij} = p^R_{ij,D}$ ($p^R_{ij,U}$) if t_0 is odd (even).*

Proof. Let the delegated agents employ the following candidate set of strategies: $\mathcal{U}^{ij}$ offers $p^R_{ij,U}$ in even periods and only accepts offers equal to or above $p^R_{ij,D}$ in odd periods; $\mathcal{D}^{ij}$ offers $p^R_{ij,D}$ in odd periods, and accepts offers equal to or below $p^R_{ij,U}$ in even periods.

Given passive beliefs, when an agent sees an off-equilibrium-path action by one party, it still perceives that the other parties are following their equilibrium actions.[33] Thus, if a delegated agent—e.g., $\mathcal{D}^{ij}$—sees a deviation from the above strategies by its rival—$\mathcal{U}^{ij}$ in this case—then it perceives that all other negotiations (which are all run by separate delegated agents) follow equilibrium strategies. Hence, both $\mathcal{D}^{ij}$ and $\mathcal{U}^{ij}$ assume that all other agreements immediately form regardless of what occurs in their respective bargain. By Binmore, Rubinstein, and Wolinsky (1986), the above strategies then comprise the unique equilibrium for each ij negotiation. $\square$

Note that this does not necessarily imply that there is a unique equilibrium for the delegated agent model. Indeed, consider our three supplier counterexample from Section 3.1. In this setting, if every negotiation is conducted by delegated agents by the downstream manufacturer, and each agent believes that all other agreements will form at Rubinstein prices (i.e., at prices $\delta/(1+\delta)$ it t_0 odd), then their agreement will also form at this price. However, there exists another equilibrium in which no agreements are ever formed: e.g., if each delegated agent for the downstream manufacturer believes that no other agreements will form, then upstream suppliers always demanding $p_{i1,U} > 0$ in each even period (and the manufacturer rejecting any offer greater than 0), and the manufacturer offering a price of $p_{i1,D} < 0$ in each odd period (and suppliers rejecting any offer less than 0) comprise equilibrium strategies.

While we do not prove that there exists a unique equilibrium of the delegated agent model, the proof of Theorem A.1 nevertheless implies that conditional on all agreements in $\mathcal{G}$ forming, the equilibrium outcome of agreements all forming at Rubinstein prices is unique. This result is the analog of Theorem 4.3.

B Proofs of Lemmas from Main Text

Proof of Lemma 2.2 Using l'Hospital's rule:

$$\lim_{\Lambda \to 0} \frac{\delta_{i,U}(1-\delta_{j,D})}{1-\delta_{i,U}\delta_{j,D}} = \lim_{\Lambda \to 0} \frac{e^{-r_{i,U}\Lambda}(1-e^{-r_{j,D}\Lambda})}{1-e^{-(r_{i,U}+r_{j,D})\Lambda}} = \frac{r_{j,D}}{r_{i,U}+r_{j,D}} \ ,$$

[32] Chipty and Snyder (1999) (see footnote 10) provides a sketch of this argument in the context of a single supplier negotiating with multiple buyers.

[33] This property holds for sequential equilibria (without public signals), and not just for perfect Bayesian equilibria with passive beliefs.

and

$$\lim_{\Lambda\to 0}\frac{1-\delta_{j,D}}{1-\delta_{i,U}\delta_{j,D}}=\lim_{\Lambda\to 0}\frac{1-e^{-r_{j,D}\Lambda}}{1-e^{-(r_{i,U}+r_{j,D})\Lambda}}=\frac{r_{j,D}}{r_{i,U}+r_{j,D}}\ .$$

Similarly,

$$\lim_{\Lambda\to 0}\frac{\delta_{j,D}(1-\delta_{i,U})}{1-\delta_{i,U}\delta_{j,D}}=\lim_{\Lambda\to 0}\frac{(1-\delta_{i,U})}{1-\delta_{i,U}\delta_{j,D}}=\frac{r_{i,U}}{r_{i,U}+r_{j,D}}\ ,$$

which proves the lemma. □

Proof of Lemma 2.3 Assume A.GFT. For any $ij\in\mathcal{G}$, since $0<\delta_{i,U}<1$ and $0<\delta_{j,D}<1$, note:

$$(\Delta\pi_{j,D}(\mathcal{G},\{ij\})-p^R_{ij,D})=\underbrace{\frac{(1-\delta_{i,U})}{(1-\delta_{i,U}\delta_{j,D})}}_{>0}\underbrace{(\Delta\pi_{j,D}(\mathcal{G},\{ij\})+\Delta\pi_{i,U}(\mathcal{G},\{ij\}))}_{>0\text{ by A.GFT}}\ .$$

Thus $\Delta\pi_{j,D}(\mathcal{G},\{ij\})>p^R_{ij,D}$. Also, note:

$$(\Delta\pi_{i,U}(\mathcal{G},\{ij\})+p^R_{ij,U})=\underbrace{\frac{(1-\delta_{j,D})}{(1-\delta_{i,U}\delta_{j,D})}}_{>0}\underbrace{(\Delta\pi_{j,D}(\mathcal{G},\{ij\})+\Delta\pi_{i,U}(\mathcal{G},\{ij\}))}_{>0\text{ by A.GFT}}\ .$$

Thus $\Delta\pi_{i,U}(\mathcal{G},\{ij\})>-p^R_{ij,U}>$. Adding the previous two inequalities and rearranging, we obtain:

$$\begin{aligned}p^R_{ij,U}-p^R_{ij,D}&=\Big(\Delta\pi_{j,D}(\mathcal{G},\{ij\})+\Delta\pi_{i,U}(\mathcal{G},\{ij\})\Big)\Big(\frac{(1-\delta_{i,U})}{(1-\delta_{i,U}\delta_{j,D})}+\frac{(1-\delta_{j,D})}{(1-\delta_{i,U}\delta_{j,D})}-1\Big)\\&=\frac{1}{1-\delta_{i,U}\delta_{j,D}}\Big(\Delta\pi_{j,D}(\mathcal{G},\{ij\})+\Delta\pi_{i,U}(\mathcal{G},\{ij\})\Big)\Big(1-\delta_{i,U}-\delta_{j,D}+\delta_{j,D}\delta_{i,U}\Big).\end{aligned}$$

Again, all three terms on the second line are positive; thus $p^R_{ij,U}>p^R_{ij,D}$. Finally, substituting in the definition of $\delta_{i,U}$ and $\delta_{j,D}$ into the definition of $p^R_{ij,D}$, it is straightforward to show $\partial p^R_{ij,D}(\Lambda)/\partial\Lambda<0\ \forall\ \Lambda>0$; thus, as $lim_{\Lambda\to 0}p^R_{ij,D}=p^{Nash}_{ij}$ by Lemma 2.2, it follows that $p^{Nash}_{ij}>p^R_{ij,D}$. A similar approach can be used to show that $p^{Nash}_{ij}<p^R_{ij,U}$. □

Proof of Lemma 3.6 Assume A.GFT. We prove the lemma using the following four claims:

1. A.SCDMC $\Rightarrow$ A.WCDMC

   We prove A.SCDMC(b) (for downstream firms) implies A.WCDMC holds for downstream firms; the proof that A.SCDMC(a) (for upstream firms) implies A.WCDMC holds for upstream firms is symmetric and omitted.

   A.SCDMC(b) states: $\pi_{j,D}(\mathcal{A}\cup\mathcal{B}\cup\{ij\})-\pi_{j,D}(\mathcal{A}'\cup\mathcal{B})\geq\Delta\pi_{j,D}(\mathcal{G},\{ij\})$ for all $ij\in\mathcal{G}$, $\mathcal{B}\subseteq\mathcal{G}_{-i,U}$, and $\mathcal{A},\mathcal{A}'\subseteq\mathcal{G}_{i,U}\setminus\{ij\}$. For the case where $\mathcal{A}=\mathcal{A}'$, A.SCDMC implies:

   $$\Delta\pi_{j,D}(\mathcal{A}\cup\mathcal{B},\{ij\})\geq\Delta\pi_{j,D}(\mathcal{G},\{ij\})\qquad\forall\ ij\in\mathcal{G},\mathcal{B}\subseteq\mathcal{G}_{-i,U},\mathcal{A}\subseteq\mathcal{G}_{i,U}\setminus\{ij\}\ .\tag{5}$$

   Index agreements in $\mathcal{A}$ from $k=1,\cdots,|\mathcal{A}|$, and let a_k represent the kth agreement in $\mathcal{A}$. This allows us to create a sequence of sets of agreements, starting at $\mathcal{B}\equiv\mathcal{G}\setminus\mathcal{A}$, in which we add in each agreement one at a time, given by $\mathcal{D}_0\equiv\mathcal{B}$, and $\mathcal{D}_k=\mathcal{D}_{k-1}\cup\{a_k\}$ for $k=1,\cdots,|\mathcal{A}|$. Then, note that for any $\mathcal{A}\subseteq\mathcal{G}_{j,D}$:

   $$\begin{aligned}\Delta\pi_{j,D}(\mathcal{G},\mathcal{A})=\Delta\pi_{j,D}(\mathcal{A}\cup\mathcal{B},\mathcal{A})&=\sum_{k=1}^{|\mathcal{A}|}\Delta\pi_{j,D}(\mathcal{D}_k,\{a_k\})\\&\geq\sum_{k=1}^{|\mathcal{A}|}\Delta\pi_{j,D}(\mathcal{G},\{a_k\})=\sum_{kj\in\mathcal{A}}\Delta\pi_{j,D}(\mathcal{G},\{kj\}),\end{aligned}$$

where the last equality on the first line follows from the index for agreements for $\mathcal{A}$, and the inequality on the second line follows from (5). This coincides with the statement of A.WCDMC for downstream firms.

2. A.WCDMC $\Rightarrow$ A.FEAS

   For all $ij \in \mathcal{G}$ and $\mathcal{A} \subseteq \mathcal{G}_{j,D}$:

$$\Delta\pi_{j,D}(\mathcal{G},\mathcal{A}) - \sum_{ij\in\mathcal{A}} p_{ij}^{Nash} \geq \sum_{kj\in\mathcal{A}} \left(\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - \frac{b_{i,U}\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - b_{j,D}\Delta\pi_{i,U}(\mathcal{G},\{ij\})}{b_{i,U}+b_{j,D}}\right)$$
$$= \sum_{kj\in\mathcal{A}} \left(\Big(\Delta\pi_{j,D}(\mathcal{G},\{ij\}) + \Delta\pi_{i,U}(\mathcal{G},\{ij\})\Big)\frac{b_{j,D}}{b_{i,U}+b_{j,D}}\right) > 0\ ,$$

   where the first inequality follows from A.WCDMC (for downstream firms) and the definition of p_{ij}^{Nash}, and the second line is positive by A.GFT. Thus $\Delta\pi_{j,D}(\mathcal{G},\mathcal{A}) > \sum_{ij\in\mathcal{A}} p_{ij}^{Nash}$, and A.FEAS holds. The proof for upstream firms is symmetric and omitted.

3. A.WCDMC $\not\Rightarrow$ A.SCDMC

   Consider a single downstream firm and three upstream suppliers. Suppose that the downstream firm profits are: 0 without any supplier, 0.25 with one supplier, 0.7 with two suppliers, and 1 with all three suppliers; assume supplier profits are always 0. This example violates A.SCDMC(b) given by (5) because the surplus to the downstream firm from having one supplier (0.25) is less than the surplus from adding the third supplier (0.3). But, it does not violate A.WCDMC, because removing two or three suppliers both result in a greater loss than the sum of the marginal values (0.75 versus 0.6 when removing two suppliers, and 1 versus 0.9 when removing all three).

4. A.FEAS $\not\Rightarrow$ A.WCDMC

   The two automobile supplier example in the paper discussed in Section 3.1 satisfies A.FEAS but not A.WCDMC when $0 \leq a < 0.5$.

□

C Proof of Theorems on Existence

C.1 Proof of Theorem 3.4

We proceed by contradiction: assume that a Nash-in-Nash limit equilibrium exists where all agreements form immediately, but A.FEAS does not hold, so that for some D_j, there exists $\mathcal{A} \subseteq \mathcal{G}_{j,D}$ such that $\Delta\pi_{j,D}(\mathcal{G},\mathcal{A}) < \sum_{ij\in\mathcal{A}} p_{ij}^{Nash}$; the proof if A.FEAS is violated for some U_i is symmetric and omitted. Let $\varepsilon = \frac{1}{|\mathcal{A}|}\Big((\sum_{ij\in\mathcal{A}} p_{ij}^{Nash}) - \Delta\pi_{j,D}(\mathcal{G},\mathcal{A})\Big)$, which is positive by assumption. By the contradictory assumption, for any t_0, there exists $\bar{\Lambda} > 0$ such that for all $\Lambda \in (0,\bar{\Lambda}]$, there is an equilibrium where all agreements in $\mathcal{G}$ form at t_0 at prices $\{p_{ij}^*\}_{ij\in\mathcal{G}}$, where $|p_{ij}^* - p_{ij}^{Nash}| < \varepsilon$ for all $ij \in \mathcal{G}$. Assume that t_0 is even, and fix $\Lambda \in (0,\bar{\Lambda}]$. Consider the following multi-period deviation: D_j rejects offers $ij \in \mathcal{A}$ at t_0 and every subsequent even period and proposes offers that are sufficiently low that they will be rejected in odd periods. By assumption, in this equilibrium, all agreements in $\mathcal{G}\setminus\mathcal{A}$ will still form at t_0. Thus, this deviation (where agreements in $\mathcal{A}$ never form) will increase D_j's payoffs by:

$$\Big(\sum_{ij\in\mathcal{A}} p_{ij}^*\Big) - \Delta\pi_{j,D}(\mathcal{G},\mathcal{A}) > \Big(\sum_{ij\in\mathcal{A}} (p_{ij}^{Nash} - \varepsilon)\Big) - \Delta\pi_{j,D}(\mathcal{G},\mathcal{A})$$
$$= \Big((\sum_{ij\in\mathcal{A}} p_{ij}^{Nash}) - \Delta\pi_{j,D}(\mathcal{G},\mathcal{A})\Big) - |\mathcal{A}| \times \varepsilon$$
$$= \Big((\sum_{ij\in\mathcal{A}} p_{ij}^{Nash}) - \Delta\pi_{j,D}(\mathcal{G},\mathcal{A})\Big) - \Big((\sum_{ij\in\mathcal{A}} p_{ij}^{Nash}) - \Delta\pi_{j,D}(\mathcal{G},\mathcal{A})\Big) = 0$$

where the inequality on the first line follows from the definition of p^*_{ij}, the second line rearranges terms, and the third line follows from substituting in the definition of ε. Thus, this deviation is profitable, yielding a contradiction. $\square$

C.2 Proof of Theorem 3.2 (Sufficiency Only) and Theorem 3.7

Assume A.GFT and *either* (i) A.WCDMC *or* (ii) A.FEAS and either A.SCDMC(a) or A.SCDMC(b). For condition set (ii), we detail the proof where A.SCDMC(a) (for upstream firms) holds; the proof when A.SCDMC(b) holds is symmetric and omitted.

We first detail our candidate equilibrium strategies:

- In every odd period, each D_j makes offers $p^R_{ij,D}$ to all firms U_i with which it has not already formed an agreement. If all price offers that it receives are equal to $p^R_{ij,D}$, U_i accepts all offers. If U_i receives exactly one non-equilibrium offer from some D_j, it accepts all other offers and rejects D_j's offer if and only if the offer is lower than $p^R_{ij,D}$. Finally, if U_i receives multiple non-equilibrium offers, it plays an arbitrary best response in its acceptance decision, respecting passive beliefs.
- In every even period with open agreements given by $\mathcal{C}$, each U_i makes offers $p_{ij,U}(\mathcal{C})$ (defined below) to all firms $D_j \in \mathcal{C}_{i,U}$. If all price offers that it receives are equal to $p_{ij,U}(\mathcal{C})$, D_j accepts all offers. If D_j receives exactly one non-equilibrium offer from some U_i and that offer is lower than $p_{ij,U}(\mathcal{C})$, then D_j still accepts all offers. If D_j receives exactly one non-equilibrium offer from some U_i and that offer is higher than $p_{ij,U}(\mathcal{C})$, then: (i) under A.WCDMC, D_j rejects U_i's offer and accepts all other offers; (ii) under A.FEAS and A.SCDMC, D_j rejects U_i's offer and plays an arbitrary best response in its acceptance decision with other offers (respecting passive beliefs). If D_j receives multiple non-equilibrium offers, it plays an arbitrary best response in its acceptance decision, respecting passive beliefs.

The prescribed strategy profile dictates that every firm makes proposals that are Rubinstein prices in odd periods, and may differ from Rubinstein prices in even periods. On the equilibrium path, all offers are accepted regardless of whether the period is odd or even. For off-equilibrium offers, acceptance decisions will depend on the exact set of offers that are made.

Construction of Even Period ($p_{ij,U}(\mathcal{C})$) Prices. We now define candidate even-period equilibrium pricing strategies $p_{ij,U}(\mathcal{C})$ iteratively as follows. For each set of open agreements $\mathcal{C} \subseteq \mathcal{G}$, consider the constraints:

$$\underbrace{\sum_{ij\in\mathcal{B}} p_{ij,U}(\mathcal{C})}_{LS} \leq \underbrace{(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{B}) + \delta_{j,D}\sum_{ij\in\mathcal{B}} p^R_{ij,D}}_{RS} \qquad \forall j \text{ s.t. } \mathcal{C}_{j,D} \neq \emptyset, \forall \mathcal{B} \subseteq \mathcal{C}_{j,D}. \tag{6}$$

where the constraint ensures that each downstream firm D_j with open agreements in $\mathcal{C}$ wishes to accept prices $p_{ij,U}(\mathcal{C})$ for any subset of agreements $\mathcal{B} \subseteq \mathcal{C}_{j,D}$ at an even period as opposed to forming those agreements in the next period at Rubinstein prices $p^R_{ij,D}$.

Step 1. Initialize $p_{ij,U}(\mathcal{C}) = p^R_{ij,D}, \forall ij$. At these values, the constraints specified by (6) are strictly satisfied:

$$\begin{aligned}\underbrace{(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{B}) + \delta_{j,D}\sum_{ij\in\mathcal{B}} p^R_{ij,D}}_{RS} &\geq (1-\delta_{j,D})\sum_{ij\in\mathcal{B}} p^{Nash}_{ij} + \delta_{j,D}\sum_{ij\in\mathcal{B}} p^R_{ij,D} \\ &> \underbrace{(1-\delta_{j,D})\sum_{ij\in\mathcal{B}} p^R_{ij,D} + \delta_{j,D}\sum_{ij\in\mathcal{B}} p^R_{ij,D}}_{=LS},\end{aligned}$$

where the inequality on the first line follows from A.FEAS, and the second line from Lemma 2.3.

Step 2. Now, for each set of open agreements $\mathcal{C}$, fix an arbitrary ordering over agreements within that set. Start with the first open agreement $ij \in \mathcal{C}$, and increase $p_{ij,U}(\mathcal{C})$ until (at least) one of the constraints given by (6) binds. Move on to the second open agreement, and do the same. Continue through all the open agreements in $\mathcal{C}$. Define the candidate set of equilibrium offers $p_{ij,U}(\mathcal{C})$ to be the offers resulting from this process. For these prices, all constraints (6) still hold. Moreover, by construction, at these prices each open agreement $ij \in \mathcal{C}$ has at least one constraint (6) that binds.

Next, we prove the following supporting Lemma.

Lemma C.1 *Candidate equilibrium prices satisfy the following properties:*

1. $p_{ij,U}(\mathcal{C}) \geq p^R_{ij,D}, \forall ij \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$.
2. $p_{ij,U}(\mathcal{C}) \leq p^R_{ij,U}, \forall ij \in \mathcal{C}, \mathcal{C} \subseteq \mathcal{G}$.
3. $p_{ij,U}(\{ij\}) = p^R_{ij,U}, \forall ij \in \mathcal{G}$.
4. *Assume A.WCDMC. Then* $p_{ij,U}(\mathcal{C}) = p^R_{ij,U}, \forall ij \in \mathcal{C}, \forall \mathcal{C} \subseteq \mathcal{G}$.
5. *All candidate equilibrium prices converge to Nash-in-Nash prices as* $\Lambda \to 0$.

Proof. We prove that each property holds in turn.

1. This follows directly from the iterative procedure: we start with $p^R_{ij,D}$ and then weakly increase prices to arrive at $p_{ij,U}(\mathcal{C})$.
2. By (6), $p_{ij,U}(\mathcal{C}) \leq (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G}, ij) + \delta_{j,D} p^R_{ij,D} = p^R_{ij,U}$.
3. By construction, $p_{ij,U}(\{ij\}) = (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G}, \{ij\}) + \delta_{j,D} p^R_{ij,D} = p^R_{ij,U}$, with the equality following again from (6).
4. Suppose, by contradiction, that A.WCDMC holds but $\exists \mathcal{C} \subseteq \mathcal{G}$ and $lm \in \mathcal{C}$ such that $p_{lm,U}(\mathcal{C}) < p^R_{lm,U}$ (Claim 2 rules out the inequality in the other direction). Then, by the construction of $p_{lm,U}(\mathcal{C})$ there must exist $\mathcal{B} \subseteq \mathcal{C}, lm \in \mathcal{B}$ for which the constraint in (6) binds. Using this $\mathcal{B}$, we arrive at a contradiction:

$$\begin{aligned}
\sum_{ij \in \mathcal{B}} p^R_{ij,U} &> \sum_{ij \in \mathcal{B}} p_{ij,U}(\mathcal{C}) \\
&= (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G}, \mathcal{B}) + \delta_{j,D} \sum_{ij \in \mathcal{B}} p^R_{ij,D} \\
&\geq (1-\delta_{j,D}) \sum_{ij \in \mathcal{B}} \Delta\pi_{j,D}(\mathcal{G}, ij) + \delta_{j,D} \sum_{ij \in \mathcal{B}} p^R_{ij,D} = \sum_{ij \in \mathcal{B}} p^R_{ij,U},
\end{aligned}$$

   where the first line follows from Claim 2 and the contradictory assumption, the second line from our choice of $\mathcal{B}$, the third line inequality from A.WCDMC, and the final equality from (1).
5. In odd periods, offers are $p^R_{ij,D}$ which converge to Nash-in-Nash prices as $\Lambda \to 0$ by Lemma 2.2. In even periods, offers satisfy $p^R_{ij,D} \leq p_{ij,U}(\mathcal{C}) \leq p^R_{ij,U}$. Since the offers lie between two sets of offers that converge to Nash-in-Nash prices, by the sandwich theorem, they also converge to Nash-in-Nash prices. □

We now prove the main result.

Lemma C.2 *The candidate strategies described above comprise an equilibrium.*

Proof. We now prove that no unilateral deviation is profitable on the part of any firm. Consider any period t where there are $\mathcal{C} \subseteq \mathcal{G}$ open agreements.

Upstream firm, t odd. Consider first an upstream firm U_i's decision of which offers to accept at an odd period t given it receives offers $\{\tilde{p}_{ij}\}_{ij\in\mathcal{C}_{i,U}}$. If U_i engages in a one-shot deviation by rejecting a subset $\mathcal{K} \subseteq \mathcal{C}_{i,U}$ of its open agreements, U_i expects that: (i) all other upstream firms will accept all of their open agreements at period t; and (ii) all non-accepted agreements $\mathcal{K}$ will form at period $t+1$ at prices $p_{ij,U}(\mathcal{K})$ (as we only consider one-shot deviations, and play is expected to follow the prescribed equilibrium strategies from $t+1$ onwards). We define the increase in U_i's payoffs from rejecting agreements $\mathcal{K} \subseteq \mathcal{C}_{i,U}$ (and following equilibrium strategies thereafter) as:

$$F(\mathcal{K}) \equiv -\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\mathcal{K}) + \sum_{ij\in\mathcal{K}}\left[\tilde{p}_{ij} - \delta_{i,U}p_{ij,U}(\mathcal{K})\right]\right], \tag{7}$$

where we omit the fact that F is implicitly a function of the firm U_i, the set of open agreements $\mathcal{C}$, the set of candidate equilibrium strategies employed by other firms, and the set of prices that U_i receives. U_i chooses to reject agreements $\hat{\mathcal{K}} = \arg\max_{\mathcal{K}\subseteq\mathcal{C}_{i,U}} F(\mathcal{K})$, as this set of rejections maximizes its payoffs.

Consider first the case where U_i receives candidate equilibrium offers $\{p^R_{ij,D}\}_{ij\in\mathcal{C}_{i,U}}$. Substituting these prices into (7), we obtain:

$$\begin{aligned}
F(\mathcal{K}) &= -\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\mathcal{K}) + \sum_{ij\in\mathcal{K}}\left[p^R_{ij,D} - \delta_{i,U}p_{ij,U}(\mathcal{K})\right]\right] \\
&\leq -\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\mathcal{K}) + \sum_{ij\in\mathcal{K}}\left[p^R_{ij,D} - \delta_{i,U}p^R_{ij,U}\right]\right] \\
&\leq -\left[\sum_{ij\in\mathcal{K}}\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + p^R_{ij,D} - \delta_{i,U}p^R_{ij,U}\right]\right] = 0\ ,
\end{aligned} \tag{8}$$

where the second line uses Lemma C.1(2) ($p_{ij,U}(\mathcal{K}) \leq p^R_{ij,U}$), the third line inequality follows from A.WCDMC or A.SCDMC (a), and the third line equality uses (2). Since $F(\emptyset) = 0$, $F(\mathcal{K})$ is maximized for $\mathcal{K} = \emptyset$. This implies that at equilibrium prices, U_i maximizes surplus by rejecting no offer, or equivalently, accepting all offers. Thus, in this case, U_i cannot gain by deviating from its candidate equilibrium strategy.

Consider next the case where U_i receives exactly one non-equilibrium offer, $\tilde{p}_{ij} \neq p^R_{ij,D}$. In this case, the increase in U_i's payoffs from rejecting agreements $\mathcal{K} \subseteq \mathcal{C}_{i,U}$ can be expressed as:

$$\begin{aligned}
F(\mathcal{K}) &= \underbrace{-\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\mathcal{K}) + \sum_{kj\in\mathcal{K}}\left[p^R_{kj,D} - \delta_{k,U}p_{kj,U}(\mathcal{K})\right]\right]}_{\leq 0 \text{ by (8)}} + 1_{\mathcal{K}}(\{ij\})(p^R_{ij,D} - \tilde{p}_{ij}) \\
&\leq 1_{\mathcal{K}}(\{ij\})(p^R_{ij,D} - \tilde{p}_{ij})\ ,
\end{aligned}$$

where $1_{\mathcal{K}}(\{ij\})$ is an indicator function for $ij \in \mathcal{K}$. Note that the increase in U_i's profits from rejecting only agreement ij can be expressed as:

$$\begin{aligned}
F(\{ij\}) &= -\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + \tilde{p}_{ij} - \delta_{i,U}p_{ij,U}(\{ij\})\right] \\
&= \underbrace{-\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + p^R_{ij,D} - \delta_{i,U}p^R_{ij,U}\right]}_{=0 \text{ by (2)}} + (p^R_{ij,D} - \tilde{p}_{ij}) \\
&= (p^R_{ij,D} - \tilde{p}_{ij})\ ,
\end{aligned}$$

where the second line follows from an application of Lemma C.1(3) ($p_{ij,U}(\{ij\}) = p^R_{ij,U}$) and a re-arranging of terms. Thus, if $\tilde{p}_{ij} \geq p^R_{ij,D}$, $F(\mathcal{K})$ is again maximized for $\mathcal{K} = \emptyset$ (accept all offers); if $\tilde{p}_{ij} < p^R_{ij,D}$, $F(\mathcal{K})$ is maximized for $\mathcal{K} = \{ij\}$ (reject only D_j's offer). Hence, upon receiving one non-equilibrium offer, U_i does not have a profitable deviation from its prescribed equilibrium strategy.

Finally, in cases with multiple out-of-equilibrium offers, the strategy profile specified above states that U_i picks an arbitrary $\hat{\mathcal{K}}$ that maximizes $F(\mathcal{K})$. By definition then, this is a best response and no unilateral deviation is profitable. Note that receiving multiple out-of-equilibrium offers is unreachable from candidate equilibrium play by any unilateral deviation.

Downstream firm, t even. Next consider a downstream firm D_j's decision of which offers to accept at an even period t given it receives offers $\{\tilde{p}_{ij}\}_{ij \in \mathcal{C}_{j,D}}$. If D_j engages in a one-shot deviation by rejecting a subset $\mathcal{K} \subseteq \mathcal{C}_{j,D}$ of its open agreements, D_j expects that: (i) all other downstream firms will accept all of their open agreements at period t; and (ii) all non-accepted offers $\mathcal{K}$ will form agreement at period $t+1$ at Rubinstein prices $p^R_{ij,D}$. We define the increase in D_j's payoffs from rejecting agreements $\mathcal{K} \subseteq \mathcal{C}_{j,D}$ (and following equilibrium strategies thereafter) as:

$$F(\mathcal{K}) \equiv -\left[(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{K}) + \sum_{ij\in\mathcal{K}} \left[-\tilde{p}_{ij} + \delta_{j,D} p^R_{ij,D}\right]\right], \tag{9}$$

where we omit the fact that F is implicitly a function of the firm D_j, the set of open agreements $\mathcal{C}$, the set of candidate equilibrium strategies employed by other firms, and the set of prices that D_j receives. D_j chooses to reject agreements $\hat{\mathcal{K}} = \arg\max_{\mathcal{K} \subseteq \mathcal{C}_{j,D}} F(\mathcal{K})$, as this set of rejections maximizes its payoffs.

Consider first the case where D_j receives candidate equilibrium offers $\{p_{ij,U}(\mathcal{C})\}_{ij\in\mathcal{C}_{j,D}}$. Substituting these prices into (9) and then applying (6), we obtain:

$$F(\mathcal{K}) = -\left[(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{K}) + \sum_{ij\in\mathcal{K}} \left[-p_{ij,U}(\mathcal{C}) + \delta_{j,D} p^R_{ij,D}\right]\right] \leq 0. \tag{10}$$

Since $F(\emptyset) = 0$, $F(\mathcal{K})$ is maximized for $\mathcal{K} = \emptyset$. This implies that at equilibrium prices, D_j maximizes surplus by rejecting no offer, or equivalently, accepting all offers. Thus, in this case, D_j cannot gain by deviating from its candidate equilibrium strategy.

Consider next the case where D_j receives exactly one non-equilibrium offer, $\tilde{p}_{ij} \neq p_{ij,U}(\mathcal{C})$. In this case, the increase in D_j's payoffs from rejecting agreements $\mathcal{K} \subseteq \mathcal{C}_{j,D}$ can be expressed as:

$$\begin{aligned} F(\mathcal{K}) &= \underbrace{-\left[(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{K}) + \sum_{ik\in\mathcal{K}} \left[-p_{ik,U}(\mathcal{C}) + \delta_{j,D} p^R_{ik,D}\right]\right]}_{\leq 0 \text{ by } (10)} + 1_{\mathcal{K}}(\{ij\})(-p_{ij,U}(\mathcal{C}) + \tilde{p}_{ij}) \\ &\leq 1_{\mathcal{K}}(\{ij\})(-p_{ij,U}(\mathcal{C}) + \tilde{p}_{ij}) \ . \end{aligned} \tag{11}$$

From (11), if $\tilde{p}_{ij} < p_{ij,U}(\mathcal{C})$, then there is no profitable deviation from the candidate equilibrium strategy of accepting all offers (as $F(\mathcal{K})$ is maximized at $\mathcal{K} = \emptyset$). If $\tilde{p}_{ij} \geq p_{ij,U}(\mathcal{C})$, there are two cases to consider:

1. Under A.WCDMC, by Lemma C.1(4), $p_{ij,U}(\mathcal{C}) = p^R_{ij,U}$. By substituting this in to (11) and then applying (2), the inequality can be shown to be an equality for $\mathcal{K} = \{ij\}$; thus, $F(\mathcal{K})$ is maximized for $\mathcal{K} = \{ij\}$ (reject only U_i's offer).

2. Otherwise, note first that if D_j rejects some set of offers $\mathcal{K}$ that does not include ij, then $F(\mathcal{K}) \leq 0$ from (11); if D_j rejects no offers, $F(\emptyset) = 0$. By rejecting a set of offers $\mathcal{K}$ that includes ij and all other offers which are included in some constraint from (6) that binds, $F(\mathcal{K}) = p_{ij,U}(\mathcal{C}) - \tilde{p}_{ij} > 0$. Thus, D_j's best response must include rejecting U_i's offer, and potentially includes rejecting other offers.

Again, there are no profitable deviations from prescribed strategies.

Finally, in cases with multiple out-of-equilibrium offers, the strategy profile specified above states that

D_j picks an arbitrary $\hat{\mathcal{K}}$ that maximizes $F(\mathcal{K})$. By definition, this is a best response and no unilateral deviation is profitable.

Upstream firm, t even. Next, we consider the decision for an upstream firm U_i of what offers to propose at an even period t. Consider the possibility that U_i deviates from the candidate equilibrium strategies and offers prices $\tilde{p}_{ij} \neq p_{ij,U}(\mathcal{C})$ for all ij in some $\mathcal{K} \subseteq \mathcal{C}_{i,U}$. By passive beliefs, each firm D_j receiving $\tilde{p}_{ij}$ perceives that it is the only one to have received an out-of-equilibrium offer. Given the above discussion regarding D's strategies in this case, if $\tilde{p}_{ij} < p_{ij,U}(\mathcal{C})$, then it will be accepted, while if $\tilde{p}_{ij} > p_{ij,U}(\mathcal{C})$, then it will be rejected, potentially along with some other offers kj. Clearly, U_i will never choose to offer $\tilde{p}_{ij} < p_{ij,U}(\mathcal{C})$, since it can always offer $p_{ij,U}(\mathcal{C})$ instead, without affecting the set of acceptances.

Thus, the only possible profitable deviation left is for U_i to offer $\tilde{p}_{ij} > p_{ij,U}(\mathcal{C})$ for all ij in $\mathcal{K}$. Given the candidate equilibrium strategies, these offers will be rejected at period t, and then accepted at $t+1$ at prices $p^R_{ij,D}$. Let $a_1, \ldots, a_{|\mathcal{K}|}$ denote the downstream firms with offers in $\mathcal{K}$ and let C_k denote the set of offers rejected by D_{a_k} following its deviant offer from U_i. Given the downstream firms' strategies, it will be the case each downstream firm D_{a_k} will reject ia_k, and $ia_k \in C_k$. Note that under A.WCDMC, given the prescribed equilibrium strategy profiles each downstream firm D_{a_k} will *only* reject ia_k, and $C_k = \{ia_k\}$; under A.FEAS, a downstream firm may also reject additional agreements.

Then, the decrease in U_i's surplus from raising the price on some set of offers $\mathcal{K} \subseteq \mathcal{C}_{i,U}$ is:

$$
\begin{aligned}
&\sum_{ij\in\mathcal{K}} [p_{ij,U}(\mathcal{C}) - \delta_{i,U} p^R_{ij,D}] + (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G}, C_1 \cup \ldots \cup C_{|K|}) \qquad (12)\\
&\quad = \sum_{ij\in\mathcal{K}} \left[p_{ij,U}(\mathcal{C}) - \delta_{i,U} p^R_{ij,D}\right] + (1-\delta_{i,U}) \sum_{k=1}^{|K|} \Delta\pi_{i,U}(\mathcal{G} \setminus (C_{k+1} \cup \ldots \cup C_{|K|}), C_k)\\
&\quad \geq \sum_{ij\in\mathcal{K}} \left[p_{ij,U}(\mathcal{C}) - \delta_{i,U} p^R_{ij,D}\right] + (1-\delta_{i,U}) \sum_{k=1}^{|K|} \Delta\pi_{i,U}(\mathcal{G}, \{ia_k\})\\
&\quad = \sum_{ij\in\mathcal{K}} \left[p_{ij,U}(\mathcal{C}) - \delta_{i,U} p^R_{ij,D} + (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G}, ij)\right]\\
&\quad \geq \sum_{ij\in\mathcal{K}} \left[p^R_{ij,D} - \delta_{i,U} p^R_{ij,U} + (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G}, ij)\right] = 0 \ ,
\end{aligned}
$$

implying that such a deviation is not profitable. In (12), the second term of the second line splits the change in surplus from the postponed agreements by downstream firms. The third line then either follows directly from A.WCDMC (as $C_k = \{ia_k\}$), or applies A.SCDMC to each element of the second term, which can be done since each element can be expressed as the difference in U_i's profits between when D_{a_k} accepts all its offers and when it rejects both U_i's offer and the other offers in C_k, holding constant the fact that all other agreements are formed except for $C_{k+1} \cup \ldots C_{|K|}$.[34] The fourth line rearranges terms. The final line inequality follows from Lemma C.1(1) and Lemma 2.3, and the last equality from (2).

Downstream firm, t odd. Finally, we consider the decision for a downstream firm D_j of what offers to propose at an odd period t. Consider the possibility that D_j deviates from the candidate equilibrium strategies and offers prices different from $\tilde{p}_{ij} \neq p^R_{ij,D}$ for all ij in some $\mathcal{K} \subseteq \mathcal{C}_{j,D}$. By passive beliefs, each firm U_i receiving $\tilde{p}_{ij}$ perceives that it is the only one to have received an out-of-equilibrium offer. Given candidate equilibrium strategies, if $\tilde{p}_{ij} > p^R_{ij,D}$, then it will be accepted, while if $\tilde{p}_{ij} < p^R_{ij,D}$, then it will be rejected, with no other impact on the acceptance of offers not in $\mathcal{K}$. D_j will not deviate and offer $\tilde{p}_{ij} > p^R_{ij,D}$ to any U_i, as it can always offer $p^R_{ij,D}$ instead and do strictly better (as no other agreements are affected). The only possible profitable deviation for D_j is to offer $\tilde{p}_{ij} < p^R_{ij,D}$ for all ij in $\mathcal{K}$. Again, given the candidate equilibrium strategies, these offers will be rejected at period t, and then accepted at $t+1$ at prices $p_{ij,U}(\mathcal{K})$.

[34] Let $\mathcal{B} \equiv \mathcal{G} \setminus (C_{k+1} \cup \ldots \cup C_{|K|})$, $\mathcal{A} \equiv C_k \setminus \{ia_k\}$, and $\mathcal{A}' \equiv \emptyset$. As $\mathcal{A}$ and $\mathcal{A}'$ only differ in agreements formed by D_{a_k}, $\Delta\pi_{i,U}(\mathcal{G} \setminus (C_{k+1} \cup \ldots \cup C_{|K|}), C_k) = \pi_{i,U}(\mathcal{A} \cup \mathcal{B} \cup \{ia_k\}) - \pi_{i,U}(\mathcal{A}' \cup \mathcal{B}) \geq \Delta\pi_{i,U}(\mathcal{G}, \{ia_k\})$, where the last inequality follows from A.SCDMC.

The decrease in D_j's payoffs from engaging in such a deviation is:

$$\begin{aligned}(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{K}) - \sum_{ij\in\mathcal{K}} & \left[p^R_{ij,D} - \delta_{j,D}p_{ij,U}(\mathcal{K})\right] \\ &\geq \sum_{ij\in\mathcal{K}}\left[(1-\delta_{j,D})p^{Nash}_{ij} - p^R_{ij,D} + \delta_{j,D}p_{ij,U}(\mathcal{K})\right] \\ &> \sum_{ij\in\mathcal{K}}\left[(1-\delta_{j,D})p^R_{ij,D} - p^R_{ij,D} + \delta_{j,D}p_{ij,U}(\mathcal{K})\right] \\ &= \delta_{j,D}\sum_{ij\in\mathcal{K}}\left[p_{ij,U}(\mathcal{K}) - p^R_{ij,D}\right] \geq 0\ ,\end{aligned} \tag{13}$$

where the second line of (13) follows from A.FEAS (and implied by A.WCDMC), the third line from Lemma 2.3, the final line equality from rearranging terms, and the final line inequality from Lemma C.1(2). Thus, D_j has no profitable deviation.

Since there are no profitable one-shot deviations for any agent in both odd and even periods, the candidate set of strategies comprise an equilibrium. By Lemma C.1(5), prices at this equilibrium converge to Nash-in-Nash prices. □

C.3 Proof of Theorem 3.2 (Necessity)

We now prove that if A.WCDMC does not hold, there is no equilibrium of the model with prices $p^*_{ij} = p^R_{ij,D}$ for all odd period histories of play and $p^*_{ij} = p^R_{ij,U}$ for all even period histories of play. We proceed by contradiction: assume that such an equilibrium exists, and that there exists an upstream firm U_i and a set of agreements $\mathcal{K} \subseteq \mathcal{G}_{i,U}$ such that $\Delta\pi_{ij,U}(\mathcal{G},\mathcal{K}) < \sum_{ij\in\mathcal{K}}\Delta\pi_{i,U}(\mathcal{G},\{ij\})$ (the proof is symmetric if A.WCDMC is violated for some downstream firm D_j). Consider again the gain in one-shot surplus from U_i rejecting all $ij \in \mathcal{K}$, denoted $F(\mathcal{K})$, evaluated at period $t_0 = 1$. From the candidate equilibrium, we know that U_i has formed agreements at prices $p^R_{ij,D}$ for all $ij \in \mathcal{G}_{i,U}$. $F(\mathcal{K})$ is given by:

$$\begin{aligned}F(\mathcal{K}) &\equiv -\left[(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\mathcal{K}) + \sum_{ij\in\mathcal{K}}\left[p^R_{ij,D} - \delta_{i,U}p^R_{ij,U}\right]\right] \\ &> -\left[\sum_{ij\in\mathcal{K}}(1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + \sum_{ij\in\mathcal{K}}\left[p^R_{ij,D} - \delta_{i,U}p^R_{ij,U}\right]\right] = 0,\end{aligned}$$

where the inequality follows from the assumption that A.WCDMC does not hold and the equality from (2). Hence, it is a profitable deviation for U_i to reject the offers in $\mathcal{K}$ at period 1, implying a contradiction. □

D Proof of Theorem 4.1 (Uniqueness for No-Delay Equilibria)

Consider any no-delay equilibrium. Theorem 4.1 follows from the following two claims.

Claim A: In every odd period t with history h^t, each agreement $ij \in \mathcal{C}(h^t)$ has equilibrium price $p_{ij}(h^t) \geq p^R_{ij,D}$. In every even period with history h^t, each agreement $ij \in \mathcal{C}(h^t)$ has equilibrium price $p_{ij}(h^t) \leq p^R_{ij,U}$.

Proof of Claim A. We prove the claim by contradiction. First, suppose that there is an odd period t with history of play h^t for which $p_{ij}(h^t) < p^R_{ij,D}$. Then, U_i has a profitable one-shot deviation: reject ij and accept all its other offers. In this case, at period $t+1$, ij will be the only open agreement (as all other agreements form at t in a no-delay equilibrium). Thus, following this deviation, ij will then form at price

$p^R_{ij,U}$ at period $t+1$ by Rubinstein (1982). The gains to U_i from this deviant action will then be:

$$\delta_{i,U}p^R_{ij,U} - p_{ij}(h^t) - (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) > \delta_{i,U}p^R_{ij,U} - p^R_{ij,D} - (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) = 0,$$

where the inequality follows from the contradictory assumption and the equality follows from (2). Hence this deviation is profitable. The even period proof is symmetric and omitted.

Claim B: Fix $\varepsilon > 0$. For any odd (even) period history of play h^t, $\exists\ \bar{\Lambda} > 0$ such that if $\Lambda \leq \bar{\Lambda}$, then the equilibrium price $p_{ij}(h^t) \leq p^{Nash}_{ij} + \varepsilon$ (for even periods, $p_{ij}(h^t) \geq p^{Nash}_{ij} - \varepsilon$).

Proof of Claim B. Define $\bar{\Lambda}$ as any positive number that is small enough so that: (a) when $\Lambda \leq \bar{\Lambda}$, the maximum absolute value of profits to any firm for any subset of agreements over period length Λ is less than $\varepsilon/2$; and (b) the maximum of the absolute value of the difference between $\delta p^R_{ij,U}$ and p^{Nash}_{ij} across all agreements in $\mathcal{G}$ is also less than $\varepsilon/2$.[35]

We now prove our claim by contradiction. First, suppose that there is an odd period history of play h^t for which, for some $\Lambda \leq \bar{\Lambda}$, there is an agreement ij that is formed where $p_{ij}(h^t) > p^{Nash}_{ij} + \varepsilon$. Consider the following deviation by D_j: at h^t, D_j makes a deviant offer $\tilde{p}_{ij}$ sufficiently low that it is sure to be rejected by U_i. D_j will expect that, at period t, U_i will reject this deviant offer (and potentially some other offers), and that all offers that do not involve U_i will be accepted. Let $\mathcal{K}$ denote the set of offers that U_i rejects following this deviant offer from D_j. At period $t+1$, given that h^{t+1} is the history following D_j's period t deviant action, U_i will propose offers at equilibrium prices $\{p_{ik}(h^{t+1})\}_{\forall ik\in\mathcal{K}}$; furthermore, D_j expects that all agreements will be formed at the end of $t+1$ (given no-delay equilibrium strategies). The gain to D_j from this deviation is:

$$\begin{aligned}\underbrace{p_{ij}(h^t) - \delta_{j,D}p_{ij}(h^{t+1})}_{\text{change in payments}} - \underbrace{(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{K})}_{\text{change in flow profits}} &> p^{Nash}_{ij} + \varepsilon - \delta_{j,D}p^R_{ij,U} - (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{K}) \\ &> \varepsilon - \varepsilon/2 - \varepsilon/2 = 0,\end{aligned}$$

where the first line inequality follows from the contradictory assumption ($p_{ij}(h^t) > p^{Nash}_{ij} + \varepsilon$) and Claim A ($p_{ij}(h^t) \leq p^R_{ij,U}$), and the second line follows from the assumption that $\Lambda < \bar{\Lambda}$ (implying that $|p^{Nash}_{ij} - \delta_{j,D}p^R_{ij,U}| < \varepsilon/2$ and $|(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{K})| < \varepsilon/2$). Thus, this deviation is profitable for D_j, implying a contradiction.

We have thus shown that, for any $\varepsilon > 0$, there is a $\bar{\Lambda}$ such that for any $\Lambda < \bar{\Lambda}$, equilibrium prices in odd periods are bounded above by Nash-in-Nash prices plus ε. The even period proof is symmetric and omitted.

□

E Proof of Theorem 4.3 (Uniqueness Without Assuming Immediate Agreement)

We prove Theorem 4.3 under two sets of conditions. The first set, as stated in the main text, comprises A.GFT, A.SCDMC, A.LNEXT, and restricts consideration to common tie-breaking equilibria. We also prove that our uniqueness result holds under A.GFT, A.SCDMC, and a "no-externalities" assumption (discussed informally in Section 4.2) without restricting attention to common tie-breaking equilibria. The no-externalities assumption states that any firm's profits only depend on agreements that directly involve that firm:

Assumption E.1 (A.NEXT: No Externalities)
For upstream firms: for all $i = 1,\ldots,N$, $\mathcal{A} \subseteq \mathcal{G}_{i,U}$, and $\mathcal{B},\mathcal{B}' \subseteq \mathcal{G}^U_{-i}$,

$$\pi_{i,U}(\mathcal{A}\cup\mathcal{B}) = \pi_{i,U}(\mathcal{A}\cup\mathcal{B}').$$

[35] Such a $\bar{\Lambda}$ exists, since profits are bounded and $\lim_{\Lambda\to 0} p^R_{ij,U} = p^{Nash}_{ij}$ by Lemma 2.2.

For downstream firms: for all $j = 1, \ldots, M$, $\mathcal{A} \subseteq \mathcal{G}_{j,D}$, and $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{G}^D_{-j}$,

$$\pi_{j,D}(\mathcal{A} \cup \mathcal{B}) = \pi_{j,D}(\mathcal{A} \cup \mathcal{B}').$$

It is straightforward to prove that A.SCDMC and A.NEXT directly imply A.LNEXT.

Thus, in our subsequent proofs, we will assume that A.GFT, A.SCDMC, and A.LNEXT hold, and either restrict attention to common tie-breaking equilibria or assume A.NEXT. For the proofs in this section, we will use equilibrium to refer to common tie-breaking equilibrium when A.NEXT is not employed.

E.1 Inductive Structure and Base Case

For any history h^t with open agreements $\mathcal{C}$ at the start of period t, let $\Gamma_{\mathcal{C}}(h^t)$ be the subgame starting at period t. We prove Theorem 4.3 by induction on the set of open agreements $\mathcal{C}$ in any subgame $\Gamma_{\mathcal{C}}(h^t)$. The base case is provided by analyzing $\Gamma_{\mathcal{C}}(h^t)$ where $|\mathcal{C}| = 1$: i.e., there is only one agreement that has not yet been formed at period t.

Lemma E.2 (Base Case) *Consider any subgame $\Gamma_{\mathcal{C}}(h^t)$ for which $|\mathcal{C}| = 1$. Then $\Gamma_{\mathcal{C}}(h^t)$, where $\mathcal{C} \equiv \{ij\}$, has a unique equilibrium involving immediate agreement at t at prices $p^R_{ij,D}$ if t is odd, and $p^R_{ij,U}$ if t is even.*

Proof. With only one open agreement $ij \in \mathcal{C}$, U_i and D_j engage in a two-player Rubinstein alternating offers bargaining game over joint surplus $\Delta\pi_{i,U}(\mathcal{G}, \{ij\}) + \Delta\pi_{j,D}(\mathcal{G}, \{ij\})$. The result directly follows from Rubinstein (1982). (Note that Rubinstein (1982) only requires A.GFT.) □

We now state the inductive hypothesis and inductive step used to prove Theorem 4.3.

Inductive Hypothesis. *Consider any $\mathcal{C} \subseteq \mathcal{G}$. For any subgame $\Gamma_{\mathcal{B}}(h^t)$ for which $\mathcal{B} \subsetneq \mathcal{C}$, every equilibrium results in immediate agreement for all $ij \in \mathcal{B}$ at prices $p^R_{ij,D}$ if t is odd, and $p^R_{ij,U}$ if t is even.*

The inductive hypothesis is that any subgame of $\Gamma_{\mathcal{C}}(h^t)$ that begins with fewer open agreements than $|\mathcal{C}|$ results in immediate agreement at the Rubinstein prices.

Lemma E.3 (Inductive Step) *Consider any subgame $\Gamma_{\mathcal{C}}(h^t)$ for which $|\mathcal{C}| > 1$. Given the inductive hypothesis, every equilibrium of $\Gamma_{\mathcal{C}}(h^t)$ has immediate agreement for all $ij \in \mathcal{C}$ at prices $p^R_{ij,D}$ if t is odd, and $p^R_{ij,U}$ if t is even.*

The inductive step states that if the inductive hypothesis holds for $\Gamma_{\mathcal{C}}(h^t)$, then $\Gamma_{\mathcal{C}}(h^t)$ also results in immediate agreement for all open agreements at Rubinstein prices. Note that Lemmas E.2 (Base Case) and E.3 (Inductive Step) imply Theorem 4.3 by induction: as we have established that the theorem holds when $|\mathcal{C}| = 1$, the inductive step implies that the theorem will hold for any $\mathcal{C} \subseteq \mathcal{G}$ and history of play h^t.

To prove Lemma E.3 (and by consequence, Theorem 4.3), we first prove the simultaneity of agreements—i.e., if any open agreements are formed in a period, all open agreements are formed—at Rubinstein prices. We employ separate lemmas for two separate cases, depending on whether there are multiple receiving firms (Lemma E.5) or a single receiving firm (Lemma E.6) in a given period. Our proofs of simultaneity use A.SCDMC and restrict attention to common tie-breaking equilibria (or, alternatively, use A.NEXT). We then prove immediacy of agreement—i.e., that all open agreements form in the current period without delay—in Lemma E.7. This proof uses A.SCDMC and A.LNEXT. Establishing Lemmas E.5-E.7 proves our result.

Before proceeding, we state and prove the following lemma that we will use in our proofs:

Lemma E.4 *Assume A.GFT and A.LNEXT. Then $\forall \mathcal{C} \subseteq \mathcal{G}$, $\exists ij \in \mathcal{C}$ such that:*

$$\Delta\pi_{j,D}(\mathcal{G}, \mathcal{C}) > \sum_{hj \in \mathcal{C}_{j,D}} p^R_{hj,U}, \qquad \textit{and}$$

$$\Delta\pi_{i,U}(\mathcal{G}, \mathcal{C}) > -\sum_{ik \in \mathcal{C}_{i,U}} p^R_{ik,D}.$$

Proof. By A.LNEXT, $\forall \mathcal{C} \subseteq \mathcal{G}$, $\exists ij \in \mathcal{C}$ such that:

$$\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}) \geq \sum_{hj \in \mathcal{C}_{j,D}} \Delta\pi_{j,D}(\mathcal{G},\{hj\}), \qquad \text{and}$$
$$\Delta\pi_{i,U}(\mathcal{G},\mathcal{C}) \geq \sum_{ik \in \mathcal{C}_{i,U}} \Delta\pi_{i,U}(\mathcal{G},\{ik\}).$$

By A.GFT, $\Delta\pi_{j,D}(\mathcal{G},\{hj\}) > p^R_{hj,U}$ and $\Delta\pi_{i,U}(\mathcal{G},\{ik\}) > -p^R_{ik,D}$ for all agreements $hj, ik \in \mathcal{G}$ (see Lemma 2.3). The lemma immediately follows. $\square$

E.2 Simultaneity of Agreements

Lemma E.5 (Simultaneity of Agreements: Multiple Receiving Firms.) *Assume that the inductive hypothesis holds.*

1. *Suppose that $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ is such that there are at least two upstream firms with open agreements in $\mathcal{C}$. In any equilibrium of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where the first open agreement in $\mathcal{C}$ is formed at an odd period $t \geq \tilde{t}$, all agreements $ij \in \mathcal{C}$ must form at t and at prices $p^R_{ij,D}$.*

2. *Suppose that $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ is such that there are at least two downstream firms with open agreements in $\mathcal{C}$. In any equilibrium of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where the first open agreement in $\mathcal{C}$ is formed at an even period $t \geq \tilde{t}$, all agreements $ij \in \mathcal{C}$ must form at t and at prices $p^R_{ij,U}$.*

Proof. We prove case 1 using two claims (A and B); the proof of case 2 is symmetric and omitted.

Claim A: In any equilibrium of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where the first set of open agreements $\mathcal{B} \subseteq \mathcal{C}$, $\mathcal{B} \neq \emptyset$, are formed at an odd period $t \geq \tilde{t}$, then all open agreements in $\mathcal{C}$ also are formed at period t.

Proof of Claim A. By contradiction, assume that there is an equilibrium of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where $\mathcal{B} \neq \mathcal{C}$ so that a non-empty set of agreements does not form at period t. By the inductive hypothesis, all agreements $hk \in \mathcal{C} \setminus \mathcal{B}$ will form at period $t+1$ at prices $p^R_{hk,U}$. Consider some agreement ij that: (1) is formed at period $t+1$ following equilibrium play under $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ and (2) $\exists h \neq i$ such that U_h has an agreement which forms at period t. Such an ij must exist since $\mathcal{C}$ includes agreements for more than one upstream firm and (by the contradictory assumption) not all agreements form at period t.

Now consider the following deviation by D_j at period t: D_j offers $\tilde{p}_{ij} \equiv p^R_{ij,D} + \varepsilon$ to U_i, where $\varepsilon = (p^R_{ij,U} - p^R_{ij,D})/2 > 0$.[36] By passive beliefs, U_i expects at least one agreement to form at period t (e.g., involving U_h) upon receiving this deviant offer from D_j; by the inductive hypothesis, U_i therefore expects that all agreements that do not form at period t will form at period $t+1$. Note that:

1. Such a deviant offer will be accepted by U_i.

   By contradiction, suppose not, and U_i rejects D_j's deviant offer and instead accepted some (potentially empty) set of offers $\mathcal{A} \subseteq (\mathcal{C}_{i,U} \setminus ij)$ at period t. The gain to U_i from adding ij to $\mathcal{A}$ is strictly positive:

$$\begin{aligned}(1-\delta_{i,U})&(\pi_{i,U}((\mathcal{G}\setminus\mathcal{C}) \cup \mathcal{A} \cup \mathcal{B}_{-i,U} \cup \{ij\}) - \pi_{i,U}((\mathcal{G}\setminus\mathcal{C}) \cup \mathcal{A} \cup \mathcal{B}_{-i,U})) + \tilde{p}_{ij} - \delta_{i,U} p^R_{ij,U} \\ &= (1-\delta_{i,U})\Delta\pi_{i,U}((\mathcal{G}\setminus\mathcal{C}) \cup \mathcal{A} \cup \mathcal{B}_{-i,U} \cup \{ij\}, \{ij\}) + p^R_{ij,D} + \varepsilon - \delta_{i,U} p^R_{ij,U} \qquad (14) \\ &\geq (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + p^R_{ij,D} + \varepsilon - \delta_{i,U} p^R_{ij,U} = \varepsilon,\end{aligned}$$

   where the second line is definitional, the third line inequality follows from A.SCDMC, and the last equality follows from (2). Hence, any best response to this deviant offer by U_i must include accepting ij.

2. Such a deviation is profitable for D_j if accepted by U_i.

[36] By Lemma 2.3, $p^R_{ij,U} > p^R_{ij,D}$, so $\varepsilon > 0$.

If U_i accepts the deviant offer in addition to some set of offers $\mathcal{A} \subseteq (\mathcal{C}_{i,U} \setminus ij)$, D_j's gain from this deviant offer is:

$$\begin{aligned}
(1-\delta_{j,D})[\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}\cup\mathcal{B}_{-i,U}\cup\{ij\}) - \pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}_{i,U}\cup\mathcal{B}_{-i,U})] - \tilde{p}_{ij} + \delta_{j,D}p^R_{ij,U} \\
\geq (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - p^R_{ij,D} - \varepsilon + \delta_{j,D}p^R_{ij,U} \\
> (1-\delta_{j,D})p^R_{ij,U} - p^R_{ij,D} - \varepsilon + \delta_{j,D}p^R_{ij,U} \\
= p^R_{ij,U} - p^R_{ij,D} - \varepsilon > 0,
\end{aligned} \tag{15}$$

where the second line follows from A.SCDMC, the third line inequality follows from Lemma 2.3, the last line equality follows from rearranging terms, and the final inequality follows from Lemma 2.3 and the choice of ε.

This is a profitable deviation for D_j, yielding a contradiction. Thus, if the first agreement forms in odd period t, all agreements must form at period t.

Claim B: In any equilibrium of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where all open agreements $\mathcal{C}$ are formed at an odd period $t \geq \tilde{t}$, all agreements $ij \in \mathcal{C}$ are formed at prices $\hat{p}_{ij} = p^R_{ij,D}$.

Proof of Claim B. By contradiction, assume that all open agreements $\mathcal{C}$ are formed at period t, but $\hat{p}_{ij} \neq p^R_{ij,D}$ for some $ij \in \mathcal{C}$. Consider the following two cases:

1. Suppose $\hat{p}_{ij} < p^R_{ij,D}$ for some ij.

   Consider the deviation where U_i rejects only this offer ij at t. Since all other agreements form at period t, from the inductive hypothesis, U_i expects to form ij at $t+1$ at price $p^R_{ij,U}$. U_i's gain from this deviation is positive:

   $$\delta_{i,U}p^R_{ij,U} - \hat{p}_{ij} - (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) > \delta_{i,U}p^R_{ij,U} - p^R_{ij,D} - (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) = 0,$$

   where the last equality follows from (2), implying a contradiction.

2. Suppose $\hat{p}_{ij} > p^R_{ij,D}$ for some ij.

   Consider the deviation where D_j lowers its offer from $\hat{p}_{ij}$ to some $\tilde{p}_{ij} \in (p^R_{ij,D}, \hat{p}_{ij})$. We first show that, under either action, *every* best response for U_i must include accepting offer $\tilde{p}_{ij}$ and forming agreement ij at period t. Suppose, by contradiction, that a best response for U_i at t would be to form only agreements $\mathcal{A} \subseteq \mathcal{C}_{i,U} \setminus \{ij\}$. Similar to the logic used to derive (14), the gain to U_i from also forming ij in addition to $\mathcal{A}$ is strictly positive:

   $$\begin{aligned}
   (1-\delta_{i,U})\Delta\pi_{i,U}\big((\mathcal{G}\setminus\mathcal{C}_{i,U})\cup\mathcal{A}\cup\{ij\},\{ij\}\big) + \tilde{p}_{ij} - \delta_{i,U}p^R_{ij,U} \\
   > (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + p^R_{ij,D} - \delta_{i,U}p^R_{ij,U} = 0,
   \end{aligned}$$

   where the last equality follows from (2), implying that forming only agreements in $\mathcal{A}$ was not a best response. An analogous equation but with $\hat{p}_{ij}$ replacing $\tilde{p}_{ij}$ (not shown) also holds, thus implying that any best response for U_i at t given the candidate equilibrium strategies also must involve forming agreement ij.

   We now show that the set of best responses for U_i at t given candidate equilibrium prices coincides to the set of best responses for U_i at t given the deviation by D_j. Consider any best response set of acceptances to D_j's deviation at t. Accepting $\mathcal{A}\cup\{ij\}$, $\mathcal{A} \subseteq \mathcal{C}_{i,U}\setminus\{ij\}$, is a best response for U_i if and only if the value to U_i of accepting this set is weakly greater than the maximum value of accepting any set $\mathcal{A}'\cup\{ij\}$, $\mathcal{A}' \subseteq \mathcal{C}_{i,U}\setminus\{ij\}$. This condition can be written as:

   $$\begin{aligned}
   (1-\delta_{i,U})\pi\big((\mathcal{G}\setminus\mathcal{C}_{i,U})\cup\mathcal{A}\cup\{ij\}\big) + \sum_{ik\in\mathcal{A}}\hat{p}_{ik} + \delta_{i,U}\sum_{ik\in\mathcal{C}_{i,U}\setminus(\mathcal{A}\cup\{ij\})}p^R_{kj,U} \\
   \geq \max_{\mathcal{A}'\subseteq\mathcal{C}_{i,U}}\left\{(1-\delta_{i,U})\pi\big((\mathcal{G}\setminus\mathcal{C}_{i,U})\cup\mathcal{A}'\cup\{ij\}\big) + \sum_{ik\in\mathcal{A}'}\hat{p}_{ik} + \delta_{i,U}\sum_{ik\in\mathcal{C}_{i,U}\setminus(\mathcal{A}'\cup\{ij\})}p^R_{kj,U}\right\},
   \end{aligned}$$

(where we omit the price paid for ij since we have shown that any best response to the deviant offer requires this agreement to be formed at t). This condition is the same as the one determining whether a set $\mathcal{A} \cup \{ij\}$ is a best response for U_i at t under candidate equilibrium strategies (as we have shown earlier in Claim B of the lemma that any best response for U_i under these strategies also must involve agreement ij being formed), implying that the sets of best responses are the same. Because the sets of best responses are the same and we restrict attention to a common tie-breaking equilibrium, U_i must accept the same set of offers—i.e., all offers in $\mathcal{C}_{i,U}$—upon receiving this deviant offer from D_j as under the candidate equilibrium strategies. Thus, the deviant offer will increase profits to D_j by $\hat{p}_{ij} - \tilde{p}_{ij} > 0$, which leads to a contradiction.

If we assume A.NEXT instead of restricting attention to common tie-breaking equilibria, then having shown that U_i accepts the deviant offer $\tilde{p}_{ij}$ is sufficient for D_j to have a profitable deviation, as under A.NEXT, D_j's profits are unaffected by U_i's other acceptances.

Thus, $\hat{p}_{ij} = p^R_{ij,D}$ $\forall ij \in \mathcal{C}$ if the first open agreement in $\mathcal{C}$ forms at an odd period. $\square$

Lemma E.6 (Simultaneity of Agreements: Single Receiving Firm) *Assume that the inductive hypothesis holds.*

1. *Suppose that $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ is such that there is exactly one downstream firm, but more than one upstream firm, with open agreements in $\mathcal{C}$. In any equilibrium of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where the first agreement is formed at an even period $t \geq \tilde{t}$, all agreements $ij \in \mathcal{C}$ must form at t and at prices $p^R_{ij,U}$.*

2. *Suppose that $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ is such that there is exactly one upstream firm, but more than one downstream firm, with open agreements in $\mathcal{C}$. In any equilibrium of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where the first agreement is formed at an odd period $t \geq \tilde{t}$, all agreements $ij \in \mathcal{C}$ must form at t and at prices $p^R_{ij,D}$.*

Proof. We prove case 1 of the lemma; the proof of case 2 is symmetric and omitted.

For this lemma, we cannot apply induction in the case where the single receiving firm rejects all of its offers as the subgame beginning in the following period will have the same set of open agreements. Analyzing this case is more involved and utilizes an argument similar to Rubinstein (1982) and Shaked and Sutton (1984), where bounds on equilibrium prices are obtained by showing that the receiving firm cannot credibly reject a sufficiently generous offer in any equilibrium without the expectation of an even more generous (and infeasible) offer in a future subgame.

We start with two definitions. For any subgame $\Gamma_{\mathcal{C}}(h^t)$ and equilibrium where all agreements in $\mathcal{C}$ are eventually formed, let $\{p_{ij}(\Gamma_{\mathcal{C}}(h^t))\}_{ij \in \mathcal{C}}$ be the equilibrium prices for this game (which need not all form at t), and define $\phi_{\Gamma_{\mathcal{C}}(h^t)} \equiv \sum_{ij \in \mathcal{C}} [p^R_{ij,D} - p_{ij}(\Gamma_{\mathcal{C}}(h^t))]$ to be the *total discount* from prices $p^R_{ij,D}$ that D_j obtains in this equilibrium of this subgame.

We prove the lemma with three claims.

Claim A: For any equilibrium and subgame $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where all agreements in $\mathcal{C}$ are eventually formed, $\phi_{\Gamma_{\mathcal{C}}(h^{\tilde{t}})} \leq 0$.

Proof of Claim A. By contradiction, assume that there is an equilibrium where in some subgame $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$, all agreements in $\mathcal{C}$ are eventually formed, and the total discount is strictly positive: $\phi_{\Gamma_{\mathcal{C}}(h^{\tilde{t}})} > 0$. Without loss of generality (since all agreements eventually form), assume that at least one open agreement forms at $\tilde{t}$ in this equilibrium.

Now consider all subgames $\{\Gamma_{\mathcal{C}}(h^t)\}$ of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ (including $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ itself) where $t \geq \tilde{t}$, t is even, there are $\mathcal{C}$ open agreements at t, h^t is consistent with $h^{\tilde{t}}$ (i.e., h^t coincides with $h^{\tilde{t}}$ for all periods $\tilde{t}$ and earlier), and the first open agreement in $\mathcal{C}$ is formed at t given equilibrium strategies. Such subgames, if $t > \tilde{t}$, can be reached from $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ if all agreements in $\mathcal{C}$ are rejected in periods $\tilde{t}, \ldots, t-1$. In any such subgame $\Gamma_{\mathcal{C}}(h^t)$ where at least one agreement is formed at t, by the inductive hypothesis all agreements will be formed at latest by $t+1$. Let $\overline{\phi}$ denote the supremum of the total discount over all subgames of $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ satisfying the criteria above.[37]

[37]Note that $\overline{\phi}$ is finite since it cannot be greater than the sum of profits in the game.

Now choose subgame $\Gamma_{\mathcal{C}}(h^t)$ with at least one agreement forming at t that has a total discount very close to the supremum and strictly positive: i.e., for which $\delta_{j,D}\overline{\phi} < \phi_{\Gamma_{\mathcal{C}}(h^t)}$ and $\phi_{\Gamma_{\mathcal{C}}(h^t)} \geq \phi_{\Gamma_{\mathcal{C}}(h^{\bar{t}})} > 0$. At this subgame, fix some U_i for which (1) agreement ij forms at period t and (2) $\hat{p}_{ij} \equiv p_{ij}(\Gamma_{\mathcal{C}}(h^t)) < p^R_{ij,D}$. Such a U_i must exist by the fact that the agreements that do not form at period t form at period $t+1$ at odd-period Rubinstein prices (which have no discount) by the inductive hypothesis. Finally, denote the agreements that form at period t at this subgame as $\hat{\mathcal{A}} \cup \{ij\}$, $\hat{\mathcal{A}} \subseteq \mathcal{C} \setminus \{ij\}$.

Now consider the following deviation by U_i at period t: U_i offers $\tilde{p}_{ij} \equiv \hat{p}_{ij} + \varepsilon$, where $\varepsilon > 0$ is sufficiently small such that (1) the total discount realized by D_j if it still forms agreements $\hat{\mathcal{A}} \cup \{ij\}$ is still strictly positive and greater than $\delta_{j,D}\overline{\phi}$, and (2) $\hat{p}_{ij} + \varepsilon < p^R_{ij,U}$. Thus, by (2.3),

$$p^R_{ij,U} - \tilde{p}_{ij} + \sum_{kj \in \hat{\mathcal{A}}} (p^R_{kj,U} - \hat{p}_{kj}) > p^R_{ij,D} - \tilde{p}_{ij} + \sum_{kj \in \hat{\mathcal{A}}} (p^R_{kj,D} - \hat{p}_{kj}) > \delta_{j,D}\bar{\phi}\,, \tag{16}$$

We now show that any best response by D_j at t must include accepting ij. Suppose, by contradiction, that a best response for D_j involves accepting only offers $\mathcal{A} \subseteq \mathcal{C} \setminus \{ij\}$ at t. We consider four potential cases of equilibrium play following this candidate best response:

1. *$\mathcal{A} = \emptyset$, and no further agreements ever form (i.e., D_j rejects all offers in every subsequent even period, and makes sufficiently low offers for all open agreements in every subsequent odd period that all of its offers are rejected).*

   Consider an alternative action for D_j of accepting only agreement ij at price $\tilde{p}_{ij}$ at period t. If D_j accepts only ij, then all other agreements will form at $t+1$ by the inductive hypothesis. The gain (in period t units) to D_j from accepting only ij as opposed to following the candidate action and rejecting all offers at t is:

$$\begin{aligned}
&(1-\delta_{j,D})\Delta\pi_{j,D}\big((\mathcal{G}\setminus\mathcal{C})\cup\{ij\},\{ij\}\big) - \tilde{p}_{ij} + \delta_{j,D}\Big(\Delta\pi_{j,D}(G,\mathcal{C}) - \sum_{kj\in\mathcal{C},k\neq i} p^R_{kj,D}\Big)\\
&\qquad > (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - p^R_{ij,U} + \delta_{j,D}\Big(\Delta\pi_{j,D}(G,\mathcal{C}) - \sum_{kj\in\mathcal{C},k\neq i} p^R_{kj,D}\Big)\\
&\qquad = -\delta_{j,D}p^R_{ij,D} + \delta_{j,D}\Big(\Delta\pi_{j,D}(G,\mathcal{C}) - \sum_{kj\in\mathcal{C},k\neq i} p^R_{kj,D}\Big)\\
&\qquad = \delta_{j,D}\Big(\Delta\pi_{j,D}(G,\mathcal{C}) - \sum_{kj\in\mathcal{C}} p^R_{kj,D}\Big) > 0,
\end{aligned} \tag{17}$$

   where the second line follows from A.SCDMC and the definition of the deviant action ($\tilde{p}_{ij} \equiv \hat{p}_{ij} + \varepsilon < p^R_{ij,U}$), the third line follows from (1), and the final line from Lemma 2.3 and A.FEAS. Thus, accepting no offers at t is not a best response in this case.

2. *$\mathcal{A} = \emptyset$, and the first agreement $ij \in \mathcal{C}$ to form does so at an odd period $t + t'$, $t' \geq 1$.*

   In this case, by Lemma E.5, all agreements must form at time $t + t'$ at Rubinstein prices $p^R_{ij,D}$ (as there are multiple receiving (upstream) firms with open agreements). Consider an alternative action for D_j of accepting $\hat{\mathcal{A}} \cup \{ij\}$ at period t (i.e., original equilibrium acceptances at t). Note that the gain from following this alternative action relative to forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices is:

$$
\begin{aligned}
&(1-\delta_{j,D})\Delta\pi_{j,D}\big((\mathcal{G}\setminus\mathcal{C})\cup(\hat{\mathcal{A}}\cup\{ij\}),\hat{\mathcal{A}}\cup\{ij\}\big)-\tilde{p}_{ij}+\delta_{j,D}p^R_{ij,D}-\sum_{kj\in\hat{\mathcal{A}}}(\hat{p}_{kj}-\delta_{j,D}p^R_{kj,D})\\
&\geq(1-\delta_{j,D})\Big(\sum_{kj\in\hat{\mathcal{A}}\cup\{ij\}}\Delta\pi_{j,D}(\mathcal{G},\{kj\})\Big)-\tilde{p}_{ij}+\delta_{j,D}p^R_{ij,D}-\sum_{kj\in\hat{\mathcal{A}}}(\hat{p}_{kj}-\delta_{j,D}p^R_{kj,D})\\
&\geq\underbrace{\Big(\sum_{kj\in\hat{\mathcal{A}}\cup\{ij\}}(1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{kj\})-p^R_{kj,U}+\delta_{j,D}p^R_{kj,D}\Big)}_{=0\text{ by }(1)}+p^R_{ij,U}-\tilde{p}_{ij}+\sum_{kj\in\hat{\mathcal{A}}}(p^R_{kj,U}-\hat{p}_{kj})\\
&>\delta_{j,D}\bar{\phi}>0,
\end{aligned}
\tag{18}
$$

where the second line follows from A.SCDMC, the third line follows from rearranging terms, and the last line follows from (16) and the assumption that $\bar{\phi}>0$. Thus, the alternative action yields D_j a payoff that is strictly higher than the payoff of forming all agreements at period $t+1$ at odd-period Rubinstein prices.

Next, the gain to D_j from forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices instead of forming all agreements in $\mathcal{C}$ in some odd period $t+t'$, $t'>1$, in period $t+1$ units, is:

$$
\begin{aligned}
(1-\delta^{t'-1}_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{C})-\sum_{kj\in\mathcal{C}}p^R_{kj,D}+\delta^{t'-1}_{j,D}\sum_{kj\in\mathcal{C}}p^R_{kj,D}&\\
\geq(1-\delta^{t'-1}_{j,D})\sum_{kj\in\mathcal{C}}\Delta\pi_{j,D}(\mathcal{G},\{kj\})-\sum_{kj\in\mathcal{C}}p^R_{kj,D}+\delta^{t'-1}_{j,D}\sum_{kj\in\mathcal{C}}p^R_{kj,D}&\\
=(1-\delta^{t'-1}_{j,D})\sum_{kj\in\mathcal{C}}\big(\Delta\pi_{j,D}(\mathcal{G},\{kj\})-p^R_{kj,D}\big)>0,&
\end{aligned}
\tag{19}
$$

where the second line follows from A.WCDMC, the third line equality follows by rearranging terms, and last inequality from Lemma 2.3. Thus, accepting no offers at t is not a best response in this case.

3. *$\mathcal{A}=\emptyset$, and the first agreement $ij\in\mathcal{C}$ to form does so at an even period $t+t'$, $t'\geq 2$.*

   Let $\mathcal{B}$ denote the equilibrium set of agreements that forms at time $t+t'$ following D_j's rejection of all offers at t. For any $kj\in\mathcal{B}$, let p'_{kj} denote the equilibrium price at which the agreement forms. By the inductive hypothesis, the remaining agreements, $\mathcal{C}\setminus\mathcal{B}$ form at $t+t'+1$ (odd) at odd-period Rubinstein prices.

   Consider an alternative action for D_j of accepting $\hat{\mathcal{A}}\cup\{ij\}$ at period t. From (18), the gain to D_j from following this alternative action as opposed to forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices is strictly greater than $\delta_{j,D}\overline{\phi}$.

   The gain to D_j from forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices relative to forming agreements $\mathcal{B}$ at t' and $\mathcal{C} \setminus \mathcal{B}$ at $t'+1$, is (in period $t+1$ units):

$$
\begin{aligned}
&(1-\delta_{j,D}^{t'-1})\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}) + \delta_{j,D}^{t'-1}\left(\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}\setminus\mathcal{B}) + \sum_{kj\in\mathcal{C}\setminus\mathcal{B}} \delta_{j,D} p^R_{kj,D} + \sum_{kj\in\mathcal{B}} p'_{kj}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} \\
&\geq (1-\delta_{j,D}^{t'-1})\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}) + \delta_{j,D}^{t'-1}\left(\sum_{kj\in\mathcal{C}\setminus\mathcal{B}} \left(\Delta\pi_{j,D}(\mathcal{G},\{kj\}) + \delta_{j,D} p^R_{kj,D}\right) + \sum_{kj\in\mathcal{B}} p'_{kj}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} \\
&\geq (1-\delta_{j,D}^{t'-1})\sum_{kj\in\mathcal{C}} \Delta\pi_{j,D}(\mathcal{G},\{kj\}) + \delta_{j,D}^{t'-1}\left(\sum_{kj\in\mathcal{C}\setminus\mathcal{B}} p^R_{kj,U} + \sum_{kj\in\mathcal{B}} p'_{kj}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} \\
&> (1-\delta_{j,D}^{t'-1})\sum_{kj\in\mathcal{C}} p^R_{kj,D} + \delta_{j,D}^{t'-1}\left(\sum_{kj\in\mathcal{C}} p^R_{kj,D} - \overline{\phi}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} \\
&> -\overline{\phi},
\end{aligned}
$$

where the second line follows from A.WCDMC, the third line follows from A.WCDMC and (1), the fourth line follows from Lemma 2.3 and the definition of $\overline{\phi}$, and the final line follows from the fact that $t' \geq 2$. Thus, the gain from the alternative action relative to the supposed equilibrium strategy—now in period t units—is greater than $(\delta_{j,D} - \delta_{j,D})\overline{\phi} = 0$. Thus, accepting no offers at t is not a best response in this case.

4. *$\mathcal{A} \neq \emptyset$, and D_j forms some agreements in $\mathcal{C} \setminus \{ij\}$ at t.*

   By assumption, $ij \notin \mathcal{A}$. By the inductive hypothesis, the remaining agreements $\mathcal{C} \setminus \mathcal{A}$ form at time $t+1$ at odd-period Rubinstein prices. In this case, the gain to D_j from adding ij to the agreements in $\mathcal{A}$ instead of forming only agreements in $\mathcal{A}$, in period t units, is:

$$
\begin{aligned}
&(1-\delta_{j,D})\Delta\pi_{j,D}\big((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}\cup\{ij\},\{ij\}\big) - \tilde{p}_{ij} + \delta_{j,D} p^R_{ij,D} \\
&\quad > (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - p^R_{ij,U} + \delta_{j,D} p^R_{ij,D} = 0,
\end{aligned}
\tag{20}
$$

   where the inequality follows from A.SCDMC and the definition of the deviant action ($\tilde{p}_{ij} \equiv \hat{p}_{ij} + \varepsilon < p^R_{ij,U}$), and the equality from (1). Thus, forming agreements $\mathcal{A}$ where $ij \notin \mathcal{A}$ is not a best response in this case.

Thus, any best response by D_j must include accepting agreement ij at price $\tilde{p}_{ij}$.

Now consider any best response for D_j to the deviant offer by U_i. Accepting offers $\mathcal{A} \cup \{ij\}$, $\mathcal{A} \subseteq \mathcal{C} \setminus \{ij\}$ is a best response if and only if the value to D_j of accepting this set is weakly greater than the maximum value of accepting any set $\mathcal{A}' \cup \{ij\}$, $\mathcal{A}' \subseteq \mathcal{C} \setminus \{ij\}$. This condition can be written as:

$$
\begin{aligned}
&(1-\delta_{j,D})\pi((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}\cup\{ij\}) + \sum_{kj\in\mathcal{A}} \hat{p}_{ij} + \delta_{j,D} \sum_{kj\in\mathcal{C}\setminus(\mathcal{A}\cup\{ij\})} p^R_{kj,D} \\
&\qquad \geq \max_{\mathcal{A}'\subseteq\mathcal{C}} \left\{(1-\delta_{j,D})\pi((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}'\cup\{ij\}) + \sum_{kj\in\mathcal{A}'} \hat{p}_{ij} + \delta_{j,D} \sum_{kj\in\mathcal{C}\setminus(\mathcal{A}'\cup\{ij\})} p^R_{kj,D}\right\},
\end{aligned}
$$

where we excluded the price paid for ij since this agreement always forms at period t, and we apply the inductive hypothesis to obtain period $t+1$ prices. Using the same logic as in Lemma E.5, the condition is the same as for a set being a best response under the candidate equilibrium offers implying that the sets of best responses are the same.

Because the sets of best responses are the same and we consider a common tie-breaking equilibrium, D_j accepts the same set of offers under the deviant offer from U_i. Moreover, in both cases, all other agreements will form at period $t+1$. Thus, the deviant offer will increase profits to U_i by $\tilde{p}_{ij} - \hat{p}_{ij} > 0$, which leads to a contradiction.

By the same arguments as in Lemma E.5, assuming A.NEXT instead of restricting attention to common tie-breaking equilibrium also leads to a contradiction.

Claim B: For any equilibrium and subgame $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where the first agreement is formed at an even period $t \geq \tilde{t}$, all agreements $ij \in \mathcal{C}$ form at t.

Proof of Claim B. By contradiction, assume that $\hat{\mathcal{A}} \subsetneq \mathcal{C}$ agreements form at period t, and $\hat{\mathcal{A}} \neq \emptyset$. By the inductive hypothesis, all agreements in $\mathcal{C} \setminus \hat{\mathcal{A}}$ are formed at period $t+1$ at prices $p^R_{ij,D}$. Consider a deviation where some U_i, $ij \in \mathcal{C} \setminus \hat{\mathcal{A}}$, offers $\tilde{p}_{ij} = \frac{1}{2}(p^R_{ij,D} + p^R_{ij,U})$ at period t.

1. Such a deviant offer will be accepted by D_j.

   By contradiction, suppose not, and a best response for D_j is to accept offers $\mathcal{A} \subseteq \mathcal{C} \setminus \{ij\}$. We again consider four cases of equilibrium play following this candidate best response:

   (a) *$\mathcal{A} = \emptyset$ and no further agreements form.*

   Consider an alternative action for D_j of forming only agreement ij at price $\tilde{p}_{ij}$ instead of rejecting all offers at t; the gain (in period t units) from this alternative as opposed to the candidate action is:

$$\begin{aligned}(1-\delta_{j,D})\Delta\pi_{j,D}\big((\mathcal{G}\setminus\mathcal{C})\cup\{ij\},\{ij\}\big) - \tilde{p}_{ij} + \delta_{j,D}\Delta\pi_{j,D}(G,\mathcal{C}) - \delta_{j,D}\sum_{kj\in\mathcal{C},k\neq i} p^R_{kj,D}\\ > (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - p^R_{ij,U} + \delta_{j,D}\Delta\pi_{j,D}(G,\mathcal{C}) - \delta_{j,D}\sum_{kj\in\mathcal{C},k\neq i} p^R_{kj,D} > 0,\end{aligned}$$

   where the logic is identical to (17). Thus, accepting no offers at t is not a best response in this case.

   (b) *$\mathcal{A} = \emptyset$, and the first agreement $ij \in \mathcal{C}$ to form does so at an odd period $t+t'$, $t' \geq 1$.*

   In this case, by Lemma E.5, all agreements form at time $t+t'$ at Rubinstein prices, as there are multiple upstream (receiving) firms in an odd period. Consider an alternative action for D_j of forming only agreement ij at price $\tilde{p}_{ij}$ instead of rejecting all offers at t; the gain from following this alternative action as opposed to forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices is:

$$\begin{aligned}(1-\delta_{j,D})\Delta\pi_{j,D}\big((\mathcal{G}\setminus\mathcal{C})\cup\{ij\},\{ij\}\big) - \tilde{p}_{ij} + \delta_{j,D}p^R_{ij,D}\\ > (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - p^R_{ij,U} + \delta_{j,D}p^R_{ij,D} = 0,\end{aligned} \tag{21}$$

   where the inequality follows from A.SCDMC and the definition of the deviant offer ($\tilde{p}_{ij} < p^R_{ij,U}$), and the equality from (1). By (19), the gain to D_j from forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices as opposed to forming all agreements in any future odd period $t+t'$, $t' \geq 1$, is weakly positive. Thus, accepting no offers at t is not a best response in this case.

   (c) *$\mathcal{A} = \emptyset$ and the first agreement $ij \in \mathcal{C}$ to form does so at an even period $t+t'$, $t' \geq 2$.*

   Let $\mathcal{B}$ denote the set of equilibrium agreements that form at time $t+t'$ following D_j's rejection of all offers at t. For any $kj \in \mathcal{B}$, let p'_{kj} denote the equilibrium price at which the agreement forms. By the inductive hypothesis, the remaining agreements, $\mathcal{C} \setminus \mathcal{B}$ form at time $t+t'+1$ (odd) at odd-period Rubinstein prices.

   Consider an alternative action for D_j of forming only agreement ij at price $\tilde{p}_{ij}$ instead of rejecting all offers at t. From (21), the gain to D_j from choosing this alternative action as opposed to forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices is strictly positive. The gain to D_j from forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices as opposed to forming agreements $\mathcal{B}$ at t' and $\mathcal{C} \setminus \mathcal{B}$ at $t'+1$, in period $t+1$ units, is:

$$
\begin{aligned}
&(1-\delta_{j,D}^{t'-1})\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}) + \delta_{j,D}^{t'-1}\left(\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}\setminus\mathcal{B}) + \sum_{kj\in\mathcal{C}\setminus\mathcal{B}} \delta_{j,D} p^R_{kj,D} + \sum_{kj\in\mathcal{B}} p'_{kj}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} \\
&\geq (1-\delta_{j,D}^{t'-1})\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}) + \delta_{j,D}^{t'-1}\left(\sum_{kj\in\mathcal{C}\setminus\mathcal{B}} \left[\Delta\pi_{j,D}(\mathcal{G},\{kj\}) + \delta_{j,D} p^R_{kj,D}\right] + \sum_{kj\in\mathcal{B}} p'_{kj}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} \\
&\geq (1-\delta_{j,D}^{t'-1})\sum_{kj\in\mathcal{C}}\Delta\pi_{j,D}(\mathcal{G},\{kj\}) + \delta_{j,D}^{t'-1}\left(\sum_{kj\in\mathcal{C}\setminus\mathcal{B}} p^R_{kj,U} + \sum_{kj\in\mathcal{B}} p'_{kj}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} \\
&> (1-\delta_{j,D}^{t'-1})\sum_{kj\in\mathcal{C}} p^R_{kj,D} + \delta_{j,D}^{t'-1}\left(\sum_{kj\in\mathcal{C}} p^R_{kj,D}\right) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} = 0,
\end{aligned}
$$

where the second line follows from A.WCDMC, the third line follows from A.WCDMC and (1), the fourth line inequality follows from Lemma 2.3 and Claim A, and the final equality follows by rearranging terms. Thus, accepting no offers at t is not a best response in this case.

(d) *$\mathcal{A} \neq \emptyset$, and D_j forms some agreements in $\mathcal{C} \setminus \{ij\}$ at t.*

By assumption, $ij \notin \mathcal{A}$. By the inductive hypothesis, the remaining agreements, $\mathcal{C} \setminus \mathcal{A}$ form at time $t+1$ at odd-period Rubinstein prices. In this case, the gain to D_j from accepting offers in $\mathcal{A} \cup \{ij\}$ instead of accepting only offers in $\mathcal{A}$ at t, in period t units, is:

$$
\begin{aligned}
&(1-\delta_{j,D})\Delta\pi_{j,D}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}\cup\{ij\},\{ij\}) - \tilde{p}_{ij} + \delta_{j,D} p^R_{ij,D} \\
&\quad > (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) - p^R_{ij,U} + \delta_{j,D} p^R_{ij,D} = 0,
\end{aligned}
\tag{22}
$$

where the logic is identical to (20). Thus, forming agreements $\mathcal{A}$ where $ij \notin \mathcal{A}$ is not a best response in this case.

Thus, any best response by D_j must include accepting the deviant offer $\tilde{p}_{ij}$ from U_i.

2. Such a deviation is profitable for U_i if accepted by D_j.

Suppose that, following this deviant offer, D_j accepts agreements $\mathcal{A}' \cup \{ij\}$ at period t, where $\mathcal{A}' \subseteq \mathcal{C}\setminus\{ij\}$. By the inductive hypothesis, all remaining agreements are formed at period $t+1$ at Rubinstein prices.

The gain to U_i from this deviation is then:

$$
\begin{aligned}
&(1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}'\cup\{ij\}) - (1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\hat{\mathcal{A}}) + \tilde{p}_{ij} - \delta_{i,U} p^R_{ij,D} \\
&\qquad \geq (1-\delta_{i,U})\Delta\pi_{i,U}((\mathcal{G},\{ij\}) + \tilde{p}_{ij} - \delta_{i,U} p^R_{ij,D} \\
&\qquad > (1-\delta_{i,U})\Delta\pi_{i,U}((\mathcal{G},\{ij\}) + p^R_{ij,D} - \delta_{i,U} p^R_{ij,U} = 0,
\end{aligned}
$$

where the second line follows from A.SCDMC, the third line inequality follows from Lemma 2.3 and the definition of the deviant action, and the last equality follows from (1). Hence, U_i will find it profitable to make the deviation.

U_i has a profitable deviation, yielding a contradiction.

Claim C: For any equilibrium and subgame $\Gamma_{\mathcal{C}}(h^{\tilde{t}})$ where all agreements in $\mathcal{C}$ are formed at an even period $t \geq \tilde{t}$, they are formed at prices $\hat{p}_{ij} = p^R_{ij,U}$ for all $ij \in \mathcal{C}$.

Proof of Claim C. By contradiction, assume that all agreements in $\mathcal{C}$ are formed at period t, but $\hat{p}_{ij} \neq p^R_{ij,U}$ for some $ij \in \mathcal{C}$.

1. Suppose that $\hat{p}_{ij} > p^R_{ij,U}$ for some ij.

   Consider the deviation where D_j rejects only this offer. Since all other agreements form at period t, by the inductive hypothesis, D_j forms this agreement at price $p^R_{ij,D}$ at $t+1$. Applying (1), D_j's gain from this action are:

$$-\delta_{j,D}p^R_{ij,D} + \hat{p}_{ij} - (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G}, \{ij\}) > -\delta_{j,D}p^R_{ij,D} + p^R_{ij,U} - (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G}, \{ij\}) = 0,$$

   implying a profitable deviation and hence a contradiction.

2. Suppose $\hat{p}_{ij} < p^R_{ij,U}$ for some ij.

   Consider a deviation where U_i raises its offer from $\hat{p}_{ij}$ to some $\tilde{p}_{ij} \in (\hat{p}_{ij}, p^R_{ij,U})$. We now show that any best response set of acceptances for D_j must include accepting ij. Suppose, by contradiction, that D_j has a best response of accepting only agreements in $\mathcal{A} \subseteq \mathcal{C} \setminus \{ij\}$ at t following this deviation. We consider three cases for equilibrium play following this best response:

   (a) *$\mathcal{A} = \emptyset$ and no further agreements form.*

   The gain to D_j from accepting only ij instead of choosing this action, in period $t+1$ units, is:

$$\begin{aligned}&(1-\delta_{j,D})\Delta\pi_{j,D}((\mathcal{G}\setminus\mathcal{C})\cup\{ij\}, \{ij\}) - \tilde{p}_{ij} + \delta_{j,D}\Delta\pi_{j,D}(G,\mathcal{C}) - \delta_{j,D}\sum_{kj\in\mathcal{C},k\neq i} p^R_{ij,D}\\ &\quad > (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G}, \{ij\}) - p^R_{ij,U} + \delta_{j,D}\Delta\pi_{j,D}(G,\mathcal{C}) - \delta_{j,D}\sum_{kj\in\mathcal{C},k\neq i} p^R_{ij,D} > 0,\end{aligned}$$

   where the logic is identical to (17). Thus accepting no offers at t is not a best response in this case.

   (b) *$\mathcal{A} = \emptyset$ and the first agreement $ij \in \mathcal{C}$ to form does so at period $t+t'$, $t' \geq 1$.*

   In this case, by Claim B, all agreements form at time $t+t'$. For any $kj \in \mathcal{C}$, let p'_{kj} denote the equilibrium price at which the agreement forms.

   Consider the alternative action by D_j of accepting only ij instead of rejecting all offers at t. First note that (21) applies and so the gain from following this deviant action as opposed to forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices is strictly positive. Next, note that the gain to D_j from forming all agreements in $\mathcal{C}$ at period $t+1$ at odd-period Rubinstein prices as opposed to forming all agreements in $\mathcal{C}$ at period $t+t'$ ($t' \geq 1$), in period $t+1$ units, is:

$$\begin{aligned}&(1-\delta^{t'-1}_{j,D})\Delta\pi_{j,D}(\mathcal{G},\mathcal{C}) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} + \delta^{t'-1}_{j,D}\sum_{kj\in\mathcal{C}} p'_{kj}\\ &\qquad \geq (1-\delta^{t'-1}_{j,D})\sum_{kj\in\mathcal{C}}\Delta\pi_{j,D}(\mathcal{G},\{kj\}) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} + \delta^{t'-1}_{j,D}\sum_{kj\in\mathcal{C}} p^R_{kj,D}\\ &\qquad = (1-\delta^{t'-1}_{j,D})\sum_{kj\in\mathcal{C}}\left[\Delta\pi_{j,D}(\mathcal{G},\{kj\}) - p^R_{kj,D}\right] \geq 0,\end{aligned} \tag{23}$$

   where the second line follows from A.WCDMC and Lemma E.5 (if t' is odd) or Claim A (if t' is even), the third line equality follows by rearranging terms, and the last inequality from Lemma 2.3. Thus accepting no offers at t is not a best response in this case.

   (c) $\mathcal{A} \neq \emptyset$.

   By assumption, $ij \notin \mathcal{A}$. By the inductive hypothesis, all remaining agreements $\mathcal{C}\setminus\mathcal{A}$ form at time $t+1$ at odd-period Rubinstein prices. Applying (22), the gain to D_j from accepting offers in $\mathcal{A}\cup\{ij\}$ instead of accepting only offers in $\mathcal{A}$ at t is positive. Thus, forming agreements $\mathcal{A}$ where $ij \notin \mathcal{A}$ is not a best response in this case.

   Thus, any best response by D_j must include accepting the deviant offer $\tilde{p}_{ij}$ from U_i. Now consider any best response set of acceptances, $\mathcal{A}\cup\{ij\}$, to the deviant prices. As in Claim A, the condition is the same as for a set being a best response under the candidate equilibrium agreements implying that the sets of best responses are the same. Because the sets of best responses are the same and we consider a common tie-breaking equilibrium, D_j accepts the same set of agreements—i.e., all agreements in $\mathcal{C}$—under the deviant offer from U_i. Thus, the deviant offer will increase profits to U_i by $\tilde{p}_{ij} - \hat{p}_{ij} > 0$, which leads to a contradiction. Furthermore, as in Claim A, A.NEXT can be used instead of restricting attention to common tie-breaking equilibria in order to establish the claim.

Thus, $\hat{p}_{ij} = p^R_{ij,U} \forall ij \in \mathcal{C}$ for agreements formed at any even period.

Claims A-C prove the lemma. □

E.3 Immediacy of Agreements

Given the inductive hypothesis, Lemmas E.5–E.6 establish that in any equilibrium of any subgame $\Gamma^{\tilde{t}}_{\mathcal{C}}$ where any agreement $ij \in \mathcal{C}$ forms at period $t \geq \tilde{t}$, all agreements in $\mathcal{C}$ form at t at Rubinstein prices. We now prove that, given the inductive hypothesis, there cannot be any delay: i.e., in any equilibrium, all agreements in $\mathcal{C}$ form immediately.

Lemma E.7 (Immediacy of all agreements.) *Assume that the inductive hypothesis holds. Then, any equilibrium of $\Gamma^t_{\mathcal{C}}$ results in all agreements $ij \in \mathcal{C}$ forming at period t.*

Proof. We prove the case where t is odd; the proof of the case where t is even is symmetric and omitted.

By contradiction, consider a candidate equilibrium where no agreements are formed at period t (as, by the previous results, if any agreement is formed at period t, all agreements are formed in that period). Let agreement $ij \in \mathcal{C}$ satisfy the conditions of A.LNEXT. We consider a deviant action by D_j from this candidate equilibrium and then verify that it is profitable for D_j. Suppose D_j offers $\tilde{p}_{ij}$ satisfying $p^R_{ij,D} < \tilde{p}_{ij} < p^R_{ij,U}$ to U_i. We first show that U_i will accept this offer and then show that it will increase D_j's surplus relative to the candidate equilibrium.

Suppose that U_i accepts the offer $\tilde{p}_{ij}$. Then, by passive beliefs, it believes that this is the only agreement to be formed at period t and, by the inductive hypothesis, that the remaining agreements will form at period $t+1$. Hence, its payoffs—in period t units—from accepting the offer are:

$$\underbrace{\tilde{p}_{ij} + (1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\{ij\})}_{\text{Payoff at } t} + \underbrace{\delta_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}\setminus\{ij\}} p^R_{ik,U}\right)}_{\text{Payoff from } t+1 \text{ on}}$$

$$= \tilde{p}_{ij} + (1-\delta_{i,U})\Delta\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\{ij\},\{ij\}) + \delta_{i,U}\left(\sum_{ik\in\mathcal{C}_{i,U}\setminus\{ij\}} p^R_{ik,U} + \pi_{i,U}(\mathcal{G})\right) + (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C})$$

$$> p^R_{ij,D} + (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + \delta_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}\setminus\{ij\}} p^R_{ik,U}\right) + (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C})$$

$$= \delta_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right) + (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}),$$

where the second line adds and subtracts the $(1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C})$ term, the third line follows from A.SCDMC and the definition of $\tilde{p}_{ij}$, and the final line uses (2) and then combines the $p^R_{ij,U}$ terms in the sum.

We next show that any best response for U_i must include accepting ij. Suppose, by contradiction, that a best response for U_i involves accepting only offers $\mathcal{B} \subseteq \mathcal{C}_{i,U}\setminus\{ij\}$ at t. We consider the following four cases of equilibrium play following this candidate best response:

1. *$\mathcal{B} = \emptyset$, and no agreements in $\mathcal{C}$ are ever formed.*

In this case, the payoffs to U_i are:

$$\begin{aligned}
\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) &= \delta_{i,U}\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) \\
&< \delta_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,D}\right) + (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) \\
&< \delta_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right) + (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}),
\end{aligned}$$

where the second line follows from Lemma E.4 (which uses A.LNEXT) and the third line follows from Lemma 2.3. Since the payoffs to U_i from rejection are less than from accepting D_j's deviant offer, rejecting all offers is not a best response in this case.

2. *$\mathcal{B}=\emptyset$, and all agreements in $\mathcal{C}$ are formed in some even period $t+t'$ for $t'=1,3,5,\ldots$.*

   If U_i accepts no other offers at period t (and by passive beliefs, U_i believes that no agreements in $\mathcal{C}_{-i,U}$ are formed at t), the payoffs to U_i are:

$$\begin{aligned}
&(1-\delta^{t'}_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + \delta^{t'}_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right) \\
&= (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + (\delta_{i,U}-\delta^{t'}_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + \delta^{t'}_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right) \\
&< (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + (\delta_{i,U}-\delta^{t'}_{i,U})\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right) \\
&\qquad + \delta^{t'}_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right) \\
&= (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + \delta_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right),
\end{aligned}$$

   where the second and fourth lines follow by rearranging terms and the third line follows from Lemma E.4. Since the payoffs to U_i from rejecting all offers at t are less than from accepting D_j's deviant offer, rejecting all offers is not a best response in this case.

3. *$\mathcal{B}=\emptyset$, and all agreements in $\mathcal{C}$ are formed in some even period $t+t'$ for $t'=2,4,6,\ldots$.*

   In this case, the payoffs to U_i are:

$$\begin{aligned}
&(1-\delta^{t'}_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + \delta^{t'}_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,D}\right) \\
&< (1-\delta^{t'}_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + \delta^{t'}_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right) \\
&= (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + (\delta_{i,U}-\delta^{t'}_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + \delta^{t'}_{i,U}\left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p^R_{ik,U}\right)
\end{aligned}$$

$$
\begin{aligned}
&< (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + (\delta_{i,U} - \delta_{i,U}^{t'}) \left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p_{ik,U}^R \right) \\
&\qquad + \delta_{i,U}^{t'} \left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p_{ik,U}^R \right) \\
&= (1-\delta_{i,U})\pi_{i,U}(\mathcal{G}\setminus\mathcal{C}) + \delta_{i,U} \left(\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{C}_{i,U}} p_{ik,U}^R \right),
\end{aligned}
$$

where the second line follows from Lemma 2.3 and the remaining logic is identical to case 2. Since the payoffs to U_i from rejecting all offers at t are less than from accepting D_j's deviant offer, rejecting all offers is not a best response in this case.

4. *$\mathcal{B} \neq \emptyset$, and U_i forms some agreements in $\mathcal{C}_{i,U} \setminus \{ij\}$ at t.*

   In this case, by the inductive hypothesis, all remaining agreements $\mathcal{A} \equiv \mathcal{C} \setminus \mathcal{B}$ form in the following (even) period $t+1$ at Rubinstein prices. Thus, we can express the payoff to U_i from this action as $(1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}) + \sum_{ik\in\mathcal{B}} \hat{p}_{ik} + \delta_{i,U}\left[\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{A}_{i,U}} p_{ik,U}^R\right]$, where $\hat{p}_{ik} \forall ik \in \mathcal{B}$ are the period t candidate equilibrium prices offered to U_i. But,

$$
\begin{aligned}
&(1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}) + \sum_{ik\in\mathcal{B}} \hat{p}_{ik} + \delta_{i,U}\left[\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{A}_{i,U}} p_{ik,U}^R\right] \\
&= (1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}) + \sum_{ik\in\mathcal{B}} \hat{p}_{ik} + \delta_{i,U}\left[\pi_{i,U}(\mathcal{G}) + p_{ij,U}^R + \sum_{ik\in\mathcal{A}_{i,U}\setminus\{ij\}} p_{ik,U}^R\right] \\
&= (1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}) + \sum_{ik\in\mathcal{B}} \hat{p}_{ik} \\
&\qquad + p_{ij,D}^R + (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + \delta_{i,U}\left[\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{A}_{i,U}\setminus\{ij\}} p_{ik,U}^R\right] \\
&< (1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}) + \sum_{ik\in\mathcal{B}} \hat{p}_{ik} + \tilde{p}_{ij} + (1-\delta_{i,U})\Delta\pi_{i,U}(\mathcal{G},\{ij\}) + \delta_{i,U}\left[\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{A}_{i,U}\setminus\{ij\}} p_{ik,U}^R\right] \\
&\le (1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}) + \sum_{ik\in\mathcal{B}} \hat{p}_{ik} + \tilde{p}_{ij} + (1-\delta_{i,U})\Delta\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}\cup\{ij\},\{ij\}) \\
&\qquad + \delta_{i,U}\left[\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{A}_{i,U}\setminus\{ij\}} p_{ik,U}^R\right] \\
&= (1-\delta_{i,U})\pi_{i,U}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}\cup\{ij\}) + \sum_{ik\in\mathcal{B}} \hat{p}_{ik} + \tilde{p}_{ij} + \delta_{i,U}\left[\pi_{i,U}(\mathcal{G}) + \sum_{ik\in\mathcal{A}_{i,U}\setminus\{ij\}} p_{ik,U}^R\right],
\end{aligned}
$$

   where the second and sixth lines follow by rearranging terms, the third line follows from (2), the fourth line follows from the the definition of the deviant offer, and the fifth line follows from A.SCDMC.

   Since the final line is the value of accepting D_j's deviant offer and all agreements in $\mathcal{B}$, the payoff to U_i from accepting D_j's deviant offer and all agreements in $\mathcal{B}$ is higher than the payoff from accepting just the offers in $\mathcal{B}$. Thus, forming agreements $\mathcal{B}$, where $ij \notin \mathcal{B}$ is not a best response in this case.

Thus, any best response by U_i must include accepting the deviant offer $\tilde{p}_{ij}$ from D_j. Note that we have not ruled out the possibility that U_i may also choose to accept additional offers in $\mathcal{C}_{i,U}$ at period t upon accepting deviant offer $\tilde{p}_{ij}$; we return to this below.

Having verified that the $\tilde{p}_{ij}$ offer will be accepted by U_i, we now check that the acceptance of this deviant offer will be profitable for D_j. D_j knows that U_i is the only firm that will form agreement(s) at period t and, by the inductive hypothesis, that the remaining agreements will form at period $t+1$. However, it is possible that upon receiving the deviant offer, U_i will also accept some other offers $\mathcal{B} \subseteq \mathcal{C}_{i,U} \setminus \{ij\}$. Hence, D_j's payoff—in period t units—from making the deviant offer satisfies:

$$\begin{aligned}
&\underbrace{-\tilde{p}_{ij} + (1-\delta_{j,D})\pi_{j,D}((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{B}\cup\{ij\})}_{\text{Payoff at } t} + \underbrace{\delta_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}\setminus\{ij\}} p^R_{kj,U}\right)}_{\text{Payoff from } t+1 \text{ on}} \\
&\geq -\tilde{p}_{ij} + (1-\delta_{j,D})\Delta\pi_{j,D}((\mathcal{G},\{ij\}) + \delta_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}\setminus\{ij\}} p^R_{kj,U}\right) + (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) \\
&> -p^R_{ij,U} + (1-\delta_{j,D})\Delta\pi_{j,D}(\mathcal{G},\{ij\}) + \delta_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}\setminus\{ij\}} p^R_{kj,U}\right) + (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) \\
&= -\delta_{j,D}p^R_{ij,D} + \delta_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}\setminus\{ij\}} p^R_{kj,U}\right) + (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) \\
&> (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) + \delta_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}} p^R_{kj,U}\right),
\end{aligned}$$

where the second line applies A.SCDMC, the third line follows from the definition of $\tilde{p}_{ij}$, the fourth line uses (1), and the final line uses Lemma 2.3 and then combines the $p^R_{kj,U}$ terms in the sum.

Next, we show that the lower bound on payoffs from this deviant offer being accepted (given by the last line of the previous set of equations) is higher than the payoff from equilibrium play under the candidate equilibrium. If D_j does not deviate from equilibrium play with the deviation $\tilde{p}_{ij}$, there are three possibilities for subsequent equilibrium play with no agreements formed at t:

1. *No further agreements are formed.*

   In this case, the payoffs to D_j from the candidate equilibrium are:

$$\begin{aligned}
\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) &= \delta_{j,D}\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) + (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) \\
&< (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) + \delta_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}} p^R_{kj,U}\right),
\end{aligned}$$

   where the inequality follows from Lemma E.4. Thus, the payoffs to D_j from the candidate equilibrium are less than from accepting U_i's deviant offer in this case.

2. *All open agreements are formed in some even period $t+t'$, for $t' = 1, 3, 5, \ldots$.*

   In this case, the payoffs to D_j from the candidate equilibrium are:

$$\begin{aligned}
&(1-\delta^{t'}_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) + \delta^{t'}_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}} p^R_{kj,U}\right) \\
&= (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) + (\delta_{j,D} - \delta^{t'}_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) + \delta^{t'}_{j,D}\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}} p^R_{kj,U}\right) \\
&< (1-\delta_{j,D})\pi_{j,D}(\mathcal{G}\setminus\mathcal{C}) + (\delta_{j,D} - \delta^{t'}_{j,D})\left(\pi_{j,D}(\mathcal{G}) - \sum_{kj\in\mathcal{C}_{j,D}} p^R_{kj,U}\right)
\end{aligned}$$

$$+ \delta_{j,D}^{t'} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{i,U}} p_{ik,U}^R \right)$$

$$= (1 - \delta_{j,D}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,U}^R \right),$$

where the second and fourth lines follow by rearranging terms and the third line follows from Lemma E.4. Thus, the payoffs to D_j from the candidate equilibrium are less than from making the deviant offer in this case.

3. *All open agreements are formed in some odd period $t + t'$, for $t' = 2, 4, 6, \ldots$.*
In this case, the payoffs to D_j from the candidate equilibrium are:

$$\begin{aligned}
&(1 - \delta_{j,D}^{t'}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{t'} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,D}^R \right) \\
&= (1 - \delta_{j,D}^{t'-1}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{t'-1} \left((1 - \delta_{j,D}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \pi_{j,D}(\mathcal{G}) - \delta_{j,D} \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,D}^R \right) \\
&= (1 - \delta_{j,D}^{t'-1}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{t'-1} \left(\pi_{j,D}(\mathcal{G}) - (1 - \delta_{j,D}) \Delta \pi_{j,D}(\mathcal{G}, \mathcal{C}) - \delta_{j,D} \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,D}^R \right) \\
&< (1 - \delta_{j,D}^{t'-1}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{t'-1} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} \left[(1 - \delta_{j,D}) \Delta \pi_{j,D}(\mathcal{G}, \{kj\}) + \delta_{j,D} p_{kj,D}^R \right] \right) \\
&= (1 - \delta_{j,D}^{t'-1}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{t'-1} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,U}^R \right) \\
&= (1 - \delta_{j,D}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + (\delta_{j,D} - \delta_{j,D}^{t'-1}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{t'-1} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,U}^R \right) \\
&< (1 - \delta_{j,D}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + (\delta_{j,D} - \delta_{j,D}^{t'-1}) \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,U}^R \right) \\
&\qquad + \delta_{j,D}^{t'-1} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,U}^R \right) \\
&= (1 - \delta_{j,D}) \pi_{j,D}(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D} \left(\pi_{j,D}(\mathcal{G}) - \sum_{kj \in \mathcal{C}_{j,D}} p_{kj,U}^R \right),
\end{aligned}$$

where the second, third, and sixth lines follow by rearranging terms, the fourth line follows from A.WCDMC, the fifth line from (1), the seventh line from Lemma E.4, and the final line also by rearranging terms. Thus, the payoffs to D_j from the deviant offer are greater than its equilibrium payoffs in this case.

Thus D_j has a profitable deviation, leading to a contradiction. Hence, any equilibrium involves immediate agreement for all $ij \in \mathcal{C}$ at t. $\square$