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ABSTRACT

This paper explores price formation when sellers are privately informed both about their preferences and the quality of their asset. In equilibrium, sellers recognize that it will be harder to sell their asset at higher prices, while buyers recognize that they will get higher quality assets on average at higher prices. There are many equilibria of this model, including one in which all trade takes place at one price. Under a behavioral restriction, we find a unique semi-separating equilibrium in which trade takes place over an interval of prices. We characterize necessary and sufficient conditions for this equilibrium to be Pareto optimal. Even though the semi-separating equilibrium allows for more trading opportunities, it may be Pareto dominated and may have less trade than the one-price equilibrium.

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1 Introduction

Consider the market for a used car (Akerlof, 1970). Some of the car's attributes, such as its make and mileage, are easy to verify, while the owner may have private information about other attributes, such as the car's reliability. When the owner sets an asking price in a classified advertisement, she may perceive a tradeoff: if she asks for a higher price, it will take her longer to sell the car, but of course she will get more money when she succeeds in selling it. Given these perceptions, the price she sets will depend both on her preferences and on the attributes of the car. If she has only a weak desire for a newer model and the car is reliable, she will set a high sale price, while she will set a lower sale price and sell the car faster in the opposite circumstance.

Turn now to a used car buyer who is reading the classified advertisements. When he sees a car with a high asking price conditional on its observable attributes, he should conclude that either the car is reliable or that the seller has a weak desire to sell it. If in expectation he believes that higher prices are associated with higher quality cars, he may be willing to pay a higher price. This means that observationally identical cars can sell at heterogeneous prices, although sellers who ask for a low price will sell their car faster than those who set a high price.

Our goal in this paper is to formalize this intuition and explore its implications in a model of exchange with private information. The model economy is populated by continuum of risk-neutral investors who live for two periods. Investors are heterogeneous in their discount factor β between the periods. At the start of the first period, each investor is endowed with one unit of a perishable consumption good and one asset that produces some amount of the consumption good in the second period. Different assets produce different amounts of the consumption good δ. At the beginning of the first period each investor privately observes the quality of his asset (the amount of consumption good it will produce) and his discount factor. Next, there is trade of the first period consumption good for assets. Investors may use their consumption good to buy assets, sell their asset for the consumption good, engage in both activities, or simply consume their endowment. We allow investors to buy or sell at any price, forming beliefs about the probability that they will be able to trade at that price and about the composition of assets offered for sale at that price. Trade is rationed by the short side of the market at every price, with all traders on the long side of the market equally likely to be trade.

We prove that, under a regularity condition, this model supports an equilibrium of the sort described in the first two paragraphs. We summarize sellers' behavior by their continuation value v, the product of their discount factor β and their asset quality δ. Sellers with higher continuation values set higher prices and sell their asset with lower probability. As long as sellers with higher continuation values have higher quality assets on average, buyers rationally perceive that they will get more by paying more, and so are willing to buy at a range of different prices. In such a *semi-separating* equilibrium, identical quality assets sell at different prices, reflecting heterogeneity in the sellers' preferences, while heterogeneous assets sell at the same price if the two sellers have the same continuation value.

We also find that our model admits many other equilibria. In a *one-price* equilibrium, all trade takes place at a single price. Any seller with a continuation value below this price sells for sure at that price, while sellers with higher continuation values do not sell. We stress that the existence of a one-price equilibrium does not come from any restriction on the prices that buyers and sellers may set. There are also many other equilibria, for example equilibria in which all trade takes place at n different prices for arbitrary n.

Our model exhibits multiple equilibria because it is a signalling game. A seller's price is a noisy signal of her asset's quality. Despite this, standard equilibrium refinements in signaling games (e.g. Cho and Kreps, 1987; Banks and Sobel, 1987) are not useful for reducing the set of equilibria. This is because those refinements rely on restricting buyers' beliefs at "off-the-equilibrium-path" prices, i.e. prices that no seller sets. We show that the multiple equilibria can be supported by some sellers who actually set every relevant price.

We instead propose a restriction both on sellers' behavior and buyers' beliefs. Formally, we prove that the semi-separating equilibrium is the unique one in which (i) all investors with the same continuation value use the same strategy when setting sale prices, so on-the-equilibrium-path behavior is the same; and (ii) buyers believe that all investors with the same continuation value are equally likely to select any sale price not chosen in equilibrium, so off-the-equilibrium-path behavior is the same as well.

We then turn to the efficiency properties of equilibrium, with a particular focus on the semi-separating equilibrium. We identify two potential sources of inefficiency. First, the semi-separating equilibrium may have too much separation. In equilibrium, investors with different continuation values always set different prices and sell with different probabilities. It may be possible to raise all investors' welfare by inducing sellers to pool across prices in some intervals. Second, the semi-separating equilibrium may have too much or too little trade. By changing the identity of the marginal buyer, we may be able to raise the welfare of all investors.

We also find that either the one-price equilibrium Pareto dominates the semi-separating equilibrium or the two equilibria are not Pareto comparable. We find this result surprising, because our belief prior to writing this paper was that the one price equilibrium artificially restricted trading opportunities, which would then reduce welfare. For example, in many formulations of the one-price equilibrium, exchange at alternative prices is prohibited (Eisfeldt, 2004; Kurlat, 2013). Instead, we prove that the benefits of pooling may be substantial in this environment. In particular, an investor with the lowest continuation value is always better off in the one-price than in the semi-separating equilibrium.

Finally, we turn to comparative statics. In a numerical example, we demonstrate that an increase in heterogeneity in asset quality reduces the amount of trade, while an increase in heterogeneity in the discount factor raises the amount of trade. Thus private information per se is not bad for trade, but rather private information along the dimension that is relevant to buyers. While this is true in both the one-price and the semi-separating equilibrium, we also find that the one-price equilibrium always has more trade than the semi-separating equilibrium. Again, we find this result surprising in light of our prior beliefs. The one-price equilibrium restrict trading opportunities, yet seems to typically expand the amount of trade.

The equilibrium of this model with multidimensional private information differs from our previous work in which investors' discount factors are observable (Guerrieri and Shimer, 2014). In that model, we found that there is a unique fully separating equilibrium and that assets of higher quality trade at higher price in less liquid markets. The predictions of the two models differ along at least four dimensions. First, with multidimensional private information there is price dispersion for assets of the same quality and heterogeneous assets selling for the same price. In our prior work, there was a one-to-one mapping from asset quality to price. Second, the equilibrium payoffs in this paper are affected by the joint distribution of discount factors and asset quality, while in our prior work, equilibrium payoffs only depended on the support of the distribution and the relative supply of assets. Third, with multidimensional private information some investors both buy and sell assets. In contrast, with observable preferences, investors only participate on one side of the market. Finally, we find that typically a continuum of equilibria exist in this environment and therefore introduce a refinement to reduce the set of equilibria. Beyond that, in this paper we introduce a formal analysis of welfare, while in our prior work we simply explored the gains from particular interventions in financial markets.

Our notion of equilibrium builds on Guerrieri, Shimer and Wright (2010), which in turn builds on prior research, most notably Wilson (1980), Gale (1996), and Ellingsen (1997). All these papers share the idea that price dispersion can arise in the presence of adverse selection, as privately informed sellers can use a high sale price to signal a high quality asset, if this comes at the cost of a lower sale probability. To our knowledge, Chang (2014) is the only other paper that has explored multidimensional private information in that sort of environment. There are at least five important differences between the results in the two papers. First, Chang looks at an environment in which the role of an investor as a buyer or seller is determined exogenously. We allow investors to choose whether to buy assets, sell assets, do both, or do neither. Second, Chang assumes that sellers are heterogeneous while buyers are homogeneous. Moreover, all buyers value any asset more than the average seller does. This ensures that in equilibrium, all assets are sold with a positive probability. In our model, investors are heterogeneous, the decision to buy and sell is endogenous, and in equilibrium some assets are transferred from investors who value them more to investors who value them less. As a result, we find that some investors may choose not to attempt to sell their assets in equilibrium.[1] Third, under an analogous parameter restriction to ours, Chang only characterizes a semi-separating equilibrium, while we prove that in our economy there can generically be a continuum of equilibria, including a one price equilibrium. We stress the comparison between the one-price and semi-separating equilibrium throughout our analysis. Fourth, Chang (2014) characterizes equilibria when the parameter restriction fails. Our paper does not do this. Fifth, we analyze efficiency properties of the semi-separating equilibrium and state necessary and sufficient conditions for local Pareto optimality, while Chang (2014) focuses on the analysis of a number of specific policies.

There is a related line of research that studies how optimal mechanisms can allow for separation when sellers are privately informed, in the spirit of Maskin and Tirole (1992). In DeMarzo and Duffie (1999), sellers can commit to retain a portion of an asset in order to signal its quality. In a similar spirit, in Chari, Shourideh and Zetlin-Jones (2013), buyers offer sellers a menu of contracts, inducing sellers of high quality assets to sell a small amount of their holdings at a high price. Both of these papers focus on environments in which asset quality is private information but sellers' preferences are common knowledge, while we allow for multidimensional private information. More fundamentally, we show that markets naturally achieve the same outcome through a shortage of buyers and rationing.

Daley and Green (2012) obtain a similar outcome using a different approach, again in a model with homogeneous sellers who are privately informed about their asset quality. They show that delay in a dynamic model plays a similar role to sale probabilities in our static setting. In their equilibrium, a sequence of short-lived buyers offer an increasing sequence of sale prices. Sellers with a low valuation sell quickly while those with a high valuation sell later, again dissipating some of the gains from trade. We show that the same dissipation of rents can occur in a static environment through an endogenous shortage of buyers at high prices.

Still other papers have developed models of adverse selection in which all trade occurs at a single price. In some of these papers, such as Eisfeldt (2004) and Kurlat (2013), investors

[1] Formally we model this as investors setting a high price at which they know they will be unable to sell their assets.

are not allowed to consider trading at a different price. In other papers, such as Tirole (2012) and Chiu and Koeppl (2011), the equilibrium is characterized by a pooling price for traded assets. In our preferred equilibrium, trade occurs at a range of different prices, but we also compare our one-price and semi-separating equilibria.

The paper proceeds as follows. Section 2 lays out the basic model. We define our notion of equilibrium in Section 3. In Section 4 we establish by construction that our model exhibits a continuum of equilibria, including the semi-separating and one-price equilibria. In Section 5 we refine our notion of equilibrium through the assumption that investors with identical preferences behave identically. We also establish uniqueness of the semi-separating equilibrium under this additional restriction. Section 6 finds conditions under which the semi-separating equilibrium is Pareto efficient. We also show that the one-price equilibrium may Pareto dominate the semi-separating equilibrium but the reverse is never possible. Section 7 performs comparative statics in a parametric case of the model. Again we show that the one price equilibrium has more trade than the semi-separating equilibrium. Section 8 briefly concludes with a discussion of additional reasons why the notion of equilibrium may be important.

2 Model

The economy lasts for two periods, $t = 1, 2$. It is populated by a unit measure of risk-neutral investors with heterogeneous discount factors $\beta \in [\underline{\beta}, \bar{\beta}] \subseteq \mathbb{R}_+$. Each investor is endowed with one unit of the period 1 consumption good and one unit of an asset that produces the period 2 consumption good as a dividend in period 2.[2] Assets are heterogeneous in their dividend $\delta \in [\underline{\delta}, \bar{\delta}] \subseteq \mathbb{R}_+$, measured in units of the period 2 consumption good. Both consumption goods and assets are divisible. Consumption must be nonnegative in each period.

At the beginning of period 1, each investor privately observes his type, that is, his discount factor β and the quality of his asset δ. Next, there is a market in which period 1 consumption goods and assets are exchanged. Each investor makes independent buying and selling decisions and so may engage in trade on both sides of the market, one side, or none. We assume that an investor can only buy assets using the period 1 consumption good that he holds at the start of the period, and so must consume any period 1 consumption goods he gets from selling his asset.[3] After the market meets, investors consume any remaining period 1 consumption good, $c_1 \geq 0$. In period 2, each investor consumes the dividends generated

[2]We assume for notational convenience alone that each investor has one unit of both the consumption good and the asset. We relax these assumptions in Section 6.

[3]Other assumptions are possible here. While they would change some of our calculations, we not believe that changing this "consumption-good-in-advance" constraint would alter our main results.

by the assets he holds in that period, $c_2 \geq 0$. An investor with discount factor β seeks to maximize $\mathbb{E}(c_1+\beta c_2)$, where expectations recognize that the investor may be uncertain about whether he will succeed in buying and selling assets and about the quality of the assets that he buys.

Let $G : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \to [0,1]$ denote the initial joint distribution of discount factors and asset quality, so $G(\beta, \delta)$ is the measure of investors who have a discount factor less than β and are endowed with an asset with dividend less than δ.[4] We assume G is atomless and let g denote the associated density, with $g(\beta, \delta) = 0$ when $(\beta, \delta) \notin [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}]$. Formally we assume that for any (β, δ) with $g(\beta, \delta) > 0$ there are many investors with this discount factor and asset quality. Informally we identify investors by the pair (β, δ). We also let $G^B(\beta)$ denote the marginal distribution of discount factors, $G^B(\beta) \equiv G(\beta, \bar{\delta})$.

It will be convenient to define an investor (β, δ)'s continuation value as $v \equiv \beta\delta$. The lowest and highest continuation values are $\underline{v} = \underline{\beta}\underline{\delta}$ and $\bar{v} = \bar{\beta}\bar{\delta}$. The distribution of continuation values is $H : [\underline{v}, \bar{v}] \to [0,1]$ where for all $v \geq \underline{v}$,

$$H(v) = \int_{\underline{\beta}}^{v/\underline{\delta}} \int_{\underline{\delta}}^{v/\beta} g(\beta, \delta) d\delta d\beta.$$

Also let $\Gamma : [\underline{v}, \bar{v}] \to [\underline{\delta}, \bar{\delta}]$ denote the expected dividend conditional on an investor's continuation value v. It is straightforward to prove that

$$\Gamma(v) \equiv \frac{\int_{\underline{\delta}}^{\bar{\delta}} g\left(\frac{v}{\delta}, \delta\right) d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \frac{1}{\delta} g\left(\frac{v}{\delta}, \delta\right) d\delta},$$

a function of the joint density g, and so a model primitive. We focus our analysis on the case where the following restriction holds:

Assumption 1 Γ *is continuous and increasing.*

It is easy to find distribution functions that satisfy this restriction. For example, suppose β and δ have independent Pareto distributions, $G(\beta, \delta) = (1 - \beta^{-\alpha_\beta})(1 - \delta^{-\alpha_\delta})$ with $\underline{\beta} = \underline{\delta} = 1$ and $\bar{\beta} = \bar{\delta} = \infty$ for some positive constants α_β and α_δ. Then

$$\Gamma(v) = \frac{(\alpha_\beta - \alpha_\delta)(v^{\alpha_\beta - \alpha_\delta + 1} - 1)}{(\alpha_\beta - \alpha_\delta + 1)(v^{\alpha_\beta - \alpha_\delta} - 1)},$$

continuous and increasing on $[1, \infty)$. Alternatively, suppose $G(\beta, \delta) = \beta^{\alpha_\beta}\delta^{\alpha_\delta}$ with $\underline{\beta} = \underline{\delta} = 0$

[4]The assumption that the support of G is rectangular is again for notational convenience only and can easily be relaxed.

and $\bar{\beta} = \bar{\delta} = 1$ for some positive constants α_β and α_δ. Then

$$\Gamma(v) = \frac{(\alpha_\delta - \alpha_\beta)\big(1 - v^{\alpha_\delta - \alpha_\beta + 1}\big)}{(\alpha_\delta - \alpha_\beta + 1)\big(1 - v^{\alpha_\delta - \alpha_\beta}\big)},$$

essentially the same functional form, and now continuous and increasing on $[0, 1]$. These are our leading examples in Sections 6 and 7.

3 Definition of Equilibrium

We now develop our notion of equilibrium. After each investor learns his discount factor β and the quality of his asset δ, a continuum of markets characterized by a price $p \in R_+$ opens up. Each investor makes independent buying and selling decisions. On the buying side, he has to decide whether to consume his unit of the period 1 consumption good or to use it to buy assets and, if he buys assets, he has to decide at which price, $p_b(\beta, \delta)$. On the selling side, he has to choose whether to sell his asset or not and, if he sells, he has to decide at which price, $p_s(\beta, \delta)$. We assume that each unit of asset and each unit of the period 1 consumption good can be brought to only one market, so an effort to sell (or buy) an asset at a price p is also a commitment not to sell (or buy) the asset at any other price.[5]

In making their optimal trading decisions, investors must form beliefs about the trading probability and the type of assets for sale at any potential price, even those not offered in equilibrium. Let $\Theta(p)$ denote the market tightness associated with price p, that is, the ratio of the amount of the consumption goods that buyers want to use to buy at price p, relative to the cost of the assets that sellers want to sell at price p. If $\Theta(p) < 1$, there are not enough goods to buy all the assets for sale at price p and the sellers are randomly rationed. If instead $\Theta(p) > 1$, there are more goods than needed to buy all the assets for sale at price p and the buyers are randomly rationed. Specifically, a seller who attempts to trade at price p expects to sell with probability $\min\{\Theta(p), 1\}$. Similarly, a buyer who attempts to trade at price p expects to buy with probability $\min\{\Theta(p)^{-1}, 1\}$. A seller who is rationed keeps his asset and in period 2 consumes the dividend produced by it. A buyer who is rationed consumes his period 1 consumption good.

In addition, let $\Delta(p)$ denote buyers' belief about the average dividend among the assets offered for sale at a price p. If some assets are sold at a price p, these beliefs must be consistent with the quality of assets offered for sale. Our definition of equilibrium also rules

[5] We again assume for notational convenience that each investor must choose a single buy price and a single sell price. Allowing an investor to divide his assets or consumption good and attempt to trade at different prices would not affect the set of equilibria.

out equilibria sustained by unreasonable beliefs about the quality of assets for sale in markets that are inactive.

We are now ready to define an equilibrium.

Definition 1 *An equilibrium is four functions* $p_s : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \times \mathbb{R}_+ \to \mathbb{R}_+$*,* $p_b : [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}] \times \mathbb{R}_+ \to \mathbb{R}_+$*,* $\Theta : \mathbb{R}_+ \to [0, \infty]$*, and* $\Delta : \mathbb{R}_+ \to [\underline{\delta}, \bar{\delta}]$ *satisfying the following conditions:*

1. *Optimal Selling Decision: given* Θ*, for all* (β, δ)

$$p_s(\beta, \delta) \in \arg\max_{p \geq \beta\delta} \min\{\Theta(p), 1\}(p - \beta\delta);$$

2. *Optimal Buying Decision: given* Θ *and* Δ*, for all* (β, δ)

$$p_b(\beta, \delta) \in \arg\max_{p \geq 0} \min\{\Theta(p)^{-1}, 1\}\left(\frac{\beta\Delta(p)}{p} - 1\right);$$

3. *Beliefs: For all* $p \in \mathbb{R}_+$ *with* $\Theta(p) < \infty$*,*

   (a) *if there exists a* (β, δ) *with* $p_s(\beta, \delta) = p$*,* $\Delta(p) = \mathbb{E}(\delta | p_s(\beta', \delta') = p)$*;*

   (b) *otherwise there exists a* (β_1, δ_1) *with* $\delta_1 \leq \Delta(p)$*,* $p \geq \beta_1\delta_1$*, and*

$$\min\{\Theta(p_s(\beta_1, \delta_1)), 1\}\big(p_s(\beta_1, \delta_1) - \beta_1\delta_1\big) = \min\{\Theta(p), 1\}\big(p - \beta_1\delta_1\big);$$

   and similarly a (β_2, δ_2) *with* $\delta_2 \geq \Delta(p)$*,* $p \geq \beta_2\delta_2$*, and*

$$\min\{\Theta(p_s(\beta_2, \delta_2)), 1\}\big(p_s(\beta_2, \delta_2) - \beta_2\delta_2\big) = \min\{\Theta(p), 1\}\big(p - \beta_2\delta_2\big);$$

4. *Market Clearing: for all* $p \geq 0$*,* $d\mu_b(p) = \Theta(p)\, d\mu_s(p)$*, where*

$$\mu_s(p) \equiv \iint_{p_s(\beta,\delta) \leq p} g(\beta, \delta)\, d\delta\, d\beta \text{ and } \mu_b(p) \equiv \iint_{p_b(\beta,\delta) \leq p} \frac{g(\beta, \delta)}{p_b(\beta, \delta)}\, d\delta\, d\beta$$

   are the measure of assets for sale at prices below p *and the purchasing power of goods at prices below* p*. Moreover, if there exists a* (β, δ) *with* $p_s(\beta, \delta) = p$ *and* $\Theta(p) > 0$*, then there exists a* (β', δ') *with* $p_b(\beta', \delta') = p$*; and if there exists a* (β, δ) *with* $p_b(\beta, \delta) = p$ *and* $\Theta(p) < \infty$*, then there exists a* (β', δ') *with* $p_s(\beta', \delta') = p$*.*

The first condition requires that investors make optimal selling decisions. Each seller (β, δ) must set an optimal price for his asset.[6] A seller who sets a price p only succeeds

[6] There is no loss of generality in assuming that he attempts to sell the asset. Attempting to sell at any price $p \geq \beta\delta$ always weakly dominates not selling the asset.

in selling with probability $\Theta(p)$. In this event, he gets p units of the consumption good in period 1 but gives up δ units of the consumption good in period 2, which he values at $\beta\delta$. If he fails to sell, he gains nothing. We impose for expositional convenience the restriction that sellers never set a price below their continuation value $\beta\delta$.[7]

The second condition requires that investors make optimal buying decisions. Each buyer (β, δ) sets an optimal price for buying assets.[8] A buyer who sets a price p only succeeds in buying with probability $\min\{\Theta(p)^{-1}, 1\}$. In this event, he gives up a unit of the consumption good and gets $1/p$ assets, each of which he anticipates will produce dividend $\Delta(p)$ next period. If he fails to buy, he gains nothing.

The first part of the third condition imposes that beliefs are consistent with the observed trading patterns whenever possible. If at least one seller sets a price p, then the expected dividend must be the average among the sellers who set that price. The second part of this condition describes beliefs at prices that nobody sets. Intuitively, we require that buyers must be able to rationalize the expected dividend as coming from some probability distribution over sellers, each of whom has a continuation value $\beta\delta$ less than the price and finds setting this price to be weakly optimal. This means that there must either be some investor with dividend $\Delta(p)$ who finds it optimal to set the price p, or that there must be both an investor with a higher quality asset and an investor with a lower quality asset, both of whom find this price optimal. In the latter case, appropriate weights on those two investors justify the expectation $\Delta(p)$.[9]

Finally, the last condition imposes market clearing. It requires that the buyer-seller ratio $\Theta(p)$ at any price p is equal to the ratio of the measure of the purchasing power of buyers at price p to the measure of sellers selling at that price. The last piece of this condition ensures that this holds even if both measures are zero yet a finite number of buyers or sellers sets price p. For notational convenience alone, we do not impose that the buyer-seller ratio is exactly equal to $\Theta(p)$ in this case.

[7] It is never strictly optimal for a seller (β, δ) to set a price $p < \beta\delta$, and is only weakly optimal if $\Theta(p) = 0$ and $\Theta(p') = 0$ for all $p' \geq \beta\delta$.

[8] We prove below that in any equilibrium with trade, $\Theta(p) = \infty$ at sufficiently low prices p. Therefore buyers can always be sure to consume in period 1 by setting a low price and so we do not give buyers the explicit option not to buy.

[9] In our previous research (Guerrieri, Shimer and Wright, 2010; Guerrieri and Shimer, 2014), the analogous condition defined a probability distribution over seller types at each price p. None of the results in this paper would change if we used that definition, but the one we use here is slightly simpler.

4 Examples of Equilibria

This section constructs two equilibria to illustrate the types of outcomes that are feasible in this environment. Appendix B shows that the model exhibits many more equilibria, indeed a continuum of equilibria within certain parameterized classes.

4.1 Semi-Separating Equilibrium

We start by looking for an equilibrium in which every investor sets a sale price that is a strictly increasing function of his continuation value, $p_s(\beta,\delta) = P(\beta\delta)$. Higher prices are associated with a lower buyer-seller ratio, $\Theta(p)$ decreasing, which ensures that every investor is willing to set the appropriate price. On the other hand, higher expected quality compensates buyers for higher prices, $\Delta(p)$ increasing, so investors are willing to purchase assets at a range of different prices.

A semi-separating equilibrium of this form exists if and only if the expected quality of an asset held by an investor with the lowest continuation value is positive, $\Gamma(\underline{v}) > 0$. In that case, the equilibrium is characterized by a discount factor for the marginal buyer, $\hat{\beta} \in (\underline{\beta}, \bar{\beta}]$, which is determined in equation (1) below.

To characterize the equilibrium, we first define two critical prices. The lowest price with trade is $\underline{p} \equiv \hat{\beta}\Gamma(\underline{v})$, the value that the marginal buyer places on an asset sold by the worst seller. Since $\hat{\beta} > \underline{\beta}$ and $\Gamma(\underline{v}) \geq \underline{\delta}$, $\Gamma(\underline{v}) > 0$ implies $\underline{p} > \underline{\beta}\underline{\delta} = \underline{v}$, so a seller with the lowest continuation value strictly prefers selling his asset for $\underline{p}$ rather than retaining it. The second critical price is the highest one with trade. Let $\bar{p}$ be the smallest price satisfying $\bar{p} = \hat{\beta}\Gamma(\bar{p})$, or $\bar{p} = \infty$ if there is no such price. That is, $\hat{\beta}\Gamma(v) > v$ whenever $v < \bar{p}$.

In the semi-separating equilibrium, the equilibrium buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & p < \underline{p} \\ \exp\left(-\int_{\underline{p}}^{p} \frac{1}{p' - \Gamma^{-1}(p'/\hat{\beta})} dp'\right) \text{ if } & p \in [\underline{p}, \bar{p}] \\ 0 & p > \bar{p}. \end{cases}$$

Facing this market tightness, an investor (β,δ) with continuation value $\beta\delta < \bar{p}$ maximizes his profit by setting sale price price $p_s(\beta,\delta) = \hat{\beta}\Gamma(\beta\delta)$. This can be confirmed directly from the first part of the definition of equilibrium, $p_s(\beta,\delta) = \arg\max_{p \geq \beta\delta} \min\{\Theta(p), 1\}(p - \beta\delta)$.

An investor (β,δ) with a higher continuation value, $\beta\delta \geq \bar{p}$, cannot sell his asset at any price satisfying $p \geq v$ and $\Theta(p) > 0$. Although such an investor is indifferent between all sale prices at which he cannot sell their asset, his behavior still matters in equilibrium since it influences buyers' beliefs. We assume that such an investor sets price $p_s(\beta,\delta) =$

$\max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$.

Turn next to the investor's belief about the quality of asset offered at each price. At prices $p < \underline{p}$, buyers are unable to find sellers, $\Theta(p) = \infty$, and so beliefs are undefined. Intermediate prices, $p \in [\underline{p}, \bar{p}]$, are offered only by investors (β, δ) with $\beta\delta = \Gamma^{-1}(p/\hat{\beta})$. Since the average quality asset held by these sellers is $\Gamma(\beta\delta) = p/\hat{\beta}$, part 3(a) of the definition of equilibrium imposes $\Delta(p) = p/\hat{\beta}$ when $p \in [\underline{p}, \bar{p}]$. Finally, at still higher prices, $\Delta(p) \leq p/\hat{\beta}$. Such beliefs are rational, since by construction an investor with continuation value $\beta\delta > \bar{p}$ always sets a price $p_s(\beta, \delta) \geq \hat{\beta}\Gamma(\beta\delta)$.[10]

Given these beliefs, we now use part 2 of the definition of equilibrium. An investor with discount factor $\beta > \hat{\beta}$ maximizes his profit by buying at any price $p \in [\underline{p}, \bar{p}]$, weakly prefers buying at those prices rather than any higher price, and strictly prefers buying at these prices rather than a lower price where there are no sellers. An investor with discount factor $\beta < \hat{\beta}$ prefers to offer a price $p < \underline{p}$, which ensures that he fails to buy in equilibrium.

The last piece of equilibrium is the determination of the marginal discount factor. In order to ensure that the supply of assets is equal to the demand, we require

$$1 - G^B(\hat{\beta}) = \int_{\underline{v}}^{\bar{p}} P(v)\Theta(P(v))dH(v). \tag{1}$$

The left hand side is the total supply of the period 1 consumption good brought to the market by investors with discount factors greater than $\hat{\beta}$. The right hand side is the total cost of purchasing up the assets brought to the market by investors with continuation values $v < \bar{p}$. We prove in Section 5 that there is a unique solution to this equation. Finally, we allocate buyers with $\beta > \hat{\beta}$ to markets so as to equate supply and demand at each price, in accordance with part 4 of the definition of equilibrium.

In summary, in a semi-separating equilibrium, investors face a tradeoff between the probability of selling their asset and the sale price. Investors with a higher continuation value choose a higher sale price because they are less concerned about the consequences of failing to sell their asset. Buyers understand this and rationally anticipate getting a higher quality asset on average when they offer a higher buy price, leaving them indifferent across a range of different prices. Thus heterogeneous assets sell at heterogeneous prices.

Figure 1 illustrates investors' behavior in this equilibrium. Investors are divided into four groups. Patient investors with a high quality asset buy other assets. Impatient investors with a low quality asset try to sell their asset. There are also patient investors with a low quality asset who try to sell their asset and buy other assets; and somewhat impatient investors with

[10]Part 3(a) of the definition of equilibrium, together with the assumption that $p_s(\beta, \delta) = \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$ imposes additional restrictions on $\Delta(p)$, but these are unimportant for our analysis.

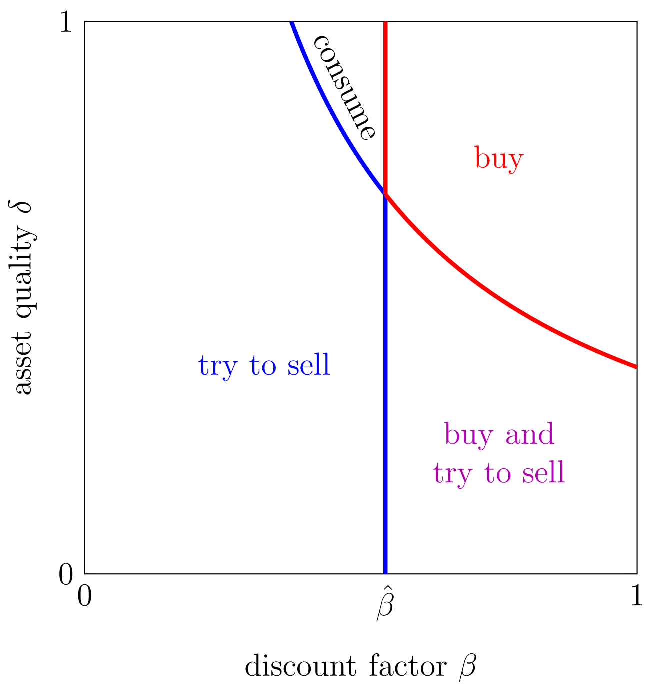

Figure 1: Behavior in a semi-separating equilibrium.

a high quality asset who neither buy nor sell asset but simply consume their endowment in each period.

4.2 One-Price Equilibrium

We next construct equilibria in which all trade takes place at a single price. A one-price equilibrium is characterized by two numbers, the trading price p^* and the identity of the marginal buyer $\hat{\beta} \in (\underline{\beta}, \bar{\beta})$. We find two equations that characterize these variables and construct an associated equilibrium.

In a one-price equilibrium, an investor can purchase an asset at any price greater than or equal to p^* and can sell an asset at any price less than or equal to p^*:

$$\Theta(p) = \begin{cases} \infty \\ 1 \\ 0 \end{cases} \Leftrightarrow p \lesseqqgtr p^*.$$

Part 1 of the definition of equilibrium implies that, taking $\Theta(p)$ as given, an investor (β, δ) with a continuation value $\beta\delta \leq p^*$ will choose to sell for $p_s(\beta, \delta) = p^*$. Investors with a higher continuation value, $\beta\delta > p^*$, set a higher sale price. To support the equilibrium, we choose one such price, $p_s(\beta,\delta) = \bar{\beta}\delta$ if $\beta\delta > p^*$. We stress that in this equilibrium, some sellers' prices depend only on the quality of their assets, not on their continuation values.

Turn next to buyers' beliefs. Part 3(a) of the definition of equilibrium implies that buyers expect

$$\Delta(p^*) = \frac{\int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p^*/\beta} \delta g(\beta,\delta)\, d\delta\, d\beta}{\int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\delta}}^{p^*/\beta} g(\beta,\delta)\, d\delta\, d\beta},$$

the average quality asset held by investors with a continuation value below p^*. At $p > p^*$, the beliefs are also pinned down by condition 3(a): $\Delta(p) = p/\bar{\beta}$ whenever $p \in (p^*, \bar{v}]$. Finally, we assume $\Delta(p) = \bar{\delta}$ when $p > \bar{v}$, consistent with condition 3(b).

Now turn to part 2 of the definition of equilibrium. Let $\hat{\beta} = p^*/\Delta(p^*)$. As long as $\hat{\beta} \le \bar{\beta}$, a buyer with discount factor $\beta > \hat{\beta}$ finds it strictly optimal to buy at price p^*, while buyers with lower discount factors find it better to offer a price $p < p^*$ at which they cannot buy.

Finally, we close the model using the market clearing condition, part 4 of the definition of equilibrium:

$$1 - G^B(\hat{\beta}) = p^* H(p^*), \tag{2}$$

The left hand side is the amount of the period 1 consumption good held by investors with discount factors greater than $\hat{\beta}$ and the right hand side is the cost of buying the assets held by investors with continuation value less than p^*. A one price equilibrium is a pair $(\hat{\beta}, p^*)$ solving $\hat{\beta}\Delta(p^*) = p^*$ and equation (2). Depending on functional forms, one or more one-price equilibrium may exist.

Eisfeldt (2004) and Kurlat (2013) assume that all trade occurs at price p^*. They restrict trading opportunities so a seller has no technology for selling his asset at a price different than p^*. We allow sellers to set such prices, yet all trade occurs at p^* in a one-price equilibrium. Our approach clarifies that the existence of a one-price equilibrium is sensitive to buyers' beliefs $\Delta(p)$ at prices $p > p^*$. It might be most natural to think that all sellers with continuation value just above p^* set a price just above p^*. If that were the case, buyers would anticipate being able to purchase an asset with expected quality just above $\Delta(p^*)$ at such prices. Since the expected quality of an asset for sale at p^* is strictly less than this—it is the average quality of an asset held by investors with continuation values less than or equal to p^*—buyers would find it more profitable to pay this higher price, breaking the one-price equilibrium.

Instead, we support the one-price equilibrium through buyers' belief that sellers with a continuation value just above p^* will set a price just above p^* only if they have the lowest quality asset consistent with the continuation value. This pushes down buyers' beliefs and supports the equilibrium. Moreover, these beliefs are consistent with equilibrium behavior.

In equilibrium, some sellers do set a price just above p^*, justifying the beliefs. Thus standard signaling game refinements, which are based on ruling out unreasonable out-of-equilibrium beliefs, have no bite in our environment.

4.3 Other Equilibria

Once one understands how to construct the one-price equilibrium, it is easy to construct many other equilibria. For example, we show in Appendix B that our model admits a continuum of one price equilibria, each characterized by a sale price p_1, a marginal buyer $\hat{\beta}$, and a sale probability $\theta_1 < 1$. Buyers do not deviate to a higher price because they believe that they will only encounter sellers with low quality assets relative to their continuation value, as we have discussed above. At lower prices, the sale probability is higher than θ_1, eventually reaching 1 at some $p_0 < p_1$. The sale probability $\Theta(p)$ in this interval leaves the seller with the lowest continuation value indifferent about charging any price $p \in [p_0, p_1]$ and keeps sellers with higher continuation values at the equilibrium price p_1. Finally, buyers prefer not to buy at a lower price because they again anticipate getting lower quality assets.

Building on this logic, we show that our model also admits a continuum of equilibrium with n prices for any positive n. And our model admits equilibria that combine intervals with semi-separation and mass points attracting a positive measure of buyers and sellers. In short, many things can happen in our model, depending on how we model sellers' behavior on the equilibrium path and buyers' beliefs off the equilibrium path.

5 Equilibrium Selection

5.1 Restrictions on Sellers' Behavior

We now introduce a restriction on behavior that ensures a unique outcome. Investors with the same continuation value have the same preferences. We impose that they behave the same as well:

Assumption 2 *All investors with (β, δ) such that $\beta\delta = v$ are restricted to set the same selling price and buyers believe that they will do so on and off the equilibrium path.*

This assumption has two important implications for the equilibrium definition. First, since investors with the same continuation value set the same selling price, there exists a function $P : [\underline{v}, \bar{v}] \to \mathbb{R}_+$ such that $p_s(\beta, \delta) = P(\beta\delta)$ for all (β, δ). This restricts part 1 of the equilibrium definition.

Second, it imposes an additional constraint on how buyers rationalize the quality of assets available at a price that is not charged in equilibrium. If they believe that some investor (β, δ) would be inclined to offer that price, then they must believe that all investors with the same continuation value would be equally inclined to offer that price. This implies that part 3 of the equilibrium definition is modified as follows: for all $p \in \mathbb{R}_+$ with $\Theta(p) < \infty$,

(a) if there exists a $v \in [\underline{v}, \bar{v}]$ with $P(v) = p$, $\Delta(p) = \mathbb{E}(\delta | P(v') = p)$;

(b) otherwise there exists a v_1 with $\Gamma(v_1) \leq \Delta(p)$, $p \geq v_1$, and

$$\min\{\Theta(P(v_1)), 1\}(P(v_1) - v_1) = \min\{\Theta(p), 1\}(p - v_1);$$

and similarly a v_2 with $\Gamma(v_2) \geq \Delta(p)$, $p \geq v_2$, and

$$\min\{\Theta(P(v_2)), 1\}(P(v_2) - v_2) = \min\{\Theta(p), 1\}(p - v_2).$$

Once we modified part 1 of the equilibrium definition as specified above, part 3(a) is the same and is included only for expositional convenience. Part 3(b) instead imposes an additional and important restriction on beliefs at prices that are not posted in equilibrium.

One way to think about this restriction is to imagine what would happen if a single buyer offered a price p that was not previously offered in the market. Some sellers would respond by offering some assets at that price, driving down the buyer-seller ratio until it achieved the value $\Theta(p)$ described in part 3(b) of the definition of equilibrium. At this buyer-seller ratio, only a small number of investors would find it optimal to sell assets at that price. The assumption states that if buyers believe that some seller with continuation value v offers assets at that price with some probability, then he must believe that all sellers with the same continuation value will offer assets at that price with the same probability. This means that the average quality of assets offered by sellers with continuation value v at any price that they find optimal is $\Gamma(v)$, a tighter restriction than imposed in part 3(b) of the definition of equilibrium in section 3. The restriction seems reasonable because sellers with the same continuation value have the same cardinal preferences over prices and so should be expected to behave in the same way.[11]

[11] Chang (2014) avoids this issue by defining a seller's type to be his continuation value, rather than the separate components (β, δ). This hard-wires the restriction in Assumptions 2 into the definition of equilibrium. We choose not to do that, highlighting instead that many equilibria exist if we do not impose this restriction.

5.2 Unique Equilibrium

We prove that there is a unique equilibrium under Assumption 2. The equilibrium depends on $\Gamma(\underline{v})$, the expected quality asset held by an investor with the lowest continuation value. If this is zero, the unique equilibrium has no trade (Proposition 1). Otherwise the equilibrium has trade at a continuum of prices (Proposition 2).

We start by introducing a new object, the inverse of the function P, which is useful throughout our proof. For all p, let $\mathbb{V} : \mathbb{R}_+ \rightrightarrows [\underline{v}, \bar{v}]$ denote the set of sellers' continuation values v for which the price $p \geq v$ is weakly optimal:

$$v \in \mathbb{V}(p) \Leftrightarrow p = \arg\max_{p' \geq v} \min\{\Theta(p'), 1\}(p' - v).$$

If $\Theta(p) < \infty$, part 3 of the definition of equilibrium ensures that $\mathbb{V}(p)$ is nonempty, but this is not true in general. For example, $\mathbb{V}(p) = \varnothing$ whenever there exists a $p' > p$ with $\Theta(p') \geq 1$.

We now proceed to establish our uniqueness result through a series of Lemmas.

Lemma 1 *Take any* $p_1 < p_2$, $v_1 \in \mathbb{V}(p_1)$, *and* $v_2 \in \mathbb{V}(p_2)$. *If* $\Theta(p_1) \geq 0$ *then* $v_1 \leq v_2$. *Moreover,* $\mathbb{V}(p)$ *is convex and closed for all* p.

All proofs are in Appendix A. The proof of this result relies on a simple revealed preference argument. Sellers with higher continuation values are more willing to accept the risk of not selling their asset in return for a given increase in the price.

Lemma 2 *Take any* $p_1 < p_2$. *If* $\Theta(p_1) < \infty$, $\Theta(p_2) < 1$. *If* $\Theta(p_1) = 0$, $\Theta(p_2) = 0$.

The proof of this result also uses revealed preference. If a seller could sell for sure at a high price p_2, he would never attempt to sell at a low price p_1. And if he can sell with positive probability at a high price, he would never attempt to sell at a lower price if the sale probability is zero.

Lemma 3 *Impose Assumptions 1 and 2. Take any* p *with* $0 < \Theta(p) < \infty$. *Then* $p \geq \underline{v}$ *and there exists a* $V(p) \in [\underline{v}, \min\{p, \bar{v}\}]$ *such that* $\mathbb{V}(p) = \{V(p)\}$ *and hence* $\Delta(p) = \Gamma(V(p))$. *Moreover,* V *and* Δ *are continuous.*

This proof relies on buyer's behavior. If an interval of sellers offered the same price p, then a buyer would prefer to buy at a slightly higher price, knowing that only sellers with higher continuation values and hence better quality assets on average would attempt to sell at that price. In other words, a buyer would gain by offering a slightly higher price. Therefore each price corresponds to at most one seller type.

Lemma 4 *Impose Assumptions 1 and 2. Take any $p_1 < p_2$ with $\Theta(p_1) < \infty$ and $\Theta(p_2) > 0$. Then $\Delta(p_1) = \Gamma(V(p_1)) < \Gamma(V(p_2)) = \Delta(p_2)$.*

This proof also relies on buyer's behavior. If an interval of prices corresponded to a single seller type, then a buyer would be unwilling to buy from a slightly better seller type. Therefore each seller type has one optimal price.

Lemma 5 *Impose Assumptions 1 and 2. Take any $p_1 < p_2$ with $\Theta(p_1) < \infty$ and $\Theta(p_2) > 0$. Then*

$$\frac{-\Theta(p_2)}{p_1 - V(p_1)} > \frac{\Theta(p_2) - \Theta(p_1)}{p_2 - p_1} \text{ and } \frac{\Theta(p_2) - \min\{\Theta(p_1), 1\}}{p_2 - p_1} > \frac{-\min\{\Theta(p_1), 1\}}{p_2 - V(p_2)}. \tag{3}$$

In particular, if $0 < \Theta(p) < 1$ and $p > V(p)$, $\Theta'(p) = -\Theta(p)/(p - V(p))$.

This proof uses the fact that each price p has a unique seller type $V(p)$ that finds that price optimal, preferring it to slightly higher or lower prices. Manipulating the implied inequalities gives the slope of the buyer-seller ratio Θ.

This then leads to our first main result.

Proposition 1 *Impose Assumptions 1 and 2. If $\Gamma(\underline{v}) = 0$, $\Theta(p) = 0$ for all $p > 0$, so there is no trade at any positive price.*

If there were any trade at a positive price, an investor with the lowest continuation value would attempt to sell at that price. The previous lemmas implied that he would be the only seller at that price and so buyers would refuse to pay the price, a contradiction.

The second result handles the other case, when the expected asset held by a seller with the lowest continuation value has a positive payoff.

Proposition 2 *Impose Assumptions 1 and 2 and $\Gamma(\underline{v}) > 0$. There is a critical buyer type $\hat{\beta} \in (\underline{\beta}, \bar{\beta})$. Given this threshold, define $\underline{p} = \hat{\beta}\Gamma(\underline{v}) > \underline{v}$ and let $\bar{p}$ be the smallest price satisfying $p \geq \hat{\beta}\Gamma(p)$. Then for all $v \in [\underline{v}, \bar{p})$, $P(v) = \hat{\beta}\Gamma(v)$, $\Delta(P(v)) = \Gamma(v)$, and*

$$\Theta(P(v)) = \exp\left(-\int_{\underline{v}}^{v} \frac{\hat{\beta}\Gamma'(\tilde{v})}{\hat{\beta}\Gamma(\tilde{v}) - \tilde{v}} d\tilde{v}\right), \tag{4}$$

with $\Theta(p) = \infty$ if $p < \hat{\beta}\Gamma(\underline{v})$ and $\Theta(p) = 0$ if $p > \bar{p}$. Moreover, $\hat{\beta}$ is uniquely determined by the market clearing condition (1).

This case uses the previous Lemmas to show that a unique semi-separating equilibrium exists, the one we described in Section 4.1. The seller with the lowest continuation value sells for sure at a low price, while sellers with higher continuation values, up to some threshold $\bar{p}$, sell with lower probabilities at higher prices. Sellers with still higher continuation values fail to sell their assets.

6 Efficiency

This section examines whether the semi-separating equilibrium is Pareto efficient among an appropriate set of incentive-compatible and feasible allocations. We break our analysis into five pieces. In Section 6.1, we define the relevant set of allocations. In Section 6.2, we show that there is no scope for a Pareto improvement by changing only investors' buying behavior. Section 6.3 characterizes necessary and sufficient conditions for a Pareto improvement by changing only investors' selling behavior. Section 6.4 develops necessary and sufficient conditions for the semi-separating equilibrium to be locally Pareto efficient, so no small change in behavior on both the buying and selling side of the market makes all investors better off. We use examples throughout to illustrate that the semi-separating equilibrium may or may not be efficient. Finally, Section 6.5 compares welfare in the semi-separating and one-price equilibria. We prove that the one-price equilibrium may Pareto dominate the semi-separating equilibrium or that the two are Pareto incomparable.

In our analysis in this section, we find it useful to relax one assumption. We no longer assume that all investors are both buyers and sellers. We think of an investor endowed with $e \in \{0, 1\}$ units of the period 1 consumption good and with $a \in \{0, 1\}$ units of the asset. Some investors may only have the consumption good and some may only have the asset. We call an investor with $e = 1$ a (potential) buyer and an investor with $a = 1$ a (potential) seller. Let $G^B(\beta)$ and $g^B(\beta)$ denote the distribution and density of discount factors among buyers, $H(v)$ and $h(v)$ denote the distribution and density of continuation values among sellers, $\Gamma(v)$ denote the average quality of assets held by sellers with continuation value v, and π^B and π^S denote the measure of buyers and sellers. For notational convenience, we normalize $\pi^S = 1$. Our baseline model assumes $e = a = 1$ and links these objects through the cumulative distribution function $G(\beta, \delta)$, while the generalization allows these to be separate objects.

This generalization of the model would only change the market clearing condition in our positive analysis, but it has an important effect on our normative analysis. First, in constructing feasible allocations, we treat buyers and sellers as separate investors, not allowing information attained from one side of the market to influence outcomes on the other side of the market.[12] Second, we define a Pareto improving allocation to be one that makes all

[12] We impose this restriction on the equilibrium as well. A buyer would prefer not to buy an asset from a seller who is also buying assets, since this indicates that his discount factor is high and hence asset quality

investors better off, including investors who are only buyers or only sellers. This rules out some potential Pareto improvements in which investors are hurt on one side of the market but the losses are offset by gains on the other side of the market. Thus our approach enlarges the set of Pareto optimal allocations relative to the benchmark in which all investors are either buyers or sellers.

6.1 Incentive-Compatible and Feasible Allocations

We use direct revelation mechanisms to describe the set of incentive-compatible and feasible allocations. Our approach restricts the set of mechanisms in a manner similar to the decentralized equilibrium.

In the decentralized equilibrium, a buyer consumes less than her endowment in period 1 in order to obtain higher expected consumption in period 2. In the mechanism design problem, each buyer reports her discount factor β to the mechanism and receives consumption $c_1^B(\beta)$ in period 1 and $c_2^B(\beta)$ in period 2. The mechanism must be incentive compatible, so a buyer prefers to report her true type β rather than misreporting it as some other $\tilde{\beta}$:

$$u^B(\beta) = c_1^B(\beta) - 1 + \beta c_2^B(\beta) \geq c_1^B(\tilde{\beta}) - 1 + \beta c_2^B(\tilde{\beta}) \tag{5}$$

for all β and $\tilde{\beta}$, where $u^B(\beta)$ is the buyer's gain from trade. In addition, the mechanism must satisfy the buyer's participation constraint, $u^B(\beta) \geq 0$ for all β.

In the decentralized equilibrium, a seller receives consumption in period 1 in return for giving up his asset. In the mechanism design problem, each seller reports his continuation value v to the mechanism, getting expected consumption $c^S(v)$ in period 1 and giving up his asset with probability $\omega(v)$.[13] Again, the mechanism must be incentive compatible, so a seller prefers to report his true type v rather than misreporting it as some other $\tilde{v}$:

$$u^S(v) = c^S(v) - v\omega(v) \geq c^S(\tilde{v}) - \omega(\tilde{v})v$$

for all v and $\tilde{v}$, where $u^S(v)$ is the seller's gain from trade. In addition, the mechanism must satisfy the seller's participation constraint, $u^S(v) \geq 0$ for all v.

Standard arguments imply that the seller's mechanism is incentive compatible if and only

is low. Our definition of equilibrium assumes that trades are hidden and so this information is unavailable.

[13] We assume that a seller only reports his continuation value, rather than both his discount factor and his asset quality. This is consistent with the restrictions on equilibrium behavior embedded in Assumption 2. A mechanism that could separately elicit a seller's asset quality and discount factor might perform better still.

if $\omega(v) \in [0,1]$ is nonincreasing and

$$c^S(v) = \int_v^{\bar{v}} \omega(x)dx + v\omega(v) + k$$

for some constant k. Since the seller cannot consume a negative amount in the first period, we require $k \geq 0$ to ensure $c^S(\bar{v}) \geq 0$. Given that $\omega(v)$ is nonincreasing, this guarantees that $c^S(v) \geq 0$ for all v. Substituting this back into the expression for $u^S(v)$ in the previous paragraph gives

$$u^S(v) = \int_v^{\bar{v}} \omega(x)dx + k, \tag{6}$$

which is nonnegative for all v since ω and k are nonnegative.

We next turn to feasibility, which hinges on the costs of these mechanisms. We start with the buyer's mechanism. In period 1, each of the π^B buyers is endowed with 1 unit of the consumption good and consumes $c_1^B(\beta)$ units of the consumption good. Allowing for free disposal, the cost is therefore

$$C_1^B \geq \pi^B \int_{\underline{\beta}}^{\bar{\beta}} (c_1^B(\beta) - 1)dG_1(\beta). \tag{7}$$

In period 2, the buyers have no endowment and receive $c_2^B(\beta)$ units of the consumption good. Thus the cost is simply

$$C_2^B \geq \pi^B \int_{\underline{\beta}}^{\bar{\beta}} c_2^B(\beta)dG_1(\beta). \tag{8}$$

Now turn to the cost of the seller's mechanism. In period 1, the sellers receive $c^S(v)$ units of the consumption good, so the cost is

$$\begin{aligned} C_1^S &\geq \int_{\underline{v}}^{\bar{v}} c^S(v)h(v)dv \\ &= \int_{\underline{v}}^{\bar{v}} \left(\int_v^{\bar{v}} \omega(x)dx \right) h(v)dv + \int_{\underline{v}}^{\bar{v}} v\omega(v)h(v)dv + k \\ &= \int_{\underline{v}}^{\bar{v}} \omega(v)(H(v) + vh(v))dv + k, \end{aligned} \tag{9}$$

where the second line uses incentive compatibility and the third line uses integration by parts. The total cost of the mechanism in period 2 is negative, given by the amount of dividends collected from the sellers:

$$C_2^S \geq -\int_{\underline{v}}^{\bar{v}} \omega(v)\Gamma(v)h(v)dv. \tag{10}$$

The buyers' and sellers' mechanisms are feasible if total costs are zero in each period, $C_1^B + C_1^S = C_2^B + C_2^S = 0$.

6.2 Buyer Efficiency

We say an allocation is *buyer efficient* if it is incentive-compatible, feasible, and Pareto optimal for buyers among all the incentive-compatible, feasible allocations with the same buyer cost (C_1^B, C_2^B). We prove that any buyer efficient allocation is characterized by a threshold $\hat{\beta}$. Buyers with discount factor $\beta < \hat{\beta}$ consume only in the first period, while buyers with $\beta > \hat{\beta}$ consume only in the second period. A buyer with discount factor $\hat{\beta}$ is indifferent between consuming in the two periods.

Proposition 3 *Let b and $\hat{\beta}$ solve*

$$C_1^B = \pi^B\big(bG^B(\hat{\beta}) - 1\big) \text{ and } C_2^B = \frac{\pi^B(1 - G^B(\hat{\beta}))b}{\hat{\beta}}. \tag{11}$$

If this defines $b \geq 1$, then any buyer efficient allocation has

$$c_1^B(\beta) = \begin{cases} b & \text{if } \beta < \hat{\beta} \\ 0 & \text{if } \beta > \hat{\beta} \end{cases} \quad \text{and} \quad c_2^B(\beta) = \begin{cases} 0 & \text{if } \beta < \hat{\beta} \\ b/\hat{\beta} & \text{if } \beta > \hat{\beta} \end{cases}.$$

Otherwise there is no incentive-compatible, feasible allocation with cost (C_1^B, C_2^B).

Our proof relies on the fact that private information is unimportant for the buyer's problem and so any buyer efficient allocation is equivalent to a competitive equilibrium of an exchange economy among the buyers.

A corollary of this result is that any equilibrium of our model is buyer efficient. This is not surprising, since there is no interesting information problem on the buyer's side of the market. A buyer is privately informed about her discount factor, but a seller does not care about the buyer's discount factor when they trade. This contrasts with the seller's side of the market, since a buyer cares about a seller's expected valuation v when they trade. We turn to the seller's problem next.

6.3 Seller Efficiency

An allocation is *seller efficient* if it is incentive compatible, feasible, and Pareto optimal for sellers among all the incentive-compatible, feasible allocations with the same seller cost (C_1^S, C_2^S).

A semi-separating equilibrium is not necessarily seller efficient. Suppose $H(v) + vh(v)$ is decreasing and $\Gamma(v)h(v)$ is increasing on some interval $[v_1, v_2] \subset [\underline{v}, \bar{p})$.[14] Consider an alternative allocation which coincides with the equilibrium trading probability $\Theta(P(v))$ outside of this interval but takes on its average value inside the interval:

$$\omega(v) = \begin{cases} \int_{v_1}^{v_2} \Theta(P(v))dv/(v_2 - v_1) & \text{if } v \in [v_1, v_2] \\ \Theta(P(v)) & \text{if } v \notin [v_1, v_2]. \end{cases}$$

Also set the constant $k = 0$.

Since $\Theta(P(v))$ is strictly decreasing on $[v_1, v_2]$, equation (6) implies that sellers with continuation value in the interior of this interval are strictly better off under the alternative policy, while sellers with any other continuation value are indifferent between the two policies. Moreover, since $H(v)+vh(v)$ is also decreasing on $[v_1, v_2]$, Chebyshev's inequality for integrals (Gradshteyn and Ryzhik, 2000, p. 1055), together with the construction of $\omega(v)$, implies

$$\int_{v_1}^{v_2} \Theta(P(v))(H(v) + vh(v))dv \geq \int_{v_1}^{v_2} \omega(v)(H(v) + vh(v))dv.$$

Therefore, equation (9) implies the alternative policy is cost feasible in period 1. Similarly, since $\Gamma(v)h(v)$ is increasing on this interval, Chebyshev's inequality for integrals implies

$$\int_{v_1}^{v_2} \Theta(P(v))\Gamma(v)h(v)dv \leq \int_{v_1}^{v_2} \omega(v)\Gamma(v)h(v)dv.$$

Therefore equation (10) implies the alternative policy is cost feasible in period 2.

More generally, we look for necessary and sufficient conditions for the semi-separating equilibrium to be Pareto efficient. Our approach involves finding conditions under which the semi-separating equilibrium maximizes the Pareto-weighted sum of utilities for some Pareto weights.

Proposition 4 *Impose Assumptions 1 and 2 and* $\Gamma(\underline{v}) > 0$*, so there exists a unique semi-separating equilibrium in which all investors with* $v < \bar{p}$ *trade with positive probability. The semi-separating equilibrium is seller efficient if and only if there exist non-negative numbers* ψ_1 *and* ψ_2 *satisfying the following conditions:*

- $\psi_1 \geq 1$,

[14]Indeed, we can construct examples in which this is true globally. Set $\underline{v} = \underline{a}^2$ and $\bar{v} = \bar{a}^2$. Assume that $h(v) = Av^{-\alpha}$, where A is such that $\int h = 1$, and $\Gamma(v) = B + C/[1 + D \exp\{-\xi(v - \underline{a}\bar{a})\}]$, where $\bar{p} > \underline{a}\bar{a}$, and the constants B, C, D are such that $\Gamma(\underline{v}) = \underline{a}$, $\Gamma(\underline{a}\bar{a}) = 2\underline{a}\bar{a}/(\underline{a} + \bar{a})$, and $\Gamma(\bar{v}) = \bar{a}$. One can verify that $H(v) + vh(v)$ is strictly decreasing and $\Gamma(v)h(v)$ is strictly increasing for all $v \in [\underline{v}, \bar{v}]$.

- $J(\underline{v}) = 0,$
- $J(v)$ *nondecreasing for* $v \in [\underline{v}, \bar{p}]$,
- $J(\bar{p}) = 1$, *and*
- $\int_{\bar{p}}^{v} J(x)dx/(v - \bar{p}) \geq 1$ *for* $v > \bar{p}$,

where $J(v) \equiv \psi_1(H(v) + vh(v)) - \psi_2 \Gamma(v)h(v)$.

When $H(v) + vh(v)$ is decreasing and $\Gamma(v)h(v)$ is increasing for all $v \in [v_1, v_2] \subset [\underline{v}, \bar{p}]$, $J(v)$ is decreasing on this interval for any $\psi_1 \geq 1$ and $\psi_2 \geq 0$. The Proposition thus confirms that the semi-separating equilibrium is inefficient in this case.

An example illustrates how the conditions in this Proposition work more generally. Assume β and δ have independent Pareto distributions, $G(\beta, \delta) = (1 - \beta^{-\alpha_\beta})(1 - \delta^{-\alpha_\delta})$ with $\underline{\beta} = \underline{\delta} = 1$ and $\bar{\beta} = \bar{\delta} = \infty$ for some positive constants α_β and α_δ. To keep the functional forms simple, also assume $\alpha_\beta = \alpha_\delta + 1$. Then

$$\Gamma(v) = \frac{1+v}{2},\ B(v) = \frac{v \log v}{v - 1}, \text{ and } H(v) = 1 - (\alpha_\delta + 1)v^{-\alpha_\delta} + \alpha_\delta v^{-\alpha_\delta - 1}.$$

Proposition 2 applies since Γ is increasing and continuous. Moreover, for all v, either $H(v)+ vh(v)$ is increasing or $\Gamma(v)h(v)$ is decreasing, or both. Therefore the logic used in our first example in this section is inapplicable.

The semi-separating equilibrium allocation depends on the number of buyers π^B since that affects the identity of the marginal buyer $\hat{\beta}$. Moreover, Proposition 2 states that the supremum of successful sale prices is the smallest solution to $\bar{p} = \hat{\beta}\Gamma(\bar{p})$, so $\bar{p} = \hat{\beta}/(2 - \hat{\beta})$ if $1 \leq \hat{\beta} < 2$ and $\bar{p} = \infty$ if $\hat{\beta} \geq 2$. We take advantage of these dependencies by treating $\hat{\beta}$ (or $\bar{p}$) as a parameter in this section. In the semi-separating equilibrium, sellers trade with probability

$$\Theta(P(v)) = \begin{cases} \left(\frac{\hat{\beta} + v(\hat{\beta} - 2)}{2(\hat{\beta} - 1)}\right)^{\frac{\hat{\beta}}{2 - \hat{\beta}}} & \hat{\beta} > 2 \text{ or } \left(\hat{\beta} < 2 \text{ and } v < \hat{\beta}/(2 - \hat{\beta})\right) \\ 0 & \text{if} \quad \hat{\beta} < 2 \text{ and } v \geq \hat{\beta}/(2 - \hat{\beta}) \\ e^{1-v} & \hat{\beta} = 2, \end{cases}$$

independent of α_δ.

First assume $0 < \alpha_\delta \leq 2$.[15] If $\hat{\beta} < 2$, set $\psi_1 = \psi_2 = \left(H(\bar{p}) + \bar{p}h(\bar{p}) - \Gamma(\bar{p})h(\bar{p})\right)^{-1} > 1$;

[15] With $\alpha_\delta \leq 1$, total dividends held by sellers are infinite, $\int_1^\infty \Gamma(v)H'(v)\,dv = \infty$, which might seem worrisome for constructing an equilibrium. Nevertheless, total dividends sold are bounded above by the dividends of the worst asset: $\int_1^\infty \Theta(P(v))\Gamma(v)H'(v)\,dv < 1$ for any value of $\hat{\beta}$, so the market clearing condition can hold.

otherwise set $\psi_1 = \psi_2 = 1$. It is easy to verify that $J(v)$ is increasing with $J(1) = 0$ and $J(\bar{p}) = 1$, so that the semi-separating equilibrium is seller efficient.

If instead $\alpha_\delta > 2$ and $\hat{\beta} \geq 2$ (so $\bar{p} = \infty$), then the semi-separating equilibrium is not seller efficient. If $\psi_1 < \psi_2$, $J'(1) < J(1) = 0$, which implies $J(v)$ is negative at values of v slightly above 1, inconsistent with a seller-efficient allocation. If $\psi_1 \geq \psi_2$, $J(v)$ is decreasing at sufficiently large values of v, again inconsistent with a seller efficient allocation when $\bar{p} = \infty$. To construct a Pareto improvement in this example, it is not enough to pool a single group of sellers. That will always either reduce some sellers' utility or raise costs in one of the periods. Instead, we must pool investors within two separate intervals.[16]

6.4 Local Pareto Efficiency

In the previous two sections, we asked whether it is possible to improve the welfare first of buyers and then of sellers without affecting the other group of investors, i.e. taking the costs C_1^B, C_2^B, C_1^S, and C_2^S as given. This section examines the possibility of achieving a Pareto improvement by moving costs across periods in a manner consistent with the resource constraint.

To understand the scope for this, we need to understand how buyers' and sellers' utility is affected by changes in the costs. We focus here on marginal changes in the costs, starting from a semi-separating equilibrium. We say an allocation is *locally Pareto efficient* if it is buyer- and seller-efficient and if no small resource-feasible change in the costs generates a Pareto improvement.

Proposition 5 *Impose Assumptions 1 and 2 and $\Gamma(\underline{v}) > 0$, so there exists a unique semi-separating equilibrium. Assume the semi-separating equilibrium allocation is seller efficient. If there exists a (ψ_1, ψ_2) satisfying the conditions in Proposition 4 and*

$$\frac{G^B(\hat{\beta}) + \hat{\beta} g^B(\hat{\beta})}{g(\hat{\beta})} > \frac{\psi_2}{\psi_1} > \frac{\hat{\beta}^2 g^B(\hat{\beta})}{1 - G^B(\hat{\beta}) + \hat{\beta} g^B(\hat{\beta})},$$

then the semi-separating equilibrium is locally Pareto efficient. Otherwise it is not locally efficient.

[16] Consider the parametric example with $\alpha_\delta = 3$ and $\hat{\beta} = 2$, so that $\Theta(P(v)) = e^{1-v}$. Set $\omega(v) = 100(1 - e^{-0.01}) = 0.995017$ for $v \in [1, 1.01]$, $\omega(v) = (e^{-7.3} - e^{-10.3})/3 = 2.13969 \times 10^{-4}$ for $v \in [8.3, 11.3]$, and $\omega(v) = e^{1-v}$ otherwise. This flattens $\Theta(P(v))$ at its average value within these two intervals. By construction $\int_v^\infty \omega(x)dx$ increases when $v \in (1, 1.01) \cup (8.3, 11.3)$ and is otherwise unchanged, hence the perturbations result in a Pareto improvement. Moreover, while either perturbation alone would raise costs in one of the periods, the two perturbations together reduce costs in both periods.

The Lagrange multipliers ψ_1 and ψ_2 give the marginal value of funds to the sellers in each period, thus their ratio is the marginal rate of substitution of funds across the two periods. The first ratio involving $G^B(\hat{\beta})$ is the marginal rate of substitution for active buyers, those with $\beta > \hat{\beta}$. The last ratio is the marginal rate of substitution for inactive buyers, those with $\beta < \hat{\beta}$. If the marginal rate of substitution for sellers lies in between these two marginal rates of substitution, there is no way to make all investors better off by reallocating resources across periods.

To see how to apply this Proposition, we build on our previous example with independent Pareto distributions. Assume $0 \leq \alpha_\delta \leq 1$. For any $\psi_2 > \psi_1$, $J'(1) < 0$, so there is no associated seller-efficient allocation, while any ratio $\psi_2/\psi_1 \geq 0$ gives us valid Pareto weights for the semi-separating equilibrium. Therefore the semi-separating equilibrium is locally Pareto efficient if and only if $\frac{\hat{\beta}^2 g^B(\hat{\beta})}{1-G^B(\hat{\beta})+\hat{\beta}g^B(\hat{\beta})} < 1$.[17] Since these conditions hinge on the value of $G^B(\hat{\beta})$ and $g^B(\hat{\beta})$, they may or may not hold in any particular economy.

6.5 Comparison with One-Price Equilibrium

We have demonstrated that the semi-separating equilibrium may be Pareto inefficient. This section briefly considers related properties of the one-price equilibrium and compares the welfare properties of the two equilibria. First, we argue that the one-price equilibrium may also be Pareto inefficient and second, that it may Pareto-dominate the semi-separating equilibrium, but the reverse is never possible.

The basic approach in Proposition 4 can be applied to judging the efficiency of an incentive-compatible, feasible allocation, not just the semi-separating equilibrium. The one-price equilibrium, where all trade occurs at price p^*, is seller efficient if and only if there exist numbers $\psi_1 \geq 1$ and $\psi_2 \geq 0$ and a nondecreasing function $\Lambda : [\underline{v}, \bar{v}] \to [0,1]$ such that $\Phi(v) = \int_{\underline{v}}^{v} \big(\Lambda(x) - \psi_1(H(x) + xh(x)) + \psi_2\Gamma(x)h(x)\big)dx$ is maximized at $v = p^*$. A necessary condition for this is that $H(v) + vh(v)$ is increasing or $\Gamma(v)h(v)$ is decreasing in a neighborhood of $v = p^*$; otherwise $\Phi(v)$ is increasing in a neighborhood of p^*. Since we can construct examples in which $H(v) + vh(v)$ is decreasing and $\Gamma(v)h(v)$ is increasing globally (see footnote 14), we know that the one-price equilibrium may not be seller efficient.

Turn next to the comparison between the one-price (O) and semi-separating (S) equilibrium. We first prove that the semi-separating equilibrium never Pareto dominates the one-price equilibrium. Observe first that buyer's utility, $u^B(\beta) = \max\{0, \beta/\hat{\beta} - 1\}$, is decreasing in $\hat{\beta}$. Therefore if buyers are weakly better off in the semi-separating equilibrium,

[17] If $1 < \alpha_\delta \leq 2$, there is also a lower bound on the ratio ψ_2/ψ_1 for generating valid Pareto weights, say $\psi_2/\psi_1 \geq \bar{\psi}$, where $\bar{\psi} \in (0, 1]$. We therefore also require $\frac{G^B(\hat{\beta})+\hat{\beta}g^B(\hat{\beta})}{g(\hat{\beta})} > \bar{\psi}$ in order for the semi-separating equilibrium to be locally Pareto efficient.

the marginal buyers must be ordered via $\hat{\beta}_S \leq \hat{\beta}_O$. Now turn to the utility of the seller with the lowest continuation value. In the semi-separating equilibrium, she sells for sure at a price $\underline{p} = \hat{\beta}_S \Gamma(\underline{v})$, and so this is her utility. In the one-price equilibrium, she sells for sure at a price $p^* = \hat{\beta}_O \Delta_O(p^*)$, where $\Delta_O(p^*)$ is the average quality of assets held by sellers with continuation value below p^*. Since sellers with higher continuation values have higher quality assets on average (Assumption 1), $\Gamma(\underline{v}) < \Delta_O(p^*)$. It follows that whenever buyers are weakly better off in the semi-separating equilibrium, $\underline{p} < p^*$ and so the seller with the lowest continuation value is better off in the one-price equilibrium.

On the other hand, we can construct examples in which the one-price equilibrium Pareto dominates the semi-separating equilibrium. A particularly easy case is one in which $\Gamma(\underline{v}) = 0$, so there is no trade in the semi-separating equilibrium. Since autarky is always feasible, any equilibrium with trade Pareto dominates this autarkic equilibrium. It is easy to construct examples of this sort. Assume β and δ are uniformly distributed on the unit square. Then $\Gamma(v) = -(1-v)/\log v$, while $\Delta(p^*) = (1 - p^*/2)/(1 - \log p^*)$ in any one-price equilibrium. While the unique semi-separating equilibrium has no trade, there is a Pareto-superior one-price equilibrium with $\hat{\beta} = 0.763$ and $p^* = 0.308$.

This result is perhaps surprising. Our intuition for constructing the semi-separating equilibrium was that it would allow for additional trades that could not happen in a one-price equilibrium. Holding everything else equal, creating more trading opportunities should make all investors better off. But everything else is not held equal. The additional trading opportunities erode trading probabilities for all but the lowest type of seller, leading in this extreme case to all trades breaking down.

7 Comparative Statics

This section uses a parameterized example to compare the amount of trade in the semi-separating and one-price equilibria, as well as to examine how the two equilibria respond to changes in the distribution of fundamentals, i.e. preferences and asset quality. Throughout we assume that the distribution of discount factors is Pareto with median 1 and tail parameter α_β, while the distribution of asset quality is an independent Pareto with mean 1 and tail parameter α_δ,

$$G(\beta, \delta) = \left(1 - (k_\beta/\beta)^{\alpha_\beta}\right)\left(1 - (k_\delta/\delta)^{\alpha_\delta}\right)$$

for $\beta \geq k_\beta$ and $\delta \geq k_\delta$, where $\alpha_\beta > 0$, $k_\beta = 2^{-1/\alpha_\beta}$, $\alpha_\delta > 1$, and $k_\delta = 1 - 1/\alpha_\delta$.

We choose these values of the constants k_β and k_δ to obtain a convenient symmetric information benchmark. If δ were observable, half of the investors would sell all their assets

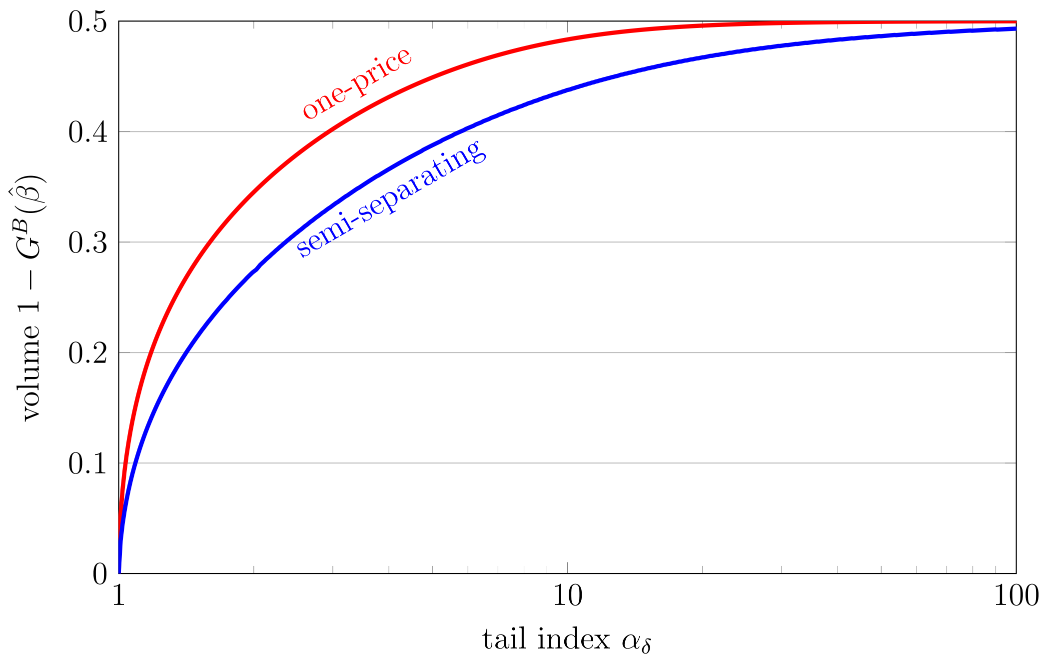

Figure 2: This figure shows the volume of trade as we vary the tail index in the asset quality distribution, α_δ. Both β and δ have independent Pareto distributions. The tail index in the preference distribution, α_β, is fixed at 2, while the median value of β and the mean value of δ are fixed at 1.

to the other half of investors in a competitive equilibrium, regardless of the value of α_β and α_δ. To show this, suppose the marginal investor has the median discount factor, $\hat{\beta} = 1$. The price of any asset in period 1 is then equal to the asset's period 2 dividend. Since the mean dividend is 1, sellers hold assets bearing 1 unit of dividends on average, so the cost of buying up all the assets held by sellers is $\frac{1}{2}$. Buyers hold 1 unit of period 1 consumption good, so they in fact have just enough of the consumption good to buy all the assets held by the sellers.

Figure 2 examines how a change in the tail index on dividends α_δ affects the equilibrium amount of trade, holding fixed $\alpha_\beta = 2$. Note that when $\alpha_\delta > 2$, the variance of the dividend is $1/\alpha_\delta(\alpha_\delta - 2)$, decreasing in α_δ, while at smaller values of α_δ the variance is infinite. Thus higher values of α_δ correspond to situations in which dividends vary less across individuals and so private information is less important.

The figure shows that as the variance of the dividend increases, the volume of trade, measured as the amount of period 1 consumption good exchanged for assets, declines in both equilibria. In the extreme when $\alpha_\delta = 1$, $k_\delta = \underline{v} = \Gamma(\underline{v}) = 0$, and all trade breaks down in both the semi-separating and one-price equilibria.

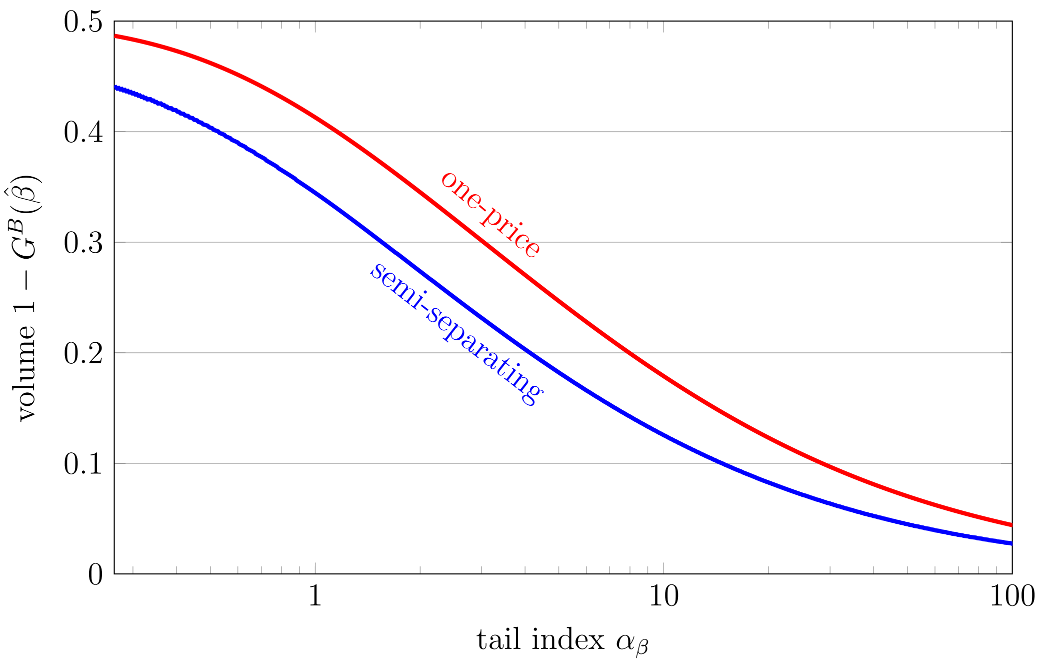

Figure 3: This figure shows the volume of trade as we vary the tail index in the preference distribution, α_β. Both β and δ have independent Pareto distributions. The tail index in the asset quality distribution, α_δ, is fixed at 2, while the median value of β and the mean value of δ are fixed at 1.

There are both partial and general equilibrium effects driving these comparative statics. In partial equilibrium, holding the identity of the marginal buyer fixed, an increase in the variance of the dividend worsens the private information problem. In both equilibria, buyers believe that sellers with low continuation values are increasingly likely to hold low quality assets, reducing equilibrium prices and hence the set of sellers willing to trade. In general equilibrium, this raises the identity of the marginal buyer $\hat{\beta}$, which raises prices. This makes sellers more willing to part with their asset. Nevertheless, the general equilibrium effect on volume can never be strong enough to fully offset the partial equilibrium; if it were, prices would fall and the general equilibrium effect would reinforce the partial equilibrium, a contradiction.

Figure 3 examines how a change in the tail index on discount factors α_β affects the equilibrium amount of trade, holding fixed $\alpha_\delta = 2$. Again, higher values of α_β are associated with a more compressed distribution. Thus the figure shows that when the gains from trade are larger, there is more trade.

When α_β is smaller than 1, the mean value of β is infinite, but there are still limits on the amount of trade. Intuitively, while a small number of individuals have an arbitrarily large preference for second period consumption, those individuals have bounded wealth and so a limited influence on prices. Only when α_β converges to 0 does trade approach the frictionless benchmark. In this limit, the density of $G^B(\beta)$ near the marginal investor is vanishingly small. This means that if private information reduced trade, the identity of the marginal investor would have to increase substantially, pushing up prices and hence the enlarging the set of willing sellers.

Both figures show that trading volume is lower in the semi-separating equilibrium than in the one-price equilibrium. In some cases, the gap in trading volumes is large. As with the welfare comparison between the two equilibria, this might appear surprising. We were motivated to construct the semi-separating equilibrium through an intuition that the one-price equilibrium limited trade opportunities by not allowing sellers to commit to trade probabilistically. In equilibrium, however, the possibility of committing to probabilistic trade allows for complete separation, which reduces the amount of trade below that in the one-price equilibrium.

8 Conclusion

This paper develops and analyzes a semi-separating equilibrium in an environment in which sellers have multidimensional private information and buyers only care about one dimension of the private information. While previous research has focused on how buyers and sellers can design institutions so as to allow sellers to signal the quality of their asset, we have argued that market economies can achieve the same outcome through an endogenous shortage of buyers. Sellers who are eager to sell set a low price and sell with a high probability, while less motivated sellers set a high price and sell with a low or zero probability. Buyers are willing to pay a range of prices, knowing that they will be rewarded with high quality on average when they pay a high price. In particular, the marginal buyer is indifferent about buying any of the assets offered for sale.

We have argued already that the notion of equilibrium affects both the efficiency of the resulting allocation and the amount of trade in the allocation. To our surprise, we found that there is less trade in the semi-separating equilibrium than in the one-price equilibrium and that the semi-separating equilibrium can never Pareto dominate the one-price equilibrium.

We believe that the notion of equilibrium matters for two reasons that we have not yet discussed. First, it would be interesting to understand how private information persists in a dynamic setting. Suppose, for example, that only the initial owner of an asset can observe its quality, but future owners know the price they paid for the asset. In a one-price equilibrium, most information about asset quality is lost in the secondary market since prices are a coarse aggregator of information. In a semi-separating equilibrium, buyers receive more nuanced information since different buyers pay different prices. Therefore, in a semi-separating equilibrium in which past transactions prices are not observed by other market participants, private information can get transmitted to secondary asset markets.

Second, the response to policy interventions is likely to depend on the nature of the equilibrium. For example, a small amount of bad assets can have a big effect on a semi-separating equilibrium, in an extreme case leading to a breakdown in trade if $\underline{v} = \Gamma(0) = 0$. An asset purchase program that removes these assets from the market can then have a big impact on asset prices and trading volumes. In contrast, a small intervention is unlikely to substantially alter a one-price equilibrium, since the equilibrium by its nature depends on all the inframarginal traders. While our model is too stylized to be calibrated, we believe these lessons are likely to carry over to quantitatively serious models of private information.

References

Akerlof, George A., "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 1970, *84* (3), 488–500.

Banks, Jeffrey S. and Joel Sobel, "Equilibrium Selection in Signaling Games," *Econometrica*, 1987, *55* (3), 647–661.

Chang, Briana, "Adverse Selection and Liquidity Distortion," 2014. University of Wisconsin Mimeo.

Chari, V.V., Ali Shourideh, and Ariel Zetlin-Jones, "Reputation and Persistence of Adverse Selection in Secondary Loan Markets," 2013. Forthcoming, *American Economic Review.*

Chiu, Jonathan and Thorsten V. Koeppl, "Trading Dynamics with Adverse Selection and Search: Market Freeze, Intervention and Recovery," 2011. Bank of Canada Mimeo.

Cho, In-Koo and David M. Kreps, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 1987, *102* (2), 179–221.

Daley, Brendan and Brett Green, "Waiting for News in the Dynamic Market for Lemons," *Econometrica*, 2012, *80* (4), 1433–1504.

DeMarzo, Peter and Darrell Duffie, "A Liquidity-Based Model of Security Design," *Econometrica*, 1999, *67* (1), 65–99.

Eisfeldt, Andrea L., "Endogenous Liquidity in Asset Markets," *Journal of Finance*, 2004, *59* (1), 1–30.

Ellingsen, Tore, "Price Signals Quality: The Case of Perfectly Inelastic Demand," *International Journal of Industrial Organization*, 1997, *16* (1), 43–61.

Gale, Douglas, "Equilibria and Pareto Optima of Markets with Adverse Selection," *Economic Theory*, 1996, *7* (2), 207–235.

Gradshteyn, I.S. and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 2000.

Guerrieri, Veronica and Robert Shimer, "Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality," *American Economic Review*, 2014, *104*, 1875–1908.

_ , _ , and Randall Wright, "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 2010, *78* (6), 1823–1862.

Kurlat, Pablo, "Lemons Markets and the Transmission of Aggregate Shocks," *American Economic Review*, 2013, *103* (4), 1463–1489.

Maskin, Eric and Jean Tirole, "The Principal-Agent Relationship with an Informed Principal, II: Common Values," *Econometrica*, 1992, *60* (1), 1–42.

Tirole, Jean, "Overcoming Adverse Selection: How Public Intervention Can Restore Market Functioning," *American Economic Review*, 2012, *102* (1), 29–59.

Wilson, Charles, "The Nature of Equilibrium in Markets with Adverse Selection," *Bell Journal of Economics*, 1980, *11* (Spring), 108–130.

A Omitted Proofs

Proof of Lemma 1. By the definition of $\mathbb{V}$, a seller with continuation value v_1 weakly prefers p_1 to p_2:

$$\min\{\Theta(p_1), 1\}(p_1 - v_1) \geq \min\{\Theta(p_2), 1\}(p_2 - v_1), \tag{12}$$

Similarly, a seller with continuation value v_2 either finds the price p_1 suboptimal because $p_1 < v_2$ or prefers p_2 to p_1:

$$\min\{\Theta(p_2), 1\}(p_2 - v_2) \geq \min\{\Theta(p_1), 1\}(p_1 - v_2). \tag{13}$$

First suppose $p_1 < v_2$. Since $v_1 \in \mathbb{V}(p_1)$, $v_1 \le p_1$, proving $v_1 < v_2$. Second suppose $p_1 \ge v_2$ so inequality (13) holds. If $\Theta(p_2) = 0$, $\Theta(p_1) > 0$ implies $p_1 = v_2$ so again $v_1 \le v_2$. If instead $\Theta(p_2) > 0$, multiply inequalities (12) and (13) and simplify to get $(p_2 - p_1)(v_2 - v_1) \ge 0$, which proves that $v_2 \ge v_1$.

To prove $\mathbb{V}(p)$ is convex, take any p and $v_1 < v_2$ with $v_1, v_2 \in \mathbb{V}(p)$. Fix any $\tilde{v} \in (v_1, v_2)$ and $\tilde{p}$ such that $\tilde{v} \in \mathbb{V}(\tilde{p})$. Note that such a $\tilde{p}$ must exist; set $\tilde{p} = p_s(\beta, \delta)$ for any $\beta\delta = \tilde{v}$. If $p < \tilde{p}$, $v_2 \in \mathbb{V}(p)$ and $\tilde{v} \in \mathbb{V}(\tilde{p})$ contradicts the first part of the lemma. If $\tilde{p} < p$, $v_1 \in \mathbb{V}(p)$ and $\tilde{v} \in \mathbb{V}(\tilde{p})$ contradicts the first part of the lemma. Therefore $p_s(\beta, \delta) = p$ for all $\beta\delta \in (v_1, v_2)$.

To prove $\mathbb{V}(p)$ is closed, suppose there exists a sequence $\{v_n\} \to v$ with $v_n \in \mathbb{V}(p)$ for all n but $v \notin \mathbb{V}(p)$. Since $p \ge v_n$ for all n, $p \ge v$ as well. The definition of $\mathbb{V}$ then implies that there exists a $\tilde{p} \ge v$ with

$$\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v) - \min\{\Theta(p), 1\}(p - v) \equiv \varepsilon > 0$$

But since $\{v_n\} \to v$, there exists an N such that for all $n > N$,

$$\big(\min\{\Theta(\tilde{p}), 1\} - \min\{\Theta(p), 1\}\big)(v_n - v) < \varepsilon.$$

Using the definition of ε, this implies $\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v_n) > \min\{\Theta(p), 1\}(p - v_n)$, and in particular $\tilde{p} > v_n$, which contradicts $v_n \in \mathbb{V}(p)$. ■

Proof of Lemma 2. Suppose there is a $p_1 < p_2$ with $\Theta(p_1) < \infty$. Part 3 of the definition of equilibrium implies that there is a (β, δ) who finds p_1 an optimal sale price. In particular, $p_1 \ge \beta\delta$ and p_1 gives weakly higher profits than p_2:

$$\min\{\Theta(p_1), 1\}(p_1 - \beta\delta) \ge \min\{\Theta(p_2), 1\}(p_2 - \beta\delta).$$

First, suppose $\Theta(p_1) = 0$. Since the left hand side is zero, the right hand side must be as well. Given that $p_2 > p_1 \ge \beta\delta$, it must be that $\Theta(p_2) = 0$. Second, suppose $\Theta(p_2) \ge 1$. Then

$$\min\{\Theta(p_2), 1\}(p_2 - \beta\delta) = p_2 - \beta\delta.$$

Also, since $p_1 \ge \beta\delta$,

$$p_1 - \beta\delta \ge \min\{\Theta(p_1), 1\}(p_1 - \beta\delta).$$

Now combining the inequalities implies $p_1 \geq p_2$, a contradiction. $\blacksquare$

Proof of Lemma 3. Since $\Theta(p) < \infty$, part 3 of the definition of equilibrium implies that there is a (β, δ) who finds p an optimal sale price and in particular $p \geq \beta\delta \geq \underline{v}$. If $p = 0$, these inequalities imply $\beta\delta = \underline{v} = 0$ as well, so $\mathbb{V}(0) = \{V(0)\}$, where $V(0) = 0$.

Otherwise, $p > 0$ and in order to find a contradiction, suppose $\mathbb{V}(p) \neq \{\beta\delta\}$. Then Lemma 1 implies it must be an interval, $\mathbb{V}(p) = [v_1, v_2]$. Lemma 1 also implies that p is the only optimal sale price for $v \in (v_1, v_2)$: $p_s(\beta, \delta) = p$ if $\beta\delta \in (v_1, v_2)$, while any investor who finds a lower (higher) sale price optimal must have a lower (higher) continuation value. Then part 3(a) of the definition of equilibrium implies

$$\Delta(p) = \frac{\int_{v_1}^{v_2} \Gamma(v) h(v) dv}{\int_{v_1}^{v_2} h(v) dv},$$

where $h(v)$ again is the density of continuation values. Monotonicity of Γ (Assumption 1) and the restrictions on sellers' behavior (Assumption 2) imply

$$\Gamma(v_1) < \Delta(p) < \Gamma(v_2) \leq \Delta(p')$$

for any price $p' > p$.

We can use this to prove that no buyer finds setting the price p optimal. If there were such a buyer, he must have $\beta\Delta(p) \geq p$ by part 2 of the definition of equilibrium. So take any

$$p' \in \left(p, \frac{p\Delta(p')}{\Delta(p)}\right),$$

a nonempty interval since $\Delta(p') > \Delta(p)$ and $p > 0$. Since $\Theta(p) < \infty$, Lemma 2 implies $\Theta(p') < 1$. Then

$$\begin{aligned} \min\{\Theta(p)^{-1}, 1\}\left(\frac{\beta\Delta(p)}{p} - 1\right) &\leq \frac{\beta\Delta(p)}{p} - 1 \\ &< \frac{\beta\Delta(p')}{p'} - 1 = \min\{\Theta(p')^{-1}, 1\}\left(\frac{\beta\Delta(p')}{p'} - 1\right). \end{aligned}$$

The first inequality uses $\min\{\Theta(p)^{-1}, 1\} \leq 1$ and $\beta\Delta(p) \geq p$. The second uses $p' < p\Delta(p')/\Delta(p)$. The equality uses $\Theta(p') < 1$.

We now have a contradiction. The measure of buyers setting price p is zero, $d\mu_b(p) = 0$, while the measure of sellers setting price p is positive, $d\mu_s(p) = \int_{v_1}^{v_2} h(v) dv$. This is inconsistent with part 4 of the definition of equilibrium, $d\mu_b(p) = \Theta(p) d\mu_s(p)$.

Now suppose V has a discontinuity at p. Using the arguments in Lemma 1, all $v \in \left(\liminf_{p' \to p} V(p'), \limsup_{p' \to p} V(p')\right)$ must find price p optimal, which contradicts the first part of this result. Finally, $\Delta(p) = \Gamma(V(p))$ by Assumption 2 and continuity of Δ follows from Assumption 1. ■

Proof of Lemma 4. Since $\Theta(p_1) < \infty$, Lemma 2 implies $\Theta(p) < 1$ for all $p > p_1$. And since $\Theta(p_2) > 0$, the same Lemma implies $\Theta(p) > 0$ for all $p < p_2$. Then Lemma 3 implies $\mathbb{V}(p) = \{V(p)\}$, a singleton, and Lemma 1 implies $V(p)$ is weakly increasing on this interval, $V(p_1) \leq V(p_2)$.

Now to find a contradiction, suppose $v = V(p_1) = V(p_2)$ and let $p_3 = \max p$ such that $v = V(p)$. By definition, $v \leq p_1 < p_2 \leq p_3$. Moreover, for any price $p \in (p_1, p_3)$, the fact that sellers with continuation value v find p an optimal price implies

$$\Theta(p) = \min\{\Theta(p_1), 1\} \frac{p_1 - v}{p - v}. \tag{14}$$

If $v = p_1$, $\Theta(p_2) = 0$, a contradiction. Therefore $v < p_1$ and $\Theta(p) > 0$.

Next note that if $v = \bar{v}$, $\beta\Gamma(v) \leq v$ for all β. Since $v < p_1$, $\beta\Gamma(v) < p$, and so part 2 of the definition of equilibrium implies that no buyer is willing to pay any price that a seller with continuation value v sets, contradicting the market clearing condition in part 4 of the definition of equilibrium. Therefore $v < \bar{v}$.

Now fix a continuation value $\tilde{v} > v$ that satisfies the following two restrictions: (1) $\Gamma(\tilde{v}) < \Gamma(v)p_3/p$ and (2) $\tilde{v} < p_3$. The first restriction is feasible since Γ is continuous by Assumption 1 and $p < p_3$ by assumption. The second restriction is feasible because $v < p_3$. Finally, let $\tilde{p}$ denote an optimal price for a seller with continuation value $\tilde{v}$, say $\tilde{p} = p_s(\tilde{\beta}, \tilde{\delta})$ for some $\tilde{\beta}\tilde{\delta} = \tilde{v}$. By Lemma 3, $\tilde{v} > v$ implies $\tilde{p} > p_3$.

The fact that a seller with continuation value $\tilde{v}$ sets price $\tilde{p}$ implies in particular that

$$\Theta(\tilde{p})(\tilde{p} - \tilde{v}) \geq \Theta(p_3)(p_3 - \tilde{v}).$$

Since $\Theta(p_3) > 0$ and $p_3 > \tilde{v}$, the right hand side is positive. The left hand side must therefore be as well, so in particular $\Theta(\tilde{p}) > 0$, so some buyers offer price $\tilde{p}$.

Now consider the value to a buyer of offering p rather than $\tilde{p}$:

$$\min\{\Theta(p)^{-1}, 1\}\left(\frac{\beta\Gamma(v)}{p} - 1\right) = \frac{\beta\Gamma(v)}{p} - 1 > \frac{\beta\Gamma(\tilde{v})}{\tilde{p}} - 1 = \min\{\Theta(\tilde{p})^{-1}, 1\}\left(\frac{\beta\Gamma(\tilde{v})}{\tilde{p}} - 1\right)$$

The first equality holds because $\Theta(p) < 1$. The inequality holds because $\Gamma(v)/p > \Gamma(\tilde{v})/p_3 >$

$\Gamma(\tilde{v})/\tilde{p}$, first by construction of $\tilde{v}$, second by $\tilde{p} > p_3$. The second equality holds because $\Theta(\tilde{p}) < 1$ as well. But then $\tilde{p}$ is not an optimal price for any buyer, inconsistent with part 4 of the definition of equilibrium.

This contradiction implies $V(p_1) < V(p_2)$. Finally, Assumption 1 implies $\Delta(p) = \Gamma(V(p))$ is therefore also strictly increasing. ■

Proof of Lemma 5. Lemma 4 implies

$$\min\{\Theta(p_1), 1\}(p_1 - V(p_1)) > \min\{\Theta(p_2), 1\}(p_2 - V(p_1)),$$

since an investor with continuation value $V(p_1)$ only finds the price p_1 optimal. Note that this implies $p_1 > V(p_1)$ as well. Since $\Theta(p_1) < \infty$, Lemma 2 implies $\Theta(p_2) < 1$, while $p_1 > V(p_1)$ implies $\Theta(p_1)(p_1 - V(p_1)) \geq \min\{\Theta(p_1), 1\}(p_1 - V(p_1))$. Combining these inequalities yields the first inequality in condition (3).

Similarly, Lemma 4 implies

$$\min\{\Theta(p_2), 1\}(p_2 - V(p_2)) > \min\{\Theta(p_1), 1\}(p_1 - V(p_2)).$$

Again note that $\Theta(p_2) < 1$ but now $\Theta(p_1) > 1$ is possible. This then leads to the second inequality in condition (3).

Now suppose $0 < \Theta(p) < 1$ and consider an arbitrary sequence of prices $\{\tilde{p}\}$ with $0 < \Theta(\tilde{p}) < 1$ and converging to p. Since V is continuous by Lemma 3, $V(\tilde{p}) \to V(p)$ as well. At every point in the sequence, condition (3) implies

$$\Theta(p)\frac{p - V(p)}{\tilde{p} - V(p)} \geq \Theta(\tilde{p}) \geq \Theta(p)\frac{p - V(\tilde{p})}{\tilde{p} - V(\tilde{p})}.$$

The two bounds converge to $\Theta(p)$, proving that $\Theta(\tilde{p}) \to \Theta(p)$. Finally, condition (3) also implies

$$\frac{-\Theta(\tilde{p})}{p - V(p)} \geq \frac{\Theta(\tilde{p}) - \Theta(p)}{\tilde{p} - p} \geq \frac{-\Theta(p)}{\tilde{p} - V(\tilde{p})}.$$

Again both bounds converge to $-\Theta(p)/(p - V(p))$, establishing the result. ■

Proof of Proposition 1. We first construct an equilibrium with $\Theta(p) = 0$ for all $p > 0$. In this equilibrium, $P(v) = \hat{\beta}\Gamma(v)$ for all v and some $\hat{\beta} \geq \bar{\beta}$, so $\Delta(p) = p/\hat{\beta}$ for all $p \leq \hat{\beta}\Gamma(\bar{v})$ and $\Delta(p) = \Gamma(\bar{v})$ otherwise. These sale prices are optimal given $\Theta(p)$. This seller behavior implies that $\beta\Delta(p)/p \leq \beta/\hat{\beta} \leq 1$ for all $p > 0$ and so $p_b(\beta, \delta) = 0$ for all $(\beta, \delta) \in [\underline{\beta}, \bar{\beta}] \times [\underline{\delta}, \bar{\delta}]$.

Finally, market clearing implies that $\Theta(0) = \infty$. This is an equilibrium consistent with the restriction in Assumption 2.

Now to find a contradiction, suppose there is an equilibrium with $\Theta(p) > 0$ for some $p > 0$. Note that $\Gamma(\underline{v}) = 0$ implies $\underline{\delta} = 0$ and hence $\underline{v} = 0$. Sellers' optimality implies $P(v) > 0$ for all $v < p$ and in particular $P(\underline{v}) > 0$. Moreover, $\Theta(p) < 1$ for all $p > P(\underline{v})$, or a seller with the lowest continuation value would set a higher price. Now fix a $v > \underline{v}$ such that $\Gamma(v) < P(\underline{v})/\bar{\beta}$; this must exist because Γ is continuous by Assumption 1. There is an optimal price for a seller with continuation value v, $P(v)$, and $\mathbb{V}(P(v)) = \{v\}$ by Lemma 3. Therefore $\Delta(P(v)) = \Gamma(v)$. Moreover, that a seller with continuation value v prefers $P(v)$ to $P(\underline{v})$ implies $\Theta(P(v)) > 0$. But then the payoff of a buyer purchasing at price $P(v)$ is $\min\{\Theta(P(v))^{-1}, 1\}(\beta\Delta(P(v))/P(v) - 1) < 0$, since $\Theta(P(v)) < 1$ and $\Delta(P(V)) = \Gamma(v) < P(\underline{v})/\bar{\beta} < P(v)/\bar{\beta} \leq P(v)/\beta$. Since no buyer finds it optimal to buy at a price p, $\Theta(p) = 0$ for all $p > 0$ whenever $\Gamma(\underline{v}) = 0$. ■

Proof of Proposition 2. We first rule out the possibility of an equilibrium in which $\Theta(p) = \infty$ for $p < \bar{p}$ and $\Theta(p) = 0$ for $p > \bar{p}$ for some $\bar{p} \geq 0$. If $\bar{p} > \underline{v}$ and $\Theta(\bar{p}) < 1$, $P(v)$ is undefined for $v \in [\underline{v}, \bar{p})$, so this cannot be an equilibrium. If $\bar{p} > \underline{v}$ and $\Theta(\bar{p}) \geq 1$, $P(v) = \bar{p}$ for $v \in [\underline{v}, \bar{p})$, which contradicts Lemma 3. Therefore $\underline{v} \geq \bar{p}$. Then since $\underline{v} = \underline{\beta}\underline{\delta}$ and $\Gamma(\underline{v}) \geq \underline{\delta}$, $\underline{\beta}\Gamma(\underline{v}) \geq \bar{p}$. Now consider the buyer's problem. At any price $p > \bar{p}$, the buyer can buy for sure. At prices $p < \beta\Gamma(\underline{v})$, the buyer makes profit conditional on buying. If $\beta > \underline{\beta}$, this defines a nonempty interval where buyers would make profit buying, and so cannot be an equilibrium.

Now using Lemma 2, there are thresholds $\underline{p} < \bar{p}$ where $\Theta(p) = \infty$ if $p < \underline{p}$, $\Theta(p) = 0$ if $p > \bar{p}$, and $\Theta(p) \in (0, 1)$ if $p \in (\underline{p}, \bar{p})$. The differential equation for Θ in Lemma 5 then applies in this range, giving

$$\Theta(p) = \lambda \exp\left(-\int_{\underline{p}}^{p} \frac{1}{\tilde{p} - V(\tilde{p})} dp\right) \tag{15}$$

for all $p \in (\underline{p}, \bar{p})$ and some constant of integration $\lambda > 0$. In addition, Lemma 2 ensures that $\lambda \leq 1$ so that $\Theta(p) < 1$ for all $p > \underline{p}$. And if $\lambda < 1$, an investor with continuation value $v = V(p)$ where $\underline{p} < p < (\underline{p} - \Theta(p)p)/(1 - \Theta(p))$ would earn higher profits selling with probability 1 at price $\underline{p} - \varepsilon$ for some sufficiently small ε, rather than selling at price p, a contradiction. Therefore $\lambda = 1$.

Turn now to the buyers' problem. Buyers know that only a seller with continuation value $V(p)$ sells at price p, so $\Delta(p) = \Gamma(V(p))$. For buyers to be willing to purchase at all prices $p \in [\underline{p}, \bar{p}]$, it must be the case that $p/\Delta(p) = \hat{\beta}$ for some constant $\hat{\beta}$, or equivalently $\Gamma(V(p)) = p/\hat{\beta}$. Substituting this into equation (15) and changing the variable of integration gives (4) for $v \in (\underline{v}, \bar{p})$. Moreover, we can extend this to $v = \underline{v}$ since buyers must be willing to purchase assets from the sellers with the lowest continuation value as well, which requires that $\Theta(\underline{p}) \leq 1$.

Given equation (4), sellers with continuation value $v < \bar{p}$ set price $P(v) = \hat{\beta}\Gamma(v)$, while all sellers with higher continuation values are indifferent about all prices $p > \bar{p}$ and in particular are willing to set prices such that $P(v) \geq \hat{\beta}\Gamma(v)$. This ensures that buyers with $\beta > \hat{\beta}$ are indifferent about buying at any price $p \in [\underline{p}, \bar{p}]$ and prefer those prices to higher prices. Buyers with lower continuation values set lower prices and do not succeed in buying. To find an equilibrium, we simply allocate the buyers to the different prices in a way that ensures the appropriate buyer-seller ratio at each price. This is feasible if the total wealth of buyers with $\beta > \hat{\beta}$ is exactly enough to purchase the assets sold by sellers with $v \in [\underline{v}, \bar{p}]$:

$$\int_{\underline{\delta}}^{\bar{\delta}} \int_{\hat{\beta}}^{\bar{\beta}} g(\beta, \delta)\, d\beta\, d\delta = \int_{\underline{\delta}}^{\bar{\delta}} \int_{\underline{\beta}}^{\bar{p}/\delta} P(\beta\delta)\Theta(P(\beta\delta))g(\beta, \delta)\, d\beta\, d\delta \tag{16}$$

The left hand side is obviously decreasing in $\hat{\beta}$, equal to 0 when $\hat{\beta} = \bar{\beta}$ and 1 when $\hat{\beta} = \underline{\beta}$. The right hand side is increasing in $\hat{\beta}$. To prove this, note first that $\bar{p}$, defined as the smallest solution to $\bar{p} \geq \hat{\beta}\Gamma(\bar{p})$ is nondecreasing in $\hat{\beta}$. So are $P(v) = \hat{\beta}\Gamma(v)$ and $\Theta(P(v))$ defined in equation (4). Finally, when $\hat{\beta} = \underline{\beta}$, $\bar{p} = \underline{v}$. If $\underline{\beta} = 0$, $\underline{v} = 0$ and we can immediately verify $0 = \underline{\beta}\Gamma(0)$ for any value of $\Gamma(0)$. If $\underline{\beta} > 0$, $\Gamma(\underline{v}) = \underline{\delta}$, since only an investor with the worst quality asset has the lowest continuation value. Therefore $\underline{v} = \underline{\beta}\Gamma(\underline{v})$. This argument ensures that the right hand side of equation (16) is smaller than the left hand side at $\hat{\beta} = \underline{\beta}$ and larger at $\hat{\beta} = \bar{\beta}$, giving a unique interior solution. ■

Proof of Proposition 3. The proposed allocation is incentive compatible, has $c_1^B(\beta)$ and $c_2^B(\beta)$ nonnegative, and satisfies the feasibility constraints (7) and (8). Now consider a competitive equilibrium of an economy in which each individual with $\beta < \hat{\beta}$ has an endowment of b in period 1 and 0 in period 2, while each individual with $\beta \geq \hat{\beta}$ has an endowment of 0 in period 1 and $b/\hat{\beta}$ in period 2. It is easy to verify the equilibrium involves no trade. The first welfare theorem implies this allocation is Pareto optimal among all allocations satisfying the two feasibility constraints. It is therefore Pareto optimal among the smaller set of allocations that also satisfy the incentive constraint (5). ■

Proof of Proposition 4. To start, assume that the semi-separating equilibrium is seller efficient. This means that there are nondecreasing integrated Pareto weights $\Lambda(v)$ with $\Lambda(\underline{v}) \geq 0$ and $\Lambda(\bar{v}) = 1$,[18] such that the allocation maximizes the Pareto-weighted sum of seller utilities,

$$\int_{\underline{v}}^{\bar{v}} u^S(v)d\Lambda(v),$$

among all incentive compatible and feasible allocations. Eliminate $u^S(v)$ using equation (6) and perform integration-by-parts to rewrite the Pareto-weighted sum of utilities as

$$\int_{\underline{v}}^{\bar{v}} \omega(v)(\Lambda(v) - \Lambda(\underline{v}))dv + k. \tag{17}$$

Any seller-efficient allocation maximizes (17) subject to $\omega(v) \in [0,1]$ nonincreasing, $k \geq 0$, and the two resource constraints (9) and (10) for some nondecreasing integrated Pareto weights $\Lambda(v)$.

Write the Lagrangian of the Pareto-weighted maximization problem, placing nonnegative multipliers ψ_1 and ψ_2 on the two constraints (9) and (10):

$$\mathcal{L} = \int_{\underline{v}}^{\bar{v}} \omega(v)\phi(v)dv + (1-\psi_1)k + \psi_1 C_1^S + \psi_2 C_2^S \tag{18}$$

subject to $k \geq 0$, and $\omega(v) \in [0,1]$ nonincreasing, where $\phi(v) \equiv \Lambda(v) - \Lambda(\underline{v}) - J(v)$ with

$$J(v) \equiv \psi_1(H(v) + vh(v)) - \psi_2\Gamma(v)h(v).$$

The Lagrangian is linear in k, which implies that $\psi_1 \geq 1$; otherwise raising k would increase the Lagrangian without bound. In addition, integration by parts implies

$$\int_{\underline{v}}^{\bar{v}} \omega(v)\phi(v)dv = \omega(\bar{v})\Phi(\bar{v}) - \int_{\underline{v}}^{\bar{v}} \Phi(v)d\omega(v),$$

where $\Phi(v) \equiv \int_{\underline{v}}^{v} \phi(x)dx$. Therefore the Lagrangian is also linear in $d\omega(v)$, which implies that $\omega(v)$ is constant at any v that does not maximize $\Phi(v)$. In the semi-separating equilibrium, $\Theta(P(v))$ is strictly decreasing for all $v \in [\underline{v}, \bar{p}]$. Therefore if the equilibrium is Pareto efficient, all values of v in this interval must maximize $\Phi(v)$. We use this to characterize the conditions for Pareto efficiency.

Now assume there is a pair (ψ_1, ψ_2) such that the five conditions in the statement of the proposition hold. Set $\Lambda(v) = J(v)$ for $v \in [\underline{v}, \bar{p}]$ and $\Lambda(v) = 1$ for $v > \bar{p}$. The first

[18] The integrated Pareto weight $\Lambda(v)$ is the sum of the Pareto weights on sellers with continuation value less than or equal to v, so the Pareto weight on v is $d\Lambda(v)$.

condition ensure that $k = 0$ is optimal with these Pareto weights and Lagrange multipliers. The next three conditions ensure that $\Lambda(\underline{v}) = 0$, $\Lambda(v)$ is nondecreasing, and $\Lambda(\bar{p}) = 1$, so $d\Lambda(v)$ are valid Pareto weights. By construction $\phi(v) = \Phi(v) = 0$ for all $v \in [\underline{v}, \bar{p}]$ and $\Phi(v) = \int_{\bar{p}}^{v}(1 - J(x))dx \leq 0$ for all $v > \bar{p}$ using the final condition. Therefore any function $\omega(v)$ that is strictly decreasing on $[\underline{v}, \bar{p}]$ and 0 at higher values of v maximizes the Lagrangian. In particular, the semi-separating equilibrium is Pareto optimal.

Conversely, suppose there is no pair (ψ_1, ψ_2) satisfying these five conditions. If the first condition failed, the Lagrangian would not have a maximum and so the semi-separating equilibrium allocation would not maximize it. If any of the next three conditions failed, any nondecreasing Pareto weight $\Lambda(v)$ would have $\phi(v) = \Lambda(v) - \Lambda(\underline{v}) - J(v) \neq 0$ for some $v \in [\underline{v}, \bar{p}]$; therefore not all $v \in [\underline{v}, \bar{p}]$ would maximize $\Phi(v)$ and any solution to the Lagrangian must have $d\omega(v)$ constant at such v, inconsistent with the semi-separating equilibrium allocation. And if the fifth condition failed, $\Phi(v) > 0$ at some $v > \bar{v}$, so again any solution to the Lagrangian must have $d\omega(v) = 0$ at all $v \leq \bar{v}$, inconsistent with the semi-separating equilibrium allocation. ∎

Proof of Proposition 5. Proposition 3 describes the buyer efficient allocation. Buyers' utility is

$$u^B(\beta) = \begin{cases} b - 1 & \text{if } \beta < \hat{\beta} \\ \beta b/\hat{\beta} - 1 & \text{if } \beta \geq \hat{\beta}, \end{cases}$$

where b and $\hat{\beta}$ depend on C_1^B and C_2^B through equation (11). Implicitly differentiating this expression, we get that a change in (C_1^B, C_2^B) of magnitude (dC_1^B, dC_2^B) raises the utility of buyers with $\beta < \hat{\beta}$ if and only if

$$dC_1^B + \frac{\hat{\beta}^2 g^B(\hat{\beta})}{1 - G^B(\hat{\beta}) + \hat{\beta} g^B(\hat{\beta})} dC_2^B > 0$$

The same change raises the utility of buyers with $\beta > \hat{\beta}$ if and only if

$$\frac{g^B(\hat{\beta})}{G^B(\hat{\beta}) + \hat{\beta} g^B(\hat{\beta})} dC_1^B + dC_2^B > 0.$$

Note that

$$\frac{G^B(\hat{\beta}) + \hat{\beta} g^B(\hat{\beta})}{g^B(\hat{\beta})} \geq \frac{\hat{\beta}^2 g^B(\hat{\beta})}{1 - G^B(\hat{\beta}) + \hat{\beta} g^B(\hat{\beta})},$$

as can be confirmed algebraically. This means that if buyers $\beta < \hat{\beta}$ like the perturbation

(dC_1^B, dC_2^B) with $dC_2^B \geq 0$, all buyers like the perturbation. And if buyers $\beta > \hat{\beta}$ like the perturbation (dC_1^B, dC_2^B) with $dC_2^B \leq 0$, all buyers like the perturbation.

Next, a feasible change in the costs satisfies $dC_1^S = -dC_1^B$ and $dC_2^S = -dC_2^B$ and so in particular $\psi_1 dC_1^S + \psi_2 dC_2^S = -\psi_1 dC_1^B - \psi_2 dC_2^B$. Proposition 4 then implies that if $\psi_1 dC_1^B + \psi_2 dC_2^B < 0$ for any (ψ_1, ψ_2) consistent with the conditions in the Proposition, the equilibrium is not locally Pareto efficient.

Putting these results together, the equilibrium is locally Pareto efficient if there exists a (ψ_1, ψ_2) consistent with the conditions in Proposition 4 such that

1. for any $dC_2^B > 0$, $dC_1^B + \frac{\hat{\beta}^2 g^B(\hat{\beta})}{1-G^B(\hat{\beta})+\hat{\beta}g^B(\hat{\beta})} dC_2^B < 0$ or $\psi_1 dC_1^B + \psi_2 dC_2^B \geq 0$, and

2. for any $dC_2^B < 0$, $\frac{g^B(\hat{\beta})}{G^B(\hat{\beta})+\hat{\beta}g^B(\hat{\beta})} dC_1^B + dC_2^B < 0$ or $\psi_1 dC_1^B + \psi_2 dC_2^B \geq 0$.

Part (1) holds if and only if $\frac{\psi_2}{\psi_1} > \frac{\hat{\beta}^2 g^B(\hat{\beta})}{1-G^B(\hat{\beta})+\hat{\beta}g^B(\hat{\beta})}$, while part (2) holds if and only if $\frac{G^B(\hat{\beta})+\hat{\beta}g^B(\hat{\beta})}{g(\hat{\beta})} > \frac{\psi_2}{\psi_1}$. ■

B Other Equilibria

We illustrate the full multiplicity of equilibria through a parametric example. Assume $G(\beta, \delta) = \beta\delta^2$ on $[0,1]^2$, so $\underline{v} = 0$, $\bar{v} = 1$, $\Gamma(v) = \frac{1+v}{2}$, and $H(v) = v(2-v)$.

B.1 Other Semi-Separating Equilibria

In section 4.1, we identified an equilibrium in which the sale price (probability) is a continuous, strictly increasing (decreasing) function of an investor's continuation value. We start by showing that there is a continuum of such equilibria.

These equilibria are indexed by the identity of the seller with the highest continuation value, $\bar{p} \in [0.456, 1]$. Given $\bar{p}$, let $\underline{p} = \bar{p}/(1+\bar{p})$, $\hat{\beta} = 2\underline{p}$, and

$$\hat{\theta} = \frac{(1-\bar{p})(2+\bar{p})(3+\bar{p})}{4\bar{p}^2(6-\bar{p})}. \tag{19}$$

The restriction on the range of $\bar{p}$ ensures that $\hat{\theta} \in [0,1]$. In such an equilibrium, the buyer-seller ratio is

$$\Theta(p) = \begin{cases} \infty & \text{if } p < \hat{\theta}\underline{p} \\ \hat{\theta}\underline{p}/p & \text{if } p \in [\hat{\theta}\underline{p}, \underline{p}) \\ \hat{\theta}\left(\frac{\bar{p}-p}{\underline{p}\bar{p}}\right)^{\bar{p}} & \text{if } p \in [\underline{p}, \bar{p}] \\ 0 & \text{if } p > \bar{p}, \end{cases}$$

while the expected quality of assets offered for sale at prices above $\hat{\theta}\underline{p}$ is $\Delta(p) \leq p/\hat{\beta}$, with equality if $p \in [\underline{p}, \bar{p}]$.

To prove this is an equilibrium, we need to discuss buying and selling behavior. Start with selling. For any investor (β, δ) with continuation value with $\beta\delta \in (0, \bar{p})$, the unique optimal selling price is $p_s(\beta, \delta) = \hat{\beta}\Gamma(\beta\delta)$. For investors with the lowest continuation value, $\beta\delta = 0$, any $p_s(\beta, \delta) \in [\hat{\theta}\underline{p}, \underline{p}]$ is optimal; we assume $p_s(\beta, \delta) = \underline{p}$. For investors with higher continuation values, $\beta\delta \geq \bar{p}$, any $p_s \geq \beta\delta$ is optimal; we assume $p_s(\beta\delta) = \beta\delta$.

Given these beliefs,

$$\Delta(p) = \begin{cases} 0 & \text{if } p \in [\hat{\theta}\underline{p}, \underline{p}) \\ p/\hat{\beta} & \text{if } p \in [\underline{p}, \bar{p}] \\ (1+p)/2 & \text{if } p > \bar{p}. \end{cases}$$

Note that we are free to assign any beliefs at prices $p \in [\hat{\theta}\underline{p}, \underline{p})$, since all investors with $\beta = 0$ find such prices optimal. We choose to assign beliefs that only those investors who have $\delta = 0$ set these prices. Given these beliefs, optimal buying behavior sets any price $p_b(\beta, \delta) \in [\underline{p}, \bar{p}]$ if $\beta \geq \hat{\beta}$ and any prices $p_b(\beta, \delta) < \hat{\theta}\underline{p}$ if $\beta < \hat{\beta}$.

Finally, we can verify that equation (19) ensures that the goods market clears.

Building on this logic, we can construct a continuum of semi-separating equilibria whenever the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by investors with the lowest continuation value. If the support of (β, δ) is a rectangle, this requires that the lowest continuation value is zero, but otherwise it may hold more generally.

B.2 Other One-Price Equilibria

The same logic supports a continuum of one-price equilibria with rationing at the equilibrium trading price. Equilibria are now characterized by three numbers, the equilibrium trading price p_1, the probability of trade at that price $\theta_1 \in [0, 1]$, and the discount factor of the marginal buyer $\hat{\beta}$, but only two equations. First, the marginal buyer must be indifferent about buying all the assets offered for sale at p_1:

$$p_1 = \hat{\beta}\frac{3 - p_1^2}{3(2 - p_1)},$$

where $(3 - p_1^2)/3(2 - p_1)$ is the average quality of assets held by investors with continuation value $v < p_1$. Second, the goods market must clear:

$$1 - \hat{\beta} = \theta_1 p_1^2(2 - p_1),$$

where $p_1(2-p_1)$ is the fraction of sellers at the price p_1. There is a solution to these equations with $\theta_1 \in [0, 1]$ if $p_1 \in [0.426, 0.634]$, giving

$$\theta_1 = \frac{3 - 6p_1 + 2p_1^2}{p_1^2(2 - p_1)(3 - p_1^2)}$$

and

$$\hat{\beta} = \frac{3p_1(2 - p_1)}{3 - p_1^2}.$$

In such an equilibrium, the buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & \text{if } p < \theta_1 p_1 \\ \theta_1 p_1/p & \text{if } p \in [\theta_1 p_1, p_1] \\ 0 & \text{if } p > p_1, \end{cases}$$

while the expected quality of assets for sale relative to the price is maximized at p_1.

To construct an equilibrium of this sort, we again discuss buying and selling behavior. All investors (β, δ) with continuation value $\beta\delta < p_1$ set price p_1 in equilibrium, while those with higher continuation values set price $p_s(\beta, \delta) = \delta$. This pins down buyers' beliefs at prices above p_1. At prices between $\theta_1 p_1$ and p_1, rational beliefs requires that investors anticipate meeting sellers with zero continuation value. To support the equilibrium, we assume that they anticipate meeting sellers with zero-quality assets:

$$\Delta(p) = \begin{cases} 0 & \text{if } p < p_1 \\ \frac{3-p_1^2}{3(2-p_1)} & \text{if } p = p_1 \\ p & \text{if } p > p_1 \end{cases}$$

One can verify that $\Delta(p)/p$ is maximized at p_1 for all $p_1 \leq 0.634$, so buyers with $\beta \geq \hat{\beta}$ in fact prefer to pay this single price: $p_b(\beta, \delta) = p_1$ if $\beta \geq \hat{\beta}$ and $p_b(\beta, \delta) = 0$ otherwise. Finally, one can verify that the goods market clears.

Again, this logic shows how to construct a continuum of one-price equilibria whenever the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by investors with the lowest continuation value.

B.3 n-Price Equilibria

Our model also admits an n-dimensional set of n-price equilibria. Denote the prices by $p_1 < \cdots < p_n$; in equilibrium all trade occurs at these prices. Also let $\theta_1 > \cdots > \theta_n$ denote the buyer-seller ratios at these prices, with $\theta_1 \in (0,1]$. Let $v_1 < \cdots < v_n$ denote the n critical continuation values who are indifferent between neighboring prices (so v_i is indifferent between setting prices p_i and p_{i+1} and v_n is indifferent between setting price p_n and setting a higher price at which she cannot sell). Finally, let $\hat{\beta}$ denote the discount factor of the marginal buyer. This gives us a total of $3n+1$ variables. These must satisfy $2n+1$ equations. The first n equations come from the indifference conditions of the marginal sellers:

$$\theta_i(p_i - v_i) = \theta_{i+1}(p_{i+1} - v_i) \text{ for } i \in \{1, \ldots, n-1\}$$

and $p_n = v_n$. The next n equations come from the marginal buyer's indifference about buying at any price. With our functional forms, this gives

$$p_i = \hat{\beta}\frac{3 - v_{i-1}^2 - v_{i-1}v_i - v_i^2}{3(2 - v_{i-1} - v_i)},$$

where $v_0 = 0$. The fraction is the average value of the assets held by investors with continuation value $v \in [v_{i-1}, v_i]$. Finally, the goods market must clear:

$$1 - \hat{\beta} = \sum_{i=1}^{n} \theta_i p_i (v_i(2 - v_i) - v_{i-1}(2 - v_{i-1})),$$

where $v_i(2 - v_i) - v_{i-1}(2 - v_{i-1})$ is the measure of investors who set price p_i, those with continuation values $v \in [v_{i-1}, v_i]$.

In equilibrium, the buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & p < p_0 \\ \frac{\theta_i(p_i - v_i)}{p - v_i} \text{ if} & p \in [p_i, p_{i+1}],\ i \in \{0, \ldots, n-1\} \\ 0 & p > p_n, \end{cases}$$

where $p_0 = \theta_1 p_1$ and $\theta_0 = 1$. Given this structure, only sellers with continuation value v_i find prices $p \in (p_i, p_{i+1})$ optimal for $i \in \{0, \ldots, n-1\}$. To support the equilibrium, we assume buyers anticipate that investors (β, δ) with $\beta = 1$ and $\delta = v_i$ set these prices. Finally, investors with $\beta\delta > p_n$ and $\delta = p$ set price $p > p_n$. This pins down buyers' beliefs. The remainder of the construction of equilibrium is now standard.

In our parametric example, first suppose $\theta_1 = 1$. We find that for any value of $\theta_2 \in [0, 0.832]$, it is possible to construct an equilibrium with trade at two prices. Higher values of θ_2 are associated with lower values of p_1 (falling from 0.426 to 0.371), lower values of $p_2 = v_2$ (falling from 0.527 to 0.446), lower values of v_1 (falling from 0.426 to 0), and higher values of $\hat{\beta}$ (rising from 0.714 to 0.743). It does not seem possible to construct equilibria with $\theta_2 > 0.832$, because the system of equations would imply $v_1 < 0$. For lower values of θ_1, there is a smaller interval of θ_2 corresponding to an equilibrium, but the interval always exists.

The possibility that $\theta_1 < 1$ again hinges on the assumption that the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by these investors. However, the remaining construction does not rely on this restriction and so appears to be completely general. For example, there are many n-price equilibria in the independent Pareto example that we use throughout the text.

Qualitatively an n-price equilibrium looks very similar to the semi-separating equilibrium. Investors with higher continuation values set weakly higher sale prices and sell with a weakly lower probability. Indeed, we conjecture that in the limit as n converges to infinity, the functions $\Theta(p)$ and $\Delta(p)$ in any n price equilibrium will be close to their values in some semi-separating equilibrium in the sense of the sup-norm.

B.4 Mixed Equilibria

Equilibria may also feature a mix of mass points and continuous distributions. For example, investors with continuation values in the interval $[0, v_1)$ may set a common price p_1, while investors with continuation values in a higher interval $[v_1, v_2]$ may set prices $P(v)$ that are strictly increasing in the continuation value. To ensure buyers are willing to pay all these prices, we require that $\hat{\beta}\Gamma(v) = P(v)$ and $\hat{\beta}\Delta(p_1) = p_1$. Since $\Delta(p_1) < \Gamma(v_1)$, this implies $p_1 < P(v_1)$, so there is also a gap in the distribution.

The same logic implies that investors with continuation value in the next interval (v_2, v_3) may set a common price p_3 satisfying $\hat{\beta}\Gamma\Delta(p_3) = p_3$; and since $\Delta(p_3) > \Gamma(v_2)$, $p_3 > P(v_2)$. Thus again there is a gap in the distribution followed by another mass point. Mass points are followed by a gap, which in turn may be followed by another mass point or by a continuous distribution. Continuous distributions are followed by a gap and then a mass point. Numerous configurations are possible.