WELFARE AND OPTIMAL TRADING FREQUENCY IN DYNAMIC DOUBLE
AUCTIONS

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ABSTRACT

This paper studies the welfare consequence of increasing trading speed in financial markets. We build and solve a dynamic trading model, in which traders receive private information of asset value over time and trade strategically with demand schedules in a sequence of double auctions. A stationary linear equilibrium and its efficiency properties are characterized explicitly in closed form. Slow trading (few double auctions per unit of time) serves as a commitment device that induces aggressive demand schedules, but fast trading allows more immediate reaction to new information. If all traders have the same speed, the socially optimal trading frequency tends to be low for scheduled arrivals of information but high for stochastic arrivals of information. If traders have heterogeneous trading speeds, fast traders prefer the highest feasible trading frequency, whereas slow traders tend to prefer a strictly lower frequency.

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1   Introduction

Trading in financial markets has become significantly faster over the last decade. Today, electronic transactions for equities, futures, and foreign exchange are typically conducted within millisecond or microseconds.[1] Electronic markets, which typically have a higher speed than manual markets, are also increasingly adapted in the over-the-counter markets for debt securities and derivatives, such as corporate bonds, interest rates swaps, and credit default swaps.[2] Exchange traded funds, which trade at a high frequency like stocks, have gained significant market shares over index mutual funds, which only allow buying and selling at the end of day.[3]

The remarkable speedup in financial markets raises important economic questions. For example, does a higher speed of trading necessarily lead to a higher social welfare, in terms of more efficient allocations of assets? What is the socially optimal frequency (if one exists) at which financial markets should operate? Moreover, given that certain investors trade at a higher speed than others, does a higher trading frequency affect fast investors and slow ones equally or differentially? Answers to these questions would provide valuable insights for the ongoing academic and policy debate on market structure, especially in the context of high-frequency trading.[4]

In this paper, we set out to investigate the welfare consequence of speeding up trading in financial markets. We build and solve a dynamic model with strategic trading, adverse selection, and imperfect competition. Specifically, in our model, a finite number ($n \geq 3$) of traders trade a divisible asset in an infinite sequence of

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[1]A millisecond is a thousandth of a second, and a microsecond is a millionth of a second. In equity markets, for example, NASDAQ reports that its system can currently handle a message within 40 microseconds; see http://www.nasdaqtrader.com/Trader.aspx?id=Latencystats. In futures market, CME reports that the median inbound latency on its system is about 52 microseconds; see http://www.cmegroup.com/globex/files/globexbrochure.pdf. EBS Market, a major electronic trading platform for foreign exchange, introduced a speed delay of up to 3 milliseconds to incoming orders in August 2013; see "EBS take new step to rein in high-frequency traders," by Wanfeng Zhou and Nick Olivari, Reuters, August 23, 2013.

[2]For example, the Dodd-Frank Act of United States Congress has mandated that standardized over-the-counter derivatives must be traded on "swap execution facilities," which are effectively electronic "mini-exchanges." These derivatives used to be traded by manual quotation and execution.

[3]See, for instance, "ETFs Gain Ground on Index Mutual Funds," by Murray Coleman, Wall Street Journal, February 20, 2014.

[4]For example, Securities and Exchange Commission (2010) provides an excellent review of economic questions and policy issues on U.S. equity market structure. In early 2014, European regulators adopted the Markets in Financial Instruments Directive (MiFID) II, which set out principles and regulations for many aspects of European financial markets. The rise of high-frequency trading is a key issue in both.

uniform-price double auctions, held at discrete time intervals. The shorter is the time interval between auctions, the higher is the speed of the market. At an exponentially-distributed time in the future, the asset pays a liquidating dividend, which, until that payment time, evolves according to a jump process. Traders receive over time informative signals of dividend shocks, as well as shocks to their private values for owning the asset. Traders also incur quadratic costs for holding inventories, which is equivalent to linearly decreasing marginal values. A trader's dividend signals, shocks to his private values, and his inventories are all his private information. In each double auction, traders submit demand schedules (i.e., a set of limit orders) and pay for their allocations at the market-clearing price. All traders take into account the price impact of their trades.

Our model incorporates many salient features of dynamic markets in practice. For example, information about the common dividend captures adverse selection, whereas private-value information and convex holding costs introduce gains from trade. These trading motives are also time-varying as news arrives over time. Moreover, the number of double auctions per unit of clock time is a simple yet realistic way to model trading frequency in dynamic markets.

A dynamic equilibrium and efficiency

The first primary result of this paper is to characterize a linear stationary equilibrium of this dynamic market and its efficiency properties. In equilibrium, a trader's optimal demand in each double auction is a linear function of the price, his signal of the dividend, his most recent private value, and his private inventory. Each coefficient is solved explicitly in closed form. Naturally, the equilibrium price in each auction is a weighted sum of the average signal of the common dividend and the average private value, adjusted for the marginal holding cost of the average inventory. Prices are martingales since the innovations in common dividend and private values have zero mean.

Because there are a finite number of traders, demand schedules in this dynamic equilibrium are not competitive. Consequently, the equilibrium allocations of assets across traders after each auction are not fully efficient, but they converge gradually and exponentially over time to the efficient allocation. This convergence remains slow and gradual even in the continuous-time limit. We show that the convergence rate per unit of clock time increases with the number of traders, the arrival intensity of the

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dividend, the variance of the private-value shocks, and the trading frequency of the market; but the convergence rate decreases with the variance of the common-value shocks, which is a measure of adverse selection.

Furthermore, equilibrium allocations also converge to the efficient level as the the number n of traders increases. A novel result from our analysis is that adverse selection—asymmetric information about the common dividend—slows down this convergence rate. Without the adverse selection, allocative inefficiency vanishes at a rate of order n^{-2}, but with adverse selection, inefficiency vanishes at a rate of order $n^{-4/3}$. In the continuous-time limit, the respective convergence rates are of orders n^{-1} and $n^{-2/3}$, respectively.

Welfare and optimal trading frequency

Our modeling framework proves to be an effective tool in answering welfare questions. Characterizing welfare and optimal trading frequency in this dynamic market is the second primary contribution of our paper.

We ask two related questions regarding trading frequency. First, what is the socially optimal trading frequency if all traders have equal speed? Second, if certain traders are faster than others, what are the trading frequencies that are optimal for fast and slow traders respectively?

The first question on homogenous speed can be readily analyzed in our benchmark model, in which all traders participate in all double auctions. We emphasize that a change of trading frequency in our model does not change the fundamental properties of the asset, such as the timing and magnitude of the dividend shocks. Increasing trading frequency involves an important tradeoff. On the one hand, a higher trading frequency allows traders to react more quickly to new information and to trade sooner toward the efficient allocation. This effect favors a faster market. On the other hand, a lower trading frequency serves as a commitment device that induces aggressive demand schedules, which leads to more efficient allocations in early rounds of trading. This effect favors a slower market. The optimal trading frequency should strike the best balance between minimizing delay in reacting to new information and maximizing the aggressiveness of demand schedules.

We show that depending on the nature of information arrivals, this tradeoff leads to different optimal trading frequencies. If new information of dividend and private values arrives at scheduled time intervals, the optimal trading frequency cannot

be higher than the frequency of new information. Hence, slow trading tends to be optimal. As the number of traders becomes sufficiently large, the optimal trading frequency coincides with the information arrival frequency. By contrast, if new information arrives at Poisson times, fast trading tends to be optimal, as traders can react sooner to unpredictable arrivals of news. In particular, as the number of traders becomes large, continuous trading becomes optimal in the limit.[5] Moreover, under natural conditions, a higher adverse selection reduces the optimal trading frequency under Poisson information arrivals. Finally, for both scheduled and stochastic arrivals of information, the more volatile is the potential change in the efficient allocation, the higher is the optimal trading frequency.

To answer the second question of welfare, we build an extension of our model to allow for heterogeneous trading speeds. In this extension, a fast trader accesses the market whenever it is open, but a flow of discrete slow traders access the market only once. Different from recent studies of high-frequency trading (see the literature section), we do not endow the fast trader with any information advantage.

We find that the fast and slow traders tend to prefer different trading frequencies. Because of his frequent access to the market, the fast trader in our model plays the endogenous role of intermediating trades among slow traders who arrive sequentially. Through this intermediation the fast trader extracts rents. A higher trading frequency reduces the number of slow traders in each double auction, making the market "thinner" and the fast trader's rents higher. We show that the optimal trading frequency for the fast trader is such that there are exactly two slow traders in each trading round, which is the highest feasible frequency under the constraint that the number of traders in each round is an integer. By contrast, slow traders typically prefer a strictly lower trading frequency (and a thicker market) because they benefit from pooling trading interests over time and providing liquidity to each other.

Relation to the literature

Our results contribute to two broad branches of literature: dynamic trading with demand schedules and the welfare effect of trading frequency.

Dynamic trading with demand schedules. Existing papers on dynamic trading with demand schedules have mostly focused on private information of inventories or

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[5]To clarify, continuous trading in our model means continuous double auctions, not a continuous limit order book. The latter is effectively a discriminatory auction, not a uniform-price auction.

private values. For example, Vayanos (1999) studies a dynamic market in which the asset fundamental value (dividend) is public information, but agents receive periodic inventory shocks. Rostek and Weretka (2014) study dynamic trading with multiple dividend payments, but traders in their model have symmetric information of the asset's fundamental. Sannikov and Skrzypacz (2014) analyze a continuous-time trading model in which traders have asymmetric risk capacity and private inventory shocks, but no asymmetric information for the common value of the asset.

Relative to these three papers, a primary distinction of our paper is that we address adverse selection, i.e., asymmetric information regarding the common value (component) of the asset.[6] Adverse selection matters for the speed of convergence to efficiency. For a fixed n, the convergence to efficiency over time is slowed down by adverse selection. As the market becomes large (i.e., $n \to \infty$), the asymptotic convergence rate with adverse selection is of order $n^{-4/3}$, slower than the convergence rate without adverse selection, n^{-2}. The n^{-2} convergence rate is also obtained by Vayanos (1999) in a model with commonly observed dividends.

Kyle, Obizhaeva, and Wang (2013) study a continuous-time trading model in which agents have pure common values but "agree to disagree" on the precision of their signals. Because trading in their model happens only in continuous time, Kyle, Obizhaeva, and Wang (2013) do not address the welfare effect of trading frequency. Their focus is non-martingale price dynamics, such as price spikes and reversals.

More broadly, our paper is related to the microstructure literature on informed trading, most notably Kyle (1985) and Glosten and Milgrom (1985), as well as many extensions.[7] This literature predominantly assumes the presence of noise traders, which makes it difficult to analyze welfare. Dynamic models based on rational expectations equilibrium (REE) assume price-taking behavior,[8] whereas traders in our model fully internalize the impacts of their trades on the equilibrium price.

Welfare consequences of trading frequency. The literature has two general approaches to modeling the welfare consequence of trading frequency. Under the first approach, all investors trade at the same speed. Under the second approach, certain

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[6]Private information regarding the common value or interdependent value of the asset is addressed in prior *static* models with demand schedules, such as Kyle (1989), Vives (2011), Rostek and Weretka (2012), and Babus and Kondor (2012), among others.

[7]Recent dynamic extensions of Kyle (1985) include Foster and Viswanathan (1996), Back, Cao, and Willard (2000), and Ostrovsky (2012), among many others.

[8]See, for example, He and Wang (1995) and Biais, Bossaerts, and Spatt (2010), based on the seminal work of Grossman and Stiglitz (1980) and Grossman (1976, 1981).

investors are faster than others. Our results contribute to both.

On homogeneous trading speed, Vayanos (1999) finds that if the inventory shocks are private information and are small, then a lower trading frequency is better for welfare.[9] Confirming Vayanos's result, we show that under scheduled information arrivals slower trading tends to be optimal. We share the intuition and underlying mechanism with Vayanos (1999) that a lower trading frequency serves as a commitment device that encourages traders to submit aggressive demand schedules.

Different from Vayanos's result, however, we characterize natural conditions under which fast trading tends to be optimal. Specifically, if new information arrives at stochastic times, too low a trading frequency prevents traders from reacting to new information quickly, thus reducing welfare. In this case, the cost of failing to react to new information can overweigh the benefit of more aggressive demand schedules. Fast trading can be optimal as a result. Under explicitly characterized conditions, the optimal trading frequency can be arbitrarily high.

Fuchs and Skrzypacz (2013) consider a bargaining-based model with many competitive one-time buyers and a single seller who has private information. They show that an "early closure" of market improves welfare relative to continuous trading. Our result differs in that we characterize explicit conditions under which the optimal trading frequency can be (arbitrarily) high.

On heterogeneous trading speeds, our paper is complementary to existing studies on the welfare consequences of high-frequency trading (HFT).[10] In the model of Biais, Foucault, and Moinas (2014), fast traders have higher transaction probabilities but also possess superior information about the value of the asset. Pagnotta and Philippon (2013) model the speed competition by multiple markets, where speed is defined as the rate of contacts among investors. Budish, Cramton, and Shim (2013) model high-frequency traders as arbitrageurs who process public information quickly and profit

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[9]Vayanos (1999) also shows that if inventory information is common knowledge, there is a continuum of equilibria. Under one of these equilibria, selected by a trembling hand refinement, welfare is increasing in trading frequency. Because our model has private information of inventories, the private-information equilibrium of Vayanos (1999) is a more appropriate benchmark for comparison. Related, Rostek and Weretka (2014) consider a market in which: (i) information of asset value is *symmetric* among agents and (ii) between consecutive dividend payments there is no news and no discounting. In this setting, they show that a higher trading frequency is better for welfare.

[10]Recent studies that model HFT, but do not address the associated welfare, include Foucault, Hombert, and Rosu (2013), Baruch and Glosten (2013), Rosu (2014), Yueshen (2014), Li (2014), Menkveld and Zoican (2014), Aït-Sahalia and Saglam (2014), and Foucault, Kozhan, and Tham (2014), among others.

from picking off stale quotes. They argue that frequent batch auctions—exactly the trading mechanism modeled in our paper—are a better market design than continuous limit order books. Hoffmann (2014) models a limit order market in which the fast traders can cancel orders (e.g. after asset value shocks) but slow traders cannot. All four papers ask the welfare question of whether investment in high-speed trading technology is socially wasteful. Jovanovic and Menkveld (2012) model high-frequency traders as informed intermediaries whose presence can reduce or create information frictions, and study to what extent high-frequency traders increase or reduce gains from trade. In the model of Cespa and Vives (2013), high-frequency traders can trade more often and may also possess superior information of the asset value. They find that HFT has an ambiguous effect on the welfare of certain types of hedgers.[11]

None of the six papers mentioned above study the welfare effect of increasing trading frequency. Moreover, the fast trader in our model has no *informational* advantage; his only advantage is having more frequent access to the market than slow traders. Still, we show that the fast trader extracts rents by intermediating trades among slow traders. Finally, our welfare question focuses on how trading frequency interacts with imperfect competition and the efficient allocations of assets, which is separate from the welfare question of investment in speed technology.

2 Dynamic Trading in Double Auctions

This section presents the dynamic trading model and characterizes the equilibrium and its properties. Main model parameters are tabulated in Appendix A for ease of reference.

2.1 Model

Timing and asset. Time is continuous, $\tau \in [0, \infty)$. A divisible asset is trading on the market. The asset pays a single liquidating dividend D at some exponential time $\mathcal{T}$ with the associated intensity $r > 0$. The random dividend time $\mathcal{T}$ is independent of all else in the model. The dividend D starts at D_0 at time $T_0 = 0$, where $D_0 \sim \mathcal{N}(0, \sigma_D^2)$. Moreover, conditional on the dividend time $\mathcal{T}$ having not arrived, the

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[11]The model of Cespa and Vives (2013) contains some infinitely risk-averse hedgers, which are equivalent to noise traders. Therefore, it is difficult to draw conclusions regarding the overall welfare effect of HFT on all hedgers in their model.

dividend is shocked at the consecutive clock times T_1, T_2, T_3, ... The shock times $\{T_k\}_{k \geq 1}$ can be deterministic or stochastic. The shocks to the dividend at each time in $\{T_k\}_{k \geq 1}$ are also i.i.d. normal with mean 0 and variance σ_D^2:

$$D_{T_k} - D_{T_{k-1}} \sim \mathcal{N}(0, \sigma_D^2). \tag{1}$$

Thus, before the dividend is paid out, the "latent" dividend process $\{D_\tau\}_{\tau \geq 0}$ follows a jump process:

$$D_\tau = D_{T_k}, \text{ if } T_k \leq \tau < T_{k+1}. \tag{2}$$

If the dividend payment time $\mathcal{T}$ happens at time τ, the realized dividend will be D_τ.

Since the expected dividend payment time is finite (with mean $1/r$), for simplicity we normalize the interest rate to be zero (i.e., there is no time discounting). Allowing time discounting does not change our qualitative results.

Trading in double auctions. There are $n \geq 3$ risk-neutral traders in this market. Trading is organized as a sequence of uniform-price divisible double auctions. The double auctions happen at clock times $\{0, \Delta, 2\Delta, 3\Delta, \ldots\}$, where $\Delta > 0$ is the length of clock time between consecutive rounds of trading. The trading frequency is the number of double auctions per unit of time, i.e., $1/\Delta$. The smaller is Δ, the higher is the trading frequency. We will refer to each trading round as a "period," indexed by $t \in \{0, 1, 2, \ldots\}$, so period-$t$ trading occurs at clock time $t\Delta$.

We denote by $z_{i,t\Delta}$ the inventory held by trader i immediately before the period-t double auction. The initial inventories $z_{i,0}$ are given exogenously, and the total initial inventory $Z \equiv \sum_i z_{i,0}$ is common knowledge. (In securities markets, Z can be interpreted as the total asset supply. In derivatives markets, Z is by definition zero.) In period t each trader submits a demand schedule $x_{i,t\Delta}(p) : \mathbb{R} \to \mathbb{R}$. The market-clearing price in period t, $p_{t\Delta}^*$, satisfies

$$\sum_{i=1}^n x_{i,t\Delta}(p_{t\Delta}^*) = 0. \tag{3}$$

In the equilibrium we characterize later, the demand schedules are strictly downward-sloping in p and the solution $p_{t\Delta}^*$ is unique. The evolution of inventory is given by

$$z_{i,(t+1)\Delta} = z_{i,t\Delta} + x_{i,t\Delta}(p_{t\Delta}^*). \tag{4}$$

After the period-t double auction, each trader i receives $x_{i,t\Delta}(p^*_{t\Delta})$ units of the assets at the price of $p^*_{t\Delta}$ per unit. (A negative $x_{i,t\Delta}(p^*_{t\Delta})$ means selling the asset.)

Information and preference. At clock time T_k, $k \in \{0, 1, 2, \cdots\}$, each trader i receives a private signal S_{i,T_k} about the dividend shock:

$$S_{i,T_k} = D_{T_k} - D_{T_{k-1}} + \epsilon_{i,T_k}, \text{ where } \epsilon_{i,T_k} \sim \mathcal{N}(0, \sigma_\epsilon^2) \text{ are i.i.d.,} \tag{5}$$

and where $D_{T_{-1}} = 0$. We also refer to T_k's as "news times." The private signals of trader i are never disclosed to anyone else.

In addition, each trader i has a private-value component $w_{i,\mathcal{T}}$ for receiving the dividend. For example, this private component can reflect tax or risk-management considerations. Specifically, at each time T_k, $k \in \{0, 1, 2, \cdots\}$, trader i receives a shock to his private value for the asset such that:

$$w_{i,T_k} - w_{i,T_{k-1}} \sim \mathcal{N}(0, \sigma_w^2), \text{ i.i.d.,} \tag{6}$$

where $w_{i,T_{-1}} = 0$. Written in continuous time, the private-value process $w_{i,\tau}$ for trader i is a jump process:

$$w_{i,\tau} = w_{i,T_k}, \text{ if } T_k \le \tau < T_{k+1}. \tag{7}$$

The private values to trader i are observed by himself and are never disclosed to anyone else.

Moreover, in an interval $[t\Delta, (t+1)\Delta)$ but before the dividend D is paid, trader i incurs a "flow cost" that is equal to $0.5\lambda z_{i,(t+1)\Delta}^2$ per unit of clock time, where $\lambda > 0$ is a commonly known constant. The quadratic flow cost is essentially a dynamic version of the quadratic cost used in the static models of Vives (2011) and Rostek and Weretka (2012). We can also interpret this flow cost as an inventory cost, which can come from regulatory capital requirements, collateral requirement, or risk-management considerations. (This inventory cost is not strictly risk aversion, however.[12]) Once the dividend is paid out in cash, the flow cost no longer applies.

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[12]While it would be desirable to find a formal link between this linear-quadratic preference and a conventional CARA-normal setting, this equivalence has not been obtained. We are not aware of studies that formally link CARA-normal setting to linear-quadratic preference under (i) a dynamic market, (ii) strategic trading with demand schedules, and (iii) adverse selection. Vayanos (1999) and Rostek and Weretka (2014) use the CARA-normal setting, although their model has no adverse selection.

By the exponential distribution of $\mathcal{T}$, conditional on the dividend is not yet paid by time $t\Delta$, the expected length of time during which the flow cost is incurred in period t is

$$(1 - e^{-r\Delta}) \int_0^\Delta \frac{re^{-r\tau}}{1 - e^{-r\Delta}} \tau \, d\tau + e^{-r\Delta}\Delta = \frac{1 - e^{-r\Delta}}{r}. \tag{8}$$

For conciseness of expressions we let $H_{i,\tau}$ be the "history" (information set) of trader i at time τ:

$$H_{i,\tau} = \{(S_{i,T_l}, w_{i,T_l})\}_{T_l \leq \tau} \cup \{z_{t'\Delta}\}_{t'\Delta \leq \tau} \cup \{x_{i,t'\Delta}(p)\}_{t'\Delta < \tau}. \tag{9}$$

That is, $H_{i,\tau}$ contains trader i's asset value-relevant information received up to time τ, trader i's path of inventories up to time τ, and trader i's demand schedules in double auctions before time τ. Notice that by the identity $z_{i,(t'+1)\Delta} - z_{i,t'\Delta} = x_{i,t'\Delta}(p_{t'\Delta}^*)$, a trader can infer the price in the past period t' from $H_{i,\tau}$.

We define the "time-τ ex post value" of trader i at τ, conditional on trader i's history and all other traders' history, as:

$$v_{i,\tau} = w_{i,\tau} + \mathbb{E}\left[D_\tau \mid H_{i,\tau} \cup \{H_{j,\tau}\}_{j \neq i}\right]. \tag{10}$$

Thus, trader i's time-$t\Delta$ ex post value for holding the quantity $z_{i,t\Delta} + x_{i,t\Delta}(p_{t\Delta}^*)$, evaluated at time $t\Delta$ immediately after the period-t auction, is $v_{i,t\Delta}(z_{i,t\Delta} + x_{i,t\Delta}(p_{t\Delta}^*))$. We call $v_{i,\tau}$ the "time-τ ex post value" because we have conditioned on $\{H_{j,\tau}\}_{j \neq i}$, which is not observed by trader i. But $v_{i,\tau}$ is not the *realized* ex post value because the dividend is not yet paid.

We define trader i's "period-t ex post utility" immediately after the double auction at time $t\Delta$ by the recursive equation:

$$\begin{aligned}
V_{i,t\Delta} = &- x_{i,t\Delta}^* p_{t\Delta}^* + (1 - e^{-r\Delta})v_{i,t\Delta}(z_{i,t\Delta} + x_{i,t\Delta}^*) - \frac{1 - e^{-r\Delta}}{r} \cdot \frac{\lambda}{2}(z_{i,t\Delta} + x_{i,t\Delta}^*)^2 \\
&+ e^{-r\Delta}\mathbb{E}\left[V_{i,(t+1)\Delta} \mid H_{i,t\Delta} \cup \{H_{j,t\Delta}\}_{j \neq i}\right],
\end{aligned} \tag{11}$$

where $x_{i,t\Delta}^*$ is a shorthand for $x_{i,t\Delta}(p_{t\Delta}^*)$. We call $V_{i,t\Delta}$ the "period-t ex post utility" because we have conditioned on $H_{i,t\Delta} \cup \{H_{j,t\Delta}\}_{j \neq i}$, but $V_{i,t\Delta}$ is not trader i's *realized* ex post utility because future signals are yet to arrive and because we have taken an expectation over $\mathcal{T}$. We describe period-t ex post utility here purely for expositional convenience; later, when defining perfect Bayesian equilibrium (see Definition 1) we

take the expectation of the period-t ex post utility over other traders' histories.[13]

We can expand the recursive definition of $V_{i,t\Delta}$ explicitly:

$$V_{i,t\Delta} = \mathbb{E}\left[-\sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta} x_{i,t'\Delta}^* p_{t'\Delta}^* + \sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta}(1 - e^{-r\Delta}) v_{i,t'\Delta}(z_{i,t'\Delta} + x_{i,t'\Delta}^*) \right.$$
$$\left. - \frac{1 - e^{-r\Delta}}{r} \sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta} \frac{\lambda}{2} (z_{i,t'\Delta} + x_{i,t'\Delta}^*)^2 \;\middle|\; H_{i,t\Delta} \cup \{H_{j,t\Delta}\}_{j\neq i} \right]. \qquad (12)$$

2.2 Characterizing the equilibrium

Definition 1 (Perfect Bayesian Equilibrium). A perfect Bayesian equilibrium is a strategy profile $\{x_{j,t\Delta}\}_{1\leq j\leq n, t\geq 0}$, where each $x_{i,t\Delta}$ depends only on $H_{i,t\Delta}$, such that for every trader i and at every path of his information set $H_{i,t\Delta}$, trader i has no incentive to deviate from $\{x_{i,t'\Delta}\}_{t'\geq t}$. That is, for every alternative strategy $\{\tilde{x}_{i,t'\Delta}\}_{t'\geq t}$, we have:

$$\mathbb{E}\left[V_{i,t\Delta}(\{x_{i,t'\Delta}\}_{t'\geq t}, \{x_{j,t'\Delta}\}_{j\neq i, t'\geq t}) \mid H_{i,t\Delta} \right]$$
$$\geq \mathbb{E}\left[V_{i,t\Delta}(\{\tilde{x}_{i,t'\Delta}\}_{t'\geq t}, \{x_{j,t'\Delta}\}_{j\neq i, t'\geq t}) \mid H_{i,t\Delta} \right]. \qquad (13)$$

We now characterize a perfect Bayesian equilibrium. For notation simplicity, we define the "total signal" $s_{i,t\Delta}$ by

$$s_{i,T_k} \equiv \frac{\chi}{\alpha} \sum_{l=0}^{k} S_{i,T_l} + \frac{1}{\alpha} w_{i,T_k}, \qquad (14)$$

$$s_{i,\tau} = s_{i,T_k}, \text{ for } \tau \in [T_k, T_{k+1}),$$

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[13]Also note that in calculating $V_{i,t\Delta}$ we do not impose positivity restrictions on the price $p_{t\Delta}^*$ or the ex post value $v_{i,t\Delta}$. They can be negative. Indeed, the market prices of many financial and commodity derivatives—including forwards, futures and swaps—are zero upon inception and can become arbitrarily negative as market conditions change over time. A unilateral break (or "free disposal") of loss-making derivatives contracts constitutes a default and leads to the loss of posted collateral and reputation. In reality, it is not uncommon for investors to pay others (e.g. broker-dealers and market makers) to dispose of loss-making derivative positions, presumably because the negative marginal values for holding these positions exceed (in absolute value) the negative price for selling them. Moreover, not imposing positivity restriction is a standard convention in models based on normal distributions of asset values. Our results do not change if the model is adjusted by, for example, adding a constant to the dividend shocks or the private values.

where $\chi \in (0,1)$ is the unique solution to [14]

$$\frac{1/\sigma_\epsilon^2}{1/\sigma_D^2 + 1/\sigma_\epsilon^2 + (n-1)\chi^2/(\chi^2\sigma_\epsilon^2 + \sigma_w^2)} = \chi, \qquad (15)$$

and

$$\alpha \equiv \frac{\chi^2\sigma_\epsilon^2 + \sigma_w^2}{n\chi^2\sigma_\epsilon^2 + \sigma_w^2} > \frac{1}{n}. \qquad (16)$$

Proposition 1. *Suppose that $n\alpha > 2$, which is equivalent to*

$$\frac{1}{n/2 + \sigma_\epsilon^2/\sigma_D^2} < \sqrt{\frac{n-2}{n}}\frac{\sigma_w}{\sigma_\epsilon}. \qquad (17)$$

There exists a perfect Bayesian equilibrium[15] in which every trader i submits the demand schedule

$$x_{i,t\Delta}(p; s_{i,t\Delta}, z_{i,t\Delta}) = b\left(s_{i,t\Delta} - p - \frac{\lambda(n-1)}{r(n\alpha-1)}z_{i,t\Delta} + \frac{\lambda(1-\alpha)}{r(n\alpha-1)}Z\right), \qquad (18)$$

where

$$b = \frac{(n\alpha-1)r}{2(n-1)e^{-r\Delta}\lambda}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right) > 0. \qquad (19)$$

The period-t equilibrium price is

$$p_{t\Delta}^* = \frac{1}{n}\sum_{i=1}^n s_{i,t\Delta} - \frac{\lambda}{rn}Z. \qquad (20)$$

The equilibrium strategy in Proposition 1 is stationary: a trader's strategy only depends on his most recent total signal $s_{i,t\Delta}$ and his current inventory $z_{i,t\Delta}$, but does not depend explicitly on t.

We now discuss the intuition of information inference in this equilibrium, particularly that related to the total signals. The total signal s_{i,T_k} of trader i is a one-

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[14]The left-hand side of Equation (15) is decreasing in χ. It is $1/(1 + \sigma_\epsilon^2/\sigma_D^2) > 0$ if $\chi = 0$ and is $1/(1 + \sigma_\epsilon^2/\sigma_D^2 + (n-1)/(1 + \sigma_w^2/\sigma_\epsilon^2)) < 1$ if $\chi = 1$. Hence, Equation (15) has a unique solution $\chi \in \mathbb{R}$, and such solution satisfies $\chi \in (0,1)$.

[15]We specify the off-equilibrium belief as follows. Off the equilibrium path, each trader i ignores previous deviations and infers $\sum_{j\neq i} s_{i,t\Delta}$ from the demand schedules and market-clearing price in the current period t. This off-equilibrium belief is used if, for example, no news arrives between two auctions but the price changes. (On equilibrium path, price does not change if no news arrives.)

dimensional summary statistic of his asset value-relevant information $\{S_{i,T_l}, w_{i,T_l}\}_{0 \le l \le k}$. In Section B.1 (Lemma 1) we show that our construction of total signals implies that

$$\mathbb{E}\left[v_{i,T_k} \Big| H_{i,T_k} \cup \left\{\sum\nolimits_{j \ne i} s_{j,T_k}\right\}\right] = \alpha s_{i,T_k} + \frac{1-\alpha}{n-1} \sum_{j \ne i} s_{j,T_k}. \tag{21}$$

Equation (21) says that conditional on $\sum_{j \ne i} s_{j,T_k}$, trader i cares about his history H_{i,T_k} only through his total signal s_{i,T_k}. Moreover, because trader i is unable to tell apart the common dividend signals from the private values of the other traders, trader i puts a weight of $\alpha > 1/n$ on his own total signal s_{i,T_k} but a weight of $(1-\alpha)/(n-1) < 1/n$ on each of the other traders' total signal s_{j,T_k}.

Although $\sum_{j \ne i} s_{j,t\Delta}$ is not directly observed by trader i, given the equilibrium strategy in (18), trader i can infer $\sum_{j \ne i} s_{j,t\Delta}$ from the market clearing price $p_{t\Delta}^*$, as he knows $\sum_{j \ne i} z_{j,t\Delta} = Z - z_{i,t\Delta}$. Since the quantity that trader i buys or sells is contingent on the market clearing price, in equilibrium it is as if trader i knows $\sum_{j \ne i} s_{j,t\Delta}$ and updates his belief about $v_{i,t\Delta}$ given this information according to (21). Our equilibrium construction follows through by conditioning on $\sum_{j \ne i} s_{j,t\Delta}$ in each period. The details are provided in Section B.1.

The equilibrium of Proposition 1 has a few interesting properties. First, the strategies $x_{i,t\Delta}$ are optimal in an "ex post" sense. For any alternative strategy $\{\tilde{x}_{i,t'\Delta}\}_{t' \ge t}$, we have:

$$\mathbb{E}[V_{i,t\Delta}(\{x_{i,t'\Delta}\}_{t' \ge t}, \{x_{j,t'\Delta}\}_{j \ne i, t' \ge t}) \mid H_{i,t\Delta} \cup \{s_{j,T_l}\}_{j \ne i, T_l \le t\Delta}]$$
$$\ge \mathbb{E}[V_{i,t\Delta}(\{\tilde{x}_{i,t'\Delta}\}_{t' \ge t}, \{x_{j,t'\Delta}\}_{j \ne i, t' \ge t}) \mid H_{i,t\Delta} \cup \{s_{j,T_l}\}_{j \ne i, T_l \le t\Delta}]. \tag{22}$$

That is, the strategy of each trader i remains optimal even if he would observe all other traders' total signals.[16] This is because trader i cares only about the sum of others' total signals, and the sum can be inferred from the price anyway.

Second, the coefficient b captures how much additional quantity of the asset a trader is willing to buy if the price drops by one unit. Thus, a larger b corresponds to

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[16]This ex post feature is related to the ex post equilibrium of Hörner and Lovo (2009), Fudenberg and Yamamoto (2011), and Hörner, Lovo, and Tomala (2012). Bergemann and Valimaki (2010) consider the ex post implementation of the socially efficient allocation in a dynamic market. Other work in the ex post implementation literature includes Crémer and McLean (1985), Bergemann and Morris (2005), and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006), among others. Perry and Reny (2005) construct an ex post equilibrium in a multi-unit ascending-price auction with interdependent values.

Figure 1: Aggressiveness of demand schedules as a function of Δ
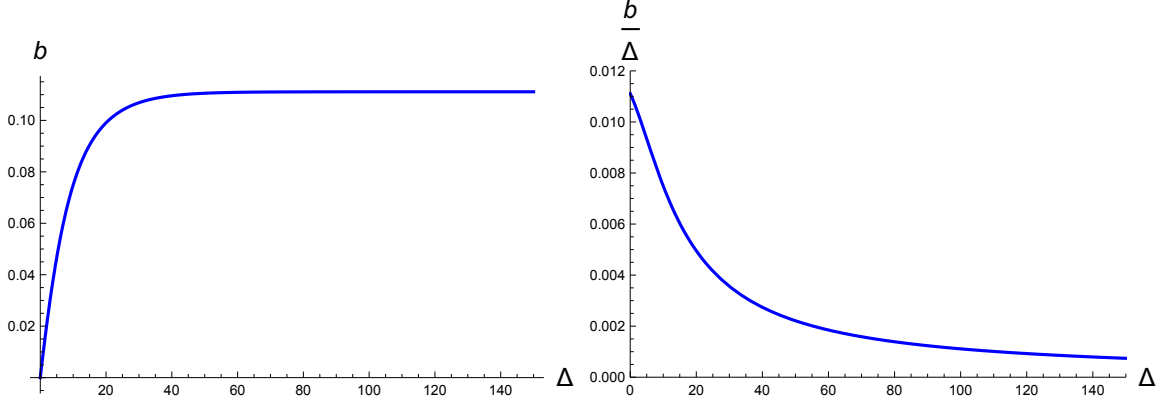
a more aggressive demand schedule. One interesting property is that b is increasing in Δ; that is, less frequent trading encourages more aggressive demand schedules *per period*. If we take into account the number of period per unit of clock time, however, the frequency-adjusted aggressiveness, b/Δ, is decreasing in Δ. These facts can be proved by direct calculation. Figure 1 provides an illustration. These comparative statics are important building blocks for determining the efficiency of the equilibrium, as we elaborate in Proposition 3, as well as the optimal trading frequency, as we investigate in Section 3.

Third, the market-clearing price $p_{t\Delta}^*$ is a martingale and aggregates the most recent total signals $\{s_{i,t\Delta}\}$, which combine common-value signals and private values. This property has a flavor of rational expectations equilibrium. The adjustment term $\lambda Z/(nr)$ in $p_{t\Delta}^*$ results from the flow cost of holding inventory. Since the total signals are martingales, the price is also a martingale. In addition, although each trader learns from $p_{t\Delta}^*$ the average total signal $\sum_i s_{i,t\Delta}/n$ in period t, he does not learn the total signal or inventory of any other *individual* trader. Nor does a trader perfectly distinguish the common-value component and the private-value component of the price. Thus, private information is not fully revealed after each round of trading. Moreover, because new shocks to the common dividend and private values may arrive by the clock time $(t+1)\Delta$ of the next double auction, a period-$(t+1)$ strategy that depends explicitly on the lagged price $p_{t\Delta}^*$ is generally not optimal.

Fourth, the negativity of the second-order condition requires $n\alpha > 2$. We show in the proof that $n\alpha > 2$ is equivalent to the condition (17). All else equal, condition (17) holds if n is sufficiently large, if signals of dividend shocks are sufficiently precise

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(i.e. σ_ϵ^2 is small enough), if new information on the common dividend is not too volatile (i.e. σ_D^2 is small enough), or if shocks to private values are sufficiently volatile (i.e. σ_w^2 is large enough).[17] All these conditions reduce adverse selection. Condition (17) essentially requires that adverse selection regarding the common dividend is not "too large" relative to the gains from trade.

Finally, the equilibrium of Proposition 1 is unique under natural restrictions on the strategies.

Proposition 2. *The equilibrium from Proposition 1 is the unique perfect Bayesian equilibrium in the following class of strategies:*

$$x_{i,t\Delta}(p) = \sum_{T_l \leq t\Delta} a_l S_{i,T_l} + a_w w_{i,t\Delta} - bp + dz_{i,t\Delta} + f, \qquad (23)$$

where $\{a_l\}_{l\geq 0}, a_w, b, d$ and f are constants.

2.3 Efficiency

We now study the allocative efficiency (or inefficiency) in the equilibrium of Proposition 1.

2.3.1 The competitive benchmark

As an efficiency benchmark, we consider a competitive equilibrium in which all traders take the price as given. To solve for a competitive equilibrium in this dynamic market, let us first conjecture that the competitive equilibrium price in every period t is:

$$p_{t\Delta}^c = \frac{1}{n} \sum_{j=1}^{n} s_{j,t\Delta} - \frac{\lambda}{rn} Z. \qquad (24)$$

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[17]As a special case, if $\sigma_\epsilon^2 = 0$ and $\sigma_w^2 > 0$, we have public information about the dividends, which implies $\chi = 1$ and $\alpha = 1$ from Equation (15). If $\sigma_D^2 = 0$ and $\sigma_w^2 > 0$, we have the pure private value case, which implies $\chi = 0$ and $\alpha = 1$. Each trader i's equilibrium strategy in the pure private value case and in the public dividend information case have the same coefficients on $s_{i,t\Delta}$, p, $z_{i,t\Delta}$, and Z.

Taking the prices in (24) as given, a trader i in period t solves:

$$\max_{\{x_{i,t'\Delta}\}_{t'\geq t}} \mathbb{E}\left[\sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta}\left((1-e^{-r\Delta})\left(v_{i,t'\Delta}(z_{i,t'\Delta}^c + x_{i,t'\Delta}(p_{t'\Delta}^c)) - \frac{\lambda}{2r}(z_{i,t'\Delta}^c + x_{i,t'\Delta}(p_{t'\Delta}^c))^2\right)\right.\right.$$

$$\left.\left. - p_{t'\Delta}^c \cdot x_{i,t'\Delta}(p_{t'\Delta}^c)\right) \,\middle|\, H_{i,t\Delta}\right] \tag{25}$$

$$\text{subject to: } z_{i,(t'+1)\Delta}^c = z_{i,t'\Delta}^c + x_{i,t'\Delta}(p_{t'\Delta}^c)$$

$$p_{t'\Delta}^c = \frac{1}{n}\sum_{j=1}^{n} s_{j,t'\Delta} - \frac{\lambda}{rn}Z.$$

Since trader i conditions his quantity on the price, given the conjectured price in (24) he can condition on $\sum_{j\neq i} s_{j,t'\Delta}$; that is, trader i solves:

$$\max_{\{x_{i,t'\Delta}\}_{t'\geq t}} \sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta}\mathbb{E}\left[(1-e^{-r\Delta})\left(v_{i,t'\Delta}(z_{i,t'\Delta}^c + x_{i,t'\Delta}(p_{t'\Delta}^c)) - \frac{\lambda}{2r}(z_{i,t'\Delta}^c + x_{i,t'\Delta}(p_{t'\Delta}^c))^2\right)\right.$$

$$\left. - p_{t'\Delta}^c \cdot x_{i,t'\Delta}(p_{t'\Delta}^c) \,\middle|\, H_{i,t'\Delta} \cup \left\{\sum_{j\neq i} s_{j,t'\Delta}\right\}\right], \tag{26}$$

under the same constraints as in (25). Optimal strategies obtained by solving (26) are also optimal in the sense of (25).

The maximization problem in (26) can be solved period by period. It is straightforward to show that the solution is: for every $t' \geq t$,

$$x_{i,t'\Delta}^c(p_{t'\Delta}^c) = -z_{i,t'\Delta}^c + \frac{r}{\lambda}\left(\mathbb{E}\left[v_{i,t'\Delta}\,\middle|\, H_{i,t'\Delta} \cup \left\{\sum_{j\neq i} s_{j,t'\Delta}\right\}\right] - p_{t'\Delta}^c\right)$$

$$= -z_{i,t'\Delta}^c + \frac{r(n\alpha - 1)}{\lambda(n-1)}(s_{i,t'\Delta} - p_{t'\Delta}^c) + \frac{1-\alpha}{n-1}Z, \tag{27}$$

where we have used Equations (21) and (24) in the second line. Hence, the competitive equilibrium strategy is

$$x_{i,t\Delta}^c(p; s_{i,t\Delta}, z_{i,t\Delta}) = \frac{r(n\alpha - 1)}{\lambda(n-1)}\left(s_{i,t\Delta} - p - \frac{\lambda(n-1)}{r(n\alpha-1)}z_{i,t\Delta} + \frac{\lambda(1-\alpha)}{r(n\alpha-1)}Z\right). \tag{28}$$

It is easy to verify that if every trader uses the above strategy, the market-clearing price is indeed (24).

The post-trading allocation in the competitive equilibrium in period t is:

$$z_{i,(t+1)\Delta}^c = z_{i,t\Delta}^c + x_{i,t\Delta}^c(p_{t\Delta}^c; s_{i,t\Delta}, z_{i,t\Delta}^c) = \frac{r(n\alpha-1)}{\lambda(n-1)}\left(s_{i,t\Delta} - \frac{1}{n}\sum_{j=1}^n s_{j,t\Delta}\right) + \frac{1}{n}Z. \quad (29)$$

We refer to this allocation as the "competitive allocation" or the "efficient allocation". The competitive allocation maximizes the social welfare in period t given the realization of total signals:

$$\{z_{i,(t+1)\Delta}^c\} \in \operatorname*{argmax}_{\{z_i\}} \sum_{i=1}^n \left(\mathbb{E}[v_{i,t\Delta}z_i \mid \{s_{j,t\Delta}\}_{1\leq j\leq n}] - \frac{\lambda}{2r}(z_i)^2\right). \quad (30)$$

For the simplicity of notation, we also define

$$z_{i,t\Delta}^e \equiv z_{i,(t+1)\Delta}^c. \quad (31)$$

That is, $z_{i,t\Delta}^e$ is the allocation that would obtain if traders play the competitive equilibrium in the period-t double auction.

2.3.2 Efficiency of the equilibrium of Proposition 1

We see that the perfect Bayesian equilibrium demand schedule in (18) from Proposition 1 is a scaled version of the competitive equilibrium demand schedule in (28), with the scaling factor

$$\frac{b}{\frac{r(n\alpha-1)}{\lambda(n-1)}} = 1 + \frac{(n\alpha-1)(1-e^{-r\Delta}) - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}}{2e^{-r\Delta}} < 1. \quad (32)$$

This feature is the familiar "demand reduction" in the literature on static divisible auctions (see Ausubel, Cramton, Pycia, Rostek, and Weretka 2011). Thus, the equilibrium allocation of Proposition 1 is not as efficient as the competitive allocation. Nonetheless, the following proposition shows that the allocation under the perfect Bayesian equilibrium of Proposition 1 converges exponentially over time to the competitive allocation.

Let us denote by $\{z_{i,t\Delta}^*\}$ the path of inventories obtained by the equilibrium strat-

egy $x_{i,t\Delta}$ of Proposition 1:

$$z^*_{i,(t+1)\Delta} = z^*_{i,t\Delta} + x_{i,t\Delta}(p^*_{t\Delta}; s_{i,t\Delta}, z^*_{i,t\Delta}), \ t \geq 0 \tag{33}$$

where $z^*_{i,0} = z_{i,0}$.

Proposition 3. *Given any $0 \leq \underline{t} \leq \bar{t}$, if $s_{i,t\Delta} = s_{i,\underline{t}\Delta}$ for all i and all $t \in \{\underline{t}, \underline{t} + 1, \ldots, \bar{t}\}$, then the equilibrium inventories $z^*_{i,t\Delta}$ satisfy: for every i,*

$$z^*_{i,t\Delta} - z^c_{i,(\underline{t}+1)\Delta} = (1+d)^{t-\underline{t}}(z^*_{i,\underline{t}\Delta} - z^c_{i,(\underline{t}+1)\Delta}), \quad \forall t \in \{\underline{t} + 1, \underline{t} + 2, \ldots, \bar{t} + 1\}, \tag{34}$$

where $\{z^c_{i,(\underline{t}+1)\Delta}\}$ is the the period-$\underline{t}$ competitive allocation defined in (29), and $d \in (-1, 0)$ is the coefficient of $z_{i,t\Delta}$ in the equilibrium strategy (18).

Moreover, we define the convergence rate of equilibrium allocations to the competitive levels per unit of clock time to be $R \equiv -\log[(1+d)^{1/\Delta}]$. This convergence rate R is increasing in the number n of traders, the arrival intensity r of dividend, and the clock-time frequency of trading $1/\Delta$. The convergence rate R is increasing with the variance σ^2_w of the private-value shocks and is decreasing with the variance σ^2_D of the common-value shocks.

Proposition 3 reveals that a sequence of double auctions is an effective mechanism to dynamically achieve allocative efficiency. Since $0 < 1 + d < 1$ in (34), the equilibrium allocations converge *exponentially* over time to the competitive allocation that is associated with the most recent total signals. As the proof of Proposition 3 makes clear, the exponential convergence result is driven by (i) the linearity and the stationarity of the perfect Bayesian equilibrium strategy, and (ii) at the competitive inventory level, the perfect Bayesian equilibrium strategy implies buying and selling zero additional unit. While the convergence is exponential in time, it is not instantaneous. Exponential convergence of this kind is previously obtained in the dynamic model of Vayanos (1999) under the assumption that common-value information is public.

The intuition for the comparative statics of Proposition 3 is simple. A larger n makes traders more competitive, and a larger r makes them more impatient. Both effects encourage aggressive bidding and speed up convergence. The effect of σ^2_D is slightly more subtle. A large σ^2_D implies a large uncertainty of a trader about the common asset value and a severe adverse selection; hence, in equilibrium the trader

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reduces his demand or supply relative to the fully competitive market. Therefore, a higher σ_D^2 implies less aggressive bidding and slower convergence to the competitive allocation. The effect of σ_w^2 is the opposite: a higher σ_w^2 implies a larger gain from trade, and hence more aggressive bidding and faster convergence to the competitive allocation. The effect of common value uncertainty in reducing the convergence speed to efficiency is confirmed by Sannikov and Skrzypacz (2014) in a continuous-time trading model.

Finally, a higher trading frequency increases the convergence speed in clock time, even though it makes traders more patient and thus less aggressive in each trading period. A higher trading frequency, however, does not necessarily lead to a higher level of social welfare, as we show in Section 3.

The comparative static of the speed of convergence with respect to σ_ϵ^2 is ambiguous. As σ_ϵ^2 increases, the normalized variances $\sigma_D^2/\sigma_\epsilon^2$ and $\sigma_w^2/\sigma_\epsilon^2$ both decrease. A decrease in $\sigma_D^2/\sigma_\epsilon^2$ increases the speed of convergence, while a decrease in $\sigma_w^2/\sigma_\epsilon^2$ decreases the speed of convergence. (Endogenous parameters α and χ, and hence the speed of convergence, depend only on the normalized variances.)

2.4 Continuous-time limit

In this subsection we examine the limit of the equilibrium in Proposition 1 as $\Delta \to 0$, that is, as trading becomes continuous in clock time.

Proposition 4. *As $\Delta \to 0$, the equilibrium of Proposition 1 converges to the following perfect Bayesian equilibrium:*

1. *Trader i's equilibrium strategy is represented by a process $\{x_{i,\tau}^\infty\}_{\tau \geq 0}$. At the clock time τ, $x_{i,\tau}^\infty$ specifies trader i's* rate *of order submission and is defined by*

$$x_{i,\tau}^\infty(p; s_{i,\tau}, z_{i,\tau}) = b^\infty \left(s_{i,\tau} - p - \frac{\lambda(n-1)}{r(n\alpha-1)} z_{i,\tau} + \frac{\lambda(1-\alpha)}{r(n\alpha-1)} Z \right), \qquad (35)$$

where

$$b^\infty = \frac{r^2(n\alpha-1)(n\alpha-2)}{2\lambda(n-1)}. \qquad (36)$$

Given a clock time $\tau > 0$, in equilibrium the total amount of trading by trader

i in the clock-time interval $[0, \tau]$ is

$$z_{i,\tau}^* - z_{i,0} = \int_{\tau'=0}^{\tau} x_{i,\tau'}^{\infty}(p_{\tau'}^*; s_{i,\tau'}, z_{i,\tau'}^*) \, d\tau'. \tag{37}$$

2. The equilibrium price at any clock time τ is

$$p_\tau^* = \frac{1}{n} \sum_{i=1}^{n} s_{i,\tau} - \frac{\lambda}{nr} Z. \tag{38}$$

3. Given any $0 \leq \underline{\tau} < \overline{\tau}$, if $s_{i,\tau} = s_{i,\underline{\tau}}$ for all i and all $\tau \in [\underline{\tau}, \overline{\tau}]$, then the equilibrium inventories $z_{i,\tau}^*$ in this interval satisfy:

$$z_{i,\tau}^* - z_{i,\underline{\tau}}^e = e^{-\frac{1}{2} r(n\alpha - 2)(\tau - \underline{\tau})} \left(z_{i,\underline{\tau}}^* - z_{i,\underline{\tau}}^e \right), \tag{39}$$

where

$$z_{i,\underline{\tau}}^e \equiv \frac{r(n\alpha - 1)}{\lambda(n-1)} \left(s_{i,\underline{\tau}} - \frac{1}{n} \sum_{j=1}^{n} s_{j,\underline{\tau}} \right) + \frac{1}{n} Z \tag{40}$$

is the efficient allocation at clock time $\underline{\tau}$ (cf. Equation (29)).

Proof. The proof follows by directly calculating the limit of Proposition 1 as $\Delta \to 0$ using L'Hopitâl's rule. $\square$

Proposition 4 reveals that even if trading occurs continuously, in equilibrium the competitive allocation is not reached instantaneously. The delay comes from traders' price impact and the associated demand reduction. This feature is also obtained by Vayanos (1999). Although submitting aggressive orders allows a trader to achieve his desired allocation sooner, aggressive bidding also moves the price against the trader and increases his trading cost. Facing this tradeoff, each trader uses a finite rate of order submission in the limit. As in Proposition 3, the rate of convergence to the competitive allocation in Proposition 4, $r(n\alpha - 2)/2$, is increasing in n, r, and σ_w^2 but decreasing in σ_D^2.[18]

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[18]The proof of Proposition 3 shows that $\partial(n\alpha)/\partial\sigma_w^2 > 0$, $\partial(n\alpha)/\partial\sigma_D^2 < 0$, and $\partial(n\alpha)/\partial n > 0$.

2.5 Inefficiency in large markets

To further explore the effect of adverse selection for allocative efficiency, and to compare with the literature (in particular with Vayanos (1999)), we consider the rate at which inefficiency vanishes as the number of traders becomes large, with and without adverse selection. Adverse selection exists if $\sigma_D^2 > 0$ and $\sigma_\epsilon^2 > 0$. For fixed $\sigma_\epsilon^2 > 0$ and $\sigma_w^2 > 0$, we compare the convergence rate in the case of a fixed $\sigma_D^2 > 0$ to that in the case of $\sigma_D^2 = 0$.

We define inefficiency as the difference between the total utilities in the perfect Bayesian equilibrium of Proposition 1 and the total utilities in the competitive equilibrium:

$$X_1(\Delta) \equiv \mathbb{E}\left[\sum_{t=0}^{\infty}(e^{-rt\Delta} - e^{-r(t+1)\Delta}) \sum_{i=1}^{n} \left(\left(v_{i,t\Delta} z^*_{i,(t+1)\Delta} - \frac{\lambda}{2r}(z^*_{i,(t+1)\Delta})^2 \right) \right. \right. \tag{41}$$
$$\left. \left. - \left(v_{i,t\Delta} z^c_{i,(t+1)\Delta} - \frac{\lambda}{2r}(z^c_{i,(t+1)\Delta})^2 \right) \right) \right]$$

where $\{z^*_{i,(t+1)\Delta}\}$ is the path of inventory given by the perfect Bayesian equilibrium in Proposition 1, and

$$z^c_{i,(t+1)\Delta} = z^e_{i,t\Delta} \equiv \frac{r(n\alpha - 1)}{\lambda(n-1)}\left(s_{i,t\Delta} - \frac{1}{n}\sum_{j=1}^{n} s_{j,t\Delta} \right) + \frac{1}{n}Z \tag{42}$$

is the competitive/efficient inventory given the period-t total signals.

Proposition 5. *Suppose the news times $\{T_k\}_{k\geq1}$ either satisfies $T_k = k\gamma$ for a constant $\gamma > 0$ or is a homogeneous Poisson process. Suppose also that $\sigma_\epsilon^2 > 0$ and $\sigma_w^2 > 0$, and that there exists a constant $C > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z^e_{i,0})^2] \leq C. \tag{43}$$

Then, the following convergence results hold:

1. *If $\sigma_D^2 > 0$, then as $n \to \infty$:*

   $$\frac{X_1(\Delta)}{n} \text{ converges to zero at the rate } n^{-4/3} \text{ for any } \Delta > 0,$$

   $$\lim_{\Delta \to 0} \frac{X_1(\Delta)}{n} \text{ converges to zero at the rate } n^{-2/3}.$$

2. *If $\sigma_D^2 = 0$, then as $n \to \infty$:*

   $$\frac{X_1(\Delta)}{n} \text{ converges to zero at the rate } n^{-2} \text{ for any } \Delta > 0,$$

   $$\lim_{\Delta \to 0} \frac{X_1(\Delta)}{n} \text{ converges to zero at the rate } n^{-1}.$$

The convergence rates under $\sigma_D^2 = 0$ (i.e. pure private values) are also obtained in the model of Vayanos (1999), who is the first to show that convergence rates differ between discrete-time trading and continuous-time trading. Relative to the results of Vayanos (1999), Proposition 5 reveals that the rate of convergence is slower if traders are subject to adverse selection. For any fixed $\Delta > 0$ and as $n \to \infty$, the inefficiency $X_1(\Delta)/n$ vanishes at the rate of $n^{-4/3}$ if $\sigma_D^2 > 0$, but the corresponding rate is n^{-2} if $\sigma_D^2 = 0$. If one first takes the limit of $\Delta \to 0$, then the convergence rates as n becomes large are $n^{-2/3}$ and n^{-1} with and without adverse selection, respectively. Interestingly, the asymptotic rates do not depend on the size of σ_D^2 but only depend on whether σ_D^2 is positive or not.

3   Welfare and Optimal Trading Frequency under Homogeneous Trading Speed

In this section and the next, we use the model framework developed in Section 2 to analyze the welfare implications of trading speed. In this section we study the effect of trading frequency on welfare and characterize the optimal trading frequency if all traders participate in all double auctions, i.e., if traders have homogeneous trading speed. In the next section we study the case of heterogeneous trading speeds in the sense that fast traders participate in all double auctions but slow traders only participate periodically.

The main result of this section is that the optimal trading frequency depends

critically on the nature of new information (i.e., the shocks to dividends and private values). If new information arrives at deterministic and scheduled intervals, then slow trading (i.e., a large Δ) tends to be optimal. If new information arrives stochastically according to a Poisson process, then fast trading (i.e., a small Δ) tends to be optimal. Throughout this section we conduct the analysis based on the perfect Bayesian equilibrium of Proposition 1, which requires the parameter condition $n\alpha > 2$.

Let us define the path of inventories implied by the perfect Bayesian equilibrium $x_{i,t\Delta}$ of Proposition 1: $z_{i,0}^* = z_{i,0}$, and

$$z_{i,\tau}^* = z_{i,t\Delta}^* + x_{i,t\Delta}(p_{t\Delta}^*; s_{i,t\Delta}, z_{i,t\Delta}^*), \text{ for } \tau \in (t\Delta, (t+1)\Delta], \tag{44}$$

for every integer $t \geq 0$, where $x_{i,t\Delta}$ is the equilibrium strategy in Proposition 1. The inventory path is discontinuous as it "jumps" after trading in each period.

We then define the equilibrium welfare as the sum of expected utilities over all traders:

$$W(\Delta) = \mathbb{E}\left[\sum_{i=1}^{n}(1 - e^{-r\Delta})\sum_{t=0}^{\infty}e^{-rt\Delta}\left(v_{i,t\Delta}z_{i,(t+1)\Delta}^* - \frac{\lambda}{2r}(z_{i,(t+1)\Delta}^*)^2\right)\right] \tag{45}$$

$$= \mathbb{E}\left[\sum_{i=1}^{n}\int_{\tau=0}^{\infty}re^{-r\Delta}\left(v_{i,\tau}z_{i,\tau}^* - \frac{\lambda}{2r}(z_{i,\tau}^*)^2\right)d\tau\right]. \tag{46}$$

Note that if the initial inventory $\{z_{i,0}\}_{1\leq i\leq n}$ is symmetrically distributed, then all traders are symmetric, and hence each trader's ex ante expected utility is simply $W(\Delta)/n$. Thus, the welfare criterion based on $W(\Delta)$ is equivalent to Parento dominance: If $W(\Delta_1) > W(\Delta_2)$ for $\Delta_1 \neq \Delta_2$, then each trader i's ex ante expected utility is higher under Δ_1 than that under Δ_2.

We denote by Δ^* the optimal trading interval that maximizes the welfare $W(\Delta)$. The optimal trading frequency is then $1/\Delta^*$.

3.1 Scheduled Arrivals of New Information

We first consider scheduled information arrivals. In particular, we suppose that shocks to the common dividend and shocks to private values occur at regularly spaced clock times $T_k = k\gamma$ for a positive constant γ, where $k \geq 0$ are integers. Examples of scheduled information include macroeconomic data releases and corporate earnings

announcements.

Proposition 6. *Suppose $T_k = k\gamma$ for a positive constant γ. Then $W(\Delta) < W(\gamma)$ for any $\Delta < \gamma$. That is, $\Delta^* \geq \gamma$.*

Proposition 6 shows that if new information repeatedly arrives at scheduled times, then the optimal trading frequency cannot be higher than the frequency of information arrivals. The intuition for this result is simple. For a large Δ, traders have to wait for a long time before the next round of trading. So they submit relatively aggressive demand schedules (and hence mitigate the "demand reduction") whenever they have the opportunity to trade, which leads to a relatively efficient allocation early on. That is, a large Δ serves as a commitment device to encourage aggressive trading immediately. If Δ is small, traders know that they can trade again soon. Consequently, they trade less aggressively in each round of double auction and end up holding relatively inefficient allocations in early rounds. We show that if $\Delta < \gamma$, then a larger Δ leads to a higher welfare.[19]

To further illustrate the intuition of Proposition 6, we plot in Figure 2 the welfare measures for the special case that new information only arrives once, at time 0 (that is, $\gamma = \infty$). The left-hand plot shows the distance between a generic trader's equilibrium inventory and the efficient inventory (defined by the period-0 information) as a function of time, under the two extreme trading frequencies of $\Delta = \infty$ (trade only once at time 0) and $\Delta = 0$ (continuous trading). If trading happens only once, then all traders submit aggressive demand schedules, and the trader's inventory immediately jumps toward the efficient level by a discrete amount. This jump is partial, however. By contrast, under continuous trading, the trader's inventory converges to the efficient level gradually and continually over time, reaching the efficient inventory in the limit. Thus, faster trading implies more inefficiency in early rounds, and slower trading implies more inefficiency in late rounds.

The right-hand plot of Figure 2 shows that the welfare $W(\Delta)$ is strictly increasing in Δ, that is, welfare is improved by slowing down trades. Intuitively, mis-allocation

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[19]In Proposition 6, we have implicitly assumed that the first round of trading always starts at clock time zero, immediately after the arrival of time-zero signals. This assumption is without loss of generality: we can show that the equilibrium welfare from starting at time 0 and trading at frequency $1/\gamma$ always dominates the equilibrium welfare from starting at some time $\tau_0 > 0$ and trading at some frequency $1/\Delta \geq 1/\gamma$; the proof is similar to that in Section B.4.2 and is available upon request. Intuitively, there is no reason to delay trading after the information arrival because in each trading period the equilibrium strategy always gives a weakly higher utility than that from not trading.

Figure 2: Trading frequency and welfare when information arrives only at time 0
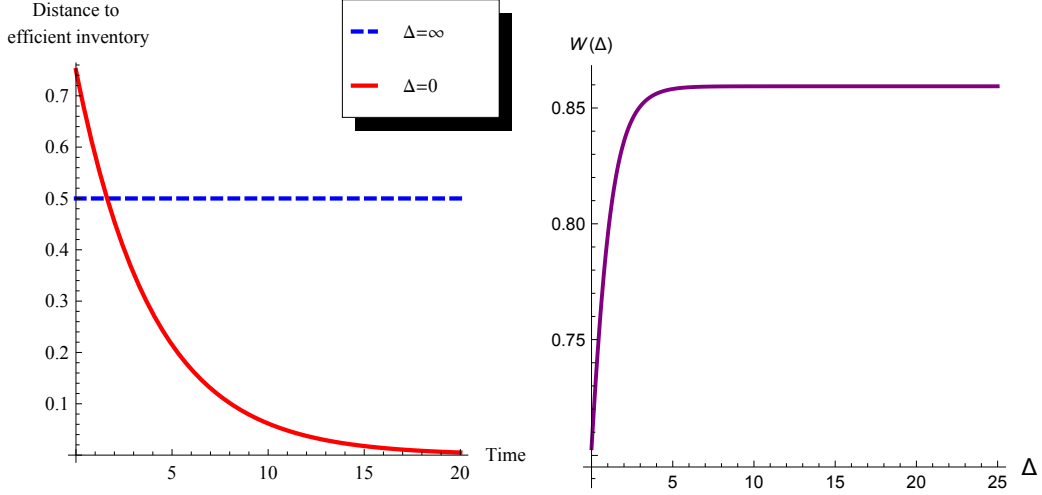
of assets is more costly in early rounds than in late rounds because the cost is convex in the inventory level. The more general result of Proposition 6 has the same intuition.

Proposition 6 establishes an upper bound of trading frequency if information arrives at scheduled intervals. Next, we try to characterize Δ^* more explicitly. As Δ increases beyond γ, the traders face a tradeoff: a large $\Delta > \gamma$ gives the benefit of a commitment device, but incurs the cost that traders cannot react quickly to new information. To further characterize Δ^* we must first define some parameters that quantify the magnitude of the new information given by the dividend shocks.

From Equation (29) we write the path of efficient inventory in *continuous time* as

$$z_{i,\tau}^e = \frac{r(n\alpha - 1)}{\lambda(n-1)} \left(s_{i,\tau} - \frac{1}{n} \sum_{j=1}^{n} s_{j,\tau} \right) + \frac{1}{n} Z, \text{ for every } \tau \geq 0. \qquad (47)$$

The inventory in this path instantaneously adjusts to the new total signals at any time τ. Since the signals are martingales, $\{z_{i,\tau}^e\}_{\tau \geq 0}$ also forms a martingale (adapted to the total signals $\{s_{j,\tau}\}_{1 \leq j \leq n, \tau \geq 0}$) for each i.

We define:

$$\sigma_z^2 \equiv \sum_{i=1}^{n} \mathbb{E}[(z_{i,T_k}^e - z_{i,T_{k-1}}^e)^2] = \left(\frac{r(n\alpha - 1)}{\lambda(n-1)}\right)^2 \frac{(n-1)(\chi^2(\sigma_D^2 + \sigma_\epsilon^2) + \sigma_w^2)}{\alpha^2} > 0, \quad (48)$$

$$\sigma_0^2 \equiv \sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2]. \qquad (49)$$

The first variance σ_z^2 describes the extent to which each arrival of new information changes the efficient inventories among traders. The second variance σ_0^2 describes the distance between the time-0 initial inventory and the efficient inventory given time-0 information. If $z_{i,0} = Z/n$ for every trader i (all traders are ex ante identical), then we have $\sigma_0^2 = \sigma_z^2$.[20] In general, σ_0^2 can be larger or smaller than σ_z^2, depending on how inefficient the initial inventories $\{z_{i,0}\}$ are given the total signals $\{s_{i,0}\}$ in period 0.

The social welfare $W(\Delta)$ is hard to analyze for a generic $\Delta > \gamma$. For analytical tractability but at no cost of economic intuition, for the case of $\Delta > \gamma$ we restrict attention to $\Delta = l\gamma$ for a positive integer l. Let $l^* \in \mathrm{argmax}_{l \in \mathbb{Z}_+} W(l\gamma)$.

Proposition 7. *Suppose $T_k = k\gamma$ for a positive constant γ. The optimal l^* weakly increases with σ_0^2/σ_z^2 from $l^* = 1$ (when $\sigma_0^2/\sigma_z^2 = 0$) to ∞ (as $\sigma_0^2/\sigma_z^2 \to \infty$).[21] Moreover, if σ_0^2/σ_z^2 remains bounded as $n \to \infty$, then $l^* = 1$ when n is sufficiently large.*

Proposition 7 reveals that the optimal Δ^* is strictly greater than γ if σ_0^2 is sufficiently large in comparison to σ_z^2. By the same intuition of Proposition 6, a large σ_0^2 favors slow trading because traders would send aggressive demand schedules that quickly reduce the inefficiency in the initial allocations $\{z_{i,0}\}$. That said, slow trading also prevents traders from reacting immediately to new information shocks, and the cost for this slow reaction is increasing in σ_z^2. If σ_0^2/σ_z^2 is large, the benefit of slow trading dominates the cost, leading to a low optimal trading frequency. If σ_0^2/σ_z^2 is small, the cost of slow trading dominates the benefit, leading to a high optimal trading frequency. In the special case that the initial allocation $\{z_{i,0}\}$ happens to be efficient given the time-0 information, the optimal $l^* = 1$.

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[20]To see this, note that Z/n is the efficient allocation to each trader if the initial total signals of all traders are zero. For each trader i, the total signal $s_{i,0}$ received at time 0 has the same distribution as the innovation $s_{i,T_k} - s_{i,T_{k-1}}$. Thus, $z_{i,0}^e - Z/n$ has the same distribution as $z_{i,T_k}^e - z_{i,T_{k-1}}^e$ for any $k \geq 1$, i.e., $\sigma_0^2 = \sigma_z^2$ if $z_{i,0} = Z/n$ for all i.

[21]In the special case when $\mathrm{argmax}_{l \in \mathbb{Z}_+} W(l\gamma)$ is not a singleton set, Proposition 7 still holds with any arbitrary selection of $l^* \in \mathrm{argmax}_{l \in \mathbb{Z}_+} W(l\gamma)$.

Figure 3: Scheduled news arrivals, optimal l^* as functions of σ_0^2/σ_z^2 and n (given parameters $r = \lambda = \gamma = \sigma_D^2 = \sigma_\epsilon^2 = \sigma_w^2 = 1$)
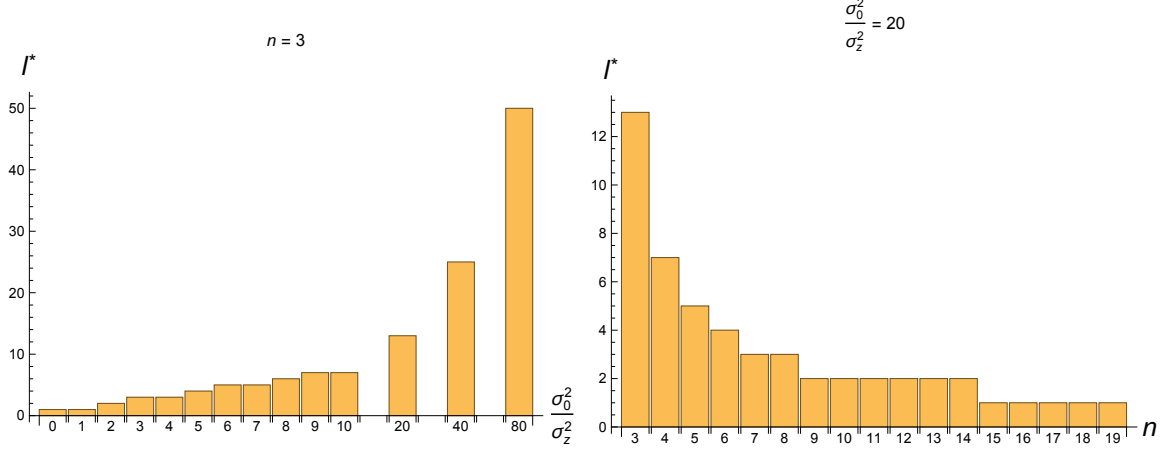
Proposition 7 also implies that for $\Delta > \gamma$ the commitment benefit induced by slow trading loses its bite if the number n of traders is large enough. All else equal, as n increases, the market becomes increasingly competitive, and the inefficiency associated with strategic trading and demand reduction shrinks. In the limit $n \to \infty$, as long as σ_z^2/σ_0^2 is positive, the allocation efficiency is entirely determined by how fast traders can react to new information. Thus, the optimal $l^* = 1$.

In Figure 3 we illustrate Proposition 7 by plotting l^* as functions of σ_z^2/σ_0^2 and n in an example.

3.2 Stochastic Arrivals of New Information

We now turn to stochastic arrivals of information. Examples of stochastic news include unexpected corporate announcements (e.g. mergers and acquisitions), regulatory actions, and geopolitical events. There are many possible specifications for stochastic information arrivals, and it is technically hard to calculate the optimal trading frequency for all of them. Instead, we analyze the simple yet natural case of a Poisson process. We expect the economic intuition of the results to apply to more general signal structures.

Suppose that the timing of the dividend shocks $\{T_k\}_{k\geq 1}$ follows a homogeneous Poisson process with intensity $\mu > 0$. (The first shock still arrives at time $T_0 = 0$.) Within a time interval τ, there are, in expectation, $\tau\mu$ dividend shocks. Since the time interval between two consecutive dividend shocks has the expectation $1/\mu$, μ is

analogous to $1/\gamma$ from Section 3.1.

Proposition 8. *Suppose that $\{T_k\}_{k \geq 1}$ is a Poisson process with intensity μ. The following comparative statics hold:*

1. *The optimal Δ^* strictly increases with σ_0^2/σ_z^2 from 0 (when $\sigma_0^2/\sigma_z^2 = 0$) to ∞ (as $\sigma_0^2/\sigma_z^2 \to \infty$).*

2. *If $\sigma_0^2 > 0$, then Δ^* strictly decreases with μ from ∞ (as $\mu \to 0$) to 0 (as $\mu \to \infty$).*

3. *If σ_0^2/σ_z^2 remains bounded as $n \to \infty$, then $\Delta^* \to 0$ as $n \to \infty$.*

4. *If $z_{i,0} = Z/n$ for every trader i (i.e., $\sigma_0^2 = \sigma_z^2$), then Δ^* strictly decreases with n and σ_w^2, and strictly increases with σ_D^2.*

Part 1 of Proposition 8 includes an interesting special case: if $\sigma_0^2 = 0$, $W(\Delta)$ is a strictly decreasing function of Δ, and the optimal $\Delta^* = 0$. As we see before, if the initial inventories $\{z_{i,0}\}$ are already efficient, there is no commitment benefit of trading slowly from time 0. Increasing trading frequency then involves the following tradeoff: traders can react faster to new information, but once the new information arrives, they trade less aggressively in each round. It turns out that the benefit of faster reaction to new information always dominates, and the intuition is as follows. If traders wish to rebalance their positions in a period but the market is closed, they incur the full inefficiency cost in this period; if traders can rebalance but reduce the aggressiveness of their demand schedules, they only incur a partial inefficiency cost in this period. A partial inventory adjustment toward the efficient level—no matter how small—is better than no adjustment at all. Thus, it is optimal to keep the market open continuously, i.e., $\Delta^* = 0$.

Figure 4 illustrates the welfare consequence of stochastic arrivals of new information, for the special case that $\sigma_0^2 = 0$ and that there is only one arrival of news at an exponentially distributed time. Since $\sigma_0^2 = 0$, the first trade starts after the arrival of news. The left-hand plot in Figure 4 show the total utility of all traders conditional on no dividend payout before the first trade (which is the same as the right-hand plot from Figure 2) and the probability that there is no dividend payout before the first trade (and hence reaching the first trade), both as functions of Δ.[22] Crucially, the left-hand plot shows that the total utility after the first trade increases with Δ at a

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[22]Suppose that the exponential news arrives with an intensity μ. The probability of reaching the

Figure 4: Trading frequency and welfare under an exponential arrival of information
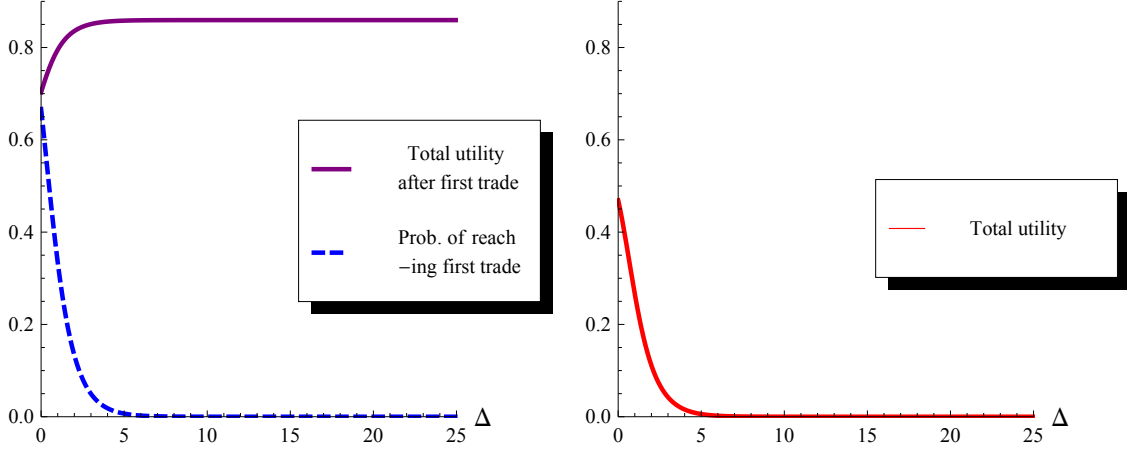
strictly slower rate than the probability of a first trade decreases with Δ. Hence the overall utility (the product of the two functions in the left-hand plot) is a decreasing function of Δ, as shown in the right-hand plot of Figure 4 (here we normalize the utility before the first trade to zero).

If $\sigma_0^2 > 0$, a lower trading frequency encourages aggressive trading in each round and reduces the inefficiency of initial inventories. As in the case of scheduled news arrivals, the optimal Δ^* is higher the larger is σ_0^2/σ_z^2. Moreover, the more imminent is new information (a higher μ), the more benefits traders receive by reacting more quickly to new information, and the higher is the optimal trading frequency.

The effects of σ_0^2/σ_z^2 and μ for optimal trading frequency are illustrated in the left-hand plot of Figure 5. For presentation convenience, in the figures we combine σ_0^2/σ_z^2 and μ into one composite parameter, $\sigma_0^2/(\mu\sigma_z^2)$, which fully determines Δ^* as we show in Section B.4.4.

In addition, as the market becomes large (as $n \to \infty$), the commitment benefit of a low trading frequency vanishes, and continuous trading becomes optimal. We illustrate the effect of n in the right-hand plot of Figure 5.

first trade (the blue dash curve in the left-hand plot of Figure 4) is

$$\sum_{t=0}^{\infty} e^{-(t+1)\Delta r}\left(e^{-t\Delta\mu} - e^{-(t+1)\Delta\mu}\right) = e^{-r\Delta}\frac{1 - e^{-\mu\Delta}}{1 - e^{-(r+\mu)\Delta}}. \tag{50}$$

Figure 5: Optimal Δ^* as functions of $\frac{\sigma_0^2}{\mu\sigma_z^2}$ and n for Poisson news arrivals (given parameters $r = \lambda = \sigma_D^2 = \sigma_\epsilon^2 = \sigma_w^2 = 1$)
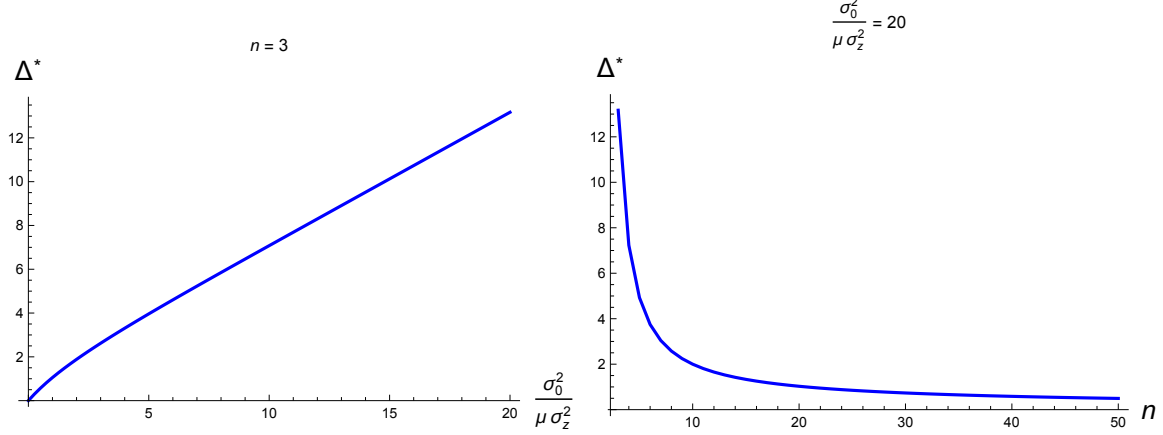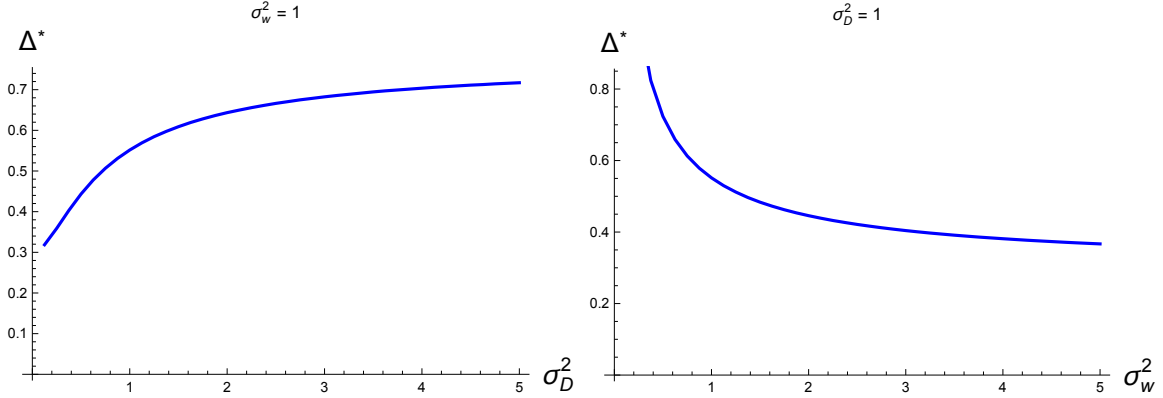

Figure 6: Optimal Δ^* as functions of σ_D^2 and σ_w^2 for Poisson news arrivals (given parameters $r = \lambda = \mu = \sigma_e^2 = 1$, $n = 5$ and $z_{i,0} = Z/n$ for every i)

Finally, Part 4 of Proposition 8 characterizes the effect of adverse selection on the optimal trading frequency, under the natural condition that all traders are ex ante identical (i.e., $\sigma_0^2 = \sigma_z^2$).[23] Because a higher adverse selection (i.e., a larger σ_D^2) reduces the aggressiveness of demand schedules, to counterbalance this negative effect the optimal Δ^* must adjust upward, all else equal. The effect of a higher σ_w^2 is the opposite: if trading is driven more by fluctuating private values, traders would submit more aggressive demand schedules, and the optimal Δ^* goes down. The comparative

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[23]A sufficient condition for Part 4 of Proposition 8 is that σ_0^2/σ_z^2 is a constant. Conditional on σ_0^2/σ_z^2 being a constant, the most natural, well-motivated condition is that $\sigma_0^2/\sigma_z^2 = 1$, i.e., all traders are ex ante identical.

statics with respect to σ_D^2 and σ_w^2 are illustrated in Figure 6. The comparative statics with respect to n is intuitive; a more competitive market has less needs to slow down in order to encourage aggressive demand schedules and realize gains from trade.

4   Heterogeneous Trading Speeds

In this section we extend our model to study the trading strategy and welfare if traders have heterogeneous speeds. In our model, speed is defined by how frequent a trader accesses the market.

4.1   Model and equilibrium

As before trading happens at times $\{0, \Delta, 2\Delta, \ldots\}$. There is a single fast trader who trades in every period. The remaining are slow traders who arrive at the market at the uniform rate of $M > 0$ per unit of clock time and trade only once each. Thus, between times $(t-1)\Delta$ and $t\Delta$, $n_S \equiv M\Delta$ slow traders arrive at the market. These traders wait to the next trading round at time $t\Delta$ and incur the associated delay costs. For concreteness, the fast trader can be interpreted as a representative market maker or high-frequency trader who accesses the market whenever possible, and the slow traders can be interpreted as individual investors or small institutions who trade infrequently. Moreover, we assume that each slow trader trades as soon as possible upon arrival.

Strictly speaking, Δ must take values in $\{1/M, 2/M, 3/M, \ldots\}$ for $M\Delta$ to be an integer, but for expositional simplicity we will solve the trading strategies and welfare for generic Δ and only later use the integer constraint when necessary.

The model with heterogeneous trading speeds is analytically harder because of the asymmetry between fast and slow traders. For this reason we use a simpler information structure. Specifically, fast and slow traders receive no signals about the dividend D; thus, they all value the dividend at $\mathbb{E}[D]$, which we normalize to be zero. Besides tractability, symmetric information regarding the asset fundamental value sets our model of this section orthogonal to recent models of high-frequency trading in which fast traders also have information advantage.[24] Moreover, each slow trader

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[24]See, for example, Biais, Foucault, and Moinas (2014), Jovanovic and Menkveld (2012), and Hoffmann (2014).

has an i.i.d. private value $w_{j,t\Delta} \sim \mathcal{N}(0, \sigma_w^2)$, whereas the fast trader has no private value. Finally, all slow traders start with zero inventory. The fast trader starts with zero inventory at time 0 but gradually accumulates inventory by trading with slow traders over time.

In each trading period $t \in \{1, 2, 3, \dots\}$, n_S slow traders and one fast trader trade in a double auction. (No slow trader has arrived at time 0, so there is no trading at time 0.) Let $x_{j,t\Delta}(p)$ be slow trader j's demand schedule in period t, and let $x_{F,t\Delta}(p)$ be the fast trader's demand schedule in period t. The market-clearing price $p_{t\Delta}^*$ in period t is given by:

$$\sum_{j=1}^{n_S} x_{j,t\Delta}(p_{t\Delta}^*) + x_{F,t\Delta}(p_{t\Delta}^*) = 0. \tag{51}$$

Conditional on no dividend payout before period $t \geq 1$, the utility of a slow trader j who trades in period t is:

$$V_{j,t\Delta} = -x_{j,t\Delta}^* p_{t\Delta}^* + \sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta} \left((1 - e^{-r\Delta})(\mathbb{E}[D] + w_{j,t\Delta})x_{j,t\Delta}^* - \frac{1 - e^{-r\Delta}}{r} \cdot \frac{\lambda}{2}(x_{j,t\Delta}^*)^2 \right)$$

$$= -x_{j,t\Delta}^* p_{t\Delta}^* + w_{j,t\Delta} x_{j,t\Delta}^* - \frac{\lambda}{2r}(x_{j,t\Delta}^*)^2, \tag{52}$$

where $x_{j,t\Delta}^* \equiv x_{j,t\Delta}(p_{t\Delta}^*)$.

Conditional on no dividend payout before period $t \geq 1$, the fast trader's utility is:

$$V_{F,t\Delta} = \sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta} \left(-x_{F,t'\Delta}^* p_{t'\Delta}^* + (1 - e^{-r\Delta})\mathbb{E}[D](z_{F,t'\Delta} + x_{F,t'\Delta}^*) - \frac{1 - e^{-r\Delta}}{r} \cdot \frac{\lambda}{2}(z_{F,t'\Delta} + x_{F,t'\Delta}^*)^2 \right)$$

$$= \sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta} \left(-x_{F,t'\Delta}^* p_{t'\Delta}^* - \frac{1 - e^{-r\Delta}}{r} \cdot \frac{\lambda}{2}(z_{F,t'\Delta} + x_{F,t'\Delta}^*)^2 \right), \tag{53}$$

where $x_{F,t'\Delta}^* \equiv x_{F,t'\Delta}(p_{t'\Delta}^*)$, $z_{F,0} = z_{F,\Delta} = 0$ and

$$z_{F,(t'+1)\Delta} = z_{F,t'\Delta} + x_{F,t'\Delta}^*. \tag{54}$$

The utilities in Equations (52) and (53) are the same as that in Equation (12) with homogeneous trading speed. The definition of perfect Bayesian equilibrium is also the same as that in Definition 1.

Proposition 9. *Suppose that $n_S \equiv M\Delta > 1$. There exists a perfect Bayesian equilib-*

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rium in which every slow trader j in every period $t \geq 1$ submits the demand schedule

$$x_{j,t\Delta}(p; w_{j,t\Delta}) = b_S(w_{j,t\Delta} - p), \tag{55}$$

and the fast trader submits the demand schedule

$$x_{F,t\Delta}(p; z_{F,t\Delta}) = b_F\left(-p - \frac{\lambda_F}{r}z_{F,t\Delta}\right), \tag{56}$$

where $b_S > 0$, $b_F > 0$ and $\lambda_F > 0$ are the unique positive numbers that satisfy

$$b_S = \frac{b_F + (n_S - 1)b_S}{1 + (b_F + (n_S - 1)b_S)\lambda/r}, \tag{57}$$

$$b_F = \frac{n_S b_S}{1 + n_S b_S \lambda_F/r}, \tag{58}$$

$$\lambda_F = \frac{1}{1 - e^{-r\Delta}\left(1 - \frac{n_S b_S}{b_F + n_S b_S} \cdot \frac{\lambda_F b_F}{r}\right)^2} \cdot \left(\lambda(1 - e^{-r\Delta}) + \frac{2e^{-r\Delta}\lambda_F^2 b_F^2 n_S b_S}{r(b_F + n_S b_S)^2}\right). \tag{59}$$

Moreover, we have $\lambda_F < \lambda$ and $b_F > b_S$.

In [Lemma 6](#) in the appendix we prove that there is always a unique positive solution (b_S, b_F, λ_F) to Equations (57), (58) and (59).

The derivation of the slow trader's equilibrium strategy is easy. Under the conjectured strategy that in period t the other slow traders use (55) and the fast trader uses (56), slow trader j's first order condition (by differentiating $V_{j,t\Delta}$ in (52) with respect to $p_{t\Delta}^*$) is:

$$-x_{j,t\Delta}(p_{t\Delta}^*) + (b_F + (n_S - 1)b_S)\left(w_{j,t\Delta} - p_{t\Delta}^* - \frac{\lambda}{r}x_{j,t\Delta}(p_{t\Delta}^*)\right) = 0, \tag{60}$$

i.e.,

$$x_{j,t\Delta}(p_{t\Delta}^*) = \frac{b_F + (n_S - 1)b_S}{1 + (b_F + (n_S - 1)b_S)\lambda/r}(w_{j,t\Delta} - p_{t\Delta}^*), \tag{61}$$

which implies Equations (55) and (57) for the slow trader.

Since $-\lambda z_{F,t\Delta}/r$ is the fast trader's marginal value at the beginning of period t and is analogous to $w_{j,t\Delta}$ of slow trader j, the fast trader's equilibrium strategy in (56) and (58) is similar to that of the slow trader, with an important difference that the flow cost λ_F characterizing the fast trader's strategy is endogenously determined

with b_F and b_S in Equation (59). Lemma 6 in the appendix shows that we always have $\lambda_F < \lambda$, so in equilibrium the fast trader trades in every period *as if* he trades only once and faces a flow cost scaling factor λ_F that is smaller than his actual flow cost scaling factor λ. The fast trader has a lower effective flow cost because he can rebalance his inventory over time. A smaller flow cost, in turn, implies that the fast trader is more aggressive in trading than the slow trader: $b_F > b_S$.

Proposition 10. *In the equilibrium of Proposition 9, the fast trader's starting inventory in period t is*

$$z_{F,t\Delta}^* = \frac{-b_F}{b_F + n_S b_S} \sum_{t'=1}^{t-1} \left(1 - \frac{n_S b_S \lambda_F b_F}{(b_F + n_S b_S) r} \right)^{t-1-t'} \sum_{j=1}^{n_S} b_S w_{j,t'\Delta}, \tag{62}$$

the period-t equilibrium price is

$$p_{t\Delta}^* = \frac{b_S}{b_F + n_S b_S} \sum_{j=1}^{n_S} w_{j,t\Delta} \tag{63}$$

$$+ \frac{b_F}{b_F + n_S b_S} \sum_{t'=1}^{t-1} \frac{n_S b_S \lambda_F b_F}{(b_F + n_S b_S) r} \left(1 - \frac{n_S b_S \lambda_F b_F}{(b_F + n_S b_S) r} \right)^{t-1-t'} \sum_{j=1}^{n_S} \frac{w_{j,t'\Delta}}{n_S},$$

and the amount of trading by the fast trader in period t is

$$x_{F,t\Delta}(p_{t\Delta}^*; z_{t\Delta}^*) = \frac{-b_F}{b_F + n_S b_S} \sum_{j=1}^{n_S} b_S w_{j,t\Delta} \tag{64}$$

$$+ \frac{\lambda_F n_S b_S}{r} \left(\frac{b_F}{b_F + n_S b_S} \right)^2 \sum_{t'=1}^{t-1} \left(1 - \frac{n_S b_S \lambda_F b_F}{(b_F + n_S b_S) r} \right)^{t-1-t'} \sum_{j=1}^{n_S} b_S w_{j,t'\Delta}.$$

The starting inventory of fast trader in period t, $z_{F,t\Delta}^*$, has a simple intuition. In any period $t' < t$, the fast trader adds an inventory equal to a constant multiple of the slow traders' total private values, $\sum_{j=1}^{n_S} w_{j,t'\Delta}$. For example, if slow traders have high private values in period t', the fast trader provides liquidity by selling. In the next period, the fast trader offloads a fraction $\frac{n_S b_S \lambda_F b_F}{(b_F + n_S b_S) r}$ of this inventory and takes on a new inventory as determined by the slow traders' private values in period $t' + 1$, and so on. Calculation shows that the starting inventory of the fast trader for period t equals to the sum of a geometrically weighted total private values of slow traders in each of the previous period, adjusted by a constant. The equilibrium price

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$p^*_{t\Delta}$ and the equilibrium trading amount simply follow from market clearing and the equilibrium strategies. In contrast to the case of homogeneous trading speed, the equilibrium price here is not a martingale as it is a geometrically weighted average of the current and past private values.

4.2 Welfare under heterogeneous trading speeds

Because of the asymmetry between the fast and slow traders, we separate their utilities in the calculation of welfare. Given the equilibrium strategies $x_{F,t\Delta}$ and $x_{j,t\Delta}$ in Proposition 9, the welfare of the fast and slow traders are, respectively:

$$W_F(\Delta) = \mathbb{E}\left[-\frac{\lambda(1 - e^{-r\Delta})}{2r} \sum_{t=1}^{\infty} e^{-rt\Delta}(z^*_{F,(t+1)\Delta})^2 - \sum_{t=1}^{\infty} e^{-rt\Delta} x_{F,t\Delta}(p^*_{t\Delta}; z^*_{F,t\Delta})p^*_{t\Delta} \right],$$
(65)

$$W_S(\Delta) = \mathbb{E}\left[\sum_{t=1}^{\infty} e^{-rt\Delta} \sum_{j=1}^{n_S} \left((w_{j,t\Delta} - p^*_{t\Delta})x_{j,t\Delta}(p^*_{t\Delta}; w_{j,t\Delta}) - \frac{\lambda}{2r} x_{j,t\Delta}(p^*_{t\Delta}; w_{j,t\Delta})^2 \right) \right],$$
(66)

where $\{z^*_{F,t\Delta}\}_{t\geq 1}$ is the fast trader's inventories in equilibrium.

We are interested in Δ^*_F that maximizes the fast trader's welfare and in Δ^*_S that maximizes the slow traders' welfare.

Proposition 11. *For any $r > 0$, $\lambda > 0$ and $M > 0$, $W_F(\Delta)$ strictly decreases in Δ whenever $\Delta \geq 2/M$. Thus, the optimal Δ^*_F that maximizes $W_F(\Delta)$ satisfies $\Delta^*_F \leq 2/M$.*

Proposition 11 reveals that the fast trader's preferred trading frequency allows no more than two slow traders in each round. If we impose the natural restriction that $\Delta^*_F M$ must be an integer, then there are exactly two slow traders in each round. (Recall that Proposition 9 requires $n_S > 1$, so a linear equilibrium with one fast trader and one slow trader is infeasible.[25]) Intuitively, a fast trader prefers a high-frequency (and thin) market because he obtains a higher rent by intermediating trades among slow traders across time. Note that because the fast trader's preferred trading

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[25]In models of double auctions, a linear equilibrium typically does not exist with two traders (see also Kyle (1989), Vives (2011) and Rostek and Weretka (2012)).

frequency is already as high as feasible, the slow traders' preferred frequency must be weakly lower than the fast trader's preferred frequency.

Pinning down the slow traders' preferred trading frequency analytically turns out to be more difficult. Nonetheless, there is a clear and intuitive tradeoff. On the one hand, a higher trading frequency allows slow traders to quickly realize the gains from trade induced by private values. On the other hand, a higher trading frequency reduces the number of traders in each round, which makes the market thinner and increases the price-impact costs. The optimal trading frequency for slow traders should strike the best balance between the two effects.

Figure 7 shows the fast and slow traders' welfare as functions of Δ. In the left-hand graph of Figure 7 we plot $W_F(\Delta)$ and $W_S(\Delta)/M$ for $r = 0.5$, $M = 10$ and $\lambda = 1$. As predicted by Proposition 11, the fast trader's welfare $W_F(\Delta)$ peaks at a $\Delta < 0.2$ and rapidly decreases with Δ if $\Delta > 0.2$. The slow traders' welfare, $W_S(\Delta)$, peaks around $\Delta = 0.67$, implying 6 or 7 slow traders in each round, and decreases slowly with Δ when $\Delta > 0.67$. The slow traders clearly prefer a lower trading frequency than the fast trader does.

In the right-hand graph of Figure 7 we plot $W_S(\Delta)/M$ for different values of r, fixing $M = 10$ and $\lambda = 1$. The graph shows that the Δ_S^* that maximizes $W_S(\Delta)$ increases as r decreases. If $r = 1$, Δ_S^* is around 0.5, implying about 5 slow traders in each round; if $r = 0.05$, Δ_S^* is around 1.9, implying about 19 slow traders in each round. As intuition suggests, the more imminent is the dividend payment (a higher r), the more costly it is for the slow traders to delay trades, and the higher is the preferred trading frequency of slow traders.

Given the different optimal speeds preferred by the fast trader and the slow ones, a natural question arises: how strong is the fast trader's incentive to speed up the market if he is able to? As a proxy for this incentive, we plot in Figure 8 the ratio $W_F(\Delta_F^*)/W_F(\Delta_S^*)$ and the difference $W_F(\Delta_F^*) - W_F(\Delta_S^*)$ as functions of r (two left-hand plots) and M (two right-hand plots). They measure the fast trader's proportional and absolute increase in utility if the market speeds up from the slow traders' preferred frequency Δ_S^* to the fast trader's preferred frequency Δ_F^*. The patterns are intuitive: speeding up the market gives the fast trader more benefit if the asset is expected to trade for a longer time (a smaller r) or if slow traders arrive at a higher rate (a larger M). Both effects increase the fast trader's rents from intermediating trades among the slow traders across time.

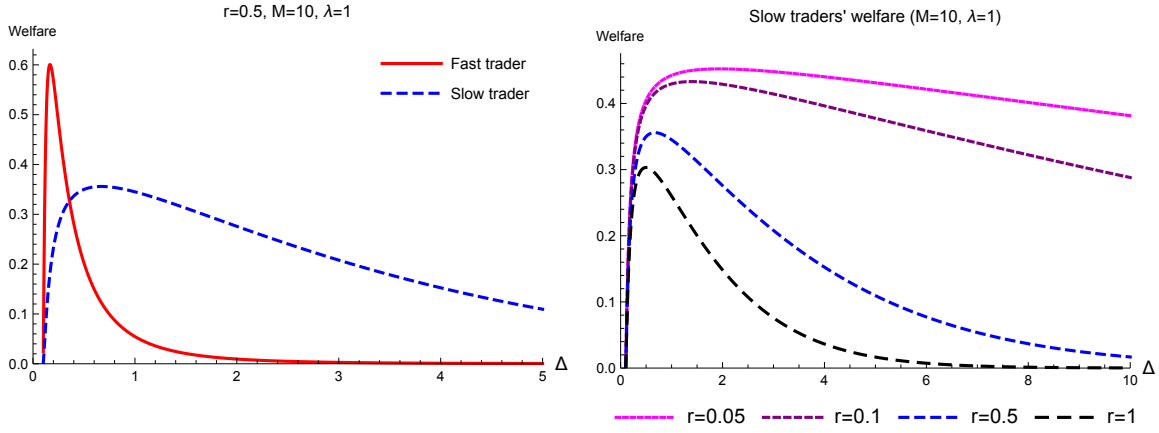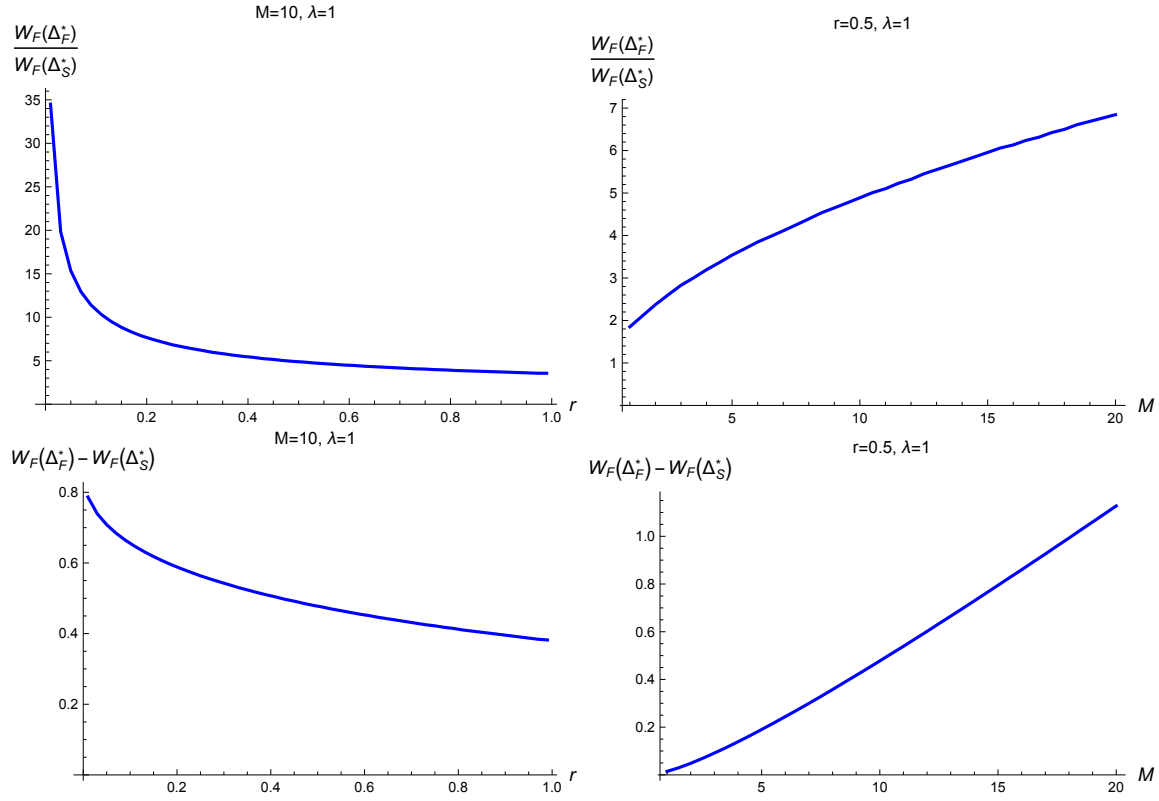Figure 7: $W_F(\Delta)$ and $W_S(\Delta)/M$ as functions of Δ

Figure 8: Incentives for the fast trader to speed up the market

5   Conclusion

In this paper, we study a dynamic model in which a finite number of traders receive private information over time and trade strategically with demand schedules in a sequence of double auctions. We characterize a stationary linear equilibrium in closed form. The equilibrium price aggregates a weighted sum of the common value information and private values information, but the two components cannot be separated from the price. Due to imperfect competition, the equilibrium allocation is not fully efficient, but it converges to the efficient allocation exponentially over time. The presence of adverse selection—asymmetric information regarding the common-value component of the asset—slows down this convergence speed. As the number n of traders increases, the asymptotic convergence rate to efficiency is of order $n^{-4/3}$ with adverse selection and of order n^{-2} without adverse selection; the corresponding rates in the continuous-time limit are of orders $n^{-2/3}$ and n^{-1}, respectively.

We use this modeling framework to study the optimal trading frequency that maximizes welfare. Trading frequency is measured as the number of double auctions per unit of clock time. A higher trading frequency reduces the aggressiveness of demand schedules, but allows more immediate reactions to new information. If traders have homogeneous trading speed, slow trading tends to be optimal for scheduled information arrivals but fast trading tends to be optimal for stochastic information arrivals. Moreover, the optimal trading frequency under stochastic information arrivals is higher if the arrival rate of fundamental information is higher. If traders have heterogeneous speeds (i.e., different frequencies of accessing the market), the fast trader extracts rents by intermediating trades among slow traders across time. As a result, the fast trader prefers the highest feasible trading frequency, whereas slow traders tend to prefer a strictly lower frequency.

A   List of Model Variables

Variable	Explanation
Exogenous Variables	
t	Discrete trading period , $t \in \{0,1,2,3,\dots\}$
τ	Continuous clock time, $\tau \in [0,\infty)$
Δ	Length of each trading period
$\mathcal{T}, r$	The clock time $\mathcal{T}$ of dividend payment has an exponential distribution with intensity $r > 0$.
$\{T_k\}_{k\in\{0,1,2,\dots\}}$	Times of shocks to the common dividend and private values
D_{T_k}	The common dividend value immediately after the k-th shock
σ_D^2	Each dividend shock $D_{T_k} - D_{T_{k-1}}$ has the distribution $\mathcal{N}(0,\sigma_D^2)$.
S_{i,T_k}	Trader i's signal of the k-th dividend shock
σ_ϵ^2	The noise in trader i's dividend signal regarding the k-th dividend shock, $S_{i,T_k} - (D_{T_k} - D_{T_{k-1}})$, has the distribution $\mathcal{N}(0,\sigma_\epsilon^2)$.
w_{i,T_k}	Trader i's private value for the asset immediately after the k-th shock
σ_w^2	Shocks to each trader i's private value, $w_{i,T_k} - w_{i,T_{k-1}}$, has the distribution $\mathcal{N}(0,\sigma_w^2)$.
λ	Before the dividend is paid, the flow cost for holding asset position q is $0.5\lambda q^2$ per unit of clock time for each trader.
Z	The total inventory held by all traders, $Z \equiv \sum_{1\le j\le n} z_{j,0}$
γ	Time interval of scheduled information arrivals
μ	Intensity of stochastic information arrivals
Endogenous Variables	
$z_{i,t\Delta}$	Trader i's inventory level right before the period-t double auction
$x_{i,t\Delta}(p)$	Trader i's demand schedule in the period-t double auction
$p_{t\Delta}^*$	The equilibrium price in period-t double auction
$H_{i,t\Delta}$	Trader i's history (information set) up to time $t\Delta$ but before the period-t double auction, defined in (9)
s_{i,T_k}	Trader i's total signal right after the k-th shock, defined in (14)
$v_{i,t\Delta}$	Trader i's expected value for the asset, evaluated right after the period-t double auction, given $H_{i,t\Delta} \cup \{H_{j,t\Delta}\}_{j\neq i}$
$V_{i,t\Delta}$	The period-t ex post utility of trader i, evaluated right after the period-t double auction, given $H_{i,t\Delta} \cup \{H_{j,t\Delta}\}_{j\neq i}$
χ, α	Constants defined before Proposition 1
$z_{i,(t+1)\Delta}^c$	The competitive allocation immediately after period-t auction
$z_{i,\tau}^e$	The continuous-time version of competitive allocation
σ_z^2, σ_0^2	Constants defined in Equations (48) and (49)
$W(\Delta)$	Welfare under homogeneous speed and trading interval Δ
$W_F(\Delta)$	The fast trader's welfare under heterogeneous speeds
$W_S(\Delta)$	Slow traders' welfare under heterogeneous speeds
Δ_F^*, Δ_S^*	The fast and slow traders' preferred frequencies under heterogeneous speeds

B   Proofs

B.1   Proof of Proposition 1

Before constructing the equilibrium, we first prove that our construction of total signal in (14) implies Equation (21).

Lemma 1. *For any constant x, we have:*

$$\mathbb{E}\left[v_{i,T_k}\Big|H_{i,T_k}\cup\left\{\sum_{j\neq i}\left(x\sum_{l=0}^{k}S_{j,T_l}+w_{j,T_k}\right)\right\}\right] \tag{67}$$

$$=w_{i,T_k}+\frac{1/(x^2\sigma_\epsilon^2)}{1/(x^2\sigma_D^2)+1/(x^2\sigma_\epsilon^2)+(n-1)/(x^2\sigma_\epsilon^2+\sigma_w^2)}\sum_{l=0}^{k}S_{i,T_l}$$

$$+\frac{1/(x^2\sigma_\epsilon^2+\sigma_w^2)}{1/(x^2\sigma_D^2)+1/(x^2\sigma_\epsilon^2)+(n-1)/(x^2\sigma_\epsilon^2+\sigma_w^2)}\cdot\frac{1}{x}\left(\sum_{j\neq i}\left(x\sum_{l=0}^{k}S_{j,T_l}+w_{j,T_k}\right)\right).$$

Proof. Define

$$\tilde{S}_{i,T_l}\equiv xS_{i,T_l}+w_{i,T_l}-w_{i,T_{l-1}}. \tag{68}$$

By the projection theorem for multivariate normal distribution:

$$\mathbb{E}\left[D_{T_l}-D_{T_{l-1}}\mid S_{i,T_l},\sum_{j\neq i}\tilde{S}_{j,T_l}\right] \tag{69}$$

$$=(x\sigma_D^2,(n-1)x\sigma_D^2)\cdot\begin{pmatrix}x^2(\sigma_D^2+\sigma_\epsilon^2)&(n-1)x^2\sigma_D^2\\(n-1)x^2\sigma_D^2&(n-1)(x^2(\sigma_D^2+\sigma_\epsilon^2)+\sigma_w^2)+(n-1)(n-2)x^2\sigma_D^2\end{pmatrix}^{-1}$$

$$\cdot\left(xS_{i,T_l},\sum_{j\neq i}\tilde{S}_{j,T_l}\right)'.$$

We compute:

$$\begin{pmatrix}x^2(\sigma_D^2+\sigma_\epsilon^2)&(n-1)x^2\sigma_D^2\\(n-1)x^2\sigma_D^2&(n-1)(x^2(\sigma_D^2+\sigma_\epsilon^2)+\sigma_w^2)+(n-1)(n-2)x^2\sigma_D^2\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}(n-1)(x^2\sigma_\epsilon^2+\sigma_w^2)+(n-1)^2x^2\sigma_D^2&-(n-1)x^2\sigma_D^2\\-(n-1)x^2\sigma_D^2&x^2(\sigma_D^2+\sigma_\epsilon^2)\end{pmatrix}$$

$$\cdot\frac{1}{(n-1)x^2(x^2\sigma_\epsilon^2+\sigma_w^2)(\sigma_D^2+\sigma_\epsilon^2)+(n-1)^2x^4\sigma_D^2\sigma_\epsilon^2},$$

and

$$\mathbb{E}\left[D_{T_l} - D_{T_{l-1}} \mid S_{i,T_l}, \sum_{j \neq i} \tilde{S}_{j,T_l}\right]$$

$$= \frac{(n-1)x^2\sigma_D^2(x^2\sigma_\epsilon^2 + \sigma_w^2)S_{i,T_l} + (n-1)x^3\sigma_D^2\sigma_\epsilon^2 \sum_{j \neq i} \tilde{S}_{j,T_l}}{(n-1)x^2(x^2\sigma_\epsilon^2 + \sigma_w^2)(\sigma_D^2 + \sigma_\epsilon^2) + (n-1)^2 x^4 \sigma_D^2 \sigma_\epsilon^2}$$

$$= \frac{(1/x^2\sigma_\epsilon^2)S_{i,T_l} + (1/(x^2\sigma_\epsilon^2 + \sigma_w^2))\frac{1}{x}\sum_{j \neq i} \tilde{S}_{j,T_l}}{1/(x^2\sigma_\epsilon^2) + 1/(x^2\sigma_D^2) + (n-1)/(x^2\sigma_\epsilon^2 + \sigma_w^2)}.$$

Summing the above equation across $l \in \{0, 1, \ldots, k\}$ and adding w_{i,T_k} gives Equation (67). $\qquad \square$

By Equation (67), we have

$$\mathbb{E}\left[v_{i,T_k} \Big| H_{i,T_k} \cup \left\{\sum_{j \neq i}\left(\chi \sum_{l=0}^{k} S_{j,T_l} + w_{j,T_k}\right)\right\}\right]$$

$$= w_{i,T_k} + \chi \sum_{l=0}^{k} S_{i,T_l} + \frac{1/(\chi^2\sigma_\epsilon^2 + \sigma_w^2)}{1/(\chi^2\sigma_D^2) + 1/(\chi^2\sigma_\epsilon^2) + (n-1)/(\chi^2\sigma_\epsilon^2 + \sigma_w^2)} \cdot \frac{1}{\chi}\left(\sum_{j \neq i}\left(\chi \sum_{l=0}^{k} S_{j,T_l} + w_{j,T_k}\right)\right)$$

$$= \alpha s_{i,T_k} + \frac{1-\alpha}{n-1}\sum_{j \neq i} s_{j,T_k},$$

where in the second line we used the definition of χ in Equation (15), and in the third line we used the definition of s_{i,T_k} in (14), and the definition of α:

$$\alpha \equiv \frac{1}{1 + \frac{(n-1)/(\chi^2\sigma_\epsilon^2 + \sigma_w^2)}{1/(\chi^2\sigma_D^2) + 1/(\chi^2\sigma_\epsilon^2) + (n-1)/(\chi^2\sigma_\epsilon^2 + \sigma_w^2)} \cdot \frac{1}{\chi}} = \frac{\chi^2\sigma_\epsilon^2 + \sigma_w^2}{n\chi^2\sigma_\epsilon^2 + \sigma_w^2}. \tag{70}$$

We conjecture that traders use the following linear, symmetric and stationary strategy:

$$x_{j,t\Delta}(p; s_{j,t\Delta}, z_{j,t\Delta}) = as_{j,t\Delta} - bp + dz_{j,t\Delta} + fZ. \tag{71}$$

This conjecture implies the market-clearing prices of

$$p_{t\Delta}^* = \frac{a}{nb}\sum_{j=1}^{n} s_{j,t\Delta} + \frac{d + nf}{nb}Z. \tag{72}$$

Fix a history $H_{i,t\Delta}$ and a realization of $\sum_{j \neq i} s_{j,t\Delta}$. We use the single-deviation principle to construct an equilibrium strategy (71): under the conjecture that other traders $j \neq i$ use strategy (71) in every period $t' \geq t$, and that trader i returns to strategy (71) in period $t' \geq t + 1$, we verify that trader i has no incentive to deviate from strategy (71) in period t.[26]

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[26]For a description of the single-deviation principle, also called "one-stage deviation principle",

If trader i uses an alternative demand schedule in period t, he faces the residual demand $-\sum_{j\neq i} x_{j,t\Delta}(p_{t\Delta})$ and is effectively choosing a price $p_{t\Delta}$ and getting $x_{i,t\Delta}(p_{t\Delta}) = -\sum_{j\neq i} x_{j,t\Delta}(p_{t\Delta})$. Therefore, by differentiating

$$\mathbb{E}\left[V_{i,t\Delta} \mid H_{i,t\Delta} \cup \left\{\sum_{j\neq i} s_{j,t\Delta}\right\}\right]$$

where $V_{i,t\Delta}$ is defined in (12), with respect to $p_{t\Delta}$ and evaluating it at $p_{t\Delta} = p_{t\Delta}^*$ in (72), we obtain the following first order condition in period t of trader i:

$$\mathbb{E}\Bigg[(n-1)b \cdot \Bigg((1 - e^{-r\Delta}) \sum_{k=0}^{\infty} e^{-rk\Delta} \frac{\partial(z_{i,(t+k)\Delta} + x_{i,(t+k)\Delta})}{\partial x_{i,t\Delta}} \left(v_{i,(t+k)\Delta} - \frac{\lambda}{r}(z_{i,(t+k)\Delta} + x_{i,(t+k)\Delta})\right)$$
$$- \sum_{k=0}^{\infty} e^{-rk\Delta} \frac{\partial x_{i,t\Delta}}{\partial x_{i,0}} p_{t\Delta}^*\Bigg) - \sum_{k=0}^{\infty} e^{-rk\Delta} x_{i,(t+k)\Delta} \frac{\partial p_{(t+k)\Delta}^*}{\partial p_{t\Delta}} \Bigg| H_{i,t\Delta} \cup \left\{\sum_{j\neq i} s_{j,t\Delta}\right\}\Bigg] = 0,$$
$$(73)$$

where we write $x_{i,(t+k)\Delta} = x_{i,(t+k)\Delta}(p_{(t+k)\Delta}^*; s_{i,(t+k)\Delta}, z_{i,(t+k)\Delta})$ for the strategy $x_{i,(t+k)\Delta}(\cdot)$ defined in (71), and by definition $z_{i,(t+k+1)\Delta} = z_{i,(t+k)\Delta} + x_{i,(t+k)\Delta}$.

Since all traders follow the conjectured strategy in (71) from period $t+1$ and onwards, we have the following evolution of inventories: for any $k \geq 1$,

$$z_{i,(t+k)\Delta} + x_{i,(t+k)\Delta} = as_{i,(t+k)\Delta} - bp_{(t+k)\Delta}^* + fZ + (1+d)z_{i,(t+k)\Delta} \tag{74}$$
$$= (as_{i,(t+k)\Delta} - bp_{(t+k)\Delta}^* + fZ) + (1+d)(as_{i,(t+k-1)\Delta} - bp_{(t+k-1)\Delta}^* + fZ)$$
$$+ \cdots + (1+d)^{k-1}(as_{i,(t+1)\Delta} - bp_{(t+1)\Delta}^* + fZ) + (1+d)^k(x_{i,t\Delta} + z_{i,t\Delta}).$$

The evolution of prices and inventories, given by Equations (72) and (74), reveals that by changing the demand or price in period t, trader i has the following effects on inventories and prices in period $t+k$, $k \geq 1$:

$$\frac{\partial(z_{i,(t+k)\Delta} + x_{i,(t+k)\Delta})}{\partial x_{i,t\Delta}} = (1+d)^k, \tag{75}$$

$$\frac{\partial x_{i,(t+k)\Delta}}{\partial x_{i,t\Delta}} = (1+d)^{k-1}d, \tag{76}$$

$$\frac{\partial p_{(t+k)\Delta}^*}{\partial p_{t\Delta}} = \frac{\partial p_{(t+k)\Delta}^*}{\partial x_{i,t\Delta}} = 0. \tag{77}$$

As we verify later, the equilibrium value of d satisfies $-1 < d < 0$, so the partial derivatives (75) and (76) converge.

---

see Theorem 4.1 and 4.2 of Fudenberg and Tirole (1991). We can apply their Theorem 4.2 because the payoff function in our model, which takes the form of a "discounted" sum of period-by-period payoffs, satisfies the required "continuity at infinity" condition.

The first order condition in (73) simplifies to:

$$\mathbb{E}\left[(n-1)b\left((1-e^{-r\Delta})\sum_{k=0}^{\infty}e^{-rk\Delta}(1+d)^k\left(v_{i,(t+k)\Delta}-\frac{\lambda}{r}(z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta})\right)\right.\right.$$
$$\left.\left.-p_{t\Delta}^*-\sum_{k=1}^{\infty}e^{-rk\Delta}(1+d)^{k-1}d\,p_{(t+k)\Delta}^*\right)-x_{i,t\Delta}\,\middle|\,H_{i,t\Delta}\cup\left\{\sum_{j\neq i}s_{j,t\Delta}\right\}\right]=0,$$

(78)

where we have (cf. Lemma 1, Equations (72) and (74)):

$$\mathbb{E}[p_{i,(t+k)\Delta}^*\mid H_{i,t\Delta}\cup\{s_{j,\tau}\}_{j\neq i,0\leq\tau\leq t\Delta}]=p_{t\Delta}^*,$$

(79)

$$\mathbb{E}[v_{i,(t+k)\Delta}\mid H_{i,t\Delta}\cup\{s_{j,\tau}\}_{j\neq i,0\leq\tau\leq t\Delta}]=\mathbb{E}\left[v_{i,t\Delta}\mid H_{i,t\Delta}\cup\left\{\sum_{j\neq i}s_{j,t\Delta}\right\}\right]$$
$$=\alpha s_{i,t\Delta}+\frac{1-\alpha}{n-1}\sum_{j\neq i}s_{j,t\Delta},$$

(80)

$$\mathbb{E}[z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta}\mid H_{i,t\Delta}\cup\{s_{j,\tau}\}_{j\neq i,0\leq\tau\leq t\Delta}]$$
$$=(as_{i,t\Delta}-bp_{t\Delta}^*+fZ)\left(\frac{1}{-d}-\frac{(1+d)^k}{-d}\right)+(1+d)^k(x_{i,t\Delta}+z_{i,t\Delta}).$$

(81)

Substituting Equations (72), (79), (80) and (81) into the first-order condition (78) and using the notation $\bar{s}_{t\Delta}=\sum_{1\leq j\leq n}s_{j,t\Delta}/n$, we get:

$$(n-1)b(1-e^{-r\Delta})\left[\frac{1}{1-e^{-r\Delta}(1+d)}\left(\alpha s_{i,t\Delta}+\frac{1-\alpha}{n-1}\sum_{j\neq i}s_{j,t\Delta}-\left(\frac{a}{b}\bar{s}_{t\Delta}+\frac{d+nf}{nb}Z\right)\right)\right.$$
$$-\sum_{k=0}^{\infty}\frac{\lambda}{r}e^{-rk\Delta}(1+d)^k\left(\frac{1}{-d}-\frac{(1+d)^k}{-d}\right)\left(as_{i,t\Delta}-b\left(\frac{a}{b}\bar{s}_{t\Delta}+\frac{d+nf}{nb}Z\right)+fZ\right)$$
$$\left.-\frac{\lambda}{(1-e^{-r\Delta}(1+d)^2)r}(x_{i,t\Delta}+z_{i,t\Delta})\right]-x_{i,t\Delta}=0.$$

(82)

Rearranging the terms gives:

$$\left(1 + \frac{(n-1)b(1-e^{-r\Delta})\lambda}{(1-e^{-r\Delta}(1+d)^2)r}\right)x_{i,t\Delta} \tag{83}$$

$$= (n-1)b(1-e^{-r\Delta})\left[\frac{1}{1-e^{-r\Delta}(1+d)}\left(\frac{n\alpha-1}{n-1}s_{i,t\Delta} + \frac{n-n\alpha}{n-1}\bar{s}_{t\Delta} - \frac{a}{b}\bar{s}_{t\Delta}\right)\right.$$

$$-\frac{\lambda e^{-r\Delta}(1+d)}{r(1-(1+d)e^{-r\Delta})(1-(1+d)^2e^{-r\Delta})}a(s_{i,t\Delta} - \bar{s}_{t\Delta})$$

$$-\frac{\lambda}{(1-e^{-r\Delta}(1+d)^2)r}z_{i,t\Delta}$$

$$\left. -\left(\frac{1}{1-e^{-r\Delta}(1+d)}\left(\frac{d+nf}{nb} + \frac{\lambda}{rn}\right) - \frac{\lambda}{(1-(1+d)^2e^{-r\Delta})nr}\right)Z\right].$$

On the other hand, substituting Equation (72) into the conjectured strategy (71) gives:

$$x_{i,t\Delta} = a(s_{i,0} - \bar{s}_0) + dz_{i,0} - \frac{d}{n}Z. \tag{84}$$

We match the coefficients in Equation (84) with those in Equation (83). First of all, we clearly have

$$a = b. \tag{85}$$

We also obtain two equations for b and d:

$$\left(1 + \frac{(n-1)b(1-e^{-r\Delta})\lambda}{(1-e^{-r\Delta}(1+d)^2)r}\right) = \frac{(1-e^{-r\Delta})(n\alpha-1)}{1-e^{-r\Delta}(1+d)} - \frac{(n-1)b(1-e^{-r\Delta})e^{-r\Delta}(1+d)\lambda}{(1-(1+d)e^{-r\Delta})(1-(1+d)^2e^{-r\Delta})r},$$

$$\left(1 + \frac{(n-1)b(1-e^{-r\Delta})\lambda}{(1-e^{-r\Delta}(1+d)^2)r}\right)d = -\frac{(n-1)b(1-e^{-r\Delta})\lambda}{(1-e^{-r\Delta}(1+d)^2)r}. \tag{86}$$

There are two solutions to the above system of equations. One of them leads to unbounded inventories, so we drop it.[27] The other solution leads to converging inventories and is given by

$$b = \frac{(n\alpha-1)r}{2(n-1)e^{-r\Delta}\lambda}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right), \tag{87}$$

$$d = -\frac{1}{2e^{-r\Delta}}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right). \tag{88}$$

---

[27] This dropped solution to (86) has the property of $(1+d)e^{-r\Delta} < -1$, which leads to an unbounded path of inventories (cf. Equation (74)) and utilities.

Lastly, matching the coefficient of Z gives:

$$f = -\frac{d}{n} - \frac{b\lambda}{nr}. \tag{89}$$

Under the condition $n\alpha > 2$, we can show that $b > 0$ and $-1 < d < 0$, that is, the demand schedule is downward-sloping in price and the inventories evolutions (75)–(76) converge.

By Equation (15), the condition $n\alpha > 2$ is equivalent to the condition

$$\chi^2 < \frac{(n-2)\sigma_w^2}{n\sigma_\epsilon^2} \tag{90}$$

which is equivalent to the following condition on the fundamentals:

$$\frac{1}{n/2 + \sigma_\epsilon^2/\sigma_D^2} < \sqrt{\frac{n-2}{n}}\frac{\sigma_w}{\sigma_\epsilon}. \tag{91}$$

Finally, we verify the second-order condition. Under the linear strategy in (71) with $b > 0$, differentiating the first-order condition (73) with respect to p_0 gives

$$(n-1)b(1-e^{-r\Delta})\left(-\frac{\lambda}{r}(n-1)b\sum_{k=0}^{\infty}e^{-rk\Delta}(1+d)^{2k} - 1\right) - (n-1)b < 0. \tag{92}$$

This completes the construction of a perfect Bayesian equilibrium.

B.2   Proof of Proposition 2

Suppose that every trader i use the strategy:

$$x_{i,t\Delta}(p) = \sum_{T_l \leq t\Delta} a_l S_{i,T_l} + a_w w_{i,t\Delta} - bp + dz_{i,t\Delta} + f, \tag{93}$$

where $\{a_l\}_{l\geq 0}, a_w, b, d$ and f are constants. We show that for everyone using (93) to be a perfect Bayesian equilibrium (PBE), the constants must be the ones given by Proposition 1. We divide our arguments into two steps.

Step 1. Define $x_l \equiv a_l/a_w$. [28] As a first step, we show that if (93) is a symmetric PBE, then we must have $x_l = \chi$ for every l, where χ is defined in Equation (15).

Suppose that $(t-1)\Delta \in [T_{k'}, T_{k'+1})$ and $t\Delta \in [T_k, T_{k+1})$, so there are $k - k' \geq 1$ dividend shocks between time $(t-1)\Delta$ and time $t\Delta$. [29] Without loss of generality, assume $k' = 0$. Since all other traders $j \neq i$ are using strategy (93), by computing

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[28] Clearly, we cannot have $a_w = 0$, since players use their private values in any equilibrium.

[29] In period $t = 0$, we take $D_{T_{-1}} = w_{i,T_{-1}} = 0$.

the difference $p_{t\Delta}^* - p_{(t-1)\Delta}^*$, trader i can infer from the period-t price the value of

$$\sum_{j\neq i}\sum_{l=1}^{k} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}.$$

By the projection theorem for normal distribution, we have

$$\mathbb{E}\left[D_{T_k} - D_{T_0} \mid H_{i,t\Delta} \cup \left\{\sum_{j\neq i}\sum_{T_l \leq t\Delta} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}\right\}\right] \tag{94}$$

$$= \mathbb{E}\left[D_{T_k} - D_{T_0} \mid \{S_{i,T_l}\}_{l=1}^{k} \cup \left\{\sum_{j\neq i}\sum_{l=1}^{k} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}\right\}\right]$$

$$= \mathbf{u}\,\boldsymbol{\Sigma}^{-1} \cdot \left(S_{i,T_1}, \ldots, S_{i,T_k}, \sum_{j\neq i}\sum_{l=1}^{k} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}\right)',$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of $\left(S_{i,T_1}, \ldots, S_{i,T_k}, \sum_{j\neq i}\sum_{l=1}^{k} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}\right)$:
for $1 \leq l \leq k+1$ and $1 \leq m \leq k+1$,

$$\boldsymbol{\Sigma}_{l,m} = \begin{cases} \sigma_D^2 + \sigma_\epsilon^2 & 1 \leq l = m \leq k \\ 0 & 1 \leq l \neq m \leq k \\ (n-1)(\sum_{l'=1}^{k} x_{l'}^2)(\sigma_D^2 + \sigma_\epsilon^2) + (n-1)k\sigma_w^2 & l = m = k+1 \\ \quad +(n-1)(n-2)\sum_{l'=1}^{k} x_{l'}^2 \sigma_D^2 & \\ (n-1)x_l \sigma_D^2 & 1 \leq l \leq k, m = k+1 \end{cases}, \tag{95}$$

and $\boldsymbol{\Sigma}_{k+1,l} = \boldsymbol{\Sigma}_{l,k+1}$. And $\mathbf{u}$ is a row vector of covariances between
$\left(S_{i,T_1}, \ldots, S_{i,T_k}, \sum_{j\neq i}\sum_{l=1}^{k} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}\right)$ and $D_{T_k} - D_{T_0}$:

$$\mathbf{u} = (\sigma_D^2, \ldots, \sigma_D^2, (n-1)\sum_{l=1}^{k} x_l \sigma_D^2). \tag{96}$$

Therefore, we have

$$\mathbb{E}\left[v_{i,t\Delta} \mid H_{i,t\Delta} \cup \left\{\sum_{j\neq i}\sum_{T_l \leq t\Delta} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}\right\}\right] \tag{97}$$

$$= w_{i,T_k} + \mathbb{E}[D_{T_0} \mid \{S_{i,T_0}\} \cup \left\{\sum_{j\neq i} x_0 S_{j,T_0} + w_{j,0}\right\}]$$

$$+ \mathbf{u}\,\boldsymbol{\Sigma}^{-1} \cdot \left(S_{i,T_1}, \ldots, S_{i,T_k}, \sum_{j\neq i}\sum_{l=1}^{k} x_l S_{j,T_l} + w_{j,T_l} - w_{j,T_{l-1}}\right)'.$$

Since we look for a symmetric equilibrium in which everyone plays (93), trader i's
conditional value in (97) must place a weight of x_l on S_{i,T_l}, $1 \leq l \leq k$, which implies
that

$$\mathbf{u}\,\boldsymbol{\Sigma}^{-1} = \mathbf{x}, \tag{98}$$

where $\mathbf{x} = (x_1, \ldots, x_k, y)$ and y is an arbitrary number. Clearly, Equation (98) is equivalent to

$$\mathbf{u} = \mathbf{x}\,\mathbf{\Sigma},$$

which implies (from the first k entries of the row vector)

$$\sigma_D^2 = x_l(\sigma_D^2 + \sigma_\epsilon^2) + y(n-1)x_l\sigma_D^2, \quad 1 \le l \le k,$$

i.e.,

$$x_1 = \cdots = x_k = \frac{\sigma_D^2}{\sigma_D^2 + \sigma_\epsilon^2 + y(n-1)\sigma_D^2}.$$

Now define $x \equiv x_1 = \cdots = x_k$. Applying Lemma 1 to the conditional value in (97) implies that for the conditional value in (97) to place a weight of x on S_{i,T_l}, $1 \le l \le k$, we must have $x = \chi$.

Step 2. Given Step 1, we can rewrite the strategy (93) as

$$x_{i,t\Delta}(p) = a_w \cdot \alpha\, s_{i,t\Delta} - bp + dz_{i,t\Delta} + f, \tag{99}$$

where $s_{i,t\Delta}$ is the total signal defined in (14) and α is defined in (16). The equilibrium construction in Section B.1 then uniquely determines the values of a_w, b, d and f. This concludes the proof of Proposition 2.

B.3 Proof of Proposition 3

We first prove the convergence to competitive allocation. Conditional on the total signals staying the same from period $\underline{t}$ to $\bar{t}$, the efficient allocation in each of these periods is also the same and is given by

$$z_{i,(\underline{t}+1)\Delta}^c = \frac{r(n\alpha - 1)}{\lambda(n-1)}\left(s_{i,\underline{t}\Delta} - \frac{1}{n}\sum_{j=1}^n s_{j,\underline{t}\Delta}\right) + \frac{1}{n}Z. \tag{100}$$

We rewrite the perfect Bayesian equilibrium strategy (18) as

$$x_{i,\underline{t}\Delta}(p; s_{i,\underline{t}\Delta}, z_{i,\underline{t}\Delta}) = as_{i,\underline{t}\Delta} - bp + dz_{i,\underline{t}\Delta} + fZ. \tag{101}$$

It is easy to verify that at the competitive allocation $z_{i,(\underline{t}+1)\Delta}^c$, trader i trades zero unit in equilibrium:

$$x_{i,\underline{t}\Delta}(p_{\underline{t}\Delta}^*; s_{i,\underline{t}\Delta}, z_{i,(\underline{t}+1)\Delta}^c) = 0; \tag{102}$$

one way to see this is to note that the perfect Bayesian equilibrium strategy is a scaled version of the competitive equilibrium strategy (Equation (32)).

That is, for every $i \in \{1, 2, \ldots, n\}$,

$$z^c_{i,(\underline{t}+1)\Delta} = \frac{as_{i,t\Delta} - bp^*_{i,t\Delta} + fZ}{-d}. \tag{103}$$

By definition, we have

$$\begin{aligned}
z^*_{i,(\underline{t}+1)\Delta} &= z^*_{i,t\Delta} + x_{i,t\Delta}(p^*_{i,t\Delta}; s_{i,t\Delta}, z^*_{i,t\Delta}) \\
&= as_{i,t\Delta} - bp^*_{i,t\Delta} + (1+d)z^*_{i,t\Delta} + fZ \\
&= (-d)z^c_{i,(\underline{t}+1)\Delta} + (1+d)z^*_{i,t\Delta},
\end{aligned}$$

where the last equality follows from (103). This proves (34) for $\bar{t} = \underline{t}$. The case of $\bar{t} > \underline{t}$ follows by induction.

Now we prove the comparative statics. We write

$$1 + d = \frac{1}{2e^{-r\Delta}} \left(\sqrt{(n\alpha - 1)^2(1 - e^{-r\Delta})^2 + 4e^{-r\Delta}} - (n\alpha - 1)(1 - e^{-r\Delta}) \right), \tag{104}$$

and

$$\eta \equiv n\alpha - 1. \tag{105}$$

We first note that $\frac{\partial(1+d)}{\partial \eta} < 0$.

1. The comparative statics with respect to r follow by straightforward calculations showing that $\frac{\partial(1+d)}{\partial r} < 0$.

2. As σ_D^2 increases, the left-hand side of Equation (15) increases, and hence the solution χ to (15) increases, which means that $n\alpha$ decreases because according to Equation (16) $n\alpha$ is a decreasing function of χ^2. Thus, $\frac{\partial \eta}{\partial \sigma_D^2} < 0$, and $\frac{\partial(1+d)}{\partial \sigma_D^2} > 0$.

3. As σ_w^2 increases, the left-hand side of Equation (15) increases, and hence the solution χ to (15) increases; by (15) this means that σ_w^2/χ^2 must increase as well. Thus, $n\alpha$ increases because according to Equation (16) $n\alpha$ is an increasing function of σ_w^2/χ^2. Hence, $\frac{\partial \eta}{\partial \sigma_w^2} > 0$ and $\frac{\partial(1+d)}{\partial \sigma_w^2} < 0$.

4. We can rewrite Equation (15) as

$$\frac{1}{\frac{1}{\alpha} + \frac{\sigma_\epsilon^2}{\sigma_D^2}} = \chi, \tag{106}$$

and Equation (16) as

$$\chi = \sqrt{\frac{1 - \alpha}{n\alpha - 1}} \frac{\sigma_w}{\sigma_\epsilon}, \tag{107}$$

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and hence

$$\frac{1}{\frac{n}{\eta+1} + \frac{\sigma_\epsilon^2}{\sigma_D^2}} = \sqrt{\frac{n-\eta-1}{n\eta}} \frac{\sigma_w}{\sigma_\epsilon}. \tag{108}$$

From Equation (108) is it straightforward to show that η must increase with n. Thus, $1 + d$ decreases in n.

5. For the comparative statics with respect to Δ, we find that

$$\frac{\partial(\log(1+d)/\Delta)}{\partial\Delta} = -\frac{1}{\Delta^2}\left(r\Delta \frac{\eta\sqrt{\eta^2(e^{r\Delta}-1)^2 + 4e^{r\Delta}} - \eta^2(e^{r\Delta}-1) - 2}{\sqrt{\eta^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\left(\sqrt{\eta^2(e^{r\Delta}-1)^2 + 4e^{r\Delta}} - \eta(e^{r\Delta}-1)\right)}\right.$$

$$\left. + \log\left(\frac{1}{2}\left(\sqrt{\eta^2(e^{r\Delta}-1)^2 + 4e^{r\Delta}} - \eta(e^{r\Delta}-1)\right)\right)\right) > 0.$$

B.4 Proofs of Propositions 5, 6, 7 and 8

We first establish some general properties of the equilibrium welfare, before specializing to the rate that inefficiency vanishes as $n \to \infty$ (Section B.4.1), and to the optimal trading frequency given scheduled (Section B.4.2 and Section B.4.3) and stochastic (Section B.4.4) arrivals of new information.

The following lemma relates the amount of inefficiency associated with an inventory allocation to the square distance between that allocation and the competitive allocation:

Lemma 2. *Let $\{z_i^e\}$ be the competitive allocation given the total signals $\{s_i\}$:*

$$\{z_i^e\} = \operatorname*{argmax}_{\{z_i'\}} \sum_{i=1}^{n}\left(\left(\alpha s_i + \frac{1-\alpha}{n-1}\sum_{j\neq i}s_j\right)(z_i') - \frac{\lambda}{2r}(z_i')^2\right) \text{ subject to: } \sum_{i=1}^{n} z_i' = Z.$$

For any profile of inventories $(z_1, z_2, \ldots, z_n)$ satisfying $\sum_{i=1}^{n} z_i = Z$, we have:

$$\sum_{i=1}^{n}\left(\left(\alpha s_i + \frac{1-\alpha}{n-1}\sum_{j\neq i}s_j\right)z_i^e - \frac{\lambda}{2}(z_i^e)^2\right) - \sum_{i=1}^{n}\left(\left(\alpha s_i + \frac{1-\alpha}{n-1}\sum_{j\neq i}s_j\right)z_i - \frac{\lambda}{2}(z_i)^2\right)$$

$$= \frac{\lambda}{2r}\sum_{i=1}^{n}(z_i - z_i^e)^2. \tag{109}$$

Proof. First, by the definition of competitive allocation, we have

$$\left(\alpha s_i + \frac{1-\alpha}{n-1}\sum_{j\neq i}s_j\right) - \lambda z_i^e = \nu \tag{110}$$

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for every i, where ν is the Lagrange multiplier for the constraint $\sum_{i=1}^{n} z_i^e = Z$ in the maximization problem of $\{z_i^e\}$.

Since $(z_i)^2 = (z_i^e)^2 + 2z_i^e(z_i - z_i^e) + (z_i - z_i^e)^2$, we have:

$$\sum_{i=1}^{n}\left(\left(\alpha s_i + \frac{1-\alpha}{n-1}\sum_{j\neq i} s_j\right) z_i - \frac{\lambda}{2r}(z_i)^2\right) \tag{111}$$

$$=\sum_{i=1}^{n}\left(\left(\alpha s_i + \frac{1-\alpha}{n-1}\sum_{j\neq i} s_j\right) z_i^e - \frac{\lambda}{2r}(z_i^e)^2\right) + \sum_{i=1}^{n}\left(\left(\alpha s_i + \frac{1-\alpha}{n-1}\sum_{j\neq i} s_j\right) - \frac{\lambda}{r}z_i^e\right)(z_i - z_i^e)$$

$$- \frac{\lambda}{2r}\sum_{i=1}^{n}(z_i - z_i^e)^2.$$

The middle term in the right-hand side of (111) is zero because of (110) and of $\sum_{i=1}^{n}(z_i - z_i^e) = Z - Z = 0$. This proves the lemma. $\qquad\square$

By Lemma 2 we write $W(\Delta)$ as:

$$W(\Delta) = \mathbb{E}\left[\int_{\tau=0}^{\infty} re^{-r\tau}\sum_{i=1}^{n}\left(v_{i,\tau}z_{i,\tau}^e - \frac{\lambda}{2r}(z_{i,\tau}^e)^2\right) d\tau\right] - X(\Delta), \tag{112}$$

where

$$X(\Delta) = \mathbb{E}\left[\int_{\tau=0}^{\infty} re^{-r\tau}\frac{\lambda}{2r}\sum_{i=1}^{n}(z_{i,\tau}^e - z_{i,\tau}^*)^2\, d\tau\right] \tag{113}$$

is the amount of inefficiency associated with the equilibrium path of inventories. Since the first term on the right-hand side of (112) is the welfare of competitive allocation in continuous time and is hence independent of Δ, the optimal trading frequency is determined by the comparative statics of $X(\Delta)$ with respect to Δ.

Lemma 3. *Suppose that $\tau \in (t\Delta, (t+1)\Delta)$. Then we have*

$$\mathbb{E}[(z_{i,\tau}^* - z_{i,\tau}^e)^2] = \mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)^2] + \mathbb{E}[(z_{i,t\Delta}^e - z_{i,\tau}^e)^2]. \tag{114}$$

Proof. Recall that $z_{i,\tau}^* = z_{i,(t+1)\Delta}^*$ for $\tau \in (t\Delta, (t+1)\Delta)$ because trading does not happen in $(t\Delta, (t+1)\Delta)$. Thus, we can rewrite, for any $\tau \in (t\Delta, (t+1)\Delta)$,

$$\mathbb{E}[(z_{i,\tau}^* - z_{i,\tau}^e)^2] = \mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)^2] + \mathbb{E}[(z_{i,t\Delta}^e - z_{i,\tau}^e)^2] + \mathbb{E}[2(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)(z_{i,t\Delta}^e - z_{i,\tau}^e)]. \tag{115}$$

Since $z_{i,(t+1)\Delta}^*$ is measurable with respect to $\{H_{j,t\Delta}\}_{1\leq j\leq n}$, $\tau > t\Delta$, and $\{z_{i,\tau'}^e\}_{\tau'\geq 0}$ is a martingale, we have $\mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)(z_{i,t\Delta}^e - z_{i,\tau}^e)] = 0$ by the law of iterated expectations. $\qquad\square$

By Lemma 3, we can further decompose $X(\Delta)$ into two terms:

$$X(\Delta) = X_1(\Delta) + X_2(\Delta), \tag{116}$$

where

$$X_1(\Delta) = (1 - e^{-r\Delta}) \cdot \frac{\lambda}{2r} \sum_{i=1}^{n} \sum_{t=0}^{\infty} e^{-rt\Delta} \mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)^2] \tag{117}$$

and

$$X_2(\Delta) = \frac{\lambda}{2r} \sum_{i=1}^{n} \sum_{t=0}^{\infty} \int_{\tau=t\Delta}^{(t+1)\Delta} r e^{-r\tau} \mathbb{E}[(z_{i,t\Delta}^e - z_{i,\tau}^e)^2] \, d\tau. \tag{118}$$

The expression $X_2(\Delta)$ is purely in terms of the competitive inventories $z_{i,t\Delta}^e$ and is relatively easy to analyze because $z_{i,\tau}^e$ depends only on the total signals at time τ. In contrast, $X_1(\Delta)$ is a function of the equilibrium inventories $z_{i,\tau}^*$ which depends on all total signals on and before time τ. The next lemma simplifies the equilibrium inventory terms in $X_1(\Delta)$.

Lemma 4.

$$\mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)^2] = (1+d)^{2(t+1)} \mathbb{E}[(z_{i,0}^* - z_{i,0}^e)^2] + \sum_{t'=0}^{t-1} (1+d)^{2(t-t')} \mathbb{E}[(z_{i,(t'+1)\Delta}^e - z_{i,t'\Delta}^e)^2] \tag{119}$$

Proof. From Proposition 3 (where $z_{i,t\Delta}^e = z_{i,(t+1)\Delta}^c$), we have

$$z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e = (1 + d)(z_{i,t\Delta}^* - z_{i,t\Delta}^e). \tag{120}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)^2] &= (1 + d)^2 \mathbb{E}[(z_{i,t\Delta}^* - z_{i,t\Delta}^e)^2] \\
&= (1 + d)^2 \mathbb{E}[(z_{i,t\Delta}^* - z_{i,(t-1)\Delta}^e)^2] + (1 + d)^2 \mathbb{E}[(z_{i,t\Delta}^e - z_{i,(t-1)\Delta}^e)^2] \\
&\quad - 2(1 + d)^2 \mathbb{E}[(z_{i,t\Delta}^* - z_{i,(t-1)\Delta}^e)(z_{i,t\Delta}^e - z_{i,(t-1)\Delta}^e)]. \tag{121}
\end{aligned}$$

Because $z_{i,t\Delta}^*$ is measurable with respect to $\{H_{j,(t-1)\Delta}\}_{1 \le j \le n}$ and $\{z_{i,\tau}^e\}_{\tau \ge 0}$ is a martingale, $\mathbb{E}[(z_{i,t\Delta}^* - z_{i,(t-1)\Delta}^e)(z_{i,t\Delta}^e - z_{i,(t-1)\Delta}^e)] = 0$ by the law of iterated expectations. The rest follows by induction. $\square$

Finally, Lemma 5 expresses $X_1(\Delta)$ in terms of the competitive inventories, similar to $X_2(\Delta)$.

Lemma 5.

$$X_1(\Delta) = \frac{\lambda(1 + d)}{2r(n\alpha - 1)} \left(\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2] + \sum_{i=1}^{n} \sum_{t=0}^{\infty} e^{-r(t+1)\Delta} \mathbb{E}[(z_{i,(t+1)\Delta}^e - z_{i,t\Delta}^e)^2] \right). \tag{122}$$

Proof. By Lemma 4, we have

$$X_1(\Delta) \tag{123}$$

$$= \frac{\lambda(1 - e^{-r\Delta})}{2r} \sum_{i=1}^{n} \sum_{t=0}^{\infty} e^{-rt\Delta} \left((1+d)^{2(t+1)} \mathbb{E}[(z_{i,0}^* - z_{i,0}^e)^2] + \sum_{t'=0}^{t-1} (1+d)^{2(t-t')} \mathbb{E}[(z_{i,(t'+1)\Delta}^e - z_{i,t'\Delta}^e)^2] \right)$$

$$= \frac{\lambda}{2r} \frac{(1 - e^{-r\Delta})(1+d)^2}{1 - (1+d)^2 e^{-r\Delta}} \sum_{i=1}^{n} \mathbb{E}[(z_{i,0}^* - z_{i,0}^e)^2]$$

$$+ \frac{1 - e^{-r\Delta}}{r} \frac{\lambda}{2} \sum_{i=1}^{n} \sum_{t'=0}^{\infty} \mathbb{E}[(z_{i,(t'+1)\Delta}^e - z_{i,t'\Delta}^e)^2] \sum_{t=t'+1}^{\infty} e^{-rt\Delta} (1+d)^{2(t-t')}$$

$$= \frac{\lambda}{2r} \frac{(1 - e^{-r\Delta})(1+d)^2}{1 - (1+d)^2 e^{-r\Delta}} \sum_{i=1}^{n} \mathbb{E}[(z_{i,0}^* - z_{i,0}^e)^2]$$

$$+ \frac{\lambda}{2r} \frac{(1 - e^{-r\Delta})(1+d)^2}{1 - (1+d)^2 e^{-r\Delta}} \sum_{i=1}^{n} \sum_{t'=0}^{\infty} \mathbb{E}[(z_{i,(t'+1)\Delta}^e - z_{i,t'\Delta}^e)^2] e^{-r(t'+1)\Delta}.$$

We can simplify the constant in the above equations by a direct calculation:

$$e^{-r\Delta}(1+d)^2 \tag{124}$$

$$= \frac{2(n\alpha - 1)^2(1 - e^{-r\Delta})^2 + 4e^{-r\Delta} - 2(n\alpha - 1)(1 - e^{-r\Delta})\sqrt{(n\alpha - 1)^2(1 - e^{-r\Delta})^2 + 4e^{-r\Delta}}}{4e^{-r\Delta}}$$

$$= 1 - (n\alpha - 1)(1 - e^{-r\Delta})(1+d),$$

which implies:

$$\frac{(1 - e^{-r\Delta})(1+d)^2}{1 - (1+d)^2 e^{-r\Delta}} = \frac{1+d}{n\alpha - 1}. \tag{125}$$

Finally, by construction: $z_{i,0}^* = z_{i,0}$ for every bidder i. $\qquad\square$

B.4.1 Proofs of Proposition 5

Suppose that $T_0 = 0$ and $\{T_k\}_{k \geq 1}$ is a homogeneous Poisson process with intensity $\mu > 0$. (The proof for scheduled information arrivals $T_k = k\gamma$ is analogous and omitted.)

Lemma 5 then implies that

$$\frac{X_1(\Delta)}{n} = \frac{\lambda(1 + d(\Delta))}{2r(n\alpha - 1)} \cdot \left(\frac{\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2]}{n} + \frac{e^{-r\Delta}\mu\Delta}{1 - e^{-r\Delta}} \frac{\sum_{i=1}^{n} \mathbb{E}[(z_{i,T_k}^e - z_{i,T_{k-1}}^e)^2]}{n} \right),$$

$$\tag{126}$$

where for any $k \geq 1$,

$$\frac{\sum_{i=1}^{n} \mathbb{E}[(z_{i,T_k}^e - z_{i,T_{k-1}}^e)^2]}{n} = \left(\frac{r(n\alpha - 1)}{\lambda(n-1)}\right)^2 \frac{(n-1)(\chi^2(\sigma_D^2 + \sigma_\epsilon^2) + \sigma_w^2)}{n\alpha^2}, \qquad (127)$$

by Equation (48).

Equation (127) tends to a positive constant as $n \to \infty$ (since $\chi \to 0$ as $n \to \infty$), and $\lim_{\Delta \to 0} \frac{e^{-r\Delta}\mu\Delta}{1-e^{-r\Delta}} = \frac{\mu}{r}$. By assumption, $\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2]/n$ is bounded as $n \to \infty$. Thus, for $\lim_{n\to\infty} X_1(\Delta)/n$ it suffices to analyze

$$\frac{1 + d(\Delta)}{n\alpha - 1} = \frac{1}{2e^{-r\Delta}}\left(\sqrt{(1 - e^{-r\Delta})^2 + \frac{4e^{-r\Delta}}{(n\alpha - 1)^2}} - (1 - e^{-r\Delta})\right). \qquad (128)$$

Suppose $\sigma_D^2 > 0$. Equation (108) (where $\eta \equiv n\alpha - 1$) implies that $n\alpha$ is of order $n^{2/3}$ as $n \to \infty$. To see this, first note that $\eta \to \infty$ and $\eta/n \to 0$ as $n \to \infty$, for otherwise the left-hand side and right-hand side of Equation (108) cannot match. Suppose that as n becomes large, η is of order n^y for some $y < 1$. The left-hand side of Equation (108) is of order n^{y-1}, and the right-hand side is of order $n^{-y/2}$. Thus, $y = 2/3$.

For any fixed $\Delta > 0$, it is straightforward to use Taylor expansion to calculate that, as n becomes large,

$$\frac{1 + d(\Delta)}{n\alpha - 1} = \frac{1}{1 - e^{-r\Delta}}(n\alpha - 1)^{-2} + O((n\alpha - 1)^{-4}).$$

Therefore, $(1 + d(\Delta))/(n\alpha - 1)$ and hence $X_1(\Delta)/n$ are of order $n^{-4/3}$.

But if we first take the limit $\Delta \to 0$, we clearly have

$$\lim_{\Delta \to 0} \frac{1 + d(\Delta)}{n\alpha - 1} = \frac{1}{n\alpha - 1},$$

so $\lim_{\Delta \to 0}(1 + d(\Delta))/(n\alpha - 1)$ and hence $\lim_{\Delta \to 0} X_1(\Delta)/n$ are of order $n^{-2/3}$.

If $\sigma_D^2 = 0$, then $n\alpha = n$. The same calculation as above shows that $X_1(\Delta)/n$ is of order n^{-2} for a fixed $\Delta > 0$ but is of order n^{-1} if we first take the limit $\Delta \to 0$.

B.4.2 Proof of Proposition 6

For any $\tau > 0$, we let $\bar{t}(\tau) = \min\{t \geq 0 : t \in \mathbb{Z}, t\Delta \geq \tau\}$. That is, if new signals arrive at the clock time τ, then $\bar{t}(\tau)\Delta$ is the clock time of the next trading period (including time τ).

For any $\Delta \leq \gamma$, by the assumption of Proposition 6 there is at most one new signal profile arrival in each interval $[t\Delta, (t + 1)\Delta)$. Thus, we only need to count the changes in competitive allocation between period $\bar{t}((k-1)\gamma)$ and $\bar{t}(k\gamma)$, for $k \in \mathbb{Z}_+$.

Using this fact, we can rewrite $X_1(\Delta)$ and $X_2(\Delta)$ as:

$$X_1(\Delta) = \frac{\lambda(1+d)}{2r(n\alpha-1)} \left(\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2] + \sum_{i=1}^{n}\sum_{k=1}^{\infty} e^{-r\bar{t}(k\gamma)\Delta} \mathbb{E}[(z_{i,k\gamma}^e - z_{i,(k-1)\gamma}^e)^2] \right)$$

$$= \frac{\lambda(1+d)}{2r(n\alpha-1)} \left(\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2] + \sum_{i=1}^{n}\sum_{k=1}^{\infty} e^{-rk\gamma} \mathbb{E}[(z_{i,k\gamma}^e - z_{i,(k-1)\gamma}^e)^2] \right)$$

$$- \frac{\lambda(1+d)}{2r(n\alpha-1)} \sum_{i=1}^{n}\sum_{k=1}^{\infty} (e^{-rk\gamma} - e^{-r\bar{t}(k\gamma)\Delta}) \mathbb{E}[(z_{i,k\gamma}^e - z_{i,(k-1)\gamma}^e)^2]. \tag{129}$$

and

$$X_2(\Delta) = \frac{\lambda}{2r} \sum_{i=1}^{n}\sum_{t=0}^{\infty} \int_{\tau=t\Delta}^{(t+1)\Delta} re^{-r\tau} \mathbb{E}[(z_{i,t\Delta}^e - z_{i,\tau}^e)^2] \, d\tau \tag{130}$$

$$= \frac{\lambda}{2r} \sum_{i=1}^{n}\sum_{k=1}^{\infty} (e^{-rk\gamma} - e^{-r\bar{t}(k\gamma)\Delta}) \mathbb{E}[(z_{i,k\gamma}^e - z_{i,(k-1)\gamma}^e)^2].$$

Note that all the expectations in the expressions of $X_1(\Delta)$ and $X_2(\Delta)$ do not depend on Δ. To make clear the dependence of d on Δ, we now write $d = d(\Delta)$. Since $(1 + d(\Delta))/(n\alpha - 1) < 1$, we have for any $\Delta < \gamma$:

$$X(\Delta) = X_1(\Delta) + X_2(\Delta) \tag{131}$$

$$> \frac{\lambda(1+d(\Delta))}{2r(n\alpha-1)} \left(\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2] + \sum_{i=1}^{n}\sum_{k=1}^{\infty} e^{-rk\gamma} \mathbb{E}[(z_{i,k\gamma}^e - z_{i,(k-1)\gamma}^e)^2] \right)$$

$$> \frac{\lambda(1+d(\gamma))}{2r(n\alpha-1)} \left(\sum_{i=1}^{n} \mathbb{E}[(z_{i,0} - z_{i,0}^e)^2] + \sum_{i=1}^{n}\sum_{k=1}^{\infty} e^{-rk\gamma} \mathbb{E}[(z_{i,k\gamma}^e - z_{i,(k-1)\gamma}^e)^2] \right)$$

$$= X(\gamma),$$

where the last inequality holds because $d(\Delta)$ decreases with Δ (which can be verified by taking derivative the $d'(\Delta)$) and where the last equality holds because $\bar{t}(k\gamma)\Delta = k\gamma$ if $\gamma = \Delta$. Therefore, we have $W(\Delta) < W(\gamma)$ for any $\Delta < \gamma$. This proves Proposition 6.

Notice that for this lower bound of $\Delta^* \geq \gamma$ we make no use of the assumption that $\mathbb{E}[(z_{i,k\gamma}^e - z_{i,(k-1)\gamma}^e)^2]$ is a constant independent of k.

B.4.3 Proof of Proposition 7

If $\Delta = l\gamma$, where $l \geq 1$ is an integer, we have:

$$X_1(l\gamma) = \frac{\lambda(1 + d(l\gamma))}{2r(n\alpha - 1)}\left(\sigma_0^2 + \sum_{t=0}^{\infty} e^{-r(t+1)l\gamma}l\sigma_z^2\right) = \frac{\lambda(1 + d(l\gamma))}{2r(n\alpha - 1)}\left(\sigma_0^2 + \frac{e^{-rl\gamma}}{1 - e^{-rl\gamma}}l\sigma_z^2\right),$$

$$(132)$$

$$\begin{aligned}
X_2(l\gamma) &= \frac{\lambda}{2r}\frac{1}{1 - e^{-rl\gamma}}\left((e^{-\gamma r} - e^{-2\gamma r}) + 2(e^{-2\gamma r} - e^{-3\gamma r}) + \cdots + (l - 1)(e^{-(l-1)\gamma r} - e^{-l\gamma r})\right)\sigma_z^2 \\
&= \frac{\lambda}{2r}\frac{1}{1 - e^{-rl\gamma}}\left(e^{-\gamma r} + e^{-2\gamma r} + e^{-3\gamma r} + \cdots + e^{-(l-1)\gamma r} - (l - 1)e^{-l\gamma r}\right)\sigma_z^2 \\
&= \frac{\lambda}{2r}\frac{1}{1 - e^{-rl\gamma}}\left(\frac{1 - e^{-rl\gamma}}{1 - e^{-\gamma r}} - 1 - (l - 1)e^{-rl\gamma}\right)\sigma_z^2 \\
&= \frac{\lambda}{2r}\left(\frac{1}{1 - e^{-\gamma r}} - 1 - l\frac{e^{-rl\gamma}}{1 - e^{-rl\gamma}}\right)\sigma_z^2.
\end{aligned}$$

$$(133)$$

Hence, if $\Delta = l\gamma$, $l \in \mathbb{Z}_+$, we have:

$$X(l\gamma) = \frac{\lambda(1 + d(l\gamma))}{2r(n\alpha - 1)}\sigma_0^2 - \frac{\lambda}{2r}\left(1 - \frac{1 + d(l\gamma)}{n\alpha - 1}\right)\frac{le^{-rl\gamma}}{1 - e^{-rl\gamma}}\sigma_z^2 + \frac{\lambda e^{-\gamma r}}{2r(1 - e^{-\gamma r})}\sigma_z^2.$$

$$(134)$$

By taking derivative, we can show that the function (involved in the first term in (134))

$$\frac{1 + d(\Delta)}{n\alpha - 1} = \frac{1}{2e^{-r\Delta}}\left(\sqrt{(1 - e^{-r\Delta})^2 + \frac{4e^{-r\Delta}}{(n\alpha - 1)^2}} - (1 - e^{-r\Delta})\right)$$

is strictly decreasing in Δ. And since

$$\left(1 - \frac{1 + d(\Delta)}{n\alpha - 1}\right)\frac{\Delta e^{-r\Delta}}{1 - e^{-r\Delta}} = \frac{(n\alpha - 1)(1 + e^{-r\Delta}) - \sqrt{(n\alpha - 1)^2(1 - e^{-r\Delta})^2 + 4e^{-r\Delta}}}{2(n\alpha - 1)e^{-r\Delta/2}} \cdot \frac{\Delta e^{-r\Delta/2}}{1 - e^{-r\Delta}},$$

and both terms on the right-hand side are strictly decreasing in Δ (easily verified with derivatives), the above function (involved in the second term in (134)) is also strictly decreasing in Δ.

Without loss of generality, suppose that $\sigma_z^2 = 1$. The above discussion implies that the function $X(l\gamma)$ satisfies strictly decreasing difference condition with respect to variables l and $\sigma_0^2 = \sigma_0^2/\sigma_z^2$:

$$\frac{\partial^2 X(l\gamma)}{\partial l\,\partial\sigma_0^2} < 0.$$

$$(135)$$

Therefore, by a well-known result from monotone comparative statics (Theorem 5

in Milgrom and Shannon (1994)), l^* that minimizes $X(l\gamma)$ is weakly increasing with $\sigma_0^2 = \sigma_0^2/\sigma_z^2$. As $\sigma_0^2/\sigma_z^2 \to 0$, the second term in (134) dominates, and hence $X(l\gamma)$ is minimized at $l^* = \infty$. As $\sigma_0^2/\sigma_z^2 \to \infty$, the first term in (134) dominates, and hence $X(l\gamma)$ is minimized at $l^* = 1$.

Finally, as n tends to infinity, the proof of Proposition 5 implies that $n\alpha$ tends to infinity as well. As $n\alpha \to \infty$, $(1 + d(l\gamma))/(n\alpha - 1) \to 0$ for every $l \in \mathbb{Z}_+$, and by assumption σ_0^2/σ_z^2 remains bounded, so the second term in (134) dominates, and hence $X(l\gamma)$ is minimized at $l^* = 1$.

B.4.4   Proof of Proposition 8

Since dividend shocks arrive according to a Poisson process with the intensity μ, we have

$$\sum_{i=1}^{n} \mathbb{E}[(z_{i,\tau}^e - z_{i,t\Delta}^e)^2] = (\tau - t\Delta)\mu\sigma_z^2, \quad \tau \in [t\Delta, (t+1)\Delta), \tag{136}$$

$$\sum_{i=1}^{n} \mathbb{E}[(z_{i,(t+1)\Delta}^e - z_{i,t\Delta}^e)^2] = \Delta\mu\sigma_z^2. \tag{137}$$

Substituting the above two expressions into (118) and (122), we have:

$$X_1(\Delta) = \frac{\lambda(1 + d)}{2r(n\alpha - 1)} \left(\sigma_0^2 + \sum_{t=0}^{\infty} e^{-r(t+1)\Delta}\Delta\mu\sigma_z^2 \right) \tag{138}$$

$$= \frac{\lambda(1 + d)}{2r(n\alpha - 1)} \left(\sigma_0^2 + \frac{\Delta e^{-r\Delta}}{1 - e^{-r\Delta}}\mu\sigma_z^2 \right)$$

and

$$X_2(\Delta) = \frac{\lambda}{2r} \sum_{t=0}^{\infty} e^{-rt\Delta} \int_{\tau=0}^{\Delta} re^{-r\tau}\tau\mu\sigma_z^2 \, d\tau = -\frac{\lambda}{2r}\frac{\Delta e^{-r\Delta}}{1 - e^{-r\Delta}}\mu\sigma_z^2 + \frac{\lambda}{2r^2}\mu\sigma_z^2. \tag{139}$$

Therefore,

$$X(\Delta) = \frac{\lambda(1 + d)}{2r(n\alpha - 1)}\sigma_0^2 - \frac{\lambda}{2r} \left(1 - \frac{1 + d}{n\alpha - 1} \right) \frac{\Delta e^{-r\Delta}}{1 - e^{-r\Delta}}\mu\sigma_z^2 + \frac{\lambda}{2r^2}\mu\sigma_z^2. \tag{140}$$

We note that the above is the same expression as Equation (134) in the proof of Proposition 7; therefore, the comparative statics with respect to σ_0^2/σ_z^2 and μ has the same proof as that in Proposition 7, with the modification that here Δ is not restricted to integer multiple of γ, and hence the comparative statics is strict. The result for $n \to \infty$ also has the same proof as that in Proposition 7.

Suppose that $z_{i,0} = Z/n$ for every trader i. Then we have $\sigma_0^2 = \sigma_z^2$. Without loss

of generality, suppose that $\sigma_0^2 = \sigma_z^2 = 1$ (since this does not change Δ^*). We will prove that $\frac{\partial^2 X}{\partial(n\alpha)\partial\Delta} > 0$ whenever the first order condition $\frac{\partial X}{\partial\Delta} = 0$ holds. The implicit function theorem then implies our comparative statics with respect to to n, σ_w^2 and σ_D^2, since $\frac{\partial(n\alpha)}{\partial n} > 0$, $\frac{\partial(n\alpha)}{\partial\sigma_w^2} > 0$ and $\frac{\partial(n\alpha)}{\partial\sigma_D^2} < 0$ by the proof of Proposition 3.

Substitute in the expression for $1 + d$, we are led to:

$$X(\delta) = \frac{\sqrt{(1-\delta)^2 + 4\delta y} - (1-\delta)}{2\delta}A - \frac{(1+\delta) - \sqrt{(1-\delta)^2 + 4\delta y}}{2\delta}\frac{\delta}{1-\delta}(-\log(\delta))B + C,$$

(141)

where

$$\delta \equiv e^{-r\Delta}, \; y \equiv \frac{1}{(n\alpha - 1)^2}, \; A \equiv \frac{\lambda}{2r}, \; B \equiv \frac{\lambda}{2r^2}\mu, \; C \equiv \frac{\lambda}{2r^2}\mu.$$

(142)

We compute:

$$\frac{\partial X}{\partial\delta} = \frac{1}{2(1-\delta)^2\delta^2\sqrt{(1-\delta)^2 + 4\delta y}}$$

$$\cdot \left(A(1-\delta)^2 \left(\sqrt{(1-\delta)^2 + 4\delta y} - (1-\delta) - 2\delta y \right) \right.$$

$$+ B\delta(1-\delta) \left((1+\delta)\sqrt{(1-\delta)^2 + 4\delta y} - (1-\delta)^2 - 4\delta y \right)$$

$$\left. + 2B\delta^2 \log(\delta) \left(\sqrt{(1-\delta)^2 + 4\delta y} - (1+\delta)y \right) \right)$$

$$= \frac{A + B\delta}{2\sqrt{(1-\delta)^2 + 4\delta y}}$$

(143)

$$\cdot \left(\left(\frac{1}{\delta^2} + \frac{2B(1-\delta+\log(\delta))}{(A+B\delta)(1-\delta)^2} \right) \left(\sqrt{(1-\delta)^2 + 4\delta y} - (1+\delta)y \right) - \frac{1-\delta}{\delta^2}(1-y) \right).$$

$$\frac{\partial^2 X}{\partial y\partial\delta} = \frac{A + B\delta}{2}((1-\delta)^2 + 4\delta y)^{-1/2}\left(-\left(\frac{1}{\delta^2} + \frac{2B(1-\delta+\log(\delta))}{(A+B\delta)(1-\delta)^2} \right)(1+\delta)\left(1 - \frac{2\delta y}{(1-\delta)^2 + 4\delta y} \right) \right.$$

$$\left. - \frac{1-\delta}{\delta^2}\left(-1 - \frac{(1-y)2\delta}{(1-\delta)^2 + 4\delta y} \right) \right).$$

(144)

Since

$$\frac{1}{\delta^2} + \frac{2B(1-\delta+\log(\delta))}{(A+B\delta)(1-\delta)^2} = \frac{\frac{1-\delta}{\delta^2}(1-y)}{\sqrt{(1-\delta)^2 + 4\delta y} - (1+\delta)y}$$

(145)

from the first order condition $\frac{\partial X}{\partial\delta} = 0$, to show $\frac{\partial^2 X}{\partial y\partial\delta} > 0$ at the first order condition it

suffices to show that for every $\delta \in (0,1)$ and $y \in (0,1)$:

$$2\delta y + 1 + \delta^2 > \frac{(1-y)(1+\delta)((1-\delta)^2 + 2\delta y)}{\sqrt{(1-\delta)^2 + 4\delta y} - (1+\delta)y}. \tag{146}$$

It is straightforward to show that the above equation is equivalent to $1 + y^2 > 2y$ for $y \in (0,1)$ and $\delta \in (0,1)$. This completes the proof.

B.5   Proof of Proposition 9

We have derived the slow traders' equilibrium strategies in the text following Proposition 9, so here we focus on characterizing the fast trader's equilibrium strategy.

We conjecture that the fast trader uses the strategy

$$x_{F,t\Delta}(p; z_{F,t\Delta}) = -b_F p + d_F z_{F,t\Delta}. \tag{147}$$

Without loss of generality let us specialize to period 1 with an arbitrary inventory $z_{F,\Delta}$. Assuming that the fast trader uses strategy (147) from period 2 and onwards, and that the slow traders use their equilibrium strategy (55) in every period, we will construct strategy (147) such that the fast trader has no incentive to deviate from strategy (147) in period 1. By the single deviation principle, this gives the equilibrium strategy of the fast trader.

Under our assumption about the traders' strategies from period 2 and onwards, we have for $t \geq 1$:

$$p^*_{(t+1)\Delta} = \frac{1}{b_F + n_S b_S} \left(\sum_{j=1}^{n_S} b_S w_{j,(t+1)\Delta} + d_F z_{F,(t+1)\Delta} \right), \tag{148}$$

and

$$\begin{aligned} z_{F,(t+2)\Delta} &= z_{F,(t+1)\Delta} + x_{F,(t+1)\Delta} \\ &= z_{F,(t+1)\Delta} - b_F p^*_{(t+1)\Delta} + d_F z_{F,(t+1)\Delta} \\ &= z_{F,(t+1)\Delta} + \frac{-b_F}{b_F + n_S b_S} \sum_{j=1}^{n_S} b_S w_{j,(t+1)\Delta} + \frac{-b_F}{b_F + n_S b_S} d_F z_{F,(t+1)\Delta} + d_F z_{F,(t+1)\Delta} \\ &= \frac{-b_F}{b_F + n_S b_S} \sum_{j=1}^{n_S} b_S w_{j,(t+1)\Delta} + \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F \right) z_{F,(t+1)\Delta}. \end{aligned} \tag{149}$$

Therefore, the fast trader in period 1 has the following effect on the future prices and

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inventories:

$$\frac{\partial(z_{F,(t+1)\Delta} + x_{F,(t+1)\Delta})}{\partial x_{F,\Delta}} = \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F\right)^t, \tag{150}$$

$$\frac{\partial x_{F,(t+1)\Delta}}{\partial x_{F,\Delta}} = \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F\right)^{t-1} \frac{n_S b_S}{b_F + n_S b_S} d_F, \tag{151}$$

$$\frac{\partial p^*_{(t+1)\Delta}}{\partial x_{F,\Delta}} = \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F\right)^{t-1} \frac{d_F}{b_F + n_S b_S}. \tag{152}$$

The fast trader's first order condition at period 1 is:

$$-x_{F,\Delta} + n_S b_S \left[\frac{1 - e^{-r\Delta}}{r} \sum_{t=0}^{\infty} e^{-rt\Delta} \mathbb{E}\left[-\lambda(z_{F,(t+1)\Delta} + x_{F,(t+1)\Delta}) \frac{\partial(z_{F,(t+1)\Delta} + x_{F,(t+1)\Delta})}{\partial x_{F,\Delta}} \mid z_{F,\Delta}, x_{F,\Delta} \right] \right.$$

$$\tag{153}$$

$$- p^*_\Delta - \sum_{t=1}^{\infty} e^{-rt\Delta} \mathbb{E}\left[\frac{\partial x_{F,(t+1)\Delta}}{\partial x_{F,\Delta}} p^*_{(t+1)\Delta} \mid z_{F,\Delta}, x_{F,\Delta} \right]$$

$$\left. - \sum_{t=1}^{\infty} e^{-rt\Delta} \mathbb{E}\left[\frac{\partial p^*_{(t+1)\Delta}}{\partial x_{F,\Delta}} x_{F,(t+1)\Delta} \mid z_{F,\Delta}, x_{F,\Delta} \right] \right] = 0,$$

i.e.,

$$-x_{F,\Delta} + n_S b_S \left[\frac{1 - e^{-r\Delta}}{r} \sum_{t=0}^{\infty} e^{-rt\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F\right)^t (-\lambda \mathbb{E}[z_{F,(t+1)\Delta} + x_{F,(t+1)\Delta} \mid z_{F,\Delta}, x_{F,\Delta}]) \right.$$

$$\tag{154}$$

$$- p^*_\Delta - \sum_{t=1}^{\infty} e^{-rt\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F\right)^{t-1} \frac{n_S b_S}{b_F + n_S b_S} d_F \mathbb{E}[p^*_{(t+1)\Delta} \mid z_{F,\Delta}, x_{F,\Delta}]$$

$$\left. - \sum_{t=1}^{\infty} e^{-rt\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F\right)^{t-1} \frac{d_F}{b_F + n_S b_S} \mathbb{E}[x_{F,(t+1)\Delta} \mid z_{F,\Delta}, x_{F,\Delta}] \right] = 0,$$

i.e.,

$$-x_{F,\Delta} + n_S b_S \left[\frac{1 - e^{-r\Delta}}{r} \sum_{t=0}^{\infty} e^{-rt\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F \right)^{2t} (-\lambda(z_{F,\Delta} + x_{F,\Delta})) \right.$$

(155)

$$-p_{\Delta}^* - \sum_{t=1}^{\infty} e^{-rt\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F \right)^{2(t-1)} \frac{n_S b_S d_F}{b_F + n_S b_S} \cdot \frac{d_F}{b_F + n_S b_S} (z_{F,\Delta} + x_{F,\Delta})$$

$$\left. - \sum_{t=1}^{\infty} e^{-rt\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F \right)^{2(t-1)} \frac{d_F}{b_F + n_S b_S} \cdot \frac{n_S b_S d_F}{b_F + n_S b_S} (z_{F,\Delta} + x_{F,\Delta}) \right] = 0,$$

i.e.,

$$-x_{F,\Delta} + n_S b_S \left[-\frac{1 - e^{-r\Delta}}{r} \cdot \frac{\lambda}{1 - e^{-r\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F \right)^2} (z_{F,\Delta} + x_{F,\Delta}) \right.$$

(156)

$$\left. -p_{\Delta}^* - \frac{2 e^{-r\Delta}}{1 - e^{-r\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F \right)^2} \cdot \frac{d_F^2 n_S b_S}{(b_F + n_S b_S)^2} (z_{F,\Delta} + x_{F,\Delta}) \right] = 0,$$

i.e.,

$$-x_{F,\Delta} + n_S b_S \left[-\frac{1}{1 - e^{-r\Delta} \left(1 + \frac{n_S b_S}{b_F + n_S b_S} d_F \right)^2} \right.$$

(157)

$$\left. \cdot \left(\frac{\lambda(1 - e^{-r\Delta})}{r} + \frac{2 e^{-r\Delta} d_F^2 n_S b_S}{(b_F + n_S b_S)^2} \right) (z_{F,\Delta} + x_{F,\Delta}) - p_{\Delta}^* \right] = 0.$$

Matching the coefficients with (147), we have

$$b_F = \frac{n_S b_S}{1 + n_S b_S \lambda_F / r},$$

(158)

and

$$d_F = -\frac{\lambda_F}{r} b_F,$$

(159)

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where

$$\lambda_F \equiv \frac{1}{1 - e^{-r\Delta}\left(1 + \frac{n_S b_S}{b_F + n_S b_S}d_F\right)^2} \cdot \left(\lambda(1 - e^{-r\Delta}) + \frac{2re^{-r\Delta}d_F^2 n_S b_S}{(b_F + n_S b_S)^2}\right) \quad (160)$$

$$= \frac{1}{1 - e^{-r\Delta}\left(1 - \frac{n_S b_S}{b_F + n_S b_S} \cdot \frac{\lambda_F b_F}{r}\right)^2} \cdot \left(\lambda(1 - e^{-r\Delta}) + \frac{2e^{-r\Delta}\lambda_F^2 b_F^2 n_S b_S}{r(b_F + n_S b_S)^2}\right).$$

And recall that

$$b_S = \frac{b_F + (n_S - 1)b_S}{1 + (b_F + (n_S - 1)b_S)\lambda/r}. \quad (161)$$

Lemma 6. *Suppose that $n_S > 1$. There is a unique set of $\lambda_F > 0$, $b_S > 0$ and $b_F > 0$ that solve Equations* (158)*,* (160) *and* (161)*. In this solution we always have $\lambda_F < \lambda$.*

Proof. Taking λ_F as given and solving Equations (158) and (161), we get unique positive solutions $b_S(\lambda_F)$ and $b_F(\lambda_F)$ by Lemma 7.[30]

By expanding the denominator on the right-hand side and moving it to the left-hand side, we can simplify Equation (160) to:

$$\frac{\lambda - \lambda_F}{\lambda_F} = \frac{e^{-r\Delta}}{1 - e^{-r\Delta}}\left(2 - \frac{\lambda_F b_F(\lambda_F)}{r}\right)\frac{\lambda_F b_F(\lambda_F)}{r}\frac{n_S^2 b_S(\lambda_F)^2}{(b_F(\lambda_F) + n_S b_S(\lambda_F))^2}. \quad (163)$$

Equation (158) implies that $\lambda_F b_F(\lambda_F)/r < 1$. Thus, any solution λ_F to Equation (163) satisfies $\lambda_F < \lambda$.

Lemma 7 shows that $(2 - \lambda_F b_F(\lambda_F)/r)\lambda_F b_F(\lambda_F)/r$ is increasing in λ_F; and that $n_S b_S(\lambda_F)/(b_F(\lambda_F) + n_S b_S(\lambda_F))$ is also increasing in λ_F. Hence the right-hand side of Equation (163) is increasing in λ_F, while the left-hand side of Equation (163) is decreasing in λ_F. It is easy to show (using Lemma 7) that $\lim_{\lambda_F \to 0} \lambda_F b_F(\lambda_F)/r = 0$, and hence the right-hand side of Equation (163) is close to 0 when λ_F is close to zero, while the left-hand side of Equation (163) is negative when λ_F is close to zero. Therefore, Equation (163) has a unique positive solution λ_F. $\qquad\square$

Finally, it is easy to check that given $\lambda_F < \lambda$, Equations (158) and (161) imply that $b_F > b_S$.

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[30]In fact, we have the following explicit solution:

$$b_F = r\frac{\lambda(2n_S - 1) + \lambda_F n_S(3n_S - 2) - \sqrt{\lambda_F^2(n_S - 2)^2 n_S^2 + \lambda^2(1 - 2n_S)^2 + 2\lambda\lambda_F n_S(n_S(2n_S - 3) + 2)}}{2\lambda_F n_S(\lambda + \lambda_F n_S)},$$

$$b_S = r\frac{\lambda(1 - 2n_S) + \lambda_F(n_S - 2)n_S + \sqrt{\lambda_F^2(n_S - 2)^2 n_S^2 + \lambda^2(1 - 2n_S)^2 + 2\lambda\lambda_F n_S(n_S(2n_S - 3) + 2)}}{2\lambda\lambda_F(n_S - 1)n_S}.$$

$$(162)$$

Lemma 7. *Suppose $n_S > 1$. For any $\lambda > 0$ and $\lambda_F > 0$, there exist unique $b_S > 0$ and $b_F > 0$ that satisfy*

$$b_S = \frac{(n_S - 1)b_S + b_F}{1 + ((n_S - 1)b_S + b_F)\lambda/r}, \quad b_F = \frac{n_S b_S}{1 + n_S b_S \lambda_F/r}. \tag{164}$$

Moreover, as λ_F increases (holding all else constant), for the (b_S, b_F) that satisfies the above equation: $(2 - b_F \lambda_F/r)b_F \lambda_F/r$ strictly increases and $b_F/(n_S b_S + b_F)$ strictly decreases.

Proof. For the simplicity of notation, let us use $n_1 \equiv n_S$, $n_2 \equiv 1$, $\lambda_1 \equiv \lambda/r$ and $\lambda_2 \equiv \lambda_F/r$.

We first want to show the existence and uniqueness of $b_i > 0$, $i \in \{1, 2\}$, that satisfy

$$b_i + (\lambda_i b_i - 1)(B - b_i) = 0, \tag{165}$$

where $B \equiv n_1 b_1 + n_2 b_2$. Solving for b_i in (165), we get:

$$b_i = \frac{2 + \lambda_i B - \sqrt{\lambda_i^2 B^2 + 4}}{2\lambda_i}, \tag{166}$$

(The quadratic equation has two solutions, but only the smaller one is the correct solution.[31]) Thus, B must solve the following equation:

$$B = \sum_{i=1}^{2} n_i \frac{2 + \lambda_i B - \sqrt{\lambda_i^2 B^2 + 4}}{2\lambda_i}. \tag{167}$$

To show that (167) has a unique positive solution B, we rationalize the numerators of (167) and rewrite it as

$$0 = B\left(-1 + \sum_{i=1}^{2} n_i \frac{2}{2 + B\lambda_i + \sqrt{\lambda_i^2 B^2 + 4}}\right).$$

Under the conjecture that $B > 0$, we have

$$0 = f(B) \equiv -1 + \sum_{i=1}^{2} n_i \frac{2}{2 + B\lambda_i + \sqrt{\lambda_i^2 B^2 + 4}}. \tag{168}$$

It is straightforward to see that $f'(B) < 0$, $f(0) = \frac{n_1 + n_2}{2} - 1 > 0$, and $f(B) \to -1$ as $B \to \infty$. Thus, Equation (168) (and hence Equation (167)) has a unique positive solution B.

We now turn to the second statement of the lemma. When λ_i increases (holding

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[31]If $b_i = \frac{2 + \lambda_i B + \sqrt{\lambda_i^2 B^2 + 4}}{2\lambda_i}$, then we would have $b_i > B$, which contradicts the definition of B.

all else constant), the B that solves (168) must decrease, which means that $B\lambda_j$ $(j \neq i)$ must decrease and hence $B\lambda_i$ must increase for the B that solves (168); thus $b_i\lambda_i(2-b_i\lambda_i) = \frac{2+\lambda_i B-\sqrt{\lambda_i^2 B^2+4}}{2} \cdot \frac{2-\lambda_i B+\sqrt{\lambda_i^2 B^2+4}}{2}$ must increase and $\frac{b_i}{B} = \frac{2+\lambda_i B-\sqrt{\lambda_i^2 B^2+4}}{2B\lambda_i}$ must decrease. $\square$

B.6 Proof of Proposition 11

Define

$$\kappa \equiv 1 - \frac{n_S b_S \lambda_F b_F}{(b_F + n_S b_S)r}. \tag{169}$$

From the characterization of the equilibrium inventory and price in Proposition 10, we have:

$$\mathbb{E}[(z_{F,(t+1)\Delta}^*)^2] = \left(\frac{b_F}{b_F + n_S b_S}\right)^2 n_S b_S^2 \sigma_w^2 \frac{1-\kappa^{2t}}{1-\kappa^2} \tag{170}$$

$$\mathbb{E}[x_{F,t\Delta}(p_{t\Delta}^*; z_{F,t\Delta}^*) \cdot p_{t\Delta}^*] = -\frac{b_F n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2} + \frac{\lambda_F^2 b_F^4 n_S^2 b_S^3 \sigma_w^2}{(b_F + n_S b_S)^4 r^2} \frac{1-\kappa^{2(t-1)}}{1-\kappa^2}. \tag{171}$$

Therefore, the fast trader gets:

$$\begin{aligned}
W_F(\Delta) &= \mathbb{E}\left[-\frac{\lambda(1-e^{-r\Delta})}{2r} \sum_{t=1}^{\infty} e^{-rt\Delta}(z_{F,(t+1)\Delta}^*)^2 - \sum_{t=1}^{\infty} e^{-rt\Delta} x_{F,t\Delta}(p_{t\Delta}^*; z_{F,t\Delta}^*) p_{t\Delta}^* \right] \\
&= -\frac{\lambda(1-e^{-r\Delta})}{2r} \left(\frac{b_F}{b_F + n_S b_S}\right)^2 n_S b_S^2 \sigma_w^2 \frac{\frac{1}{1-e^{-r\Delta}} - \frac{1}{1-e^{-r\Delta}\kappa^2}}{1-\kappa^2} . \\
&\quad + \frac{e^{-r\Delta}}{1-e^{-r\Delta}} \frac{b_F n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2} - \frac{\lambda_F^2 b_F^4 n_S^2 b_S^3 \sigma_w^2}{(b_F + n_S b_S)^4 r^2} e^{-r\Delta} \frac{\frac{1}{1-e^{-r\Delta}} - \frac{1}{1-e^{-r\Delta}\kappa^2}}{1-\kappa^2} \\
&= -\frac{b_F^2 n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2 r} \left(\frac{\lambda(1-e^{-r\Delta})}{2} + \frac{\lambda_F^2 b_F^2 n_S b_S e^{-r\Delta}}{(b_F + n_S b_S)^2 r} \right) \frac{e^{-r\Delta}}{(1-e^{-r\Delta})(1-e^{-r\Delta}\kappa^2)} \\
&\quad + \frac{e^{-r\Delta}}{1-e^{-r\Delta}} \frac{b_F n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2}. \tag{172}
\end{aligned}$$

Applying Condition ([59](#)) to the last equation, we get:

$$
\begin{aligned}
W_F(\Delta) &= -\frac{b_F^2 n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2 r} \frac{\lambda_F (1 - e^{-r\Delta} \kappa^2)}{2} \frac{e^{-r\Delta}}{(1 - e^{-r\Delta})(1 - e^{-r\Delta} \kappa^2)} + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} \frac{b_F n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2} \\
&= -\frac{b_F^2 n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2 r} \frac{\lambda_F}{2} \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} + \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} \frac{b_F n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2} \\
&= \frac{e^{-r\Delta}}{1 - e^{-r\Delta}} \frac{b_F n_S b_S^2 \sigma_w^2}{(b_F + n_S b_S)^2} \left(1 - \frac{\lambda_F b_F}{2r} \right) \\
&= \frac{r(\lambda - \lambda_F) \sigma_w^2}{2 n_S \lambda_F^2},
\end{aligned}
\tag{173}
$$

where the last line follows from Equation ([163](#)).

Applying the explicit expressions for $b_F(\lambda_F)$ and $b_S(\lambda_S)$ in Equation ([162](#)), we have

$$
\begin{aligned}
F(n_S, \lambda_F) &\equiv \left(2 - \frac{\lambda_F b_F(\lambda_F)}{r} \right) \frac{\lambda_F b_F(\lambda_F)}{r} \frac{n_S^2 b_S(\lambda_F)^2}{(b_F(\lambda_F) + n_S b_S(\lambda_F))^2} \tag{174} \\
&= \frac{(2n_S - 1)\lambda + n_S^2 \lambda_F - \sqrt{\lambda_F^2 (n_S - 2)^2 n_S^2 + \lambda^2 (1 - 2n_S)^2 + 2\lambda \lambda_F n_S (n_S(2n_S - 3) + 2)}}{2(\lambda + n_S \lambda_F)} \\
&= \frac{(2n_S - 1) + n_S x - \sqrt{x^2 (n_S - 2)^2 + (1 - 2n_S)^2 + 2x(n_S(2n_S - 3) + 2)}}{2(1 + x)} \equiv F(n_S, x),
\end{aligned}
$$

where $x \equiv n_S \lambda_F / \lambda < n_S$.

We can rewrite Equation ([163](#)) as

$$
\frac{n_S - x}{x} = \frac{1}{e^{r\Delta} - 1} F(n_S, x).
\tag{175}
$$

Taking log of both sides in Equation ([175](#)) and differentiating with respect to Δ, we get (recall $n_S = M\Delta$):

$$
\left(-\frac{1}{n_S - x} - \frac{1}{x} \right) \frac{dx}{d\Delta} + \frac{M}{n_S - x} = -\frac{r e^{r\Delta}}{e^{r\Delta} - 1} + \frac{\partial \log(F)}{\partial n_S} M + \frac{\partial \log(F)}{\partial x} \frac{dx}{d\Delta},
$$

i.e.,

$$
\frac{dx}{d\Delta} = \frac{\frac{r e^{r\Delta}}{e^{r\Delta} - 1} + \frac{1}{n_S - x} M - \frac{\partial \log(F)}{\partial n_S} M}{\frac{1}{n_S - x} + \frac{1}{x} + \frac{\partial \log(F)}{\partial x}},
\tag{176}
$$

and

$$\frac{d\log(W_F)}{d\Delta} = \left(-\frac{1}{n_S - x} - \frac{2}{x}\right)\frac{dx}{d\Delta} + \frac{M}{n_S - x} \tag{177}$$

$$\leq M\left(\left(-\frac{1}{n_S - x} - \frac{2}{x}\right)\frac{\frac{1}{n_S} + \frac{1}{n_S - x} - \frac{\partial\log(F)}{\partial n_S}}{\frac{1}{n_S - x} + \frac{1}{x} + \frac{\partial\log(F)}{\partial x}} + \frac{1}{n_S - x}\right),$$

since $W_F(\Delta) = \frac{r(n_S - x)\sigma_w^2}{2\lambda x^2}$ and $\frac{re^{r\Delta}}{e^{r\Delta}-1} \geq 1/\Delta = M/n_S$.

We calculate:

$$\frac{\partial\log(F)}{\partial x} = \frac{-1 + 2n_S + (3n_S - 2)x + \sqrt{x^2(n_S - 2)^2 + (1 - 2n_S)^2 + 2x(n_S(2n_S - 3) + 2)}}{2x(1 + x)\sqrt{x^2(n_S - 2)^2 + (1 - 2n_S)^2 + 2x(n_S(2n_S - 3) + 2)}} > 0,$$

$$\frac{\partial\log(F)}{\partial n_S} = \frac{3 - 2n_S - (n_S - 2)x + \sqrt{x^2(n_S - 2)^2 + (1 - 2n_S)^2 + 2x(n_S(2n_S - 3) + 2)}}{2(n_S - 1)\sqrt{x^2(n_S - 2)^2 + (1 - 2n_S)^2 + 2x(n_S(2n_S - 3) + 2)}} > 0. \tag{178}$$

To show that $\frac{d\log(W_F)}{d\Delta} < 0$, it suffices to show that the second line of (177) is negative, which is equivalent to

$$3 - \frac{x}{n_S} - (2n_S - x)\frac{\partial\log(F)}{\partial n_S} > x\frac{\partial\log(F)}{\partial x}. \tag{179}$$

Using the expressions in (178), it is straightforward to show that (179) holds whenever $0 \leq x \leq n_S$ and $n_S \geq 2$.

Therefore, the optimal $\Delta_F^* \leq 2/M$.

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