ASSET MANAGEMENT CONTRACTS AND EQUILIBRIUM PRICES

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ABSTRACT

We study how the agency relationship between investors and asset managers affects equilibrium prices. We begin with a static contracting model, in which the optimal contract bounds managers' portfolio risk regardless of their private information. We embed that model into an equilibrium asset-pricing model with noise traders and overlapping generations of investors and managers. Risk limits generate an inverted risk-return relationship: overvalued assets have high volatility because managers buy them during bull markets to meet risk limits. Because overvalued assets have higher share price and volatility, risk limits are more constraining when trading against overvaluation, biasing the aggregate market upward.

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1 Introduction

Financial markets have become highly institutionalized. For example, individual investors were holding directly 47.9% of U.S. stocks in 1980, but only 21.5% in 2007, with most of the remainder held by financial institutions such as mutual funds, pension funds and insurance companies (French (2008)). Financial institutions account for an even larger share of the market for bonds, derivatives and commodities.

Investment decisions in financial institutions are made by professional managers on behalf of the investors owning the assets. This generates an agency problem, as the objectives of investors and managers may not coincide. How does the agency problem manifest itself in the contracts that investors and managers write and in the portfolios that managers choose? What are the implications for equilibrium asset prices? Does the agency problem render financial markets less efficient, and if so in what ways? These are the questions that we address in this paper.

We first develop a model of optimal contracting between investors and managers that combines (i) moral hazard arising from managers' effort to acquire information and (ii) adverse selection arising from managers' preferences and the private information they may acquire. The optimal contract involves risk limits: the risk of the portfolio chosen by managers is kept within bounds, even when the optimal level of risk given the private information that managers may acquire exceeds the bounds. Investors constrain their managers in that way because the latter may not acquire information and gamble for a high fee. Risk limits are pervasive in the asset-management industry, and are often referred to as tracking-error constraints. They can concern risk measured in absolute terms or relative to a benchmark portfolio.[1]

We next embed our contracting model into a continuous-time equilibrium asset-pricing model with one riskless and multiple risky assets. The prices of the risky assets are influenced by random demand by noise traders. High demand raises prices and causes overvaluation, while low demand causes undervaluation. Managers can observe noise-trader demand and thus the direction of the

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[1]Risk limits relative to a benchmark portfolio are common for pension funds, insurance companies, endowments, sovereign wealth funds, institutional asset managers, and mutual funds. They can bound a managed portfolio's tracking error (standard deviation of the difference between the portfolio's return and the return of a benchmark portfolio), or the difference between the weight that the managed portfolio allocates to each asset class, geographical area, or industry sector, and the corresponding benchmark weight. Risk limits in absolute terms are more common for hedge funds and trading desks in broker-dealer firms. For a discussion of tracking-error constraints and their implications for financial markets, see the 2003 report by the Committee on the Global Financial System (BIS (2003)). According to that report, bounds on tracking error are on average 1% for actively-managed bond portfolios and between 2-6% for actively-managed stock portfolios (p.20). The Norwegian Sovereign Wealth Fund (NBIM), one of the largest institutional investors globally, reports the following regarding its tracking-error constraint: "The Ministry of Finance has set limits for how much risk NBIM may take in its active management of the fund. The most important limit is expressed as expected relative volatility (tracking error) and puts a ceiling on how much the return on the fund may be expected to deviate from the return on the benchmark portfolio. The expected tracking error limit is 125 basis points, or 1.25%." (https://www.nbim.no/en/investments/investment-risk/)

mispricing.

We show two main results. First, risk limits generate an inverted risk-return relationship: over-valued assets have low expected return and high volatility, while undervalued assets have high expected return and low volatility. The high volatility of overvalued assets stems from an amplification effect. Positive news to asset fundamentals cause managers' positions to become larger and risk limits to become more binding. In response to the binding limits, managers cut down on their positions. In the case of overvalued assets, this amounts to *buying* the assets because managers are shorting them to begin with (or, in an extension of our model, they are under-weighting them relative to a benchmark portfolio). Buying pressure causes prices to rise, amplifying the initial shock. Our amplification effect differs from those commonly emphasized in the literature because it concerns distortions during bubbles rather than crises.[2] The potential of risk limits to induce distortions and amplification during bubbles has been noted in the policy debate.[3] Risk-return inversion has been documented in a large empirical literature.[4]

The second result concerns the aggregate price distortion that risk limits generate. Because risk limits can prevent managers from absorbing noise-trader demand, they cause overvalued assets to become more overvalued and undervalued assets to become more undervalued. We show that the positive distortions dominate, biasing the aggregate market upward. Indeed, since overvalued assets have higher share price and volatility than undervalued ones, risk limits bind more severely for a short position in the former than for a long position of an equal number of shares in the latter. A common theme of both of our main results is that corrective forces in asset markets may

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[2]For amplification effects during crises see, for example, Bernanke and Gertler (1989), Geanakoplos (1997), Holmstrom and Tirole (1997) and Kiyotaki and Moore (1997) in macroeconomic settings, and Shleifer and Vishny (1997), Gromb and Vayanos (2002) and He and Krishnamurthy (2012, 2013) in finance settings.

[3]For example, the BIS (2003) report notes: "Overvalued assets/stocks tend to find their way into major indices, which are generally capitalization-weighted and therefore will more likely include overvalued securities than undervalued securities. Asset managers may therefore need to buy these assets even if they regard them as overvalued; otherwise they risk violating agreed tracking errors." (p.19). In a similar spirit, a 2015 IMF working paper (Jones (2015)) notes: "Another source of friction capable of amplifying bubbles stems from the captive buying of securities in momentum-biased market capitalization-weighted financial benchmarks. Underlying constituents that rise most in price will see their benchmark weights increase irrespective of fundamentals, inducing additional purchases from fund managers seeking to minimize benchmark tracking error. As a case in point, the 1980s Heisei bubble saw Japan's share of the MSCI World equity market capitalization soar from 21% in 1983 to 51% by 1989, while during the 1990s technology bubble, the technology sector weighting in the S&P500 rose from 5% in 1993 to 34% by 2000." (p.21).

[4]Haugen and Baker (1996) and Ang, Hodrick, Xing, and Zhang (2006) document that expected return is negatively related to volatility in the cross-section of U.S. stocks. The latter paper also documents a negative relationship between expected return and the idiosyncratic component of volatility. Since volatility is negatively related to expected return, it is also negatively related to CAPM alpha, which is expected return adjusted for CAPM beta, i.e., for systematic risk. Black (1972), Black, Jensen, and Scholes (1972), and Frazzini and Pedersen (2014) document that alpha is negatively related to beta in the cross-section of U.S. stocks, and Asness, Frazzini, and Pedersen (2014) find a similar relationship at the industry-sector level. The relationship between expected return and beta is almost flat during 1926-2012 (Frazzini and Pedersen (2014)), and turns negative during the second half of the sample (Baker, Bradley, and Wurgler (2011)). Alternative explanations for the inverted risk-return relationship include leverage constraints (Black (1972), Frazzini and Pedersen (2014)) and disagreement (Hong and Sraer (2016)). The leverage explanation can account for the negative relationship between beta and alpha but not for that between beta and expected return. Our explanation can account for both relationships.

be weaker during bubbles than during crises.

Section 2 develops our model of optimal contracting. A risk-averse investor can employ a manager to invest in one riskless and one risky asset. Investment takes place in one period and the assets pay off in the next and final period. The manager is either risk-averse and able to observe a private signal about the payoff of a risky asset by incurring a private cost, or risk-neutral and unable to observe the signal at any cost. If the investor employs the manager, then they agree on a fee that can be any non-negative and increasing function of the investor's final wealth. Our model builds on Vayanos (2018), who assumes symmetric return distributions. We extend his analysis to asymmetric distributions because such distributions arise in our equilibrium asset-pricing model.

To induce the risk-averse manager type to observe the signal, the investor must make the fee sufficiently high when wealth is high, i.e., when the manager has taken a large position in the risky asset and that position has performed well. Such a fee, however, can induce the risk-neutral type, who is uninformed, to also take a large position and gamble. We show that if the cost of observing the signal is sufficiently high, then the investor cannot induce the risk-neutral type to take less risk than even the best-informed risk-averse type (i.e., the type with most extreme signal realization). Intuitively, when the risk-neutral type takes less risk, she receives a lower fee conditional on her position performing well (because the position is smaller) and a higher fee conditional on her position performing poorly. Hence, her fee becomes less risky, and this attracts the risk-averse type, who the investor must expose to a high level of risk so that she observes the signal.

Since the uninformed risk-neutral type pools with the best-informed risk-averse type, the investor designs the fee so that the common position of both types in the risky asset is strictly below the optimal position given the latter type's information. The common position involves pooling by an interval of risk-averse types. Hence, it becomes independent of the signal when the signal realization is sufficiently extreme.

Section 3 develops a frictionless continuous-time equilibrium asset-pricing model, which Section 4 uses to embed our contracting model in. The model of Section 3 has one risky asset, but is extended to multiple assets in Section 4. The dividend flow of the risky asset follows a square-root process, the riskless rate is constant, and there are overlapping generations of risk-averse investors living over infinitesimal intervals. We show that this novel combination of assumptions yields a closed-form solution for the equilibrium price of the risky asset. The price is an affine increasing function of the dividend flow. The square-root specification ensures that the volatility of the asset's return per share, and hence of a position in the asset, also increases in the dividend flow. This property is critical for our subsequent analysis.

Section 4 modifies the model of Section 3 by assuming that some investors are experts, who

observe noise-trader demand and invest in the risky asset without a manager, and some are non-experts. It uses a limit version of our contracting model, when the uncertainty between the two periods is small, to deduce the optimal contract between non-experts and their managers. The risk limit depends on the distribution of expected returns, which is endogenously derived in equilibrium (and depends on the risk limit).

The equilibrium in Section 4 involves an unconstrained region, where the risk limit does not bind, and a constrained region where it binds. An increase in the asset's dividend flow moves the equilibrium towards the constrained region because the volatility of an asset position increases. The price in each region is characterized by a second-order ordinary differential equation (ODE), with smooth-pasting between regions. The solution is no longer affine or closed-form. Yet, by exploiting the structure of the ODEs, we prove existence of a solution and a number of key properties (e.g., monotonicity, convexity, comparison to the case of no risk limits), which in turn we use to prove our main results.

Section 5 extends our model to risk limits specified relative to a benchmark portfolio. Such risk limits do not come out of our contracting model, which restricts the fee to depend only on the investor's wealth. We show that asset prices have the exact same properties as in Section 4, provided that we consider positions relative to the benchmark portfolio. In particular, risk limits yield high volatility for assets that managers underweight relative to the benchmark portfolio (rather than short) and low volatility for assets that they overweight (rather than long).

The agency problem in asset management and its implications for managers' portfolio choice and equilibrium asset prices are the subject of a large theoretical literature. One strand of the literature focuses on managers' reputation concerns.[5] A second and related strand focuses on investors' decisions to invest with managers as a function of managers' past performance.[6] Our paper belongs to a third strand that studies contracts between investors and managers.[7] Our focus on risk limits and their asymmetric effects across overvalued and undervalued assets is new to the literature.

Some of the papers on asset-management contracts take prices as given and study how contracts can address the combination of (i) moral hazard arising from managers' effort to acquire

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[5]See, for example, Froot, Scharfestein, and Stein (1993), Dow and Gorton (1997), Dasgupta and Prat (2008), Dasgupta, Prat, and Verardo (2011), and Guerrieri and Kondor (2012).

[6]See, for example, Shleifer and Vishny (1997), Berk and Green (2004), Vayanos (2004), Basak, Pavlova, and Shapiro (2007), He and Krishnamurthy (2012, 2013), Kaniel and Kondor (2012), Vayanos and Woolley (2013), and Makarov and Plantin (2015).

[7]Papers within the third strand assume or derive contracts that are performance-contingent. While the fees paid by investors to asset-management firms often depend only on assets under management and not on performance, salaries paid by the firms to their manager-employees depend on performance evaluated in absolute and/or relative terms. This is noted in the BIS (2003) report (p.22-23). It is documented more extensively in Ma, Tang, and Gómez (forthcoming), who show that for 79% of the mutual funds in their sample, managers receive bonuses that are performance-contingent.

information and (ii) adverse selection arising from the private information that managers acquire. Stoughton (1993) shows that rendering managers' linear fee more sensitive to performance induces them to choose a less risky portfolio but does not change their incentives to acquire information. Admati and Pfleiderer (1997) rely on that observation to show that benchmarking distorts managers' portfolio choice without encouraging them to acquire more information. Li and Tiwari (2009) show that benchmarking can improve information-acquisition incentives when the fee has an option-like component. Dybvig, Farnsworth, and Carpenter (2010) study information acquisition and portfolio choice under general non-linear contracts. Their work builds on Demski and Sappington (1987), in which managers' action is not explicitly portfolio choice. He and Xiong (2013) show that investors may limit managers' choice of assets to enhance their information-acquisition incentives. Our contracting model has similarities with a two-state version of Dybvig, Farnsworth, and Carpenter (2010). The key difference is that we introduce the uninformed risk-neutral type, whose presence gives rise to risk limits.[8]

Other papers determine equilibrium asset prices given contracts. In Kapur and Timmermann (2005), managers receive a fee assumed to be linear in the fund's performance in absolute terms and relative to a benchmark. Benchmarking is beneficial when managers have limited liability, and raises the price of the benchmark portfolio. The same effect of benchmarking on prices arises in Cuoco and Kaniel (2011) when the fee is assumed to be linear, but can reverse when the fee has option-like components. Sato (2016) derives optimal contracts in a model with overlapping generations of managers who can abscond with a fraction of fund value. He shows that asset expected returns rise with the extent of delegation. Cvitanic and Xing (2018) derive optimal contracts in a continuous-time model with infinitely-lived managers and a similar form of moral hazard. They show that contracts are linear in absolute performance, relative performance, and a measure of quadratic variation, and that mispricings worsen when the agency problem becomes more severe.[9] Gorton, He, and Huang (2010), Huang (2018) and Sockin and Xiaolan (2018) assume moral hazard on information acquisition and show that externalities arise because contracts depend on the informativeness of equilibrium prices, which in turn depends on contracts.[10] Contracts in all of the above papers do not involve risk limits.

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[8]For papers that study contract choice taking prices as given and that do not assume or explicitly model moral hazard on information acquisition, see, for example, Bhattacharya and Pfleiderer (1985), Starks (1987), Das and Sundaram (2002), Palomino and Prat (2003), Ou-Yang (2003) and Cadenillas, Cvitanic, and Zapatero (2007).

[9]Cvitanic and Xing (2018) build on an earlier version of our paper (Buffa, Vayanos, and Woolley (2014)), considering general contracts and focusing on the case where the asset's dividend flow follows an Ornstein-Uhlenbeck rather than a square-root process. Other papers on benchmarking include Buffa and Hodor (2018), who consider managers with heterogeneous benchmarks, and Kashyap, Kovrijnykh, Li, and Pavlova (2018) who examine how the effects of benchmarking on prices feed into real investment. See also Brennan (1993), Basak and Pavlova (2013), and Qiu (2017), where managers' relative-performance concerns arise because of preferences rather than explicit contracts.

[10]Kyle, Ou-Yang, and Wei (2011) explore moral hazard on information acquisition in a model where managers are not price-takers. Equilibrium models in which managers observe private signals and that effort involves no moral hazard include Garcia and Vanden (2009), Malamud and Petrov (2014) and Garleanu and Pedersen (2018).

2 Risk Limits in a Static Contracting Model

2.1 Model

There are two periods 0 and 1. The riskless rate is zero. A risky asset pays D per share in period 1 and trades at price S per share in period 0. We assume that D takes the values $S + d$ and $S - d$, where $d > 0$. The prior probabilities of these outcomes are π_0 and $1 - \pi_0$, respectively.

An investor can invest in the risky asset by employing an asset manager. The investor has negative exponential utility over consumption in period 1, with coefficient of absolute risk aversion ρ. The manager can be risk-averse or risk-neutral. A risk-averse manager has negative exponential utility over period 1 consumption, with coefficient of absolute risk aversion $\bar{\rho}$, and can observe an informative signal about the asset payoff by incurring a private cost K. A risk-neutral manager has linear utility over period 1 consumption, and cannot observe the signal at any cost. The probability that the manager is risk-neutral is $\lambda \in (0, 1)$. Both the risk-averse and the risk-neutral manager have an outside option of zero. Our assumed heterogeneity across managers captures the idea that investors are concerned that managers may take excessive risk relative to their information. This is because the risk-neutral manager does not observe the signal but may take more risk than a risk-averse manager who observes it.

We denote the risk-averse manager's posterior probability of $S+d$ by π, and assume that π takes values in an interval $[\pi_{min}, \pi_{max}]$ with $\pi_{min} < \frac{1}{2} < \pi_{max}$, and with positive density $h(\pi)$. Setting $\bar{\pi} \equiv \max\{1 - \pi_{min}, \pi_{\max}\} > \frac{1}{2}$ and $h(\pi) = 0$ for the additional values of π, we take the interval to be $[1 - \bar{\pi}, \bar{\pi}]$. We refer to the risk-averse manager with posterior π for $S + d$ as the risk-averse type π. When not making reference to a specific posterior, e.g., before the signal is observed, we refer to the risk-averse manager as the risk-averse type. We likewise refer to the risk-neutral manager as the risk-neutral type.

If the investor employs the manager, then they agree on a contract in period 0. The contract specifies a fee $f(W)$ that is paid to the manager in period 1 and can depend on the investor's gross-of-fee wealth W in that period. Given the fee, the risk-averse type chooses whether or not to observe the signal, and the risk-averse and risk-neutral types choose a position of z shares for the investor.

We allow the fee $f(W)$ to be a general function of W subject to two restrictions. The first restriction is that the fee must be non-negative, i.e., the manager has limited liability. The second restriction, to which we refer as monotonicity, applies only to values of W that can be obtained in equilibrium. The fee levels corresponding to any two such values $W_1 > W_2$ must satisfy $f(W_1) - f(W_2) \geq \epsilon(W_1 - W_2)$, where ϵ is a positive constant that does not depend on (W_1, W_2). The

constant ϵ can be arbitrarily small, and in fact we focus on the limit of the optimal fee when ϵ goes to zero. In that limit, monotonicity requires only that the fee is non-decreasing in W across values of W that can be obtained in equilibrium.

A non-decreasing fee is economically appealing because it ensures that the manager does not engage in (unmodeled) activities that reduce W, e.g., costly round-trip transactions, so to raise her fee. If, in addition, these activities yield a slight benefit to the manager, then the fee must be strictly increasing. We impose the lower bound ϵ on the fee's slope to rule out that the investor can induce the risk-averse and risk-neutral types to choose different positions z just by exploiting their indifference over z holding the fee level constant.

If the investor does not employ the manager, then he pays no fee and cannot invest in the risky asset. Employing the manager is always optimal for the investor. Indeed, the investor can replicate the outcome of not employing the manager by employing her, setting the fee to zero, and inducing the risk-averse and risk-neutral types to choose a zero position by exploiting their indifference over z. This fee satisfies monotonicity because the only value of W that is obtained in equilibrium is zero. From now on we assume that the investor employs the manager. We also assume that the optimal contract induces the risk-averse type to observe the signal, and we determine a sufficient condition for this property to hold at the end of this section.

We set the investor's wealth in period 0 equal to zero. This assumption is without loss of generality because the investor has negative exponential utility. The investor's gross-of-fee wealth W in period 1 is given by the budget constraint

$$W = z(D - S). \tag{2.1}$$

Wealth in period 1 is equal to the capital gains between periods 0 and 1.

The investor chooses the fee $f(W)$ to maximize his utility. He is subject to the manager's incentive-compatibility (IC) constraints on whether or not to observe the signal and what position z to choose. He must also ensure that the fee satisfies non-negativity and monotonicity. Non-negativity ensures that the manager's individual rationality (IR) constraint is satisfied.

When writing the (IC) constraints, we can focus on values of W that can be obtained in equilibrium. These are the values for which the monotonicity property $f(W_1) - f(W_2) \geq \epsilon(W_1 - W_2)$ must hold. Positions that give rise to other values of W can be made dominated by extending the fee function $f(W)$ to be equal to zero for those values. The fee function can alternatively be extended so that it is non-decreasing for all W. Under either extension, the (IC) constraints can concern only (i) the positions chosen by the risk-averse and risk-neutral types, and (ii) the opposites of those positions because they yield the same values of W as the original positions with possibly different

probabilities.

The non-decreasing extension of the fee function is in Vayanos (2018). That paper develops our contracting model in the simpler cases where π is distributed symmetrically around $\frac{1}{2}$ and takes two values or a continuum of values. We extend his analysis to the continuum asymmetric case because the equilibrium model of Section 3, in which we embed the model of this section, has those features.

2.2   Optimal Contract

We denote by $z(\pi)$ the position chosen by the risk-averse type π, and by $\hat{z}$ the position chosen by the risk-neutral type. We denote by

$$U(\pi, z) \equiv - \left[\pi e^{-\bar{\rho}f(zd)} + (1 - \pi)e^{-\bar{\rho}f(-zd)} \right] \tag{2.2}$$

the utility of the risk-averse type π when she chooses position z, and set $U(\pi) \equiv U(\pi, z(\pi))$. We denote by

$$\hat{U}(z) \equiv \pi_0 f(zd) + (1 - \pi_0)f(-zd) \tag{2.3}$$

the utility of the risk-neutral type when she chooses position z, and set $\hat{U} \equiv \hat{U}(\hat{z})$. Using this notation, we next state the incentive compatibility (IC) constraints.

A first (IC) constraint concerns the decision of the risk-averse type whether or not to observe the signal. Observing the signal is optimal if

$$e^{\bar{\rho}K} \int_{1-\bar{\pi}}^{\bar{\pi}} U(\pi)h(\pi)d\pi \geq U(\pi_0). \tag{2.4}$$

The left-hand side of (2.4) is the utility when the risk-averse type observes the signal. That utility involves an integral over all possible posteriors π that the risk-averse type may have. The integral is multiplied by $e^{\bar{\rho}K}$ because the private cost K of observing the signal is subtracted from consumption. The right-hand side of (2.4) is the utility when the risk-averse type does not observe the signal. Without the signal, the risk-averse type's probability for $S + d$ is the prior π_0.

The remaining (IC) constraints concern the choice of positions by the risk-averse type after she has observed her signal, and by the risk-neutral type. The risk-averse type π must prefer the position $z(\pi)$ to positions chosen by other risk-averse types $\pi' \in [1 - \bar{\pi}, \bar{\pi}]$ and to the position $\hat{z}$ chosen by the risk-neutral type. She must also prefer $z(\pi)$ to the opposites of those positions. This

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yields the constraint

$$U(\pi) \geq \max \left\{ \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} U(\pi, z(\pi')), U(\pi, \hat{z}) \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} U(\pi, -z(\pi')), U(\pi, -\hat{z}) \right\}. \qquad (2.5)$$

The risk-neutral type prefers the position $\hat{z}$ to positions chosen by risk-averse types and to the opposite of those positions and of $\hat{z}$ if

$$\hat{U} \geq \max \left\{ \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} \hat{U}(z(\pi')), \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} \hat{U}(-z(\pi')), \hat{U}(-\hat{z}) \right\}. \qquad (2.6)$$

We denote by

$$\Delta(\pi) \equiv f(z(\pi)d) - f(-z(\pi)d)$$

the difference in fee levels across the payoff realizations $S + d$ and $S - d$ for the risk-averse type π. We denote by

$$\Delta(\hat{\pi}) \equiv f(\hat{z}d) - f(-\hat{z}d)$$

the corresponding quantity for the risk-neutral type. Lemma 2.1 derives necessary and sufficient conditions for the part of the (IC) constraint (2.5) that concerns the risk-averse types to hold.

Lemma 2.1. *The (IC) constraint*

$$U(\pi) \geq \max \left\{ \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} U(\pi, z(\pi')), \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} U(\pi, -z(\pi')) \right\} \qquad (2.7)$$

holds for all $\pi \in [1 - \bar{\pi}, \bar{\pi}]$ if and only if the following conditions (i)-(iii) hold:

(i) $\Delta(\pi)$ is non-decreasing.

(ii) If $\Delta(\pi)$ is continuous at π, then $U(\pi)$ is differentiable at π and

$$U'(\pi) = -U(\pi) \frac{e^{\bar{\rho}\Delta(\pi)} - 1}{\pi + (1 - \pi)e^{\bar{\rho}\Delta(\pi)}}. \qquad (2.8)$$

If instead $\Delta(\pi)$ is discontinuous at π, then $U(\pi)$ has left- and right-derivatives at π, which are given by substituting the left- and right-limits of $\Delta(\pi)$, respectively, into (2.8).

(iii) $U(\pi) = U(1 - \pi)$.

The function $\Delta(\pi)$ has the following additional properties:

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(iv) *By redefining $z(\pi)$ for a measure-zero set of types to another position that gives those types the same utility, we can assume $\Delta(\pi) = -\Delta(1-\pi)$ for $\pi \neq \frac{1}{2}$.*

(v) *$\Delta(\pi) \geq 0$ for $\pi \in (\frac{1}{2}, \bar{\pi}]$, and $\Delta(\pi) \leq 0$ for $\pi \in [1-\bar{\pi}, \frac{1}{2})$.*

Property (i) of Lemma 2.1 is a sorting condition. When the risk-averse type is more optimistic that state $S + d$ will occur, the fee that she receives in that state relative to state $S - d$ must be higher. The fee difference $\Delta(\pi)$ is the sorting device. Property (ii) is an envelope condition. If the fee levels $f(z(\pi)d)$ and $f(-z(\pi)d)$ are differentiable in π, then the envelope condition follows by differentiating the utility $U(\pi) \equiv U(\pi, z(\pi))$ with respect to π and using the first-order condition for $z(\pi)$. The envelope condition does not require, however, differentiability or even continuity of $f(z(\pi)d)$ and $f(-z(\pi)d)$.

Property (iii) follows because the utility of the risk-averse type π choosing a position z is the same as that of type $1 - \pi$ choosing $-z$. Hence, neither type can achieve a larger utility than the other. Utility for type π and position z is the same as for type $1 - \pi$ and position $-z$ because in both cases wealth takes the values zd and $-zd$ with probabilities π and $1 - \pi$, respectively. The same argument implies that by possibly redefining positions in cases of indifference, we can assume $z(\pi) = -z(1 - \pi)$ and hence $\Delta(\pi) = -\Delta(1 - \pi)$. While the redefinition could, in principle, concern a large set of types and hence change the investor's payoffs, Lemma 2.1 shows that a redefinition over a measure-zero set suffices to yield $\Delta(\pi) = -\Delta(1 - \pi)$ (Property (iv)). Lemma 2.2 derives properties of $z(\pi)$ that parallel those of $\Delta(\pi)$.

Lemma 2.2. *The positions $(z(\pi), \hat{z})$ have the following properties:*

(i) *$z(\pi)$ is non-decreasing.*

(ii) *$z(\pi) = -z(1 - \pi)$ for $\pi \neq \frac{1}{2}$.*

(iii) *$z(\pi) \geq 0$ for $\pi \in (\frac{1}{2}, \bar{\pi}]$, and $z(\pi) \leq 0$ for $\pi \in [1-\bar{\pi}, \frac{1}{2})$.*

Properties (i) and (ii) of Lemma 2.2 follow by combining their counterparts from Lemma 2.1 with the monotonicity of the fee. The argument is as follows. The monotonicity of the fee implies that the fee difference Δ increases strictly with position size z: a larger position z yields a larger capital gain in state $S+d$ and hence a larger fee in that state, while also yielding a larger capital loss in state $S-d$ and hence a smaller fee in that state. As a consequence, any monotonicity property of Δ extends to z. For example, since $\Delta(\pi)$ is non-decreasing (Property (i) in Lemma 2.1), $z(\pi)$ must also be non-decreasing: if $z(\pi)$ were decreasing, then $\Delta(\pi)$ would have to be decreasing. Moreover,

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since $\Delta(\pi)$ and $\Delta(1 - \pi)$ are opposites (Property (iv) in Lemma 2.1), $z(\pi)$ and $z(1 - \pi)$ must also be opposites: if, for example, $z(\pi)$ exceeded $-z(1 - \pi)$, then $\Delta(\pi)$ would also have to exceed $-\Delta(1 - \pi)$. Property (iii) follows from Properties (i) and (ii): since $z(\pi)$ is non-decreasing and $z(\pi) = -z(1 - \pi)$, $z(\pi)$ must be non-negative for $\pi > \frac{1}{2}$ and non-positive for $\pi < \frac{1}{2}$.

We next turn to the position $\hat{z}$ of the risk-neutral type. Since that type does not observe the signal, the investor wants her to choose a position that is less extreme than the position $z(\bar{\pi})$ of the most optimistic risk-averse type $\bar{\pi}$ and the position $z(1 - \bar{\pi}) = -z(\bar{\pi})$ of the most pessimistic risk-averse type $1 - \bar{\pi}$. The investor can ensure that $\hat{z}$ does not exceed $z(\bar{\pi})$ in absolute value by setting the fee $f(W)$ to zero for $W > z(\bar{\pi})d$. The investor may be unable, however, to ensure that $\hat{z}$ is (strictly) smaller than $z(\bar{\pi})$ in absolute value. In that case, the risk-neutral type is pooled with the most optimistic or most pessimistic risk-averse types. Pooling gives rise, in turn, to risk limits.

Because of the monotonicity of the fee, the investor can avoid pooling if he can induce the risk-neutral type to accept a fee difference $\hat{\Delta}$ that is smaller than $\Delta(\bar{\pi})$ in absolute value. Two countervailing effects determine the possibility of pooling. On the one hand, the most optimistic and the most pessimistic risk-averse types are better informed than the risk-neutral type, and hence tend to prefer a more extreme fee difference. This favors separation. On the other hand, risk-aversion induces a preference for a less extreme fee difference. This favors pooling.

The pooling effect dominates when the cost K of observing the signal is sufficiently high. This is because to induce the risk-averse type to observe the signal, the investor must pay her more when she achieves a favorable wealth outcome. When K is high, the fee difference between favorable and unfavorable wealth outcomes must be large. This can leave the risk-averse type exposed to more risk than she finds optimal given her information and risk-aversion. Hence, any attempt by the investor to induce a less extreme position by the risk-neutral type, by paying her less conditional on good performance and more conditional on bad performance, would attract the risk-averse type.

The condition on K under which the pooling effect dominates involves not only a lower bound but also an upper bound. This is because if K is too high, then the investor cannot induce the risk-averse type to observe the signal regardless of the fee difference. Lemma 2.3 states the pooling condition. The lower bound on K is implied by the inequality in the right-hand side and the upper bound by the inequality in the left-hand side. We set $\bar{\pi}_0 \equiv \max\{\pi_0, 1 - \pi_0\}$ and $\bar{h}(\pi) \equiv h(\pi) + h(1 - \pi)$.

Lemma 2.3. *When the pooling condition*

$$1 - \frac{\int_{\bar{\pi}_0}^{\bar{\pi}}(\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi}{1 - \bar{\pi}_0} < e^{-\bar{\rho}K} < 1 - \frac{(\bar{\pi} - \bar{\pi}_0)\int_{\bar{\pi}_0}^{\bar{\pi}}(\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi}{\bar{\pi}(1 - 2\bar{\pi}_0) + \bar{\pi}_0^2} \tag{2.9}$$

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holds, $|\hat{z}| \geq z(\bar{\pi})$.

When the pooling condition holds, the position $z(\bar{\pi})$ of the most optimistic risk-averse type $\bar{\pi}$ and $z(1 - \bar{\pi})$ of the most pessimistic risk-averse type $1 - \bar{\pi}$ are bounded in absolute value by the position $\hat{z}$ of the risk-neutral type. This ranking of positions yields the central result of this section. This is that the investor imposes, through the design of the fee, a binding limit on positions: a maximum long and a maximum short position that are opposites, and are smaller in absolute value than the optimal positions conditional on facing the most optimistic and most pessimistic risk-averse types, respectively. The limit is binding because the extreme positions attract the uninformed risk-neutral type. The risk-neutral type chooses the maximum long position if $\pi_0 > \frac{1}{2}$, chooses the maximum short position if $\pi_0 < \frac{1}{2}$, and is indifferent between the two if $\pi_0 = \frac{1}{2}$.

Because the limit is binding, it involves pooling by risk-averse types: the maximum long position is chosen by an interval of optimistic risk-averse types, and the maximum short position is chosen by an interval of pessimistic risk-averse types. Indeed, since the maximum long and maximum short positions are smaller in absolute value than the optimal positions conditional on facing the most optimistic and most pessimistic risk-averse types, respectively, they are also smaller than the optimal positions conditional on facing slightly less optimistic and slightly less pessimistic risk-averse types. The investor thus sets the positions of the latter types equal to the maximum long and maximum short position, respectively. Pooling is characterized by an interval $[\pi^*, \bar{\pi}]$ of optimistic risk-averse types who choose the maximum long position, and an interval $[1 - \bar{\pi}, 1 - \pi^*]$ of pessimistic risk-averse types who choose the maximum short position.

Theorem 2.1 determines the positions $(z(\pi), \hat{z})$ under the optimal contract when the pooling condition (2.9) holds and the lower bound ϵ on the fee's slope goes to zero. The optimal positions are expressed in terms of the fee difference $\Delta(\pi)$, which Theorem 2.1 does not characterize. Theorem 2.1 also assumes that the investor finds it optimal to induce the risk-averse type to observe the signal. We return to the determination of $\Delta(\pi)$ and to the optimality of information acquisition in the special case that we consider in Section 2.3.

Theorem 2.1. *Suppose that the pooling condition (2.9) holds and that the investor finds it optimal to induce the risk-averse type to observe the signal. In the limit when ϵ goes to zero:*

- *The risk-averse types $\pi \in [\pi^*, \bar{\pi}]$ pool at the common position*

$$z(\bar{\pi}) = \frac{1}{2\rho d} \log \left(\frac{\pi^*}{1 - \pi^*} \right) + \frac{\Delta(\bar{\pi})}{2d}. \tag{2.10}$$

  The risk-averse types $\pi \in [1 - \bar{\pi}, 1 - \pi^*]$ pool at $-z(\bar{\pi})$.

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- *The risk-averse types $\pi \in [1 - \pi^*, \pi^*]$ choose the distinct positions*

$$z(\pi) = \frac{1}{2\rho d} \log\left(\frac{\pi}{1 - \pi}\right) + \frac{\Delta(\pi)}{2d}. \tag{2.11}$$

- *The risk-neutral type chooses $z(\bar{\pi})$ if $\pi_0 > \frac{1}{2}$, chooses $-z(\bar{\pi})$ if $\pi_0 < \frac{1}{2}$, and is indifferent between $z(\bar{\pi})$ and $-z(\bar{\pi})$ if $\pi_0 = \frac{1}{2}$.*

- *The pooling threshold π^* is the unique solution in $(\bar{\pi}_0, \bar{\pi})$ of*

$$(1 - \lambda) \int_{\pi^*}^{\bar{\pi}} (\pi - \pi^*)\bar{h}(\pi)d\pi = \lambda(\pi^* - \bar{\pi}_0), \tag{2.12}$$

  and decreases in λ.

Equations (2.10) and (2.11) characterize the optimal positions $(z(\pi), \hat{z})$. The relevant risk exposure that a position $z(\pi)$ generates for the investor is $z(\pi) - \frac{\Delta(\pi)}{2d}$. This is because the exposure that the investor passes on to the manager through the fee has to be netted out.

Equation (2.12) determines the pooling threshold π^*. Reducing π^*, hence tightening the risk limit, lowers the positions of the risk-averse types $\pi \in [\pi^*, \bar{\pi}] \cup [1 - \bar{\pi}, 1 - \pi^*]$ further below their optimal level, in absolute value. The left-hand side of (2.12) reflects this cost of the risk limit. On the other hand, reducing π^* brings the position of the uninformed risk-neutral type closer to its optimal level, which corresponds to the prior belief π_0. The right-hand side of (2.12) reflects this benefit of the risk limit. The optimal π^* equates the cost to the benefit. An increase in the probability λ of the risk-neutral type raises the benefit and lowers the cost, hence tightening the risk limit.

2.3 Special Case

To embed the static contracting model into the continuous-time equilibrium model of the following sections, we specialize it in two ways. First, we take the risk-aversion coefficient $\bar{\rho}$ of the manager to be large relative to that of the investor. This assumption captures the idea that the investor is large relative to the manager, or equivalently that agents are identical but there are many investors for any given manager.[11] Second, we take uncertainty, measured by d, to be small. This assumption

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[11]Suppose that the investor is an aggregate of N individual investors, each with risk-aversion coefficient ρ, and that the manager has also risk-aversion coefficient ρ. Define one share of the risky asset to pay off $(S + d)N$ or $(S - d)N$ instead of $S + d$ or $S - d$. The risk-aversion coefficient of the investor group is $\frac{\rho}{N}$. Redefine the numeraire so that one new unit is N old units. The payoff of one share then becomes $S + d$ or $S - d$. Absolute risk-aversion coefficients are multiplied by N, so the risk aversion of the investor group becomes ρ and that of the manager becomes $N\rho \equiv \bar{\rho}$. Moreover, the cost of observing the signal becomes $\frac{K}{N}$, which is why we take it to be inversely proportional to N in what follows.

corresponds to periods being short, and allows us to bring the model to continuous time. We take the probabilities $(\pi_0, \bar{\pi})$ to be $(\frac{1}{2}(1 + \mu_0 d), \frac{1}{2}(1 + \bar{\mu}d))$, the risk-aversion coefficient $\bar{\rho}$ of the manager to be $N\rho$, and the cost K of observing the signal to be $\frac{kd}{N\rho}$, where N is large, d is small, and $(\mu_0, \bar{\mu}, k, \rho)$ are held constant. We define μ by $\pi \equiv \frac{1}{2}(1 + \mu d)$, denote by $h(\mu)$ the continuous density of μ in the interval $[-\bar{\mu}, \bar{\mu}]$, and set $\bar{h}(\mu) \equiv h(\mu) + h(-\mu)$. We treat $(z(\pi), \Delta(\pi))$ as functions of μ rather than π, and refer to risk-averse types accordingly.

Since the (IC) constraints (2.4) and (2.5) involve the product of $\bar{\rho}$ with the fee, they imply that the fee is of order $\frac{1}{N}$. Hence, when N is large, the fee is negligible relative to the investor's wealth. When, in addition, N is of order larger than $\frac{1}{d}$, the fee's contribution $\frac{\Delta(\mu)}{2d}$ to the optimal positions $(z(\mu), \hat{z})$ is also negligible, and hence the investor's risk exposure gross and net of the fee is approximately the same. Building on these observations, Proposition 2.1 determines a condition under which the investor induces the risk-averse type to observe the signal, and computes the asymptotic behavior of the optimal positions $(z(\mu), \hat{z})$.

Proposition 2.1. *Suppose that* $(\pi_0, \bar{\pi}) = (\frac{1}{2}(1 + \mu_0 d), \frac{1}{2}(1 + \bar{\mu}d))$, $\bar{\rho} = N\rho$, *and* $K = \frac{kd}{N\rho}$, *where* d *is small and* N *is large. In the limit when* ϵ *goes to zero,*

- *The investor induces the risk-averse type to observe the signal if* N *is of order larger than* $\frac{1}{d^2}$. *The pooling condition (2.9) holds if* $k < 2 \int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \bar{h}(\mu) d\mu$.

- *Under these conditions,*

    - *The common position of the risk-averse types* $\mu \in [\mu^*, \bar{\mu}]$ *is* $z(\bar{\mu}) = \frac{\mu^*}{\rho} + o(1)$, *where* $o(x)$ *are terms of order smaller than* x. *The common position of the risk-averse types* $\mu \in [-\bar{\mu}, -\mu^*]$ *is* $-z(\bar{\mu}) = -\frac{\mu^*}{\rho} + o(1)$.

    - *The positions* $z(\mu)$ *of the risk-averse types* $\mu \in [1 - \mu^*, \mu^*]$ *are* $z(\mu) = \frac{\mu}{\rho} + o(1)$.

    - *The pooling threshold* μ^* *(defined as* $\pi^* = \frac{1}{2}(1 + \mu^* d)$*) is the unique solution in* $(|\mu_0|, \bar{\mu})$ *of*

$$(1 - \lambda) \int_{\mu^*}^{\bar{\mu}} (\mu - \mu^*) \bar{h}(\mu) d\mu = \lambda(\mu^* - |\mu_0|). \tag{2.13}$$

Proposition 2.1 implies that when N is large and d is small, the static contracting model is equivalent to a simple reduced-form model without a manager. With probability $1 - \lambda$ in the latter model, the investor chooses his optimal position as if he knows the parameter μ that characterizes the asset's expected return, but he is subject to a risk limit. The investor's optimal position conditional on μ is $\frac{\mu}{\rho}$, and the risk limit requires that the position does not exceed $\frac{\mu^*}{\rho}$ in absolute

value. With probability λ, the investor chooses one of the positions $\frac{\mu^*}{\rho}$ and $-\frac{\mu^*}{\rho}$. We use the reduced-form model in the equilibrium analysis in Section 4.

3   A Dynamic Asset-Pricing Model Without Risk Limits

3.1   Model

Time t is continuous and goes from zero to infinity. The riskless rate is exogenous and equal to $r > 0$. A risky asset pays a dividend flow D_t per share and is in supply of θ shares. The price S_t per share of the risky asset is determined endogenously in equilibrium. The supply θ can result from the asset issuer and from noise traders. We allow θ to take both positive and negative values. Negative values arise when the demand by noise traders exceeds the supply by the asset issuer.

The risky asset's return per share in excess of the riskless rate is

$$dR_t^{sh} \equiv D_t dt + dS_t - r S_t dt, \tag{3.1}$$

and the risky asset's return per dollar in excess of the riskless rate is

$$dR_t \equiv \frac{dR_t^{sh}}{S_t} = \frac{D_t dt + dS_t}{S_t} - r dt. \tag{3.2}$$

For simplicity, we refer to dR_t^{sh} and dR_t as share return and dollar return, respectively, omitting that they are in excess of the riskless rate. The share return is convenient when deriving the equilibrium. The dollar return is more commonly used, and we focus on it when showing properties of return moments. We refer to the dollar return simply as return.

The dividend flow D_t follows a square-root process

$$dD_t = \kappa \left(\bar{D} - D_t \right) dt + \sigma \sqrt{D_t} dB_t, \tag{3.3}$$

where $(\kappa, \bar{D}, \sigma)$ are positive constants and dB_t is a Brownian motion. The square-root specification (3.3) allows for closed-form solutions, while also ensuring that dividends remain positive. A property of the square-root specification that is key for our analysis is that the volatility of dividends per share (i.e., of D_t) increases with the dividend level. This property is realistic: if a firm becomes larger and keeps the number of its shares constant, then its dividends per share become more uncertain in absolute terms (but not necessarily as fraction of the firm's size).[12]

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[12]Dividends are often assumed to follow a geometric Brownian motion (GBM). Under the GBM specification, the volatility of dividends per share is proportional to the dividend level. Hence, the volatility of dividends per share increases with the dividend level, exactly as under the square-root specification. The two specifications have different

There are overlapping generations of investors living over infinitesimal periods. Each generation forms a continuum with measure one. An investor belonging to the generation born at time t invests in the riskless and in the risky asset. He receives the proceeds of his investment at time $t + dt$, consumes, and then dies. The investor has negative exponential utility over consumption at time $t + dt$, with coefficient of absolute risk aversion ρ. We denote by z_t the investor's position in the risky asset, expressed in terms of number of shares. Without loss of generality, we set the investor's wealth at time t equal to zero. The investor's position in the riskless asset thus is $-z_t S_t$, and his wealth at time $t + dt$ is given by the budget constraint

$$dW_t = z_t(D_t dt + dS_t) - r z_t S_t dt = z_t dR_t. \tag{3.4}$$

Wealth at time $t + dt$ is equal to the capital gains between t and $t + dt$. These are, in turn, equal to the number of shares z_t times the share return dR_t.

The investor chooses the number of shares z_t to maximize $-\exp(-\rho dW_{t+dt})$ subject to (3.4). Given that uncertainty is Brownian, the investor's objective is equivalent to

$$\mathbb{E}_t(dW_t) - \frac{\rho}{2}\mathbb{V}\mathrm{ar}_t(dW_t), \tag{3.5}$$

a mean-variance objective over infinitesimal changes in wealth. The equilibrium price S_t of the risky asset must be such that the solution to the investor's maximization problem is $z_t = \theta$.

3.2   Equilibrium

The equilibrium price S_t is a function of the dividend flow D_t, which is the only state variable in the model. Denoting this function by $S(D_t)$, we can write the share return dR_t^{sh} as

$$
\begin{aligned}
dR_t^{sh} &= D_t dt + dS(D_t) - rS(D_t)dt \\
&= \left[D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) \right] dt + \sigma\sqrt{D_t}S'(D_t)dB_t,
\end{aligned} \tag{3.6}
$$

where the second step follows from (3.3) and Ito's lemma.

Using the budget constraint (3.4), we can write the investor's objective as

$$z_t \mathbb{E}_t(dR_t^{sh}) - \frac{\rho}{2}z_t^2 \mathbb{V}\mathrm{ar}_t(dR_t^{sh}).$$

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implications for the volatility of dividends per share as *fraction* of the dividend level. Under the GBM specification that quantity is independent of the dividend level, while under the square-root specification it decreases with the dividend level. We adopt the square-root over the GBM specification because of tractability.

The first-order condition with respect to z_t is

$$\mathbb{E}_t(dR_t^{sh}) = \rho z_t \text{Var}_t(dR_t^{sh}). \tag{3.7}$$

The expected share return $\mathbb{E}_t(dR_t^{sh})$ is the drift term in (3.6), and the share return variance $\text{Var}_t(dR_t^{sh})$ is the square of the diffusion term. Moreover, market clearing implies $z_t = \theta$. Making these substitutions in (3.7), we find the following ordinary differential equation (ODE) for the function $S(D_t)$:

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \rho\theta\sigma^2 D_t S'(D_t)^2. \tag{3.8}$$

The ODE (3.8) is second-order and non-linear, and must be solved over $(0, \infty)$. We require that its solution $S(D_t)$ has a derivative that converges to finite limits at zero and infinity. This yields one boundary condition at zero and one at infinity.

We look for an affine solution to the ODE (3.8):

$$S(D_t) = a_0 + a_1 D_t, \tag{3.9}$$

where (a_0, a_1) are constant coefficients. This function satisfies the boundary conditions since its derivative is constant. Substituting this function into (3.8) and identifying terms, we can compute (a_0, a_1).

Proposition 3.1. *Suppose $\theta > -\frac{(r+\kappa)^2}{4\rho\sigma^2}$. The equilibrium price S_t of the risky asset is given by (3.9) with*

$$a_0 = \frac{\kappa}{r}a_1\bar{D}, \tag{3.10}$$

$$a_1 = \frac{2}{(r + \kappa) + \sqrt{(r + \kappa)^2 + 4\rho\theta\sigma^2}}. \tag{3.11}$$

The intuition for (3.10) and (3.11) is as follows. The coefficient a_1 is the sensitivity of the price S_t to changes in the dividend flow D_t. Consider a unit increase in D_t. If the supply θ of the risky asset is equal to zero, then (3.11) implies that the price S_t increases by $a_1 = \frac{1}{r+\kappa}$. This is the present value of the increase in future expected dividends discounted at the riskless rate r. Indeed, a unit increase in D_t raises the expected dividend flow $E_t(D_{t'})$ at time $t' > t$ by $e^{-\kappa(t'-t)}$. Hence, the present value of future expected dividends increases by

$$\int_t^\infty e^{-\kappa(t'-t)} e^{-r(t'-t)} dt' = \frac{1}{r + \kappa}.$$

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If the supply θ of the risky asset is positive, then the price S_t increases by $a_1 < \frac{1}{r+\kappa}$ in response to a unit increase in D_t. This is because the increase in D_t not only raises expected dividends, but also makes them riskier due to the square-root specification of D_t. Moreover, this risk raises the discount rate when θ is positive because in equilibrium the investors hold the θ shares, so the risky asset covaries positively with their wealth. If instead $\theta < 0$, then the risk lowers the discount rate, and the price S_t increases by $a_1 > \frac{1}{r+\kappa}$. Equation (3.11) implies that a_1 decreases in θ, and this effect is stronger if the volatility parameter σ and the investors' risk-aversion coefficient ρ are larger.

The coefficient a_0 is equal to the price level when the dividend flow D_t is zero. If the mean-reversion parameter κ were equal to zero, and hence the dividend flow were to stay at zero forever, then a_0 would be equal to zero. Because, however, κ is positive, and hence the dividend flow returns with certainty to positive values, a_0 is positive. Moreover, a_0 inherits properties of a_1 since the larger a_1 is, the more the price increases when the dividend flow becomes positive. In particular, a_0 decreases in the supply θ of the risky asset.

Since an increase in θ lowers a_1, it makes the price less sensitive to changes in the dividend flow D_t. Since it also lowers a_0, it lowers the price for any value of D_t. Proposition 3.2 derives the effect of θ on the expected return and the return volatility of the risky asset.

Proposition 3.2. *An increase in θ raises the asset's conditional expected return $\mathbb{E}_t(dR_t)$ and leaves the return's conditional volatility $\sqrt{\mathbb{V}\mathrm{ar}_t(dR_t)}$ unaffected. The effects on the unconditional values of these variables, $\mathbb{E}(dR_t)$ and $\sqrt{\mathbb{V}\mathrm{ar}(dR_t)}$, are the same as on the conditional values.*

Recall from (3.2) that the return of the risky asset is

$$dR_t = \frac{D_t}{S_t}dt + \frac{dS_t}{S_t} - rdt.$$

The volatility of that return is caused by the term $\frac{dS_t}{S_t}$, i.e., the capital gains per dollar invested. Since an increase in θ lowers the sensitivity a_1 of the price S_t to changes in the dividend flow D_t, it makes the capital gains $dS_t = a_1 dD_t$ per share less volatile. At the same time, the share price $S_t = a_0 + a_1 D_t$ also decreases. Because θ has the same percentage effect on a_0 and a_1, the capital gains $\frac{dS_t}{S_t}$ per dollar invested do not change, and neither does return volatility $\sqrt{\mathbb{V}\mathrm{ar}_t(dR_t)}$. On the other hand, expected return $\mathbb{E}(dR_t)$ increases because of the term $\frac{D_t}{S_t}dt$, i.e., the dividends per dollar invested. An increase in θ does not affect the dividend flow D_t per share but lowers the share price S_t.[13]

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[13]The price remains affine and the comparative statics in Proposition 3.2 still hold when there is an infinitely lived representative investor with negative exponential utility over intertemporal consumption, rather than overlapping

4 A Dynamic Asset-Pricing Model With Risk Limits

4.1 Model

The model is as described in Section 3 except that investors are divided into experts and non-experts. The optimization problem of experts is as in Section 3, and we denote by z_{1t} the optimal number of shares. The behavior of non-experts follows instead the reduced-form model derived in Section 2.3. With probability $1 - \lambda$, a non-expert solves the same optimization problem as an expert but is subject to a risk limit. We denote by z_{2t} the optimal number of shares coming out of the constrained problem. With probability λ, a non-expert chooses a position right at the risk limit. The mass of non-experts is $x \in [0, 1)$.

To specify the risk limit and explain why the reduced-form model is applicable, we recall the static contracting model of Section 2 from which the reduced-form model derives. The payoff of the risky asset can take two values yielding opposite capital gains d and $-d$ per share. The investor cannot invest in the risky asset on his own and must rely on a manager. The manager can observe a signal about the probabilities of the two values.

Suppose, in line with the model of Section 2, that a non-expert cannot invest in the risky asset on his own and must rely on a manager. Suppose additionally that a non-expert observes neither the asset supply θ nor the dividend flow D_t, and the same is true for a manager who does not incur the observation cost. These agents use the unconditional distribution of (θ, D_t).[14] On the other hand, a manager who incurs the cost observes (θ, D_t), and so does an expert. We assume that the observation cost is such that non-experts choose to induce risk-averse manager types to observe (θ, D_t). The supply θ can be uncertain because noise-trader demand is random.

Under the above assumptions, we can embed the static contracting model of Section 2 into the continuous-time equilibrium model of Section 3 if we can discretize the latter so that (i) the asset payoff can take two values in each period, and (ii) the resulting capital gains are opposite and independent of (θ, D_t). Ensuring (i) is straightforward, as a Brownian motion can be discretized to take two values in each period. To ensure (ii), we use two degrees of freedom. First, by redefining

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generations of investors living over infinitesimal periods. This is shown in an earlier version of our paper (Buffa, Vayanos, and Woolley (2014)). A natural question is how the comparative statics extend to the case where the investor has constant relative rather than constant absolute risk aversion utility. We have investigated this question in a three-period binomial model with one riskless and one risky asset and with utility over terminal consumption. We assume that noise traders cause the supply θ of the risky asset available to the investor to change in the first period. The investor's wealth remains the same before and after he trades with them, while his holdings B_0 of the riskless asset change. An increase in θ lowers the price of the risky asset and raises its expected return, as in Proposition 3.2. The return volatility, which is independent of θ in Proposition 3.2, takes the same value for $\theta = 0$ and for θ such that $B_0 = 0$, and is inverse hump-shaped in-between.

[14]We can rule out that these agents learn from the price S_t by assuming that non-experts do not observe S_t and that managers must trade before observing S_t via a market order (i.e., a price-inelastic demand function).

an asset share, we can multiply the two values of the capital gains (per share) by the same scalar and render their difference independent of (θ, D_t). The redefinition does not affect the contracting problem since a non-expert contracts with his manager only on his wealth and not on the share price. Second, we can change one of the two values, holding their difference constant, so that it becomes independent of (θ, D_t), while also changing the probabilities so that the asset's expected return does not change.

We next determine the risk limit that the model of Section 2 implies when it is embedded into the model of Section 3 in the way described above. The risk limit in Section 2.3 is of the form $|z_t| \leq \frac{\mu^*}{\rho}$. The quantity z_t represents redefined shares, for which the difference between the two values of the capital gains is independent of (θ, D_t). Using (3.6) and taking dB_t to be plus or minus $\sqrt{dt}$, the difference between the two values of the capital gains before the redefinition is $2\sigma\sqrt{D_t}S'(D_t)\sqrt{dt}$. Hence, if the difference after the redefinition is set to $2\sqrt{dt}$, one share corresponds to $\sigma\sqrt{D_t}S'(D_t)$ redefined shares, and the risk limit is

$$|z_{2t}|\sigma\sqrt{D_t}S'(D_t) \leq \frac{\mu^*}{\rho}. \tag{4.1}$$

Intuitively, (4.1) requires that the volatility (standard deviation) of the manager's position cannot exceed a threshold.

The risk-limit parameter μ^* is determined from (2.13), which involves the distribution $h(\mu)$ of μ. To determine $h(\mu)$, we note that the expected return per share in the contracting model of Section 2.3 is

$$\mathbb{E}(\hat{R}^{sh}) = \frac{1}{2}(1 + \mu d)d + \frac{1}{2}(1 - \mu d)(-d) = \mu d^2$$

and the variance of the return per share is $\mathbb{V}\mathrm{ar}(\hat{R}^{sh}) = d^2 - \mu^2 d^4$. Hence, for small d, $\mu = \frac{\mathbb{E}(\hat{R}^{sh})}{\mathbb{V}\mathrm{ar}(\hat{R}^{sh})}$. Since the return per share $\hat{R}^{sh}$ concerns redefined shares, $\sigma\sqrt{D_t}S'(D_t)$ of which correspond to one share,

$$\mu = \frac{\mathbb{E}_t\left(\frac{dR_t^{sh}}{\sigma\sqrt{D_t}S'(D_t)}\right)}{\mathbb{V}\mathrm{ar}_t\left(\frac{dR_t^{sh}}{\sigma\sqrt{D_t}S'(D_t)}\right)} = \frac{D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t)}{\sigma\sqrt{D_t}S'(D_t)}, \tag{4.2}$$

where the second step follows from (3.6). The distribution $h(\mu)$ of μ is determined by (4.2). It depends on the unconditional distribution of (θ, D_t) and on the equilibrium form of the price function $S(D_t)$ (which depends on θ).

In the model of Section 2.3, the risk-neutral manager type chooses the maximum long position if the unconditional expectation μ_0 of μ is positive, chooses the maximum short position if $\mu_0 < 0$,

and is indifferent between the two if $\mu_0 = 0$. For the equilibrium analysis in this section, it is convenient that the risk-neutral type is equally likely to choose either position even when $\mu_0 \neq 0$. Under this condition, the positions chosen by risk-neutral types cancel in the aggregate, as half are equal to the maximum long position and half to the maximum short position. The market-clearing condition becomes $(1 - x)z_{1t} + (1 - \lambda)xz_{2t} = \theta$, and involves only the position z_{1t} of experts and the position z_{2t} of non-experts who employ risk-averse manager types.

To ensure that risk-neutral types invest in the manner described above, we assume that each of them has prior belief $\pi_0 = \frac{1}{2}(1 + |\mu_0|d)$ or $\pi_0 = \frac{1}{2}(1 - |\mu_0|d)$, and the two priors are independent across types and equally likely. (Thus, half of the risk-neutral types have a wrong prior.) The model of Section 2 can accommodate this extension. Corollary A.1 shows that Proposition 2.1 carries through provided that μ_0 in (2.13) is replaced by zero.

In Sections 4.2 and 4.3 we derive properties of the equilibrium for given values of θ and μ^*. The endogenous determination of μ^*, as a function of λ and other exogenous parameters, is not essential for the qualitative properties of the equilibrium. The same is true in Section 4.4, where we determine properties of the equilibrium that involve expectations over θ. The endogenous determination of μ^* becomes essential in Section 4.5 where we examine how the equilibrium changes when λ changes. This is because μ^* changes endogenously as part of the equilibrium.

4.2 Equilibrium

The first-order condition of an expert is (3.7), with z_{1t} replacing z_t. To determine the position z_{2t} of a non-expert, we distinguish cases depending on whether the risk limit binds or not. We occasionally refer to the position of a non-expert as being chosen by a manager, even though managers are absent from the reduced-form model derived in Section 2.3.

Consider first the unconstrained region, where the risk limit (4.1) does not bind. The first-order condition of a non-expert is (3.7), with z_{2t} replacing z_t. Since the first-order conditions of an expert and a non-expert are identical, $z_{1t} = z_{2t}$. Setting $z_{1t} = z_{2t}$ into the market-clearing condition, we find

$$z_{1t} = z_{2t} = \frac{\theta}{1 - \lambda x}. \tag{4.3}$$

Substituting z_{1t} from (4.3) into the first-order condition of an expert, we find the ODE

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \frac{\rho\theta}{1 - \lambda x}\sigma^2 D_t S'(D_t)^2. \tag{4.4}$$

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Substituting z_{2t} from (4.3) into (4.1), we find that the unconstrained region is defined by

$$\frac{|\theta|}{1-\lambda x}\sigma\sqrt{D_t}S'(D_t) \le \frac{\mu^*}{\rho}. \tag{4.5}$$

Consider next the constrained region, where (4.1) holds as an equality and the risk limit binds. Using the market-clearing condition to write z_{1t} as a function of z_{2t}, and substituting into the first-order condition of an expert, we find

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \rho\frac{\theta - (1-\lambda)xz_{2t}}{1-x}\sigma^2 D_t S'(D_t)^2. \tag{4.6}$$

If $\theta > 0$, then the asset's expected return is positive and so is z_{2t}. Conversely, if $\theta < 0$, then $z_{2t} < 0$. Using these observations to substitute z_{2t} from (4.1), which holds as an equality in the constrained region, into (4.6), we find the ODE

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \frac{\rho\theta}{1-x}\sigma^2 D_t S'(D_t)^2 - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{D_t}S'(D_t). \tag{4.7}$$

The function $\text{sgn}(\theta)$ is the sign function, equal to one if $\theta > 0$ and to minus one if $\theta < 0$. The constrained region is defined by the opposite inequality to (4.5), i.e.,

$$\frac{|\theta|}{1-\lambda x}\sigma\sqrt{D_t}S'(D_t) > \frac{\mu^*}{\rho}. \tag{4.8}$$

The price function $S(D_t)$ solves the ODE (4.4) in the unconstrained region (4.5), and (4.7) in the constrained region (4.8). The two ODEs are second-order and non-linear, and must be solved as a system over $(0, \infty)$. As in Section 3.2, we require that $S'(D_t)$ converges to finite limits at zero and infinity.

Since $S'(D_t)$ converges to a finite limit at zero, values of D_t close to zero belong to the unconstrained region (4.5). Conversely, since $S'(D_t)$ converges to a finite limit at infinity, values of D_t close to infinity belong to the constrained region (4.8). Hence, the unconstrained and constrained regions are separated by at least one boundary point and more generally by an odd number of such points. At a boundary point D^*, the values of $S(D^*)$ implied by the two ODEs must be equal, and the same is true for the values of $S'(D^*)$. These are the smooth-pasting conditions. The boundary points must be solved together with the ODEs. This makes the problem a free-boundary one.

The ODE (4.4) has an affine solution same as the one derived in Proposition 3.1, with $\frac{\theta}{1-\lambda x}$ replacing θ. That solution, however, does not satisfy the ODE (4.7). Hence, it represents the equilibrium price only when non-experts do not impose a risk limit by setting μ^* to infinity rather

than to the optimal finite value given in Proposition 2.1. While a closed-form solution to the ODEs (4.4) and (4.7) for finite μ^* is not available, we can prove existence of a solution and a number of key properties. Our proof approach follows that in Kondor and Vayanos (2018), although the specific arguments differ.[15]

Theorem 4.1. *Suppose* $\theta > -\frac{(1-x)(r+\kappa)^2}{4\rho\sigma^2}$ *and* $\kappa\bar{D} > \frac{\sigma^2}{4}$. *A solution* $S(D_t)$ *to the system of ODEs (4.4) in the unconstrained region (4.5), and (4.7) in the constrained region (4.8), with a derivative that converges to finite limits at zero and infinity, exists and has the following properties:*

- *It is increasing.*

- *It lies below the affine solution derived for* $\mu^* = \infty$ *when* $\theta > 0$, *and above it when* $\theta < 0$.

- *Its derivative* $S'(D_t)$ *lies below the derivative of the affine solution derived for* $\mu^* = \infty$ *when* $\theta > 0$, *and above it when* $\theta < 0$.

- *It is concave when* $\theta > 0$, *and convex when* $\theta < 0$.

- *The unconstrained and constrained regions are separated by only one boundary point* D^*.

Theorem 4.1 confirms that an increase in the dividend flow D_t raises the price S_t. It also shows that the risk limit exacerbates the effects that noise-trader demand has on the price. Indeed, consider the case where non-experts do not impose a risk limit by setting $\mu^* = \infty$. The price is then given by the affine solution in Proposition 3.1, with $\frac{\theta}{1-\lambda x}$ replacing θ. Under that solution, the price is higher when $\theta < 0$, corresponding to high noise-trader demand, than when $\theta > 0$, corresponding to low noise-trader demand. Consider next the case where non-experts set μ^* to its optimal finite value. Theorem 4.1 shows that the price increases uniformly (lies above the affine solution) for $\theta < 0$, and decreases uniformly (lies below the affine solution) for $\theta > 0$. Hence, the difference between the price for $\theta < 0$ and $\theta > 0$ increases: the asset becomes even more expensive when noise-trader demand is high, and becomes even cheaper when noise-trader demand is low.

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[15]A key difficulty in proving existence is that a solution must be found over the open interval $(0, \infty)$, with a boundary condition at each end. To address this difficulty, we start with a compact interval $[\epsilon, M] \subset (0, \infty)$ and show that there exists a unique solution to the ODEs with one boundary condition at ϵ and one at M. The boundary conditions are derived from the limits of $S'(D_t)$ at zero and infinity. In the case of M, for example, the requirement that $S'(D_t)$ has a finite limit at infinity determines that limit uniquely, and we set $S'(M)$ equal to that value. To construct the solution over $[\epsilon, M]$, we use $S'(M)$ and an arbitrary value for $S''(M)$ as initial conditions for the ODEs at M, and show that there exists a unique $S''(M)$ so that the boundary condition at ϵ is satisfied. Showing uniqueness uses continuity of solutions with respect to the initial conditions, as well as a monotonicity property with respect to the initial conditions that follows from the structure of our ODEs. We next show that when ϵ converges to zero and M to infinity, the solution over $[\epsilon, M]$ converges to a solution over $(0, \infty)$. The monotonicity property of solutions with respect to the initial conditions is key to the convergence proof because it yields monotonicity of the solution with respect to ϵ and M.

Intuitively, the risk limit exacerbates the effects that noise-trader demand has on the price because it can prevent the managers employed by non-experts from absorbing that demand.

The risk limit exacerbates the effects of noise-trader demand not only on the price level but also on the price sensitivity to changes in the dividend flow D_t. Recall from Proposition 3.1 that in the absence of a risk limit ($\mu^* = \infty$), the price is more sensitive to D_t when $\theta < 0$ than when $\theta > 0$. Theorem 4.1 shows that in the presence of a risk limit (μ^* finite), the price becomes even more sensitive to D_t when $\theta < 0$ and even less sensitive when $\theta > 0$. These effects are driven by the same forces as the non-linearities, to which we next turn.

Theorem 4.1 shows that the price is non-linear in D_t: it is (strictly) concave for $\theta > 0$ and (strictly) convex for $\theta < 0$. These effects are driven by the trading of the managers employed by non-experts in response to their risk limit. Suppose that $\theta > 0$ and D_t is in the constrained region. Following an increase in D_t, the non-experts' long positions go up in value and their volatility rises. The risk limit induces the managers employed by non-experts to cut those positions. They thus sell some shares of the asset to experts, and these sales *dampen* the price rise. The dampening effect is weaker when D_t is smaller and in the unconstrained region because it concerns not actual sales but an expectation that sales might occur in the future. The price increase is thus smaller for larger D_t, resulting in concavity. Conversely, suppose that $\theta < 0$ and D_t is in the constrained region. Following an increase in D_t, the managers employed by non-experts cut their short positions, and thus buy from experts. These purchases *amplify* the price rise. The amplification effect is weaker when D_t is smaller and in the unconstrained region, resulting in convexity.

Because of the amplification effect, the price is more sensitive to changes in D_t than in the absence of a risk limit when $\theta < 0$. Conversely, because of the dampening effect, the price is less sensitive to changes in D_t than in the absence of a risk limit when $\theta > 0$.

Figure 1 illustrates the properties of the price shown in Theorem 4.1 using a numerical example. The figure's left panel plots the price as a function of D_t. The thin lines represent the price in the absence of a risk limit ($\mu^* = \infty$) and the thick lines the price with a risk limit (μ^* finite). In each case, the solid blue line corresponds to a positive value of θ and the dashed red line to the opposite negative value. The figure's right panel plots the position of non-experts using the same conventions. Besides confirming the properties shown in Theorem 4.1, the figure shows that the risk limit has a larger effect on prices and positions when $\theta < 0$ than when $\theta > 0$. We return to this point in Section 4.4, where we analyze overvaluation bias.

Figure 1: Effect of Risk Limit on Prices and Positions
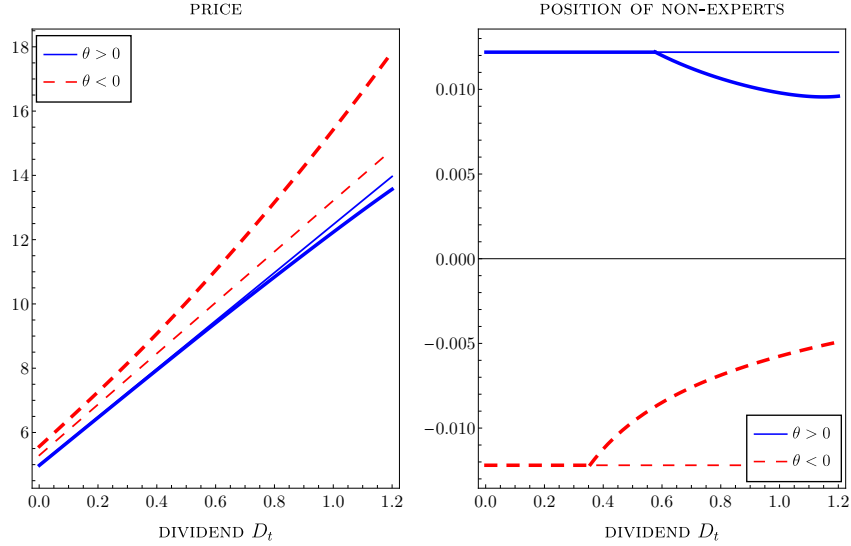
Asset price S_t (left panel) and position z_{2t} of non-experts (right panel) as a function of the dividend flow D_t. The thin lines in each panel represent the price in the absence of a risk limit ($\mu^* = \infty$) and the thick lines the price with a risk limit (μ^* finite). In each case, the solid blue line corresponds to a positive value of supply θ and the dashed red line to the opposite negative value. Parameter values are: $r = 0.03$, $\kappa = 0.1$, $\bar{D} = 0.2$, $\sigma = 0.2$, $\theta \in \{0.01, -0.01\}$, $\rho = 1$, $x = 0.9$, $\lambda = 0.2$.

4.3   Risk-Return Inversion

In the model of Section 3, in which there is no risk limit, the supply θ has no effect on the asset's return volatility (Proposition 3.2). This result no longer holds with a risk limit: volatility is higher when $\theta < 0$, corresponding to high noise-trader demand, than when $\theta > 0$, corresponding to low noise-trader demand.

Proposition 4.1. *Under the assumptions in Theorem 4.1, both the conditional and the unconditional volatility of the asset's return are:*

- *Higher when $\theta < 0$ than when $\theta > 0$.*

- *Higher than under the affine solution derived for $\mu^* = \infty$ when $\theta < 0$, and lower when $\theta > 0$.*

The intuition for Proposition 4.1 is related to the convexity and concavity results of Theorem 4.1. For D_t close to zero, the risk limit is far from binding, and volatility is independent of θ, as in the case of no risk limit. For large values of D_t, the risk limit binds, and forces the managers employed by non-experts to trade when D_t changes. As explained after Theorem 4.1, trading amplifies movements in D_t when $\theta < 0$, and dampens them when $\theta > 0$. The amplification effect

causes volatility to be higher when $\theta < 0$ than in the risk limit's absence. Conversely, the dampening effect causes volatility to be lower when $\theta > 0$ than in the risk limit's absence, and hence also lower than when $\theta < 0$. The same comparisons hold for smaller values of D_t because of the expectation that the risk limit might bind in the future.

Since the asset's expected return is positive when $\theta > 0$ and negative when $\theta < 0$, Proposition 4.1 implies a negative relationship between volatility and expected return: expected return is low and volatility is high when $\theta < 0$, and conversely expected return is high and volatility is low when $\theta > 0$. High volatility goes together with overvaluation (low expected return) because they are both driven by high noise-trader demand. Indeed, to accommodate the high demand, investors hold short positions. Moreover, some of these positions have to be unwound because of the risk limit when the market goes up, yielding amplification and high volatility.

A negative relationship between volatility and expected return runs counter to the prediction of standard theories that investors should earn a higher return as compensation for bearing more risk. A negative relationship has been documented empirically within asset classes, and is known as the *volatility anomaly*. Haugen and Baker (1996) and Ang, Hodrick, Xing, and Zhang (2006) document the volatility anomaly in the cross-section of U.S. stocks.

The negative relationship between volatility and expected return in our model holds as a comparative-statics result rather than as a cross-sectional result because there is only one risky asset. We can, however, extend our model to multiple risky assets and derive a cross-sectional result. The simplest way to perform the extension is to assume that dividend flows are independent across assets, and that the risk limit for non-experts applies asset-by-asset rather than across their entire portfolio. The contracting model of Section 2 would yield an asset-by-asset risk limit when extended to multiple assets if each manager can acquire information on only one asset and a non-expert hires one manager per asset. Deriving a risk limit across an entire portfolio would require deriving optimal contracts in the model of Section 2 when each manager can acquire information on multiple assets.

Formally, the extension is as follows. There are N risky assets instead of one. Asset $n = 1, .., N$ pays a dividend flow D_{nt} per share and is in supply of θ_n shares. The dividend flow D_{nt} follows the square-root process

$$dD_{nt} = \kappa_n \left(\bar{D}_n - D_{nt} \right) dt + \sigma_n \sqrt{D_{nt}} dB_{nt}, \tag{4.9}$$

which generalizes (3.3), and the Brownian motions $\{dB_{nt}\}_{n=1,..,N}$ are independent. Experts and non-experts can invest in all N risky assets. The position z_{2nt} of a non-expert in asset n is subject

to the risk limit

$$|z_{2nt}|\sigma\sqrt{D_{nt}}S'(D_{nt}) \leq \frac{\mu_n^*}{\rho}, \tag{4.10}$$

which generalizes (4.1).

The multi-asset extension yields a replica of the one-asset model: the price for each asset is given by Proposition 3.1 in the case of no risk limits, and by Theorem 4.1 in the case of risk limits. In the multi-asset extension, the negative relationship between volatility and expected return becomes a cross-sectional result.

An additional advantage of the multi-asset extension is that we can use it to study how expected return relates to CAPM beta.[16] The CAPM implies a positive relationship between beta and expected return. Empirically, however, the relationship is flat or negative, a fact known as the *beta anomaly.* Black (1972), Black, Jensen, and Scholes (1972), and Frazzini and Pedersen (2014) document a flat relationship in the cross-section of U.S. stocks. Baker, Bradley, and Wurgler (2011) find that the relationship turns negative in recent decades.

Our multi-asset extension yields a negative relationship between beta and expected return. This is because with independent dividend flows, an asset's beta is proportional to the asset's return variance times the asset price. Assets with $\theta < 0$ have high beta because they have both high price (Theorem 4.1) and high return variance (Proposition 4.1).[17]

Proposition 4.2. *In the multi-asset extension of our model, suppose* $\theta_n > -\frac{(1-x)(r+\kappa_n)^2}{4\rho\sigma_n^2}$ *and* $\kappa_n\bar{D}_n > \frac{\sigma_n^2}{4}$ *for all* $n = 1, .., N$. *An asset* n *with* $\theta_n < 0$ *has higher conditional and unconditional CAPM beta than an otherwise identical asset* n' *with* $\theta_{n'} > 0$.

Figure 2 illustrates the properties of return moments shown in this section using a numerical example. The figure's left panel plots the conditional expected return as a function of D_t. The thin lines represent the expected return in the absence of a risk limit ($\mu^* = \infty$) and the thick lines the expected return with a risk limit (μ^* finite). In each case, the solid blue line corresponds to a positive value of θ and the dashed red line to the opposite negative value. The figure's middle panel plots the conditional return volatility as a function of D_t using the same conventions. The figure's right panel plots the conditional CAPM betas in a two-asset model as a function of D_t,

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[16] An asset's CAPM beta is the covariance between the asset's return and the return of the market portfolio, divided by the variance of the market portfolio's return. With only one asset, the market portfolio coincides with the asset and the beta is one.

[17] The negative relationship between beta and expected return would arise even in the absence of a risk limit: the return variance would be independent of θ (Proposition 3.2), but the price would be higher for low-θ assets (Proposition 3.1).

Figure 2: Effect of Risk Limit on Return Moments
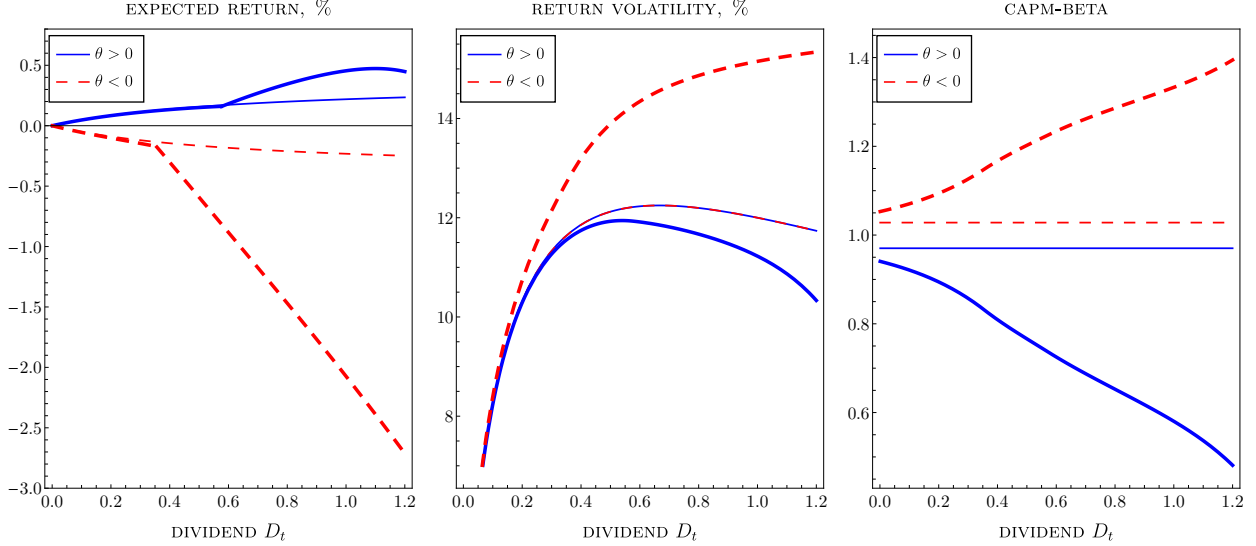
Conditional expected return $\mathbb{E}_t(dR_t)$ (left panel), conditional return volatility $\sqrt{\mathbb{V}\mathrm{ar}_t(dR_t)}$ (middle panel), and conditional CAPM beta $\frac{\mathbb{C}\mathrm{ov}_t(dR_{1t}, dR_{Mt})}{\mathbb{V}\mathrm{ar}(dR_{Mt})}$ (right panel) as a function of the dividend flow D_t. The thin lines in each panel represent the return moments in the absence of a risk limit ($\mu^* = \infty$) and the thick lines the return moments with a risk limit (μ^* finite). In each case, the solid blue line corresponds to a positive value of supply θ and the dashed red line to the opposite negative value. Beta is computed in a two-asset model as a function of D_t, considering a realization where $D_{1t} = D_{2t} = D_t$ and taking the market portfolio to consist of an equal number of shares in each asset. Parameter values are as in Figure 1.

considering a realization where $D_{1t} = D_{2t} = D_t$. We use the same conventions as in the other two panels, and take the market portfolio to consist of an equal number of shares in each asset.[18] The figure confirms the negative relationship between return volatility and beta on one hand, and expected return on the other.

4.4 Overvaluation Bias

Proposition 4.1 shows that the risk limit exacerbates the effects that noise-trader demand has on the price level: the asset becomes even more expensive when noise-trader demand is high ($\theta < 0$) and even cheaper when noise-trader demand is low ($\theta > 0$). In this section we show that these effects of the risk limit do not cancel on average, but there is a bias towards overvaluation.

Suppose that the supply θ of the asset can take either a positive value or the opposite negative value, with each outcome being equally likely. Figure 3 plots the average price, taking expectations over the two values of θ, as a function of D_t. In the multi-asset extension of our model, the average

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[18]We are assuming that asset issuers supply an equal number of shares of each asset, and that an empiricist who constructs the market portfolio observes that supply only and not its combination with noise-trader demand.

Figure 3: Effect of Risk Limit on Average Price
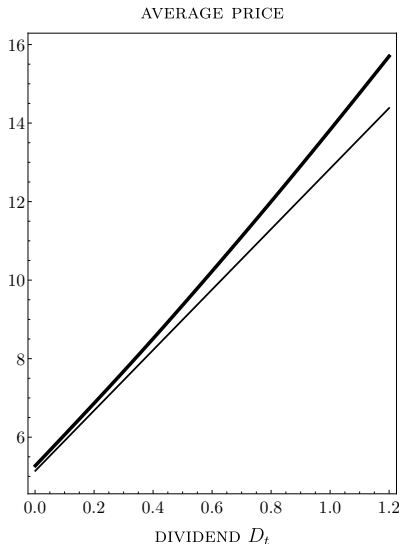
Average price $\frac{1}{N}\sum_{n=1}^{N} S_{nt}$ as a function of the dividend flow D_t. The thin line represents the average price in the absence of a risk limit ($\mu^* = \infty$), and the thick line represents the same quantity with a risk limit (μ^* finite). Half of the assets are in positive supply and the other half are in the opposite negative supply. The parameters of the dividend-flow process are the same across assets, and we consider a realization where $D_{nt} = D_t$ for all $n = 1, ..N$. The parameters of the dividend-flow process are the same across assets. Parameter values are as in Figure 1.

price can be interpreted as the cross-sectional average of prices. The multi-asset interpretation requires that half of the assets are in positive supply and the other half are in the opposite negative supply, that the parameters of the dividend-flow process are the same across assets, and that we consider a realization where $D_{nt} = D_t$ for all $n = 1, ..N$. We adopt the multi-asset interpretation in the rest of this section.

The thin line in Figure 3 represents the average price in the absence of a risk limit ($\mu^* = \infty$), and the thick line represents the same quantity with a risk limit (μ^* finite). Imposing the risk limit raises the average price: overvalued assets ($\theta < 0$, low expected return) appreciate by more than undervalued assets ($\theta > 0$, high expected return) depreciate.

Key to the price asymmetry is that the risk limit binds more severely for overvalued assets than for undervalued ones. Indeed, recall from (4.1) that the risk limit constrains the volatility of a manager's asset position. Since the share price and volatility per share are larger for overvalued assets than for an undervalued ones, the risk limit binds more severely for a short position in the former than for a long position of an equal number of shares in the latter.

The risk limit's asymmetric effect on positions yields the price asymmetry. With a binding risk limit, non-experts hold smaller short positions, in terms of number of shares, in overvalued assets

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than they hold long positions in undervalued ones. Market clearing requires that if the supply θ is opposite across the two types of assets, experts hold larger short positions than long positions. For experts to be induced to do so, overvaluation must be severe.

The right panel of Figure 1 confirms the risk limit's asymmetric effect on positions. The risk limit for overvalued assets ($\theta < 0$) becomes binding at $D_t = 0.356$. For undervalued assets ($\theta > 0$) instead, the risk limit becomes binding at $D_t = 0.576$. Moreover, for any $D_t > 0.576$, the discrepancy between the position of non-experts with and without the risk limit is larger for overvalued than for undervalued assets.

4.5 Comparative Statics

Some of the results in Sections 4.2, 4.3 and 4.4 compare the equilibrium in which non-experts impose the optimal risk limit to the equilibrium in which they (suboptimally) impose no limit. All exogenous parameters are held constant in those comparisons. This section compares instead equilibria in which the risk limit is always set to its optimal value, and changes because exogenous parameters change. The parameter we focus on is the fraction λ of risk-neutral manager types.

An increase in λ has two effects on equilibrium prices. The *risk-limit effect* is that non-experts tighten the risk limit to account for the higher probability that their manager is uninformed. The *informed-trading effect* is that because the manager is uninformed with higher probability, she is more likely to trade suboptimally and not absorb noise-trader demand. The risk-limit effect derives from Proposition 2.1: (2.13) shows that μ^* decreases when λ increases. The informed-trading effect enters through (4.4): an increase in λ raises the aggregate risk aversion parameter $\frac{\rho}{1-\lambda x}$ of the agents absorbing noise-trader demand. The comparisons in Sections 4.2, 4.3 and 4.4 isolate the risk-limit effect relative to the case $\lambda = 0$. This is because the aggregate risk aversion parameter $\frac{\rho}{1-\lambda x}$ is held constant, and a comparison is made with the case where $\mu^* = \infty$, a value that can be derived from (2.13) by setting $\lambda = 0$ and $\bar{\mu} = \infty$.

Figure 4 illustrates the risk-limit and informed-trading effects. The left panel plots the price, and the middle panel plots the conditional expected return, both as a function of D_t. The thin solid blue and dashed red lines correspond to the base case, in which $\lambda = 0.2$. The dotted blue and red lines correspond to $\lambda = 0.5$ and isolate the risk-limit effect. The thick solid blue and thick dashed red lines also correspond to $\lambda = 0.5$ and show the combined effect. The figure confirms that the risk-limit and the informed-trading effect work in the same direction, exacerbating the impact of noise-trader demand. It shows additionally that the risk-limit effect is the stronger of the two.

The endogenous determination of the risk limit plays a critical role in the comparative statics. Because the risk-limit and the informed-trading effect render the asset more mispriced in equilib-

Figure 4: Comparative statics
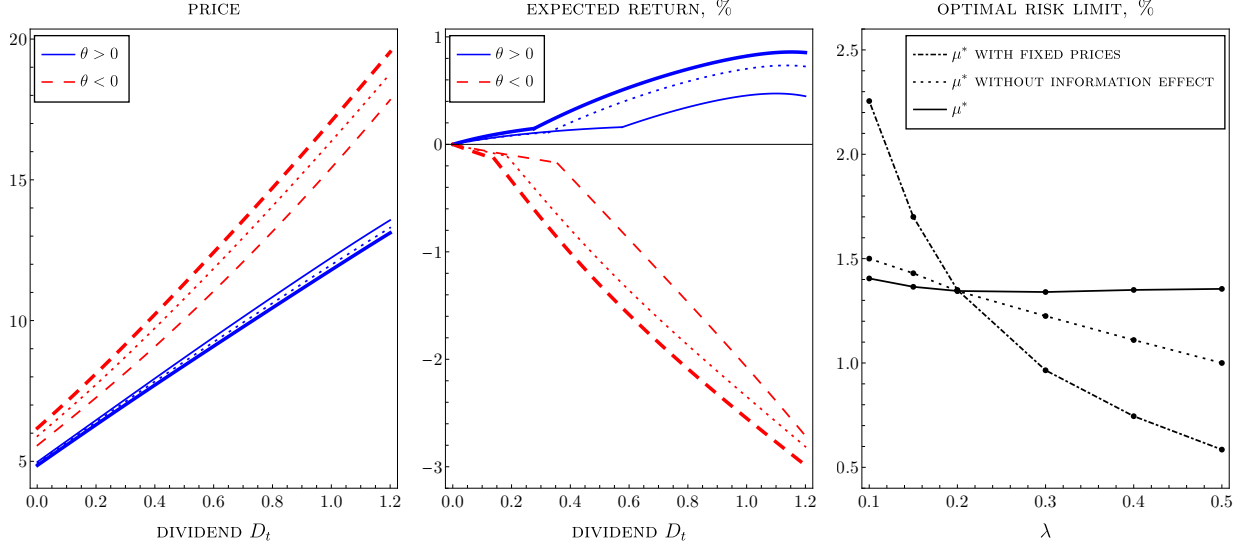
Price S_t (left panel), conditional expected return $\mathbb{E}_t(dR_t)$ (middle panel), and risk-limit parameter μ^* (right panel) for different values of the fraction λ of risk-neutral manager types. The thin solid blue and dashed red lines correspond to the base case, in which $\lambda = 0.2$. The dotted blue and red lines correspond to $\lambda = 0.5$ and isolate the risk-limit effect. The thick solid blue and thick dashed red lines also correspond to $\lambda = 0.5$ and show the combined effect. The blue lines correspond to a positive value of supply θ and the red lines correspond to the opposite negative value. The dashed-dotted line in the right panel shows the impact of λ on μ^* when prices are held constant. The dotted line shows the impact when equilibrium prices change because of the risk-limit effect, and the solid line shows the impact when prices change because of the combined effect. Values for parameters other than λ are as in Figure 1.

rium, investors partly relax the risk limit to benefit from the larger mispricing. This mitigates partly the two effects, as shown in the right panel of Figure 4. When prices are held constant, an increase in λ from 0.2 to 0.5 lowers μ^* sharply (dashed-dotted line). The decline in μ^* becomes smaller when investors respond to the change in equilibrium prices caused by the risk-limit effect (dotted line). The decline in μ^* becomes almost non-existent when investors respond to the change in prices caused by the combined effect (solid line).

5 Benchmarks

In the contracting model of Section 2, the manager's fee can depend only on the investor's wealth W and not on asset payoffs (except through W). This can describe situations where investors do not observe the payoffs of the assets in which managers invest. For example, investors may not be familiar with the types of assets (or strategies) in which hedge funds invest, and indices for those assets may not be readily available. In other situations, however, managers invest in a well-defined

set of assets for which indices are available. For example, equity mutual funds hold long positions in the stock market. In such situations, managers' fees can depend on indices or other asset payoff information, in addition to the managers' return. In particular, investors can pay managers based on how their return compares to a stock-market index.

Extending the model of Section 2 to the case where the manager's fee can depend both on the investor's wealth and on asset payoffs, and embedding it into the equilibrium model of Section 3, introduces new complications. For example, by observing the asset payoffs, the investor can learn about their volatility, and set a laxer risk limit when volatility is high. In this section we do not determine the optimal contract, but employ a simple extension of the reduced-form model derived in Section 2.3 to determine equilibrium prices. We show that the analysis of Section 4 carries through essentially unchanged.

We assume that with probability $1 - \lambda$, a non-expert solves the same optimization problem as an expert but is subject to the risk limit

$$|z_{2t} - \eta|\sigma\sqrt{D_t}S'(D_t) \leq \frac{\mu^*}{\rho}, \tag{5.1}$$

where z_{2t} is the non-expert's position in terms of number of shares, and η is a non-negative constant. With probability λ, a non-expert chooses a position right at the risk limit, and is equally likely to be at the upper or at the lower limit. The average position in that case is η. We assume that choice outcomes are independent across non-experts. The market-clearing condition is $(1 - x)z_{1t} + (1 - \lambda)xz_{2t} + \lambda x\eta = \theta$, where z_{1t} denotes the position of an expert.

Intuitively, (5.1) requires that the volatility of the manager's position in the risky asset, *relative to a benchmark position η*, is bounded. The risk limit (4.1) in Section 4 is a special case of (5.1) with $\eta = 0$. When $\eta > 0$, the benchmark position is long. Setting η positive rather than zero is likely to be better for the investor if his optimal position in the risky asset under his prior beliefs is long. Suppose, for example, that η is a known supply by the asset issuer and $\eta - \theta$ is a random mean-zero demand by noise traders. (The total supply is $\eta - (\eta - \theta) = \theta$.) If the investor does not observe the noise-trader demand, then the optimal position under his prior beliefs is equal (or close) to η.

To derive the ODE in the unconstrained region, we note that the market-clearing condition implies that the common position of experts and non-experts is

$$z_{1t} = z_{2t} = \frac{\theta - \lambda x\eta}{1 - \lambda x}. \tag{5.2}$$

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Substituting z_{1t} from (5.2) into the first-order condition of an expert, we find the ODE

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \frac{\rho(\theta - \lambda x \eta)}{1 - \lambda x}\sigma^2 D_t S'(D_t)^2. \tag{5.3}$$

Substituting z_{2t} from (5.2) into (5.1), which holds as a strict inequality in the unconstrained region, we find that the unconstrained region is defined by

$$\frac{|\theta - \eta|}{1 - \lambda x}\sigma\sqrt{D_t}S'(D_t) \leq \frac{\mu^*}{\rho}. \tag{5.4}$$

To derive the ODE in the constrained region, we use the market-clearing condition to write z_{1t} as a function of z_{2t}, and substitute into the first-order condition of an expert. This yields

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \rho\frac{\theta - (1 - \lambda)x z_{2t} - \lambda x \eta}{1 - x}\sigma^2 D_t S'(D_t)^2. \tag{5.5}$$

We next note that $z_{2t} - \eta$ has the same sign as $\theta - \eta$. Indeed, since non-experts are prevented from choosing the same position as experts by the risk limit, and since that limit is not binding if z_{2t} is sufficiently close to η, $z_{1t} - \eta$ and $z_{2t} - \eta$ have the same sign. The market-clearing condition written as $(1 - x)(z_{1t} - \eta) + (1 - \lambda)x(z_{2t} - \eta) = \theta - \eta$ then implies that $z_{2t} - \eta$ and $\theta - \eta$ have the same sign. Using that observation to substitute z_{2t} from (5.1), which holds as an equality in the constrained region, into (5.5), we find the ODE

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \frac{\rho(\theta - x\eta)}{1 - x}\sigma^2 D_t S'(D_t)^2$$
$$- \frac{\text{sgn}(\theta - \eta)(1 - \lambda)x\mu^*}{1 - x}\sigma\sqrt{D_t}S'(D_t). \tag{5.6}$$

The constrained region is defined by the opposite inequality to (5.4), i.e.,

$$\frac{|\theta - \eta|}{1 - \lambda x}\sigma\sqrt{D_t}S'(D_t) > \frac{\mu^*}{\rho}. \tag{5.7}$$

The results of Theorem 4.1 and Propositions 4.1 and 4.2 carry through provided that all comparisons between θ and zero are replaced by ones between θ and η. Our numerical solutions indicate that the overvaluation bias shown in Section 4.4 carries through as well provided that the two equally likely values of θ average to η rather than to zero.

Proposition 5.1. *Suppose that $\theta > -\frac{(1-x)(r+\kappa)^2}{4\rho\sigma^2} + x\eta$ and $\kappa\bar{D} > \frac{\sigma^2}{4}$. A solution $S(D_t)$ to the system of ODEs (5.3) in the unconstrained region (5.4), and (5.6) in the constrained region (5.7), with a derivative that converges to finite limits at zero and infinity, exists. It has the same properties*

as in Theorem 4.1 and Propositions 4.1 and 4.2 provided that all comparisons between θ and zero are replaced by ones between θ and η.

When $\theta < \eta$, the risk limit renders the asset more expensive because it can induce managers employed by non-experts to hold a larger long position: the managers' unconstrained position, given by (5.2), is smaller than η when $\theta < \eta$, and the risk limit brings it closer to η. The risk limit also renders the price more sensitive to changes in D_t because of the amplifying effect of the trading that it induces: managers employed by non-experts are forced to buy the asset when D_t increases. Because of the amplifying effect, the price becomes convex and the return becomes more volatile. The converse results hold when $\theta < \eta$.

6   Conclusion

We study how the agency relationship between investors and asset managers affects equilibrium asset prices. We first develop a static contracting model that combines (i) moral hazard arising from managers' effort to acquire information and (ii) adverse selection arising from managers' preferences and the private information they may acquire. We show that the optimal contract involves risk limits: the risk of the portfolio chosen by managers is kept within bounds, even when the optimal level of risk given the private information that managers may acquire exceeds the bounds. Investors constrain their managers in that way because the latter may not acquire information and gamble for a high fee.

We next embed the contracting model into an equilibrium asset-pricing model with noise traders and overlapping generations of investors and managers. The frictionless version of that model is to our knowledge new to the literature. It yields a simple closed-form solution for asset prices, and generates more realistic properties than the tractable CARA-normal alternative, e.g., prices and dividends are always positive, and the volatility of asset returns per share increases in the dividend flow.

We show two main results. First, risk limits generate an inverted risk-return relationship: overvalued assets have low expected return and high volatility, while undervalued assets have high expected return and low volatility. The high volatility of overvalued assets arises because managers buy them during bull markets to meet risk limits. Unlike previous literature on amplification effects, amplification in our model happens during bubbles rather than crises. Our second result also concerns distortions during bubbles. Risk limits cause overvalued assets to become more overvalued and undervalued assets to become more undervalued. Yet, because overvalued assets have higher share price and volatility, risk limits are more constraining when trading against overvaluation,

biasing the aggregate market upward.

Our analysis suggests that risk limits (or tracking-error constraints as they are often referred to in the asset management industry) can have important effects on managers' portfolio policies and equilibrium asset prices. Empirical research has started to investigate these effects. For example, Christoffersen and Simutin (2017) find that mutual-fund managers who manage pension-fund assets, and hence face greater pressure to meet benchmarks, hold a larger fraction of their portfolios in high-beta stocks and achieve lower alphas. This is consistent with our results that overvaluation is associated with high beta, and that more constrained managers hold more shares in overvalued assets and fewer shares in undervalued ones. Lines (2016) finds that mutual-fund managers shift their portfolio weights towards those of the benchmark when volatility rises, putting downward price pressure on overweight stocks and upward pressure on underweight stocks. This is consistent with the amplification effect that we derive.

Extending the empirical investigation by bringing in proxies for noise-trader demand could yield sharper tests of the theoretical mechanisms. Such proxies could include flows into mutual funds, or restricted mandates by institutional investors not to invest in some industry sectors. Empirical studies have documented that high demand according to these proxies is associated with low future returns.[19] Our analysis implies additionally that high demand should be associated with high volatility, and that the trading of managers with tight risk limits should be contributing to this.

Another promising extension concerns the normative and policy implications. While each investor in our model seeks to limit the risk taken by his manager, the combined effect of those efforts is to raise the volatility of overvalued assets. Would a regulator or a social planner internalize this effect and impose a laxer risk limit? More generally, how would privately optimal risk limits compare to socially optimal ones? Our model can help address these questions because it provides an explicit contractual problem that risk limits solve, and captures the two-way feedback from risk limits to equilibrium asset prices.

---

[19]Frazzini and Lamont (2008) argue that noise-trader demand ("dumb money" in their terminology) can be proxied by flows into mutual funds, as these predict low long-horizon returns for the stocks bought by the funds. In a similar spirit, Coval and Stafford (2007) find that that stocks sold by mutual funds that experience extreme outflows earn high long-horizon returns, while stocks bought by funds that experience extreme inflows earn low returns. Hong and Kacperczyk (2009) find that stocks in "sin industries" (alcohol, gaming and tobacco) are less held by institutions, presumably because of restricted mandates, and earn higher returns. An alternative proxy for noise-trader demand could be holdings by controlling shareholders, e.g., in family firms.

Appendix

A Proofs for Section 2

Proof of Lemma 2.1. In this and subsequent proofs we denote by

$$\Gamma(\pi) \equiv \pi_0 f(z(\pi)d) + (1 - \pi_0) f(-z(\pi)d)$$

the expected fee for the risk-averse type π under the prior probabilities $(\pi_0, 1 - \pi_0)$, and by

$$\hat{\Gamma} \equiv \pi_0 f(\hat{z}d) + (1 - \pi_0) f(-\hat{z}d)$$

the same quantity for the risk-neutral type. Using the definitions of $(\Delta(\pi), \hat{\Delta}, \Gamma(\pi), \hat{\Gamma})$, we can write the utility of the risk-averse type π when she chooses positions $z(\pi')$, $z(\pi)$ and $\hat{z}$ as

$$U(\pi, z(\pi')) = -\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1 - \pi) e^{\bar{\rho}\pi_0 \Delta(\pi')}\right] e^{-\bar{\rho}\Gamma(\pi')}, \tag{A.1}$$

$$U(\pi) = U(\pi, z(\pi)) = -\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1 - \pi) e^{\bar{\rho}\pi_0 \Delta(\pi)}\right] e^{-\bar{\rho}\Gamma(\pi)}, \tag{A.2}$$

$$U(\pi, \hat{z}) = -\left[\pi e^{-\bar{\rho}(1-\pi_0)\hat{\Delta}} + (1 - \pi) e^{\bar{\rho}\pi_0 \hat{\Delta}}\right] e^{-\bar{\rho}\hat{\Gamma}}, \tag{A.3}$$

respectively, and the utility of the risk-neutral type when she chooses positions $\hat{z}$ and $z(\pi')$ as

$$\hat{U} = \hat{\Gamma}, \tag{A.4}$$

$$\hat{U}(z(\pi')) = \Gamma(\pi'), \tag{A.5}$$

respectively.

We next show that the (IC) constraint (2.7) implies Property (i). Equation (A.1) implies that the risk-averse type π prefers $z(\pi)$ to $z(\pi')$ if

$$-\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1 - \pi) e^{\bar{\rho}\pi_0 \Delta(\pi)}\right] e^{-\bar{\rho}\Gamma(\pi)} \geq -\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1 - \pi) e^{\bar{\rho}\pi_0 \Delta(\pi')}\right] e^{-\bar{\rho}\Gamma(\pi')}. \tag{A.6}$$

Conversely, the risk-averse type π' prefers $z(\pi')$ to $z(\pi)$ if

$$-\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1 - \pi') e^{\bar{\rho}\pi_0 \Delta(\pi')}\right] e^{-\bar{\rho}\Gamma(\pi')} \geq -\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1 - \pi') e^{\bar{\rho}\pi_0 \Delta(\pi)}\right] e^{-\bar{\rho}\Gamma(\pi)}. \tag{A.7}$$

Multiplying (A.6) and (A.7) by minus one, to make their sides positive, and then multiplying each side of (A.6) by the corresponding side of (A.7), we find

$$\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)}\right]\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi')}\right]$$

$$\leq \left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi')}\right]\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi)}\right]$$

$$\Leftrightarrow (\pi - \pi')\left[e^{\bar{\rho}[\pi_0\Delta(\pi)-(1-\pi_0)\Delta(\pi')]} - e^{\bar{\rho}[\pi_0\Delta(\pi')-(1-\pi_0)\Delta(\pi)]}\right] \geq 0$$

$$\Leftrightarrow (\pi - \pi')\left[e^{\rho(\Delta(\pi)-\Delta(\pi'))} - 1\right] \geq 0. \tag{A.8}$$

Equation (A.8) implies that if $\pi > \pi'$ then $\Delta(\pi) \geq \Delta(\pi')$. Hence, $\Delta(\pi)$ is non-decreasing.

We next show that the (IC) constraint (2.7) implies Property (ii). Consider first a point π at which $\Delta(\pi)$ is continuous. Equations (A.6) and (A.7) imply

$$\frac{\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)}}{\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi')}} \geq e^{-\bar{\rho}\Gamma(\pi')} \geq \frac{\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)}}{\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi')}}.$$
$$\tag{A.9}$$

Since $\Delta(\pi)$ is continuous at π, both fractions in (A.9) converge to $e^{-\bar{\rho}\Gamma(\pi)}$ when π' goes to π. Equation (A.9) then implies that $e^{-\bar{\rho}\Gamma(\pi')}$ converges to the same limit. Hence, $\Gamma(\pi)$ is continuous at π. Equation (A.2) implies

$$\frac{U(\pi) - U(\pi')}{\pi - \pi'}$$
$$= \frac{\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi')}\right]e^{-\bar{\rho}\Gamma(\pi')} - \left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)}}{\pi - \pi'}.$$
$$\tag{A.10}$$

Combining (A.10) with (A.6), we find

$$\frac{U(\pi) - U(\pi')}{\pi - \pi'}$$
$$\geq \frac{\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi')}\right]e^{-\bar{\rho}\Gamma(\pi')} - \left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi')}\right]e^{-\bar{\rho}\Gamma(\pi')}}{\pi - \pi'}$$
$$= \left[e^{\bar{\rho}\pi_0\Delta(\pi')} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')}\right]e^{-\bar{\rho}\Gamma(\pi')}. \tag{A.11}$$

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Combining (A.10) with (A.7), we find

$$
\begin{aligned}
&\frac{U(\pi) - U(\pi')}{\pi - \pi'} \\
&\leq \frac{\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)} - \left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)}}{\pi - \pi'} \\
&= \left[e^{\bar{\rho}\pi_0\Delta(\pi)} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)}. \tag{A.12}
\end{aligned}
$$

Since $(\Delta(\pi), \Gamma(\pi))$ are continuous at π, the right-hand side of (A.11) converges to

$$
\left[e^{\bar{\rho}\pi_0\Delta(\pi)} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)}
$$

when π' goes to π. Equations (A.11) and (A.12) then imply that $\frac{U(\pi)-U(\pi')}{\pi-\pi'}$ converges to the same limit. Using (A.2), we can write that limit as

$$
-U(\pi)\frac{e^{\bar{\rho}\pi_0\Delta(\pi)} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)}}{\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)}} = -U(\pi)\frac{e^{\bar{\rho}\Delta(\pi)} - 1}{\pi + (1-\pi)e^{\bar{\rho}\Delta(\pi)}}.
$$

Hence, $U(\pi)$ is differentiable at π, with $U'(\pi)$ given by (2.8).

Consider next a point π at which $\Delta(\pi)$ is discontinuous. Since $\Delta(\pi)$ is non-decreasing, $\Delta(\pi)$ has left- and right-limits at π, which we denote by $\Delta(\pi^-)$ and $\Delta(\pi^+)$, respectively. Equation (A.9) written for $\pi' < \pi$ implies that $\Gamma(\pi)$ has a left-limit $\Gamma(\pi^-)$ at π, given by

$$
e^{-\bar{\rho}\Gamma(\pi^-)} = \frac{\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)}\right]e^{-\bar{\rho}\Gamma(\pi)}}{\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi^-)}}. \tag{A.13}
$$

Consider next $\pi' < \pi'' < \pi$ and the (IC) constraint that the risk-averse type π' prefers $z(\pi')$ to $z(\pi'')$. Taking the limit of that equation when π'' goes to π, we find

$$
-\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi')}\right]e^{-\bar{\rho}\Gamma(\pi')} \geq -\left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi^-)}\right]e^{-\bar{\rho}\Gamma(\pi^-)}. \tag{A.14}
$$

Combining (A.10) with (A.13) and (A.14), we obtain the following counterpart of (A.12):

$$
\begin{aligned}
&\frac{U(\pi) - U(\pi')}{\pi - \pi'} \\
&\leq \frac{\left[\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi^-)}\right]e^{-\bar{\rho}\Gamma(\pi^-)} - \left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi^-)}\right]e^{-\bar{\rho}\Gamma(\pi^-)}}{\pi - \pi'} \\
&= \left[e^{\bar{\rho}\pi_0\Delta(\pi^-)} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)}\right]e^{-\bar{\rho}\Gamma(\pi^-)}. \tag{A.15}
\end{aligned}
$$

Since $(\Delta(\pi), \Gamma(\pi))$ have left-limits at π, the right-hand side of (A.11) converges to

$$\left[e^{\bar{\rho}\pi_0\Delta(\pi^-)} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)} \right] e^{-\bar{\rho}\Gamma(\pi^-)}$$

when π' goes to π from the left. Equations (A.11) and (A.15) then imply that $\frac{U(\pi)-U(\pi')}{\pi-\pi'}$ converges to the same limit. Using (A.2) and (A.13), we can write that limit as

$$-U(\pi)\frac{e^{\bar{\rho}\pi_0\Delta(\pi^-)} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)}}{\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi^-)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi^-)}} = -U(\pi)\frac{e^{\bar{\rho}\Delta(\pi^-)} - 1}{\pi + (1-\pi)e^{\bar{\rho}\Delta(\pi^-)}}.$$

Hence, $U(\pi)$ has a left-derivative at π, with $U'(\pi)$ given by substituting $\Delta(\pi^-)$ in (2.8). The argument for the right-derivative is identical.

We next show that the (IC) constraint (2.7) implies Property (iii). Equation (2.2) implies

$$U(\pi, -z) = U(1-\pi, z). \tag{A.16}$$

Using (A.16), we can write the condition that the risk-averse type π prefers $z(\pi)$ to $-z(1-\pi)$ as

$$U(\pi) \geq U(\pi, -z(1-\pi)) = U(1-\pi, z(1-\pi)) = U(1-\pi). \tag{A.17}$$

The same derivation for the risk-averse type $1-\pi$ yields $U(1-\pi) \geq U(\pi)$, and hence $U(\pi) = U(1-\pi)$. Equations (A.17) and $U(\pi) = U(1-\pi)$ imply that the risk-averse type π is indifferent between $z(\pi)$ and $-z(1-\pi)$.

We next show that Properties (i), (ii) and (iii) imply the (IC) constraint (2.7). For this result and subsequent proofs we use

$$U(\pi) = U(\bar{\pi})\exp\left[\int_\pi^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right], \tag{A.18}$$

where

$$H(\Delta, \pi) \equiv \frac{e^{\bar{\rho}\Delta} - 1}{\pi + (1-\pi)e^{\bar{\rho}\Delta}}.$$

Equation (A.18) follows by integrating the ordinary differential equation (ODE) (2.8). The integration proof must account for possible points of discontinuity of $\Delta(\pi)$. Since $\Delta(\pi)$ is non-decreasing, its discontinuity points are at most countable, and the same is true for the discontinuity points of $\pi \to H(\Delta(\pi), \pi)$. Hence, $\pi \to H(\Delta(\pi), \pi)$ is measurable, and the integral $\int_\pi^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'$ in (A.18) is well-defined. Since $U(\pi)$ and $\int_\pi^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'$ have left- and right-derivatives, the function $K(\pi) \equiv U(\pi)\exp\left[-\int_\pi^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right]$ has also left- and right-derivatives. Moreover, Prop-

erty (ii) implies that the left- and right-derivatives of $K(\pi)$ are zero for all π. Hence, $K(\pi) = U(\bar{\pi})$, which implies (A.18).

Combining (A.2) and (A.18), we find

$$e^{-\bar{\rho}\Gamma(\pi)} = -\frac{U(\bar{\pi})\exp\left[\int_\pi^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right]}{\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)}}. \tag{A.19}$$

Substituting $e^{-\bar{\rho}\Gamma(\pi)}$ and $e^{-\bar{\rho}\Gamma(\pi')}$ from (A.19) into (A.6), we find that (A.6) is equivalent to

$$\begin{aligned}
&U(\bar{\pi})\exp\left[\int_\pi^{\bar{\pi}} H(\Delta(\pi''), \pi'')d\pi''\right] \\
&\geq U(\bar{\pi})\exp\left[\int_{\pi'}^{\bar{\pi}} H(\Delta(\pi''), \pi'')d\pi''\right] \frac{\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi')}}{\pi' e^{-\bar{\rho}(1-\pi_0)\Delta(\pi')} + (1-\pi')e^{\bar{\rho}\pi_0\Delta(\pi')}} \\
&\Leftrightarrow 1 \leq \exp\left[\int_{\pi'}^{\pi} H(\Delta(\pi''), \pi'')d\pi''\right] \frac{\pi + (1-\pi)e^{\bar{\rho}\Delta(\pi')}}{\pi' + (1-\pi')e^{\bar{\rho}\Delta(\pi')}},
\end{aligned} \tag{A.20}$$

where the second step follows by dividing both sides by

$$U(\bar{\pi})\exp\left[\int_\pi^{\bar{\pi}} H(\Delta(\pi''), \pi'')d\pi''\right],$$

which is negative. Since $H(\Delta, \pi)$ is increasing in Δ, and $\Delta(\pi)$ is non-decreasing,

$$\begin{aligned}
\exp\left[\int_{\pi'}^{\pi} H(\Delta(\pi''), \pi'')d\pi''\right] &\geq \exp\left[\int_{\pi'}^{\pi} H(\Delta(\pi'), \pi'')d\pi''\right] \\
&= \exp\left[\left[\log\left(\pi'' + (1-\pi'')e^{\bar{\rho}\Delta(\pi')}\right)\right]_\pi^{\pi'}\right] \\
&= \frac{\pi' + (1-\pi')e^{\bar{\rho}\Delta(\pi')}}{\pi + (1-\pi)e^{\bar{\rho}\Delta(\pi')}}.
\end{aligned} \tag{A.21}$$

Equation (A.21) implies that (A.20) holds for all (π, π'). Hence, $U(\pi) \geq \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} U(\pi, z(\pi'))$ for all $\pi \in [1-\bar{\pi}, \bar{\pi}]$. To show $U(\pi) \geq \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} U(\pi, -z(\pi'))$ for all $\pi \in [1-\bar{\pi}, \bar{\pi}]$, we note that

$$\begin{aligned}
&U(\pi) \geq U(\pi, -z(\pi')) = U(1-\pi, z(\pi')) \\
&\Leftrightarrow U(1-\pi) \geq U(1-\pi, z(\pi')),
\end{aligned} \tag{A.22}$$

where the first step follows from (A.16) and the second from Property (iii). Equation (A.22) holds because of equation $U(\pi) \geq \max_{\pi' \in [1-\bar{\pi}, \bar{\pi}]} U(\pi, z(\pi'))$ written for $1-\pi$ instead of π. Hence, the (IC) constraint (2.7) holds.

We next show that the (IC) constraint (2.7) implies Property (iv). Suppose that $\Delta(\pi)$ is continuous at π. Property (ii) implies that $U(\pi)$ is differentiable at π. Property (iii) implies that

$U(\pi)$ is also differentiable at $1 - \pi$, with $U'(1 - \pi) = -U'(\pi)$. Combining the latter equation with (2.8), we find

$$
\begin{aligned}
& -U(1 - \pi)H(\Delta(1 - \pi), 1 - \pi) = U(\pi)H(\Delta(\pi), \pi) \\
& \Leftrightarrow H(-\Delta(1 - \pi), \pi) = H(\Delta(\pi), \pi) \\
& \Leftrightarrow \Delta(\pi) = -\Delta(1 - \pi),
\end{aligned}
$$

where the second step follows from Property (iii) and because the definition of $H(\Delta, \pi)$ implies

$$
H(\Delta, 1 - \pi) = -H(-\Delta, \pi), \tag{A.23}
$$

and the third step follows because $H(\Delta, \pi)$ is increasing in Δ. Suppose next that $\Delta(\pi)$ is discontinuous at π, and that $\Delta(\pi) \neq -\Delta(1 - \pi)$. Since the risk-averse type π is indifferent between $z(\pi)$ and $-z(1 - \pi)$, we can redefine $z(\pi)$ for $\pi \in [1 - \bar{\pi}, \frac{1}{2})$ to $-z(1 - \pi)$, preserving the (IC) constraint (2.7). Under this redefinition, $\Delta(\pi) = -\Delta(1 - \pi)$. Since the points of discontinuity of $\Delta(\pi)$ are at most countable, the redefinition concerns a measure-zero set of types.

We finally show that the (IC) constraint (2.7) implies Property (v). Property (v) follows from Properties (i) and (iv): since $\Delta(\pi)$ is non-decreasing and $\Delta(\pi) = -\Delta(1 - \pi)$, $\Delta(\pi)$ must be non-negative for $\pi > \frac{1}{2}$ and non-positive for $\pi < \frac{1}{2}$. $\qquad\square$

Proof of Lemma 2.2. To show Property (i), suppose by contradiction that $z(\pi) > z(\pi')$ for $\pi < \pi'$. Using the definition of $\Delta(\pi)$, we find

$$
\begin{aligned}
\Delta(\pi) &\equiv f(z(\pi)d) - f(-z(\pi)d) \\
&= f(z(\pi)d) - f(z(\pi')d) + f(z(\pi')d) - f(-z(\pi')d) + f(-z(\pi')d) - f(-z(\pi)d) \\
&\geq 2\epsilon(z(\pi) - z(\pi'))d + f(z(\pi')d) - f(-z(\pi')d) \\
&> f(z(\pi')d) - f(-z(\pi')d) \equiv \Delta(\pi'), \tag{A.24}
\end{aligned}
$$

where the third step follows from fee monotonicity and the fourth from $z(\pi) > z(\pi')$. Equation (A.24) yields $\Delta(\pi) > \Delta(\pi')$, which is a contradiction because $\Delta(\pi)$ is non-decreasing.

To show Property (ii), we use a similar argument. Suppose by contradiction that $z(\pi) >$

$-z(1-\pi)$. Using the definition of $\Delta(\pi)$, we find

$$
\begin{aligned}
\Delta(\pi) &\equiv f(z(\pi)d) - f(-z(\pi)d) \\
&= f(z(\pi)d) - f(-z(1-\pi)d) + f(-z(1-\pi)d) - f(z(1-\pi)d) + f(z(1-\pi)d) - f(-z(\pi)d) \\
&\geq 2\epsilon(z(\pi) + z(1-\pi))d + f(-z(1-\pi)d) - f(z(1-\pi)d) \\
&> f(-z(1-\pi)d) - f(z(1-\pi)d) \equiv -\Delta(1-\pi), \quad\quad\quad\quad\quad\quad\quad\quad\text{(A.25)}
\end{aligned}
$$

where the third step follows from fee monotonicity and the fourth from $z(\pi) > -z(1-\pi)$. Equation (A.24) yields $\Delta(\pi) > -\Delta(1-\pi)$, which is a contradiction because $\Delta(\pi) = -\Delta(1-\pi)$. We can likewise derive a contradiction by assuming $z(\pi) < -z(1-\pi)$. Hence, $z(\pi) = -z(1-\pi)$.

Property (iii) follows from Properties (i) and (ii): since $z(\pi)$ is non-decreasing and $z(\pi) = -z(1-\pi)$, $z(\pi)$ must be non-negative for $\pi > \frac{1}{2}$ and non-positive for $\pi < \frac{1}{2}$. $\qquad\square$

To prove Lemma 2.3, we first prove the following lemma.

Lemma A.1. *The (IC) constraint (2.4) is equivalent to*

$$
\int_{\frac{1}{2}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right] \bar{h}(\pi)d\pi \leq e^{-\bar{\rho}K}, \quad\quad\quad\quad\quad\quad\quad\quad\text{(A.26)}
$$

and yields the following bounds on K and $\Delta(\bar{\pi})$:

$$
e^{-\bar{\rho}K} > 1 - \frac{\int_{\bar{\pi}_0}^{\bar{\pi}}(\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi}{1 - \bar{\pi}_0}, \quad\quad\quad\quad\quad\quad\quad\quad\text{(A.27)}
$$

$$
e^{\bar{\rho}\Delta(\bar{\pi})} \geq \frac{\bar{\pi}_0\left(1 - e^{-\bar{\rho}K}\right) + \int_{\bar{\pi}_0}^{\bar{\pi}}(\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi}{\int_{\bar{\pi}_0}^{\bar{\pi}}(\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi - (1 - \bar{\pi}_0)\left(1 - e^{-\bar{\rho}K}\right)}, \quad\quad\quad\quad\quad\quad\quad\quad\text{(A.28)}
$$

where $\bar{\pi}_0 \equiv \max\{\pi_0, 1 - \pi_0\}$.

Proof of Lemma A.1. Substituting $U(\pi)$ from (A.18) into (2.4), we find

$$
e^{\bar{\rho}K}\int_{1-\bar{\pi}}^{\bar{\pi}} U(\bar{\pi})\exp\left[\int_{\pi}^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right] h(\pi)d\pi \geq U(\bar{\pi})\exp\left[\int_{\pi_0}^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right].
$$

Dividing both sides by

$$
e^{\bar{\rho}K}U(\bar{\pi})\exp\left[\int_{\pi_0}^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right],
$$

which is negative, we find

$$\int_{1-\bar{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\pi_0} H(\Delta(\pi'),\pi')d\pi'\right] h(\pi)d\pi \le e^{-\bar{\rho}K}. \tag{A.29}$$

To show that (A.29) is equivalent to (A.26), suppose first $\pi_0 \ge \frac{1}{2}$. We write the left-hand side of (A.29) as

$$\int_{1-\bar{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'\right] h(\pi)d\pi$$

$$= \int_{1-\bar{\pi}}^{\frac{1}{2}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'\right] h(\pi)d\pi + \int_{\frac{1}{2}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'\right] h(\pi)d\pi$$

$$= \int_{\frac{1}{2}}^{\bar{\pi}} \exp\left[\int_{1-\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'\right] h(1-\pi)d\pi + \int_{\frac{1}{2}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'\right] h(\pi)d\pi, \tag{A.30}$$

where the first step follows because $\bar{\pi}_0 = \pi_0$ for $\pi_0 \ge \frac{1}{2}$, and the third step follows from the change of variable π to $1-\pi$. To simplify (A.30), we note that

$$\int_{1-\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi' = \int_{1-\pi}^{\frac{1}{2}} H(\Delta(\pi'),\pi')d\pi' + \int_{\frac{1}{2}}^{\pi} H(\Delta(\pi'),\pi')d\pi' + \int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'$$

$$= \int_{\frac{1}{2}}^{\pi} H(\Delta(1-\pi'),1-\pi')d\pi' + \int_{\frac{1}{2}}^{\pi} H(\Delta(\pi'),\pi')d\pi' + \int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'$$

$$= -\int_{\frac{1}{2}}^{\pi} H(-\Delta(1-\pi'),\pi')d\pi' + \int_{\frac{1}{2}}^{\pi} H(\Delta(\pi'),\pi')d\pi' + \int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'$$

$$= -\int_{\frac{1}{2}}^{\pi} H(\Delta(\pi'),\pi')d\pi' + \int_{\frac{1}{2}}^{\pi} H(\Delta(\pi'),\pi')d\pi' + \int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'$$

$$= \int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi', \tag{A.31}$$

where the second step follows from the change of variable π' to $1-\pi'$, the third step follows from (A.23), and the fourth step follows from Property (iv) of Lemma 2.1. Using (A.31), we write (A.30) as

$$\int_{\frac{1}{2}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'),\pi')d\pi'\right] \bar{h}(\pi)d\pi.$$

Therefore, when $\pi_0 \ge \frac{1}{2}$, (A.29) is equivalent to (A.26). To reach the same conclusion when $\pi_0 < \frac{1}{2}$,

we write the left-hand side of (A.29) as

$$\int_{1-\bar{\pi}}^{\bar{\pi}} \exp\left[-\int_{1-\pi}^{1-\pi_0} H(\Delta(1-\pi'), 1-\pi')d\pi'\right] h(\pi)d\pi$$

$$= \int_{1-\bar{\pi}}^{\bar{\pi}} \exp\left[\int_{1-\pi}^{1-\pi_0} H(-\Delta(1-\pi'), \pi')d\pi'\right] h(\pi)d\pi$$

$$= \int_{1-\bar{\pi}}^{\bar{\pi}} \exp\left[\int_{1-\pi}^{1-\pi_0} H(\Delta(\pi'), \pi')d\pi'\right] h(\pi)d\pi$$

$$= \int_{1-\bar{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{1-\pi_0} H(\Delta(\pi'), \pi')d\pi'\right] h(1-\pi)d\pi$$

$$= \int_{1-\bar{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right] h(1-\pi)d\pi, \tag{A.32}$$

where the first step follows from the change of variable π' to $1-\pi'$, the second step follows from (A.23), the third step follows from Property (iv) of Lemma 2.1, the fourth step follows from the change of variable π to $1-\pi$, and the fifth step follows because $\bar{\pi}_0 = 1-\pi_0$ for $\pi_0 < \frac{1}{2}$. This brings us to the case $\pi_0 \geq \frac{1}{2}$, with $h(1-\pi)$ replacing $h(\pi)$.

To derive the bounds (A.27) and (A.28), we derive a lower bound for $\exp\left[\int_\pi^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right]$ by distinguishing two cases for π. For $\pi \in [\bar{\pi}_0, \bar{\pi}]$,

$$\exp\left[\int_\pi^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right] \geq \exp\left[\int_\pi^{\bar{\pi}_0} H(\Delta(\bar{\pi}), \pi')d\pi'\right]$$

$$= \exp\left[\left[\log\left(\pi' + (1-\pi')e^{\bar{\rho}\Delta(\bar{\pi})}\right)\right]_{\bar{\pi}_0}^\pi\right]$$

$$= \frac{\pi + (1-\pi)e^{\bar{\rho}\Delta(\bar{\pi})}}{\bar{\pi}_0 + (1-\bar{\pi}_0)e^{\bar{\rho}\Delta(\bar{\pi})}} \tag{A.33}$$

where the first step follows because $H(\Delta, \pi)$ is increasing in Δ, and $\Delta(\pi)$ is non-decreasing. For $\pi \in (\frac{1}{2}, \bar{\pi}_0)$,

$$\exp\left[\int_\pi^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right] \geq \exp\left[\int_\pi^{\bar{\pi}_0} H(0, \pi')d\pi'\right] = 1 \tag{A.34}$$

because $H(\Delta, \pi)$ is increasing in Δ, and $\Delta(\pi)$ is non-negative for $\pi > \frac{1}{2}$. Combining (A.26), (A.33) and (A.34), we find

$$\int_{\frac{1}{2}}^{\bar{\pi}_0} \bar{h}(\pi)d\pi + \int_{\bar{\pi}_0}^{\bar{\pi}} \frac{\pi + (1-\pi)e^{\bar{\rho}\Delta(\bar{\pi})}}{\bar{\pi}_0 + (1-\bar{\pi}_0)e^{\bar{\rho}\Delta(\bar{\pi})}} \bar{h}(\pi)d\pi \leq e^{-\bar{\rho}K}. \tag{A.35}$$

Grouping together terms in $e^{\bar{\rho}\Delta(\bar{\pi})}$ and using

$$\int_{\frac{1}{2}}^{\bar{\pi}_0} \bar{h}(\pi)d\pi + \int_{\bar{\pi}_0}^{\bar{\pi}} \bar{h}(\pi)d\pi = \int_{\frac{1}{2}}^{\bar{\pi}} \bar{h}(\pi)d\pi = \int_{\frac{1}{2}}^{\bar{\pi}} [h(\pi) + h(1-\pi)]d\pi = \int_{1-\bar{\pi}}^{\bar{\pi}} h(\pi)d\pi = 1,$$

we find

$$\bar{\pi}_0 \left(1 - e^{-\bar{\rho}K}\right) + \int_{\bar{\pi}_0}^{\bar{\pi}} (\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi$$

$$\leq e^{\bar{\rho}\Delta(\bar{\pi})} \left[\int_{\bar{\pi}_0}^{\bar{\pi}} (\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi - (1 - \bar{\pi}_0)\left(1 - e^{-\bar{\rho}K}\right) \right]. \tag{A.36}$$

Since the left-hand side of (A.36) is positive, (A.36) can hold only if the term in square brackets in the right-hand side is positive. The latter condition is equivalent to the upper bound (A.27) on K. Assuming that (A.27) holds, dividing both sides of (A.36) by the term in square brackets in the right-hand side yields the lower bound (A.28) on $\Delta(\bar{\pi})$. $\qquad\square$

Proof of Lemma 2.3. In this and subsequent proofs we denote by

$$G_\pi(\Delta) \equiv - \left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta} + (1-\pi)e^{\bar{\rho}\pi_0\Delta} \right]$$

the utility of the risk-averse type π when she receives $(1-\pi_0)\Delta$ under payoff realization $S + d$ and $-\pi_0\Delta$ under payoff realization $S - d$. The function $G_\pi(\Delta)$ is concave, maximum at

$$\Delta^*(\pi) \equiv \frac{1}{\bar{\rho}} \log \left(\frac{\pi(1-\pi_0)}{(1-\pi)\pi_0} \right), \tag{A.37}$$

increasing for $\Delta < \Delta^*(\pi)$, and decreasing for $\Delta > \Delta^*(\pi)$. We likewise set

$$\bar{G}_\pi(\Delta) \equiv - \left[\pi e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta} + (1-\pi)e^{\bar{\rho}\bar{\pi}_0\Delta} \right],$$

$$\bar{\Delta}^*(\pi) \equiv \frac{1}{\bar{\rho}} \log \left(\frac{\pi(1-\bar{\pi}_0)}{(1-\pi)\bar{\pi}_0} \right). \tag{A.38}$$

The function $G_\pi(\Delta(\pi))$ characterizes how the expected utility $U(\pi)$ of risk-averse type π depends on the fee difference $\Delta(\pi)$, holding the expected fee $\Gamma(\pi)$ under the prior probabilities $(\pi_0, 1-\pi_0)$ constant. Indeed, using $G_\pi(\Delta)$, we can write (A.2) as $U(\pi) = G_\pi(\Delta(\pi))e^{-\rho\Gamma(\pi)}$. Holding $\Gamma(\pi)$ constant, $U(\pi)$ is hump shaped in $\Delta(\pi)$ and maximum for $\Delta^*(\pi)$.

Using (A.2), (A.3) and $G_\pi(\Delta)$, we can write the constraint $U(\pi) \geq U(\pi, \hat{z})$, which is included

in (2.5), as

$$- \left[\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\pi_0\Delta(\pi)} \right] e^{-\rho\Gamma(\pi)} \geq - \left[\pi e^{-\bar{\rho}(1-\pi_0)\hat{\Delta}} + (1-\pi)e^{\bar{\rho}\pi_0\hat{\Delta}} \right] e^{-\rho\hat{\Gamma}}$$

$$\Leftrightarrow G_\pi(\Delta(\pi))e^{-\rho\Gamma(\pi)} \geq G_\pi(\hat{\Delta})e^{-\rho\hat{\Gamma}}. \tag{A.39}$$

Using (A.4) and (A.5), we can write the constraint $\hat{U} \geq \hat{U}(z(\pi'))$, which is included in (2.6), as

$$\hat{\Gamma} \geq \Gamma(\pi'). \tag{A.40}$$

We next explain the role of the two inequalities in (2.9). The inequality in the left-hand side is (A.27). The inequality in the right-hand side is equivalent to the lower bound (A.28) on $\Delta(\bar{\pi})$ exceeding

$$\min\{\Delta^*(\bar{\pi}), -\Delta^*(1-\bar{\pi})\} = \min\left\{ \frac{1}{\bar{\rho}}\log\left(\frac{\bar{\pi}(1-\pi_0)}{(1-\bar{\pi})\pi_0}\right), \frac{1}{\rho}\log\left(\frac{\bar{\pi}\pi_0}{(1-\bar{\pi})(1-\pi_0)}\right) \right\}$$

$$= \frac{1}{\rho}\log\left(\frac{\bar{\pi}(1-\bar{\pi}_0)}{(1-\bar{\pi})\bar{\pi}_0}\right) = \bar{\Delta}^*(\bar{\pi}). \tag{A.41}$$

Indeed, multiplying by $\bar{\rho}$ and taking the exponential, we find that the latter condition is equivalent to

$$\frac{\bar{\pi}_0\left(1 - e^{-\bar{\rho}K}\right) + \int_{\bar{\pi}_0}^{\bar{\pi}}(\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi}{\int_{\bar{\pi}_0}^{\bar{\pi}}(\pi - \bar{\pi}_0)\bar{h}(\pi)d\pi - (1-\bar{\pi}_0)\left(1 - e^{-\bar{\rho}K}\right)} > \frac{\bar{\pi}(1-\bar{\pi}_0)}{(1-\bar{\pi})\bar{\pi}_0}. \tag{A.42}$$

Rearranging (A.42), we find the inequality in the left-hand side of (2.9).

To show that $|\hat{z}| \geq z(\bar{\pi})$, we proceed in two steps. The first step is to suppose that $|\hat{z}| < z(\bar{\pi})$ and show that there exists $\hat{\pi} \in [\bar{\pi}_0, \bar{\pi})$ such that $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$ and

$$F(\pi) \equiv \exp\left[\int_\pi^{\hat{\pi}} H(\Delta(\pi'), \pi')d\pi'\right] - \frac{\pi e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\bar{\pi}_0\Delta(\pi)}}{\hat{\pi}e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\hat{\pi})} + (1-\hat{\pi})e^{\bar{\rho}\bar{\pi}_0\Delta(\hat{\pi})}} \geq 0 \tag{A.43}$$

for all $\pi \in [\hat{\pi}, \bar{\pi}]$. The second step is to show that these properties together with (2.9) yield a violation of (A.26) and hence a contradiction.

The intuition for the first step is that if $|\hat{z}| < z(\bar{\pi})$ then the investor exposes the risk-neutral type to a lower level of risk than the optimal level $\Delta^*(\bar{\pi})$ or $\Delta^*(1-\bar{\pi})$ of the extreme risk-averse type $\bar{\pi}$ or $1-\bar{\pi}$. Hence, there exists an intermediate risk-averse type $\hat{\pi}$ or $1-\hat{\pi}$ whose optimal level of risk coincides with the level to which the risk-neutral type is exposed. The intuition for the second step is that because the investor exposes type $\hat{\pi}$ or $1-\hat{\pi}$ to her optimal level of risk, he cannot expose more extreme types to high risk levels (otherwise they would mimic type $\hat{\pi}$ or $1-\hat{\pi}$),

and hence cannot provide sufficient incentives to observe the signal. The proof of the second step requires a variational argument because while it is possible to expose some more extreme types to high risk levels, it is the average exposure that matters, and bounding it requires optimizing over the function $\Delta(\pi)$.

Step 1: Suppose that $|\hat{z}| < z(\bar{\pi})$. We distinguish cases according to how π_0 compares with $\frac{1}{2}$.

When $\pi_0 > \frac{1}{2}$, $\hat{z}$ is non-negative, and fee monotonicity implies $\Delta(\bar{\pi}) > \hat{\Delta} \geq 0$. (A negative position $\hat{z}$ is dominated by the opposite positive position $-\hat{z}$ because it yields the same wealth outcomes $\hat{z}d$ and $-\hat{z}d$ but with worse probabilities: the probability of $\hat{z}d < 0$ is $\pi_0 > \frac{1}{2}$ under $\hat{z}$ and $1 - \pi_0$ under $-\hat{z}$.) Condition $\pi_0 > \frac{1}{2}$ also implies $\Delta^*(\bar{\pi}) < -\Delta^*(1 - \bar{\pi})$, as can be seen by the derivation of (A.41). Since (2.9) implies that the lower bound (A.28) on $\Delta(\bar{\pi})$ exceeds $\min\{\Delta^*(\bar{\pi}), -\Delta^*(1 - \bar{\pi})\} = \Delta^*(\bar{\pi})$, $G_{\bar{\pi}}(\Delta)$ is decreasing for $\Delta \in [\Delta^*(\bar{\pi}), \Delta(\bar{\pi})]$, and hence is more negative for $\Delta(\bar{\pi})$ than for all $\Delta \in [\Delta^*(\bar{\pi}), \Delta(\bar{\pi}))$. The fee difference $\hat{\Delta}$, which is smaller than $\Delta(\bar{\pi})$ because of fee monotonicity, cannot be in $[\Delta^*(\bar{\pi}), \Delta(\bar{\pi}))$, and hence is smaller than $\Delta^*(\bar{\pi})$. Indeed, if $\hat{\Delta} \in [\Delta^*(\bar{\pi}), \Delta(\bar{\pi}))$, then $G_{\bar{\pi}}(\Delta(\bar{\pi})) < G_{\bar{\pi}}(\hat{\Delta}) < 0$ and $\hat{\Gamma} \geq \Gamma(\bar{\pi})$ (implied by (A.40)) yield a violation of (A.39). Since $\Delta^*(\pi_0) = 0 \leq \hat{\Delta} < \Delta^*(\bar{\pi})$ and $\Delta^*(\pi)$ is continuous and increasing, there exists a unique $\hat{\pi} \in [\pi_0, \bar{\pi}]$ such that $\Delta^*(\hat{\pi}) = \hat{\Delta}$. Since $\Delta^*(\hat{\pi}) = \hat{\Delta}$ is the only maximizer of $G_{\hat{\pi}}(\Delta)$, and since (A.40) implies $\hat{\Gamma} \geq \Gamma(\hat{\pi})$, (A.39) implies $\Delta(\hat{\pi}) = \Delta^*(\hat{\pi}) = \hat{\Delta}$ and $\Gamma(\hat{\pi}) = \hat{\Gamma}$. Combining $\Gamma(\hat{\pi}) = \max_{\pi \in [\pi_0, \bar{\pi}]} \Gamma(\pi)$ and (A.19), and noting that $U(\bar{\pi}) < 0$, we find

$$\frac{\exp\left[\int_{\pi}^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right]}{\pi e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)} + (1 - \pi)e^{\bar{\rho}\pi_0\Delta(\pi)}} \geq \frac{\exp\left[\int_{\hat{\pi}}^{\bar{\pi}} H(\Delta(\pi'), \pi')d\pi'\right]}{\hat{\pi} e^{-\bar{\rho}(1-\pi_0)\Delta(\hat{\pi})} + (1 - \hat{\pi})e^{\bar{\rho}\pi_0\Delta(\hat{\pi})}} \tag{A.44}$$

for all $\pi \in [\hat{\pi}, \bar{\pi}]$. Since $\bar{\pi}_0 = \pi_0$ (which follows from $\pi_0 > \frac{1}{2}$), $\hat{\pi} \in [\bar{\pi}_0, \bar{\pi})$. Moreover, (A.37) and (A.38) imply $\bar{\Delta}^*(\pi) = \Delta^*(\pi)$, and hence $\Delta(\hat{\pi}) = \Delta^*(\hat{\pi})$ implies $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$. Furthermore, (A.44) implies (A.43).

When $\pi_0 < \frac{1}{2}$, $\hat{z}$ is non-positive, and fee monotonicity implies $-\Delta(\bar{\pi}) = \Delta(1 - \bar{\pi}) < \hat{\Delta} \leq 0$. Condition $\pi_0 < \frac{1}{2}$ also implies $\Delta^*(\bar{\pi}) > -\Delta^*(1-\bar{\pi})$, as can be seen by the derivation of (A.41). Since (2.9) implies that the lower bound (A.28) on $\Delta(\bar{\pi}) = -\Delta(1-\bar{\pi})$ exceeds $\min\{\Delta^*(\bar{\pi}), -\Delta^*(1-\bar{\pi})\} = -\Delta^*(1 - \bar{\pi})$, $G_{1-\bar{\pi}}(\Delta)$ is increasing for $\Delta \in [\Delta(1 - \bar{\pi}), \Delta^*(1 - \bar{\pi})]$, and hence is more negative for $\Delta(1-\bar{\pi})$ than for all $\Delta \in (\Delta(1 - \bar{\pi}), \Delta^*(1 - \bar{\pi})]$. A similar argument as in the case $\pi_0 > \frac{1}{2}$ implies that $\hat{\Delta}$, which exceeds $\Delta(1 - \bar{\pi})$ because of fee monotonicity, cannot be in $(\Delta(1 - \bar{\pi}), \Delta^*(1 - \bar{\pi})]$, and hence exceeds $\Delta^*(1-\bar{\pi})$. Following a similar argument as in the case $\pi_0 > \frac{1}{2}$, we can construct $1 - \hat{\pi} \in (1 - \bar{\pi}, \pi_0]$ such that $\Delta(1 - \hat{\pi}) = \Delta^*(1 - \hat{\pi}) = \hat{\Delta}$ and $\Gamma(1 - \hat{\pi}) = \hat{\Gamma}$. Combining $\Gamma(1 - \hat{\pi}) =$

$\max_{\pi \in [1-\bar\pi, \pi_0]} \Gamma(\pi)$ and (A.19), and noting that $U(\bar\pi) < 0$, we find

$$\frac{\exp\left[\int_\pi^{\bar\pi} H(\Delta(\pi'), \pi')d\pi'\right]}{\pi e^{-\bar\rho(1-\pi_0)\Delta(\pi)} + (1-\pi)e^{\bar\rho\pi_0\Delta(\pi)}} \geq \frac{\exp\left[\int_{1-\hat\pi}^{\bar\pi} H(\Delta(\pi'), \pi')d\pi'\right]}{(1-\hat\pi)e^{-\bar\rho(1-\pi_0)\Delta(1-\hat\pi)} + \hat\pi e^{\bar\rho\pi_0\Delta(1-\hat\pi)}} \tag{A.45}$$

for all $\pi \in [1-\bar\pi, \pi_0]$. Since $\bar\pi_0 = 1-\pi_0$ (which follows from $\pi_0 < \frac{1}{2}$), $\hat\pi \in [\bar\pi_0, \bar\pi)$. Moreover, (A.37) and (A.38) imply $\bar\Delta^*(1-\pi) = -\Delta^*(\pi)$, and hence $\Delta(\hat\pi) = -\Delta(1-\hat\pi) = -\Delta^*(1-\hat\pi)$ implies $\Delta(\hat\pi) = \bar\Delta^*(\hat\pi)$. Furthermore, (A.45) implies

$$\exp\left[\int_\pi^{1-\hat\pi} H(\Delta(\pi'), \pi')d\pi'\right] - \frac{\pi e^{-\bar\rho\bar\pi_0\Delta(\pi)} + (1-\pi)e^{\bar\rho(1-\bar\pi_0)\Delta(\pi)}}{(1-\hat\pi)e^{-\bar\rho\bar\pi_0\Delta(1-\hat\pi)} + \hat\pi e^{\bar\rho(1-\bar\pi_0)\Delta(1-\hat\pi)}} \geq 0$$

for all $\pi \in [1-\bar\pi, 1-\hat\pi]$, or equivalently

$$\exp\left[\int_{1-\pi}^{1-\hat\pi} H(\Delta(\pi'), \pi')d\pi'\right] - \frac{(1-\pi)e^{-\bar\rho\bar\pi_0\Delta(1-\pi)} + \pi e^{\bar\rho(1-\bar\pi_0)\Delta(1-\pi)}}{(1-\hat\pi)e^{-\bar\rho\bar\pi_0\Delta(1-\hat\pi)} + \hat\pi e^{\bar\rho(1-\bar\pi_0)\Delta(1-\hat\pi)}} \geq 0 \tag{A.46}$$

for all $\pi \in [\hat\pi, \bar\pi]$. Making the change of variable π' to $1-\pi'$, we can write (A.46) as

$$\exp\left[\int_{\hat\pi}^{\pi} H(\Delta(1-\pi'), 1-\pi')d\pi'\right] - \frac{(1-\pi)e^{-\bar\rho\bar\pi_0\Delta(1-\pi)} + \pi e^{\bar\rho(1-\bar\pi_0)\Delta(1-\pi)}}{(1-\hat\pi)e^{-\bar\rho\bar\pi_0\Delta(1-\hat\pi)} + \hat\pi e^{\bar\rho(1-\bar\pi_0)\Delta(1-\hat\pi)}} \geq 0$$

$$\Leftrightarrow \exp\left[-\int_{\hat\pi}^{\pi} H(-\Delta(1-\pi'), \pi')d\pi'\right] - \frac{(1-\pi)e^{-\bar\rho\bar\pi_0\Delta(1-\pi)} + \pi e^{\bar\rho(1-\bar\pi_0)\Delta(1-\pi)}}{(1-\hat\pi)e^{-\bar\rho\bar\pi_0\Delta(1-\hat\pi)} + \hat\pi e^{\bar\rho(1-\bar\pi_0)\Delta(1-\hat\pi)}} \geq 0$$

$$\Leftrightarrow \exp\left[\int_{\pi}^{\hat\pi} H(\Delta(\pi'), \pi')d\pi'\right] - \frac{(1-\pi)e^{\bar\rho\bar\pi_0\Delta(\pi)} + \pi e^{-\bar\rho(1-\bar\pi_0)\Delta(\pi)}}{(1-\hat\pi)e^{\bar\rho\bar\pi_0\Delta(\hat\pi)} + \hat\pi e^{-\bar\rho(1-\bar\pi_0)\Delta(\hat\pi)}} \geq 0, \tag{A.47}$$

where the second step follows from (A.23) and the third step follows from Property (iv) of Lemma 2.1. Equation (A.47) implies (A.43).

When $\pi_0 = \frac{1}{2}$, $\hat z$ can have any sign. If $\hat z$ is non-negative, then we follow the argument in the case $\pi_0 > \frac{1}{2}$. If $\hat z$ is non-positive, then we follow the argument in the case $\pi_0 < \frac{1}{2}$. In both cases we use the properties $\Delta^*(\bar\pi) = -\Delta^*(1-\bar\pi)$ and $\bar\pi_0 = \pi_0 = 1-\pi_0$, which are implied by $\pi_0 = \frac{1}{2}$.

Step 2: Consider the problem of minimizing

$$\int_{\hat\pi}^{\bar\pi} \exp\left[\int_\pi^{\hat\pi} H(\Delta(\pi'), \pi')d\pi'\right] \bar h(\pi)d\pi \tag{A.48}$$

with respect to $\Delta(\pi)$ that is defined over $[\hat\pi, \bar\pi]$, is left-continuous with right-limits, and satisfies

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$F(\pi) \geq 0$ and $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$. The Lagrangian for this problem is

$$\int_{\hat{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\hat{\pi}} H(\Delta(\pi'), \pi')d\pi'\right] \bar{h}(\pi)d\pi - \int_{\hat{\pi}}^{\bar{\pi}} F(\pi)\mu(\pi)d\pi$$

$$= \int_{\hat{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\hat{\pi}} H(\Delta(\pi'), \pi')d\pi'\right] [\bar{h}(\pi) - \mu(\pi)]d\pi$$

$$+ \int_{\hat{\pi}}^{\bar{\pi}} \frac{\pi e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\bar{\pi}_0\Delta(\pi)}}{\hat{\pi}e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\hat{\pi})} + (1-\hat{\pi})e^{\bar{\rho}\bar{\pi}_0\Delta(\hat{\pi})}}\mu(\pi)d\pi,$$

and yields the first-order condition

$$- H_\Delta(\Delta(\pi), \pi)\int_{\pi}^{\bar{\pi}} \exp\left[\int_{\pi'}^{\hat{\pi}} H(\Delta(\pi''), \pi'')d\pi''\right] [\bar{h}(\pi') - \mu(\pi')]d\pi'$$

$$+ \bar{\rho}\mu(\pi)\frac{(1-\pi)\bar{\pi}_0 e^{\bar{\rho}\bar{\pi}_0\Delta(\pi)} - \pi(1-\bar{\pi}_0)e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)}}{\hat{\pi}e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\hat{\pi})} + (1-\hat{\pi})e^{\bar{\rho}\bar{\pi}_0\Delta(\hat{\pi})}} = 0. \tag{A.49}$$

If $F(\pi) > 0$ for π in an open interval (π_1, π_2), then for π in that interval $\mu(\pi) = 0$ and (A.49) becomes

$$\int_{\pi}^{\bar{\pi}} \exp\left[\int_{\pi'}^{\hat{\pi}} H(\Delta(\pi''), \pi'')d\pi''\right] [\bar{h}(\pi') - \mu(\pi')]d\pi' = 0. \tag{A.50}$$

Differentiating (A.50) in (π_1, π_2), and using $\mu(\pi) = 0$, $\bar{h}(\pi) > 0$ and $F(\pi) > 0$, we find

$$\exp\left[\int_{\pi}^{\hat{\pi}} H(\Delta(\pi'), \pi')d\pi'\right] = 0 > \frac{\pi e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\pi)} + (1-\pi)e^{\bar{\rho}\bar{\pi}_0\Delta(\pi)}}{\hat{\pi}e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\hat{\pi})} + (1-\hat{\pi})e^{\bar{\rho}\bar{\pi}_0\Delta(\hat{\pi})}},$$

a contradiction. Hence, the function $\Delta(\pi)$ that minimizes (A.48) satisfies $F(\pi) = 0$ in a set that is dense in $[\hat{\pi}, \bar{\pi}]$. Since $\Delta(\pi)$ is left-continuous and $F(\hat{\pi}) = 0$, this set coincides with $[\hat{\pi}, \bar{\pi}]$. Suppose next, by contradiction, that $\Delta(\pi)$ is discontinuous at a point π_1. Since $\Delta(\pi)$ is left-continuous, $\pi_1 \neq \bar{\pi}$. Since, in addition, $F(\pi_1) = F(\pi_1^+) = 0$ implies $\bar{G}_\pi(\Delta(\pi_1)) = \bar{G}_\pi(\Delta(\pi_1^+))$, and since $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$, $\pi_1 \neq \hat{\pi}$. Hence, $\pi_1 \in (\hat{\pi}, \bar{\pi})$. Combining (A.49) for π_1 and for $\pi_1' > \pi_1$, we find

$$\int_{\pi_1}^{\pi_1'} \exp\left[\int_{\pi'}^{\hat{\pi}} H(\Delta(\pi''), \pi'')d\pi''\right] [\bar{h}(\pi') - \mu(\pi')]d\pi'$$

$$+ \frac{\mu(\pi_1)\bar{G}'_{\pi_1}(\Delta(\pi_1))}{H_\Delta(\Delta(\pi_1), \pi_1)\bar{G}_{\hat{\pi}}(\Delta(\hat{\pi}))} - \frac{\mu(\pi_1')\bar{G}'_{\pi_1'}(\Delta(\pi_1'))}{H_\Delta(\Delta(\pi_1'), \pi_1')\bar{G}_{\hat{\pi}}(\Delta(\hat{\pi}))} = 0. \tag{A.51}$$

If $\Delta(\pi_1) < \Delta(\pi_1^+)$, in which case $\Delta(\pi_1) < \bar{\Delta}^*(\pi_1) < \Delta(\pi_1^+)$, $\bar{G}'_{\pi_1}(\Delta(\pi_1)) > 0$ and $\bar{G}'_{\pi_1}(\Delta(\pi_1^+)) < 0$, then the second term in (A.50) is non-negative because $\mu(\pi_1) \geq 0$, and the third term is non-negative because $\mu(\pi_1') \geq 0$. If $\mu(\pi_1^+) > 0$, then the third term is positive in the limit when π_1' goes to π_1. Since the first term goes to zero in that limit, (A.50) is violated, a contradiction. If instead

$\mu(\pi_1^+) = 0$, then the first term is positive close to the limit because $\bar{h}(\pi_1) > 0$, and hence (A.50) is again violated, a contradiction. Therefore, $\Delta(\pi_1) > \bar{\Delta}^*(\pi_1) > \Delta(\pi_1^+)$. If $\Delta(\pi)$ is discontinuous at an additional point π_2', then the same reasoning implies $\Delta(\pi_2') > \bar{\Delta}^*(\pi_2') > \Delta(\pi_2'^+)$. Assume without loss of generality that $\pi_2' > \pi_1$, and take the infimum π_2 of the discontinuity points π_2' that exceed π_1. The infimum π_2 must strictly exceed π_1 because otherwise taking the limit in $\Delta(\pi_2') > \bar{\Delta}^*(\pi_2')$ yields $\Delta(\pi_1^+) \geq \bar{\Delta}^*(\pi_1)$. Hence, $\Delta(\pi)$ is continuous in the non-empty interval (π_1, π_2). Differentiating $F(\pi) = 0$ in that interval, we find

$$
\begin{aligned}
& H(\Delta(\pi), \pi) \exp\left[\int_\pi^{\hat{\pi}} H(\Delta(\pi'), \pi') d\pi'\right] + \frac{e^{\bar{\rho}\pi_0 \Delta(\pi)} - e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)}}{\hat{\pi} e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\hat{\pi})} + (1-\hat{\pi})e^{\bar{\rho}\bar{\pi}_0 \Delta(\hat{\pi})}} \\
& - \bar{\rho}\Delta'(\pi) \frac{(1-\pi)\bar{\pi}_0 e^{\bar{\rho}\pi_0 \Delta(\pi)} - \pi(1-\bar{\pi}_0)e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)}}{\hat{\pi} e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\hat{\pi})} + (1-\hat{\pi})e^{\bar{\rho}\bar{\pi}_0 \Delta(\hat{\pi})}} = 0 \\
\Leftrightarrow\ & \bar{\rho}\Delta'(\pi) \frac{(1-\pi)\bar{\pi}_0 e^{\bar{\rho}\pi_0 \Delta(\pi)} - \pi(1-\bar{\pi}_0)e^{-\bar{\rho}(1-\pi_0)\Delta(\pi)}}{\hat{\pi} e^{-\bar{\rho}(1-\bar{\pi}_0)\Delta(\hat{\pi})} + (1-\hat{\pi})e^{\bar{\rho}\bar{\pi}_0 \Delta(\hat{\pi})}} = 0, \quad\quad\quad (A.52)
\end{aligned}
$$

where the second step follows by using $F(\pi) = 0$ and the definition of $H(\Delta, \pi)$. Equation (A.52) with the initial condition $\Delta(\pi_1^+) < \bar{\Delta}^*(\pi_1)$ implies $\Delta(\pi) = 0$ for $\pi \in (\pi_1, \pi_2)$, and hence $\Delta(\pi_1^+) = \Delta(\pi_2)$. This is a contradiction. Indeed, if π_2 is a discontinuity point, then $\Delta(\pi_2) > \bar{\Delta}^*(\pi_2) > \bar{\Delta}^*(\pi_1) > \Delta(\pi_1^+)$. If instead, π_2 is a continuity point and hence a limit of discontinuity points, then taking the limit in $\Delta(\pi_2') > \bar{\Delta}^*(\pi_2')$ yields $\Delta(\pi_2^+) \geq \bar{\Delta}^*(\pi_2)$, which implies $\Delta(\pi_2) = \Delta(\pi_2^+) \geq \bar{\Delta}^*(\pi_2) > \bar{\Delta}^*(\pi_1) > \Delta(\pi_1^+)$. Therefore, $\Delta(\pi)$ can have at most one discontinuity point π_1. Equation (A.52) with the initial condition $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$ implies, however, that $\Delta(\pi)$ is of the form $\Delta(\pi) = \bar{\Delta}^*(\pi)$ for $\pi \in [\hat{\pi}, \pi_3]$ and $\Delta(\pi) = \Delta(\pi_3)$ for $\pi \in (\pi_3, \pi_1]$, where $\pi_3 \in [\hat{\pi}, \pi_1]$. This implies $\Delta(\pi_1) \leq \bar{\Delta}^*(\pi_1)$, a contradiction. Therefore, $\Delta(\pi)$ is continuous over the entire interval $[\hat{\pi}, \bar{\pi}]$. Equation (A.52) with the initial condition $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$ implies that $\Delta(\pi)$ is of the form $\Delta(\pi) = \bar{\Delta}^*(\pi)$ for $\pi \in [\hat{\pi}, \pi_3]$ and $\Delta(\pi) = \bar{\Delta}^*(\pi_3)$ for $\pi \in (\pi_3, \bar{\pi}]$, where $\pi_3 \in [\hat{\pi}, \bar{\pi}]$. The solution that minimizes (A.48) corresponds to $\pi_3 = \bar{\pi}$. That solution, which is non-decreasing, also minimizes (A.48) over the set of non-decreasing functions that satisfy $F(\pi) \geq 0$ and $\Delta(\hat{\pi}) = \Delta^*(\hat{\pi})$. Indeed, a non-decreasing function has a countable set of discontinuity points, and left- and right-limits at those points. By setting its value at the discontinuity points to its left-limit, we can transform it into a left-continuous function with right-limits. Since this operation is performed at a countable set of points, the resulting function yields an identical value for (A.48).

Using our solution of the minimization problem, we can show that (A.26) is violated for any $\Delta(\pi)$ that is defined over $[1 - \bar{\pi}, \bar{\pi}]$, is non-decreasing, and satisfies $\Delta(\pi) = -\Delta(1 - \pi)$, $F(\pi) \geq 0$

and $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$. We write (A.26) as

$$\int_{\frac{1}{2}}^{\hat{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right] \bar{h}(\pi)d\pi$$

$$+ \exp\left[\int_{\hat{\pi}}^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right] \int_{\hat{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\hat{\pi}} H(\Delta(\pi'), \pi')d\pi'\right] \bar{h}(\pi)d\pi \le e^{-\bar{\rho}K}. \tag{A.53}$$

Since (A.48) is minimized for $\Delta(\pi) = \bar{\Delta}^*(\pi)$,

$$\int_{\hat{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\hat{\pi}} H(\Delta(\pi'), \pi')d\pi'\right] \bar{h}(\pi)d\pi$$

$$\ge \int_{\hat{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\hat{\pi}} H(\bar{\Delta}^*(\pi'), \pi')d\pi'\right] \bar{h}(\pi)d\pi$$

$$\ge \int_{\hat{\pi}}^{\bar{\pi}} \exp\left[\int_{\pi}^{\hat{\pi}} H(\bar{\Delta}^*(\bar{\pi}), \pi')d\pi'\right] \bar{h}(\pi)d\pi, \tag{A.54}$$

where the second step follows because $\bar{\Delta}^*(\pi)$ is increasing and $H(\Delta, \pi)$ is increasing in Δ. Moreover, for $\pi \in [\bar{\pi}_0, \hat{\pi}]$,

$$\exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\pi'), \pi')d\pi'\right] \ge \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\Delta(\hat{\pi}), \pi')d\pi'\right]$$

$$= \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\bar{\Delta}^*(\hat{\pi}), \pi')d\pi'\right]$$

$$\ge \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\bar{\Delta}^*(\bar{\pi}), \pi')d\pi'\right], \tag{A.55}$$

where the first step follows because $H(\Delta, \pi)$ is increasing in Δ, the second step follows because $\Delta(\hat{\pi}) = \bar{\Delta}^*(\hat{\pi})$, and the third step follows because $\bar{\Delta}^*(\pi)$ is increasing and $H(\Delta, \pi)$ is increasing in Δ. Substituting (A.34), (A.54) and (A.55) into (A.53), we find

$$\int_{\frac{1}{2}}^{\bar{\pi}_0} \bar{h}(\pi)d\pi + \int_{\bar{\pi}_0}^{\bar{\pi}} \exp\left[\int_{\pi}^{\bar{\pi}_0} H(\bar{\Delta}^*(\bar{\pi}), \pi')d\pi'\right] \bar{h}(\pi)d\pi \le e^{-\bar{\rho}K}$$

$$\Leftrightarrow \int_{\frac{1}{2}}^{\bar{\pi}_0} \bar{h}(\pi)d\pi + \int_{\bar{\pi}_0}^{\bar{\pi}} \frac{\pi + (1-\pi)e^{\bar{\rho}\bar{\Delta}^*(\bar{\pi})}}{\bar{\pi}_0 + (1-\bar{\pi}_0)e^{\bar{\rho}\bar{\Delta}^*(\bar{\pi})}} \bar{h}(\pi)d\pi \le e^{-\bar{\rho}K}, \tag{A.56}$$

where the second step follows as in (A.33). Since (A.56) is identical to (A.35) with $\bar{\Delta}^*(\bar{\pi})$ replacing $\Delta(\bar{\pi})$, the argument that follows (A.35) implies that $\bar{\Delta}^*(\bar{\pi})$ exceeds the lower bound (A.28) on $\Delta(\bar{\pi})$. This contradicts (2.9). $\qquad\square$

Proof of Theorem 2.1. If the investor induces the risk-averse type to observe the signal, then he chooses positions $(z(\pi), \hat{z})$ and fee levels $(f(z(\pi)d), f(-z(\pi)d), f(\hat{z}d), f(-\hat{z}d))$ for $\pi \in [1 - \bar{\pi}, \bar{\pi}]$

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to maximize the utility

$$U = -\,(1-\lambda)\int_{1-\bar{\pi}}^{\bar{\pi}}\left[\pi e^{-\rho[z(\pi)d-f(z(\pi)d)]}+(1-\pi)e^{-\rho[-z(\pi)d-f(-z(\pi)d)]}\right]h(\pi)d\pi$$
$$-\,\lambda\left[\pi_0 e^{-\rho[\hat{z}d-f(\hat{z}d)]}+(1-\pi_0)e^{-\rho[-\hat{z}d-f(-\hat{z}d)]}\right]. \tag{A.57}$$

The investor is subject to the (IC) constraints (2.4), (2.6), (2.7),

$$U(\pi)\geq\max\left\{U(\pi,\hat{z}),U(\pi,-\hat{z})\right\}, \tag{A.58}$$

and the non-negativity and monotonicity of the fee. We refer to this optimization problem as $(\mathcal{P})$.

We can simplify $(\mathcal{P})$ using symmetry. Making the change of variable π to $1-\pi$, we can write the first half of the integral in (A.57) as

$$\int_{1-\bar{\pi}}^{\frac{1}{2}}\left[\pi e^{-\rho[z(\pi)d-f(z(\pi)d)]}+(1-\pi)e^{-\rho[-z(\pi)d-f(-z(\pi)d)]}\right]h(\pi)d\pi$$
$$=\int_{\frac{1}{2}}^{\bar{\pi}}\left[(1-\pi)e^{-\rho[z(1-\pi)d-f(z(1-\pi)d)]}+\pi e^{-\rho[-z(1-\pi)d-f(-z(1-\pi)d)]}\right]h(1-\pi)d\pi$$
$$=\int_{\frac{1}{2}}^{\bar{\pi}}\left[(1-\pi)e^{-\rho[-z(\pi)d-f(-z(\pi)d)]}+\pi e^{-\rho[z(\pi)d-f(z(\pi)d)]}\right]h(1-\pi)d\pi,$$

where the second step follows from Property (ii) of Lemma 2.2. Properties (ii) and (iii) of Lemma 2.2 imply that the (IC) constraint (2.7) reduces to

$$U(\pi)\geq\max\left\{\max_{\pi'\in(\frac{1}{2},\bar{\pi}]}U(\pi,z(\pi')),U\left(\pi,\left|z\left(\frac{1}{2}\right)\right|\right)\right\} \tag{A.59}$$

for all $\pi\in[\frac{1}{2},\bar{\pi}]$. Likewise, the (IC) constraint (A.58) reduces to

$$U(\pi)\geq U(\pi,|\hat{z}|) \tag{A.60}$$

for all $\pi\in[\frac{1}{2},\bar{\pi}]$. Distinguishing the cases $\pi_0>\frac{1}{2}$, where $\bar{\pi}_0=\pi_0$ and $\hat{z}\geq 0$; $\pi_0<\frac{1}{2}$, where $\bar{\pi}_0=1-\pi_0$ and $\hat{z}\leq 0$; and $\pi_0=\frac{1}{2}$; we find

$$\pi_0 e^{-\rho[\hat{z}d-f(\hat{z}d)]}+(1-\pi_0)e^{-\rho[-\hat{z}d-f(-\hat{z}d)]}=\bar{\pi}_0 e^{-\rho[|\hat{z}|d-f(|\hat{z}|d)]}+(1-\bar{\pi}_0)e^{-\rho[-|\hat{z}|d-f(-|\hat{z}|d)]}.$$

We likewise find that the (IC) constraint (2.6) reduces to

$$\hat{U} = \bar{\pi}_0 f(|\hat{z}|d) + (1 - \bar{\pi}_0) f(-|\hat{z}|d)$$

$$\geq \max \left\{ \max_{\pi' \in (\frac{1}{2}, \bar{\pi}]} [\bar{\pi}_0 f(z(\pi)d) + (1 - \bar{\pi}_0) f(-z(\pi)d)], \left[\bar{\pi}_0 f\left(\left| z\left(\frac{1}{2}\right) \right| d \right) + (1 - \bar{\pi}_0) f\left(-\left| z\left(\frac{1}{2}\right) \right| d \right) \right] \right\}.$$

$$(A.61)$$

Hence, the problem $(\mathcal{P})$ reduces to maximizing

$$U = -(1 - \lambda) \int_{\frac{1}{2}}^{\bar{\pi}} \left[\pi e^{-\rho[z(\pi)d - f(z(\pi)d)]} + (1 - \pi) e^{-\rho[-z(\pi)d - f(-z(\pi)d)]} \right] \bar{h}(\pi) d\pi$$

$$- \lambda \left[\bar{\pi}_0 e^{-\rho[|\hat{z}|d - f(|\hat{z}|d)]} + (1 - \bar{\pi}_0) e^{-\rho[-|\hat{z}|d - f(-|\hat{z}|d)]} \right] \tag{A.62}$$

over $(z(\pi), |\hat{z}|)$ and $(f(z(\pi)d), f(-z(\pi)d), f(|\hat{z}|d), f(-|\hat{z}|d))$ for $\pi \in [\frac{1}{2}, \bar{\pi}]$, subject to (2.4), (A.59), (A.60), (A.61), $z(\pi) \geq 0$, and the non-negativity and monotonicity of the fee. Given a solution $z(\pi)$ for $\pi \in [\frac{1}{2}, \bar{\pi}]$, we define $z(\pi)$ for $\pi \in [1 - \bar{\pi}, \frac{1}{2})$ by $z(\pi) = -z(1 - \pi)$.

When the pooling condition (2.9) holds, the problem $(\mathcal{P})$ reduces to maximizing (A.62) over $(z(\pi), |\hat{z}|)$ and $(f(z(\pi)d), f(-z(\pi)d), f(|\hat{z}|d), f(-|\hat{z}|d))$ for $\pi \in [\frac{1}{2}, \bar{\pi}]$ subject to the following constraints: (i) (A.26), (ii) $\Delta(\pi)$ is non-decreasing, (iii) $U(\pi)$ is given by (A.18), (iv) $z(\pi) \geq 0$, (v) $|\hat{z}| = z(\bar{\pi})$, (vi) $f(-z(\bar{\pi})d) = 0$, and (vii) fee monotonicity. Indeed, Lemma A.1 shows that the (IC) constraint (2.4) is equivalent to (i). Lemma 2.1 shows that the (IC) constraint (A.59) for $\pi \in [\frac{1}{2}, \bar{\pi}]$ is equivalent to (ii) and (iii). Lemma 2.3 implies (v) because it is suboptimal for the investor to induce the uninformed risk-neutral type to choose a more extreme position than that of types $\bar{\pi}$ and $1 - \bar{\pi}$. (For a detailed proof that $|\hat{z}| > z(\bar{\pi})$ is suboptimal in the symmetric case $h(\pi) = h(1 - \pi)$, see Vayanos (2018).) Lemma 2.2 shows that (ii) and (vii) imply that $z(\pi)$ is non-decreasing. Constraint (vii) and $z(\pi)$ non-decreasing imply that non-negativity reduces to (vi). The (IC) constraint (A.60) follows from (v) and the (IC) constraint (A.59). The (IC) constraint (A.61) follows from Lemma 2.3 and (v).

Using $(\Delta(\pi), \Gamma(\pi))$ and (v), we can write (A.62) as

$$U = -(1 - \lambda) \int_{\frac{1}{2}}^{\bar{\pi}} \left[\pi e^{-\rho[z(\pi)d - (1 - \pi_0)\Delta(\pi)]} + (1 - \pi) e^{\rho[z(\pi)d - \pi_0 \Delta(\pi)]} \right] e^{\rho \Gamma(\pi)} \bar{h}(\pi) d\pi$$

$$- \lambda \left[\bar{\pi}_0 e^{-\rho[z(\bar{\pi})d - (1 - \pi_0)\Delta(\bar{\pi})]} + (1 - \bar{\pi}_0) e^{\rho[z(\bar{\pi})d - \pi_0 \Delta(\bar{\pi})]} \right] e^{\rho \Gamma(\bar{\pi})}. \tag{A.63}$$

The problem $(\mathcal{P})$ reduces to maximizing (A.63) over $(z(\pi), \Delta(\pi), \bar{\Gamma}(\pi))$ for $\pi \in [\frac{1}{2}, \bar{\pi}]$, subject to (i), (ii), (iii), (iv), (vi) and (vii). Since $\Delta(\pi) \equiv f(z(\pi)d) - f(-z(\pi)d)$, we must impose the additional constraint (viii) $\Delta(\pi)$ is non-negative, $\Delta(\pi) = 0$ when $z(\pi) = 0$, and $\Delta(\pi)$ is constant in any

interval where $z(\pi)$ is constant.

The limit when ϵ goes to zero of the solution to $(\mathcal{P})$ is the solution to maximizing (A.63) over $(z(\pi), \Delta(\pi), \bar{\Gamma}(\pi))$ for $\pi \in [\frac{1}{2}, \bar{\pi}]$, subject to (i), (ii), (iii), (iv), (vi), (viii) and $z(\pi)$ non-decreasing. The reason why we can replace (vii) by $z(\pi)$ non-decreasing when deriving the limit is that for all $\epsilon > 0$ (ii) and (vii) imply $z(\pi)$ non-decreasing, but for $\epsilon = 0$ (vii) is implied by $z(\pi)$ non-decreasing and (A.59) (or equivalently (ii) and (iii)) and is hence redundant.

Using (A.19) and $\Gamma(\bar{\pi}) = \pi_0 \Delta(\bar{\pi})$, which follow from (iii) and (vi), respectively, we can write (A.63) as

$$
\begin{aligned}
U = - \; & (1 - \lambda) \int_{\frac{1}{2}}^{\bar{\pi}} \left[\pi e^{-\rho[z(\pi)d - \Delta(\pi)]} + (1 - \pi) e^{\rho z(\pi) d} \right] \\
& \times \exp\left[-\frac{\rho}{\bar{\rho}} \int_{\pi}^{\bar{\pi}} H(\Delta(\pi'), \pi') d\pi' \right] \left[\frac{\pi e^{-\bar{\rho}\Delta(\pi)} + 1 - \pi}{\pi e^{-\bar{\rho}\Delta(\bar{\pi})} + 1 - \bar{\pi}} \right]^{\frac{\rho}{\bar{\rho}}} \bar{h}(\pi) d\pi \\
& - \lambda \left[\bar{\pi}_0 e^{-\rho[z(\bar{\pi})d - \Delta(\bar{\pi})]} + (1 - \bar{\pi}_0) e^{\rho z(\bar{\pi}) d} \right].
\end{aligned}
\tag{A.64}
$$

The problem $(\mathcal{P})$ reduces to maximizing (A.64) over $(z(\pi), \Delta(\pi))$ for $\pi \in [\frac{1}{2}, \bar{\pi}]$, subject to (i), (ii), (iv), (viii) and $z(\pi)$ non-decreasing. In other words, (A.64) must be maximized over non-negative and non-decreasing $(z(\pi), \Delta(\pi))$, subject to the (IC) constraint (A.26) and the constraint that $\Delta(\pi) = 0$ when $z(\pi) = 0$ and that $\Delta(\pi)$ is constant in any interval where $z(\pi)$ is constant.

Without loss of generality, we can assume $z(\frac{1}{2}) = 0$ and hence $\Delta(\frac{1}{2}) = 0$. Indeed, if $z(\frac{1}{2}) > 0$, then we can set $z(\frac{1}{2})$ and $\Delta(\frac{1}{2})$ to zero. Since the density $h(\pi)$ is continuous, this change does not affect the (IC) constraint (A.26) and the investor's utility (A.64). Moreover, the functions $(z(\pi), \Delta(\pi))$ remain non-negative and non-decreasing, and the constraint that $\Delta(\pi) = 0$ when $z(\pi) = 0$ and that $\Delta(\pi)$ is constant in any interval where $z(\pi)$ is constant, remains satisfied.

We next consider

$$
\hat{\pi} \equiv \inf\{\pi : z(\pi) = z(\bar{\pi}) \, \forall \, \pi \in [\hat{\pi}, \bar{\pi}]\},
$$

and show that for the solution to $(\mathcal{P})$, $\hat{\pi}$ is equal to π^* defined in (2.12), and $z(\bar{\pi})$ is given by (2.10). We proceed by contradiction. We assume that these properties are not true, and show that the investor can raise his utility by changing $z(\pi)$ while leaving $\Delta(\pi)$ the same. Since $\Delta(\pi)$ does not change, it remains non-negative and non-deceasing, and the (IC) constraint (A.26) remains satisfied. Moreover, under all the changes that we consider, $z(\pi)$ remains non-negative and non-decreasing, and $\Delta(\pi)$ remains equal to zero when $z(\pi) = 0$ and remains constant in any interval where $z(\pi)$ is constant. Note that since $z(\pi)$ is constant in $[\hat{\pi}, \bar{\pi}]$, $\Delta(\pi)$ is also constant in that interval.

Equation (2.12) defines $\pi^* \in (\bar{\pi}_0, \bar{\pi})$ uniquely because (i) the left-hand side decreases in π^* and the right-hand side increases in π^*, (ii) the left-hand side is positive for $\pi^* = \bar{\pi}_0$ and the right-hand side is zero for that value, and (iii) the left-hand side is zero for $\pi^* = \bar{\pi}$ and the right-hand side is positive for that value. Since, in addition, the left-hand side decreases in λ and the right-hand side increases in λ, π^* decreases in λ. Equation (2.12) also implies

$$
\begin{aligned}
&(1-\lambda) \int_{\pi^*}^{\bar{\pi}} \pi \bar{h}(\pi) d\pi + \lambda \bar{\pi}_0 \\
&= \frac{1-\lambda}{\pi^* - \bar{\pi}_0} \left((\pi^* - \bar{\pi}_0) \int_{\pi^*}^{\bar{\pi}} \pi \bar{h}(\pi) d\pi + \bar{\pi}_0 \int_{\pi^*}^{\bar{\pi}} (\pi - \pi^*) \bar{h}(\pi) d\pi \right) \\
&= \frac{(1-\lambda)\pi^*}{\pi^* - \bar{\pi}_0} \int_{\pi^*}^{\bar{\pi}} (\pi - \bar{\pi}_0) \bar{h}(\pi) d\pi,
\end{aligned}
\tag{A.65}
$$

and

$$
\begin{aligned}
&(1-\lambda) \int_{\pi^*}^{\bar{\pi}} (1-\pi) \bar{h}(\pi) d\pi + \lambda (1 - \bar{\pi}_0) \\
&= \frac{1-\lambda}{\pi^* - \bar{\pi}_0} \left((\pi^* - \bar{\pi}_0) \int_{\pi^*}^{\bar{\pi}} (1-\pi) \bar{h}(\pi) d\pi + (1 - \bar{\pi}_0) \int_{\pi^*}^{\bar{\pi}} (\pi - \pi^*) \bar{h}(\pi) d\pi \right) \\
&= \frac{(1-\lambda)(1-\pi^*)}{\pi^* - \bar{\pi}_0} \int_{\pi^*}^{\bar{\pi}} (\pi - \bar{\pi}_0) \bar{h}(\pi) d\pi.
\end{aligned}
\tag{A.66}
$$

Suppose that $\hat{\pi} < \pi^*$, and consider first the case where

$$
z(\bar{\pi}) \le \frac{1}{2\rho d} \log \left(\frac{(1-\lambda) \int_{\hat{\pi}}^{\bar{\pi}} \pi \bar{h}(\pi) d\pi + \lambda \bar{\pi}_0}{(1-\lambda) \int_{\hat{\pi}}^{\bar{\pi}} (1-\pi) \bar{h}(\pi) d\pi + \lambda (1 - \bar{\pi}_0)} \right) + \frac{\Delta(\bar{\pi})}{2d}.
\tag{A.67}
$$

If the investor replaces $z(\pi)$ for all $\pi \in [\pi^*, \bar{\pi}]$ by $z(\bar{\pi}) + \phi$, for small $\phi > 0$, then his utility (A.63) becomes

$$
\begin{aligned}
U = -(1-\lambda) &\left\{ \int_{\frac{1}{2}}^{\pi^*} \left[\pi e^{-\rho[z(\pi)d - (1-\pi_0)\Delta(\pi)]} + (1-\pi) e^{\rho[z(\pi)d - \pi_0\Delta(\pi)]} \right] e^{\rho\Gamma(\pi)} \bar{h}(\pi) d\pi \right. \\
&+ \left[e^{-\rho[(z(\bar{\pi})+\phi)d - (1-\pi_0)\Delta(\bar{\pi})]} \int_{\pi^*}^{\bar{\pi}} \pi \bar{h}(\pi) d\pi + e^{\rho[(z(\bar{\pi})+\phi)d - \pi_0\Delta(\bar{\pi})]} \int_{\pi^*}^{\bar{\pi}} (1-\pi) \bar{h}(\pi) d\pi \right] e^{\rho\Gamma(\bar{\pi})} \left. \right\} \\
&- \lambda \left[\bar{\pi}_0 e^{-\rho[(z(\bar{\pi})+\phi)d - (1-\pi_0)\Delta(\bar{\pi})]} + (1 - \bar{\pi}_0) e^{\rho[(z(\bar{\pi})+\phi)d - \pi_0\Delta(\bar{\pi})]} \right] e^{\rho\Gamma(\bar{\pi})}.
\end{aligned}
\tag{A.68}
$$

Equation (A.68) implies

$$
\begin{aligned}
\left. \frac{\partial U}{\partial \phi} \right|_{\phi=0} = \rho d &\left\{ e^{-\rho[z(\bar{\pi})d - (1-\bar{\pi}_0)\Delta(\bar{\pi})]} \left((1-\lambda) \int_{\pi^*}^{\bar{\pi}} \pi \bar{h}(\pi) d\pi + \lambda \bar{\pi}_0 \right) \right. \\
&\left. - e^{\rho[z(\bar{\pi})d - \bar{\pi}_0\Delta(\bar{\pi})]} \left((1-\lambda) \int_{\pi^*}^{\bar{\pi}} (1-\pi) \bar{h}(\pi) d\pi + \lambda (1 - \bar{\pi}_0) \right) \right\} e^{\rho\Gamma(\bar{\pi})}.
\end{aligned}
$$

Hence, utility increases if

$$\left.\frac{\partial U}{\partial \phi}\right|_{\phi=0} > 0 \Leftrightarrow z(\bar{\pi}) < \frac{1}{2\rho d} \log\left(\frac{(1-\lambda)\int_{\pi^*}^{\bar{\pi}} \pi \bar{h}(\pi)d\pi + \lambda \bar{\pi}_0}{(1-\lambda)\int_{\pi^*}^{\bar{\pi}}(1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\bar{\pi}_0)}\right) + \frac{\Delta(\bar{\pi})}{2d}. \tag{A.69}$$

Equation (A.67) implies (A.69) if

$$\frac{(1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}} \pi \bar{h}(\pi)d\pi + \lambda \bar{\pi}_0}{(1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}}(1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\bar{\pi}_0)} < \frac{(1-\lambda)\int_{\pi^*}^{\bar{\pi}} \pi \bar{h}(\pi)d\pi + \lambda \bar{\pi}_0}{(1-\lambda)\int_{\pi^*}^{\bar{\pi}}(1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\bar{\pi}_0)}$$

$$\Leftrightarrow \frac{\int_{\hat{\pi}}^{\pi^*} \pi \bar{h}(\pi)d\pi + \frac{\pi^*}{\pi^*-\bar{\pi}_0}\int_{\pi^*}^{\bar{\pi}}(\pi-\bar{\pi}_0)\bar{h}(\pi)d\pi}{\int_{\hat{\pi}}^{\pi^*}(1-\pi)\bar{h}(\pi)d\pi + \frac{1-\pi^*}{\pi^*-\bar{\pi}_0}\int_{\pi^*}^{\bar{\pi}}(\pi-\bar{\pi}_0)\bar{h}(\pi)d\pi} < \frac{\pi^*}{1-\pi^*}, \tag{A.70}$$

where the equivalence follows from (A.65) and (A.66). Equation (A.70) is equivalent to

$$\frac{\int_{\hat{\pi}}^{\pi^*} \pi \bar{h}(\pi)d\pi}{\int_{\hat{\pi}}^{\pi^*}(1-\pi)\bar{h}(\pi)d\pi} < \frac{\pi^*}{1-\pi^*},$$

which holds because $\hat{\pi} < \pi^*$.

Consider next the case where

$$z(\bar{\pi}) > \frac{1}{2\rho d} \log\left(\frac{(1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}} \pi \bar{h}(\pi)d\pi + \lambda \bar{\pi}_0}{(1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}}(1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\bar{\pi}_0)}\right) + \frac{\Delta(\bar{\pi})}{2d}. \tag{A.71}$$

In the sub-case $\hat{\pi} > \frac{1}{2}$, consider a small $\eta > 0$ and suppose that the investor replaces $z(\pi)$ for all $\pi \in [\hat{\pi} - \eta, \bar{\pi}]$ by $(1-\phi)z(\pi) + \phi z(\hat{\pi} - \eta)$, for small $\phi > 0$. Under this change, $z(\pi)$ decreases for all $\pi \in [\hat{\pi} - \eta, \bar{\pi}]$ because $z(\pi) \geq z(\hat{\pi} - \eta)$. Moreover, the decrease is strict for all $\pi \in (\hat{\pi}, \bar{\pi}]$ because $z(\pi) = z(\bar{\pi}) > z(\hat{\pi} - \eta)$. The investor's utility (A.63) becomes

$$
\begin{aligned}
U = & -(1-\lambda)\left\{\int_{\frac{1}{2}}^{\hat{\pi}-\eta}\left[\pi e^{-\rho[z(\pi)d-(1-\pi_0)\Delta(\pi)]} + (1-\pi)e^{\rho[z(\pi)d-\pi_0\Delta(\pi)]}\right]e^{\rho\Gamma(\pi)}\bar{h}(\pi)d\pi \right.\\
& + \int_{\hat{\pi}-\eta}^{\hat{\pi}}\left[\pi e^{-\rho[[(1-\phi)z(\pi)+\phi z(\hat{\pi}-\eta)]d-(1-\pi_0)\Delta(\pi)]} + (1-\pi)e^{\rho[[(1-\phi)z(\pi)+\phi z(\hat{\pi}-\eta)]d-\pi_0\Delta(\pi)]}\right]e^{\rho\Gamma(\pi)}\bar{h}(\pi)d\pi \\
& + \left[e^{-\rho[[(1-\phi)z(\bar{\pi})+\phi z(\hat{\pi}-\eta)]d-(1-\pi_0)\Delta(\bar{\pi})]}\int_{\hat{\pi}}^{\bar{\pi}} \pi \bar{h}(\pi)d\pi \right.\\
& \left.\left. + e^{\rho[[(1-\phi)z(\bar{\pi})+\phi z(\hat{\pi}-\eta)]d-\pi_0\Delta(\bar{\pi})]}\int_{\hat{\pi}}^{\bar{\pi}}(1-\pi)\bar{h}(\pi)d\pi\right]e^{\rho\Gamma(\bar{\pi})}\right\} \\
& - \lambda\left[\bar{\pi}_0 e^{-\rho[[(1-\phi)z(\bar{\pi})+\phi z(\hat{\pi}-\eta)]d-(1-\pi_0)\Delta(\bar{\pi})]} + (1-\bar{\pi}_0)e^{\rho[[(1-\phi)z(\bar{\pi})+\phi z(\hat{\pi}-\eta)]d-\pi_0\Delta(\bar{\pi})]}\right]e^{\rho\Gamma(\bar{\pi})}.
\end{aligned}
\tag{A.72}
$$

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Equation (A.72) implies

$$\frac{\partial U}{\partial \phi}\bigg|_{\phi=0} = (1-\lambda)\rho d \int_{\hat{\pi}-\eta}^{\hat{\pi}} [z(\hat{\pi}-\eta) - z(\pi)] \left[\pi e^{-\rho[z(\pi)d-(1-\pi_0)\Delta(\pi)]} + (1-\pi)e^{\rho[z(\pi)d-\pi_0\Delta(\pi)]} \right] e^{\rho\Gamma(\pi)}\bar{h}(\pi)d\pi$$

$$+ \rho d[z(\hat{\pi}-\eta) - z(\bar{\pi})] \left[e^{-\rho[z(\bar{\pi})d-(1-\pi_0)\Delta(\bar{\pi})]} \left((1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}} \pi\bar{h}(\pi)d\pi + \lambda\bar{\pi}_0 \right) \right.$$

$$\left. - e^{\rho[z(\bar{\pi})d-\pi_0\Delta(\bar{\pi})]} \left((1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}} (1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\pi_0) \right) \right] e^{\rho\Gamma(\bar{\pi})}. \tag{A.73}$$

Since (A.71) implies

$$e^{-\rho[z(\bar{\pi})d-(1-\pi_0)\Delta(\bar{\pi})]} \left((1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}} \pi\bar{h}(\pi)d\pi + \lambda\bar{\pi}_0 \right)$$

$$- e^{\rho[z(\bar{\pi})d-\pi_0\Delta(\bar{\pi})]} \left((1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}} (1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\pi_0) \right) < 0,$$

the second term in (A.73) is non-zero and dominates the first term for small η. Since, in addition, $z(\bar{\pi}) > z(\hat{\pi}-\eta)$, the second term in (A.73) is positive. Hence, $\frac{\partial U}{\partial \phi}\big|_{\phi=0} > 0$.

In the sub-case $\hat{\pi} = \frac{1}{2}$, suppose that the investor replaces $z(\pi)$ for all $\pi \in (\hat{\pi}, \bar{\pi}]$ by $z(\bar{\pi}) - \phi$, for small $\phi > 0$. The investor's utility (A.63) becomes

$$U = -(1-\lambda) \left[e^{-\rho[(z(\bar{\pi})-\phi)d-(1-\pi_0)\Delta(\bar{\pi})]} \int_{\frac{1}{2}}^{\bar{\pi}} \pi\bar{h}(\pi)d\pi + e^{\rho[(z(\bar{\pi})-\phi)d-\pi_0\Delta(\bar{\pi})]} \int_{\frac{1}{2}}^{\bar{\pi}} (1-\pi)\bar{h}(\pi)d\pi \right] e^{\rho\Gamma(\bar{\pi})}$$

$$- \lambda \left[\bar{\pi}_0 e^{-\rho[(z(\bar{\pi})-\phi)d-(1-\pi_0)\Delta(\bar{\pi})]} + (1-\bar{\pi}_0)e^{\rho[(z(\bar{\pi})-\phi)d-\pi_0\Delta(\bar{\pi})]} \right] e^{\rho\Gamma(\bar{\pi})}. \tag{A.74}$$

Equation (A.74) implies

$$\frac{\partial U}{\partial \phi}\bigg|_{\phi=0} = -\rho d \left[e^{-\rho[z(\bar{\pi})d-(1-\pi_0)\Delta(\bar{\pi})]} \left((1-\lambda)\int_{\frac{1}{2}}^{\bar{\pi}} \pi\bar{h}(\pi)d\pi + \lambda\bar{\pi}_0 \right) \right.$$

$$\left. - e^{\rho[z(\bar{\pi})d-\pi_0\Delta(\bar{\pi})]} \left((1-\lambda)\int_{\frac{1}{2}}^{\bar{\pi}} (1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\bar{\pi}_0) \right) \right] e^{\rho\Gamma(\bar{\pi})}.$$

Since (A.71) implies that the term in square brackets is negative, $\frac{\partial U}{\partial \phi}\big|_{\phi=0} > 0$.

Suppose next that $\hat{\pi} > \pi^*$, and consider first the case where (A.71) holds. The investor can raise his utility through the same change in $z(\pi)$ as in the case where $\hat{\pi} < \pi^*$, (A.71) holds, and $\hat{\pi} > \frac{1}{2}$.

Consider next the case where (A.67) holds. Consider a small $\eta > 0$, and suppose that the investor replaces $z(\pi)$ for all $\pi \in [\hat{\pi}-\eta, \bar{\pi}]$ by $(1-\phi)z(\pi) + \phi z(\hat{\pi}-\eta)$, for small $\phi < 0$. Under this change, $z(\pi)$ increases for all $\pi \in [\hat{\pi}-\eta, \bar{\pi}]$ because $z(\pi) \geq z(\hat{\pi}-\eta)$. Moreover, the increase is strict

for all $\pi \in (\hat{\pi}, \bar{\pi}]$ because $z(\pi) = z(\bar{\pi}) > z(\hat{\pi} - \eta)$. The investor's utility (A.63) becomes (A.72), and its partial derivative with respect to ϕ is (A.73). When (A.67) holds as a strict inequality, it implies together with $z(\bar{\pi}) > z(\hat{\pi} - \eta)$ that the second term in (A.73) is negative. Hence, $\frac{\partial U}{\partial \phi}\big|_{\phi=0} < 0$, and utility increases. When (A.67) holds as an equality, the second term in (A.73) is zero, and utility increases if

$$\frac{\partial U}{\partial \phi}\bigg|_{\phi=0} < 0 \Leftrightarrow (1-\lambda)\rho d \int_{\hat{\pi}-\eta}^{\hat{\pi}} [z(\hat{\pi}-\eta)-z(\pi)] \left[\pi e^{-\rho[z(\pi)d-(1-\pi_0)\Delta(\pi)]} + (1-\pi)e^{\rho[z(\pi)d-\pi_0\Delta(\pi)]}\right] e^{\rho\Gamma(\pi)}\bar{h}(\pi)d\pi < 0.$$
$$(A.75)$$

For small η, (A.75) is equivalent to

$$\hat{\pi}e^{-\rho[z(\hat{\pi}^-)d-(1-\pi_0)\Delta(\hat{\pi}^-)]} + (1-\hat{\pi})e^{\rho[z(\hat{\pi}^-)d-\pi_0\Delta(\hat{\pi}^-)]} < 0 \Leftrightarrow z(\hat{\pi}^-) < \frac{1}{2\rho d}\log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) + \frac{\Delta(\hat{\pi}^-)}{2d}.$$

Hence, utility may not increase only if (A.67) holds as an equality and

$$z(\hat{\pi}^-) \geq \frac{1}{2\rho d}\log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) + \frac{\Delta(\hat{\pi}^-)}{2d}. \qquad (A.76)$$

When (A.67) holds as an equality

$$\begin{aligned}
z(\bar{\pi}) &= \frac{1}{2\rho d}\log\left(\frac{(1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}}\pi\bar{h}(\pi)d\pi + \lambda\bar{\pi}_0}{(1-\lambda)\int_{\hat{\pi}}^{\bar{\pi}}(1-\pi)\bar{h}(\pi)d\pi + \lambda(1-\bar{\pi}_0)}\right) + \frac{\Delta(\bar{\pi})}{2d} \\
&= \frac{1}{2\rho d}\log\left(\frac{\int_{\hat{\pi}}^{\pi^*}\pi h(\pi)d\pi + \frac{\pi^*}{\pi^*-\bar{\pi}_0}\int_{\pi^*}^{\bar{\pi}}(\pi-\bar{\pi}_0)\bar{h}(\pi)d\pi}{\int_{\hat{\pi}}^{\pi^*}(1-\pi)h(\pi)d\pi + \frac{1-\pi^*}{\pi^*-\bar{\pi}_0}\int_{\pi^*}^{\bar{\pi}}(\pi-\bar{\pi}_0)\bar{h}(\pi)d\pi}\right) + \frac{\Delta(\bar{\pi})}{2d} \\
&< \frac{1}{2\rho d}\log\left(\frac{\pi^*}{1-\pi^*}\right) + \frac{\Delta(\bar{\pi})}{2d}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.77)
\end{aligned}$$

where the second step follows from (A.65) and (A.66), and the third step follows from $\pi^* < \hat{\pi}$. Equations (A.76), (A.77), $z(\bar{\pi}) = z(\hat{\pi}^+) \geq z(\hat{\pi}^-)$, and $\pi^* < \hat{\pi}$ imply $\Delta(\bar{\pi}) = \Delta(\hat{\pi}^+) > \Delta(\hat{\pi}^-)$, i.e., $\Delta(\pi)$ is discontinuous at $\hat{\pi}$. We can, however, rule out such a discontinuity. Hence, (A.67) cannot hold as an equality together with (A.76), completing our proof that $\hat{\pi}$ cannot exceed π^*.

To rule out a discontinuity of $\Delta(\pi)$ at $\hat{\pi} < \bar{\pi}$, we consider a small $\eta > 0$, and assume that the investor replaces $\Delta(\pi)$ by $\Delta(\hat{\pi}^-) + \phi^-$ for all $\pi \in [\hat{\pi}-\eta, \hat{\pi})$, and by $\Delta(\bar{\pi}) - \phi^+$ for all $\pi \in (\hat{\pi}, \hat{\pi}+\eta]$, where (ϕ^-, ϕ^+) are small and chosen so that (A.26) holds. Using (A.64) to compute the change in the investor's utility, we show that utility increases. (We use (A.64) rather than (A.63) because it accounts for the change in $\Gamma(\pi)$ induced by the change in $\Delta(\pi)$.) The intuition why utility increases is that the investor exposes types $\pi > \hat{\pi}$ or $\pi < 1 - \hat{\pi}$ to a high level of risk because $\Delta(\bar{\pi}) > \bar{\Delta}^*(\bar{\pi})$. Reducing risk for types slightly above $\hat{\pi}$ and below $1 - \hat{\pi}$, while raising it for types

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slightly below $\hat{\pi}$ and above $1 - \hat{\pi}$, allows the investor to compensate the risk-averse type less while preserving incentives to observe the signal. For a detailed proof that $\Delta(\pi)$ is continuous at $\hat{\pi}$ in the symmetric case $h(\pi) = h(1 - \pi)$, see Vayanos (2018). The proof establishes more generally that $\Delta(\pi)$ is continuous at any $\pi > \frac{1}{2}$.

To rule out a discontinuity of $\Delta(\pi)$ at $\bar{\pi}$, we assume that the investor lowers $\Delta(\bar{\pi})$ by a small $\phi > 0$. Since $\Delta(\pi)$ for $\pi < \bar{\pi}$ remains the same, so does the left-hand side of the (IC) constraint (A.26). Moreover, the only effect on the integral in (A.64) is through the term $\bar{\pi} e^{-\bar{\rho}\Delta(\bar{\pi})} + 1 - \bar{\pi}$, which increases. Hence, the term in (A.64) that corresponds to the risk-averse type increases. Since the term that corresponds to the risk-neutral type also increases, utility increases.

Since $\hat{\pi}$ cannot exceed π^* and cannot be smaller than π^*, it is equal to π^*. If

$$z(\bar{\pi}) > \frac{1}{2\rho d} \log \left(\frac{2(1 - \lambda) \int_{\pi^*}^{\bar{\pi}} \pi h(\pi) d\pi + \frac{\lambda}{2}}{2(1 - \lambda) \int_{\pi^*}^{\bar{\pi}} (1 - \pi) h(\pi) d\pi + \frac{\lambda}{2}} \right) + \frac{\Delta(\bar{\pi})}{2d}, \quad (A.78)$$

then the investor can raise his utility through the same change in $z(\pi)$ as in the case where $\hat{\pi} < \pi^*$, (A.71) holds, and $\hat{\pi} > \frac{1}{2}$. If instead (A.78) holds as a strict inequality in the other direction, then the investor can raise his utility through the same change in $z(\pi)$ as in the case where $\hat{\pi} > \pi^*$ and (A.67) holds. Hence, (A.78) holds as an equality, which means from (A.65) and (A.66) that $z(\bar{\pi})$ is given by (2.10).

To show that $z(\pi)$ is given by (2.11) for $\pi \in (\frac{1}{2}, \pi^*)$, we maximize (A.63) point-wise over $z(\pi)$, without requiring that $z(\pi)$ is non-negative and non-decreasing, and that $\Delta(\pi)$ is equal to zero when $z(\pi) = 0$ and is constant in any interval where $z(\pi)$ is constant. This point-wise maximization yields (2.11). Since $\Delta(\pi)$ is non-negative and non-decreasing, (2.11) implies that $z(\pi)$ is positive and increasing for $\pi \in (\frac{1}{2}, \pi^*)$. The properties that $z(\pi)$ is non-negative and non-decreasing extend to the larger interval $[\frac{1}{2}, \bar{\pi}]$: in the case of $\frac{1}{2}$ because $z(\frac{1}{2}) = 0$, and in the case of $[\pi^*, \bar{\pi}]$ because (2.10), (2.11) and $\Delta(\pi^{*-}) \leq \Delta(\bar{\pi})$ imply $z(\pi^{*-}) \leq z(\bar{\pi})$. Since $z(\pi)$ is positive and increasing for $\pi \in (\frac{1}{2}, \pi^*)$, the constraint that $\Delta(\pi)$ is equal to zero when $z(\pi) = 0$ and is constant in any interval where $z(\pi)$ is constant is trivially satisfied. $\square$

Proof of Proposition 2.1. We first show that the pooling condition (2.9) holds and that the pooling threshold μ^* is given by (2.13). Since $\pi_0 = \frac{1}{2}(1 + \mu_0 d)$,

$$\bar{\pi}_0 = \max\{\pi_0, 1 - \pi_0\} = \max \left\{ \frac{1}{2}(1 + \mu_0 d), \frac{1}{2}(1 - \mu_0 d) \right\} = \frac{1}{2}(1 + |\mu_0| d). \quad (A.79)$$

Making the change of variable $\pi = \frac{1}{2}(1 + \mu d)$ in the integrals in (2.9), and using $\bar{\pi} = \frac{1}{2}(1 + \bar{\mu} d)$,

(A.79) and $K = \frac{kd}{N\rho} = \frac{kd}{\bar{\rho}}$, we can write (2.9) as

$$1 - \frac{\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu}{\frac{1}{2}(1 - |\mu_0|d)} d < e^{-kd} < 1 - \frac{(\bar{\mu} - |\mu_0|) \int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu}{(1 + \bar{\mu}d)|\mu_0|d + \frac{1}{4}(1 + |\mu_0|d)^2} d^2$$

$$\Leftrightarrow 2 \left(\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu \right) d + o(d) > kd + o(d) > 4 \left((\bar{\mu} - |\mu_0|) \int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu \right) d^2 + o(d^2).$$

(A.80)

For small d, (A.80) holds if $k < 2 \int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu$. Making the same substitutions in (2.12), and setting $\pi^* = \frac{1}{2}(1 + \mu^* d)$, we find (2.13).

We next show that $\Delta(\bar{\mu})$ is of order $\frac{1}{N}$. Making the same substitutions as above in (A.28), we find

$$\Delta(\bar{\mu}) \geq \frac{1}{N\rho} \log \left[\frac{\frac{1}{2}(1 + |\mu_0|d) \left(1 - e^{-kd}\right) + \left(\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu \right) d}{\left(\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu \right) d - \frac{1}{2}(1 - |\mu_0|d) \left(1 - e^{-kd}\right)} \right]$$

(A.81)

$$= \frac{1}{N\rho} \left\{ \log \left[\frac{\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu + \frac{k}{2}}{\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|) \, \bar{h}(\mu) d\mu - \frac{k}{2}} \right] + o(1) \right\}.$$

(A.82)

Therefore, $\Delta(\bar{\mu})$ is bounded below by a term of order $\frac{1}{N}$. Suppose next, by contradiction, that $\Delta(\bar{\mu})$ is of order larger than $\frac{1}{N}$. Since

$$\exp \left[-\frac{\rho}{\bar{\rho}} \int_{\pi}^{\bar{\pi}} H(\Delta(\pi'), \pi') d\pi' \right] \left[\frac{\pi e^{-\bar{\rho}\Delta(\pi)} + 1 - \pi}{\bar{\pi} e^{-\bar{\rho}\Delta(\bar{\pi})} + 1 - \bar{\pi}} \right]^{\frac{\rho}{\bar{\rho}}}$$

$$= \exp \left[-\frac{2d}{N} \int_{\mu}^{\bar{\mu}} \frac{e^{N\rho\Delta(\mu')} - 1}{1 + \mu'd + (1 - \mu'd)e^{N\rho\Delta(\mu')}} d\mu' \right] \left[\frac{(1 + \mu d)e^{-N\rho\Delta(\mu)} + 1 - \mu d}{(1 + \bar{\mu}d)e^{-N\rho\Delta(\bar{\mu})} + 1 - \bar{\mu}d} \right]^{\frac{1}{N}}$$

is equal to one plus a term of order $\frac{1}{N}$, (A.64) implies that the investor's utility U is lower than the upper bound

$$\max_{z(\mu)_{\mu \in (0, \mu^*)}, z(\bar{\mu})} \frac{1}{2} \left\{ -(1 - \lambda) \int_0^{\mu^*} \left[(1 + \mu d)e^{-\rho z(\mu)d} + (1 - \mu d)e^{\rho z(\mu)d} \right] \bar{h}(\mu) d\mu \right.$$

$$- \left((1 - \lambda) \int_{\mu^*}^{\bar{\mu}} (1 + \mu d) \bar{h}(\mu) d\mu + \lambda(1 + |\mu_0|d) \right) e^{-\rho z(\bar{\mu})d}$$

$$\left. - \left((1 - \lambda) \int_{\mu^*}^{\bar{\mu}} (1 - \mu d) \bar{h}(\mu) d\mu + \lambda(1 - |\mu_0|d) \right) e^{\rho z(\bar{\mu})d} \right\}$$

(A.83)

by a term of order larger than $\frac{1}{N}$. Consider next the function $\hat{\Delta}(\mu) = \frac{X(\mu)}{\bar{\rho}} = \frac{X(\mu)}{N\rho}$, where $X(\mu)$ is equal to zero for $\mu \in (0, |\mu_0|)$ and to a value $X(\bar{\mu})$ independent of N for $\mu \in (|\mu_0|, \bar{\mu}]$. Equation

(A.64) and Theorem 2.1 imply that under that under $\hat{\Delta}(\mu)$, the investor's utility U is lower than the upper bound in (A.83) by a term of order $\frac{1}{N}$. Moreover, (A.82) and Lemma A.1 imply that if

$$X(\bar{\mu}) > \log\left[\frac{\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|)\,\bar{h}(\mu)d\mu + \frac{k}{2}}{\int_{|\mu_0|}^{\bar{\mu}} (\mu - |\mu_0|)\,\bar{h}(\mu)d\mu - \frac{k}{2}}\right],$$

then (A.26) is satisfied. Hence, $\Delta(\mu)$ is dominated by $\hat{\Delta}(\mu)$, a contradiction.

We next determine the asymptotic behavior of $z(\mu)$ for $\mu \in (0, \mu^*)$ and of $z(\bar{\mu})$. Since the function $\Delta(\mu)$ is non-negative and non-decreasing, and since $\Delta(\bar{\mu})$ is of order $\frac{1}{N}$, $\Delta(\mu)$ for $\mu < \bar{\mu}$ is of order equal to or smaller than $\frac{1}{N}$. Therefore, $\frac{\Delta(\mu)}{2d}$ is of order equal to or smaller than $\frac{1}{Nd}$, which is of order smaller than one because N is assumed to be of order larger than $\frac{1}{d^2}$. Since $\frac{\Delta(\mu)}{2d}$ is of order smaller than one, and

$$\frac{1}{2\rho d}\log\left(\frac{\pi}{1-\pi}\right) = \frac{1}{2\rho d}\log\left(\frac{1+\mu d}{1-\mu d}\right) = \frac{\mu}{\rho} + o(1),$$

(2.10) and (2.11) imply that the asymptotic behavior of $z(\mu)$ for $\mu \in (0, \mu^*)$ and of $z(\bar{\mu})$ is as in the proposition.

We finally show that the investor finds it optimal to induce the risk-averse type to observe the signal. When the risk-averse type does not observe the signal, the investor's utility is bounded above by the utility of paying the manager a zero fee and having her choose the position z that is optimal given the prior π_0. That utility is

$$\max_z \frac{1}{2}\left\{-(1+\mu_0 d)e^{-\rho z d} - (1-\mu_0 d)e^{\rho z d}\right\} = -\frac{1}{2}\sqrt{1 - \mu_0^2 d^2}. \tag{A.84}$$

The utility (A.84) is lower than the upper bound (A.83) by a term of order d^2. Indeed, the upper bound in (A.83) exceeds

$$\max_z \frac{1}{2}\left\{-\left((1-\lambda)\int_0^{\bar{\mu}}(1+\mu d)\bar{h}(\mu)d\mu + \lambda(1+|\mu_0|d)\right)e^{-\rho z d}\right.$$
$$\left. -\left((1-\lambda)\int_0^{\bar{\mu}}(1-\mu d)\bar{h}(\mu)d\mu + \lambda(1-|\mu_0|d)\right)e^{\rho z d}\right\}, \tag{A.85}$$

because constraining $z(\mu)$ to be constant in $(0, \bar{\mu}]$ yields a lower maximum than without the constraint. Since

$$\int_0^{\bar{\mu}} \bar{h}(\mu)d\mu = \int_0^{\bar{\mu}} [h(\mu) + h(-\mu)]d\mu = \int_{-\bar{\mu}}^{\bar{\mu}} h(\mu)d\mu = 1,$$

(A.85) is equal to

$$
\max_z \frac{1}{2} \left\{ - \left[1 + \left((1-\lambda) \int_0^{\bar{\mu}} \mu \bar{h}(\mu) d\mu + \lambda |\mu_0| \right) d \right] e^{-\rho z d} \right.
$$
$$
\left. - \left[1 - \left((1-\lambda) \int_0^{\bar{\mu}} \mu \bar{h}(\mu) d\mu + \lambda |\mu_0| \right) d \right] e^{\rho z d} \right\}
$$
$$
= -\frac{1}{2} \sqrt{ 1 - \left((1-\lambda) \int_0^{\bar{\mu}} \mu \bar{h}(\mu) d\mu + \lambda |\mu_0| \right)^2 d^2 }.
\tag{A.86}
$$

Since, in addition,

$$
\int_0^{\bar{\mu}} \mu \bar{h}(\mu) d\mu = \int_0^{\bar{\mu}} \mu [h(\mu) + h(-\mu)] d\mu = \int_{-\bar{\mu}}^{\bar{\mu}} |\mu| h(\mu) d\mu > |\mu_0|,
$$

(A.86) exceeds (A.84) by a term of order d^2. Therefore, (A.83) exceeds (A.84) by a term of order equal to or larger than d^2. Since the investor's utility when the risk-averse type observes the signal is lower than (A.83) by a term of order $\frac{1}{N}$, which is assumed smaller than $\frac{1}{d^2}$, it exceeds the utility when the risk-averse type does not observe the signal. $\qquad\square$

We end this section by proving a corollary on the modification of the model of Section 2.3 presented at the end of Section 4.1.

Corollary A.1. *Suppose that the model is as in Section 2.3, except that each risk-neutral type has prior belief $\pi_0^+ = \frac{1}{2}(1 + |\mu_0|d)$ or $\pi_0^- = \frac{1}{2}(1 - |\mu_0|d)$, with the two prior beliefs being independent across types and equally likely. Proposition 2.1 carries through provided that (2.13) is replaced by*

$$
(1-\lambda) \int_{\mu^*}^{\bar{\mu}} (\mu - \mu^*) \bar{h}(\mu) d\mu = \lambda \mu^*.
\tag{A.87}
$$

Proof of Corollary A.1. Since the prior beliefs π_0^+ and π_0^- yield the same $\bar{\pi}_0$, Lemma 2.3 implies that (2.9) yields pooling under both beliefs. Hence, under (2.9), the risk-neutral type chooses $z(\bar{\pi})$ under prior belief π_0^+ and $-z(\bar{\pi})$ under prior belief π_0^-. Since π_0^+ and π_0^- are equally likely, $z(\bar{\pi})$ and $-z(\bar{\pi})$ are also equally likely, and the investor's objective (A.62) is replaced by

$$
U = - (1-\lambda) \int_{\frac{1}{2}}^{\bar{\pi}} \left[\pi e^{-\rho[z(\pi)d - f(z(\pi)d)]} + (1-\pi) e^{-\rho[-z(\pi)d - f(-z(\pi)d)]} \right] \bar{h}(\pi) d\pi
$$
$$
- \lambda \left[\frac{1}{2} e^{-\rho[|\hat{z}|d - f(|\hat{z}|d)]} + \frac{1}{2} e^{-\rho[-|\hat{z}|d - f(-|\hat{z}|d)]} \right],
\tag{A.88}
$$

with $|\hat{z}| = z(\bar{\pi})$. The difference between (A.88) and (A.62) is that $\bar{\pi}_0$ is replaced by $\frac{1}{2}$. The rest of the proof of Theorem 2.1 and Proposition 2.1 carry through with that change. Hence, (2.12) holds

with $\bar{\pi}_0$ replaced by $\frac{1}{2}$, and (2.13) holds with $|\mu_0|$ replaced by zero. $\qquad \square$

B Proofs for Section 3

Proof of Proposition 3.1. Substituting the affine price function (3.9) into the ODE (3.8), we find

$$D_t + \kappa(\bar{D} - D_t)a_1 - r(a_0 + a_1 D_t) = \rho\theta\sigma^2 D_t a_1^2. \tag{B.1}$$

Equation (B.1) is affine in D_t. Identifying the terms that are linear in D_t yields the equation

$$\rho\theta\sigma^2 a_1^2 + (r + \kappa)a_1 - 1 = 0. \tag{B.2}$$

Equation (B.2) is quadratic in a_1. When $\theta > 0$, the left-hand side is increasing for positive values of a_1, and (B.2) has a unique positive solution, given by (3.11). When $\theta < 0$, the left-hand side is hump-shaped for positive values of a_1, and (B.2) has either two positive solutions, or one positive solution, or no solution. Condition $\theta > -\frac{(r+\kappa)^2}{4\rho\sigma^2}$ in Proposition 3.1 ensures that two positive solutions exist when $\theta < 0$. Equation (3.11) gives the smaller of the two solutions, which is the continuous extension of the unique positive solution when $\theta > 0$. Identifying the constant terms yields the equation

$$\kappa\bar{D}a_1 - ra_0 = 0,$$

whose solution is (3.10). $\qquad \square$

Proof of Proposition 3.2. Substituting the price from (3.9) into (3.6), we find that the asset's share return is

$$\begin{aligned}
dR_t^{sh} &= \left[D_t + \kappa(\bar{D} - D_t)a_1 - r(a_0 + a_1 D_t)\right] dt + \sigma\sqrt{D_t}a_1 dB_t \\
&= \rho\theta\sigma^2 D_t a_1^2 dt + \sigma\sqrt{D_t}a_1 dB_t, \tag{B.3}
\end{aligned}$$

where the second step follows from (B.1). Substituting the share return from (B.3) and the price

from (3.9) into (3.2), we find that the asset's (dollar) return is

$$
\begin{aligned}
dR_t &= \frac{\rho\theta\sigma^2 D_t a_1^2 dt + \sigma\sqrt{D_t}a_1 dB_t}{a_0 + a_1 D_t} \\
&= \frac{\rho\theta\sigma^2 D_t a_1 dt + \sigma\sqrt{D_t}dB_t}{\frac{\kappa}{r}\bar{D} + D_t} \\
&= \frac{\frac{\rho\theta\sigma^2 D_t dt}{r+\kappa+\sqrt{(r+\kappa)^2+4\rho\theta\sigma^2}} + \sigma\sqrt{D_t}dB_t}{\frac{\kappa}{r}\bar{D} + D_t},
\end{aligned}
\tag{B.4}
$$

where the second step follows from (3.10) and the third step follows from (3.11).

The conditional expected return is the drift coefficient in (B.4) times dt,

$$
\mathbb{E}_t(dR_t) = \frac{\rho\theta\sigma^2 D_t dt}{\left(r + \kappa + \sqrt{(r+\kappa)^2 + 4\rho\theta\sigma^2}\right)\left(\frac{\kappa}{r}\bar{D} + D_t\right)}.
$$

It is increasing in θ because the function

$$
Y(\theta) \equiv \frac{\theta}{A + \sqrt{B + C\theta}}
$$

for positive constants (A, B, C) is increasing in θ. (The derivative of $Y(\theta)$ has the same sign as

$$
A + \sqrt{B + C\theta} - \frac{C}{2\sqrt{B + C\theta}}\theta = A + \frac{1}{\sqrt{B + C\theta}}\left(B + \frac{C\theta}{2}\right).
$$

This expression is positive for $B + C\theta > 0$, a condition which is required for the term in the square root to be positive.) The unconditional expected return is the unconditional expectation of the conditional expected return,

$$
\mathbb{E}(dR_t) = \mathbb{E}\left(\mathbb{E}_t(dR_t)\right),
$$

because of the law of iterative expectations. Since $\mathbb{E}_t(dR_t)$ is increasing in θ for any given D_t, $\mathbb{E}(dR_t)$ is increasing in θ.

The return's conditional volatility is the diffusion coefficient in (B.4) times $\sqrt{dt}$,

$$
\sqrt{\mathbb{V}\mathrm{ar}_t(dR_t)} = \frac{\sigma\sqrt{D_t}dt}{\frac{\kappa}{r}\bar{D} + D_t}.
\tag{B.5}
$$

It is independent of θ. The return's unconditional variance is the unconditional expectation of the return's conditional variance,

$$
\mathbb{V}\mathrm{ar}(dR_t) = \mathbb{E}\left(\mathbb{V}\mathrm{ar}_t(dR_t)\right).
\tag{B.6}
$$

Since $\mathbb{Var}_t(dR_t)$ is independent of θ for any given D_t, $\mathbb{Var}(dR_t)$ is independent of θ, and so is the return's unconditional volatility $\sqrt{\mathbb{Var}_t(dR_t)}$. Equation (B.6) is implied by the law of total variance

$$\mathbb{Var}(dR_t) = \mathbb{E}\left(\mathbb{Var}_t(dR_t)\right) + \mathbb{Var}\left(\mathbb{E}_t(dR_t)\right) \tag{B.7}$$

and because in continuous time the second term in the right-hand side of (B.7) is negligible relative to the first: the second term is of order dt^2 while the first is of order dt. □

C Proofs for Section 4

Proof of Theorem 4.1. We prove the theorem through a series of lemmas. Lemma C.1 shows existence of a solution to the ODE system in a compact interval and with initial conditions at the one end of the interval.

Lemma C.1. [Existence in compact interval with conditions at one boundary] *Consider $\epsilon > 0$ and $M > \epsilon$ sufficiently large. A solution $S(D_t)$ to the system of ODEs (4.4) in the unconstrained region (4.5), and (4.7) in the constrained region (4.8), with the initial conditions*

$$S'(M) = \frac{2}{(r+\kappa) + \sqrt{(r+\kappa)^2 + 4\frac{\rho\theta}{1-x}\sigma^2}}, \tag{C.1}$$

$$S(M) = \frac{1}{r}\left((\kappa\bar{D} + rM)S'(M) + \frac{1}{2}\sigma^2 M\Phi + \frac{sgn(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M)\right), \tag{C.2}$$

exists, either in the entire interval $[\epsilon, M]$, or in a maximal interval $(\hat{\epsilon}, M]$ with $\hat{\epsilon} \geq \epsilon$. In the latter case $\lim_{D_t \to \hat{\epsilon}}|S'(D_t)| = \infty$.

Proof of Lemma C.1. The ODEs (4.4) and (4.7) satisfy the conditions of the Cauchy-Lipschitz theorem for any $D_t > 0$. To show this for the ODE (4.4), we write it as a system of two first-order ODEs:

$$S'(D_t) = T(D_t),$$
$$T'(D_t) = \frac{2}{\sigma^2 D_t}\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 D_t T(D_t)^2 - D_t - \kappa(\bar{D} - D_t)T(D_t) + rS(D_t)\right).$$

The function

$$(D_t, S, T) \longrightarrow \left(\begin{array}{c} T \\ \frac{2}{\sigma^2 D_t}\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 D_t T^2 - D_t - \kappa(\bar{D} - D_t)T + rS\right) \end{array}\right)$$

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is continuously differentiable for $(D_t, S, T) \in (0, \infty) \times (-\infty, \infty) \times (-\infty, \infty)$. Hence, it is locally Lipschitz in that set, and the Cauchy-Lipschitz theorem implies that for any $(D_t, S, T) \in (0, \infty) \times (-\infty, \infty) \times (-\infty, \infty)$, the ODE (4.4) has a unique solution in a neighborhood of D_t with initial conditions $S(D_t) = S$ and $S'(D_t) = T$. The same argument establishes local existence of a solution to the ODE (4.7).

Consider the solution to the ODE (4.7) with initial conditions (C.1) and (C.2). The value of $S(M)$ in (C.2) is implied from the ODE (4.7) by setting $S''(M) = \Phi$. Indeed, (C.2) is equivalent to

$$S(M) = \frac{1}{r}\left(M + \kappa(\bar{D} - M)S'(M) + \frac{1}{2}\sigma^2 M\Phi - \frac{\rho\theta}{1-x}\sigma^2 MS'(M)^2 + \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M)\right)$$

(C.3)

because the value of $S'(M)$ in (C.1) solves the equation

$$\frac{\rho\theta}{1-x}\sigma^2 S'(M)^2 + (r+\kappa)S'(M) - 1 = 0.$$

(C.4)

Equation (C.4) is quadratic in $S'(M)$. When $\theta > 0$, the left-hand side is increasing for positive values of $S'(M)$, and (C.4) has a unique positive solution, given by (C.1). When $\theta < 0$, the left-hand side is hump-shaped for positive values of $S'(M)$, and (C.4) has either two positive solutions, or one positive solution, or no solution. Condition $\theta > -\frac{(1-x)(r+\kappa)^2}{4\rho\sigma^2}$ in Theorem 4.1 ensures that two positive solutions exist when $\theta < 0$. Equation (C.1) gives the smaller of the two solutions, which is the continuous extension of the unique positive solution when $\theta > 0$.

Since $S'(M)$ is independent of M, (4.8) is met for M sufficiently large. Continuity then implies that the solution to the ODE (4.7) with initial conditions (C.1) and (C.2) lies in the constrained region (4.8) in a neighborhood to the left of M. We extend the solution maximally to the left of M, up to a point m_1 where either the solution explodes ($\lim_{D_t \to m_1}|S'(D_t)| = \infty$) or condition (4.8) that defines the constrained region is violated in a neighborhood to the left of m_1. In the second case, we extend the solution to the left of m_1 by using the ODE (4.4) instead of (4.7). If the first derivative of $\sqrt{D_t}S'(D_t)$ at m_1 is non-zero, then it has to be positive because (4.8) is violated to the left of m_1, and the extended solution lies in the unconstrained region (4.5) in a neighborhood to the left of m_1, by continuity. (Extending the solution to the left of m_1 by using the ODE (4.4) instead of (4.7) yields the same first derivative of $\sqrt{D_t}S'(D_t)$, i.e., the first derivatives of $\sqrt{D_t}S'(D_t)$ from the left, using (4.4), and the right, using (4.7), coincide. The first derivatives of $S(D_t)$ from the left and the right coincide because the first derivative from the right is used as initial condition when extending the solution to the left. The second derivatives of $S(D_t)$ from the left and the right coincide because the first derivatives coincide and (4.5) holds with equality at m_1. The result, used

next in the proof, that higher-order derivatives of $\sqrt{D_t}S'(D_t)$ from the left and the right coincide if all lower-order derivatives are zero uses a similar argument and differentiation of (4.4) and (4.7).) If the first derivative of $\sqrt{D_t}S'(D_t)$ at m_1 is zero, then the second derivative must also be zero because otherwise (4.8) would not be violated to the left of m_1. If the third derivative of $\sqrt{D_t}S'(D_t)$ at m_1 is non-zero, then it has to be positive because (4.8) is violated to the left of m_1, and the extended solution lies in the unconstrained region (4.5) in a neighborhood to the left of m_1, by continuity. Proceeding in this manner for higher-order derivatives, we conclude that the extended solution (using the ODE (4.4) instead of (4.7) to the left of m_1) may not lie in the unconstrained region (4.5) in a neighborhood to the left of m_1 only if all n-th order derivatives of $\sqrt{D_t}S'(D_t)$ at m_1, for $n \geq 1$, are zero. Writing, however, the ODE (4.7) in terms of the function $U(D_t) \equiv \sqrt{D_t}S'(D_t)$ taking the $(n+1)$-th order derivative of the resulting equation at m_1, and using $U(m_1) > 0$ and $\frac{d^{n+1}}{dD_t^{n+1}}[U(D_t)]_{D_t=m_1} = 0$ for all $n \geq 0$, we find

$$\frac{d^{n+1}}{dD_t^{n+1}}\left[D_t + \kappa(\bar{D} - D_t)\frac{U(D_t)}{\sqrt{D_t}} + \frac{1}{2}\sigma^2\sqrt{D_t}\left(U'(D_t) - \frac{1}{2D_t}U(D_t)\right) - rS(M)\right.$$
$$\left. -r\int_M^{D_t}\frac{U(D_t')}{\sqrt{D_t'}}dD_t'\right]_{D_t=m_1} = \frac{d^{n+1}}{dD_t^{n+1}}\left[\frac{\rho\theta}{1-x}\sigma^2 U(D_t)^2 - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma U(D_t)\right]_{D_t=m_1}$$
$$\Rightarrow \frac{d^{n+1}}{dD_t^{n+1}}\left[\kappa\frac{\bar{D} - D_t}{\sqrt{D_t}} - \frac{1}{4}\sigma^2\frac{1}{\sqrt{D_t}}\right]_{D_t=m_1} - r\frac{d^n}{dD_t^n}\left[\frac{1}{\sqrt{D_t}}\right]_{D_t=m_1} = 0$$

for all $n \geq 0$, a contradiction. Hence, the extended solution lies in the unconstrained region (4.5) in a neighborhood to the left of m_1. We extend that solution maximally to the left of m_1, up to a point m_2 where either the solution explodes ($\lim_{D_t \to m_2}|S'(D_t)| = \infty$) or where condition (4.5) is violated in a neighborhood to the left of m_2. In the second case, we extend the solution to the left of m_1 by using the ODE (4.7) instead of (4.4). Repeating this process yields a solution to the system of ODEs (4.4) in the unconstrained region (4.5), and (4.7) in the constrained region (4.8), with initial conditions (C.1) and (C.2), which either is defined in $[\epsilon, M]$ or explodes at an $\hat{\epsilon} \geq \epsilon$. $\square$

Lemma C.2 shows that the solution derived in Lemma C.1 is either increasing in D_t or is decreasing and then increasing.

Lemma C.2. [Monotonicity] *For the solution derived in Lemma C.1, either $S'(D_t) > 0$ for all D_t, or there exists $m < M$ such that $S'(D_t) > 0$ for all $D_t \in (m, M]$, $S'(D_t) < 0$ for all $D_t < m$, and $S(m) > 0$.*

Proof of Lemma C.2. Since $S'(M) > 0$, $S'(D_t) > 0$ for D_t smaller than and close to M. Suppose that there exists $D_t < M$ such that $S'(D_t) \leq 0$, and consider the supremum m within that set.

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The definition of m implies $S'(D_t) > 0$ for all D_t in the non-empty set (m, M), $S'(m) = 0$, and $S''(m) \geq 0$. If $S''(m) = 0$, then differentiation of (4.4) and (4.7) at m yields $S'''(m) < 0$, which contradicts $S'(D_t) > S'(m) = 0$ for $D_t > m$. Hence, $S''(m) > 0$, which in turn implies $S'(D_t) < 0$ for D_t smaller than and close to M

Suppose next, by contradiction, that there exists $D_t < m$ such that $S'(D_t) \geq 0$, and consider the supremum m_1 within that set. The definition of m_1 implies that $S'(D_t) < 0$ for all D_t in the non-empty set (m_1, m), $S'(m_1) = 0$, and $S''(m_1) \leq 0$.

Substituting $S'(m) = 0$ and $S''(m) > 0$ in (4.4) and (4.7), we find that in both cases

$$m - rS(m) < 0. \tag{C.5}$$

Likewise, substituting $S'(m_1) = 0$ and $S''(m_1) \leq 0$ in (4.4) and (4.7), we find

$$m_1 - rS(m_1) \geq 0. \tag{C.6}$$

Equations (C.5) and (C.6) imply

$$S(m) - S(m_1) > \frac{m - m_1}{r} > 0,$$

which contradicts $S'(D_t) < 0$ for all $D_t \in (m_1, m)$. Hence, either $S'(D_t) > 0$ for all D_t, or there exists $m < M$ such that $S'(D_t) > 0$ for all $D_t \in (m, M]$ and $S'(D_t) < 0$ for all $D_t < m$. In the latter case, (C.5) implies $S(m) > \frac{m}{r} > 0$. $\qquad \square$

Lemma C.3 shows a monotonicity property of the solution with respect to the initial conditions. If a solution $S_1(D_t)$ lies below another solution $S_2(D_t)$ at M, and their first derivatives are equal at M, then $S_1(D_t)$ lies below $S_2(D_t)$ for all $D_t < M$, while the comparison reverses for the first derivatives.

Lemma C.3. [Monotonicity over initial conditions] *Consider two solutions $S_1(D_t)$ and $S_2(D_t)$ derived in Lemma C.1 for Φ_1 and $\Phi_2 > \Phi_1$, respectively. For all $D_t < M$, $S_1(D_t) < S_2(D_t)$ and $S_1'(D_t) > S_2'(D_t)$.*

Proof of Lemma C.3. Equation (C.1) implies $S_1'(M) = S_2'(M)$. Equations (C.3) and $\Phi_1 < \Phi_2$ imply $S_1(M) < S_2(M)$ and $S_1''(M) < S_2''(M)$. Combining the latter inequality with $S_1'(M) = S_2'(M)$, we find $S_1'(D_t) > S_2'(D_t)$ for D_t smaller than and close to M. Moreover, by continuity, $S_1(D_t) < S_2(D_t)$ for D_t smaller than and close to M. Suppose, by contradiction, that there exists $D_t < M$ such that $S_1(D_t) \geq S_2(D_t)$ or $S_1'(D_t) \leq S_2'(D_t)$, and consider the supremum m within that

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set. The definition of m implies $S_1(D_t) < S_2(D_t)$ and $S_1'(D_t) > S_2'(D_t)$ for all D_t in the non-empty set (m, M), and $S_1(m) = S_2(m)$ or $S_1'(m) = S_2'(m)$.

Since $S_1(M) < S_2(M)$ and $S_1'(D_t) > S_2'(D_t)$ for all $D_t \in (m, M)$, $S_1(m) < S_2(m)$. Hence, $S_1'(m) = S_2'(m)$. Equations (4.4) and (4.7) both imply, however, that since $S_1(m) < S_2(m)$, $S_1''(m) < S_2''(m)$. Hence, $S_1'(D_t) < S_2'(D_t)$ for D_t close to and larger than m, a contradiction. $\square$

Lemma C.4 derives properties of the solution for $\Phi = 0$. For this and subsequent results, we use the function $Z(D_t)$ defined by

$$Z(D_t) \equiv (\kappa\bar{D} + rD_t)S'(D_t) - rS(D_t).$$

Lemma C.4. [Solution for $\Phi = 0$] *The solution $S(D_t)$ derived in Lemma C.1 has the following properties for $\Phi = 0$:*

- *When $\theta > 0$, the solution satisfies $Z(\epsilon) < 0$ if it can be defined in $[\epsilon, M]$, and satisfies $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = -\infty$ and $\lim_{D_t \to \hat{\epsilon}} S(D_t) > 0$ if it explodes at $\hat{\epsilon} \geq \epsilon$.*

- *When $\theta < 0$, the solution can be defined in $[\epsilon, M]$, and satisfies $Z(\epsilon) > 0$.*

Proof of Lemma C.4. We start with the case $\theta > 0$. Suppose first that there exists $D_t < M$ such that $S'(D_t) \leq 0$. Lemma C.2 implies that there exists a unique $m < M$ such that $S'(D_t) > 0$ for all $D_t \in (m, M]$, $S'(D_t) < 0$ for all $D_t < m$, and $S(m) > 0$. Hence, if the solution can be defined in $[\epsilon, M]$, it satisfies

$$(\kappa\bar{D} + r\epsilon)S'(\epsilon) \leq 0 < rS(m) \leq rS(\epsilon),$$

which implies $Z(\epsilon) < 0$. If instead the solution explodes at $\hat{\epsilon} \geq \epsilon$, it satisfies $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = -\infty$ and $\lim_{D_t \to \hat{\epsilon}} S(D_t) > S(m) > 0$.

Suppose next that $S'(D_t) > 0$ for all $D_t \leq M$. We will show that the solution is strictly convex, can be defined in $[\epsilon, M]$, and satisfies $Z(\epsilon) < 0$. We first show that $S'''(M) < 0$. We write the ODE (4.7) as

$$\frac{1}{2}\sigma^2 S''(D_t) = \frac{\rho\theta}{1-x}\sigma^2 S'(D_t)^2 - \frac{\operatorname{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{D_t}}S'(D_t) - 1 + \frac{rS(D_t) - \kappa\bar{D}S'(D_t)}{D_t} + \kappa S'(D_t).$$

$$\text{(C.7)}$$

Differentiating both sides, we find

$$
\frac{1}{2}\sigma^2 S'''(D_t) = 2\frac{\rho\theta}{1-x}\sigma^2 S'(D_t)S''(D_t) - \frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\left(\frac{1}{\sqrt{D_t}}S''(D_t) - \frac{1}{2D_t^{\frac{3}{2}}}S'(D_t)\right)
$$

$$
+ \frac{rS'(D_t) - \kappa\bar{D}S''(D_t)}{D_t} - \frac{rS(D_t) - \kappa\bar{D}S'(D_t)}{D_t^2} + \kappa S''(D_t). \tag{C.8}
$$

Setting $D_t = M$ in (C.8) and using $S''(M) = \Phi = 0$, we find

$$
\frac{1}{2}\sigma^2 S'''(M) = \frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2M^{\frac{3}{2}}}S'(M) + \frac{rS'(M)}{M} - \frac{rS(M) - \kappa\bar{D}S'(M)}{M^2}
$$

$$
= -\frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2M^{\frac{3}{2}}}S'(M) < 0, \tag{C.9}
$$

where the second step follows by substituting $S(M)$ from (C.2) and using again $\Phi = 0$.

Since $S'''(M) < 0$ and $S''(M) = 0$, $S''(D_t) > 0$ for D_t smaller than and close to M. Suppose, by contradiction, that there exists $D_t < M$ such that $S''(D_t) \leq 0$, and consider the supremum m within that set. The definition of m implies that $S''(D_t) > 0$ for all D_t in the non-empty set (m, M), $S''(m) = 0$, and $S'''(m) \geq 0$.

Suppose that m lies in the constrained region. Setting $D_t = m$ in (C.8), and using $S''(m) = 0$ and $S'''(m) \geq 0$, we find

$$
\frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2m^{\frac{3}{2}}}S'(m) + \frac{rS'(m)}{m} - \frac{rS(m) - \kappa\bar{D}S'(m)}{m^2} \geq 0
$$

$$
\Leftrightarrow -\frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2m^{\frac{3}{2}}}S'(m) + \frac{1}{m}\left(\frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right) \geq 0, \quad \text{(C.10)}
$$

where the second step follows by substituting $S(m)$ from (4.7) and using again $S''(m) = 0$. The contradiction follows because both terms in the left-hand side of (C.10) are negative. The first term is negative because $S'(m) > 0$. The second term is negative because (i) $S''(D_t) > 0$ for all $D_t \in (m, M)$ implies $S'(m) < S'(M)$, and (ii) the latter inequality together with $S'(m) > 0$ imply that the left-hand side of (C.4) becomes negative when $S'(M)$ is replaced by $S'(m)$.

Suppose next that m lies in the unconstrained region. The ODE (4.4) yields the following counterpart of (C.8):

$$
\frac{1}{2}\sigma^2 S'''(D_t) = 2\frac{\rho\theta}{1-\lambda x}\sigma^2 S'(D_t)S''(D_t) + \frac{rS'(D_t) - \kappa\bar{D}S''(D_t)}{D_t} - \frac{rS(D_t) - \kappa\bar{D}S'(D_t)}{D_t^2} + \kappa S''(D_t).
$$

$$
\tag{C.11}
$$

Setting $D_t = m$ in (C.11), and using $S''(m) = 0$, $S'''(m) \geq 0$, and (4.4), we find the following

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counterpart of (C.10):

$$\frac{1}{m}\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right) \geq 0. \tag{C.12}$$

The contradiction follows because (i) $S''(D_t) > 0$ for all $D_t \in (m, M)$ implies $S'(m) < S'(M)$, (ii) the latter inequality together with $S'(m) > 0$ imply that the left-hand side of (C.4) becomes negative when $S'(M)$ is replaced by $S'(m)$, and (iii) the left-hand side of (C.4) being negative, $\lambda \in (0, 1)$ and $\theta > 0$ imply that the left-hand side of (C.12) is negative. Since $S''(D_t) > 0$ for all $D_t < M$, $S(D_t)$ is strictly convex.

If the solution explodes at $\hat\epsilon \geq \epsilon$, then convexity implies $\lim_{D_t \to \hat\epsilon} S'(D_t) = -\infty$, contradicting $S'(D_t) > 0$ for all D_t. Hence, the solution can be defined in $[\epsilon, M]$. Moreover, convexity implies

$$rS(\epsilon) \geq rS(M) + r(\epsilon - M)S'(M)$$
$$= (\kappa\bar D + r\epsilon)S'(M) + \frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M), \tag{C.13}$$

where the second step follows by substituting $S(M)$ from (C.2) and using $\Phi = 0$. Equation (C.13) implies $Z(\epsilon) < 0$ because $S'(M) > 0$ and $S'(M) > S'(\epsilon)$.

We next consider the case $\theta < 0$. We will show that the solution is strictly concave, can be defined in $[\epsilon, M]$, and satisfies $Z(\epsilon) > 0$. Equation (C.9) implies $S'''(M) > 0$. Since $S'''(M) > 0$ and $S''(M) = 0$, $S''(D_t) < 0$ for D_t smaller than and close to M. Suppose, by contradiction, that there exists $D_t < M$ such that $S''(D_t) \geq 0$, and consider the supremum m within that set. The definition of m implies that $S''(D_t) < 0$ for all D_t in the non-empty set (m, M), $S''(m) = 0$, and $S'''(m) \leq 0$.

Suppose that m lies in the unconstrained region. Since $S''(m) = 0$ and $S'''(m) \leq 0$, (C.12) holds as an inequality in the opposite direction, i.e.,

$$\frac{1}{m}\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right) \leq 0. \tag{C.14}$$

Unlike in the case $\theta > 0$, (C.14) does not yield a contradiction when combined with the comparison between $S'(m)$ and $S'(M)$. This is because the left-hand side of (C.4) is hump-shaped for positive values of $S'(M)$, rather than increasing. It increases until the mid-point between the two positive roots, and then decreases to $-\infty$.

To derive a contradiction, we examine the behavior of $Z(D_t)$ in (m, M). Since

$$Z'(D_t) = (\kappa\bar D + rD_t)S''(D_t) < 0 \tag{C.15}$$

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for all $D_t \in (m, M)$, $Z(D_t)$ is decreasing in (m, M). Moreover, (C.2) and $\Phi = 0$ imply

$$Z(M) = (\kappa\bar{D} + rM)S'(M) - rS(M) = -\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M), \qquad (C.16)$$

and (4.4) and $S''(m) = 0$ imply

$$Z(m) = (\kappa\bar{D} + rm)S'(m) - rS(m) = m\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right). \qquad (C.17)$$

Since $Z(D_t)$ is decreasing,

$$Z(m) > Z(M)$$
$$\Leftrightarrow m\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right) > -\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M) > 0, \quad (C.18)$$

where the second step follows from (C.16) and (C.17). Equation (C.18) contradicts (C.14).

Suppose next that m lies in the constrained region. Since $S''(m) = 0$ and $S'''(m) \le 0$, (C.10) holds as an inequality in the opposite direction, i.e.,

$$-\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2m^{\frac{3}{2}}}S'(m) + \frac{1}{m}\left(\frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right) \le 0. \qquad (C.19)$$

Equations (4.7) and $S''(m) = 0$ imply

$$Z(m) = (\kappa\bar{D} + rm)S'(m) - rS(m)$$
$$= m\left(\frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right) - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{m}S'(m). \qquad (C.20)$$

Equation (C.15) implies

$$Z(m) = Z(M) - \int_m^M (\kappa\bar{D} + rD_t)S''(D_t)dD_t$$
$$\Rightarrow Z(m) \ge Z(M) - \int_m^M rmS''(D_t)dD_t$$
$$\Leftrightarrow Z(m) > Z(M) + rm[S'(m) - S'(M)]$$
$$\Leftrightarrow m\left(\frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1\right) - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{m}S'(m)$$
$$> -\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M) + rm[S'(m) - S'(M)], \qquad (C.21)$$

where the last step follows from (C.16) and (C.20). Combining (C.19) and (C.21), we find

$$-\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{\sqrt{m}}{2}S'(m) > -\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M) + rm[S'(m) - S'(M)]$$

$$\Leftrightarrow \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{\sqrt{m}}{2}S'(m) - rm[S'(m) - S'(M)] > \frac{(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M). \tag{C.22}$$

The left-hand side of (C.22) is linear in $S'(m)$. Since $S''(D_t) < 0$ for all $D_t \in (m, M)$, $S'(m)$ is bounded below by $S'(M)$. To derive an upper bound for $S'(m)$, we note that since $S'(m) > S'(M)$, (C.21) implies

$$\frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 + (r+\kappa)S'(m) - 1 + \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m}}S'(m) > 0.$$

Hence, $S'(m)$ is smaller than the larger positive root of the quadratic equation

$$\frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 + \left(r + \kappa + \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m}}\right)S'(m) - 1 = 0,$$

which is

$$-\frac{\left(r + \kappa + \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m}}\right) + \sqrt{\left(r + \kappa + \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m}}\right)^2 + 4\frac{\rho\sigma^2\theta}{1-x}}}{2\frac{\rho\sigma^2\theta}{1-x}}.$$

This root is, in turn, smaller than

$$-\frac{\left(r + \kappa + \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m}}\right) + \left(r + \kappa + \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m}}\right)\sqrt{1 + \frac{4\frac{\rho\sigma^2\theta}{1-x}}{(r+\kappa)^2}}}{2\frac{\rho\sigma^2\theta}{1-x}} = S^* + \frac{B}{\sqrt{m}},$$

where S^* is the larger positive root of (C.4) and

$$B \equiv -\frac{\frac{(1-\lambda)x\mu^*}{1-x}\sigma + \frac{(1-\lambda)x\mu^*}{1-x}\sqrt{1 + \frac{4\frac{\rho\sigma^2\theta}{1-x}}{(r+\kappa)^2}}}{2\frac{\rho\sigma^2\theta}{1-x}} > 0.$$

When $S'(m)$ in (C.22) is set to $S'(M)$, the left-hand side is smaller than the right-hand side. When $S'(m)$ in (C.22) is set to the upper bound $S^* + \frac{B}{\sqrt{m}}$, the left-hand side is a quadratic function of $\sqrt{m}$, with the coefficient of $(\sqrt{m})^2 = m$ being $-r[S^* - S'(M)] < 0$. It is, therefore, bounded above, and smaller than the right-hand side for sufficiently large M. Hence, (C.22) does not hold, a contradiction. Since $S''(D_t) < 0$ for all $D_t < M$, $S(D_t)$ is strictly concave.

If the solution explodes at $\hat{\epsilon} \geq \epsilon$, then concavity implies $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = \infty$. The right-hand side of (4.4) and (4.7) is of order $S'(D_t)^2$ for D_t close to $\hat{\epsilon}$. The left-hand side, however, does not

exceed

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) - rS(D_t)$$
$$\leq D_t + \kappa(\bar{D} - D_t)S'(D_t) - rS(M) - r(D_t - M)S'(D_t),$$

where both the first and the second steps follow from concavity. Hence, the left-hand side is bounded by a term of order $S'(D_t)$, a contradiction. Therefore, the solution does not explode and can be defined in $[\epsilon, M]$. Equation $Z(\epsilon) > 0$ holds because $Z(D_t)$ is decreasing and $Z(M) > 0$ from (C.16). □

Lemma C.5 derives properties of the solution for $|\Phi|$ large.

Lemma C.5. [Solution for large $|\Phi|$] *The solution $S(D_t)$ derived in Lemma C.1 has the following properties:*

- *When $\theta > 0$ and Φ is negative and large, the solution can be defined in $[\epsilon, M]$, and satisfies $Z(\epsilon) > 0$.*

- *When $\theta < 0$ and Φ is positive and large, the solution satisfies $Z(\epsilon) < 0$ if it can be defined in $[\epsilon, M]$, and satisfies $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = -\infty$ and $\lim_{D_t \to \hat{\epsilon}} S(D_t) > 0$ if it explodes at $\hat{\epsilon} \geq \epsilon$.*

Proof of Lemma C.5. We start with the case $\theta > 0$. Suppose that Φ is negative and sufficiently large so that $S(M)$ defined by (C.2) is negative. We will show that $S'(D_t) > 0$ and $S''(D_t) < 0$ for all D_t. Both inequalities hold by continuity for D_t smaller than and close to M. Suppose, by contradiction, that there exists $D_t < M$ such that $S'(D_t) \leq 0$ or $S''(D_t) \geq 0$, and consider the supremum m within that set. The definition of m implies $S'(D_t) > 0$ and $S''(D_t) < 0$ for all D_t in the non-empty set (m, M), and $S'(D_t) = 0$ or $S''(D_t) = 0$.

Since $S'(M) > 0$ and $S''(D_t) < 0$ for all $D_t \in (m, M)$, $S'(m) > 0$. Hence, $S''(m) = 0$. Since, in addition, $S(M) < 0$ and $S'(D_t) > 0$ for all $D_t \in (m, M)$, $S(m) < 0$. Setting $D_t = m$ in (C.8) and (C.11), and using $S(m) < 0$, $S'(m) > 0$ and $S''(m) = 0$, we find $S'''(m) > 0$. Hence, $S''(D_t) > 0$ for D_t close to and larger than m, a contradiction. Therefore, $S'(D_t) > 0$ and $S''(D_t) < 0$ for all D_t.

Since the solution is concave, we can use the same argument as in the proof of Lemma C.4 in the case $\theta < 0$, to show that the solution does not explode at $\hat{\epsilon} \geq \epsilon$. Hence, the solution can be defined in $[\epsilon, M]$. It satisfies $Z(\epsilon) > 0$ because $S(\epsilon) < 0$ and $S'(\epsilon) > 0$.

We next consider the case $\theta < 0$. We will show, by contradiction, that there exists $D_t < M$ such that $S'(D_t) \leq 0$. Existence of such a D_t will imply, from Lemma C.2, existence of a unique

$m < M$ such that $S'(D_t) > 0$ for all $D_t \in (m, M]$, $S'(D_t) < 0$ for all $D_t < m$, and $S(m) > 0$. Hence, if the solution can be defined in $[\epsilon, M]$, it satisfies

$$(\kappa \bar{D} + r\epsilon)S'(\epsilon) \le 0 < rS(m) \le rS(\epsilon),$$

which implies $Z(\epsilon) < 0$. If instead the solution explodes at $\hat{\epsilon} \ge \epsilon$, it satisfies $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = -\infty$ and $\lim_{D_t \to \hat{\epsilon}} S(D_t) > S(m) > 0$.

To derive the contradiction, we assume that $S'(D_t) > 0$ for all $D_t \le M$, and will show that $S''(D_t)$ is bounded below by $\frac{\Phi}{2}$. Continuity yields the bound $S''(D_t) \ge \frac{\Phi}{2}$ for D_t smaller than and close to M because $S'(M) = \Phi$. Suppose, by contradiction, that there exists D_t such that $S''(D_t) < \frac{\Phi}{2}$, and consider the supremum within that set. The definition of m implies $S''(m) > \frac{\Phi}{2}$ for all D_t in the non-empty set (m, M), and $S''(m) = \frac{\Phi}{2}$.

If m lies in the constrained region, (4.7) implies

$$
\begin{aligned}
\frac{1}{2}\sigma^2 S''(m) &= \frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 - \frac{\operatorname{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m}}S'(m) - 1 + \frac{rS(m) - \kappa(\bar{D}-m)S'(m)}{m} \\
&\ge \frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 - 1 + \frac{rS(M) + rS'(M)(m-M) - \kappa(\bar{D}-m)S'(m)}{m} \\
&= \frac{\rho\theta}{1-x}\sigma^2 S'(m)^2 - 1 \\
&\quad + \frac{\kappa\bar{D}(S'(M) - S'(m)) + \frac{1}{2}\sigma^2 M\Phi + \frac{\operatorname{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M) + rmS'(M) + \kappa mS'(m)}{m} \\
&> \frac{\rho\theta}{1-x}\sigma^2 S'(M)^2 - 1 + \frac{\frac{1}{2}\sigma^2 M\Phi - \frac{(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M)}{m}, \quad\quad\quad\text{(C.23)}
\end{aligned}
$$

where the second step follows from $S'(m) > 0$ and because convexity implies

$$S(m) \ge S(M) + S'(M)(m - M),$$

the third step follows by substituting $S(M)$ from (C.2), and the fourth step follows because $S'(M) > S'(m) > 0$. Since for sufficiently large Φ,

$$\frac{1}{2}\sigma^2 M\Phi - \frac{(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{M}S'(M) > 0,$$

the right-hand side of (C.23) is bounded below by

$$\frac{\rho\theta}{1-x}\sigma^2 S'(M)^2 - 1 + \frac{1}{2}\sigma^2\Phi - \frac{(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{M}}S'(M),$$

which, in turn, is bounded below by $\frac{1}{4}\sigma^2\Phi$ for sufficiently large Φ. Hence, (C.23) implies that $S''(m)$

exceeds $\frac{\Phi}{2}$, a contradiction. If m lies in the unconstrained region, we can follow the same steps to derive a counterpart of (C.23) using (4.4), and then derive a contradiction. Hence, $S''(D_t) \geq \frac{\Phi}{2}$ for all $D_t \leq M$.

If the solution explodes at $\hat{\epsilon} \geq \epsilon$, then convexity implies $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = -\infty$. This is ruled out, however, by $S'(D_t) > 0$ for all $D_t \leq M$. Hence, the solution can be defined in $[\epsilon, M]$. Since, however, $S''(D_t)$ is bounded below by $\frac{\Phi}{2}$, and $S'(M)$ is independent of Φ, $S'(\epsilon)$ is negative for sufficiently large Φ. This contradicts our assumption that $S'(D_t) > 0$ for all $D_t \leq M$, and establishes that there exists $D_t < M$ such that $S'(D_t) \leq 0$. $\qquad \square$

Taken together, Lemmas C.4 and C.5 show that for two extreme values of Φ ($\Phi = 0$ and $|\Phi|$ large) the solution lies on two different "sides" of the equation $Z(\epsilon) = 0$, which we use as boundary condition at ϵ. Lemma C.6 uses these results and a continuity argument to show that there exists Φ such that $Z(\epsilon) = 0$ holds. It also uses the monotonicity property of the solution shown in Lemma C.3 to establish that this Φ is unique.

Lemma C.6. [Existence in compact interval with conditions at both boundaries] *Consider an interval $[\epsilon, M]$, with ϵ sufficiently small and M sufficiently large. A solution $S(D_t)$ to the system of ODEs (4.4) in the unconstrained region (4.5), and (4.7) in the constrained region (4.8), with the boundary conditions (C.1) and $Z(\epsilon) = 0$ exists in $[\epsilon, M]$ and is unique. Moreover, $S''(M) < 0$ when $\theta > 0$, and $S''(M) > 0$ when $\theta < 0$.*

Proof of Lemma C.6. We denote by $Z_\Phi(\epsilon)$ the value of $Z(\epsilon)$ for the solution $S(D_t)$ derived in Lemma C.1. If $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = -\infty$ for $\hat{\epsilon} \geq \epsilon$, in which case $\lim_{D_t \to \hat{\epsilon}} S(D_t) > 0$, we set $Z_\Phi(\epsilon) = -\infty$. If $\lim_{D_t \to \hat{\epsilon}} S'(D_t) = \infty$ for $\hat{\epsilon} \geq \epsilon$, in which case $\lim_{D_t \to \hat{\epsilon}} S(D_t)$ is finite or $-\infty$, we set $Z_\Phi(\epsilon) = \infty$.

Lemma C.3 implies that for $\Phi_1 < \Phi_2$, $Z_{\Phi_1}(\epsilon) > Z_{\Phi_2}(\epsilon)$ if $Z_{\Phi_1}(\epsilon)$ and Z_{Φ_2} are finite, $Z_{\Phi_2}(\epsilon) = -\infty$ if $Z_{\Phi_1}(\epsilon) = -\infty$, and $Z_{\Phi_1}(\epsilon) = \infty$ if $Z_{\Phi_2}(\epsilon) = \infty$. Hence, $Z_\Phi(\epsilon)$ is equal to ∞ in an interval $(-\infty, \underline{\Phi}]$, is finite and decreasing in an interval $(\underline{\Phi}, \bar{\Phi})$, and is equal to $-\infty$ in the remaining interval $[\bar{\Phi}, \infty)$. Continuity of the solution with respect to the initial conditions implies that $Z_\Phi(\epsilon)$ is continuous in Φ in $(\underline{\Phi}, \bar{\Phi})$. Moreover, if $\underline{\Phi}$ is finite, then $\lim_{\substack{\Phi \to \underline{\Phi} \\ \Phi > \underline{\Phi}}} Z_\Phi(\epsilon) = \infty$, and if $\bar{\Phi}$ is finite, then $\lim_{\substack{\Phi \to \bar{\Phi} \\ \Phi < \bar{\Phi}}} Z_\Phi(\epsilon) = -\infty$.

When $\theta > 0$, Lemma C.4 implies that $Z_0(\epsilon)$ is negative and possibly equal to $-\infty$, and Lemma C.5 implies that $Z_\Phi(\epsilon)$ is positive and finite for Φ negative and large. Hence, $\underline{\Phi} = -\infty$. If $\bar{\Phi} > 0$, then continuity and monotonicity of $Z_\Phi(\epsilon)$ in $(-\infty, 0]$, $\lim_{\Phi \to -\infty} Z_\Phi(\epsilon) > 0$, and $Z_0(\epsilon) < 0$ imply that there exists a unique $\Phi \in (-\infty, 0)$ such that $Z_\Phi(\epsilon) = 0$. If $\bar{\Phi} \leq 0$, then continuity and

monotonicity of $Z_\Phi(\epsilon)$ in $(-\infty, \bar{\Phi})$, $\lim_{\Phi \to -\infty} Z_\Phi(\epsilon) > 0$, and $\lim_{\substack{\Phi \to \bar{\Phi} \\ \Phi < \bar{\Phi}}} Z_\Phi(\epsilon) = -\infty$ imply that there exists a unique $\Phi \in (-\infty, \bar{\Phi})$ such that $Z_\Phi(\epsilon) = 0$. In both cases, there exists a unique $\Phi \in (-\infty, 0)$ such that $Z_\Phi(\epsilon) = 0$. The solution $S(D_t)$ derived in Lemma C.1 for this Φ satisfies $Z(\epsilon) = 0$ and $S''(M) = \Phi < 0$.

When $\theta < 0$, Lemma C.4 implies that $Z_0(\epsilon)$ is positive and finite, and Lemma C.5 implies that $Z_\Phi(\epsilon)$ is negative and possibly equal to $-\infty$ for Φ positive and large. Hence, $\underline{\Phi} < 0$ and $\bar{\Phi} > 0$. If $\bar{\Phi} < \infty$, then continuity and monotonicity of $Z_\Phi(\epsilon)$ in $[0, \bar{\Phi})$, $Z_0(\epsilon) > 0$ and $\lim_{\substack{\Phi \to \bar{\Phi} \\ \Phi < \bar{\Phi}}} Z_\Phi(\epsilon) = -\infty$ imply that there exists a unique $\Phi \in (0, \bar{\Phi})$ such that $Z_\Phi(\epsilon) = 0$. If $\bar{\Phi} = \infty$, then continuity and monotonicity of $Z_\Phi(\epsilon)$ in $[0, \infty)$, $Z_0(\epsilon) > 0$ and $\lim_{\Phi \to \infty} Z_\Phi(\epsilon) < 0$ imply that there exists a unique $\Phi \in (0, \infty)$ such that $Z_\Phi(\epsilon) = 0$. In both cases, there exists a unique $\Phi \in (0, \infty)$ such that $Z_\Phi(\epsilon) = 0$. The solution $S(D_t)$ derived in Lemma C.1 for this Φ satisfies $Z(\epsilon) = 0$ and $S''(M) = \Phi > 0$. $\qquad\square$

Lemmas C.7-C.11 show properties of the solution derived in Lemma C.6. Lemma C.7 shows that the solution is increasing in D_t.

Lemma C.7. [Monotonicity and Positivity] *For the solution derived in Lemma C.6, $S(D_t) > 0$ and $S'(D_t) > 0$ for all $D_t \in [\epsilon, M]$.*

Proof of Lemma C.7. The solution derived in Lemma C.6 coincides with that derived in Lemma C.1 for a specific value of Φ. Hence, Lemma C.2 implies that either $S'(D_t) > 0$ for all D_t, or there exists $m < M$ such that $S'(D_t) > 0$ for all $D_t \in (m, M]$, $S'(D_t) < 0$ for all $D_t < m$, and $S(m) > 0$. In the second case, $S'(\epsilon) \le 0$ and $S(\epsilon) > 0$, contradicting $Z(\epsilon) = 0$.

Since $S'(\epsilon) > 0$, $Z(\epsilon) = 0$ implies $S(\epsilon) > 0$. Combining $S(\epsilon) > 0$ with $S'(D_t) > 0$ for all D_t, we find $S(D_t) > 0$ for all D_t. $\qquad\square$

Lemma C.8 shows that the solution lies below the affine solution derived for $\mu^* = \infty$ when $\theta > 0$, and above it when $\theta < 0$.

Lemma C.8. [Comparison with the affine solution] *Consider the solution derived in Lemma C.6, and the affine solution $a_0 + a_1 D_t$ derived for $\mu^* = \infty$. When $\theta > 0$, $S(D_t) < a_0 + a_1 D_t$ for all $D_t \in [\epsilon, M]$, and when $\theta < 0$, $S(D_t) > a_0 + a_1 D_t$ for all $D_t \in [\epsilon, M]$.*

Proof of Lemma C.8. To derive the affine solution for $\mu^* = \infty$, we substitute the affine price function (3.9) into the ODE (4.4) and identify terms. Identifying the terms that are linear in D_t

yields the equation

$$\frac{\rho\theta}{1 - \lambda x}\sigma^2 a_1^2 + (r + \kappa)a_1 - 1 = 0. \tag{C.24}$$

Identifying the constant terms yields (3.10). When $\theta > 0$, (C.24) has the unique positive solution, given by

$$a_1 = \frac{2}{(r + \kappa) + \sqrt{(r + \kappa)^2 + 4\frac{\rho\theta}{1-\lambda x}\sigma^2}}. \tag{C.25}$$

When $\theta < 0$, Condition $\theta > -\frac{(1-x)(r+\kappa)^2}{4\rho\sigma^2}$ in Theorem 4.1 ensures that (C.24) has two positive solutions. Equation (C.25) gives the smaller of the two solutions, which is the continuous extension of the unique positive solution when $\theta > 0$.

To prove the lemma, we start with the case $\theta > 0$, and consider the problem of maximizing

$$V(D_t) \equiv S(D_t) - (a_0 + a_1 D_t),$$

over the compact set $[\epsilon, M]$. The result in the lemma will follow if we show that the maximum value $V_{\max}$ of $V(D_t)$ is negative. Using (3.10), we can write $V(D_t)$ as

$$V(D_t) = S(D_t) - \frac{a_1}{r}(\kappa\bar{D} + rD_t).$$

Suppose first that $V(D_t)$ is maximized at $D_t = M$. Using (C.2), we can write $V(M)$ as

$$V(M) = \frac{1}{r}\left((\kappa\bar{D} + rM)(S'(M) - a_1) + \frac{1}{2}\sigma^2 M\Phi + \frac{\text{sgn}(\theta)(1 - \lambda)x\mu^*}{1 - x}\sigma\sqrt{M}S'(M)\right). \tag{C.26}$$

Equations (C.1) and (C.25) imply that $S'(M)$ and a_1 are independent of M, and that $S'(M) < a_1$. Since, in addition $\Phi < 0$, (C.26) implies that $V_{\max} = V(M) < 0$ for M sufficiently large.

Suppose next that $V(D_t)$ is maximized at an interior point $m \in (\epsilon, M)$ that lies in the constrained region. The first- and second-order conditions of the maximization problem are $S'(m) = a_1$

and $S''(m) \le 0$. Setting $D_t = m$ in (4.7) and using $S'(m) = a_1$ and $S''(m) \le 0$, we find

$$m + \kappa(\bar{D} - m)a_1 - rS(m) \ge \frac{\rho\theta}{1-x}\sigma^2 m a_1^2 - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{m}a_1$$

$$\Rightarrow m + \kappa(\bar{D} - m)a_1 - rS(m) > \frac{\rho\theta}{1-x}\sigma^2 m a_1^2 - \frac{(1-\lambda)x\rho\theta}{(1-x)(1-\lambda x)}\sigma^2 m a_1^2$$

$$\Leftrightarrow m + \kappa(\bar{D} - m)a_1 - rS(m) > \frac{\rho\theta}{1-\lambda x}\sigma^2 m a_1^2$$

$$\Leftrightarrow (\kappa\bar{D} + rm)a_1 - rS(m) > 0$$

$$\Leftrightarrow V_{\max} = V(m) < 0, \tag{C.27}$$

where the second step follows from (4.8) and the fourth step follows from (C.24).

Suppose next that $V(D_t)$ is maximized at an interior point $m \in (\epsilon, M)$ that lies in the unconstrained region. Setting $D_t = m$ in (4.4) and using $S'(m) = a_1$ and $S''(m) \le 0$, we find

$$m + \kappa(\bar{D} - m)a_1 - rS(m) \ge \frac{\rho\theta}{1-\lambda x}\sigma^2 m a_1^2$$

$$\Leftrightarrow (\kappa\bar{D} + rm)a_1 - rS(m) \ge 0$$

$$\Leftrightarrow V_{\max} = V(m) \le 0, \tag{C.28}$$

To show that (C.28) holds as a strict inequality, we proceed by contradiction. If (C.28) holds as an equality, then $S(m)$ and $S'(m)$ are the same as under the affine solution $S(D_t) = \frac{a_1}{r}(\kappa\bar{D} + rD_t)$. Hence, the solution derived in Lemma C.6 coincides with the affine solution in an interval in the unconstrained region that includes m and that has a boundary with the constrained region at an $m_1 \ge m$. Setting $D_t = m_1$ in (C.8) and using $S(m_1) = \frac{a_1}{r}(\kappa\bar{D} + rm_1)$, $S'(m_1) = a_1$ and $S''(m_1) = 0$, we find that the third derivative of $S(D_t)$ from the right at m_1 is

$$\frac{1}{2}\sigma^2 S'''(m_1) = \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2D_t^{\frac{3}{2}}}S'(D_t) > 0. \tag{C.29}$$

Since $S'''(m_1) > 0$, $S''(D_t)$ is positive in a neigborhood to the right of m_1, and hence $S'(D_t)$ exceeds a_1. This means that $V(D_t)$, which is equal to zero for all $D_t \in [m, m_1]$ because $S(D_t)$ coincides with the affine solution, increases to the right of m_1, a contradiction since $V(D_t)$ would then be maximized in the constrained region.

If $V(D_t)$ is maximized at ϵ, then $S'(\epsilon) \le a_1$ and hence,

$$V_{\max} = V(\epsilon) = S(\epsilon) - \frac{a_1}{r}(\kappa\bar{D} + r\epsilon) = \frac{1}{r}(\kappa\bar{D} + r\epsilon)(S'(\epsilon) - a_1) \le 0, \tag{C.30}$$

where the second step follows from $Z(\epsilon) = 0$. To show that (C.30) holds as a strict inequality, we

follow the same argument as in the case where $V(D_t)$ is maximized at an interior point m in the unconstrained region.

The argument in the case $\theta < 0$ is symmetric. We consider the problem of minimizing $V(D_t)$ over $[\epsilon, M]$, and show that the minimum value $V_{\min}$ of $V(D_t)$ is positive.

Suppose first that $V(D_t)$ is minimized at $D_t = M$. Equations (C.1) and (C.25) imply that $S'(M)$ and a_1 are independent of M, and that $S'(M) > a_1$. Since, in addition, $\Phi > 0$, (C.26) implies that $V_{\min} = V(M) > 0$ for M sufficiently large.

Suppose next that $V(D_t)$ is maximized at an interior point $m \in (\epsilon, M)$ that lies in the constrained region. The first- and second-order conditions of the maximization problem are $S'(m) = a_1$ and $S''(m) \geq 0$. Setting $D_t = m$ in (4.7), using $S'(m) = a_1$ and $S''(m) \leq 0$, and proceeding as in the derivation of (C.27), we find $V_{\max} = V(m) > 0$.

Suppose next that $V(D_t)$ is maximized at an interior point $m \in (\epsilon, M)$ that lies in the unconstrained region. Setting $D_t = m$ in (4.4), using $S'(m) = a_1$ and $S''(m) \geq 0$, and proceeding as in the derivation of (C.28), we find $V_{\max} = V(m) \geq 0$. To show that (C.28) holds as a strict inequality, we follow the same argument as in the case $\theta < 0$ and find that (C.29) implies $S'''(m_1) < 0$. This implies that $V(D_t)$, which is equal to zero for all $D_t \in [m, m_1]$, decreases to the right of m_1, a contradiction since $V(D_t)$ would then be minimized in the constrained region.

If $V(D_t)$ is maximized at ϵ, then $S'(\epsilon) \geq a_1$, and hence (C.30) implies $V_{\min} = V(\epsilon) \geq 0$. To show that (C.30) holds as a strict inequality, we follow the same argument as in the case where $V(D_t)$ is maximized at an interior point m in the unconstrained region.

Note that since $Z(\epsilon)$ implies

$$S(\epsilon) - \frac{a_1}{r}(\kappa\bar{D} + r\epsilon) = \frac{1}{r}(\kappa\bar{D} + r\epsilon)(S'(\epsilon) - a_1),$$

Lemma C.8 implies that $S'(\epsilon) < a_1$ when $\theta > 0$, and $S'(\epsilon) > a_1$ when $\theta < 0$. $\qquad\square$

Lemma C.9 shows that the constrained and the unconstrained regions have a single boundary and hence do not alternate. Proving this result requires condition $\kappa\bar{D} > \frac{\sigma^2}{4}$ of Theorem 4.1. This condition is required in all subsequent lemmas as well because they build on Lemma C.9, but is not used in all previous lemmas.

Lemma C.9. [Single boundary between unconstrained and constrained region] *There exists $m \in [\epsilon, M]$ such that the unconstrained region is $[\epsilon, m]$ and the constrained region is $(m, M]$.*

Proof of Lemma C.9. The constrained region includes a neighborhood to the left of M, for

sufficiently large M, as shown in Lemma C.1. The unconstrained region includes a neighborhood to the right of ϵ, for sufficiently small ϵ. This is because $S'(\epsilon)$ is bounded above uniformly for all values of ϵ sufficiently small. When $\theta > 0$, the upper bound is a_1. When $\theta < 0$, Lemma C.5 implies that Φ is bounded above because otherwise $Z(\epsilon) < 0$. The upper bound on Φ implies one on $S(M)$ from (C.2), which in turn implies one on $S(\epsilon)$ from Lemma C.7, which in turn implies one on $S'(\epsilon)$ from $Z(\epsilon) = 0$.

Consider the non-empty set of $m > \epsilon$ such that $[\epsilon, m]$ lies in the unconstrained region, and the supremum m_1 of that set. Consider the non-empty set of $m > m_1$ such that (m_1, m) lies in the constrained region, and the supremum m_2 of that set. Suppose, by contradiction, that $m_2 < M$, in which case the unconstrained region begins again at m_2. Consider, in that case, the non-empty set of $m > \epsilon$ such that $[m_2, m]$ lies in the unconstrained region, and the supremum m_3 of that set. Since the constrained region includes a neighborhood to the left of M, $m_3 < M$.

Since (4.5) holds as an equality at m_i, $i = 1, 2, 3$,

$$\sqrt{m_i} S'(m_i) = \frac{\mu^*(1 - \lambda x)}{\rho \sigma |\theta|}. \tag{C.31}$$

Since (4.5) holds to the left of m_i, $i = 1, 3$, and (4.8) holds to the right of m_i, the derivative of $\sqrt{D_t} S'(D_t)$ is non-negative for $D_t = m_i$, and hence

$$\sqrt{m} S''(m_i) + \frac{1}{2\sqrt{m_i}} S'(m_i) \geq 0 \Leftrightarrow m_i S''(m_i) \geq -\frac{S'(m_i)}{2} = -\frac{\mu^*(1 - \lambda x)}{2\rho \sigma |\theta|} \frac{1}{\sqrt{m_i}} \quad \text{for} \quad i = 1, 3, \tag{C.32}$$

where the last step follows from (C.31). Conversely, since (4.8) holds to the left of m_2, and (4.5) holds to the right of m_2, the derivative of $\sqrt{D_t} S'(D_t)$ is non-positive for $D_t = m_2$, and hence

$$m_2 S''(m_2) \leq -\frac{S'(m_2)}{2} = -\frac{\mu^*(1 - \lambda x)}{2\rho \sigma |\theta|} \frac{1}{\sqrt{m_2}}. \tag{C.33}$$

Since (4.8) holds in (m_1, m_2),

$$S(m_2) - S(m_1) = \int_{m_1}^{m_2} S'(D_t) dD_t > \int_{m_1}^{m_2} \frac{\mu^*(1 - \lambda x)}{\rho \sigma |\theta|} \frac{1}{\sqrt{D_t}} dD_t = \frac{2\mu^*(1 - \lambda x)}{\rho \sigma |\theta|} (\sqrt{m_2} - \sqrt{m_1}). \tag{C.34}$$

Conversely, since (4.5) holds in (m_2, m_3),

$$S(m_3) - S(m_2) = \int_{m_2}^{m_3} S'(D_t) dD_t \leq \int_{m_2}^{m_3} \frac{\mu^*(1 - \lambda x)}{\rho \sigma |\theta|} \frac{1}{\sqrt{D_t}} dD_t = \frac{2\mu^*(1 - \lambda x)}{\rho \sigma |\theta|} (\sqrt{m_3} - \sqrt{m_2}).$$

Since (4.5) holds as an equality at m_i, $i = 1, 2, 3$, these points satisfy both (4.4) and (4.7). Setting $D_t = m_i$ in (4.4) and using (C.31), we find

$$m_i + \kappa(\bar{D} - m_i)\frac{\mu^*(1 - \lambda x)}{\rho\sigma|\theta|}\frac{1}{\sqrt{m_i}} + \frac{1}{2}\sigma^2 m_i S''(m_i) - rS(m_i) = \frac{(\mu^*)^2(1 - \lambda x)}{\rho\theta}. \qquad (C.36)$$

Subtracting (C.36) for m_2 from the same equation for m_1, we find

$$m_1 - m_2 + \frac{\mu^*(1 - \lambda x)}{\rho\sigma|\theta|}\left[\kappa\bar{D}\left(\frac{1}{\sqrt{m_1}} - \frac{1}{\sqrt{m_2}}\right) - \kappa(\sqrt{m_1} - \sqrt{m_2})\right]$$
$$+ \frac{1}{2}\sigma^2\left[m_1 S''(m_1) - m_2 S''(m_2)\right] - r[S(m_1) - S(m_2)] = 0,$$
$$\Rightarrow m_1 - m_2 + \frac{\mu^*(1 - \lambda x)}{\rho\sigma|\theta|}\frac{m_2 - m_1}{\sqrt{m_1} + \sqrt{m_2}}\left(\kappa\bar{D}\frac{1}{\sqrt{m_1 m_2}} + \kappa\right)$$
$$+ \frac{\mu^*(1 - \lambda x)}{\rho\sigma|\theta|}\frac{\sigma^2}{4}\left(\frac{1}{\sqrt{m_2}} - \frac{1}{\sqrt{m_1}}\right) + \frac{2\mu^*(1 - \lambda x)}{\rho\sigma|\theta|}r(\sqrt{m_2} - \sqrt{m_1}) < 0$$
$$\Rightarrow \frac{\mu^*(1 - \lambda x)}{\rho\sigma|\theta|}\frac{1}{\sqrt{m_1} + \sqrt{m_2}}\left(\frac{\kappa\bar{D} - \frac{\sigma^2}{4}}{\sqrt{m_1 m_2}} + \kappa + 2r\right) - 1 < 0, \qquad (C.37)$$

where the second step follows from (C.32), (C.33) and (C.34), and the third step follows by dividing throughout by $m_2 - m_1 > 0$. Subtracting (C.36) for m_3 from the same equation for m_2, and using (C.32), (C.33) and (C.34), we similarly find

$$\frac{\mu^*(1 - \lambda x)}{\rho\sigma|\theta|}\frac{1}{\sqrt{m_2} + \sqrt{m_3}}\left(\frac{\kappa\bar{D} - \frac{\sigma^2}{4}}{\sqrt{m_2 m_3}} + \kappa + 2r\right) - 1 \geq 0. \qquad (C.38)$$

Condition $\kappa\bar{D} - \frac{\sigma^2}{4} > 0$ of Theorem 4.1 ensures that because $m_3 > m_1$, the left-hand side of (C.37) is larger than the left-hand side of (C.38). This is a contradiction because the former should be negative and the latter non-negative. Therefore, $m_2 = M$, and the lemma holds by setting $m = m_1$. $\qquad \square$

Lemma C.10 shows that the solution is concave when $\theta > 0$, and convex when $\theta < 0$.

Lemma C.10. [Concavity/convexity] *The solution derived in Lemma C.6 satisfies $S''(D_t) < 0$ for all $D_t \in [\epsilon, M]$ when $\theta > 0$, and $S''(D_t) > 0$ for all $D_t \in [\epsilon, M]$ when $\theta < 0$.*

Proof of Lemma C.10. We start with the case $\theta > 0$. Lemma C.6 shows that $S''(M) < 0$.

Moreover, setting $D_t = \epsilon$ in (4.4) and solving for $S''(\epsilon)$, we find

$$
\begin{aligned}
\frac{1}{2}\sigma^2 \epsilon S''(\epsilon) &= \frac{\rho\theta}{1-\lambda x}\sigma^2 \epsilon S'(\epsilon)^2 - \kappa(\bar{D}-\epsilon)S'(\epsilon) + rS(\epsilon) - \epsilon \\
&= \frac{\rho\theta}{1-\lambda x}\sigma^2 \epsilon S'(\epsilon)^2 + (r+\kappa)\epsilon S'(\epsilon) - \epsilon \\
&= \epsilon\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 S'(\epsilon)^2 + (r+\kappa)S'(\epsilon) - 1\right) < 0,
\end{aligned}
\tag{C.39}
$$

where the second step follows from $Z(\epsilon) = 0$, and the last step because $S'(\epsilon) < a_1$.

Suppose, by contradiction, that there exists $D_t \in (\epsilon, M)$ such that $S''(D_t) \geq 0$, and consider the infimum m_1 within that set. Since $S''(\epsilon) < 0$, $m_1 > \epsilon$. The definition of m_1 implies $S''(D_t) < 0$ for all $D_t \in (\epsilon, m_1)$, $S''(m_1) = 0$ and $S'''(m_1) \geq 0$.

Suppose that m_1 lies in the unconstrained region. Setting $D_t = m_1$ in (C.11), and using $S''(m_1) = 0$, $S'''(m_1) \geq 0$ and (4.4), we find (C.12), written for m_1 instead of m. The contradiction follows because (i) $S''(D_t) < 0$ for all $D_t \in (\epsilon, m_1)$ implies $S'(m_1) < S'(\epsilon) < a_1$, (ii) the latter inequality together with $S'(m_1) > 0$ imply that the left-hand side of (C.24) becomes negative when a_1 is replaced by $S'(m_1)$.

Suppose next that m_1 lies in the constrained region and that $S'''(m_1) > 0$. Since $S''(m_1) = 0$, $S'''(m_1) > 0$ implies that $S''(D_t) > 0$ for D_t close to and larger than m_1. We denote by m_2 the supremum of the set of m such that $S''(D_t) > 0$ for all $D_t \in (m_1, m)$. Since $S''(M) < 0$, $m_2 < M$. The definition of m_2 implies $S''(D_t) < 0$ for all $D_t \in (m_1, m_2)$, $S''(m_2) = 0$ and $S'''(m_2) \leq 0$. Setting $D_t = m_1$ in (C.8), and using $S''(m_1) = 0$, $S'''(m_1) \geq 0$ and (4.7), we find (C.10), written for m_1 instead of m. Multiplying both sides by $\frac{m_1}{S'(m_1)} > 0$, we rewrite that equation as

$$
-\frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2\sqrt{m_1}} + \frac{\rho\theta}{1-x}\sigma^2 S'(m_1) + r + \kappa - \frac{1}{S'(m_1)} \geq 0.
\tag{C.40}
$$

Since m_2 exceeds m_1, Lemma C.9 implies that it lies in the constrained region. Setting $D_t = m_2$ in (C.8), and using $S''(m_1) = 0$, $S'''(m_1) \leq 0$ and (4.7), we find (C.19), written for m_2 instead of m. Multiplying both sides by $\frac{m_2}{S'(m_2)} > 0$, we rewrite that equation as

$$
-\frac{\mathrm{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2\sqrt{m_2}} + \frac{\rho\theta}{1-x}\sigma^2 S'(m_2) + r + \kappa - \frac{1}{S'(m_2)} \leq 0.
\tag{C.41}
$$

Since $m_2 > m_1$ and $S'(m_2) > S'(m_1)$, the left-hand side of (C.41) is larger than the left-hand side of (C.40). This is a contradiction because the former should be non-positive and the latter non-negative.

Suppose finally that m_1 lies in the constrained region and that $S'''(m_1) = 0$. If there exists

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$D_t > m_1$ such that $S''(D_t) > 0$, then the same argument as in the case where $S'''(m_1) > 0$ yields a contradiction. If $S''(D_t) \leq 0$ for all $D_t > m_1$, then $S''(m_1) = S'''(m_1) = 0$ implies $S''''(m_1) \leq 0$. To derive a contradiction, we differentiate twice (4.7) at $D_t = m_1$. Using $S''(m_1) = S'''(m_1) = 0$, we find

$$\frac{1}{2}\sigma^2 m_1 S''''(m_1) = \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{4m_1^{\frac{3}{2}}}S'(m_1) > 0. \tag{C.42}$$

Hence, $S''(D_t) < 0$ for all $D_t \in [\epsilon, M]$.

We next consider the case $\theta < 0$. Lemma C.6 shows that $S''(M) > 0$. Moreover, setting $D_t = \epsilon$ in (4.4), solving for $S''(\epsilon)$, and using $Z(\epsilon) = 0$, we find the following counterpart of (C.39)

$$\frac{1}{2}\sigma^2\epsilon S''(\epsilon) = \epsilon\left(\frac{\rho\theta}{1-\lambda x}\sigma^2 S'(\epsilon)^2 + (r+\kappa)S'(\epsilon) - 1\right). \tag{C.43}$$

We will show that $S''(\epsilon) > 0$, ruling out the cases $S'(\epsilon) < 0$ and $S'(\epsilon) = 0$ by contradiction arguments.

Suppose, by contradiction, that $S''(\epsilon) < 0$. We denote by m_1 the supremum of the set of m such that $S''(D_t) < 0$ for all $D_t \in [\epsilon, m)$. Since $S''(M) > 0$, $m_1 < M$. The definition of m_1 implies $S''(D_t) < 0$ for all $D_t \in [\epsilon, m_1)$, $S''(m_1) = 0$ and $S'''(m_1) \geq 0$. Equations $Z(\epsilon) = 0$, (C.15) and $S''(D_t) < 0$ for all $D_t \in [\epsilon, m_1)$ imply $Z(m_1) < 0$.

If m_1 lies in the unconstrained region, (4.4) and $S''(m_1) = 0$ imply (C.17), written for m_1 instead of m. Moreover, setting $D_t = m_1$ in (C.11), and using $S''(m_1) = 0$, $S'''(m_1) \geq 0$ and (4.4), we find (C.12), written for m_1 instead of m. The two equations yield a contradiction when combined with $Z(m_1) < 0$.

If m_1 lies in the constrained region, (4.7) and $S''(m_1) = 0$ imply (C.20), written for m_1 instead of m. Moreover, setting $D_t = m_1$ in (C.8), and using $S''(m_1) = 0$, $S'''(m_1) \geq 0$ and (4.7), we find (C.10), written for m_1 instead of m. The two equations yield a contradiction when combined with $Z(m_1) < 0$, as can be seen by multiplying the latter equation by $-m_1^2$ and adding it to the former equation.

Suppose next by contradiction that $S''(\epsilon) = 0$. Since $S'(\epsilon) > a_1$, (C.43) implies that $S'(\epsilon)$ is equal to the larger positive root of (C.24), which we denote by a_1^*. Hence, $S'(\epsilon)$ is the same as under the affine solution $S(D_t) = \frac{a_1^*}{r}(\kappa\bar{D} + rD_t)$. The same is true for $S(\epsilon)$ because of $Z(\epsilon) = 0$. Hence, the solution derived in Lemma C.6 coincides with the affine solution in an interval in the unconstrained region that includes ϵ and that has a boundary with the constrained region at an $m_1 \geq m$. Proceeding as in the proof of Lemma C.8, we find that the third derivative of $S(D_t)$ from the right at m_1 is negative, and hence $S''(D_t)$ is negative in a neigborhood to the right of m_1. Since

$Z(m_1) = 0$, we can then use the previous argument to derive a contradiction. This establishes that $S''(\epsilon) > 0$.

Suppose, by contradiction, that there exists $D_t \in (\epsilon, M)$ such that $S''(D_t) \leq 0$, and consider the infimum m_1 within that set. Since $S''(\epsilon) > 0$, $m_1 > \epsilon$. The definition of m_1 implies $S''(D_t) > 0$ for all $D_t \in (\epsilon, m_1)$, $S''(m_1) = 0$ and $S'''(m_1) \leq 0$. Equations $Z(\epsilon) = 0$, (C.15) and $S''(D_t) > 0$ for all $D_t \in [\epsilon, m_1)$ imply $Z(m_1) > 0$.

Suppose that m_1 lies in the unconstrained region. Equations (4.4) and $S''(m_1) = 0$ imply (C.17), written for m_1 instead of m. Moreover, setting $D_t = m_1$ in (C.11), and using $S''(m_1) = 0$, $S'''(m_1) \leq 0$ and (4.4), we find (C.14), written for m_1 instead of m. The two equations yield a contradiction when combined with $Z(m_1) > 0$.

Suppose next that m_1 lies in the constrained region and that $S'''(m_1) < 0$. Since $S''(m_1) = 0$, $S'''(m_1) < 0$ implies that $S''(D_t) < 0$ for D_t close to and larger than m_1. We denote by m_2 the supremum of the set of m such that $S''(D_t) < 0$ for all $D_t \in (m_1, m)$. Since $S''(M) > 0$, $m_2 < M$. The definition of m_2 implies $S''(D_t) < 0$ for all $D_t \in (m_1, m_2)$, $S''(m_2) = 0$ and $S'''(m_2) \geq 0$. Setting $D_t = m_1$ in (C.8), and using $S''(m_1) = 0$, $S'''(m_1) \leq 0$ and (4.7), we find (C.19), written for m_1 instead of m. Multiplying both sides by m_1, we rewrite that equation as

$$-\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2\sqrt{m_1}}S'(m_1) + \frac{\rho\theta}{1-x}\sigma^2 S'(m_1)^2 + (r+\kappa)S'(m_1) - 1 \leq 0. \tag{C.44}$$

Since m_2 exceeds m_1, Lemma C.9 implies that it lies in the constrained region. Setting $D_t = m_2$ in (C.8), and using $S''(m_1) = 0$, $S'''(m_1) \geq 0$ and (4.7), we find (C.10), written for m_2 instead of m. Multiplying both sides by m_2, we rewrite that equation as

$$-\frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2\sqrt{m_2}}S'(m_2) + \frac{\rho\theta}{1-x}\sigma^2 S'(m_2)^2 + (r+\kappa)^2 - 1 \geq 0. \tag{C.45}$$

Since $S''(D_t) < 0$ for all $D_t \in (m_1, m_2)$, $Z(m_2) < Z(m_1)$ Using (C.20) to compute $Z(m_1)$ and $Z(m_2)$, we find

$$\begin{aligned}
&m_1\left(\frac{\rho\theta}{1-x}\sigma^2 S'(m_1)^2 + (r+\kappa)S'(m_1) - 1\right) - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{m_1}S'(m_1) \\
&> m_2\left(\frac{\rho\theta}{1-x}\sigma^2 S'(m_2)^2 + (r+\kappa)S'(m_2) - 1\right) - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\sqrt{m_2}S'(m_2) \\
&\Rightarrow \frac{\rho\theta}{1-x}\sigma^2 S'(m_1)^2 + (r+\kappa)S'(m_1) - 1 - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m_1}}S'(m_1) \\
&> \frac{\rho\theta}{1-x}\sigma^2 S'(m_2)^2 + (r+\kappa)S'(m_2) - 1 - \frac{\text{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{\sqrt{m_2}}S'(m_2),
\end{aligned} \tag{C.46}$$

where the second step follows by multiplying the left-hand side, which is positive since $Z(m_1) > 0$,

by $\frac{m_2}{m_1} > 1$. Multiplying (C.44) by -1 and adding to the sum of (C.45) and (C.46), we find

$$-\frac{\operatorname{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2\sqrt{m_1}}S'(m_1) > -\frac{\operatorname{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{2\sqrt{m_2}}S'(m_2),$$

a contradiction since $m_2 > m_1$ and $S'(m_2) < S'(m_1)$.

Suppose finally that m_1 lies in the constrained region and that $S'''(m_1) = 0$. If there exists $D_t > m_1$ such that $S''(D_t) < 0$, then the same argument as in the case where $S'''(m_1) < 0$ yields a contradiction. If $S''(D_t) \geq 0$ for all $D_t > m_1$, then $S''(m_1) = S'''(m_1) = 0$ implies $S''''(m_1) \geq 0$. To derive a contradiction, we differentiate twice (4.7) at $D_t = m_1$. Using $S''(m_1) = S'''(m_1) = 0$, we find

$$\frac{1}{2}\sigma^2 m_1 S''''(m_1) = \frac{\operatorname{sgn}(\theta)(1-\lambda)x\mu^*}{1-x}\sigma\frac{1}{4m_1^{\frac{3}{2}}}S'(m_1) < 0. \tag{C.47}$$

Hence, $S''(D_t) > 0$ for all $D_t \in [\epsilon, M]$. $\qquad\square$

Lemma C.11 shows that the derivative of the solution lies below the derivative of the affine solution derived for $\mu^* = \infty$ when $\theta > 0$, and above it when $\theta < 0$.

Lemma C.11. [Comparison with the derivative of the affine solution] *Consider the solution derived in Lemma C.6, and the affine solution $a_0 + a_1 D_t$ derived for $\mu^* = \infty$. When $\theta > 0$, $S'(D_t) < a_1$ for all $D_t \in [\epsilon, M]$, and when $\theta < 0$, $S(D_t) > a_1$ for all $D_t \in [\epsilon, M]$.*

Proof of Lemma C.11. When $\theta > 0$, the result follows because the solution is concave and $S'(\epsilon) < a_1$. When $\theta < 0$, the result follows because the solution is convex and $S'(\epsilon) > a_1$. $\qquad\square$

Lemma C.12 shows that if a solution to the system of ODEs exists in $(0, \infty)$ and its derivative converges to finite limits at zero and infinity, then these limits are almost uniquely determined.

Lemma C.12. [Limits at zero and infinity] *Consider a solution $S(D_t)$ to the system of ODEs (4.4) in the unconstrained region (4.5), and (4.7) in the constrained region (4.8), defined in $(0, \infty)$. Suppose that $S'(D_t)$ converges to finite limits at zero and infinity, denoted by $S'(0)$ and $S'(\infty)$, respectively. Then $S'(\infty)$ is a root of (C.4), and $S'(0)$ satisfies $Z(0) \equiv \kappa\bar{D}S'(0) - rS(0) = 0$, where $S(0)$ denotes the limit of $S(D_t)$ at zero.*

Proof of Lemma C.12. We start with the limit at zero. Since $\lim_{D_t \to 0} S'(D_t)$ exists and is finite, the same is true for $\lim_{D_t \to 0} S(D_t)$. (The latter limit is $S(D_t) - \int_0^{D_t} S'(\hat{D}_t)\, d\hat{D}_t$ for any given D_t).

Since $\lim_{D_t \to 0} S'(D_t)$ exists and is finite, values of D_t close to zero lie in the unconstrained region. Moreover, since $\lim_{D_t \to 0} S'(D_t)$ and $\lim_{D_t \to 0} S(D_t)$ exist and are finite, taking the limit of both sides of the ODE (4.4) when D_t goes to zero implies that $\lim_{D_t \to 0} D_t S''(D_t)$ exists and is finite. If the latter limit differs from zero, then $|S''(D_t)| \geq \frac{\ell}{D_t}$ for $\ell > 0$ and for all D_t smaller than a sufficiently small $\eta > 0$. Since, however, for $D_t < \eta$,

$$S'(D_t) = S'(\eta) + \int_\eta^{D_t} S''(\hat{D}_t) d\hat{D}_t \Rightarrow |S'(D_t) - S'(\eta)| \geq \int_{D_t}^\eta \frac{\ell}{\hat{D}_t} d\hat{D}_t = \ell \log\left(\frac{\eta}{D_t}\right),$$

$\lim_{D_t \to 0} S'(D_t)$ would be plus or minus infinity, a contradiction. Hence, $\lim_{D_t \to 0} D_t S''(D_t) = 0$. Taking the limit of (4.4) when D_t goes to zero, and using $\lim_{D_t \to 0} S'(D_t) = S'(0)$, $\lim_{D_t \to 0} S'(D_t) = S(0)$, $\lim_{D_t \to 0} D_t S''(D_t) = 0$, and the finiteness of $S'(0)$ and $S(0)$, we find $Z(0) = 0$.

We next consider the limit at infinity. Since $\lim_{D_t \to \infty} S'(D_t)$ exists and is finite, it is equal to $\lim_{D_t \to \infty} \frac{S(D_t)}{D_t}$. This follows by writing $\frac{S(D_t)}{D_t}$ as

$$\frac{S(D_t)}{D_t} = \frac{S(0) + \int_0^{D_t} S'\left(\hat{D}_t\right) d\hat{D}_t}{D_t},$$

and noting that $\lim_{D_t \to \infty} \frac{S(0)}{D_t} = 0$ and $\lim_{D_t \to \infty} \frac{\int_0^{D_t} S'(\hat{D}_t) d\hat{D}_t}{D_t} = \lim_{D_t \to \infty} S'(D_t)$.

Since $\lim_{D_t \to \infty} S'(D_t)$ exists and is finite, large values of D_t lie in the constrained region. Dividing both sides of the ODE (4.7) by D_t, taking the limit when D_t goes to infinity, and using the existence and finiteness of $\lim_{D_t \to \infty} S'(D_t)$ and $\lim_{D_t \to \infty} \frac{S(D_t)}{D_t}$, we find that $\lim_{D_t \to \infty} S''(D_t)$ exists and is finite. If the latter limit differs from zero, then $|S''(D_t)| \geq \ell > 0$ for $\ell > 0$ and for all D_t sufficiently large, implying that $\lim_{D_t \to 0} S'(D_t)$ would be plus or minus infinity, a contradiction. Hence, $\lim_{D_t \to 0} S''(D_t) = 0$. Taking the limit of (4.7) when D_t goes to infinity, and using $\lim_{D_t \to \infty} S'(D_t) = \lim_{D_t \to 0} \frac{S(D_t)}{D_t} = S(\infty)$, $\lim_{D_t \to 0} S''(D_t) = 0$, and the finiteness of $S'(\infty)$, we find that $S'(\infty)$ is a root of (C.4). $\qquad \square$

Lemma C.13 shows that a solution to the system of ODEs with a derivative that converges to finite limits at zero and infinity exists in $(0, \infty)$, and has the properties in Lemmas C.7-C.11.

Lemma C.13. [**Existence in** $(0, \infty)$] *A solution $S(D_t)$ to the system of ODEs (4.4) in the unconstrained region (4.5), and (4.7) in the constrained region (4.8), with a derivative that converges to finite limits at zero and infinity exists in $(0, \infty)$, and has the properties in Lemmas C.7-C.11.*

Proof of Lemma C.13. We will construct the solution in $(0, \infty)$ as the simple limit of solutions in compact intervals $[\epsilon, M]$. We denote by $S_{\epsilon, M}(D_t)$ the solution derived in Lemma C.6, and by

$\Phi_{\epsilon,M}$ and $Z_{\epsilon,M}(D_t)$ the corresponding values of Φ and $Z(D_t)$.

We start with the case $\theta > 0$, and first derive the limit when ϵ goes to zero, holding M constant. Consider $\epsilon_1 > \epsilon_2 > 0$, and suppose, by contradiction, that $\Phi_{\epsilon_2,M} > \Phi_{\epsilon_1,M}$. Indeed, suppose, by contradiction, that $\Phi_{\epsilon_2,M} \leq \Phi_{\epsilon_1,M}$. Lemma C.3 then implies $S_{\epsilon_2,M}(\epsilon_1) \leq S_{\epsilon_1,M}(\epsilon_1)$ and $S'_{\epsilon_2,M}(\epsilon_1) \geq S'_{\epsilon_1,M}(\epsilon_1)$, which in turn imply $Z_{\epsilon_2,M}(\epsilon_1) \geq Z_{\epsilon_1,M}(\epsilon_1) = 0$. This is a contradiction because $S''_{\epsilon_2,M}(D_t) < 0$ and $Z_{\epsilon_2,M}(\epsilon_2) = 0$ imply $Z_{\epsilon_2,M}(\epsilon_1) < 0$. Hence, $\Phi_{\epsilon_2,M} > \Phi_{\epsilon_1,M}$, and Lemma C.3 implies $S_{\epsilon_2,M}(D_t) > S_{\epsilon_1,M}(D_t)$ and $S'_{\epsilon_2,M}(D_t) < S'_{\epsilon_1,M}(D_t)$ for all $D_t \in (\epsilon_1, M)$.

Since for given $D_t \in (0, M)$, the function $\epsilon \to S_{\epsilon,M}(D_t)$, defined for $\epsilon < D_t$, is increasing in ϵ and is bounded above by the affine solution derived for $\mu^* = \infty$ (Lemma C.8), it converges to a finite limit $S_M(D_t)$ when ϵ goes to zero. Likewise, since for given D_t, the function $\epsilon \to S'_{\epsilon,M}(D_t)$, defined for $\epsilon < D_t$, is decreasing in ϵ and is bounded below by zero (Lemma C.7), it converges to a finite limit $\hat{S}_M(D_t)$ when ϵ goes to zero.

The simple limit $S_M(D_t)$ of $S_{\epsilon,M}(D_t)$ is differentiable, and its derivative is the simple limit $\hat{S}_M(D_t)$ of $S'_{\epsilon,M}(D_t)$. To show this result, we use the intermediate value theorem together with a uniform bound on $S''_{\epsilon,M}(D_t)$. The function $S_{\epsilon,M}(D_t)$ is bounded above by the affine solution $\frac{a_1}{r}(\kappa \bar{D} + r D_t)$ and below by zero (Lemma C.7). Likewise, the function $S'_{\epsilon,M}(D_t)$ is bounded above by a_1 (Lemma C.11) and below by zero. Hence, for any given D_t and neighborhood $\mathcal{N}$ around D_t, the ODEs (4.4) and (4.7) imply a bound Q on $S''_{\epsilon,M}(m)$ that is uniform over $m \in \mathcal{N}$, ϵ and M. The intermediate value theorem implies that for $m \in \mathcal{N}$,

$$\left| \frac{S_{\epsilon,M}(m) - S_{\epsilon,M}(D_t)}{m - D_t} - S'_{\epsilon,M}(D_t) \right| = \left| S'_{\epsilon,M}(m') - S'_{\epsilon,M}(D_t) \right| = \left| S''_{\epsilon,M}(m'') \right| |m' - D_t| < Q|m - D_t|,$$

where m' is between m and D_t, and m'' is between m' and D_t. Taking the limit when ϵ goes to zero, we find

$$\left| \frac{S_M(m) - S_M(D_t)}{m - D_t} - \hat{S}_M(D_t) \right| \leq Q|m - D_t|,$$

which establishes that $S_M(D_t)$ is differentiable at D_t and its derivative is $S'_M(D_t) = \hat{S}_M(D_t)$. Since $S'_{\epsilon,M}(D_t)$ and $S'_{\epsilon,M}(D_t)$ have simple limits, we can use the ODEs (4.4) and (4.7) to construct a simple limit for $S''_{\epsilon,M}(D_t)$, which we denote by $\hat{\hat{S}}_M(D_t)$. The same argument that establishes $S'_M(D_t) = \hat{S}_M(D_t)$ can be used to establish $\hat{\hat{S}}_M(D_t) = S''_M(D_t)$, and hence that $S_M(D_t)$ solves the system of ODEs in $(0, M]$. Since $S'_{\epsilon,M}(D_t)$ is decreasing in D_t and is bounded below by zero, its limit $S'_M(D_t)$ over ϵ is non-increasing in D_t and has the same lower bound. Hence, $S'_M(D_t)$ converges to a finite limit $S'_M(0)$ when D_t goes to zero. Using the same argument as in Lemma C.12, we can show that $Z_M(0) \equiv \kappa \bar{D} S'_M(0) - r S_M(0) = 0$, where $S_M(0)$ denotes the limit of $S_M(D_t)$

when D_t goes to zero.

Since $S_M(D_t)$, $S'_M(D_t)$ and $S''_M(D_t)$ are the simple limits of $S_{\epsilon,M}(D_t)$, $S'_{\epsilon,M}(D_t)$ and $S''_{\epsilon,M}(D_t)$, respectively, the properties in Lemmas C.7, C.8, C.10 and C.11 hold as weak inequalities for all $D_t \in (0, M]$. Following similar arguments as in these Lemmas and using $Z_M(0) = 0$, we can show that the inequalities are strict.

We next derive the limit when M goes to infinity. Consider $M_2 > M_1$. Since $S''_{M_2}(D_t) < 0$ and $S'_{M_2}(M_1) = S'_{M_1}(M_1)$, $S'_{M_2}(M_1) > S'_{M_1}(M_1)$. Suppose, by contradiction, that $S_{M_2}(M_1) \leq S_{M_1}(M_1)$. Equations $S_{M_2}(M_1) \leq S_{M_1}(M_1)$ and $S'_{M_2}(M_1) > S'_{M_1}(M_1)$ imply $S_{M_2}(D_t) < S_{M_1}(D_t)$ for D_t smaller than and close to M_1. The same argument as in Lemma C.3 then implies $S_{M_2}(D_t) < S_{M_1}(D_t)$ and $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ for all $D_t \in (0, M_1)$. Since $S_{M_2}(M_1) \leq S_{M_1}(M_1)$ and $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ for all $D_t \in (0, M_1)$, $S_{M_2}(0) < S_{M_1}(0)$. Combining the latter equation with $S'_{M_2}(0) \geq S'_{M_1}(0)$, which follows by taking the limit of $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ when D_t goes to zero, we find $Z_{M_2}(0) > Z_{M_1}(0)$, a contradiction since $Z_{M_2}(0) = Z_{M_1}(0) = 0$. Hence, $S_{M_2}(M_1) > S_{M_1}(M_1)$.

The inequalities $S_{M_2}(D_t) > S_{M_1}(D_t)$ and $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ hold by continuity for D_t smaller than and close to M_1. Suppose, by contradiction, that there exists $D_t \in (0, M_1)$ such that $S_{M_2}(D_t) \leq S_{M_1}(D_t)$ or $S'_{M_2}(D_t) \leq S'_{M_1}(D_t)$, and consider the supremum m within that set. The definition of m implies $S_{M_2}(D_t) > S_{M_1}(D_t)$ and $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ for all $D_t \in (m, M_1)$, and $S_{M_2}(m) = S_{M_1}(m)$ or $S'_{M_2}(m) = S'_{M_1}(m)$. Only one of the latter two equations holds since otherwise the solutions $S_{M_1}(D_t)$ and $S_{M_2}(D_t)$ would coincide. If $S_{M_2}(m) = S_{M_1}(m)$ and $S'_{M_2}(m) > S'_{M_1}(m)$, then $S_{M_2}(D_t) < S_{M_1}(D_t)$ and $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ for D_t smaller than and close to M_1. The same argument as in Lemma C.3 then implies $S_{M_2}(D_t) < S_{M_1}(D_t)$ and $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ for all $D_t \in (0, m)$. This, in turn, implies $Z_{M_2}(0) > Z_{M_1}(0)$, a contradiction. If instead $S_{M_2}(m) > S_{M_1}(m)$ and $S'_{M_2}(m) = S'_{M_1}(m)$, then the same argument as in Lemma C.3 implies $S_{M_2}(D_t) > S_{M_1}(D_t)$ and $S'_{M_2}(D_t) < S'_{M_1}(D_t)$ for all $D_t \in (0, m)$. This, in turn, implies $Z_{M_2}(0) < Z_{M_1}(0)$, a contradiction. Hence, $S_{M_2}(D_t) > S_{M_1}(D_t)$ and $S'_{M_2}(D_t) > S'_{M_1}(D_t)$ for all $D_t \in (0, M_1)$.

Since for given $D_t \in (0, \infty)$, the function $M \to S_M(D_t)$, defined for $D_t < M$, is increasing in M and is bounded above by the affine solution derived for $\mu^* = \infty$, it converges to a finite limit $S(D_t)$ when M goes to infinity. Likewise, since for given $D_t \in (0, \infty)$, the function $M \to S'_M(D_t)$, defined for $D_t < M$, is increasing in M and is bounded above by a_1, it converges to a finite limit $\hat{S}(D_t)$ when M goes to infinity. The same argument as when taking the limit over ϵ establishes that $\hat{S}(D_t) = S'(D_t)$ and that $S(D_t)$ solves the system of ODEs in $(0, \infty)$. Since $S'_M(D_t)$ is decreasing in D_t and is bounded below by zero and above by a_1, its limit $S'(D_t)$ over M is non-increasing in D_t and has the same bounds. Hence, $S'(D_t)$ converges to finite limits $S'(0)$ when D_t goes to zero and $S'(\infty)$ when D_t goes to infinity. Lemma C.12 implies that $Z(0) \equiv \kappa \bar{D} S'(0) - r S(0) = 0$, where $S(0)$ denotes the limit of $S(D_t)$ when D_t goes to zero. Lemma C.12 also implies that $S'(\infty)$ is a

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root of (C.4). Since $S'(D_t)$ is the simple limit of $S'_M(D_t)$, which is positive and increasing in M, it is also positive. Hence, $S'(\infty)$ is non-negative and equal to the unique positive root of (C.4). The same arguments as when taking the limit over ϵ establish that the properties in Lemmas C.7, C.8, C.10 and C.11 hold for all $D_t \in (0, \infty)$.

The argument in the case $\theta < 0$ is symmetric. The monotonicity of $S_{\epsilon,M}(D_t)$ and $S'_{\epsilon,M}(D_t)$ as functions of ϵ, and of $S_M(D_t)$ and $S'_M(D_t)$ as functions of M, is the opposite relative to the case $\theta > 0$. The limit $S'(\infty)$ is non-negative and equal to the smaller of the two positive roots of (C.4) because $S'(D_t)$ is bounded above by that root. The upper bound on $S'(D_t)$ follows from the same upper bound on $S'_M(D_t)$: convexity implies that $S'_M(D_t) < S'_M(M)$ for all $D_t \in (0, M)$, and $S'(M)$ is equal to the smaller positive root of (C.4). $\square$

Theorem 4.1 follows from Lemma C.13. $\square$

Proof of Proposition 4.1. Substituting the asset's share return from (3.6) into (3.2), and setting $S_t = S(D_t)$, we find that the asset's dollar return is

$$dR_t = \frac{\left[D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t)\right] dt + \sigma\sqrt{D_t}S'(D_t)dB_t}{S(D_t)} - rdt. \qquad (C.48)$$

The return's conditional volatility is the diffusion coefficient in (C.48) times $\sqrt{dt}$:

$$\sqrt{\mathbb{V}\mathrm{ar}_t(dR_t)} = \frac{\sigma\sqrt{D_t}S'(D_t)\sqrt{dt}}{S(D_t)}. \qquad (C.49)$$

The return's conditional volatility under the affine solution derived for $\mu^* = \infty$ is given by (B.5). (While a_1 is different than in Section 3, volatility is independent of a_1.) Comparing (C.49) and (B.5), we find that the return's conditional volatility is higher than under the affine solution if

$$S'(D_t)(\kappa\bar{D} + rD_t) > rS(D_t) \Leftrightarrow Z(D_t) > 0,$$

and is lower than under the affine solution if $Z(D_t) < 0$. When $\theta > 0$, $Z(0) = 0$ and concavity imply $Z(D_t) < 0$, and hence conditional volatility is lower than under the affine solution. When instead $\theta < 0$, $Z(0) = 0$ and convexity imply $Z(D_t) > 0$, and hence conditional volatility is higher than under the affine solution. The comparison of conditional volatility across the cases $\theta > 0$ and $\theta < 0$ follows from the comparison of each case with the affine solution since volatility under the affine solution is independent of θ.

Since the return's unconditional variance is the unconditional expectation of the return's conditional variance, the comparisons derived for conditional volatility carry over to unconditional

volatility. $\qquad\square$

Proof of Proposition 4.2. The conditional beta of asset n is

$$\beta_{nt} = \frac{\mathbb{C}\mathrm{ov}_t(dR_{nt}, dR_{Mt})}{\mathbb{V}\mathrm{ar}_t(dR_{Mt})}, \tag{C.50}$$

where dR_{nt} denotes the return of asset n and dR_{Mt} denotes the return of the market portfolio. Assuming that the market portfolio includes η_m shares of asset $m = 1, .., N$, its return is

$$dR_{Mt} = \frac{dR_{Mt}^{sh}}{S_{Mt}} = \frac{\sum_{m=1}^{N} \eta_m dR_{mt}^{sh}}{\sum_{m=1}^{N} \eta_m S_{mt}} = \sum_{m=1}^{N} \frac{\eta_m S_{mt}}{\sum_{m=1}^{N} \eta_m S_{mt}} dR_{mt} = \sum_{m=1}^{N} \omega_{mt} dR_{mt}, \tag{C.51}$$

where S_{Mt} denotes the market portfolio's price and

$$\omega_{mt} \equiv \frac{\eta_m S_{mt}}{\sum_{m=1}^{N} \eta_m S_{mt}}$$

denotes the weight of asset n in the market portfolio. Equation (C.50) implies that the conditional beta of asset n exceeds that of asset n' if

$$\mathbb{C}\mathrm{ov}_t(dR_{nt}, dR_{Mt}) > \mathbb{C}\mathrm{ov}_t(dR_{n't}, dR_{Mt})$$
$$\Leftrightarrow \omega_n \mathbb{V}\mathrm{ar}_t(dR_{nt}) > \omega_{n'} \mathbb{V}\mathrm{ar}_t(dR_{n't})$$
$$\Leftrightarrow \eta_n S_{nt} \mathbb{V}\mathrm{ar}_t(dR_{nt}) > \eta_{n'} S_{n't} \mathbb{V}\mathrm{ar}_t(dR_{n't}), \tag{C.52}$$

where second step follows from (C.51) and the independence of returns across assets.

Suppose next that $\theta_n < 0$ and $\theta_{n'} > 0$, and that assets n and n' are otherwise identical ($\kappa_n = \kappa_{n'}$, $\bar{D}_n = \bar{D}_{n'}$, $\sigma_n = \sigma_{n'}$, $\eta_n = \eta_{n'}$, and $D_{nt} = D_{n't}$ for a given t). Since a_1 decreases in θ (Equation (C.25)), the affine solution derived for $\mu^* = \infty$ is larger for θ_n than for θ'_n. Since, in addition, S_{nt} lies above the affine solution for θ_n, while $S_{n't}$ lies below the affine solution for θ'_n (Theorem 4.1), $S_{nt} > S_{n't}$. Since, finally, $\mathbb{V}\mathrm{ar}_t(dR_{nt}) > \mathbb{V}\mathrm{ar}_t(dR_{n't})$ (Proposition 4.1), (C.52) implies $\mathbb{C}\mathrm{ov}_t(dR_{nt}, dR_{Mt}) > \mathbb{C}\mathrm{ov}_t(dR_{n't}, dR_{Mt})$ and hence $\beta_{nt} > \beta_{n't}$.

The unconditional beta of asset n is

$$\beta_{nt} = \frac{\mathbb{C}\mathrm{ov}(dR_{nt}, dR_{Mt})}{\mathbb{V}\mathrm{ar}(dR_{Mt})} = \frac{\mathbb{E}\left(\mathbb{C}\mathrm{ov}_t(dR_{nt}, dR_{Mt})\right)}{\mathbb{E}\left(\mathbb{V}\mathrm{ar}_t(dR_{Mt})\right)},$$

Since the conditional covariance of $\mathbb{C}\mathrm{ov}_t(dR_{nt}, dR_{Mt})$ is larger for asset n than for asset n', the same is true for the unconditional covariance, and hence for the unconditional beta. $\qquad\square$

D Proofs for Section 5

Proof of Proposition 5.1. Setting $\hat{\theta} \equiv \theta - \eta$, we can write (5.3) as

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \left(\frac{\rho\hat{\theta}}{1 - \lambda x} + \rho\eta\right)\sigma^2 D_t S'(D_t)^2. \tag{D.1}$$

and (5.6) as

$$D_t + \kappa(\bar{D} - D_t)S'(D_t) + \frac{1}{2}\sigma^2 D_t S''(D_t) - rS(D_t) = \left(\frac{\rho\hat{\theta}}{1 - x} + \rho\eta\right)\sigma^2 D_t S'(D_t)^2$$
$$- \frac{\text{sgn}(\hat{\theta})(1 - \lambda)x\mu^*}{1 - x}\sigma\sqrt{D_t}S'(D_t). \tag{D.2}$$

Using (D.1) and (D.2), we can replicate the proofs of Theorem 4.1 and Propositions 4.1 and 4.2. In those of these proofs that distinguish the cases $\theta > 0$ and $\theta < 0$, we instead distinguish the cases $\hat{\theta} > 0$ and $\hat{\theta} < 0$, or equivalently $\theta > \eta$ and $\theta < \eta$.

Lemma C.1 carries through by replacing the initial conditions (C.1) and (C.2) by

$$S'(M) = \frac{2}{(r + \kappa) + \sqrt{(r + \kappa)^2 + 4\left(\frac{\rho\hat{\theta}}{1-x} + \rho\eta\right)\sigma^2}}, \tag{D.3}$$

$$S(M) = \frac{1}{r}\left((\kappa\bar{D} + rM)S'(M) + \frac{1}{2}\sigma^2 M\Phi + \frac{\text{sgn}(\hat{\theta})(1 - \lambda)x\mu^*}{1 - x}\sigma\sqrt{M}S'(M)\right). \tag{D.4}$$

The value of $S'(M)$ in (D.3) solves the quadratic equation

$$\left(\frac{\rho\hat{\theta}}{1 - x} + \rho\eta\right)\sigma^2 S'(M)^2 + (r + \kappa)S'(M) - 1 = 0, \tag{D.5}$$

which replaces (C.4). When $\frac{\rho\hat{\theta}}{1-x} + \rho\eta > 0$, the left-hand side of (D.5) is increasing for positive values of $S'(M)$, and (D.5) has a unique positive solution, given by (D.3). When $\frac{\rho\hat{\theta}}{1-x} + \rho\eta < 0$, the left-hand side is hump-shaped for positive values of $S'(M)$, and (D.5) has either two positive solutions, or one positive solution, or no solution. Condition $\theta > -\frac{(1-x)(r+\kappa)^2}{4\rho\sigma^2} + x\eta$ in Proposition 5.1 ensures that two positive solutions exist when $\frac{\rho\hat{\theta}}{1-x} + \rho\eta < 0$. Equation (D.3) gives the smaller of the two solutions, which is the continuous extension of the unique positive solution when $\frac{\rho\hat{\theta}}{1-x} + \rho\eta > 0$.

Lemmas C.2 and C.3 carry through. Lemma C.4 carries through, but in the case $\hat{\theta} < 0$ and m lies in the constrained region, we need to distinguish between two subcases: $\frac{\rho\hat{\theta}}{1-x} + \rho\eta < 0$, where we use the argument in the lemma's proof, and $\frac{\rho\hat{\theta}}{1-x} + \rho\eta < 0$, where instead we adapt the argument that concerns the case $\hat{\theta} > 0$. Lemmas C.5-C.7 carry through. Lemma C.8 carries through by

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replacing (C.25) by

$$a_1 = \frac{2}{(r + \kappa) + \sqrt{(r + \kappa)^2 + 4\left(\frac{\rho\hat{\theta}}{1 - \lambda x} + \rho\eta\right)\sigma^2}}. \tag{D.6}$$

Lemmas C.9-C.13 and Propositions 4.1 and 4.2 carry through. $\qquad\square$

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