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BARGAINING IN BILATERAL OLIGOPOLY:
AN ALTERNATING OFFERS REPRESENTATION OF THE "NASH-IN-NASH" SOLUTION

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Abstract

The concept of a Nash equilibrium in Nash bargains, proposed in Horn and Wolinsky (1988), has become the workhorse bargaining model for predicting and estimating the division of surplus in applied analysis of bilateral oligopoly. This paper proposes a non-cooperative foundation for this concept—in which agreements between each pair of firms maximizes their bilateral Nash product conditional on all other negotiated agreements—by extending the Rubinstein (1982) alternating offers model to a setting with multiple upstream and downstream firms. In our model, downstream firms make simultaneous offers to upstream firms in odd periods, and upstream firms make simultaneous offers to downstream firms in even periods. Given restrictions on underlying payoffs, we prove that there exists a perfect Bayesian equilibrium with passive beliefs that generates the "Nash-in-Nash" solution, and that this equilibrium outcome is unique.

1 Introduction

Bilateral bargaining between pairs of agents is pervasive in many economic environments. Manufacturers bargain with retailers over wholesale prices, and firms and unions negotiate the wages paid to workers. In many of these cases, negotiations are interdependent: e.g., a

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firm's profitability may depend on the prices negotiated by its competitors. Given the centrality of these environments, it is surprising that there is no clear prediction from theory for the right framework for modeling bilateral bargaining with externalities in applied analysis.

Ignoring these environments is difficult as the relevant policy questions have multiplied. For example, in 2012, hospitals in the United States received $285 billion from private insurers for their services.[1] Typically, hospitals and insurers bilaterally negotiate the prices for these services. Likewise, in the cable TV industry, the impact of consummated mergers (e.g., Comcast and NBC, approved in 2011), or proposed mergers (e.g., between Time Warner and Comcast, and between AT&T and DirecTV) hinges on changes to fees negotiated with content providers, such as ESPN or Netflix. In these sectors, prices and contracts terms are determined neither by perfect competition, nor by take it or leave it offers (as is assumed in Bertrand competition). Instead, because there are few firms on different sides of each market (hospitals and insurers, distributors and content providers), prices are negotiated.

To understand the determinants of prices in markets characterized by bilateral oligopoly, economists have recently focused on the "Nash-in-Nash" bargaining solution first proposed in Horn and Wolinsky (1988). This solution has become the workhorse bargaining model for predicting the division of surplus in many applied settings. Recent examples include Crawford and Yurukoglu (2012), Grennan (2013), Gowrisankaran, Nevo, and Town (2014), and Ho and Lee (2013), which consider sectors including cable television and inpatient hospital services. Moreover, this concept has also begun to influence regulatory policy, such as the FCC using a bargaining model similar to that proposed in this paper in its analysis of the Comcast-NBC merger (Rogerson, 2013).

The Nash-in-Nash bargaining solution is a set of transfers between all pairs of agents, such that each transfer is the solution to a bilateral Nash bargaining problem between each pair, *conditional on all other negotiated agreements*.[2] The latter emphasis is important, as there are often economic interdependencies and contracting externalities across negotiations. For instance, the value of adding an additional hospital to the network of a managed care organization (henceforth, MCO) may be lower if the MCO already contracts with several hospitals. In these cases, a bilateral Nash bargain between two firms cannot be conducted in isolation, since each negotiation depends on the outcomes of other negotiations. Since

[1] See Exhibit 1 on p. 4 of "National Health Expenditure Accounts: Methodology Paper, 2010" at `http://www.cms.gov/Research-Statistics-Data-and-Systems/Statistics-Trends-and-Reports/NationalHealthExpendData/downloads/dsm-10.pdf` accessed on September 25 2012.

[2] The solution to the Nash bargaining problem is the transfer that maximizes the Nash bargaining product, which in turn is the product of the value of each firm from agreement net of its disagreement point. The asymmetric Nash bargaining product, which we focus on, raises each firm's value net disagreement point to some power, where this exponent is often referred to as the Nash bargaining weight. The Nash bargain satisfies certain intuitive axioms; see Nash (1950) for details.

the outcome can be interpreted as a Nash equilibrium of a game where independent agents seek to maximize the Nash product of each pairwise bargain holding fixed the agreements of others, this solution has been referred to as "Nash-in-Nash."[3]

Although the Nash-in-Nash bargaining solution has been increasingly employed in recent work, it is not without limitations. In particular, Nash bargaining is a cooperative game theory concept which is embedded in a non-cooperative Nash equilibrium. Recognizing the need for an underlying non-cooperative model, Horn and Wolinsky state that, "although this will not be part of the formal model, it will sometimes be useful to think of the static model outlined above as the reduced form of an appropriate dynamic model," (p. 411). Yet, there has been little work on understanding whether the Nash-in-Nash solution could be implemented as the equilibrium payoffs of a dynamic bargaining game.

For the case of negotiations between two agents, Rubinstein (1982) shows that the Nash bargaining solution emerges as the unique subgame perfect equilibrium of an extensive form game with alternating offers as the time period between offers goes to 0. However, with multiple agents—in particular, more than one upstream or downstream firm—this model cannot be directly applied. Previous non-cooperative rationalizations of the Nash-in-Nash solution have typically been motivated by firms sending representatives to negotiate each bilateral agreement in separate, closed rooms; once negotiations start, representatives in different rooms (including those from the same firm) do not communicate with one another.[4] This particular rationalization implies that firms are not able to explicitly coordinate efforts across multiple bargains or utilize information learned in one bargain in another, and thus might be criticized for requiring agents to be "schizophrenic."

The purpose of this paper is to provide a credible non-cooperative extensive form that rationalizes the Nash-in-Nash bargaining solution without requiring firms to behave independently across bilateral bargains. By supplying a reasonable theoretical foundation for the Nash-in-Nash bargaining solution, this paper provides justification for its use in recent and ongoing applied work. Furthermore, although there exist alternative theoretical solution concepts for bargaining amongst multiple agents, the Nash-in-Nash solution has proven particularly well suited for the empirical analysis of bilateral oligopoly given its ability to nest Bertrand-Nash price setting models (hence providing a natural extension to previous approaches) and its tractability (which is critical given the complexity of combining theory

[3] This solution can also be interpreted as a *contract-equilibrium* in the spirit of Cremer and Riordan (1987).

[4] See, for instance, Crawford and Yurukoglu (2012): "Each distributor and each conglomerate sends separate representatives to each meeting. Once negotiations start, representatives of the same firm do not coordinate with each other. We view this absence of informational asymmetries as a weakness of the bargaining model," (p. 659). We spell out this argument more formally in Appendix A. See also Björnerstedt and Stennek (2007) and Inderst and Montez (2014) which provides a proof of existence in a similar setting with separate representatives.

with data in the settings analyzed).

We consider a framework that we believe is a natural extension of Rubinstein (1982) in which multiple "upstream" and "downstream" players—from now on "firms"—make simultaneous alternating offers. Each period, upstream and downstream firms earn flow payoffs which are a function of the set of agreements that have been reached; these payoffs are primitives of the analysis. An agreement consists of a payment made by a downstream firm an upstream firm for joining that downstream firm's "network," or set of contracting partners.[5] In odd periods, each downstream firm makes simultaneous offers to each upstream firm with which it has not yet reached an agreement. Each upstream firm then accepts or rejects any subset of its offers. Even periods are identical, except with upstream firms making the offers and downstream firms accepting or rejecting. Offers cannot be renegotiated after being accepted, flow payoffs are realized at the end of each period as a function of reached agreements, and agents have heterogeneous discount factors over future profits. We also do not restrict attention to stationary or "Markov" strategies, and allow for firms to condition their actions on the entire past history of offers, acceptances, and rejections.

Crucially, our model admits the possibility that firms can jointly deviate across multiple negotiations and hence optimally respond to information acquired from one of its negotiations in its other negotiations. This also implies that our game has imperfect information: within a period, any given firm does not see offers made to other firms. To proceed, we place restrictions on firm beliefs following the observation of an off-equilibrium offer and employ Perfect Bayesian Equilibrium with passive beliefs (henceforth, passive-beliefs equilibrium) as our solution concept. Passive beliefs implies that a firm i, upon receiving an off-equilibrium offer from firm j, assumes that j and all other firms still make equilibrium offers to their other contracting partners. This solution concept and refinement on beliefs has been widely used and employed in the vertical contracting literature (Hart and Tirole (1990), McAfee and Schwartz (1994); c.f. Whinston (2006)).

We make two principal restrictions on the payoff functions for our results: (i) given all other agreements have been made, the joint surplus from any two agents coming to an agreement is positive; and (ii) the marginal contribution of any bilateral agreement to a firm's payoff is weakly lower when all agreements among all firms has been reached than when any subset of agreements has been reached. Both assumptions are central for the full set of agreements to be "stable" at the proposed Nash-in-Nash bargaining solution prices. If the first assumption is violated for any bilateral pair, there would be no payment such that both

[5]We restrict our analysis to the case where the prices are lump-sum payments. E.g., if downstream firms engage in price competition for consumers, the negotiated prices with upstream firms would represent fixed fees. Because of this, only the presence of agreements, but not their prices, affect the value of other agreements.

parties would with to maintain an agreement (given all other agreements are formed). In this case, it is likely that these "unstable" agreements would not be reached (although our bargaining protocol would then potentially be applicable to a smaller set of potential agreements that may form).[6] If the second assumption is violated, then some firm may wish to drop multiple agreements: the gains to some set of agreements may be offset by the required payments (which are a function of the marginal contribution of each individual agreement) to maintain them. In settings where there may be complementarities across agreements, another surplus division protocol (e.g., multilateral bargaining, cooperative solution concepts such as the Shapley value) not predicated on bilateral bargaining may be more appropriate.

This paper has two main results. The first proves that, given the above two assumptions, there exists a passive-belief equilibrium which involves immediate agreement among all agents with negotiated prices that, as the time between periods goes to 0, converge to the Nash-in-Nash solution with Nash bargaining weights being a function of each firm's discount factor. The second proves that, with an additional assumption on underlying payoffs (or, in exchange for a weaker assumption, an equilibrium refinement on strategies), *every* passive-belief equilibrium also satisfies these properties, and hence the outcome of any equilibrium is unique.

We view the proof of our uniqueness result as our primary technical contribution. The proof proceeds by induction on the number or set of agreements which have not yet been reached at some point in time (which we call "open" agreements). We begin by noting that Rubinstein proves that any subgame with only one open agreement will result in immediate agreement at our candidate equilibrium prices (i.e., the Nash-in-Nash payments); this is our base case. Now we consider a subgame where the set of multiple open agreements is $\mathcal{C}$. The key to our result is proving our inductive step: if all equilibria for any subgame with fewer open agreements than contained in $\mathcal{C}$ yields immediate agreement at the Nash-in-Nash prices, then any equilibria where the set of open agreements is $\mathcal{C}$ also yields immediate agreement at the Nash-in-Nash prices. Once this is proven, the uniqueness result follows directly for any arbitrary game with multiple firms on both sides of the market.

We prove our inductive step in a series of cases. First, we consider subgames with open agreements involving only one downstream firm, and prove that if the first agreement happens in either an odd or even period, all open agreements must occur in that period; furthermore, we prove that there cannot be delay and any periods without an agreement being reached. Using a similar structure and techniques, we then prove that this also holds for subgames with open agreements involving only one upstream firm, and then for subgames where there

[6]We focus on bargaining and surplus division for a fixed network in this paper; endogenizing the network that is formed is outside the scope of the current analysis (c.f. Lee and Fong (2013)).

are open agreements involving multiple upstream and downstream firms.

Our paper is related to a literature on multilateral and coalitional bargaining with more than two players, which includes papers by Chatterjee, Dutta, Ray, and Sengupta (1993); Merlo and Wilson (1995); Krishna and Serrano (1996); Chae and Yang (1994) (c.f. Osborne and Rubinstein (1994); Muthoo (1999)). Our setting and extensive form game departs from this literature in at least three distinct ways. First, many previous papers allowed for only one offer at a time (e.g., a random proposer model) while our paper allows for simultaneous offers. Second, we focus on environments where agents can be divided in two distinct groups (i.e., upstream and downstream firms). Third, we restrict attention to bilateral surplus division, ruling out transfers between agents who do not have an agreement, such as side payments among firms on the same side of the market, as these would generally violate antitrust laws. We leverage these modeling choices, motivated by our focus on bilateral oligopoly, in deriving our results.

The remainder of our paper is divided as follows. Section 2 describes our extensive form bargaining protocol, equilibrium concept, and main assumptions. Section 3 and 4 are the heart of the paper, and state the main results (existence and uniqueness) and provide an overview of and intuition for our proofs. Section 5 discusses caveats and extensions of our analysis, and Section 6 concludes.

2 Model

Consider the negotiations between N upstream firms, $U_1, U_2, \ldots, U_N$, and M downstream firms, $D_1, D_2, \ldots, D_M$. Let $\mathcal{G}$ represent the set of agreements (also referred to as *contracts* or *links*) among all firms, and $\mathcal{A} \subseteq \mathcal{G}$ represent any subset of agreements. We only permit agreements between downstream and upstream firms; i.e., we only consider *bipartite* bargaining environments in which downstream firms contract with upstream firms, not with each other.[7] Denote an agreement between U_i and D_j as ij; the set of potential agreements that U_i can form as $\mathcal{G}_i^U$; and the set of agreements that D_j can form as $\mathcal{G}_j^D$.

Figure 1 provides a graphical representation of this market. In this example, $\mathcal{A} = \{11, 22, 23\}$, indicating that 3 of 9 possible agreements have been formed.

We take as primitives of the model profit functions $\{\pi_i^U(\mathcal{A})\}_{\forall i \in N, \mathcal{A} \subseteq \mathcal{G}}$ and $\{\pi_j^D(\mathcal{A})\}_{\forall j \in M, \mathcal{A} \subseteq \mathcal{G}}$, which represent the surplus realized by upstream and downstream firms for any realized set

[7]In many market settings, contractual agreements between two firms on the same side of the market can be interpreted as collusion and hence constitute per se antitrust violations. Alternatively, agreements between two firms on the same side of the market can be viewed as a horizontal merger, in which case our analysis would treat those merged firms as one entity. We do not explicitly model the determination of such mergers in this paper.

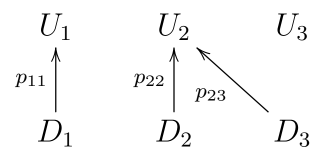

Figure 1: M Downstream Firms, N Upstream Firms Market

of agreements $\mathcal{A}$. Importantly, the payoffs from an agreement may depend on the set of other agreements reached, which allows for the possibility of contracting externalities (i.e., D_j's profits depend on D_k's agreements, $k \neq j$). We assume each upstream firm U_i and downstream firm D_j negotiate over *price* p_{ij}, which represents the lump-sum payment made from D_j to U_i for forming an agreement (e.g., in the healthcare example, an agreement would represent a hospital joining an insurer's network and serving its patients). Because we are assuming prices are lump-sum, surplus to other parties depends on the set of agreements reached but not on the negotiated prices.[8]

We model a dynamic game with infinitely many discrete periods. Periods are indexed $t = 1, 2, 3, \ldots$, and the time between periods is Λ. Payoffs for each firm are discounted. The discount factors for an upstream and a downstream firm are represented by $\delta_{i,U}$ and $\delta_{i,D}$, where $\delta_{i,k} \equiv \exp(-r_{i,k}\Lambda)$ for $k \in \{U, D\}$.[9]

The game begins in period $t_0 \geq 1$ with no agreements reached. In odd periods, each downstream firm D_j simultaneously offers contracts $\{p_{ij}\}_{ij \in \mathcal{G}_j^D}$ to each U_i with which it does not yet have an agreement; each U_i then simultaneously accepts or rejects any offers it received. In even periods, each upstream firm U_i simultaneously offers contracts $\{p_{ij}\}_{ij \in \mathcal{G}_i^U}$ to downstream firms with which it does not yet have an agreement; each D_j then simultaneously accepts or rejects any contract offers it received. If D_j accepts an offer from U_i, or U_i accepts an offer from D_j, then an agreement (or contract or link) is formed between two firms, and those two firms remain contracted with one another for the rest of the game. Each U_i receives its payment from D_j, p_{ij}, immediately in the period in which an agreement is reached.

We assume that within a period, a firm only observes the set of contracts that it offers, or that are offered to it. However, at the end of any period, we assume that all firms observe

[8] Suppose instead that profits to each firm depends on not only the set of agreements reached by all agents, $\mathcal{G}$, but also the set of prices agreed upon, $\mathbf{p} \equiv \{p_{ij}\}_{ij \in \mathcal{G}}$: i.e., payoffs to each D_j are given by $\pi_j(\mathcal{G}, \mathbf{p})$. This would be the case if, for instance, negotiated prices represented wholesale prices or linear fee contracts, and downstream firms engaged in price competition with one another. Dealing with bargaining in a context without transferable utility is difficult. Indeed, to our knowledge, this issue has not been resolved in the context of a two player, Rubinstein (1982) bargaining game, let alone the environment considered in this paper with multiple upstream and downstream firms.

[9] The model can also be recast without discounting but with an exogenous probability of breakdown occurring after the rejection of any offer as in Binmore, Rubinstein, and Wolinsky (1986).

all contracts that have been offered in that period, and which (if any) contracts that have been accepted.[10]

This implies that at the beginning of each period, every firm observes a common *history of play* h^t which contains the sequence of all actions (offers and acceptance/rejections) that have been made by every firm in each preceding period.

As an example, let $N = 2$ and $M = 1$ so that there is only 1 downstream firm. If D_1 reaches agreement with U_1 and U_2 at $t = 1$, D_1 would pay would pay p_{11} and p_{21} immediately to each upstream firm, and then earn $(1-\delta_{1,D})\pi_1^D(\{11, 21\})$ each period going forward; each U_i would immediately receive p_i, and earn profits $(1 - \delta_{i,U})\pi_i^U(\{11, 21\})$ from $t = 1$ onwards. If D_1 reached agreement with U_1 in period 1 and U_2 in period 2, then it would pay p_{11} in period 1 and p_{21} in period 2, and earn gross revenues of $(1 - \delta_{1,D})\pi_j^D(\{11\})$ in period 1 and $(1 - \delta_{1,D})\pi_1^D(\{11, 21\})$ from period 2 onwards.[11]

Two points about our model are worth noting. First, while the payoffs continue to accrue to all firms forever, the actions in the game stop at the point of the last agreement. Thus, the game can also be formulated to end in the period of last agreement, with a lump-sum payment realized by all firms at this time. Second, if $M = N = 1$, our game is equivalent to the Rubinstein (1982) alternating offers model.

2.1 Equilibrium Concept

Rubinstein (1982) considers subgame perfect equilibria of his model. Because our model has imperfect information, our solution concept is perfect Bayesian equilibrium. However, perfect Bayesian equilibrium does not place restrictions on beliefs for information sets that are not reached in equilibrium; in particular, it does not restrict beliefs of an upstream firm U_i over offers received by other firms upon receiving an out-of-equilibrium price offer from D. Following the literature on vertical contracting (Hart and Tirole, 1990; McAfee and Schwartz, 1994; Segal, 1999), we assume "passive beliefs": i.e., each firm U_i assumes that other firms receive equilibrium offers even when it observes off-equilibrium offers from D_j.

[10] Institutionally, the contracted price between U_i and D_j will generally not be observed by $U_j, j \neq i$, either for competitive or antitrust concerns. Relaxing this assumption does not change this model, as contracted prices here do not affect the surplus to be divided. All our results will hold as long as the identity of firms reaching an agreement is publicly known at the end of each period.

[11] We express profits in terms of flows, since we believe this is a more accurate depiction of many markets. In contrast, profits are paid as a lump sum in the Rubinstein model. However, our formulation is equivalent to D_1 receiving the incremental profits as a lump sum (e.g., if agreements $\mathcal{A}$ were reached in period 1 and agreements $\mathcal{B}$ were reached in period 2, then D_1 would receive $\pi_1^D(\mathcal{A})$ in period 1 and $\pi_1^D(\mathcal{A} \cup \mathcal{B}) - \pi_1^D(\mathcal{A})$ in period 2. We avoid payments between downstream and upstream firms other than lump sum transfers, or otherwise, since each party has a potentially different discount rate, loans could be made between upstream and downstream agents that lead to unbounded increases in the utilities of both parties.

Henceforth, when we refer to an "equilibrium" of this game, we are referring to a perfect Bayesian equilibrium with passive beliefs.

2.2 Nash-in-Nash and Rubinstein Payoffs

For exposition, it will be useful to define $\Delta\pi_j^D(\mathcal{A},\mathcal{B}) \equiv \pi_j^D(\mathcal{A}) - \pi_j^D(\mathcal{A}\setminus\mathcal{B})$, for $\mathcal{B}\subseteq\mathcal{A}\subseteq\mathcal{G}$. This term is the increase in profits to D_j of adding agreements in $\mathcal{B}$ to the set of agreements $\mathcal{A}\setminus\mathcal{B}$. One can think of $\Delta\pi_j^D(\mathcal{A},\mathcal{B})$ as the "marginal contribution" of agreements $\mathcal{B}$ given agreements $\mathcal{A}$ have been reached. Correspondingly, let $\Delta\pi_i^U(\mathcal{A},\mathcal{B}) \equiv \pi_i^U(\mathcal{A}) - \pi_i^U(\mathcal{A}\setminus\mathcal{B})$.

We first define the Nash-in-Nash payoffs for our game and the candidate set of prices determined in our equilibrium.

For a given set of agreements $\mathcal{G}$ and set of bargaining weights $\{b_{j,D}\}_{\forall j}$ and $\{b_{i,U}\}_{\forall i}$, the Nash-in-Nash payoffs are a vector of prices $\{p_{ij}^N\}_{i\in\{1,\ldots,N\},j\in\{1,\ldots,M\}}$ such that:

$$\begin{aligned} p_{ij}^N &= \arg\max_p [\pi_j^D(\mathcal{G}) - \pi_j^D(\mathcal{G}\setminus ij) - p]^{b_{j,D}} \times [\pi_i^U(\mathcal{G}) - \pi_i^U(\mathcal{G}\setminus ij) + p]^{b_{i,U}} \\ &= \frac{b_{i,U}\Delta\pi_j^D(\mathcal{G},ij) - b_{j,D}\Delta\pi_i^U(\mathcal{G},ij)}{b_{i,U}+b_{j,D}}\ ,\forall i=1,\ldots,N, j=1,\ldots,M. \end{aligned}$$

In words, the Nash-in-Nash payoff p_{ij}^N maximizes the Nash bargaining product between D_j and U_i given all other agreements in $\mathcal{G}$ are reached. The terms $b_{i,U}$ and $b_{j,D}$ are the bargaining weights of the Nash bargaining problem, which determine the portion of the surplus accruing to each firm.

For our analysis, we also define:

$$\begin{aligned} p_{ij,U}^R &= \frac{(1-\delta_{j,D})\Delta\pi_j^D(\mathcal{G},ij) - \delta_{j,D}(1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G},ij)}{1-\delta_{i,U}\delta_{j,D}} \\ p_{ij,D}^R &= \frac{\delta_{i,U}(1-\delta_{j,D})\Delta\pi_j^D(\mathcal{G},ij) - (1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G},ij)}{1-\delta_{i,U}\delta_{j,D}}. \end{aligned}$$

They will be the candidate even and odd offers made in equilibrium by firms; when $M=N=1$, they correspond to the Rubinstein (1982) offers made in alternating periods. As in Binmore, Rubinstein, and Wolinsky (1986), these candidate prices also converge to the Nash-in-Nash prices as the time period between offers becomes arbitrarily small:

Lemma 2.1 $\lim_{\Lambda\to 0} p_{ij,U}^R = \lim_{\Lambda\to 0} p_{ij,D}^R = p_{ij}^N$, *where* $b_{i,U} = r_{j,D}/(r_{i,U}+r_{j,D})$ *and* $b_{j,D} = r_{i,U}/(r_{i,U}+r_{j,D})$.

(All proofs are contained in the Appendix.)

Finally, note that Rubinstein payoffs make the agent that receives an offer indifferent between accepting the offer or waiting until next period and having its counteroffer accepted. In our case, in an even (upstream-proposing) period, this means that the downstream firm is indifferent between accepting and waiting, or:

$$\underbrace{(1-\delta_{j,D})\Delta\pi_j^D(\mathcal{G},ij)}_{\text{Loss in profit from waiting}} = \underbrace{p_{ij,U}^R - \delta_{j,D}p_{ij,D}^R}_{\text{Decrease in transfer payment from waiting}} . \tag{1}$$

Correspondingly, for the upstream firm in odd periods,

$$(1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G},ij) = \delta_{i,U}p_{ij,U}^R - p_{ij,D}^R. \tag{2}$$

Also, note that:

$$p_{ij,U}^R - p_{ij,D}^R = \frac{(1-\delta_{j,D})(1-\delta_{i,U})}{(1-\delta_{i,U}\delta_{j,D})}(\Delta\pi_j^D(\mathcal{G},ij) + \Delta\pi_i^U(\mathcal{G},ij)).$$

2.3 Assumptions

We now state the main assumptions that we leverage in our analysis.

Our first assumption states that the joint surplus created from U_i and D_j coming to an agreement (given all other agreements have been formed) is positive:

Assumption 2.2 (A.GFT: Gains From Trade)

$$\Delta\pi_j^D(\mathcal{G},ij) + \Delta\pi_i^U(\mathcal{G},ij) > 0 \qquad \forall i,j$$

The Gains from Trade (GFT) Assumption is necessary for all agreements to be formed and maintained in equilibrium. Since:

$$\begin{aligned}(\Delta\pi_j^D(\mathcal{G},ij) - p_{ij,D}^R) &= \frac{(1-\delta_{i,U})}{(1-\delta_{i,U}\delta_{j,D})}(\Delta\pi_j^D(\mathcal{G},ij) + \Delta\pi_i^U(\mathcal{G},ij))\\ (\Delta\pi_i^U(\mathcal{G},ij) + p_{ij,U}^R) &= \frac{(1-\delta_{j,D})}{(1-\delta_{i,U}\delta_{j,D})}(\Delta\pi_j^D(\mathcal{G},ij) + \Delta\pi_i^U(\mathcal{G},ij))\end{aligned}$$

A.GFT also implies that firms will not wish to unilaterally drop agreements at the candidate "Rubinstein prices"; i.e.,:

$$\begin{aligned}\Delta\pi_j^D(\mathcal{G},ij) &> p_{ij,U}^R > p_{ij,D}^R\\ \Delta\pi_i^U(\mathcal{G},ij) &> -p_{ij,D}^R > -p_{ij,U}^R\end{aligned} \tag{3}$$

Our next assumption states that the surplus created from an agreement between D_j and U_i is decreasing in the set of agreements already reached by all players for both D_j and U_i:

Assumption 2.3 (A.CDMC: Conditional Decreasing Marginal Contribution)

$$\begin{aligned}\Delta\pi_j^D(\mathcal{E}, ij) &\geq \Delta\pi_j^D(\mathcal{G}, ij) \qquad \forall ij \in \mathcal{E}, \forall \mathcal{E} \subseteq \mathcal{G} \\ \Delta\pi_i^U(\mathcal{E}, ij) &\geq \Delta\pi_i^U(\mathcal{G}, ij) \qquad \forall ij \in \mathcal{E}, \forall \mathcal{E} \subseteq \mathcal{G}\end{aligned}$$

Both of these assumptions are sufficient for the observed network to be "stable" at the Nash-in-Nash prices: i.e., for any set of Nash Bargaining weights, no firm would wish deviate and unilaterally drop any subset of its agreements. To see this, note that any downstream firm D_j's gain from a subset of $\mathcal{K}$ agreements at the Nash-in-Nash prices is strictly positive:

$$\begin{aligned}\Delta\pi_j^D(\mathcal{G}, \mathcal{K}) - \sum_{ij \in \mathcal{K}} p_{ij}^N &= \Delta\pi_j^D(\mathcal{G}, \mathcal{K}) - \sum_{ij \in \mathcal{K}} \frac{b_{i,U}\Delta\pi_j^D(\mathcal{G}, ij) - b_{j,D}\Delta\pi_i^U(\mathcal{G}, ij)}{b_{i,U} + b_{j,D}} \\ &> \sum_{ij \in \mathcal{K}} [\Delta\pi_j^D(\mathcal{G}, ij) - \frac{b_{i,U}\Delta\pi_j^D(\mathcal{G}, ij) - b_{j,D}\Delta\pi_i^U(\mathcal{G}, ij)}{b_{i,U} + b_{j,D}}] > 0 \qquad \forall \mathcal{K} \in \mathcal{G}_j^D\end{aligned} \tag{4}$$

where the second line follows from A.CDMC and A.GFT. Similarly it can be shown that the same holds for any upstream firm U_i:

$$\Delta\pi_i^U(\mathcal{G}, \mathcal{A}) + \sum_{ij \in \mathcal{A}} p_{ij}^N > 0 \qquad \forall \mathcal{A} \subseteq \mathcal{G}_i^U \tag{5}$$

A.CDMC is satisfied by many of the applications of the Nash-in-Nash solution concept. For instance, in Capps, Dranove, and Satterthwaite (2003) adding another hospital to the choice set increases surplus, but this increase in surplus is decreasing in the size of the network.[12]

A counterexample is useful to illustrate why A.CDMC is crucial for the full network of agreements to be stable at the Nash-in-Nash prices. Consider the following example which violates the assumption, which we call the "automobile supplier" example. Suppose that there are three upstream firms (parts suppliers) which each supply a component that is

[12] Capps, Dranove, and Satterthwaite (2003) show that the profit for an insurer is related to the ex ante surplus received by enrollees from the insurer's network of hospitals. For a logit model, the total surplus of the insurer's network $\mathcal{H}$ can be expressed as $\sum_i \log\left(\sum_{j\in\mathcal{H}} u_{ij}\right)$ where u_{ij} is the exponentiated utility (net of an i.i.d. Type I extreme value error) that patient i receives from visiting hospital j and the 'i' sum is over the patients of the insurer. The marginal contribution of some hospital $k \notin \mathcal{H}$ to the insurer's network—denoted willingness-to-pay—is thus $WTP = \sum_i \log\left(u_{ik} + \sum_{j\in\mathcal{H}} u_{ij}\right) - \sum_i \log\left(\sum_{j\in\mathcal{H}} u_{ij}\right)$, which can be shown to be decreasing as we add elements to $\mathcal{H}$. The diminishing returns property also holds more generally, e.g. with random coefficients logit models (Berry, Levinsohn, and Pakes, 1995).

indispensable for production to the downstream firm (automobile manufacturer). As the marginal contribution to total surplus of each upstream firm is the total surplus, the Nash-in-Nash payoffs with equal bargaining weights would give each upstream firm half of the total surplus. But, this would then leave the downstream firm with a negative payoff since it pays 3/2 of the total surplus to the upstream suppliers, implying the downstream firm would not wish to reach agreement at these prices with all firms. In this model, then, it is implausible that transfers will be based on marginal contributions; either a subset of agreements will be reached, or concepts based on average values, such as the Shapley Value, may be more appropriate for the determination of surplus division. As we will show, Nash-in-Nash prices make accepting agents indifferent about adding any particular agreement when all other equilibrium agreements are formed. If marginal contributions are increasing, then the accepting agents will strictly prefer to remove several agreements, implying the full network of agreements will not be an equilibrium outcome. Notice as well that the previous rationalization of the "Nash-in-Nash" solution concept using representatives negotiating each agreement in separate rooms (Crawford and Yurukoglu, 2012) does not rule out the automobile supplier example.[13]

3 Existence of Equilibrium

Our first result is that there exists an equilibrium of this game generating immediate agreement at the Rubinstein prices, which converge to the "Nash-in-Nash" prices as the time between periods goes to 0.

Theorem 3.1 (Existence.) *Assume A.GFT and A.CDMC. Then there exists an equilibrium of the bargaining game beginning at period t_0 with:*

(a) immediate agreement between all agents at t_0;

(b) equilibrium prices $p^*_{ij} = p^R_{ij,D}$ $\forall i,j$ if t_0 is odd, and $p^*_{ij} = p^R_{ij,U}$ $\forall i,j$ if t_0 is even; and

(c) $p^*_{ij} \to p^N_{ij}$ $\forall i,j$ as $\Lambda \to 0$ regardless of whether t_0 is odd or even period, where $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$ and $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$.

[13] Another, more formal example: consider a one upstream, two downstream firm example with equal discount factors, and payoffs of $\pi^D_j(\{1,2\}) = 10$, $\pi^D_j(\{1\}) = \pi^D_j(\{2\}) = 4$, and $\pi^U_i(\cdot) = 0$ $\forall i$, with $\pi^U_j = 0$. The Nash-in-Nash transfers are $p^R_i = \frac{1}{2}\left(\pi^D_j(\{1,2\}) - \pi^D_j(\{1,2 \setminus i\})\right) = \frac{1}{2}(10-4) = 3$. In even periods, our equilibrium makes D_j indifferent between dropping either U_1 or U_2 and keeping both. Since dropping one lowers surplus by 6 but dropping both lowers surplus only by 4 more (but doubles payments), this means that D_j will be strictly better off by dropping both firms and waiting to the following odd period to make an offer, which then breaks our candidate equilibrium through an inframarginal deviation. These type of inframarginal deviations are ruled out by Assumption 2.3.

The proof, contained in the appendix, first proposes a candidate equilibrium where in odd periods (downstream proposing), the downstream firms offer $p^R_{ij,D}$ to all upstream firms with which they have not yet reached an agreement. Upstream firms choose to accept any offer at or above $p^R_{ij,D}$. Likewise, in even periods (upstream proposing), upstream firms propose $p^R_{ij,U}$ to all downstream firms with which they have not yet reached an agreement, and downstream firms choose to accept any offer at or above $p^R_{ij,U}$. We then show that any one-shot deviation from these strategies on the part of either sending or receiving parties does not make them better off.

The restriction to strategies that satisfy passive beliefs puts structure on what happens following a deviation from equilibrium strategies. In particular, if firm D_j makes an out-of-equilibrium offer $\tilde{p}_{ij} < p^R_{ij,D}$ to firm U_i, then U_i believes that D_j has offered $p^R_{kj,D}$ (the equilibrium offers) to all other upstream firms U_k. This means that when U_i considers what will happen following a deviant offer, it expects all agreements to be signed, except for the one between D_j and U_i. Since, at this point, there is a single outstanding agreement to be negotiated over, this subgame is precisely the one studied by Rubinstein (1982), and it has a unique equilibrium with payments of $p^R_{ij,U}$ in the following period.

It is clear that A.GFT is essential for this set of strategies to be an equilibrium. Whenever A.GFT is violated, firms might find it profitable to drop this single agreement; i.e., engage in a marginal deviation. The role of A.CDMC is more complex, with violations of this assumption leading to cases where firms may want to drop groups of agreements.

4 Uniqueness of Equilibrium Outcome with Nash-in-Nash Transfers

The second result of our paper is that, under stronger assumptions, *every* perfect Bayesian equilibrium with passive beliefs satisfies the conditions of Theorem 3.1: i.e., agreement between all firms is immediate at the Rubinstein prices, which converge to the "Nash-in-Nash" prices as the time between periods goes to 0. If there are multiple equilibria of this game, they will only vary in prescribed behavior *off* the equilibrium path; they will all result in this same outcome on the equilibrium path.[14]

To prove that the equilibrium outcome is unique, we will use a strenghtening of our A.CDMC assumption, and an equilibrium refinement:

[14] Appendix E provides an example where there are multiple equilibria that vary in off-equilibrium-path actions, but coincide along the equilibrium path.

Assumption 4.1 (A.CDMC′: Strong Conditional Decreasing Marginal Contribution)

$$\pi_j^D(\{\mathcal{A}_i \cup ij, \mathcal{A}_{-i}\}) - \pi_j^D(\{\mathcal{A}'_i, \mathcal{A}_{-i}\}) \geq \Delta\pi_j^D(\mathcal{G}, ij) \qquad \forall ij \in \mathcal{G}; \mathcal{A}_{-i} \subseteq \mathcal{G}^U_{-i}; \mathcal{A}_i, \mathcal{A}'_i \subseteq \mathcal{G}^U_i \setminus ij$$
$$\pi_i^U(\{\mathcal{A}_j \cup ij, \mathcal{A}_{-j}\}) - \pi_i^U(\{\mathcal{A}'_j, \mathcal{A}_{-j}\}) \geq \Delta\pi_i^U(\mathcal{G}, ij) \qquad \forall ij \in \mathcal{G}; \mathcal{A}_{-j} \subseteq \mathcal{G}^D_{-j}; \mathcal{A}_j, \mathcal{A}'_j \subseteq \mathcal{G}^D_j \setminus ij$$

A.CDMC′ implies A.CDMC, and states that at any subnetwork, the marginal contribution realized by D_j for coming to an agreement with U_i is at least as much as the marginal contribution of D_j coming to agreement with U_i in the full network, even if U_i (and only U_i) were to change any of its other agreements. A similar condition holds for any upstream firm U_i's gain to an agreement with D_j.

Assumption 4.2 (A.ASR: Acceptance Strategies Refinement) *We restrict attention to equilibria in which: if any firm, given the strategies of all other firms, is weakly willing to accept an offer (holding fixed its other prescribed actions), it accepts that offer.*

A.ASR rules out equilibria in which strategies prescribe a firm (given the strategies of other firms and its other actions) rejecting an offer that it is indifferent over accepting or rejecting.

Alternatively, we also can prove our uniqueness result by utilizing a stronger assumption on underlying payoffs without imposing the additional equilibrium refinement:

Assumption 4.3 (A.LEXT: Limited Externalities)

$$\pi_i^U(\mathcal{A}_i, \mathcal{A}_{-i}) = \pi_i^U(\mathcal{A}_i, \mathcal{A}'_{-i}) \qquad \forall i;\ \forall \mathcal{A}_i \subseteq \mathcal{G}^U_i;\ \forall \mathcal{A}_{-i}, \mathcal{A}'_{-i} \subseteq \mathcal{G}^U_{-i}$$
$$\pi_j^D(\mathcal{A}_j, \mathcal{A}_{-j}) = \pi_j^D(\mathcal{A}_j, \mathcal{A}'_{-j}) \qquad \forall j;\ \forall \mathcal{A}_j \subseteq \mathcal{G}^D_j;\ \forall \mathcal{A}_{-j}, \mathcal{A}'_{-j} \subseteq \mathcal{G}^D_{-j}$$

A.LEXT states that each firm's profits depend only on its own links formed, and not those of others. It is straightforward to prove that A.LEXT and A.CDMC imply A.CDMC′, and in this sense A.LEXT is stronger than A.CDMC′.

We now state our uniqueness result:

Theorem 4.4 *Assume A.GFT, and* **either** *(i) A.CDMC′ and A.ASR; or (ii) A.CDMC and A.LEXT. Then* every *equilibrium of the bargaining game satisfies the conditions in Theorem 3.1 with immediate agreement at t_0, prices $p^*_{ij} = p^R_{ij,D}$ ($p^R_{ij,U}$) if t_0 is odd (even), and prices $p^*_{ij} \to p^N_{ij}$ as $\Lambda \to 0$, where $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$ and $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$.*

In the following subsections, we will provide a discussion of the assumptions, an example of the proof with two upstream firms and one downstream firm, and an outline of the proof in the general case with multiple upstream and downstream firms. Further details and formal proofs are contained in the appendix.

4.1 Discussion of Assumptions

The first additional restriction on payoffs, A.CDMC′ (which is either assumed or implied by A.CDMC and A.LEXT), is used to ensure that firms will not wish to strategically delay agreement in an equilibrium: for example, if D_j benefits from an agreement that U_i has with D_k, but U_i would accept D_j's offer in a given period instead of D_k, then D_j might have a strategic incentive to delay agreement with U_i. However, given A.CDMC′, D_j would not wish to do so. The difference between A.CDMC′ and A.CDMC is that when evaluating one firm's gains from a given bilateral agreement with which it is involved, the agreements of the other firm involved in the same bilateral agreement are allowed to change.

We use either A.ASR or A.LEXT to ensure that an offering firm is not paid less than its Rubinstein price, due to an off-equilibrium threat by the recipient firm to add or drop another offer it is indifferent over if a higher price is demanded. I.e., consider a candidate equilibrium in which in the first period (even) t_0, U_i offers D_j the price $\hat{p}_{ij} = p^R_{ij,U} - \varepsilon$ for $\varepsilon > 0$. In this period, assume that D_j also comes to agreement with U_k, but D_j is indifferent between accepting and rejecting this offer given agreement is also reached with U_i in this period (and given strategies for continuation play if the offer is rejected). If U_i were to engage in a deviation and demand a higher payment $\tilde{p}_{ij} = p^R_{ij,U}$, D_j could threaten to accept the deviation $\tilde{p}_{ij}$ from U_i, but reject the offer from U_k (and come to agreement with U_k in the next odd period $t_0 + 1$). Since D_j was originally indifferent over accepting and rejecting U_k's offer at t_0, such a threat is credible. Furthermore, if U_i's profits positively depend on whether or not D_j comes to agreement with U_k or not, then such a deviation may not be worthwhile if the loss in profits to U_i from D_j rejecting U_k outweigh the increase in payment. A.ASR rules out the possibility of this threat being made; on the other hand, A.LEXT rules out the possibility that U_i would be deterred by this threat (since U_i's' profits would not depend on D_j's actions with regards to U_k).[15]

Remarks. A particular setting where both A.CDMC′ and A.LEXT are satisfied is when there are $N \geq 1$ firms on one side of the market each with profits (net of transfers) that are constant (e.g., zero), and only one firm on the other side of the market with profits (net of

[15] In Appendix E, we detail an equilibrium that results in immediate agreement with a firm receiving greater than its Rubinstein price if both A.LEXT and A.ASR do not hold; however, in this equilibrium, it still is the case that as $\Lambda \to 0$, prices converge to the Nash-in-Nash prices. Whether all equilibria converge to the Nash-in-Nash prices without assuming either A.LEXT or A.ASR is an open question.

transfers) satisfying A.CDMC.

One example of A.LEXT holding is bargaining between a monopolist cable distributor and many content providers using a model such as in Crawford and Yurukoglu (2012) with lump-sum transfers instead of linear fees: since the content providers typically have zero marginal costs of providing their channels to cable operators, their profits (net of transfers) will typically not depend on the agreements of other channels.

Another example is a special case of negotiations between many hospitals and one managed care organization (MCO), similar to the model used in Capps, Dranove, and Satterthwaite (2003). Suppose that the hospital's cost function has constant marginal costs c, and can be given by $C(q) = F + cq$ (where F is a fixed cost). Moreover, suppose that the MCO reimburses hospitals for the marginal cost of treating each patient, in addition to offering them lump-sum payments for joining their network. In this case, the hospital's profits will not depend on the contracts signed by other hospitals (thus satisfying A.LEXT), and the MCO's profits will generally satisfy A.CDMC (see footnote 12).

4.2 Example: Two Upstream Firms and One Downstream Firm (2x1)

We first provide an outline of the argument in a simple example with two upstream firms U_i, U_k, and one downstream firm D_j.

Consider a subgame where there is only one open agreement between U_i and D_j: this corresponds to the Rubinstein (1982) bargaining game, and results in immediate agreement at prices $p^R_{ij,D}$ if the period is odd (downstream proposing) or $p^R_{ij,U}$ if the period is even (upstream proposing).

To show that any equilibrium of this game with two open agreements satisfies the theorem, first consider an equilibrium in which the first agreement is reached in an odd period t between U_k and D_j, and it is the only one to occur in that period. Then the subgame beginning at $t+1$ will be a Rubinstein bargaining game resulting in prices $p^R_{ij,U}$. In this case, it is straightforward to show that D_j will find it profitable to "bring up" agreement with U_i to period t as well by offering $p^R_{ij,D}$ in period t to U_i, as U_i will find it profitable to accept. Hence, we have a contradiction, and if there is an equilibrium with an agreement in an odd period, all agreements must occur in that period. Furthermore, given that this is the case, it can be shown that any equilibrium with agreement in an odd period must have prices equal to $\hat{p}_{ij} = p^R_{ij,D}$ and $\hat{p}_{kj} = p^R_{kj,D}$: if the price is too high for U_i (say), the downstream firm will have an incentive to reduce the price to $p^R_{ij,D}$; if the price is below $p^R_{ij,D}$, the U_i will wish to reject and—since the subgame beginning at the next period $t+1$ is Rubinstein bargaining again—will obtain $p^R_{ij,U}$ next period.

Next, consider an equilibrium in which only one agreement (which is the first) is reached in an even period t; again, assume that this is between U_k and D_j. We can show that if U_i offers D_j at most $p^R_{ij,U}$ at t, this will induce D_j to accept. However, since D_j may reject U_k upon accepting this offer (as nothing rules this out), we can leverage A.CDMC′ (again, either assumed or implied by A.CDMC and A.LEXT) to insure that U_i will still wish to engage in this deviation—i.e., U_i prefers to reach agreement with D_j in period t as opposed to $t+1$ regardless of whether or not D_j also comes to an agreement with U_k.[16] As before, this implies that if one agreement is reached in an even period, both agreements must be reached; this is a contradiction. To show that prices cannot be different than $p^R_{ij,U}$ in an even period where both agreements occur, first note that D_j will reject anything higher (and can obtain Rubinstein prices in the next $t+1$ odd period subgame by accepting at least one offer in the current period). Second, note that lower offers can be improved on by being raised without inducing D_j to reject. To ensure that, say, U_i would actually wish to raise an offer lower than $p^R_{ij,U}$, we leverage either A.ASR or A.LEXT to rule out the possibility (as discussed earlier) that D_j could threaten to reject U_k in response to such a deviation, thereby potentially harming U_i's profits.

Finally, an equilibrium without immediate agreement at t_0 cannot exist: if t_0 is odd, D_j will find it profitable to make offers to both U_i and U_k that will be accepted; and if t_0 is even, either U_i or U_k will find it profitable to make an early offer.

4.3 Structure of Proof

We now provide the structure of our general proof where there are $N \geq 1$ upstream firms and $M \geq 1$ downstream firms.

For any $\mathcal{C} \subseteq \mathcal{G}$, let $\Gamma^t_{\mathcal{C}}(h^t)$ represent the subgame beginning at period $t \geq t_0$ when there are still $\mathcal{C}$ "open" agreements, or agreements that have not been reached (i.e., all agreements $ij \in \mathcal{G} \setminus \mathcal{C}$ have been formed prior to period t), and history of play h^t. Recall the history at time t contains the sequence of actions, which include offers and acceptances/rejections, that have been made by all firms in all preceding periods. We will prove Theorem 4.4 by induction on $\mathcal{C}$ for any arbitrary t and history of play h^t.

The base case is provided by analyzing $\Gamma^t_{\mathcal{C}}(\cdot)$ when $|\mathcal{C}| = 1$: i.e., there is only one agreement in $\mathcal{G}$ that has not yet been reached at time t.

[16] As will be made clearer in the next subsection, we also need to prove that D_j does not want to reject both U_i and U_k in period t upon receiving an off equilibrium offer from U_i, which requires proving that D_j cannot obtain higher profits in the future upon doing so. This is more involved, and the intuition for this is provided in Section 4.4.

Proposition 4.5 (Base Case) *Let $|\mathcal{C}| = 1$ with only one open agreement: $\mathcal{C} \equiv \{ij\}$. Then the subgame $\Gamma^t_{\mathcal{C}}(h^t)$ for any $t \geq t_0$ and any history of play h^t (consistent with $\mathcal{C}$ being the set of open agreements at t) has a unique equilibrium involving immediate agreement at t with prices $\hat{p}_{ij} = p^R_{ij,D}$ if t is odd, and $\hat{p}_{ij} = p^R_{ij,U}$ if t is even.*

Proof With only one open agreement $ij \in \mathcal{C}$, D_i and U_j engage in a 2-player Rubinstein alternating offers bargaining game over joint surplus $\Delta\pi^U_i(\mathcal{G}, ij) + \Delta\pi^D_j(\mathcal{G}, ij)$, and the result directly follows from Rubinstein (1982). □

We now state the inductive hypothesis and inductive step used to prove Theorem 4.4.

Inductive Hypothesis. *Fix $\mathcal{C} \subseteq \mathcal{G}$, t, and h^t. For any $\mathcal{B} \subset \mathcal{G}$ such that $|\mathcal{B}| < |\mathcal{C}|$, any equilibrium of $\Gamma^{t'}_{\mathcal{B}}(h^{t'})$, where $t' > t$ and $h^{t'}$ contains h^t (and is consistent with $\mathcal{B}$ being the set of open agreements at t'), results in immediate agreement between U_i and D_j $\forall ij \in \mathcal{B}$ at prices $\hat{p}_{ij} = p^R_{ij,D}$ if t' is odd, and $\hat{p}_{ij} = p^R_{ij,U}$ if t' is even.*

The inductive hypothesis states that any subgame involving fewer open agreements than $|\mathcal{C}|$ results in immediate agreement at the Rubinstein prices. It implies that if any non-empty set of agreements are reached at any point during the subgame $\Gamma^t_{\mathcal{C}}(h^t)$ at period $t' \geq t$ so that only a strict subset $\mathcal{B} \subset \mathcal{C}$ of open agreements remain, then all remaining agreements $ij \in \mathcal{B}$ are reached in the subsequent period $t' + 1$ at $p^R_{ij,D}$ ($p^R_{ij,U}$) if $t' + 1$ is odd (even).

Proposition 4.6 (Inductive Step) *Assume A.GFT, and either (i) A.CDMC′ and A.ASR; or (ii) A.CDMC and A.LEXT. Consider any subgame $\Gamma^t_{\mathcal{C}}(h^t)$ where $\mathcal{C} \subseteq \mathcal{G}$, $t \geq t_0$. Given the inductive hypothesis, any equilibrium of $\Gamma^t_{\mathcal{C}}(h^t)$ results in immediate between U_i and D_j $\forall ij \in \mathcal{C}$ at prices $\hat{p}_{ij} = p^R_{ij,D}$ if t is odd, and $\hat{p}_{ij} = p^R_{ij,U}$ if t is even.*

The inductive step states that if the inductive hypothesis holds for any subgame with $\mathcal{C}$ open agreements, then this subgame also results in immediate agreement for all open agreements $ij \in \mathcal{C}$ at the Rubinstein prices.

Note that the Proposition 4.5 (Base Case) and Proposition 4.6 (Inductive Step) imply Theorem 4.4 by induction: as we have established the theorem holds when $|\mathcal{C}| = 1$, the inductive step implies that the theorem will hold for any $\mathcal{C} \subseteq \mathcal{G}$ when $|\mathcal{C}| \geq 1$.[17]

[17] The Theorem is also implied if the inductive hypothesis only held for strict subsets $\mathcal{B} \subset \mathcal{C}$ (as opposed to all subgames when there are fewer open agreements): starting with all subsets of open agreements with only two firms, we can construct larger and larger subsets of open agreements which ultimately will imply the main result holds for the initial bargaining game $\Gamma^{t_0}_{\mathcal{G}}$. I.e., given Propositions 4.5 and 4.6, the Theorem can be shown to hold for all 1×1 through $1 \times N$ and 1×1 through $M \times 1$ settings (where MxN refers to M downstream and N upstream firms). Once that is established, the Theorem can be shown to hold for all 2×2 through $2 \times N$ and 2×2 through $M \times 2$ settings. This argument can be repeated by induction in a similar fashion to obtain the result for $M \times N$.

To prove Proposition 4.6 (and by consequence, Theorem 4.4), we proceed in three steps: we first focus on subgames $\Gamma^t_{\mathcal{C}}(\cdot)$ where $\mathcal{C}$ contains only agreements involving one downstream firm (Section 4.4); we then focus on subgames where where $\mathcal{C}$ contains only agreements involving one upstream firm (Section 4.5); and finally, we focus on subgames where $\mathcal{C}$ contains more than one upstream and more than one downstream firm (Section 4.6). For expositional convenience, we will drop the history of play argument from $\Gamma^t_{\mathcal{C}}$ for the remainder of the text acknowledging that these subgames will be for any arbitrary history of play consistent with there being $\mathcal{C}$ open agreements at t (though we will still allow for history-dependent strategies to be played).

4.4 Proof of Proposition 4.6: One Downstream Firm, Many Upstream Firms

Consider any subgame $\Gamma^{\tilde{t}}_{\mathcal{C}}$ where $\mathcal{C} \subseteq \mathcal{G}$ contains only open agreements involving one downstream firm D_j, and $|C| = m$ so that there are $m > 1$ remaining agreements between D_j and m upstream firms that have not yet been reached at time $\tilde{t}$. WLOG, assume that thes upstream firms are indexed $\{1, \ldots, m\}$. Assume that the inductive hypothesis holds.

We prove Proposition 4.6 holds in this case using 4 lemmas. For these lemmas, consider a candidate equilibrium of the subgame with the first agreement $ij \in \mathcal{C}$ reached in period $t \geq \tilde{t}$, and accepted prices denoted $\{\hat{p}_{1j}, \ldots, \hat{p}_{mj}\}$. Let $\mathcal{A} \subseteq \mathcal{C}$ denote the set of agreements reached at period t. By the inductive hypothesis, all agreements $ij \in \mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ not reached at period t will reached in period $t+1$ at prices $p^R_{ij,D}$ ($p^R_{ij,U}$) if $t+1$ is odd (even).

The first two lemmas prove that all agreements $ij \in \mathcal{C}$ must occur simultaneously at prices $p^R_{ij,D}$ or $p^R_{ij,U}$ depending on whether or not the first agreement $ij \in \mathcal{C}$ occurs in an odd or an even period.

Lemma 4.7 (Odd, simultaneous.) *In any equilibrium of $\Gamma^{\tilde{t}}_{\mathcal{C}}$ with the first agreement occurring in an odd period (i.e., the downstream firms propose), all agreements must occur at the same time with $\hat{p}_{ij} = p^R_{ij,D}$ $\forall ij \in \mathcal{C}$.*

Lemma 4.8 (Even, simultaneous.) *In any equilibrium of $\Gamma^{\tilde{t}}_{\mathcal{C}}$ with the first agreement occurring in an even period (i.e., the upstream firms propose), all agreements must occur at the same time with $\hat{p}_{ij} = p^R_{ij,U}$ $\forall ij \in \mathcal{C}$.*

The proofs of these two lemmas both proceed by contradiction, assuming that the set of agreements $\mathcal{B}$ which occur in period $t+1$ is non-empty. For Lemma 4.7 (t is odd), the sole downstream firm D_j at time t can make a deviant offer $\tilde{p}_{ij} \equiv p^R_{ij,D}$ to some firm U_i, $ij \in \mathcal{B}$, that U_i will accept.[18] Furthermore, by A.GFT and A.CDMC$'$, D_j will find it profitable to make such a deviation and come to agreement with ij earlier. This is a contradiction, and proves that $\mathcal{B}$ must be empty if the first agreement is reached in an odd period. Next, proving payments must equal $p^R_{ij,D}$ is straightforward: any offer lower would be rejected by an upstream firm, as in the resultant subgame only one agreement would be outstanding, and the upstream firm would receive the (discounted) 2-player Rubinstein price in the following period; any higher offer would not be offered by D_j, as the lower offer $p^R_{ij,D}$ would be accepted by each U_i.

The proof of Lemma 4.8 (when the first agreement occurs in an even period), though similar in structure to the proof of Lemma 4.7, is more involved. We first prove that D_j cannot induce any firm U_i, $ij \in \mathcal{C}$, to accept a price lower than $p^R_{ij,D}$ in any equilibrium. To show this, we use a similar subgame perfection argument to that in Rubinstein (1982) generalized to multiple players: suppose D_j first reaches an agreement with some set of upstream firms $\{U_i : ij \in \mathcal{A}\}$, at prices $\hat{p}_{ij}$ such that $\sum_{ij\in\mathcal{A}} \left(p^R_{ij,D} - \hat{p}_{ij}\right) > 0$. As this must have occurred in an even period (by Lemma 4.7), this implies D_j would have rejected any higher offer $\tilde{p}_{ij} \equiv p^R_{ij,D} - \varepsilon > \hat{p}_{ij}$, $\varepsilon > 0$, for some $ij \in \mathcal{A}$ where $\hat{p}_{ij} < p^R_{ij,D}$; if this were not the case, U_i would have offered $\tilde{p}_{ij}$ at time t and obtain strictly higher payoffs.[19] However, for D_j to credibly reject such an offer in equilibrium, D_j must anticipate receiving a higher payoff in some subgame following the rejection; we show this implies D_j must then anticipate paying an even lower prices to some set of firms in some future subgame. Repeating the argument implies even lower and lower prices paid by D_j, so that in order for the original rejection to be supportable, in some subgame D_j must be able to pay a price to some upstream firm that the upstream firm would rather reject than accept, a contradiction.

Once the lower bound on prices has been established, we can establish the simultaneity of accepted offers. If $\mathcal{B}$ is non-empty, any firm U_i, $ij \in \mathcal{B}$, would also be able to make a deviant offer $\tilde{p}_{ij} \equiv p^R_{ij,D}$ earlier at time t to D_j. Since we have established D_j cannot pay any less than $p^R_{ij,D}$ to any upstream firm for agreement, it is straightforward to show that D_j will not wish to reject the deviation. However, without stronger assumptions on the underlying payoffs for D_j, it may be the case that D_j, upon accepting the deviant offer $\tilde{p}_{ij}$ from U_i, rejects some other offers $\mathcal{A}'$ that it had previously accepted. This may affect U_i's payoffs, and imply that U_i will not wish to engage in such a deviation.

To rule out this possibility, we leverage A.CDMC$'$. This assumption implies that even if

[18] Since each upstream firm only has one agreement not yet reached (as $\mathcal{C}$ only contains agreements involving one downstream firm), and since U_i assumes that all other agreements $\mathcal{A}$ at time t will still be accepted (given the passive beliefs assumption), showing U_i will accept this offer is straightforward.

[19] We leverage either A.LEXT or A.ASR here to ensure that U_i still wishes to offer $\tilde{p}_{ij}$ if it is accepted by D_j, as D_j may have an incentive to change its actions with regards to other agreements it is indifferent over.

D_j adjusted its other offers accepted at period t and accepts U_i's deviant offer $\tilde{p}_{ij}$, U_i will still find it profitable to engage in such a deviation; i.e., U_i would rather contract one period earlier with D_j at period t instead of $t+1$ regardless of whether or not D_j adjusts whom it contracts with at period t. Thus, this proves that any equilibrium with agreement in an even period t must have all agreements occurring in that period (and $\mathcal{B} = \emptyset$).

Finally, proving that equilibrium prices are $p^R_{ij,U}$ in an even period if all offers are accepted at the same time follows as before, with the exception that we leverage either A.ASR or A.LEXT to rule out the possibility (as discussed earlier) that D_j could keep prices below $p^R_{ij,U}$ through an off-equilibrium threat: i.e., D_j rejecting another agreement (say, from U_k) if a higher price were demanded by U_i.

The next Lemma states that agreement occurs immediately among all agents regardless of whether $\tilde{t}$ is odd or even.

Lemma 4.9 (Immediate agreement.) *Any equilibrium of $\Gamma^{\tilde{t}}_{\mathcal{C}}$ results in immediate agreement for all $ij \in \mathcal{C}$ at time $\tilde{t}$.*

The proof proceeds by contradiction: if there is an odd period in which no offers are accepted, D_j has an incentive to offer $p^R_{ij,D}$ to all upstream firms, who all accept; similarly, if there is an even period in which no offers are accepted, any U_i has an incentive to offer $p^R_{ij,U}$ to D_j, which is accepted.

Thus, given Proposition 4.5 and Lemmas 4.7-4.9, we have shown that any equilibrium for a subgame beginning at period t with $\mathcal{C}$ open agreements involving only one downstream firm has immediate agreement among all firms with prices given by either $p^R_{ij,D}$ ($p^R_{ij,U}$) $\forall ij \in \mathcal{C}$ for t odd (even).

4.5 Proof of Proposition 4.6: One Upstream Firm, Many Downstream Firms

Consider any subgame $\Gamma^{\tilde{t}}_{\mathcal{C}}$ where $\mathcal{C} \subseteq \mathcal{G}$ contains only open agreements involving one upstream firm U_i, and $|C| = n$ so that there are $n > 1$ remaining agreements that have not yet been reached at time $\tilde{t}$. WLOG, assume that thes downstream firms are indexed $\{1, \ldots, n\}$.

This case is exactly symmetric to the one downstream, many upstream firm case proved in Section 4.4, and thus the proof that Proposition 4.6 holds in any subgame with multiple downstream firms and one upstream firm follows immediately.[20]

[20] The only difference is that the sign on prices, since payments are from downstream to upstream firms, is reversed.

4.6 Proof of Proposition 4.6: Many Upstream and Many Downstream Firms

Having proven Proposition 4.6 holds for all subgames with open agreements involving either only one upstream firm or one downstream firm, we now focus on subgames $\Gamma^t_{\mathcal{C}}$ where $\mathcal{C} \subseteq \mathcal{G}$ involves more than one upstream *and* more than one downstream firm. As before, we will prove Proposition 4.6 with 4 lemmas.

Consider a candidate equilibrium for $\Gamma^t_{\mathcal{C}}$ with prices $\{\hat{p}_{ij}\}_{ij\in\mathcal{C}}$. Recall the inductive hypothesis assumes that any subgame $\Gamma^{t'}_{\mathcal{C}'}$ s.t. $t' > t$ and $\mathcal{C}' \subset \mathcal{C}$ has a unique equilibrium resulting in immediate agreement at prices $p^R_{ij,D}$ ($p^R_{ij,U}$) if t' is odd (even).

We first show that if the first agreement in $\mathcal{C}$ occurs in an odd period or even period, all agreements in $\mathcal{C}$ must also occur in that period at prices $p^R_{ij,D}$ ($p^R_{ij,U}$) if t is odd (even).

Lemma 4.10 (Odd, simultaneous.) *If the first agreement occurs in an odd period $t' \geq t$, then all agreements $ij \in \mathcal{C}$ must occur at t' with $\hat{p}_{ij} = p^R_{ij,D}$ $\forall ij \in \mathcal{C}$.*

Lemma 4.11 (Even, simultaneous.) *If the first agreement occurs in an even period $t' \geq t$, then all agreements $ij \in \mathcal{C}$ must occur at t' with $\hat{p}_{ij} = p^R_{ij,U}$ $\forall ij \in \mathcal{C}$.*

The proof of the many upstream, many downstream firm case is more straightforward than the proofs with just one firm on one side of the market. Here, there is no sharp distinction between even and odd periods, as these cases are now symmetric. In addition, what made the cases with a single upstream or downstream firm so difficult is that a single firm could hold up the entire bargaining process, since all remaining agreements are signed with a single firm. When there are many upstream and downstream firms, this is no longer the case.

The proof proceeds in two steps, similar to the proofs of lemmas 4.7and 4.8.

First, we show, by contradiction, that all agreements will occur simultaneously. Assume that agreement is not simultaneous, and that there is some non-empty set of agreements $\mathcal{A} \subset \mathcal{C}$ formed at period t which is odd (the proof with t being even follows similarly); by the inductive lemma, all remaining agreements $\mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ will be formed in the following period $t+1$ at prices $p^R_{ij,U}$. Since there are multiple upstream firms with open agreements at t, there will then be a pair of agreements $ij \in \mathcal{B}$ and $ab \in \mathcal{A}$ so that $U_i \neq U_a$: i.e., we can find an agreement formed at time t and another agreement at time $t+1$ *involving different upstream firms.* For such an agreement $ij \in \mathcal{B}$, suppose that D_j chooses to offer $p^R_{ij,D}$ at time t in order to "pull up" this agreement by one period. Even if U_i rejects this offer, U_i believes that the agreement between U_a and D_b will still be formed in period t (by passive beliefs), and the inductive lemma applies in the next period. By A.GFT and A.CDMC′, U_i

will thus accept this deviant offer from D_j; and by A.CDMC$'$, D_j will find this making this deviation profitable even if U_i decides to accept additional agreements. As a result, we have a contradiction, and agreement must be immediate.

Second, once we have proved that all agreements occur simultaneously, it is straightforward to show that all prices $\hat{p}_{ij} = p^R_{ij,D}$ in the same fashion as in lemmas 4.7 and 4.8.

We next show that all agreements $ij \in \mathcal{C}$ must occur immediately at t in any equilibrium.

Lemma 4.12 (Immediate agreement.) *Any equilibrium of* $\Gamma^{\tilde{t}}_{\mathcal{C}}$ *results in immediate agreement for all* $ij \in \mathcal{C}$ *at time* $\tilde{t}$.

The proof of this lemma mirrors the proof of Lemma 4.9. D_j can engage in the deviation of pulling up all of it's offers with U_i's up to the current period (if the current period is an downstream proposing period). By A.CDMC$'$ and A.GFT, the upstream firms U_i will accept these "early offers." Again, by symmetry, we can show that the same logic also holds if the current period is one in which upstream firms make offers.

5 Discussion

The existence of an equilibrium generating the Nash-in-Nash outcome (Theorem 3.1) relies on two assumptions (A.GFT and A.CDMC) that we believe to be natural for many bilateral bargaining environments that are often studied. As discussed in the introduction, a set of agreements will not necessarily be stable under bilateral negotiations when A.GFT and A.CDMC fail. In particular, without A.CDMC, there may be strong "complementarities" across contracting partners—e.g., the gains to agreement are increasing as more firms contract; bilateral negotiations over the marginal contributions of links can then generate prices that exceed the total contribution of a set of links, and thus induce instability. In these settings, other surplus division protocols (e.g., multilateral bargaining, cooperative solution concepts such as the Shapley value) may be more appropriate.[21]

To prove the uniquness of our equilibrium outcome, we have leveraged stronger assumptions that guarantee that equilibrium prices coincide with "Rubinstein" prices between each bilateral pair. As discussed in Section 4 and in Appendix E, although counterexample equilibria which do not satisfy this property have been found when both A.LEXT and A.ASR do not hold, their outcomes still converge to immediate agreement and Nash-in-Nash prices when $\Lambda \to 0$. Whether all equilibria satisfy this property under weaker assumptions is an open research question.

[21] See Stole and Zweibel (1996) for a non-cooperative game generating the Shapley value.

For tractability, we have considered lump sum transfers between agents which did not influence payoffs. In many setings, however, negotiations may occur over linear prices that may affect total surplus. In this case, profits may depend on not only the set of agreements reached by all agents, $\mathcal{G}$, but also the set of prices agreed upon, $\mathbf{p} \equiv \{p_{ij}\}_{ij \in \mathcal{G}}$: i.e., payoffs to each D_j are given by $\pi_j(\mathcal{G}, \mathbf{p})$. This corresponds to settings in which negotiated prices represented wholesale prices or linear fee contracts, and downstream firms engaged in price competition with one another. There are many open questions in these environments, including properties on the underying profit functions which guarantee the existence and/or uniqueness of an underlying "Nash-in-Nash" solution.

Finally, this paper has focused on the division of surplus within a fixed network.[22] Furthermore, the model does not allow for renegotiation, and assumes that once a link is formed, it is permanent. We view this model, however, as an potential input into more general settings: e.g., Lee and Fong (2013) nests the bargaining framework developed in this paper into a larger dynamic game that endogenizes the network structure, and allows for links to be formed and broken over time. Work along these lines remains an important topic for future research.

6 Conclusion

The concept of a Nash equilibrium in Nash bargains, proposed by Horn and Wolinsky (1988), has been widely used in applied settings. We develop an extensive form alternating offers game to rationalize this cooperative outcome: downstream firms make simultaneous offers in odd periods, while upstream firms make simultaneous offers in even periods, and firms can accept or reject any subset of offers that are made to them. We consider the limiting equilibrium of our game as the period length between the alternating offers goes to zero. With the assumptions that each bilateral agreement creates surplus for the firms involved, and that each firm's own marginal contribution to the network is weakly decreasing, we show that our alternating offers game has a passive beliefs equilibrium whose payoffs converge to the Nash-in-Nash payoffs. In addition, under stronger conditions on payoffs (or weaker conditions with an equilibrium refinement), we show that any equilibrium will possess these properties, and hence the equilibrium outcome is unique.

We believe that our results provide justification for the use of the Nash-in-Nash solution as a credible bargaining framework for use in applied work. Rather than requiring an assumption on the inability of firms to coordinate across its multiple negotiations, we allow in

[22] We have assumed that this fixed network is the complete network, but the model can be generalized to a case where $\mathcal{G}$ represents an arbitrary set of links that can form.

our extensive form the possibility that firms may wish to engage in deviations across multiple negotiations. We further believe that the mechanisms that we highlight in our extensive form reasonably captures aspects of bargaining games that occur in real-world industry settings.

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A An Alternative Non-cooperative Foundation for the Nash-in-Nash Bargaining Solution

In this section, we present an alternative extensive form which involves separate bilateral negotiations between representatives for each firm, and show that this representation also admits the Nash-in-Nash bargaining division as an equilibrium outcome. For this equilibrium, only A.GFT is required.

Consider the setting introduced in Section 4.6, where N upstream firms negotiate with M downstream firms. For every pair of firms U_i and D_j, $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$, U_i and D_j send individual representatives who engage in the alternating-offers bargaining protocol of Rubinstein (1982). Although each representative for each firm seeks to maximize his firm's total expected profits across all bargains, each representative does not know the state or outcome of any other bilateral bargain until his own bargain has concluded. One interpretation is that each pair of representatives from different firms are sequestered in separate bargaining rooms, and no one outside the room knows the status of the bargain until it is finished.

In this environment, there exists an equilibrium among *representatives for each firm* which yields the Nash-in-Nash bargaining outcomes:

Theorem A.1 *Assume A.GFT and every firm send representatives to all potential negotiating partners. Then there exists an equilibrium with:*

(a) immediate agreement between all representatives for each firm;

(b) equilibrium prices $\hat{p}_{ij} = p^R_{ij,D}$ $\forall i, j$ if the game begins in an odd period, and $\hat{p}_{ij} = p^R_{ij,U}$ $\forall i, j$ if the game begins in an even period; and

(c) $\hat{p}_{ij} \to p^R_{ij}; \forall i,j$ as $\Lambda \to 0$ regardless of whether the game starts in an odd or even period, where $b_{i,U} = r_{j,D}/(r_{i,U} + r_{j,D})$ and $b_{j,D} = r_{i,U}/(r_{i,U} + r_{j,D})$.

To prove the theorem, assume each pair of representatives $U_{i,j}$ and $D_{j,i}$ who negotiate between U_i and D_j employ the following candidate set of strategies: $U_{i,j}$ offers $p^R_{ij,U}$ in even periods and only accepts offers equal to or above $p^R_{ij,D}$ in odd periods; $D_{j,i}$ offers $p^R_{ij,D}$ in odd periods, and accepts offers equal to or above $p^R_{ij,U}$ in even periods. Given the equilibrium strategies of all other representatives (including those from the same firm), $U_{i,j}$ and $D_{j,i}$ believe that any off-equilibrium action made by other in their bargain does not affect or influence the outcomes of other negotiations; as such, given that agreement is expected to occur in all other negotiations, the unique equilibrium for $U_{i,j}$ and $D_{j,i}$ is the set of candidate strategies described (Rubinstein, 1982). As no agent has a profitable deviation, the set of strategies comprise an equilibrium, and the theorem is proved.

B Proof of Lemma 2.1

Proof of Lemma 2.1 Using l'Hospital's rule:

$$\lim_{\Lambda \to 0} \frac{\delta_{i,U}(1 - \delta_{j,D})}{1 - \delta_{i,U}\delta_{j,D}} = \lim_{\Lambda \to 0} \frac{\exp^{-r_{i,U}\Lambda}(1 - \exp^{-r_{j,D}\Lambda})}{1 - \exp^{-(r_{i,U}+r_{j,D})\Lambda}} = \frac{r_{j,D}}{r_{i,U} + r_{j,D}} \qquad ,$$

and

$$\lim_{\Lambda \to 0} \frac{1 - \delta_{j,D}}{1 - \delta_{i,U}\delta_{j,D}} = \lim_{\Lambda \to 0} \frac{1 - \exp^{-r_{j,D}\Lambda}}{1 - \exp^{-(r_{i,U}+r_{j,D})\Lambda}} = \frac{r_{j,D}}{r_{i,U} + r_{j,D}} \qquad .$$

Similarly, it can be shown that:

$$\lim_{\Lambda\to 0}\frac{\delta_{j,D}(1-\delta_{i,U})}{1-\delta_{i,U}\delta_{j,D}}=\lim_{\Lambda\to 0}\frac{(1-\delta_{i,U})}{1-\delta_{i,U}\delta_{j,D}}=\frac{r_{i,U}}{r_{i,U}+r_{j,D}}\qquad .$$

which proves the Lemma. □

C Proof of Theorem 3.1 (Existence.)

To prove the theorem, we first provide a candidate equilibrium profile that satisfies conditions (a)-(c) of the Theorem. We then check that any one-shot deviation from these strategies is not profitable for any firm.

Consider the following strategy profile:

- In any odd period, each D_j makes offers $p^R_{ij,D}$ to all firms U_i that have not already reached agreement. U_i accepts any offer greater than or equal to $p^R_{ij,D}$. If only one offer is less than $p^R_{ij,D}$ for some U_i, then U_i rejects this offer; otherwise, U_i plays an arbitrary best response (respecting passive beliefs).
- In even periods, each U_i which has not already reached an agreement makes offers of $p^R_{ij,U}$. D_j accepts any offer less than or equal to $p^R_{ij,U}$. If only one offer is greater than $p^R_{ij,U}$ for some D_j, then D_j rejects this offer; otherwise, D_j plays an arbitrary best response (respecting passive beliefs).

The strategy profile dictates that every firm makes proposals that are the Rubinstein offers: i.e., downstream firms offer $p^R_{ij,D}$ in odd periods, and upstream firms offer $p^R_{ij,U}$ in even periods. On the equilibrium path, all offers are accepted. For off-equilibrium offers, the strategy profile will depend on the number of off-equilibrium offer a firm receives. Consider an odd period, and assume an upstream firm receives offer(s) that do not correspond to the Rubinstein price. If only one off-equilibrium offer is received that is below the Rubinstein price by an upstream firm U_i, then U_i will reject this offer and accept all others. If multiple off-equilibrium offers are received that are below the Rubinstein price, the optimal strategy for U_i may involve accepting some of these offers and rejecting others. In the proof below, the intuition for this particular construction will be made clear.

First, it is easy to check that this strategy profile satisfies conditions (a)-(b) of the Theorem: agreement is immediate at t_0 at the Rubinstein prices. Part (c) of the theorem then follows directly from Lemma 2.1.

To prove that the proposed strategy profile is an equilibrium, we first examine one-shot deviations on the part of downstream and upstream agents during odd periods. In an odd period, we examine two types of deviations: (i) a downstream firm D_j makes deviant offers $\{\tilde{p}_{ij}:\tilde{p}_{ij}\neq p^R_{ij,D}\}_{ij\in\mathcal{A}}$ for some subset of agreements $\mathcal{A}$; and (ii) an upstream firm U_i rejects $\{p_{ij}:p_{ij}\geq p^R_{ij,D}\}_{ij\in\mathcal{B}'}$, or accepts $\{p_{ik}:p_{ik}<p^R_{ik,D}\}_{ik\in\mathcal{B}''}$ for some subsets of agreements $\mathcal{B}',\mathcal{B}''\subseteq\mathcal{G}_i$.

1. Consider deviations by a downstream firm D_j. Consider a deviation in which D_j offers $\{\tilde{p}_{ij}:\tilde{p}_{ij}>p^R_{ij,D}\}_{\forall ij\in\mathcal{A}'\subseteq\mathcal{G}_j}$, and $\{\tilde{p}_{ij}:\tilde{p}_{kj}<p^R_{kj,D}\}_{\forall kj\in\mathcal{A}''\subseteq\mathcal{G}_j}$ for some subsets $\mathcal{A}',\mathcal{A}''\subseteq\mathcal{G}_j$.
   (a) Assume D_j offers payments $\{\tilde{p}_{ij}:\tilde{p}_{ij}>p^R_{ij,D}\}$ for some subset of agreements $ij\in\mathcal{A}'\subseteq\mathcal{G}_j$. All agreements $ij\in\mathcal{A}'$ will be accepted (given candidate equilibrium strategies).

However, D_j could do strictly better by offering only $p^R_{ij,D}$ instead for all agreements $ij \in \mathcal{A}'$, as these upstream firms will still accept this lower offer; thus, any deviation where $\mathcal{A}' \neq \emptyset$ is dominated by one in which D_j offers $p^R_{ij,D}$ instead for $ij \in \mathcal{A}'$. Hence, we restrict attention to deviations where D_j does not offer more than $p^R_{ij,D}$ to any set of upstream firms: i.e., $\mathcal{A}' = \emptyset$.

(b) Now assume D_j offers payments $\{\tilde{p}_{kj} : \tilde{p}_{kj} < p^R_{kj,D}\}_{kj \in \mathcal{A}''}$ where $\mathcal{A}'' \neq \emptyset$, and offers $p^R_{ij,D}$ for all other $ij \notin \mathcal{A}''$. Given candidate equilibrium strategies, each U_{kj}, $kj \in \mathcal{A}''$, will reject the deviant offer $\tilde{p}_{kj}$ In the subsequent subgame beginning in an even period, there will only be $|\mathcal{A}''|$ upstream firms without agreement with D_j. Under the equilibrium strategies, such a subgame will result in immediate agreement in the next (even) period for all $kj \in \mathcal{A}''$ at prices $p^R_{ij,U}$. Such a deviation by D_j will be profitable if:

$$(1-\delta_{j,D})\pi^D_j(\mathcal{G} \setminus \mathcal{A}'') - \delta_{j,D} \sum_{kj \in \mathcal{A}''} p^R_{kj,U} > (1-\delta_{j,D})\pi^D_j(\mathcal{G}) - \sum_{kj, \in \mathcal{A}''} p^R_{kj,D}$$

where the LHS and RHS represents the relevant differences in flow profits and payments from the deviation as opposed to the equilibrium strategies. This expression can be rearranged as:

$$(1-\delta_{j,D}) \left[\Delta\pi^D_j(\mathcal{G}, \mathcal{A}) - \sum_{kj \in \mathcal{A}''} p^R_{kj,D} \right] + \delta_{j,D} \sum_{kj \in \mathcal{A}''} (p^R_{kj,U} - p^R_{kj,D}) < 0$$

Both terms on the LHS are positive: $\Delta\pi^D_j(\mathcal{G}, \mathcal{A}) - \sum_{kj \in \mathcal{A}''} p^R_{kj,D} > 0$ by (4), and $p^R_{ij,U} > p^R_{ij,D}$ for all $ij \in \mathcal{G}$. This implies the inequality does not hold, and any deviation in which $\mathcal{A}'' \neq \emptyset$ is not profitable.

Thus, any deviation by D_j in an odd period is not profitable.

2. Consider deviations by an upstream firm U_i.

   (a) First, consider deviations by U_i on the (candidate) equilibrium path: i.e., deviations at nodes of the game in which U_i receives equilibrium offers $p^R_{ij,D}$ from all D_j, $ij \in \mathcal{G}_i$. Such deviations will comprise U_i rejecting some subset $\mathcal{B} \subseteq \mathcal{G}_j$ of agreements. If U_i rejects $\mathcal{B} \neq \emptyset$ offers, the subsequent (odd) period will only have $|\mathcal{B}|$ downstream firms without agreement with U_i (as all other agreements will have been made under the candidate equilibrium strategies). Under the candidate equilibrium strategies, the subgame starting in the subsequent even period will result in immediate agreement at prices $p^R_{ij,U}$. Such a deviation for U_i will be profitable if:

$$(1-\delta_{i,U})\pi^U_i(\mathcal{G} \setminus \mathcal{B}) + \delta_{i,U} \sum_{ij \in \mathcal{B}} p^R_{kj,U} > (1-\delta_{i,U})\pi^U_i(\mathcal{G}) + \sum_{ij, \in \mathcal{B}} p^R_{ij,D}$$

   which can be re-expressed as:

$$\sum_{ij \in \mathcal{B}} (\delta_{i,U} p^R_{kj,U} - p^R_{ij,D}) > (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, \mathcal{B})$$

   Note the LHS is equal to $(1-\delta_{j,D}) \sum_{ij \in \mathcal{B}} \Delta\pi(\mathcal{G}, ij)$ by (1). By A.CDMC, however, the

RHS is weakly greater than the LHS, and this inequality will not hold. Thus, U_i will not wish to reject any offers $p^R_{ij,D}$ on the candidate equilibrium path.

(b) Next, consider deviations by U_i off the candidate equilibrium path. Assume U_i receives offers $\{\tilde{p}_{ij} : \tilde{p}_{ij} \geq p^R_{ij,D}\}_{ij\in\mathcal{B}'}$ and offers $\{\tilde{p}_{ik} : \tilde{p}_{ik} < p^R_{ik,D}\}_{ik\in\mathcal{B}''}$, where there is at least one offer $\tilde{p}_{il} \neq p^R_{il,D}$, $il \in \mathcal{G}_i$.

First, note that given passive beliefs, U_i does not update its beliefs on the offers made by any downstream firm to other upstream firms U_{-i}. Thus, if U_i rejects any offers $\mathcal{C} \subseteq \mathcal{G}_i$, U_i believes that the subsequent subgame beginning in an even period will only involve $|\mathcal{C}|$ downstream firms without agreement with U_i. Given candidate equilibrium strategies, those agreements $ij \in \mathcal{C}$ will be reached at prices $p^R_{ij,U}$.

i. Consider any deviation in which U_i rejects all offers $ij \in \mathcal{B}$, where for some $ij \in \mathcal{B}$, $\tilde{p}_{ij} \geq p^R_{ij,D}$. Such a deviation is (weakly) dominated by a deviation in which U_i accepts $\tilde{p}_{ij}$ instead, and rejects $\mathcal{B} \setminus ij$, if:

$$(1-\delta_{i,U})\pi^U_i(\mathcal{G} \setminus \{\mathcal{B} \setminus ij\}) + \delta_{i,U} p^R_{ij,U} < (1-\delta_{i,U})\pi^U_i(\mathcal{G} \setminus \mathcal{B}) + \tilde{p}_{ij}$$
$$(\Leftrightarrow) \qquad \delta_{i,U} p^R_{ij,U} - \tilde{p}_{ij} < (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G} \setminus \mathcal{B}, ij)$$

By (2) and since $\tilde{p}_{ij} \geq p^R_{ij,D}$, the LHS is less than or equal to $(1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ij)$. By A.CDMC, the RHS is greater than or equal to this same amount, and the inequality holds. Thus, any deviation involving U_i rejecting an offer $\tilde{p}_{ij} \geq p^R_{ij,D}$ will be weakly dominated by a deviation in which U_i does not reject the offer; hence, we will focus only on deviations in which U_i accepts all offers $\tilde{p}_{ij} \geq p^R_{ij,D}$.

ii. Consider now a deviation where U_i accepts all offers $\{\tilde{p}_{ij} : \tilde{p}_{ij} \geq p^R_{ij,D}\}_{ij\in\mathcal{B}'}$, and also accepts some offers $\{\tilde{p}_{ik} : \tilde{p}_{ik} < p^R_{ik,D}\}_{ik\in\mathcal{C}\subseteq\mathcal{B}''}$, $\mathcal{C} \neq \emptyset$. There are two cases to consider:

A. $|\mathcal{B}''| \leq 1$ and there is only one offer $\tilde{p}_{ik} < p^R_{ik,D}$ that U_i receives. U_i will be strictly better off rejecting this offer if:

$$(1-\delta_{i,U})\pi^U_i(\mathcal{G} \setminus ik) + \delta_{i,U} p^R_{ik,U} \geq (1-\delta_{i,U})\pi^U_i(\mathcal{G}) + \tilde{p}_{ik}$$
$$(\Leftrightarrow) \qquad (\delta_{i,U} p^R_{ik,U} - \tilde{p}_{ik}) \geq (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ik)$$

Since $\tilde{p}_{ik} < p^R_{ik,D}$, the LHS is greater than $(1-\delta_{i,U})\ \Delta\pi^U_i(\mathcal{G}, ik)$ (by equation (1)), and this inequality is thus satisfied. Thus, U_i will not wish to deviate.

B. $|\mathcal{B}''| > 1$ and there are multiple offers that U_i receives which are less than $p^R_{ik,D}$. Note that this subgame is not reachable via a unilateral deviation from the candidate equilibrium stratey profile, since it requires off-equilibrium offers from multiple downstream firms. As a result, the strategy profile played by U_i in this subgame does not affect the optimality of any strategy profile played by D_j on the equilibrium path.[23]

The proof for even periods is symmetric, and omitted. Since there are no profitable one-shot deviations for any agent in both odd and even periods, the candidate set of strategies is an equilibrium.

□

[23] In this subgame, it may be the case that U_i wishes to accept some subset of agreements $\mathcal{C} \subseteq \mathcal{B}''$, where $\tilde{p}_{ik} < p^R_{ik,D}$ for $ik \in \mathcal{B}''$. To see why, such a strategy is profitable as opposed to rejecting all offers in $\mathcal{B}''$ as

D Proofs for Lemmas used to Prove Theorem 4.4 (Uniqueness.)

In the following proofs, we will use both A.CDMC and A.CDMC$'$ (with the understanding that A.CDMC$'$ implies A.CDMC) to emphasize the circumstances in which the stronger assumption (A.CDMC$'$) is required. We will also use A.CDMC$'$ when only A.CDMC and A.LEXT are assumed to hold (since A.CDMC and A.LEXT imply A.CDMC$'$) as it serves to emphasize how these assumptions are leveraged.

D.1 Supporting Lemma

A corollary of A.CDMC is the following lemma, which is used in some of the subsequent proofs.

Lemma D.1 *$\Delta\pi_j^D(\mathcal{A}\cup\mathcal{B},\mathcal{A}) \geq \sum_{ij\in\mathcal{A}} \Delta\pi_j^D(\mathcal{G}, ij)$ and $\Delta\pi_i^U(\mathcal{A}\cup\mathcal{B},\mathcal{A}) \geq \sum_{ij\in\mathcal{A}} \Delta\pi_i^U(\mathcal{G}, ij)$ for all $\mathcal{A},\mathcal{B}\subseteq\mathcal{G}$, $\mathcal{A}\cap\mathcal{B}=\emptyset$.*

Proof First, index agreements in $\mathcal{A}$ from $k = 1, \cdots, |\mathcal{A}|$, where a_k is the kth agreements in $\mathcal{A}$. This allows us to create a sequence of sets of agreements in which we add in each agreement one at a time, give by $\mathcal{D}_0 \equiv B$, and $\mathcal{D}_k = \mathcal{D}_{k-1} \cup a_k$ for $k = 1, \cdots, |\mathcal{A}|$.

This allows us to decompose $\Delta\pi_j^L(\mathcal{A}\cup\mathcal{B},\mathcal{A})$, $L \in \{U, D\}$, into:

$$\Delta\pi_j^L(\mathcal{A}\cup\mathcal{B},\mathcal{A}) = \sum_{k=1}^{|\mathcal{A}|} \Delta\pi_j^L(\mathcal{D}_k, a_k)$$

Then, by A.CDMC, $\Delta\pi_j^L(\mathcal{D}_k, a_k) \geq \Delta\pi_j^L(\mathcal{G}, a_k)$ for all a_k since $\mathcal{D}_k \subseteq \mathcal{G}$. This implies that $\Delta\pi_j^L(\mathcal{A}\cup\mathcal{B},\mathcal{A}) \geq \sum_{ij\in\mathcal{A}} \Delta\pi_j^L(\mathcal{G}, ij)$.

D.2 One Downstream Firm, Many Upstream Firms

For these lemmas, consider a candidate equilibrium of the subgame $\Gamma_{\mathcal{C}}^t$ with the first agreement $ij \in \mathcal{C}$ reached in period $t \geq \tilde{t}$, and accepted prices denoted $\{\hat{p}_{1j}, \ldots, \hat{p}_{mj}\}$. Let $\mathcal{A} \subseteq \mathcal{C}$ denote the set of agreements reached at period t. By the inductive hypothesis, all agreements $ij \in \mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ not reached at period t will reached in period $t+1$ at prices $p_{ij,D}^R$ ($p_{ij,U}^R$) if $t+1$ is odd (even).

---

long as:

$$(1-\delta_{i,U})\pi_i^U(\mathcal{G}\setminus\mathcal{B}'') + \delta_{i,U}\sum_{ik\in\mathcal{C}} p_{ik,U}^R < (1-\delta_{i,U})\pi_i^U(\mathcal{G}\setminus\mathcal{B}''\cup\mathcal{C}\}) + \sum_{ik\in\mathcal{C}} \tilde{p}_{ik}$$

$$(\Leftrightarrow) \qquad \sum_{ik\in\mathcal{C}} (\delta_{i,U} p_{ik,U}^R - \tilde{p}_{ik}) < (1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G}\setminus\mathcal{B}''\cup\mathcal{C},\mathcal{C})$$

Since $\tilde{p}_{ik} < p_{ik,D}^R$, the LHS is greater than $(1-\delta_{i,U})\sum_{ik\in\mathcal{C}} \Delta\pi_i^U(\mathcal{G}, ik)$. As shown previously, if $|\mathcal{B}''| = 1$, then this inequality can never hold for $\mathcal{C} \neq \emptyset$; however, by A.CDMC, the RHS may be greater than the LHS for some subset $\mathcal{C}$ if $|\mathcal{B}''| > 1$, and thus the optimal strategy for U_i is to accept the subset of agreements $\mathcal{C} \subseteq \mathcal{B}''$ which maximizes:

$$(1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G}\setminus\mathcal{B}''\cup\mathcal{C},\mathcal{C}) - \sum_{ik\in\mathcal{C}} (\delta_{i,U} p_{ik,U}^R - \tilde{p}_{ik})$$

(where $\mathcal{C}$ can be empty). By construction, this strategy is optimal for U_i when receiving multiple offers $\{\tilde{p}_{ik} : \tilde{p}_{ik} < p_{ik,D}^R\}$.

Proof of Lemma 4.7 (Odd, simultaneous). Suppose the first agreement occurs in some odd period $t \geq \tilde{t}$. We prove all agreements occur simultaneously by contradiction.

Assume $\mathcal{B} \neq \emptyset$, implying that not all agreements in $\mathcal{C}$ are reached in period t. By the inductive hypothesis, all U_i such that $ij \in \mathcal{B}$ will reach agreement with D_j at $t+1$ at prices $p^R_{ij,U}$.

Consider the following deviation by D_j in period t: D_j offers $\tilde{p}_{ij} \equiv p^R_{ij,D} + \varepsilon$ to some U_i, $ij \in \mathcal{B}$. U_i will accept this deviation at time t if it obtains higher profits, or:

$$\begin{aligned}(1-\delta_{i,U})\pi^U_i((\mathcal{G}\setminus\mathcal{B})\cup ij) + \delta_{i,U}\pi^U_i(\mathcal{G}) + \tilde{p}_{ij} &> (1-\delta_{i,U})\pi^U_i(\mathcal{G}\setminus\mathcal{B}) + \delta_{i,U}\pi^U_i(\mathcal{G}) + \delta_{i,U}p^R_{ij,U} \\ \Leftrightarrow \qquad \tilde{p}_{ij} &> \delta_{i,U}p^R_{ij,U} - (1-\delta_{i,U})\Delta\pi^U_i((\mathcal{G}\setminus\mathcal{B})\cup ij, ij)\end{aligned}$$

which holds since $\tilde{p}_{ij} \equiv p^R_{ij,D} + \varepsilon = \delta_{i,U}p^R_{ij,U} - (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ij) + \varepsilon$ (see (2)) and $\Delta\pi^U_i((\mathcal{G}\setminus\mathcal{B})\cup ij, ij) \geq \Delta\pi^U_i(\mathcal{G}, ij)$ by A.CDMC, for for sufficiently small ε.

This deviation will be profitable for D_j if D_j's profit gains from reaching agreement with U_i one period earlier is greater than D_j's difference in payments:

$$\begin{aligned} & (1-\delta_{j,D})\Delta\pi^D_j((\mathcal{G}\setminus\mathcal{B})\cup ij, ij) > \tilde{p}_{ij} - \delta_{j,D}p^R_{ij,U} = p^R_{ij,D} - \delta_{j,D}p^R_{ij,U} + \varepsilon \\ \Leftrightarrow \quad & (1-\delta_{j,D})\Delta\pi^D_j((\mathcal{G}\setminus\mathcal{B})\cup ij, ij) > \left[(\delta_{i,U} - \delta_{j,D})p^R_{ij,U} - (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ij)\right] + \varepsilon \\ \Leftrightarrow \quad & (1-\delta_{j,D})\Delta\pi^D_j((\mathcal{G}\setminus\mathcal{B})\cup ij, ij) + (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ij) > (\delta_{i,U} - \delta_{j,D})p^R_{ij,U} + \varepsilon\end{aligned}$$

(where the second line follows from (2)). Since $\Delta\pi^D_j((\mathcal{G}\setminus\mathcal{B})\cup ij, ij) \geq \Delta\pi^D_j(\mathcal{G}, ij) > p^R_{ij,U}$ and, $\Delta\pi^U_i(\mathcal{G}, ij) > -p^R_{ij,U}$ by equation (3), this inequality holds for sufficiently small ε and the deviation is profitable for D_j; a contradiction. Thus, if the first agreement occurs in odd time t, all agreements must occur at time t.

Now suppose all agreements occur at time t (odd), but $\hat{p}_{ij} \neq p^R_{ij,D}$ for some ij. We will show this leads to a contradiction:

1. If $\hat{p}_{ij} < p^R_{ij,D}$ for some ij, U_i can reject this offer and, as all other upstream firms will agree in equilibrium at time t, obtain a price of $p^R_{ij,U}$ at $t+1$ by the inductive hypothesis. This is a profitable deviation if U_i's gains in prices exceed its profit gains from coming to agreement one period early:
$$\delta_{i,U}p^R_{ij,U} - \hat{p}_{ij} > (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ij)$$
Since the RHS is equal to $\delta_{i,U}p^R_{ij,U} - p^R_{ij,D}$ by (2), this inequality holds leading to a contradiction.

2. If $\hat{p}_{ij} > p^R_{ij,D}$ for some ij, D_j can profitably reduce its offer to $p^R_{ij,D} + \varepsilon$ for $\varepsilon \in (0, \hat{p}_{ij} - p^R_{ij,D})$; U_i will still accept if:
$$p^R_{ij,D} + \varepsilon - \delta_{i,U}p^R_{ij,U} > -(1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ij)$$
Since the RHS is equal to $p^R_{ij,D} - \delta_{i,U}p^R_{ij,U}$ by (2), this inequality holds and leads to a contradiction.

Thus, $\hat{p}_{ij} = p^R_{ij,D}$ $\forall i$ if the first agreement occurs in an odd period. $\square$

Proof of Lemma 4.8 (Even, simultaneous). This Lemma will be proven with 3 claims. First, we prove that $\hat{p}_{ij} \geq p^R_{ij,D}$ for all $ij \in \mathcal{C}$. Second, we prove all agreements $ij \in \mathcal{C}$ occur at time t. Finally, we prove all agreements occur at prices $\hat{p}_{ij} = p^R_{ij,U}$.

Claim A: Equilibrium prices $\hat{p}_{ij} \geq p^R_{ij,D}$ for all $ij \in \mathcal{C}$.

Given a candidate equilibrium, for any subgame Γ^t of $\Gamma^{\tilde{t}}_{\mathcal{C}}$ beginning at $t \geq \tilde{t}$, let ϕ represent the *total discount* from prices $p^R_{kj,D}$ that D_j can obtain in this equilibrium: i.e., $\phi_\Gamma \equiv \sum_{kj\in\mathcal{C}}(p^R_{kj,D} - \hat{p}_{kj})$. Let $\overline{\phi}$ be the maximum total discount that D_j could achieve under any equilibrium in any subgame of $\Gamma^{\tilde{t}}_{\mathcal{C}}$; $\overline{\phi}$ is finite, as no upstream firm would offer more than its own total achievable profits in any equilibrium strategy.

Assume that $\overline{\phi} > 0$ and consider the equilibrium and subgame in which this maximum discount is reached.[24] We will show that the assumption that $\overline{\phi} > 0$ leads to a contradiction, which implies that in any equilibrium in which the first agreement is reached in an even period, prices cannot be lower than $p^R_{ij,D}$ for any agreement $ij \in \mathcal{C}$.

Without loss of generality, let this subgame be denoted Γ^t $(t \geq \tilde{t})$, and assume that the period in which the first agreement occurs in this subgame is t. By Lemma 4.7, t cannot be odd since this would imply that $\phi = 0$ as all agreements would occur at Rubinstein prices. Thus, t is even.

Let $\mathcal{A} \subseteq \mathcal{C}$ denote the set of agreements reached in period t at prices $\hat{p}_{ij}$. By the inductive hypothesis, all other agreements $kj \in \mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ occur at time $t+1$ at prices $p^R_{kj,D}$. Thus, by our definition of ϕ, $\sum_{ij\in\mathcal{A}} \hat{p}_{ij} = (\sum_{ij\in\mathcal{A}} p^R_{ij,D}) - \overline{\phi}$. For these prices $\{\hat{p}_{ij}\}_{ij\in\mathcal{A}}$ to have been equilibrium offers, it must be the case that D_j would have rejected any alternative offer $\tilde{p}_{ij} > \hat{p}_{ij}$ from any U_i, $ij \in \mathcal{A}$, at time t. If not, U_i would have a profitable deviation by offering $\tilde{p}_{ij} \equiv \hat{p}_{ij} + \varepsilon$.

- We first show that if D_j accepts $\tilde{p}_{ij}$ as defined above, then U_i would wish to engage in this deviation, leading to a contradiction.

  Under A.LEXT, this is straightforward to show: U_i obtains strictly higher payments under this deviation from D_j without changing the timing of its own agreements, and U_i's profits do not depend on whether or not D_j makes changes to its other agreements.

  Under A.ASR, note that it cannot be the case that some other set of agreements $\mathcal{A}' \neq \mathcal{A}$ would be reached at period t if D_j accepted $\tilde{p}_{ij}$. By the inductive hypothesis, any agreements $\mathcal{B}' \equiv \mathcal{C} \setminus \mathcal{A}'$ not reached at t would occur in period $t+1$; as a result, if D_j would reach a different set of agreements $\mathcal{A}' \neq \mathcal{A}$ subsequent to accepting the higher deviant offer $\tilde{p}_{ij}$ at t, then it would have obtained strictly higher payoffs by reaching agreements $\mathcal{A}'$ as opposed to $\mathcal{A}$ at time t in the original candidate equilibrium; this is a contradiction.[25] Consequently, if $\tilde{p}_{ij}$ were accepted at time t, U_i would obtain the same flow profits as in the candidate equilibrium (since the same set of agreements would be reached at time t and $t+1$), but it would obtain a strictly higher price as $\tilde{p}_{ij} > \hat{p}_{ij}$.

  Thus, if D_j accepted $\tilde{p}_{ij}$, U_i would prefer to make such a deviant offer.

Thus, D_j needs to credibly reject $\tilde{p}_{ij}$ if such an offer is made.

Consider now $ij \in \mathcal{A}$ such that $p^R_{ij,D} - \hat{p}_{ij} > 0$; such an ij exists since we have assumed $\overline{\phi} > 0$, and means we can construct $\tilde{p}_{ij} \equiv \hat{p}_{ij} + \varepsilon < p^R_{ij,D}$ for some $\varepsilon > 0$. Since D_j must reject $\tilde{p}_{ij}$ at time t if it were offered by U_i, this implies D_j also must either subsequently (i) reach agreements $\mathcal{A}' \subseteq \mathcal{C} \setminus \{ij\}$ at time t, or (ii) reject *all* offers upon rejecting $\tilde{p}_{ij}$. We show now that either action by D_j leads to a contradiction.

1. Suppose D_j rejects $\tilde{p}_{ij}$, but reaches agreements $\mathcal{A}' \subseteq \mathcal{C} \setminus \{ij\}$. By the inductive hypothesis, D_j would reach all other agreements $kj \in \mathcal{B}' \equiv \mathcal{C} \setminus \mathcal{A}'$ at $t+1$ at prices $p^R_{kj,D}$. However, D_j

[24] Notice that this proof assumes that the maximum payoff $\overline{\phi}$ is achieved by some equilibrium (i.e., $\overline{\phi}$ is a maximum rather than a supremum). If this is not the case, we can consider any subgame in which the total discount from Rubinstein prices is greater than or equal to $\delta_{j,D}\overline{\phi}$ and substitute this value for $\overline{\phi}$ in the proof.

[25] A.ASR rules out the possibility that D_j is indifferent between $\mathcal{A}$ and $\mathcal{A}'$.

would rather accept $\tilde{p}_{ij}$ and reach agreements $\mathcal{A}' \cup \{ij\}$ instead of rejecting U_i's deviation if the gains to coming to an agreement earlier exceeded the additional payment required:

$$(1-\delta_{j,D})\Delta\pi_j^D(\mathcal{G} \setminus \mathcal{B}' \cup \{ij\}, ij) > \tilde{p}_{ij} - \delta_{j,D} p_{ij,D}^R$$

By assumption, the RHS is strictly less than $(1-\delta_{j,D})p_{ij,D}^R$; since $\Delta\pi_j^D(\mathcal{G} \setminus \mathcal{B}' \cup \{ij\}, ij) > \Delta\pi_j^D(\mathcal{G}, ij)$ by A.CDMC and since $\Delta\pi_j^D(\mathcal{G}, \{ij\}) > p_{ij,D}^R$ by A.GFT, this inequality holds, leading to a contradiction.

2. Suppose D_j rejects *all* offers at period t upon receiving the deviant offer $\tilde{p}_{ij}$ from U_i.

   Let Γ_{RA}^{t+1} denote the subgame following D_j's rejection of all offers at period t, and let Π^{RA} denote D_j's payoffs in this subgame (discounted to period t). Note if D_j rejects all offers in $\mathcal{C}$ at time t upon receiving out-of-equilibrium deviation $\tilde{p}_{ij}$, D_j must expect to obtain in the subsequent subgame at least as much as it would have obtained had it accepted $\tilde{p}_{ij}$ and all other offers $\hat{p}_{kj}$, $kj \in \mathcal{A} \setminus ij$, at t, but rejected all other offers. This lower bound is:

$$\begin{aligned}\underline{\Pi}^D &\equiv (1-\delta_{j,D})\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - \tilde{p}_{ij} - \sum_{kj \in \mathcal{A} \setminus ij} \hat{p}_{kj} + \delta_{j,D}\left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C} \setminus \mathcal{A}} p_{kj,D}^R\right) \\ &= (1-\delta_{j,D})\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}) - \sum_{kj \in \mathcal{A}} p_{kj,D}^R + \delta_{j,D}\left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C} \setminus \mathcal{A}} p_{kj,D}^R\right) + \overline{\phi} - \varepsilon \qquad (6)\end{aligned}$$

   where $\underline{\Pi}^D$ represents D_j's expected payoffs if D_j only accepted U_i's deviant offer at time t, and accepted all other offers $kj \in \mathcal{C} \setminus ij$ at $t+1$ at prices $p_{kj,D}^R$ (by the inductive hypothesis). For D_j to prefer rejecting all offers at t, it must be the case that $\Pi^{RA} \geq \underline{\Pi}^D$.

   In the subgame Γ_{RA}^{t+1}, the first accepted offer in $\mathcal{C}$ can occur at an odd (downstream proposing) period, or an even (upstream proposing) period. We go through each case in turn.

   (a) D_j cannot reject all offers and earn a payoff greater than $\underline{\Pi}^D$ by having the first agreement $kj \in \mathcal{C}$ reached in any subsequent odd period $t+\tau$ ($\tau \geq 1$, odd), as Lemma 4.7 implies all agreements in $\mathcal{C}$ would also be realized in the same period at prices $p_{kj,D}^R$; this would yield (discounted to period t) payoffs to D_j of:

$$\begin{aligned}\Pi^{RA} &\equiv \sum_{p=0}^{\tau-1} \delta_{j,D}^{\rho}(1-\delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}^{\tau}\left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R\right) \\ &\leq (1-\delta_{j,D})\pi_j^D(\mathcal{G} \setminus \mathcal{C}) + \delta_{j,D}\left(\pi_j^D(\mathcal{G}) - \sum_{kj \in \mathcal{C}} p_{kj,D}^R\right) \qquad (7)\end{aligned}$$

   where the last inequality is implied from A.GFT (i.e., $\Delta\pi_j^D(\mathcal{G}, \mathcal{C}) - \sum_{ij \in \mathcal{C}} p_{kj,D}^R > 0$). Using (6) and (7) implies $\underline{\Pi}^D - \Pi^{RA} > 0$ if:

$$(1-\delta_{j,D})\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}, \mathcal{A}) + \overline{\phi} - \varepsilon > \sum_{kj \in \mathcal{A}} (1-\delta_{j,D}) p_{kj,D}^R$$

   Since $\Delta\pi_j^D((\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{A}, \mathcal{A}) \geq \Delta\pi_j^D(\mathcal{G}, \mathcal{A}) \geq \sum_{kj \in \mathcal{A}} p_{kj,D}^R$ by A.GFT and A.CDMC, this last inequality holds, and D_j cannot earn higher profits by rejecting all offers at t and reaching agreement in some subsequent odd period.

(b) Thus, in order for D_j to credibly reject $\tilde{p}_{ij}$ and all other offers in $\mathcal{C}$ at time t, D_j must expect the first agreement $kj \in \mathcal{C}$ to occur in some subsequent even period $t+\tau$ ($\tau \geq 2$, even) and obtain some payoff $\Pi^{RA} > \underline{\Pi}^D$. Since the set of open agreements at $t+\tau$ is the same as at t, the same logic of rejecting all offers still holds. Thus this strategy must be supported by ever increasing future payoffs and ever decreasing payments, which ultimately leads to a contradiction.

Suppose all agreements $kj \in \mathcal{A}' \subseteq \mathcal{C}$, $\mathcal{A}' \neq \emptyset$ are reached at even period $t+\tau$ at prices p'_{kj}, and (by the inductive hypothesis) the remaining agreements $lj \in \mathcal{B}' \equiv \mathcal{C} \setminus \mathcal{A}'$ are reached in the next period $t+\tau+1$ at prices $p^R_{lj,D}$. Then D_j's payoffs (discounted to period t) are

$$\begin{aligned}\Pi^{RA} \equiv & \sum_{p=0}^{\tau-1} \delta^{\rho}_{j,D}(1-\delta_{j,D})\pi^D_j(\mathcal{G}\setminus\mathcal{C}) \\ & + \delta^{\tau}_{j,D}\left[(1-\delta_{j,D})\pi^D_j(\mathcal{G}\setminus\mathcal{B}') - \sum_{kj\in\mathcal{A}'} p'_{kj} + \delta_{j,D}\left(\pi^D_j(\mathcal{G}) - \sum_{kj\in\mathcal{B}'} p^R_{kj,D}\right)\right]\end{aligned} \tag{8}$$

Combining (6) and (8) yields:[26]

$$\begin{aligned}(\underline{\Pi}^D - \Pi^{RA}) = & (1-\delta_{j,D})\left[\Delta\pi^D(\mathcal{G}\setminus\mathcal{C}\cup\mathcal{A},\mathcal{A}) - \sum_{kj\in\mathcal{A}} p^R_{kj,D}\right] \\ & + (1-\delta_{j,D})\sum_{\rho=1}^{\tau-1}\delta^{\rho}_{j,D}\left[\Delta\pi^D_j(\mathcal{G},\mathcal{C}) - \sum_{kj\in\mathcal{C}} p^R_{kj,D}\right] \\ & + (1-\delta_{j,D})\delta^{\tau}_{j,D}\left[\Delta\pi^D_j(\mathcal{G},\mathcal{B}') - \sum_{kj\in\mathcal{C}} p^R_{kj,D} + \sum_{kj\in\mathcal{A}'} p'_{kj}\right] \\ & + \delta^{\tau+1}_{j,D}\left[\sum_{kj\in\mathcal{A}'} p'_{kj} - p^R_{kj,D}\right] + \overline{\phi} - \varepsilon\end{aligned} \tag{9}$$

We show that this expression is positive, leading to a contradiction. We go through this expression line by line. The first term is:

$$T_1 \equiv (1-\delta_{j,D})\left[\underbrace{\Delta\pi^D((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A},\mathcal{A}) - \sum_{kj\in\mathcal{A}} p^R_{kj,D}}_{\Xi_1}\right]$$

Since $\Delta\pi^D_j((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A},\mathcal{A}) \geq \sum_{kj\in\mathcal{A}}\Delta\pi^D_j(\mathcal{G},kj) > \sum_{kj\in\mathcal{A}} p^R_{kj,D}$ by A.GFT and A.CDMC, Ξ_1 is strictly positive, and thus T_1 is as well.

[26] Here, and in other expressions, we will leverage the identity: $\delta = (1-\delta)\sum_{\rho=1}^{\tau-1}\delta^{\rho} + \delta^{\tau}$.

The second term of equation 9 is:

$$T_2 \equiv (1-\delta_{j,D}) \sum_{\rho=1}^{\tau-1} \delta_{j,D}^{\rho} \left[\underbrace{\Delta\pi_j^D(\mathcal{G},\mathcal{C}) - \sum_{kj\in\mathcal{C}} p_{kj,D}^R}_{\Xi_2} \right]$$

which, again by A.GFT (see (3)) and A.CDMC, $\Xi_2 > 0$, and thus T_2 is strictly positive. The third term of equation 9 is:

$$\begin{aligned} T_3 &\equiv (1-\delta_{j,D})\delta_{j,D}^{\tau} \left[\Delta\pi_j^D(\mathcal{G},\mathcal{B}') - \sum_{kj\in\mathcal{C}} p_{kj,D}^R + \sum_{kj\in\mathcal{A}'} p'_{kj} \right] \\ &= (1-\delta_{j,D})\delta_{j,D}^{\tau} \left[\underbrace{\Delta\pi_j^D(\mathcal{G},\mathcal{B}') - \sum_{kj\in\mathcal{B}'} p_{kj,D}^R}_{\Xi_3} + \sum_{kj\in\mathcal{A}'} [p'_{kj} - p_{kj,D}^R] \right] \end{aligned}$$

and the fourth term of equation 9 is:

$$T_4 \equiv \delta_{j,D}^{\tau+1} \left[\sum_{kj\in\mathcal{A}'} p'_{kj} - p_{kj,D}^R \right] + \overline{\phi} - \varepsilon$$

Note that $\Xi_3 > 0$ by A.GFT and A.CDMC. Thus, it is straightforward to show that a sufficient condition for $T_3 + T_4 > 0$ is:

$$\varepsilon < \overline{\phi} - \delta_{j,D}^{\tau} \left[\sum_{kj\in\mathcal{A}'} [p_{kj,D}^R - p'_{kj}] \right]$$

Since $\overline{\phi}$ is the maximum discount from Rubinstein prices obtainable in *any* subgame and $\delta_{j,D} < 1$, the RHS will be strictly greater than 0: thus, there is some $\varepsilon > 0$ such that $T_1 + T_2 + T_3 + T_4 > 0$ and $\underline{\Pi}^D \leq \Pi^{RA}$, thus implying that D_j will not wish to reject all offers at time t upon receiving the deviant offer $\tilde{p}_{ij} = \hat{p}_{ij} + \varepsilon$ from U_i.

Consequently, D_j cannot credibly reject $\tilde{p}_{ij} \equiv p_{ij,D}^R + \varepsilon$ if U_i offered it at time t. Since offering $\tilde{p}_{ij}$ at time t is a profitable deviation for U_i, the original assumption that $\overline{\phi} > 0$ and that there exists some $\hat{p}_{ij} < p_{ij,D}^R$ leads to a contradiction. Hence, $\hat{p}_{ij} \geq p_{ij,D}^R$ for all $ij \in \mathcal{C}$.

Claim B: All agreements will occur simultaneously.

Let $\mathcal{A} \subseteq \mathcal{C}$ denote the set of agreements reached at period t in a candidate equilibrium. By the inductive hypothesis, all agreements in $\mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ are reached at period $t+1$ with prices $p_{ij,D}^R$.

We will show by contradiction that $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \emptyset$: i.e., if one agreement occurs at time t (even), all agreements $ij \in \mathcal{C}$ must occur at time t. Suppose not and $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \neq \emptyset$. By the inductive assumption, all U_i such that $ij \in \mathcal{B}$ will reach agreement at $t+1$ at prices $p_{ij,D}^R$.

Consider the deviation where some U_i, $ij \in \mathcal{B}$, offers $\tilde{p}_{ij} = p_{ij,D}^R$ at time t.

1. *Such a deviation will be accepted by D_j.*

   Suppose not, and D_j rejects $\tilde{p}_{ij}$.

   (a) Suppose that D_j rejects $\tilde{p}_{ij}$ but accepts some non-empty set of offers $\mathcal{A}' \subseteq \mathcal{C} \setminus ij$ at time t. However, D_j would find it more profitable to accept $\tilde{p}_{ij}$ while still accepting all agreements $kj \in \mathcal{A}'$ if D_j's profit gains from reaching agreement with U_i one period earlier is greater than D_j's difference in payments:

$$\begin{aligned} & (1-\delta_{j,D})\Delta\pi_j^D((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}'\cup ij, ij) > \tilde{p}_{ij} - \delta_{j,D}p^R_{ij,D} \\ \Leftrightarrow \quad & (1-\delta_{j,D})\Delta\pi_j^D((\mathcal{G}\setminus\mathcal{C})\cup\mathcal{A}'\cup ij, ij) > (1-\delta_{j,D})p^R_{ij,D} \end{aligned}$$

   By A.GFT and A.CDMC, this inequality holds. Thus, D_j cannot reject $\tilde{p}_{ij}$ while still accepting $\mathcal{A}'$ at time t.

   (b) Thus, the only way D_j can reject $\tilde{p}_{ij}$ is for D_j to reject all offers at time t.

   If D_j were to reject all offers at time t, the inductive hypothesis does not apply in the subgame beginning at $t+1$, and the first agreement can occur either in a subsequent odd or even period. We examine both cases.

   First, note that D_j can always accept only $\tilde{p}_{ij}$ at period t, and receive all other offers $kj \in \mathcal{C}\setminus ij$ at period $t+1$ at prices $p^R_{ij,D}$ (by the inductive hypothesis), yielding profits:

$$\begin{aligned} \tilde{\Pi}_j^D &= (1-\delta_{ij,D})\pi_j^D((\mathcal{G}\setminus\mathcal{C})\cup ij) - \tilde{p}_{ij} + \delta_{j,D}\left(\pi_j^D(\mathcal{G}) - \sum_{kj\in\mathcal{C}\setminus ij} p^R_{kj,D}\right) \\ &= (1-\delta_{j,D})\left[\left(\pi_j^D((\mathcal{G}\setminus\mathcal{C})\cup ij) - \tilde{p}_{ij}\right) + \left(\sum_{\rho=1}^{\tau-1}\delta^{\rho}_{j,D}\left[\pi_j^D(\mathcal{G}) - \tilde{p}_{ij} - \sum_{kj\in\mathcal{C}\setminus ij} p^R_{kj,D}\right]\right)\right] \\ &\quad + \delta^{\tau}_{j,D}\left(\pi_j^D(\mathcal{G}) - \tilde{p}_{ij} - \sum_{kj\in\mathcal{C}\setminus ij} p^R_{kj,D}\right) \end{aligned}$$

      i. *First agreement occurs in an odd period.* If the first accepted offer in $\mathcal{C}$ is at an odd period $t+\tau$ ($\tau \geq 1$), then Lemma 4.7 implies all agreements in $\mathcal{C}$ would also be realized in the same period at prices $p^R_{kj,D}$, yielding profits (discounted to period t):

$$\begin{aligned} \bar{\Pi}_j^{D,o} &= (1-\delta^{\tau}_{j,D})\pi_j^D(\mathcal{G}\setminus\mathcal{C}) + \delta^{\tau}_{j,D}\left(\pi_j^D(\mathcal{G}) - \sum_{kj\in\mathcal{C}} p^R_{kj,D}\right) \qquad (10) \\ &= (1-\delta_{j,D})\left[\pi_j^D(\mathcal{G}\setminus\mathcal{C}) + \left(\sum_{\rho=1}^{\tau-1}\delta^{\rho}_{j,D}\pi_j^D(\mathcal{G}\setminus\mathcal{C})\right)\right] + \delta^{\tau}_{j,D}\left(\pi_j^D(\mathcal{G}) - \sum_{kj\in\mathcal{C}} p^R_{kj,D}\right) \end{aligned}$$

      D_j would prefer to accept only $\tilde{p}_{ij}$ at period t instead if $\tilde{\Pi}_j^D - \bar{\Pi}_j^D > 0$, or:

$$\left[\Delta\pi_j^D(G\setminus\mathcal{C}\cup ij, ij) - \tilde{p}_{ij}\right] + \left[\sum_{\rho=1}^{\tau-1}\delta^{\rho}_{j,D}\left(\Delta\pi_j^D(\mathcal{G},\mathcal{C}) - \tilde{p}_{ij} - \sum_{kj\in\mathcal{C}\setminus ij} p^R_{kj,D}\right)\right] > 0$$

Each term of this expression is positive by A.CDMC and A.GFT since $\Delta\pi((G \setminus \mathcal{C}) \cup ij, ij) \geq \Delta\pi(G, ij) > p^R_{ij,D}$, and since $\Delta\pi(\mathcal{G}, \mathcal{C}) - \sum_{kj\in\mathcal{C}} p^R_{kj,D} > 0$.
Thus, D_j will not find it profitable to reject all offers and reach agreement in a subsequent odd period.

ii. *First agreement occurs in an even period.* If the first accepted offer in $\mathcal{C}$ is at an even period $t+\tau$ ($\tau \geq 2$), then the lowest possible payments it can make in equilibrium to all $ij \in \mathcal{C}$ are $\hat{p}_{ij} \geq p^R_{ij,D}$ (by Claim A). Let $\bar{\Pi}^{D,e}_j$ denote the most that D_j can achieve in any subgame in which the first agreement in $\mathcal{C}$ is reached in some future even period $t+\tau$ (discounted to period t).[27] However, it is straightforward to show that the expression for $\bar{\Pi}^{D,e}_j$ is identical to that of $\bar{\Pi}^{D,o}_j$ in (10) (for τ even instead of odd); hence, using the same analysis as before, it must be that $\tilde{\Pi}^D_j - \bar{\Pi}^{D,e}_j > 0$. Thus, D_j will not find it profitable to reject all offers and reach agreement in a subsequent even period.

Thus D_j will accept the deviation $\tilde{p}_{ij}$ from U_i.

2. *Such a deviation is profitable for U_i if D_j accepts.*

Assume if D_j accepts the deviation from U_i at period t, D_j also accepts agreements $\mathcal{A}' \subseteq \mathcal{C} \setminus ij$ at period t as well as ij; by the inductive hypothesis, D_j then reaches agreements $\mathcal{B}' \equiv \mathcal{C} \setminus [\mathcal{A}' \cup ij]$ in the following period $t+1$.

U_i will find the deviation profitable in this case iff:

$$\begin{aligned}
\tilde{p}_{ij} + (1-\delta_{i,U})\pi^U_i(\mathcal{G} \setminus \mathcal{B}') &> \delta_{i,U} p^R_{ij,D} + (1-\delta_{i,U})\pi^U_i(\mathcal{G} \setminus \mathcal{B}) \\
\Leftrightarrow \qquad \tilde{p}_{ij} - \delta_{i,U} p^R_{ij,D} &> (1-\delta_{i,U})(\pi^U_i(\mathcal{G} \setminus \mathcal{B}) - \pi^U_i(\mathcal{G} \setminus \mathcal{B}')) \\
\Leftrightarrow \qquad (1-\delta_{i,U}) p^R_{ij,D} &> (1-\delta_{i,U})(\pi^U_i(\mathcal{G} \setminus \mathcal{C} \cup \mathcal{A}) - \pi^U_i(\mathcal{G} \setminus \mathcal{C} \cup \mathcal{A}' \cup ij))
\end{aligned}$$

By A.CDMC':

$$\pi^U_i((\mathcal{G} \setminus \mathcal{C}') \cup \mathcal{A}' \cup ij)) - (\pi^U_i(\mathcal{G} \setminus \mathcal{C} \cup \mathcal{A}) \geq \Delta\pi^U_i(\mathcal{G}, ij)$$

and so the desired inequality will hold since $p^R_{ij} > -\Delta\pi^D_i(\mathcal{G}, ij)$ by A.GFT.

Hence, U_i will find it profitable to make the deviation.

Since D_j must accept the deviant offer in any equilibrium, and U_i finds it profitable to make the deviation if it is accepted, then the original candidate equilibrium is not an equilibrium; contradiction. Consequently, if the first agreement occurs in an even period, all agreements must occur simultaneously.

Claim C: If all agreements $ij \in \mathcal{C}$ occur simultaneously in an even period t, then $\hat{p}_{ij} = p^R_{ij,U} \forall ij \in \mathcal{C}$.

1. Assume that $\hat{p}_{ij} > p^R_{ij,U}$ for some ij. Consider the following deviation for D_j: D_j rejects U_i and accepts all other offers at t; D_j will then come to agreement with U_i in $t+1$ for payment $p^R_{ij,D}$ by the inductive hypothesis. This is profitable for D_j if $(1-\delta_{j,D})\Delta\pi^D_j(\mathcal{G}, ij) < \hat{p}_{ij} - \delta_{j,D} p^R_{ij,D}$. Since the LHS of this inequality is equal to $p^R_{ij,U} - \delta_{j,D} p^R_{ij,D}$ (see (1)), this inequality will hold if $\hat{p}_{ij} > p^R_{ij,U}$. Contradiction.

[27] By the inductive hypothesis, if D_j reaches any agreements at time $t+\tau$, all other agreements would occur in the following period $t+\tau+1$ at prices $p^R_{ij,D}$. However, D_j would strictly prefer reaching agreement at period $t+\tau$ instead of $t+\tau+1$ at prices $p^R_{ij,D}$ as $\Delta\pi^D_j(G, ij) \geq p^R_{ij,D}$ by A.GFT and A.CDMC.

2. Assume $\hat{p}_{ij} < p^R_{ij,U}$. Suppose that some U_i for which $\hat{p}_{ij} < p^R_{ij,U}$ makes a deviant offer $\tilde{p}_{ij} = p^R_{ij,U} - \varepsilon > \hat{p}_{ij}$. We now show that this deviation is profitable to U_i, leading to a contradiction.

   If U_i offers $\tilde{p}_{ij}$ instead of $\hat{p}_{ij}$ at time t, we show that D_j accepting all offers (including $\tilde{p}_{ij}$) at period t is more profitable than:

   (a) *D_j rejecting all offers at period t.* By Lemma 4.7 and Claim B of this lemma, if D_j rejects all offers at t, then all agreements $ij \in \mathcal{C}$ will form in the same future period (if at all). Moreover, if they occur at a future odd period, they will be for the prices $\{p^R_{kj,D}\}_{kj\in\mathcal{C}}$ (Lemma 4.7), and if they occur at an even period, they will be for prices of $\{p^R_{kj,D}\}_{kj\in\mathcal{C}}$ or higher (Claim A of this lemma). Thus, the most profitable case for D_j by rejecting all offers at t is to have agreement immediately in the following period $t+1$ and pay no more than $p^R_{kj,D}$ to all $kj \in \mathcal{C}$. In this case, the loss in profits from delay is:

   $$(1-\delta_{j,D})\Delta\pi^D_j(\mathcal{G},\mathcal{C}) \geq (1-\delta_{j,D})\sum_{kj\in C}\Delta\pi^D_j(\mathcal{G},kj) = \sum_{kj\in C}(p^R_{kj,U} - \delta_{j,D}p^R_{kj,D}), \qquad (11)$$

   from A.CDMC and (1) respectively. But the change in payments is less than $\sum_{kj\in C} p^R_{kj,U} - \delta_{j,D}p^R_{kj,D} - \varepsilon$, implying that D_j would be better off accepting all offers at time t. Contradiction.

   (b) *D_j rejecting offers $\mathcal{B} \subset \mathcal{C}$ at time t where $ij \in \mathcal{B}$.* By the inductive hypothesis, all remaining offers $kj \in \mathcal{B}$ occur in period $t+1$ at prices $\{p^R_{kj,D}\}_{kj\in\mathcal{B}}$. D_j would rather accept all offers in at time t (including deviant offer $\tilde{p}_{ij}$) if:

   $$(1-\delta_{j,D})\Delta\pi(\mathcal{G},\mathcal{B}) > \tilde{p}_{ij} - \delta_{j,D}p^R_{ij,D} + \sum_{kj\in\mathcal{B}\setminus ij}(\hat{p}_{kj} - \delta_{j,D}p^R_{kj,D})$$

   Similar to (11), the LHS can be shown to be greater than $\sum_{kj\in\mathcal{B}} p^R_{kj,U} - \delta_{j,D}p^R_{kj,D}$. Since $\tilde{p}_{ij} < p^R_{ij,U}$ and $\hat{p}_{kj} \leq p^R_{kj,U} \forall kj \in \mathcal{B}\setminus ij$, it follows that this inequality holds.

   The only case not ruled out yet is if D_j accepts the deviation from U_i at time t, but rejects some other subset of offers $\mathcal{B}, ij \notin \mathcal{B}$. Under A.LEXT, even if D_j prefered this over accepting all offers, U_i would still prefer to engage in such a deviation since it receives a higher payment and doesn't affect the timing of its own agreements. Under A.ASR, this case implies that D_j would have strictly preferred rejecting $\mathcal{B} \subset \mathcal{C}$ in the original candidate equilibrium absent a deviant offer from U_i, which yields a contradiction (as again, A.ASR rules out indifference on the part of D_j).

   Consequently, we have shown that D_j will accept the deviant offer from U_i, and U_i would prefer this deviation: under A.LEXT, the deviation doesn't affect the timing of its own agreements, and under A.ASR, D_j will still accept all offers. Thus this strictly increases U_i's profits, yielding a contradiction.

Thus, $\hat{p}_{ij} = p^R_{ij,U} \forall ij \in \mathcal{C}$ for agreements reached in an even period.

Claims A-C prove the lemma. □

Proof of Lemma 4.9 (Immediate agreement.) We prove this lemma by contradiction.

Consider a candidate equilibrium where D_j does not reach agreement with any firm at $\tilde{t}$ (else, by Lemma 4.7 and Lemma 4.8, agreement with all firms would have occurred).

1. Assume $\tilde{t}$ is odd.

   If all agreements are rejected at $\tilde{t}$, by Lemmas 4.7 and 4.8: (a) D_j can earn at most $(1-\delta_{j,D})\pi_j^D(\mathcal{G}\setminus\mathcal{C})+\delta_{j,D}(\pi_j^D(\mathcal{G})-\sum_{ij\in\mathcal{C}} p_{ij,U}^R)$ in any subsequent subgame by coming to agreement with all remaining firms in the following even period; and (b) each U_i, $ij \in \mathcal{C}$, can earn at most $(1-\delta_{i,U})\pi_i^U(\mathcal{G}\setminus\mathcal{C}) + \delta_{i,U}\pi_i^U(\mathcal{G}) + \delta_{i,U}p_{ij,U}^R$ by coming to agreement in the following period.

   Suppose D_j offered $\tilde{p}_{ij} = p_{ij,D}^R$ to all U_i, $ij \in \mathcal{C}$, at time $\tilde{t}$. Accepting $\tilde{p}_{ij}$ at time t for each U_i will be profitable if the gains in profits exceed the difference in prices:

   $$(1-\delta_{i,U})\Delta\pi_i^U((\mathcal{G}\setminus\mathcal{C})\cup ij, ij) > \delta_{i,U}p_{ij,U}^R - p_{ij,D}^R$$

   where the LHS is U_i's expected gains in profits to agreeing early versus what it could obtain in the optimal subgame (given passive beliefs, U_i only expects $\{(\mathcal{G}\setminus\mathcal{C})\cup ij\}$ agreements to be reached at time t upon accepting the deviation). This holds, since $\Delta\pi_i^U((\mathcal{G}\setminus\mathcal{C})\cup ij, ij) \geq \Delta\pi_i^U(\mathcal{G}, ij)$ by A.CDMC, and by (2).

   We now show that this deviation will be profitable for D_j. Deviating and offering $\tilde{p}_{ij}$ to all $ij \in \mathcal{C}$ at period t (which all U_i, $ij \in \mathcal{C}$ will accept) is profitable for D_j if:

   $$\begin{aligned}(1-\delta_{j,D})\Delta\pi_j^D(G,\mathcal{C}) &> \sum_{ij\in\mathcal{C}}(\tilde{p}_{ij} - \delta_{j,D}p_{ij,D}^R)\\ \Rightarrow \Delta\pi_j^D(G,\mathcal{C}) &> \sum_{ij\in\mathcal{C}} p_{ij,D}^R\end{aligned}$$

   where again the LHS is D_j's expected gains in profits to agreeing early versus what it could obtain in the optimal subgame. This is implied by A.GFT and A.CDMC. So D_j has a profitable deviation, leading to a contradiction.

2. Assume $\tilde{t}$ is even.

   If all agreements are rejected at $\tilde{t}$, by Lemmas 4.7 and 4.8: (a) D_j can earn at most $\bar{\Pi}^{D_j} = (1-\delta_{j,D})\pi_j^D(\mathcal{G}\setminus\mathcal{C}) + \delta_{j,D}\pi_j^D(\mathcal{G}) - \delta_{j,D}\sum_{ij\in\mathcal{C}} p_{ij,D}^R$ in any subsequent subgame by coming to agreement in the following period; and (b) each U_i, $ij \in \mathcal{C}$, can earn at most $(1-\delta_{i,U})\pi_i^U(\mathcal{G}\setminus\mathcal{C}) + \delta_{i,U}\pi_i^U(\mathcal{G}) + \delta_{i,U}p_{ij,U}^R$ by coming to agreement in the following period.

   Suppose U_i for some $ij \in \mathcal{C}$ offers $\tilde{p}_{ij} = p_{ij,U}^R$ in the first period.

   (a) If D_j accepts only this deviation at time t (and rejects all other offers), it earns:

   $$\tilde{\Pi}^D = (1-\delta_{j,D})(\pi_j^D(\mathcal{G}\setminus\mathcal{C}\cup ij) - p_{ij,U}^R) + \delta_{j,D}\left(\pi_j^D(\mathcal{G}) - p_{ij,U}^R - \sum_{kj\in\mathcal{C}\setminus ij} p_{kj,D}^R\right)$$

   D_j change in payoffs from accepting this deviation $\tilde{p}_{ij}$ as opposed to rejecting all offers is:

   $$\begin{aligned}&(1-\delta_{j,D})(\Delta\pi_j^D(\mathcal{G}\setminus\mathcal{C}\cup ij, ij) - p_{ij,U}^R) + \delta_{j,D}(p_{ij,D}^R - p_{ij,U}^R)\\ =&(1-\delta_{j,D})\Delta\pi_j^D(\mathcal{G}\setminus\mathcal{C}\cup ij, ij) - p_{ij,U}^R + \delta_{j,D}p_{ij,D}^R\end{aligned}$$

   which is positive since by A.CDMC $\Delta\pi_j^D((\mathcal{G}\setminus\mathcal{C})\cup ij, ij) \geq \Delta\pi_j^D(\mathcal{G}, ij)$, and $\Delta\pi_j^D(\mathcal{G}, ij) = \delta_{i,U}p_{ij,U}^R - p_{ij,D}^R$ by (1).

Furthermore, D_j will not wish to accept additional agreements $\mathcal{A}' \subseteq \mathcal{C} \setminus ij$ at time t. To see why, if D_j accepted agreements $kj \in \mathcal{A}'$ in addition to ij, it would gain:

$$\Delta\pi_j^D(\mathcal{G} \setminus C \cup \mathcal{A}' \cup ij, \mathcal{A}') - \sum_{kj \in \mathcal{A}'} (\hat{p}_{kj,t} - \delta_{j,D} p^R_{ij,D})$$

where $\hat{p}_{kj,t}$ are the offers that were rejected by D_j at time t in the candidate equilibrium. However, since $\Delta\pi_j^D(\mathcal{G} \setminus C \cup \mathcal{A}' \cup ij, \mathcal{A}') > \Delta\pi_j^D(\mathcal{G} \setminus C \cup \mathcal{A}', \mathcal{A}')$ by A.CDMC, if accepting additional agreements $\mathcal{A}'$ is profitable for D_j upon accepting deviation $\tilde{p}_{ij}$, then it would have been profitable to accept $\mathcal{A}'$ in the original candidate equilibrium, which is a contradiction. Consequently, D_j would only accept $\tilde{p}_{ij}$ at time t, and reach agreement with all other firms $kj \in \mathcal{C} \setminus ij$ in the subsequent period.

(b) U_i's payoffs from having its offer accepted at time t is:

$$(1 - \delta_{i,U})(\pi_i^U(\mathcal{G} \setminus \mathcal{C} \cup ij) + p^R_{ij,U}) + \delta_{i,U}(\pi_i^U(\mathcal{G}) + p^R_{ij,U})$$

since we have shown D_j would accept U_i's deviation at time t, but reject all other offers. So U_i's change in payoffs from having it's offer $\tilde{p}_{ij}$ accepted at time t is:

$$(1 - \delta_{i,U})(\Delta\pi_i^U((\mathcal{G} \setminus \mathcal{C}) \cup ij, ij) + p^R_{ij,D})$$

which is positive since $\Delta\pi_i^U(\mathcal{G} \setminus \mathcal{C}, ij) \geq \Delta\pi_i^U(\mathcal{G}, ij)$ by A.CDMC and $p^R_{ij,U} > \Delta\pi_i^U(\mathcal{G}, ij)$ by equation (3).

So U_i has a profitable deviation, leading to a contradiction.

Thus, any equilibrium involves immediate agreement for all $ij \in \mathcal{C}$. □

D.3 Many Upstream and Many Downstream Firms

Proof of Lemma 4.10 (Odd, simultaneous.) In the candidate equilibrium, let $\mathcal{A}$ indicate the set of agreements reached first, say in period t (odd), and $\mathcal{B} \equiv \mathcal{C} \setminus \mathcal{A}$ the set of agreements reached at some later date. By the inductive hypothesis, all agreements $ij \in \mathcal{B}$ will occur at $t+1$ at prices $p^R_{ij,U}$.

We first prove that all agreements $ij \in \mathcal{C}$ occur at the same time (i.e., $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \emptyset$), and then prove all agreements occur at the Rubinstein prices.

Claim A: All agreements occur at the same time. We prove the claim by contradiction.

Suppose all agreements are not simultaneous so that $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \neq \emptyset$. Since there are multiple upstream firms with open agreements at time t (by assumption), we can find agreements $ab \in \mathcal{A}$ and $ij \in \mathcal{B}$ s.t. $U_a \neq U_i$: i.e., we can find an agreement formed at time t and another agreement formed at $t+1$ involving different upstream firms.[28] Consider the following deviation by D_j at time t: D_j offers $\tilde{p}_{ij} \equiv p^R_{ij,D} + \varepsilon$ to U_i.

1. *Such a deviant offer will be accepted by U_i.*

[28] If we cannot find two agreements $ab \in \mathcal{A}, ij \in \mathcal{B}$ involving two different upstream parties, it must have been that there was only one upstream firm in $\mathcal{C}$ (which is ruled out by assumption); the case with one upstream firm in $\mathcal{C}$ has been covered in in Section 4.4.

To show this, assume not, and assume that U_i rejects $\tilde{p}_{ij}$ but accepts some set of agreements $\mathcal{A}'_i \subseteq \mathcal{C} \setminus ij$. Even if $\mathcal{A}'_i$ is empty, by passive beliefs U_i still anticipates ab will come to agreement at t; thus, by induction, all agreements $\mathcal{C} \setminus \mathcal{A}'$ will be formed in the next period at prices $p^R_{ij,U}$.

Instead of rejecting $\tilde{p}_{ij}$, U_i can do better by accepting $\mathcal{A}'_i \cup ij$ at time t than accepting only $\mathcal{A}'_i$ if the change in payoffs exceed the change in prices:

$$(1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G} \setminus \mathcal{B}' \cup ij, ij) > \delta_{i,U} p^R_{ij,U} - \tilde{p}_{ij}$$

(where $\mathcal{B}'_i \equiv \mathcal{C}_i \setminus \{\mathcal{A}'_i \cup ij\}$). Since $\Delta\pi^U_i(\mathcal{G} \setminus \mathcal{B} \cup ij, ij) \geq \Delta\pi^U_i(\mathcal{G}, ij)$ by A.CDMC, since $(1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, ij) = \delta_{i,U} p^R_{ij,U} - p^R_{ij,D}$ by (2), and since $\tilde{p}_{ij} = p^R_{ij,D} + \varepsilon$, this inequality holds. Contradiction, and thus U_i cannot reject the deviant offer.

2. *Such a deviation is profitable for D_j if accepted by U_i.*

   Assume if U_i accepts the deviation from D_j at time t, U_i also accepts agreements $\mathcal{A}'_i \subseteq C_i \setminus ij$ at time t as well as ij; let $\mathcal{B}'_i \equiv \mathcal{C}_i \setminus \{\mathcal{A}'_i \cup ij\}$. By the inductive hypothesis, all other agreements in $kj \in \mathcal{B}' \equiv \mathcal{B}_{-i} \cup \mathcal{B}'_i$ will be reached at time $t+1$ at prices $p^R_{kj,U}$.

   D_j will find the deviation of offering $\tilde{p}_{ij} = p^R_{ij,D} + \varepsilon$ at time t profitable if:

   $$(1-\delta_{j,D})(\pi^D_j(\mathcal{G} \setminus \mathcal{B}') - \pi^D_j(\mathcal{G} \setminus \mathcal{B})) > \tilde{p}_{ij} - \delta_{j,D} p^R_{ij,U}$$
   $$\text{for which } (1-\delta_{j,D})(\pi^D_j(\mathcal{G} \setminus \mathcal{B}') - \pi^D_j(\mathcal{G} \setminus \mathcal{B})) > (1-\delta_{j,D}) p^R_{ij,U} \text{ is sufficient.}$$

   By A.CDMC':

   $$(\pi^D_j(\mathcal{G} \setminus \mathcal{B}') - \pi^D_j(\mathcal{G} \setminus \mathcal{B})) \geq \Delta\pi^D_j(\mathcal{G}, ij), \text{ then equation (3) applies.}$$

Since there is a profitable deviation for D_j, there is a contradiction. Thus all agreements happen at the same time.

Claim B: $\hat{p}_{ij} = p^R_{ij,D}$.

Now, suppose the candidate equilibrium price $\hat{p}_{ij} > p^R_{ij,D}$ for some $ij \in \mathcal{C}$. Then, D_j can deviate and offer $\tilde{p}_{ij} \equiv p^R_{ij,D} + \varepsilon < \hat{p}_{ij}$. If U_i rejects some subset of its agreements $\mathcal{B}' \subseteq \mathcal{C}_i$, $ij \in \mathcal{B}'$, U_i anticipates receiving $p^R_{il,U}$ in the subsequent subgame from all $il \in \mathcal{B}'$ in the next period by the inductive hypothesis and passive beliefs (or Claim A if U_i rejects all agreements). Thus, at time t, U_i will accept D_j's deviation (and all other offers it receives) as opposed to rejecting some subset of offers that includes $\tilde{p}_{ij}$ if:

$$(1-\delta_{i,U})\pi^U_i(\mathcal{G}) + \tilde{p}_{ij} + \sum_{il \in \mathcal{C} \setminus ij} \hat{p}_{il,t} + \delta_{i,U}\pi^U_i(\mathcal{G}) > (1-\delta_{i,U})\pi^U_i(\mathcal{G} \setminus \mathcal{B}') + \sum_{il \in \mathcal{A}'} \hat{p}_{il,t} + \delta_{i,U}(\pi^U_i(\mathcal{G}) + \sum_{il \in \mathcal{B}'} p^R_{il,D})$$
$$\Rightarrow \qquad (1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, \mathcal{B}') > \sum_{il \in \mathcal{B}' \setminus ij} (\delta_{i,U} p^R_{il,D} - \hat{p}_{il,t}) + (\delta_{i,U} p^R_{ij,D} - \tilde{p}_{ij})$$

where $\hat{p}_{il,t}$ are the candidate equilibrium prices offered and agreed upon at time t between U_i and D_j. Note:

1. $(1-\delta_{i,U})\Delta\pi^U_i(\mathcal{G}, il) \geq \delta_{i,U} p^R_{il,D} - \hat{p}_{il}$ for all $il \in \mathcal{C}$, else U_i would have rejected $\hat{p}_{il,t}$ in the candidate equilibrium and obtained a strictly higher payoff by coming to an agreement with D_l in the subsequent period (see (2)).

2. $\Delta\pi_i^U(\mathcal{G},\mathcal{B}') \geq \sum_{il\in\mathcal{B}'} \Delta\pi_i^U(\mathcal{G}, il)$ by A.CDMC and Lemma D.1.

Thus, since $\tilde{p}_{ij} = p_{ij,D}^R + \varepsilon$, this inequality holds for $\varepsilon > 0$, and U_i would accept D_j's deviation; contradiction.

Next, suppose $\hat{p}_{ij} < p_{ij,D}^R$ for some $ij \in \mathcal{C}$. U_i can reject this offer (but accept all others) and receive $p_{ij,U}^R$ at time $t+1$ by the inductive hypothesis. This deviation is profitable for U_i if the loss in payoffs is less than the increase in obtained prices:

$$(1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G}, ij) < \delta_{i,U} p_{ij,U}^R - \hat{p}_{ij}$$

By (2), this inequality holds if $\hat{p}_{ij} < p_{ij,D}^R$. Contradiction.

Hence, for all $ij \in \mathcal{C}$, $\hat{p}_{ij} = p_{ij,D}^R$. □

Proof of Lemma 4.11 (Even, simultaneous.) The proof here is symmetric to the case considered in lemma 4.10.

Proof of Lemma 4.12 (Immediate agreement.) Suppose not, and all agreements happen at time $\hat{t} = t + \tau$, and $\tau > 0$.

- $\hat{t}$ is odd

  Consider the deviation where D_j offers all U_i with $ij \in \mathcal{C}$ an offer of $\tilde{p}_{ij} = p_{ij,D}^R + \varepsilon$ at time t. By the inductive lemma all other agreements occur at time $t+1$ with prices $p_{ij,U}^R$. Given passive beliefs, U_i best responds to only $\tilde{p}_{ij}$ being offered. So the highest payoff for U_i, if he rejects, is agreement in the next period given by $(1-\delta_{i,U})\pi_i^U(\mathcal{G}\setminus\mathcal{C}) + \delta_{i,U}\left(\pi_i^U(\mathcal{G}) + \sum_{il\in\mathcal{C}_i} p_{ij,U}^R\right)$. Thus the change in profit for U_i from accepting this deviation are:

  $$\begin{aligned}
  &(1-\delta_{i,U})\left[\Delta\pi_i^U(\mathcal{G}\setminus\mathcal{C}\cup ij, ij) + p_{ij,D}^R\right] + \delta_{i,U}\left[p_{ij,D}^R - p_{ij,U}^R\right] + \varepsilon \\
  =&(1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G}, ij) + p_{ij,D}^R - \delta_{i,U} p_{ij,U}^R \\
  &+ (1-\delta_{i,U})\left[\Delta\pi_i^U(\mathcal{G}\setminus\mathcal{C}\cup ij, ij) - \Delta\pi_i^U(\mathcal{G}, ij)\right] + \varepsilon \\
  =&(1-\delta_{i,U})\left[\Delta\pi_i^U(\mathcal{G}\setminus\mathcal{C}\cup ij, ij) - \Delta\pi_i^U(\mathcal{G}, ij)\right] + \varepsilon
  \end{aligned}$$

  Since by equation (1) $(1-\delta_{i,U})\Delta\pi_i^U(\mathcal{G}, ij) + p_{ij,D}^R - \delta_{i,U} p_{ij,U}^R = 0$. Next, $\Delta\pi_i^U(\mathcal{G}\setminus\mathcal{C}\cup ij, ij) \geq \Delta\pi_i^U(\mathcal{G}, ij)$ by A.CDMC, so U_i will accept this deviation.

  Meanwhile, D_j's highest possible payoffs are from coming to an agreement in the next period $(1-\delta_{j,D})\pi_j^D(\mathcal{G}\setminus\mathcal{C}) + \delta_{j,D}\left[\pi_j^D(\mathcal{G}) - \sum_{kj\in\mathcal{C}_j} p_{ij,D}^R\right]$ (where $\mathcal{C}_j$ indicates the set of open agreements to which D_j is a party, and $\mathcal{C}_{-j}$ indicates the set of open agreements to which D_j is not a party). Thus D_j's profits from offering this deviation are:

  $$(1-\delta_{j,D})\left[\Delta\pi_j^D(\mathcal{G}\setminus C_{-j}, \mathcal{C}_j) - \sum_{kj\in\mathcal{C}_j}(p_{kj,D}^R - \varepsilon)\right]$$

  Notice that $\Delta\pi_j^D(\mathcal{G}\setminus C_{-j}, \mathcal{C}_j) \geq \Delta\pi_j^D(\mathcal{G}, \mathcal{C}_j)$ by A.CDMC, and $\Delta\pi_j^D(\mathcal{G}, \mathcal{C}_j) > \sum_{kj\in\mathcal{C}_j} p_{kj,D}^R$ by equation (4) since D_j will not want to drop agreements at the Rubinstein prices.

- $\hat{t}$ is even

  Consider the deviation where U_i offers D_j an offer of $\tilde{p}_{ij} = p^R_{ij,U}$ at time t. The proof of this case is symmetric to the odd case.

$\square$

E Counterexamples

E.1 Counterexample to Unique Equilibrium

This subsection provides an example of a game with multiple equilibria even when the assumptions from Theorem 3.1 – A.GFT and either (i) A.CDMC′ and A.ASR or (ii) A.CDMC and A.LEXT – hold. As discussed in the text, the equilibria will differ only in their prescribed off-equilibrium play and hence realized outcomes are the same across equilibria.

Let $M = 1$ and $N = 2$ so that there is one downstream firm D_1 and two upstream firms U_1, U_2. In this case, if there are multiple and simultaneous deviations by both upstream firms in an even (upstream-proposing) period—which will reach a node off the equilibrium path—then D_1's best response may be to accept only one and not both of these deviations, and the choice of which offer to accept may be arbitrary.

Numerical Example. For notation, let $\mathcal{A}_k$ denote the network that is empty for $k = 0$; only D_1 and U_k contract for $k = 1, 2$; and is full (i.e., D_1 contracts with both upstream firms) for $k = 3$. Let $\pi^U_i(\mathcal{A}_k) = 0, \forall i, k$, so profits accrue only to D_1. Let $\pi^D_1(\mathcal{A}_0) = 0$, $\pi^D_1(\mathcal{A}_1) = \pi^D_1(\mathcal{A}_2) = 6$, and $\pi^D_1(\mathcal{A}_3) = 8$. Note that this example satisfies both A.CDMC and A.LEXT.

Suppose that $\delta_{1,U} = \delta_{1,D} = \delta_{2,D} = 0.9$. Note that $p^R_{1,U} = p^R_{2,U} \approx 1.0526$. Now consider the even-period node where U_1 and U_2 have both deviated from their equilibrium strategies and offered $p^*_1 = p^*_2 = 1$, and D_1 is deciding which offer(s) to accept. It is easy to verify that, at this node, D_1 should accept either offer but not both offers. Thus, one equilibrium involves D_1 accepting U_1's offer at this node, while another equilibrium involves D_1 accepting U_2's offer at this node.

The underlying logic is that the difference in D_1's payoffs between one and two contracts, which is 2, is smaller than the difference in D_1's payoffs between zero and one contracts, which is 6. The p^R_U payoffs are designed to make D_1 indifferent between accepting both offers and only one—but D_1 strictly prefers one contract to none at these prices.

E.2 Counterexample to Unique Equilibrium Payoffs

This subsection provides an example of a game with an equilibrium where a firm can be paid more than the Rubinstein price when A.LEXT and A.ASR do not hold.

Now, let $N = 1$ and $M = 2$ so that there is one upstream firm U_1 and two downstream firms D_1, D_2. Assume that the first period, t_0, is odd so that downstream firms make initial proposals, and that $(1 - \delta_{1,D})\Delta\pi^D_1(\mathcal{G}, \{12\}) > 0$ (which represents the one period value of the externality imposed on D_1 by U_1 coming to an agreement with D_2).

Consider the strategy profile prescribed in the proof of Theorem 3.1 in Section C, and alter it so that:

- In odd periods, D_1 offers to U_1 the payment $\hat{p}^R_{11,D} \equiv p^R_{11,D} + \varepsilon$, where:

$$\varepsilon \in (0, \min\{(1-\delta_{1,D})\Delta\pi_1^D(\mathcal{G}, \{12\}), (1-\delta_{1,D})\Delta\pi_1^D(\mathcal{G}, \{11\}) - (p^R_{11,D} - \delta_{1,D} p^R_{11,U})\}]$$

- In odd periods, U_1 accepts any offer $p_{11} \geq p^R_{11,D}$ from D_1, and rejects otherwise; however, if U_1 accepts p_{11} and $p_{11} \neq \hat{p}_{11}$, then U_1 rejects p_{12} if $p_{12} = p^R_{12,D}$.

Essentially, this change to the strategies implies that D_1 offers more than its Rubinstein price in an odd period, and that U_1 threatens to reject D_2's offer of $p^R_{12,D}$ if D_1 makes an deviant offer that is greater than or equal to the Rubinstein price, but different than $\hat{p}_{11}$. Since U_1 is indifferent over accepting and rejecting $p^R_{12,D}$ from D_2 given it accepts D_1 under the strategy profiles given, U_1's off-equilibrium threat is credible. The premiums over the Rubinstein price $p^R_{11,D}$ made in the first period can be no higher than either D_1's gain from U_1 contracting with D_2 immediately, or D_1's option of offering such a high price in period 1 so that U_1 rejects it, and then contracting with U_1 in the following period at $p^R_{11,U}$.

As long as A.GFT and A.CDMC hold for the remainder of the underlying payoffs, it is straightforward to show that this strategy profile will comprise an equilibrium. Essentially, it is sustained by the positive externality on D_1 that is generated by U_1 coming to agreement with D_2; U_1 can leverage this to extract a higher price from D_1 when negotiating in an odd period.

Nonetheless, as $\Lambda \to 0$, the outcome of this equilibrium also converges to the one detailed in the uniqueness proof: i.e., Nash-in-Nash prices for all firms, and immediate agreement.

Numerical Example. For notation, now let $\mathcal{A}_k$ denote the network that is empty for $k = 0$; only D_i and U_1 contract for $k = 1, 2$; and is full (i.e., U_1 contracts with both downstream firms) for $k = 3$. Let $\pi_1^U(\mathcal{A}_0) = \pi_1^D(\mathcal{A}_0) = \pi_2^D(\mathcal{A}_0) = \pi_1^D(\mathcal{A}_2) = \pi_2^D(\mathcal{A}_1) = 0$, $\pi_1^U(\mathcal{A}_1) = \pi_1^U(\mathcal{A}_2) = 5$, $\pi_1^U(\mathcal{A}_3) = 8$, $\pi_1^D(\mathcal{A}_1) = \pi_2^D(\mathcal{A}_2) = 1$, and $\pi_1^D(\mathcal{A}_3) = \pi_2^D(\mathcal{A}_3) = 2$.

Suppose again that $\delta_{1,U} = \delta_{1,D} = \delta_{2,D} = .9$. Then the strategies prescribed above with $\varepsilon \leq 0.1$ comprise an equilibrium.