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RESOURCE ALLOCATION AND INEFFICIENCY IN THE FINANCIAL SECTOR

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ABSTRACT

I analyze whether banks are efficient at allocating resources across intermediation activities. Competition between lenders means that resources are needed to draw borrowers into credit matches. At the same time, imperfect information between lenders and borrowers means that resources are also needed for screening. I show that the privately optimal allocation of resources is constrained inefficient. In particular, too many resources are spent on getting rather than vetting borrowers but, once properly vetted, not enough matches are retained. Uninformed lending is thus inefficiently high, informed lending is inefficiently low, and a tax on matching activities helps remedy the situation.

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1 Introduction

Understanding how agents allocate finite resources is fundamental to economics yet very little work exists on the allocation of bank resources across intermediation activities. Attracting new clients is a competitive endeavor so we often observe banks creating financial products and advertising their loan services (*matching activities*). At the same time, information frictions such as adverse selection mean that banks must also devote some resources to learning about who they attract (*screening activities*). A tradeoff arises because the marginal resource put into one activity could have instead been put into the other. Do the private and social margins line up? I show that they do not: lenders screen too little but are then overly selective about who to finance once informed. These results contrast with existing papers and have important implications for the health of the financial system.

Central to my analysis is that both matching and screening are costly. The literature on adverse selection has traditionally bundled these activities and predicted that lenders screen too much. Excessive screening, however, is difficult to reconcile with the run-up to the recent financial crisis: by all accounts, information acquisition was low and economists have turned to securitization for answers.¹ While securitization can certainly distort incentives, my results suggest it is not a prerequisite for too little screening: there are potentially deeper problems in how competing banks allocate resources across intermediation activities.

To understand the mechanisms in my model, begin with the allocation decision of an unmatched lender. By budgeting resources to both matching and screening, this lender forms an informed match with positive probability. If the borrower is desirable, then the match is retained and unavailable to other lenders until it breaks. If the borrower is undesirable, then the match is rejected and the borrower returns to the available pool. An increase in informed matching thus worsens the adverse selection problem faced by other lenders.

¹The adverse selection literature is discussed in more detail at the end of this section. The negative effect of securitization on screening is shown empirically in Keys et al (2010) and Purnanandam (2011) but, as per Gorton and Pennacchi (1995) and subsequent work, originate-to-distribute models cannot completely eliminate screening since loan sales must be incentive compatible.

The intuition behind too much screening in other papers is that more screening leads to more informed matches. Resource allocation breaks this intuition by introducing a nonmonotonicity in the relationship between screening and the number of informed matches. If lenders devote a tiny share of resources to screening, then few matches are informed and the distribution of available borrowers is largely unchanged. An increase in screening deteriorates the distribution by creating more informed matches but, as more and more resources are funneled into screening, matching becomes very low and the deterioration reverses: the fraction of informed matches is high but the number is low. Put another way, more screening can sometimes produce a better distribution than less screening.

The non-monotonicity just described is the first step away from the conventional view of too much screening. The second step is understanding how resource allocation affects the retention decisions of informed lenders. To this end, consider two screening intensities with the same effect on the distribution of available borrowers. Since the relationship between screening and the number of informed matches is non-monotone, we know that such intensities exist. The higher of the two screening intensities leaves fewer resources open for matching. This decreases the rate at which unmatched lenders contact the available distribution, increasing the expected duration until a good draw. The value of being unmatched then falls, prompting informed lenders to retain more of their matches. More screening can thus have a positive effect on the survival of informed matches.

Key to the discussion so far is that lenders choose the cutoff between desirable and undesirable borrowers once informed. This is a departure from banking models with two borrower types. By assuming one good type and one bad type, such models make the retention decision of an informed lender exogenous. With an exogenous cutoff, I show that a Walrasian interbank market prices the effect of resource allocation on the distribution of available borrowers. With an endogenous cutoff, however, one market cannot price both the distributional externality and the additional effect on informed lenders. The result is too little screening and too little informed retention. By way of corrective taxation, I also show that a matching tax which remits proceeds to the interbank market implements the constrained efficient allocation as an equilibrium. In contrast, a broader scheme that taxes lender revenues is entirely ineffective. Absent a matching tax, my model predicts that uninformed credit is inefficiently high and informed credit is inefficiently low. Lenders as a whole thus recover too little from their borrowers and the economy settles on a steady state that supports too little credit overall.

All of the results described so far are robust to the introduction of moral hazard. Indeed, allowing lenders to influence borrower effort via loan rates increases the welfare loss stemming from my externalities. The results are also robust to relationship lending under high discount factors. Relationship lending allows lenders to learn about their borrowers through repeated interactions.² Therefore, even if initial screening is unsuccessful, a lender can discover his borrower's type during the process of providing uninformed credit. On one hand, this mitigates the downside of entering an uninformed match but, on the other, it worsens the distribution of available borrowers. The discount factor governs how lenders weigh a worse distribution against the fact that learning through relationship lending is a lagged substitute for screening. I show that there is one discount factor for which the decentralized economy with relationship lending achieves constrained efficiency. For discount factors above this critical value, too little screening and too little informed retention re-emerge.

In emphasizing financial non-neutrality, my paper is broadly related to the macroeconomic literature on credit channels (e.g., Gurley and Shaw (1955), Williamson (1987), Bernanke and Gertler (1989), Kiyotaki and Moore (1997)). It also relates to a recent branch of this literature which investigates financial sector inefficiency under the asset price mechanism of Kiyotaki and Moore (1997). In Lorenzoni (2008) and Korinek (2011), for example, firms do not internalize that more leverage will require more fire sales should a negative shock hit. The result is over-borrowing ex ante and the potential for a financial crisis ex post. While I also focus on financial sector inefficiency, the externalities in my model do not

 $^{^{2}}$ For more on relationship lending, see Boot (2000), Hachem (2011), and the references therein.

rely on fire sales or asset prices more generally.³

At a microeconomic level, my paper is also related to the literature on adverse selection in banking.⁴ The seminal work of Broecker (1990) expounds screening externalities among banks who can costlessly attract borrowers via Bertrand competition. When more banks operate a free but noisy screening technology, a winner's curse problem makes any one bank less likely to provide credit. Broecker's model has since been extended to allow for screening costs (e.g., Cao and Shi (2001), Hauswald and Marquez (2006), Direr (2008), and Gehrig and Stenbacka (2011)). Whenever screening is found to be inefficient in these extensions, the inefficiency involves too much screening.⁵ Although Dell'Ariccia and Marquez (2006) model screening via separating contracts, their results also feature too much screening: any inefficiency involves the market settling on a screening rather than pooling equilibrium. In contrast, my model predicts that banks choose less than the efficient amount of screening.

While screening costs are now common in the banking literature, matching costs are still fairly rare. There are several ways to motivate costly matching. The approach I take is that banks compete for borrowers through non-price means. A similar motivation arises in Heider and Inderst (2012). However, their model abstracts from costly screening and focuses on agency problems between bank managers and loan officers. Moreover, several features of their external environment are exogenous (e.g., cost of funds, available distribution, outside option) and efficiency is not analyzed. Though not pursued here, another way to motivate costly matching is with search frictions. The key variable in standard models of random search is market tightness, defined as the ratio of available agents on either side of the market.⁶ My model is set up so that no externalities are imparted through this ratio. More

 $^{^{3}}$ For more on fire sales, see Gromb and Vayanos (2002), Allen and Gale (2005), Shleifer and Vishny (2011), and the references therein. Also see Shleifer and Vishny (2010) for a model where asset prices interact with the division of bank investment between traditional lending and securitized debt.

 $^{^4\}mathrm{See}$ Frexias and Rochet (1997) for an overview.

⁵Under certain parameters, Gehrig and Stenbacka (2011) find cycles with delayed screening but even this does not culminate in insufficient information production unless firms are assumed to die in the interim. For a model of competition externalities without screening, see Parlour and Rajan (2001).

⁶See, for example, Hosios (1990), Yashiv (2007), and the references therein.

recently, search models with heterogeneous agents have also studied market composition (e.g., Shimer and Smith (2001)). However, these models abstract from asymmetric information so the intensity with which an agent searches has no effect on his ability to assess match quality.⁷ There is thus a dearth of models that can explain how banks allocate resources between matching and screening and how this allocation then affects the macroeconomy.

The rest of the paper proceeds as follows: Section 2 describes the baseline environment and derives the core results; Section 3 extends the baseline model to include relationship lending; Section 4 extends the baseline model to include moral hazard and assess welfare losses; and Section 5 concludes. All proofs are presented in the appendix.

2 Baseline Model

Time is discrete. All agents are infinitely-lived, risk neutral, and have discount factor $\beta \in (0, 1)$. There is a continuum of firm types, $\omega \in [0, 1]$, with symmetric density function $f(\cdot)$. To ease exposition, set $f(\cdot) = 1$. Each firm has private information about its type. It also has access to a production project but requires one unit of external capital before it can undertake this project. Once undertaken, the time to project completion is geometrically distributed with parameter $\mu \in (0, 1]$. A completed (mature) project then generates θ units of output with probability $q(\omega)$ and zero units with probability $1 - q(\omega)$, where $q'(\cdot) > 0$.

Firms cannot store project output and they do not have direct access to capital so they borrow from a unit measure of ex ante identical lenders that also populates the economy. Lenders cannot produce and have no alternative use for capital. Instead, they have access to two technologies that allow them to emerge as intermediaries. The first technology is matching: lenders can create and/or advertise financial products to match firms with capital. I abstract from the exact process through which lenders generate their matches, positing instead a one-to-one matching technology that is only available to unmatched lenders. With

⁷While Becsi et al (2013) combine search and finance, screening is negated by borrower type homogeneity.

one-to-one matching and equal (unit) populations of firms and lenders each period, the technology is such that the number of unmatched lenders equals the number of unmatched firms and a lender's matching probability depends only on his own matching effort. The second technology is screening: a matched lender can investigate the quality of his match to determine whether he actually wants to extend capital. For completeness, assume lenders cannot commit to any actions that will dissuade certain borrowers from making themselves available to match. Also assume that matches are dissolved once the underlying project matures. Combined with the fact that all loans involve exactly one unit of capital, lenders thus do not have enough instruments to offer separating contracts in lieu of screening.⁸

Although lenders may want to undertake both matching and screening, it is either too costly or too time-consuming to make each activity succeed with certainty. I introduce a resource constraint to capture this. In particular, each lender is endowed with $z \in (0, \infty)$ units of non-transferable effort every period. A lender who devotes π units of effort to matching gets a borrower with probability $p(\pi)$ and discovers that borrower's type with probability $p(z - \pi)$ immediately thereafter. Relationship lending, introduced in Section 3, will provide an alternative means of learning but only after some time has elapsed. The function $p(\cdot)$ satisfies p(0) = 0, $p(\infty) = 1$, $p'(\cdot) > 0$, and $p''(\cdot) < 0$.

To flesh out the implications of a lender's resource allocation decision, examine how lenders evolve over time. Begin with a lender who is unmatched at the end of date t - 1. At the beginning of date t, the lender chooses π . If he fails to attract a borrower, then he stays unmatched throughout t and must try again in t + 1. Otherwise, he forms a match and exerts screening effort $z - \pi$. Successful screening means that the lender's information set contains the borrower's true type whereas unsuccessful screening means that it only contains the distribution from which he drew the match. Denote this distribution by $\psi(\cdot)$. Conditional on his information set, the lender must decide whether to finance the borrower

⁸The abstraction from separating contracts is less stark than may initially seem. Separation is not free: the lender has to forgo some rents to ensure incentive compatibility for all borrower types. Livshits et al (2011) capture this in a simple way by introducing a fixed cost which increases with finer separation.

he just attracted or whether to let him go and try for another borrower in t + 1. To streamline the exposition, assume matches are enforceable unless the lender can prove that the borrower's type is too low (i.e., he can only discriminate based on ω). Once retention decisions have been made, newly matched borrowers undertake production and previously matched borrowers discover whether their projects have matured.

Any output from a mature project is split so that the borrower consumes an exogenous fraction $\delta \in (0, 1)$ and repays the rest to his lender. I will endogenize δ in Section 4. Lenders can detect the presence of positive output so borrowers repay if and only if their projects are successful. For analytical tractability, assume lenders cannot observe whether a borrower was successful in previous matches without first screening him.

Finally, to close the model, I introduce a Walrasian interbank market for capital with market clearing cost r. The cost is quoted so that 1 + r is the present discounted value of a lender's gross cost of funds. Lenders who do not have enough capital to finance their matches borrow from the interbank market. For all other lenders, interbank trade is the opportunity cost of proceeding with a match. The interbank market allows us to abstract from the distribution of capital across lenders and focus instead on the aggregate capital stock. Unless otherwise indicated, attention is restricted to steady states.⁹

2.1 Optimization Problems

All lenders take as given the cost of funds r and the distribution of available borrowers $\psi(\cdot)$. Let U denote the value function of an unmatched lender and let $J(\omega) - (1+r)$ denote the expected net present value from financing a type ω borrower.

⁹In the spirit of Diamond (1984) and Rajan (1994), my model focuses on the lending function of banks. To justify the abstraction from deposit-taking decisions, one could imagine something akin to deposit insurance. By making banks look equally riskless to depositors, such insurance would dull both the need to compete for deposits and the fear of a bank run, allowing all the action to come from the lending side. Interbank capital could then be interpreted as the insured deposits held by the banking system and δ could be interpreted as the exogenous fraction of deposits withdrawn each period. That banks funded entirely by insured deposits would then just maximize expected interest income is consistent with Stein (1998).

Matched Lenders Consider first a lender who finances ω . The match matures with probability μ , in which case the lender gets an expected payoff of $\theta (1 - \delta) q(\omega)$ then begins next period unmatched. With probability $1 - \mu$, the match does not mature so the lender gets nothing this period and carries his borrower over. This yields:

$$J(\omega) = \mu \left[\theta \left(1 - \delta\right) q(\omega) + \beta U\right] + (1 - \mu) \beta J(\omega) \tag{1}$$

Unmatched Lenders Consider now an unmatched lender who devotes effort π to matching. He will get a match with probability $p(\pi)$ and discover the quality of that match with probability $p(z - \pi)$. Absent moral hazard, the difference between informed and uninformed lenders is the ability of an informed lender to only retain borrowers who yield $J(\omega) - (1 + r) \ge \beta U$. An uninformed lender cannot do this because he does not know ω . The value function of an unmatched lender is thus:

$$U = \max_{\pi \in [0,z]} \left\{ \begin{array}{l} p(\pi) p(z-\pi) \int_{0}^{1} \max \left\{ J(\omega) - (1+r) - \beta U, 0 \right\} \psi(\omega) \, d\omega \\ + p(\pi) \left[1 - p(z-\pi) \right] \int_{0}^{1} \left[J(\omega) - (1+r) - \beta U \right] \psi(\omega) \, d\omega + \beta U \end{array} \right\}$$
(2)

Optimality Conditions The following result will now be useful:

Proposition 1 There exists a unique U satisfying equations (1) and (2).

Given Proposition 1, $q'(\omega) > 0$ implies $J'(\omega) > 0$ so the informed lender's retention decision is characterized by a reservation strategy. In particular, he only retains $\omega \in [\underline{\omega}, 1]$ where:

$$\underline{\omega} \equiv \arg\min_{\omega \in [0,1]} |J(\omega) - \beta U - (1+r)|$$
(3)

Any non-trivial equilibrium will have $\underline{\omega} \in (0, 1)$ and $\pi \in (0, z)$. I shall thus proceed under the assumption that both $\underline{\omega}$ and π are interior then provide appropriate parameter restrictions. The first order condition for π , obtained from (2), can now be written quite compactly:

$$1 - p(z - \pi) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)} = \frac{\int_{\underline{\omega}}^{1} \left[q(\omega) - q(\underline{\omega})\right]\psi(\omega)d\omega}{\int_{0}^{\underline{\omega}} \left[q(\underline{\omega}) - q(\omega)\right]\psi(\omega)d\omega}$$
(4)

Notice that I have used equations (1) and (3) to substitute out $J(\cdot)$ and 1 + r. To interpret equation (4), recall that an unmatched lender who chooses a high value of π has a high probability of becoming matched but uninformed. We know from the informed retention decision that the benefit of being matched rather than unmatched comes from financing borrowers above $\underline{\omega}$ while the benefit of being informed rather than uninformed comes from weeding out borrowers below $\underline{\omega}$. The right-hand side of equation (4) captures the ratio of these benefits while the left-hand side is increasing in π under the assumptions on $p(\cdot)$. Therefore, equation (4) just says that the optimal choice of π will be high when the benefit of being matched is large relative to the benefit of being informed.

2.2 Aggregate Conditions

We now need to pin down r and $\psi(\cdot)$ conditional on lender behavior. I focus on a symmetric equilibrium where all lenders choose the same π and $\underline{\omega}$.

Distributions Let $\phi(\omega)$ denote the mass of type ω borrowers in uninformed matches and let $\lambda(\omega)$ denote the mass in informed matches. Both $\phi(\omega)$ and $\lambda(\omega)$ are beginning-of-period measures. In steady state, the inflows into each of $\phi(\omega)$ and $\lambda(\omega)$ must exactly offset the outflows. At the end of any given period, the outflows are $\mu\phi(\omega)$ and $\mu\lambda(\omega)$ respectively and the mass of unmatched type ω borrowers is thus $1 - (1 - \mu) [\lambda(\omega) + \phi(\omega)]$. Fraction $p(\pi) [1 - p(z - \pi)]$ of these unmatched borrowers will be drawn into uninformed matches next period and fraction $p(\pi) p(z - \pi)$ will be drawn into informed matches. However, any $\omega < \omega$ drawn into an informed match will be immediately rejected given the informed retention strategy derived above. Define an indicator function $\mathcal{I}(\cdot)$ which equals one if its argument is true and zero if its argument is false. We can then write:

$$\mu\phi(\omega) = \left[1 - (1 - \mu)\left[\lambda\left(\omega\right) + \phi\left(\omega\right)\right]\right]p(\pi)\left[1 - p(z - \pi)\right]$$
(5)

$$\mu\lambda\left(\omega\right) = \mathcal{I}\left(\omega \ge \underline{\omega}\right) \times \left[1 - (1 - \mu)\left[\lambda\left(\omega\right) + \phi\left(\omega\right)\right]\right] p\left(\pi\right) p\left(z - \pi\right)$$
(6)

The distribution of available borrowers can now be calculated as:

$$\psi(\omega) = \frac{1 - (1 - \mu) [\lambda(\omega) + \phi(\omega)]}{\int_0^1 [1 - (1 - \mu) [\lambda(x) + \phi(x)]] dx}$$
(7)

Interbank Market Turn next to r which adjusts to properly distribute capital across lenders. Equation (3) pins down r provided we have another equation for $\underline{\omega}$. Interbank clearing provides the additional equation. In particular, the capital needed to finance new loans equals the capital available from maturing loans. Mathematically:

$$\mu \int_{0}^{1} \left[\lambda\left(\omega\right) + \phi\left(\omega\right)\right] d\omega = \mu \theta \left(1 - \delta\right) \int_{0}^{1} q\left(\omega\right) \left[\lambda\left(\omega\right) + \phi\left(\omega\right)\right] d\omega \tag{8}$$

Equation (8) implicitly assumes that lenders return all the proceeds from maturing loans back to the interbank market. This is not necessary for the results. One could instead let lenders eat a small fraction $\varepsilon > 0$ so that only $1 - \delta - \varepsilon$ is returned. Bigger ε would mean a smaller capital stock and thus require a higher r to clear the interbank market.

2.3 Equilibrium

Combining the results of Subsections 2.1 and 2.2 will complete the characterization of the decentralized equilibrium. Begin by solving equations (5) and (6) for $\phi(\omega)$ and $\lambda(\omega)$ then substitute into the clearing condition defined by equation (8). This yields:

$$\frac{1 - p\left(z - \pi\right)}{1 - \frac{(1 - \mu)p(\pi)p(z - \pi)}{\mu + (1 - \mu)p(\pi)}} = \frac{\int_{\omega}^{1} \left[q\left(\omega\right) - \frac{1}{\theta(1 - \delta)}\right] d\omega}{\int_{0}^{\omega} \left[\frac{1}{\theta(1 - \delta)} - q\left(\omega\right)\right] d\omega}$$
(9)

Now calculate the distribution of available borrowers $\psi(\omega)$ from equation (7) and substitute into the lender optimality condition defined by equation (4). This yields:

$$\frac{1 - p\left(z - \pi\right) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}}{1 - \frac{(1 - \mu)p(\pi)p(z - \pi)}{\mu + (1 - \mu)p(\pi)}} = \frac{\int_{\underline{\omega}}^{1} \left[q\left(\omega\right) - q\left(\underline{\omega}\right)\right] d\omega}{\int_{0}^{\underline{\omega}} \left[q\left(\underline{\omega}\right) - q\left(\omega\right)\right] d\omega}$$
(10)

The decentralized equilibrium is a pair $(\pi^*, \underline{\omega}^*)$ that solves equations (9) and (10).

Proposition 2 Under Assumptions 1 and 2 below, there is an equilibrium with $\pi^* \in (0, z)$ and $\underline{\omega}^* \in (0, 1)$. This equilibrium is also unique under Assumption 3.

- **Assumption 1** $q(1) > \frac{1}{\theta(1-\delta)} > \int_0^1 q(\omega) d\omega$
- **Assumption 2** p(z) is sufficiently high
- **Assumption 3** $\frac{p'(z-\pi)}{1-p(z-\pi)} < \frac{p'(\pi)}{p(\pi)} \frac{p''(\pi)}{p'(\pi)} \frac{p''(z-\pi)}{p'(z-\pi)}$ for any $\pi \in (0, z)$

Begin with Assumption 1. Capital demand is bounded above by one since the measure of borrowers is one and each borrower gets at most one unit of capital. There are no such restrictions on lender revenue so I impose $\frac{1}{\theta(1-\delta)} > \int_0^1 q(\omega) d\omega$ to bound capital supply. I then impose $\frac{1}{\theta(1-\delta)} < q(1)$ which is necessary for at least the highest borrower type to be desirable. With $p'(\cdot) > 0$, Assumption 2 is just a lower bound on the amount of resources available to a lender each period. It prevents the resource constraint from being so tight that lenders cannot pursue enough intermediation to clear the interbank market. With $p'(\cdot) > 0$ and $p''(\cdot) < 0$, a stricter version of Assumption 3 is $p''(\cdot) \leq -\frac{p'(\cdot)^2}{1-p(\cdot)}$, interpretable as follows: if $p(\cdot)$ increases rapidly and/or approaches one, it picks up enough curvature to slow down. This ensures that lenders face economically meaningful tradeoffs when allocating finite resources and is sufficient for a unique equilibrium.

2.4 Constrained Efficiency

Consider a social planner who holds the entire capital supply and who must allocate it to firms every period in order to achieve production. So that no inefficiencies are driven by differences in assumptions, the planner faces the same technologies and constraints as lenders in the decentralized economy. In particular, the fraction of available firms reached by the planner is $p(\pi)$ and, out of these firms, he is informed about fraction $p(z - \pi)$. The worst firm he allocates to when informed is $\underline{\omega}$. The planner chooses $\pi \in [0, z]$ and $\underline{\omega} \in [0, 1]$ to maximize the total present discounted value of output, $\mathcal{W} \equiv \frac{\mu\theta}{1-\beta} \int_0^1 q(\omega) [\lambda(\omega) + \phi(\omega)] d\omega$, subject to equations (5), (6), and (8). In this context, equation (8) is the planner's aggregate feasibility constraint. Combining the planner's first order conditions for π and ω yields:

$$\frac{1 - p\left(z - \pi\right) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}}{\left(1 - \frac{(1 - \mu)p(\pi)p(z - \pi)}{\mu + (1 - \mu)p(\pi)}\right)^2} = \frac{\int_{\underline{\omega}}^1 \left[q\left(\omega\right) - q\left(\underline{\omega}\right)\right] d\omega}{\int_0^{\underline{\omega}} \left[q\left(\underline{\omega}\right) - q\left(\omega\right)\right] d\omega}$$
(11)

The constrained efficient allocation is a pair $(\hat{\pi}, \underline{\hat{\omega}})$ that solves equations (9) and (11).

Proposition 3 If μ is not too close to zero and $q''(\cdot)$ is not too low, $(\widehat{\pi}, \underline{\widehat{\omega}})$ is unique.

The above proposition restricts $q(\cdot)$ to being convex, linear, or mildly concave. In other words, some differences in borrower quality should also be visible among higher types.

2.5 Main Results

The following proposition summarizes the first main result of this paper:

Proposition 4 Invoke Assumptions 1, 2, and the conditions in Proposition 3. Any equilibrium is constrained efficient if and only if $\mu = 1$. The direction of inefficiency for $\mu < 1$ is $\pi^* > \hat{\pi}$ with $\underline{\omega}^* > \hat{\underline{\omega}}$, stated more compactly as $(\pi^*, \underline{\omega}^*) \gg (\hat{\pi}, \hat{\underline{\omega}})$.

Both the planner and the decentralized solution have equation (9) in common so any inefficiencies reflect differences in equations (10) and (11). There are no differences in these equations if $\mu = 1$, making the equilibrium constrained efficient. Notice that $\mu = 1$ results in all matches being destroyed at the end of the period. In other words, the pool from which unmatched lenders draw matches is always the initial pool. The equilibrium only being efficient in this case suggests that the externalities associated with $\mu < 1$ arise because of intertemporal match preservation. In particular, a lender who engages in both matching and screening will have a positive probability of forming an informed match. If the borrower is desirable (i.e., if $\omega \geq \underline{\omega}$), then the match is retained and unavailable to other lenders until it matures. If the borrower is undesirable (i.e., if $\omega < \underline{\omega}$), then the match is rejected and the borrower returns to the available pool. Resource allocation thus determines the probability with which a lender worsens the pool of borrowers available to other lenders. Note that the externality is on the quality of this pool: matching is one-to-one and the number of available borrowers always equals the number of available lenders so it is the likelihood of getting a desirable borrower that worsens, not the likelihood of getting any borrower at all.

The distributional externality just described involves both matching and screening yet $\pi^* > \hat{\pi}$ as per Proposition 4 suggests a negative externality associated primarily with the matching activity. The distributional intuition must therefore be made more precise. As a first step, it will be instructive to compare my model with other banking models that also feature distributional externalities (i.e., winner's curse models). In Direr (2008), for example, high screening by all lenders worsens the available pool so each individual lender does indeed have an incentive to incur screening costs. Low screening has the opposite effect, resulting in both "low screening" and "high screening" equilibria. While high screening in my model also increases the fraction of matched borrowers that are discovered, the diversion of resources from matching to screening means there are very few matches to being with. The number of matched borrowers who are discovered is therefore quite low, reducing the effect on the available borrower pool. Modeling the cost of screening as foregone matching rather than a reduced-form screening cost thus circumvents a "high screening" equilibrium.

Modeling an interbank clearing cost r and an endogenous threshold $\underline{\omega}$ – two other dimensions along which my model and standard winner's curse models differ – also has important implications. If $\underline{\omega}$ were exogenous, then r could price the distributional externality and implement constrained efficiency. Let me elaborate on this point. Exogenous $\underline{\omega}$ eliminates equation (3) as an equilibrium condition and replaces $\int_0^1 \max \{J(\omega) - (1+r) - \beta U, 0\} \psi(\omega) d\omega$ in equation (2) with $\int_{\omega}^1 [J(\omega) - (1+r) - \beta U] \psi(\omega) d\omega$ for some constant $\underline{\omega}$. The equilibrium would then be a triple $\{U, \pi, r\}$ which solves the new version of (2), the resulting first order condition for π , and interbank clearing as per equation (9). Equation (9) is common to both the planner and the market so a given $\underline{\omega}$ pins down π^* and $\hat{\pi}$ at the same point, with radjusting to deliver this point from the remaining equilibrium conditions. The adjustment mechanism can be understood as follows: $\pi^* > \hat{\pi}$ implies that too many low quality borrowers are being financed so the resulting destruction of interbank capital bids up the cost of funds r, decreasing the benefit of an average match and thus decreasing π^* .

We can now conclude that inefficiency under $\mu < 1$ involves a distributional externality as well as an inability of r to properly price this externality when $\underline{\omega}$ is endogenous. Endogeneity of $\underline{\omega}$ means that the lowest type retained by an informed lender is the type that gives him his outside option. This option is the present discounted value of being unmatched so $\underline{\omega}^* > \underline{\widehat{\omega}}$ as in Proposition 4 suggests that βU is too large. Resource allocation affects βU in two ways. First, it changes the distribution of borrowers from which unmatched lenders draw. Second, for any given distribution, the allocation is chosen to maximize the value of being unmatched. Therefore, even with an adversely selected distribution, the matching probability may be high enough to reduce the expected duration until a good draw and convince informed lenders to break more matches. Indeed, in the present environment, a high matching probability means that not a lot of screening is taking place so the distributional externality may actually be quite small. Notice the role of the matching versus screening tradeoff here: if screening were costless, then high π would be associated with a dramatic reduction in the quality of available borrowers so, even with a high matching probability, informed lenders may not want to be very selective.

Putting everything together, the negative externality associated with matching has two parts. The first part is a distributional externality which emerges most strongly when π is increased from an initially low value. Increasing π in this case helps match formation more than it hurts type discovery. The resulting boost in informed matching negatively affects the distribution of available borrowers. The second part is an outside option externality which emerges most strongly when π is increased from an initially high value. Increasing π here hurts type discovery more than it helps match formation. Although this weakens the distributional deterioration, a weaker deterioration combined with a higher matching rate increases the relative value of being unmatched and negatively affects informed retention.

Internalizing the effects of resource allocation on both the unmatched and informed problems leads the planner to prescribe $(\hat{\pi}, \underline{\hat{\omega}}) \ll (\pi^*, \underline{\omega}^*)$. In words, the planner devotes more resources to screening new matches but is less restrictive in his cutoff once informed. Effectively then, unmatched lenders are too liberal while informed lenders are too conservative. It is this tension that confounds the ability of r to implement $(\hat{\pi}, \underline{\hat{\omega}})$ as an equilibrium: π^* too high increases r by destroying interbank capital but $\underline{\omega}^*$ too high decreases r by decreasing the demand for such capital so the adjustment mechanism is stunted in either direction.¹⁰ The next proposition summarizes the aggregate implications of Proposition 4:

Proposition 5 If $\mu < 1$, then: (i) uninformed lending is too high; (ii) informed lending is too low; (iii) total lending is too low; (iv) the average default rate is constrained efficient.

The first two parts of Proposition 5 are intuitive given that unmatched lenders overdo matching relative to screening while informed lenders are too selective in the types they retain. The third part then reveals that the composition of informed versus uninformed lending results in an inefficiently small credit market overall. Both screening and total credit being inefficiently low contrasts with Dell'Ariccia and Marquez (2006). Moreover, the fact that my model can deliver these features without assuming bad economic prospects contrasts with Ruckes (2004). The joint determination of matching, screening, and retention thus provides new insight into the relationship between lending standards and lending volumes.

¹⁰Prior to this, we saw two cases where the adjustment mechanism worked: $\mu = 1$ and $\underline{\omega}$ fixed. With $\mu = 1$, there is no distributional externality so the only externality imparted by resource allocation is maximization of the unmatched value. However, just like r adjusts to price the distributional externality when $\underline{\omega}$ is held fixed, r adjusts to price in the direct effect on βU when the distributional externality is absent.

To understand the fourth part of Proposition 5, consider Figure 1 which shows the distribution of financed projects. On one hand, higher screening by the planner means that he allocates a lower fraction of his capital to low quality projects. On the other hand though, lower selectivity means that he ends up dividing the remaining fraction across a wider range of project qualities. Relative to the market then, the planner allocates lower fractions of his capital to both very good and very bad projects. Equation (9) is such that these differences exactly offset, yielding a constant average default rate. I will return to this in Section 4. For now, I just emphasize that the market has the same average default rate as the planner but less capital to lend overall because the decentralized distribution of default rates is a mean-preserving spread of the constrained efficient distribution.

2.6 Corrective Taxation

The results so far suggest a role for corrective taxation. As a benchmark, consider a scheme which taxes lender revenues at a constant rate then remits the proceeds to the interbank market. I refer to this scheme as a remitted revenue tax. Remittance ensures equation (9) is unchanged but, from the perspective of an individual lender, the tax is isomorphic to an increase in δ . Notice, however, that δ dropped out of equation (10). The two equations that define $(\pi^*, \underline{\omega}^*)$ are thus unchanged so the remitted revenue tax is entirely ineffective.

Consider now a scheme which taxes lending intensity π at a constant rate τ then remits the proceeds to the interbank market. I refer to this scheme as a remitted matching tax. Remittance again ensures that equation (9) is unchanged but, from the perspective of an individual lender, the tax subtracts $\tau\pi$ from the maximization problem in equation (2). As shown next, the matching tax succeeds where the revenue tax failed:

Proposition 6 There is a $\tau > 0$ that implements $(\widehat{\pi}, \underline{\widehat{\omega}})$ as an equilibrium.

The comparison between revenue and matching taxes reiterates the nature of the inefficiencies discussed in Subsection 2.5. In particular, addressing the underlying externalities requires

targeting the resource allocation decision: a more general tax is not enough.¹¹

3 Extension: Relationship Lending

By dissolving matches after one project, Section 2 shut out relationship lending – that is, the acquisition of information about a particular borrower over repeated interactions and the use of that information in subsequent financing. An ability to resolve information frictions over time could have important implications for the allocation of resources between matching and screening so it is natural to ask how relationship lending affects the baseline results.

A simple way to incorporate relationship lending is to distinguish between the time to project completion and the time to exogenous match separation. In the baseline model, these two events coincided and were geometrically distributed with parameter $\mu \in (0, 1]$. I now model only the time to exogenous separation as being geometrically distributed: the time to project completion will instead be fixed at one period. A lender who is still matched at the end of date t thus decides whether to finance another project for his borrower at the beginning of date t + 1 (i.e., whether to retain the borrower again). If the date t match was informed, then the lender enters date t+1 still informed. If the date t match was uninformed, then the lender date t. To ensure that the number of available borrowers still equals the number of available lenders without any further changes to the baseline model, assume learning via relationship lending occurs after the matching stage. Also assume no intertemporal commitment to keep the contracting space in line with the baseline model.

Optimization Problems I will use tildes to distinguish the endogenous objects in the relationship lending model from their baseline counterparts. Since informed lenders can now

¹¹In practice, τ could be interpreted as either a direct tax on the number of loans or a corporate governance regulation enforced by bank examiners. Governance regulations are expounded by Kashyap et al (2008) insofar as they do not distort the "search for performance" that allows banks to allocate resources. As I show, however, the decentralized allocation of internal bank resources is itself distorted so there is additional scope for bank examiners to monitor the composition of a bank's workforce and levy costs accordingly.

make multiple retention decisions over the course of one match, I will also define an explicit value function for them. In particular, let $\tilde{J}_I(\omega)$ denote the value of an informed lender who is matched with a type ω borrower. If the lender rejects, then he gets zero in the current period and begins next period unmatched. If the lender accepts (i.e., retains), then his expected payoff in the current period is $\theta (1 - \delta) q(\omega) - (1 + \tilde{r})$ and he faces the same accept/reject choice next period provided the match is not exogenously destroyed. Therefore:

$$\widetilde{J}_{I}(\omega) = \max\left\{\left[\theta\left(1-\delta\right)q\left(\omega\right) - \left(1+\widetilde{r}\right) + \beta\left(1-\mu\right)\widetilde{J}_{I}(\omega) + \beta\mu\widetilde{U}\right], \beta\widetilde{U}\right\}$$
(12)

The value of an unmatched lender, \widetilde{U} , is now given by the following Bellman equation:

$$\widetilde{U} = \max_{\pi \in [0,z]} \left\{ \begin{array}{l} p\left(\pi\right) \left[p\left(z-\pi\right)+\beta\left(1-\mu\right)\left[1-p\left(z-\pi\right)\right]\right] \int_{0}^{1} \widetilde{J}_{I}\left(\omega\right) \widetilde{\psi}\left(\omega\right) d\omega \\ +p\left(\pi\right) \left[1-p\left(z-\pi\right)\right] \int_{0}^{1} \left[\theta\left(1-\delta\right)q\left(\omega\right)-\left(1+\widetilde{r}\right)+\beta\mu\widetilde{U}\right] \widetilde{\psi}\left(\omega\right) d\omega \\ +\left[1-p\left(\pi\right)\right] \beta\widetilde{U} \end{array} \right\}$$
(13)

Retention decisions can again be shown to follow a reservation strategy. In particular, an informed lender will only retain $\omega \in [\underline{\omega}, 1]$, where $\widetilde{J}_I(\underline{\omega}) \equiv \beta \widetilde{U}$. I defer the proof until Proposition 7. What I want to highlight instead is the key difference between equation (13) and the unmatched problem in the baseline model: the weight assigned to becoming an informed lender. An unmatched lender clearly becomes informed when both his matching and screening activities succeed. This is as before and occurs with probability $p(\pi) p(z - \pi)$. Now, however, the lender also has a way of becoming informed when only his matching activity succeeds. This occurs with probability $p(\pi) [1 - p(z - \pi)]$ but is discounted by $\beta (1 - \mu)$ since relationship lending informs him next period provided the match survives. The discount factor β will thus affect an unmatched lender's assessment of how well learning through relationship lending substitutes for screening.

Aggregate Conditions The distribution of available borrowers and the interbank clearing condition are still given by equations (7) and (8) but with the following laws of motion for

uninformed and informed matches respectively:

$$\widetilde{\phi}(\omega) = \left[1 - (1 - \mu)\left[\widetilde{\lambda}(\omega) + \widetilde{\phi}(\omega)\right]\right] p(\pi) \left[1 - p(z - \pi)\right]$$
(14)

$$\mu \widetilde{\lambda} (\omega) = \mathcal{I} (\omega \ge \underline{\omega}) \times \left[(1-\mu) \, \widetilde{\phi} (\omega) + \left[1 - (1-\mu) \left[\widetilde{\lambda} (\omega) + \widetilde{\phi} (\omega) \right] \right] p(\pi) \, p(z-\pi) \right]$$
(15)

The difference between these equations and their baseline counterparts is the flow from uninformed to informed matches. With relationship lending, any uninformed matches not lost to exogenous separation become informed. Comparing to equations (5) and (6), we thus have an extra outflow $(1 - \mu) \tilde{\phi}(\omega)$ in (14) and an extra inflow $(1 - \mu) \tilde{\phi}(\omega)$ in (15). Notice that the extra inflow in (15) is subject to the retention decision $\mathcal{I}(\omega \ge \omega)$. Therefore, for the same values of π and ω , the relationship lending model will have a worse distribution of available borrowers than the baseline model.

Equilibrium The equilibrium can be synthesized as in Subsection 2.3. First, combine interbank clearing with $\tilde{\lambda}(\cdot)$ and $\tilde{\phi}(\cdot)$ to get the relationship lending version of equation (9):

$$\frac{1 - p\left(z - \pi\right)}{1 + \frac{(1 - \mu)\left[1 - p(\pi)p(z - \pi)\right]}{\mu + (1 - \mu)p(\pi)}} = \frac{\int_{\underline{\omega}}^{1} \left[q\left(\omega\right) - \frac{1}{\theta(1 - \delta)}\right] d\omega}{\int_{0}^{\underline{\omega}} \left[\frac{1}{\theta(1 - \delta)} - q\left(\omega\right)\right] d\omega}$$
(16)

Next, solve the optimization problem in equation (13). Combine the resulting first order condition for π with $\widetilde{J}_I(\underline{\omega}) \equiv \beta \widetilde{U}$ and the Bellman equation for $\widetilde{J}_I(\cdot)$. Using $\widetilde{\lambda}(\cdot)$ and $\widetilde{\phi}(\cdot)$ to substitute out $\widetilde{\psi}(\cdot)$ then yields the relationship lending version of equation (10):

$$\frac{1-p\left(z-\pi\right)+\frac{p(\pi)p'(z-\pi)}{p'(\pi)}}{\mu\left(1+\frac{(1-\mu)\left[1-p(\pi)p(z-\pi)\right]}{\mu+(1-\mu)p(\pi)}\right)} = \frac{1}{1-\beta\left(1-\mu\right)} \left(\frac{\int_{\underline{\omega}}^{1}\left[q\left(\omega\right)-q\left(\underline{\omega}\right)\right]d\omega}{\int_{0}^{\underline{\omega}}\left[q\left(\underline{\omega}\right)-q\left(\omega\right)\right]d\omega}\right)$$
(17)

The decentralized equilibrium is a pair $(\tilde{\pi}^*, \underline{\tilde{\omega}}^*)$ that solves equations (16) and (17). The difference between (9) and (16) stems solely from distributional differences (i.e., $\tilde{\lambda}(\cdot) \neq \lambda(\cdot)$ and $\tilde{\phi}(\cdot) \neq \phi(\cdot)$) while the difference between (10) and (17) also involves the discount factor. As we will see next, this introduces a special role for β .

Main Results The efficiency benchmark for the relationship lending model is similar to Subsection 2.4. In particular, consider a social planner who chooses $\pi \in [0, z]$ and $\underline{\omega} \in [0, 1]$ to maximize the total present discounted value of output subject to aggregate feasibility.¹² Using $\tilde{\lambda}(\cdot)$ and $\tilde{\phi}(\cdot)$ instead of $\lambda(\cdot)$ and $\phi(\cdot)$, this maximization yields:

$$\frac{1 - p\left(z - \pi\right) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}}{\mu\left(1 + \frac{(1 - \mu)\left[1 - p(\pi)p(z - \pi)\right]}{\mu + (1 - \mu)p(\pi)}\right)^2} = \frac{\int_{\underline{\omega}}^1 \left[q\left(\omega\right) - q\left(\underline{\omega}\right)\right] d\omega}{\int_0^{\underline{\omega}} \left[q\left(\underline{\omega}\right) - q\left(\omega\right)\right] d\omega}$$
(18)

The constrained efficient allocation is a pair $(\tilde{\widehat{\pi}}, \underline{\widetilde{\omega}})$ that solves equations (16) and (18). Neither equation depends on β so \mathcal{B} in the following proposition is explicitly defined:

Proposition 7 The assumptions in Propositions 2 and 3 suffice for $(\tilde{\pi}^*, \underline{\widetilde{\omega}}^*)$ and $(\tilde{\tilde{\pi}}, \underline{\widetilde{\widetilde{\omega}}})$ to exist uniquely. Invoking these assumptions and defining $\mathcal{B} \equiv \frac{1-p(\tilde{\tilde{\pi}})p(z-\tilde{\tilde{\pi}})}{1+(1-\mu)p(\tilde{\tilde{\pi}})[1-p(z-\tilde{\tilde{\pi}})]} \in (0,1)$:

1. If $\mu = 1$ or $\beta = \mathcal{B}$, then the equilibrium is constrained efficient

2. If
$$\mu < 1$$
 and $\beta < \mathcal{B}$, then $(\tilde{\pi}^*, \underline{\widetilde{\omega}}^*) \ll \left(\widehat{\widetilde{\pi}}, \underline{\widetilde{\widetilde{\omega}}}\right)$ but $\exists \ \widetilde{\tau} < 0$ that implements $\left(\widetilde{\widetilde{\pi}}, \underline{\widetilde{\widetilde{\omega}}}\right)$
3. If $\mu < 1$ and $\beta > \mathcal{B}$, then $(\widetilde{\pi}^*, \underline{\widetilde{\omega}}^*) \gg \left(\widetilde{\widetilde{\pi}}, \underline{\widetilde{\widetilde{\omega}}}\right)$ but $\exists \ \widetilde{\tau} > 0$ that implements $\left(\widetilde{\widetilde{\pi}}, \underline{\widetilde{\widetilde{\omega}}}\right)$

The key takeaway from Proposition 7 is existence of a discount factor that restores constrained efficiency. Recall that relationship lending changes the baseline model in two ways: it provides an alternative to screening and it worsens the distribution of available borrowers. Taken separately, a worse distribution gives unmatched lenders an incentive to choose lower π while the ability to learn without screening gives them an incentive to choose higher π . Equilibrium requires a fixed point where these two incentives exactly offset. By changing an unmatched lender's assessment of how well relationship lending substitutes for screening, β changes where the exact offset occurs. There is one β for which it occurs at the constrained efficient allocation. If β exceeds this critical value, then the inefficiency in Proposition 4 still holds. It can then be verified that the macro implications in Proposition 5 also still hold.¹³

¹²Since successful projects now produce output in one period, the objective function in Subsection 2.4 is technically scaled up by $\frac{1}{\mu}$. However, with μ constant, this has no bearing on the results.

¹³The proof is similar to the proof of Proposition 5 and is thus omitted.

4 Extension: Moral Hazard

The analysis so far has abstracted from moral hazard: lenders were entitled to an exogenous share of project output and borrowers behaved independently of this share. Rejecting borrowers with bad projects was thus the only motive to screen. A moral hazard problem on the borrower side may introduce additional screening motives (i.e., extracting more from borrowers with good projects). Do the core results of Propositions 4 and 5 change in any meaningful way? As I show below, the only qualitative difference with moral hazard is the potential for constrained *in*efficiency of the average default rate. On the quantitative side, there are additional insights about the size of the welfare loss.

4.1 Borrower Problem

Return to the environment of Section 2. A successful project still yields θ upon completion but the success probability now depends on how much effort the borrower exerts when the completion period arrives. In particular, a type ω who exerts effort $e \in [0, 1]$ succeeds with probability $e\alpha(\omega)$, where $\alpha(\cdot) \in [0, 1]$, $\alpha'(\cdot) > 0$, and $\alpha''(\cdot) \le 0$. Success allows ω to consume $\theta - R$, where R is the gross loan rate charged by the lender. However, exerting eimparts a disutility of $-c \ln(1-e)$ on the borrower, where c > 0 is a constant.¹⁴ Matches are dissolved upon project completion so each match is defined by one R. The borrower thus chooses $e \in [0, 1]$ to maximize $e\alpha(\omega) [\theta - R] + c \ln(1-e)$, resulting in the following strategy:

$$e(\omega, R) = \begin{cases} 1 - \frac{c}{\alpha(\omega)[\theta - R]} & if \quad R < \theta - \frac{c}{\alpha(\omega)} \\ 0 & if \quad R \ge \theta - \frac{c}{\alpha(\omega)} \end{cases}$$
(19)

There are two differences relative to Section 2. First, success probabilities are given by $e(\omega, R) \alpha(\omega)$ rather than $q(\omega)$. Second, the lender share of project output is R/θ rather than $1 - \delta$. The rest of the baseline environment stands so the additional insights of the moral hazard model will come entirely from making R an equilibrium object.

¹⁴This functional form rules out the corner choice of e = 1 and thus conserves on algebra.

4.2 Loan Rate Choice

Consider a lender who finances type ω at loan rate R. The lender's expected payoff in the completion period is now $e(\omega, R) \alpha(\omega) R$ instead of $\theta(1 - \delta) q(\omega)$. If informed, the lender can condition his loan rate choice on ω . Otherwise, he can only offer a pooled rate that conditions on the distribution $\psi(\cdot)$. This has two implications. First, the lender's loan rate will depend on the success of his screening activity. Second, informed and uninformed matches are no longer identical for desirable types so loan rates must be chosen subject to a borrower participation constraint. For analytical tractability, I focus on equilibria where such constraints do not bind (i.e., equilibria where the unconstrained loan rates satisfy borrower participation). Loan rate offers are take-it-or-leave-it but equation (19) imposes an important restriction: higher rates induce lower repayment probabilities so lenders must share surplus.

Proposition 8 Suppose $c \leq \frac{\theta \alpha(0)^2}{\alpha(1)}$. Any symmetric steady state with non-binding borrower participation constraints takes the form $(\pi, \underline{\omega})$. Informed lenders charge $R(\omega) = \theta - \sqrt{\frac{c\theta}{\alpha(\omega)}}$ and uninformed lenders charge $\overline{R}(\pi, \underline{\omega}) = \theta - \sqrt{\frac{c\theta}{\overline{\alpha}(\pi, \underline{\omega})}}$, where $\overline{\alpha}(\pi, \underline{\omega}) \equiv \frac{\int_0^1 \alpha(\omega) d\omega + \sigma(\pi) \int_0^{\underline{\omega}} \alpha(\omega) d\omega}{1 + \sigma(\pi) \underline{\omega}}$ and $\sigma(\pi) \equiv \frac{(1-\mu)p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)[1-p(z-\pi)]}$. Moreover, all financed borrowers exert positive effort.

To make the notation more compact, combine Proposition 8 with the expected payoff $e(\omega, R) \alpha(\omega) R$. In particular, define:

$$g(\omega) \equiv e(\omega, R(\omega)) \alpha(\omega) R(\omega) = \left(\sqrt{\theta \alpha(\omega)} - \sqrt{c}\right)^{2}$$
$$h(\omega, \overline{R}(\cdot)) \equiv e(\omega, \overline{R}(\cdot)) \alpha(\omega) \overline{R}(\cdot) = \left(\alpha(\omega) - \frac{c}{\theta - \overline{R}(\cdot)}\right) \overline{R}(\cdot)$$

Also define the difference $D(\omega, \overline{R}(\cdot)) \equiv g(\omega) - h(\omega, \overline{R}(\cdot))$. An informed lender can always charge $\overline{R}(\cdot)$ instead of $R(\omega)$ so $D(\omega, \overline{R}(\cdot)) \geq 0$ with strict inequality for at least some ω .

4.3 Incorporating into Baseline Environment

We can now derive the moral hazard versions of equations (9) and (10) that would prevail under Proposition 8. Begin with equation (9) which captured equilibrium in the interbank market. The capital needed to finance new loans is still $\mu \int_0^1 [\lambda(\omega) + \phi(\omega)] d\omega$ but the capital available from maturing loans is now $\mu \left[\int_0^1 g(\omega) \lambda(\omega) d\omega + \int_0^1 h(\omega, \overline{R}(\cdot)) \phi(\omega) d\omega \right]$ so the moral hazard version of equation (9) becomes:

$$\frac{1-p\left(z-\pi\right)}{1-\frac{(1-\mu)p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)}} = \frac{\int_{\underline{\omega}}^{1} \left[g\left(\omega\right)-1\right] d\omega}{\int_{0}^{\underline{\omega}} \left[1-g\left(\omega\right)\right] d\omega + \Delta\left(\pi,\underline{\omega}\right)}$$
(20)

where

$$\Delta\left(\pi,\underline{\omega}\right) \equiv \int_{0}^{\underline{\omega}} D\left(\omega,\overline{R}\left(\pi,\underline{\omega}\right)\right) d\omega + \left(1 - \frac{(1-\mu)p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)}\right) \int_{\underline{\omega}}^{1} D\left(\omega,\overline{R}\left(\pi,\underline{\omega}\right)\right) d\omega$$

Turn now to equation (10) which captured optimality of π . Using the expected payoffs under moral hazard, the lender value functions become:

$$U = \max_{\pi \in [0,z]} \left\{ \begin{array}{l} p(\pi) p(z-\pi) \int_0^1 \max\left\{ J_I(\omega) - (1+r) - \beta U, 0 \right\} \psi(\omega) \, d\omega \\ + p(\pi) \left[1 - p(z-\pi) \right] \int_0^1 \left[J_N(\omega) - (1+r) - \beta U \right] \psi(\omega) \, d\omega + \beta U \end{array} \right\}$$

where
$$J_I(\omega) = \mu \left[g(\omega) + \beta U \right] + (1-\mu) \beta J_I(\omega)$$

$$J_{N}(\omega) = \mu \left[h\left(\omega, \overline{R}(\cdot)\right) + \beta U \right] + (1-\mu) \beta J_{N}(\omega)$$

The informed cutoff satisfies $J_I(\underline{\omega}) \equiv \beta U + (1+r)$ and the computation of $\psi(\cdot)$ proceeds as before so the moral hazard version of equation (10) amounts to:

$$\frac{1 - p\left(z - \pi\right) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}}{1 - \frac{(1 - \mu)p(\pi)p(z - \pi)}{\mu + (1 - \mu)p(\pi)}} = \frac{\int_{\underline{\omega}}^{1} \left[g\left(\omega\right) - g\left(\underline{\omega}\right)\right] d\omega}{\int_{0}^{\underline{\omega}} \left[g\left(\underline{\omega}\right) - g\left(\omega\right)\right] d\omega + \Delta\left(\pi, \underline{\omega}\right)}$$
(21)

The decentralized equilibrium is now a pair $(\pi^*, \underline{\omega}^*)$ that solves equations (20) and (21) and is internally consistent with Proposition 8. Notice that imposing $g(\cdot) = h(\cdot) = \theta(1 - \delta) q(\cdot)$ on equations (20) and (21) recovers their baseline counterparts.

Proposition 9 Define $\overline{a} \equiv \alpha(1) - \sqrt{\frac{\alpha(1)}{\theta}}$ and assume the following:

Assumption 4
$$\frac{p'(z-\pi)}{1-p(z-\pi)} \in \left[-\frac{1}{2}\left(\frac{p''(\pi)}{p'(\pi)} + \frac{p''(z-\pi)}{p'(z-\pi)}\right), -\left(\frac{p''(\pi)}{p'(\pi)} + \frac{p''(z-\pi)}{p'(z-\pi)}\right)\right]$$
 for any $\pi \in (0, z)$

There exist scalars $\underline{a} \in (0, \overline{a}), \ \overline{p} \in (0, 1), \ \underline{p} \in (0, \overline{p}), \ \overline{c} \in \left(0, \frac{\theta \alpha(0)^2}{\alpha(1)}\right), \ and \ \underline{c} \in [0, \overline{c}) \ such that$ $<math>\alpha(0) \in (\underline{a}, \overline{a}), \ p(z) \in [\underline{p}, \overline{p}], \ and \ c \in [\underline{c}, \overline{c}] \ are \ sufficient \ for \ a \ unique \ symmetric \ steady \ state \ with \ non-binding \ borrower \ participation \ constraints.$

Assumption 4 imposes somewhat stricter conditions on the curvature of $p(\cdot)$ than Assumption 3. The parameter bounds are also somewhat stricter than simple analogs of the remaining baseline assumptions. These additional restrictions help make tractable the complexity that stems from interactions between π , $\underline{\omega}$, and $\overline{R}(\cdot)$ under moral hazard.

4.4 Comparison to Baseline Results

To proceed, we need an efficiency benchmark for the moral hazard model. Along with choosing $\pi \in [0, z]$ and $\underline{\omega} \in [0, 1]$, the planner must now divide output between consumption and capital. In particular, if he allocates one unit of capital to firm ω when informed and ω succeeds in producing θ , then $\kappa(\omega)$ is returned to the planner as capital and $\theta - \kappa(\omega)$ is consumed by the firm. A similar statement applies when the planner is uninformed but with $\overline{\kappa}$ in place of $\kappa(\omega)$. I begin by considering a planner that maximizes the total present discounted value of capital subject to aggregate feasibility. A more conventional welfare measure is taken up in Subsection 4.5 but capital maximization provides a useful starting point. To see why, recall that lenders are the only source of saving in the decentralized economy so loan rates ultimately dictate the decentralized division of output between consumption and capital (i.e., $R(\omega)$ and R play the same role as $\kappa(\omega)$ and $\overline{\kappa}$). Since lenders choose this division to maximize capital, they will automatically be inefficient if judged against a planner who maximizes net output. Notice that this inefficiency stems from borrowers not being able to store and arises even if π and $\underline{\omega}$ are treated as model parameters. Considering a capital-maximizing planner is one way to generate a constrained efficiency benchmark that is also constrained by the storage assumption. The derivation appears in the appendix and it produces the following moral hazard version of equation (11):

$$\frac{1 - p\left(z - \pi\right) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}}{\left(1 - \frac{(1 - \mu)p(\pi)p(z - \pi)}{\mu + (1 - \mu)p(\pi)}\right)^2} = \frac{\int_{\omega}^1 \left[g\left(\omega\right) - g\left(\underline{\omega}\right)\right] d\omega - \frac{(1 - \mu)p(\pi)[1 - p(z - \pi)]}{\mu}Q\left(\pi, \underline{\omega}\right)}{\int_0^{\omega} \left[g\left(\underline{\omega}\right) - g\left(\omega\right)\right] d\omega + \widehat{\Delta}\left(\pi, \underline{\omega}\right)}$$
(22)

$$where$$

$$\widehat{\Delta}\left(\pi,\underline{\omega}\right) \equiv \int_{0}^{\underline{\omega}} D\left(\omega,\overline{R}\left(\pi,\underline{\omega}\right)\right) d\omega + \left(1 - \frac{(1-\mu)p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)}\right)^{2} \int_{\underline{\omega}}^{1} D\left(\omega,\overline{R}\left(\pi,\underline{\omega}\right)\right) d\omega$$

$$Q\left(\pi,\underline{\omega}\right) \equiv \left(1 - \underline{\omega} + \frac{1 - p(z-\pi) + \frac{p(\pi)p'(z-\pi)}{p'(\pi)}}{\left(1 - \frac{(1-\mu)p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)}\right)^{2}}\underline{\omega}\right) D\left(\underline{\omega},\overline{R}\left(\pi,\underline{\omega}\right)\right) + \frac{p(\pi)p'(z-\pi)}{p'(\pi)[1-p(z-\pi)]} \int_{\underline{\omega}}^{1} D\left(\omega,\overline{R}\left(\pi,\underline{\omega}\right)\right) d\omega$$

The constrained efficient allocation is a pair $(\hat{\pi}, \underline{\hat{\omega}})$ that solves equations (20) and (22). Imposing $g(\cdot) = h(\cdot) = \theta(1 - \delta) q(\cdot)$ on equation (22) again recovers the baseline counterpart.

Proposition 10 For $\mu < 1$, the steady state of Proposition 9 is such that $(\pi^*, \underline{\omega}^*) \gg (\widehat{\pi}, \underline{\widehat{\omega}})$ but there exists a $\tau > 0$ that implements $(\widehat{\pi}, \underline{\widehat{\omega}})$.

Proposition 10 confirms the core inefficiency result in Proposition 4. Since the laws of motion for $\lambda(\cdot)$ and $\phi(\cdot)$ are also unchanged from the baseline model, it is trivial to show that the first three parts of Proposition 5 continue to hold. In particular, the moral hazard model still predicts too much uninformed credit, too little informed credit, and too little total credit. The default prediction, however, is now ambiguous. A maturing project financed at loan rate R defaults with probability $1 - e(\omega, R) \alpha(\omega)$ so the economy's average default rate is:

$$\mu \left[1 - \int_0^1 e\left(\omega, R\left(\omega\right)\right) \alpha\left(\omega\right) \frac{\lambda(\omega)}{\int_0^1 [\lambda(x) + \phi(x)] dx} d\omega - \int_0^1 e\left(\omega, \overline{R}\left(\cdot\right)\right) \alpha\left(\omega\right) \frac{\phi(\omega)}{\int_0^1 [\lambda(x) + \phi(x)] dx} d\omega \right]$$

This can be rewritten as $\mu \left[1 - \frac{\int_0^1 e(\omega, R(\omega))\alpha(\omega)\lambda(\omega)d\omega + \int_0^1 e(\omega, \overline{R}(\cdot))\alpha(\omega)\phi(\omega)d\omega}{\int_0^1 e(x, R(x))\alpha(x)R(x)\lambda(x)d\omega + \int_0^1 e(x, \overline{R}(\cdot))\alpha(x)\overline{R}(\cdot)\phi(x)dx} \right]$ since interbank clearing equates $\mu \int_0^1 [\lambda(x) + \phi(x)] dx$ to the capital available from maturing loans. In the baseline model, $R(\omega) = \overline{R}(\cdot) = \theta(1-\delta)$ so interbank clearing was enough to make the average default rate a constant. This is no longer the case in the model hazard model. Therefore, the inefficiency result of Proposition 10 will also be transmitted to defaults.

4.5 Comparative Statics

The primitives are the lender resource constraint z, the project completion rate μ , the output parameter θ , the borrower disutility parameter c, the intermediation technologies $p(\cdot)$, and the exogenous component of borrower success rates $\alpha(\cdot)$. For the rest of this section, I normalize z = 1 and adopt $p(x) = 1 - \exp(-\eta x)$ with $\eta > 0$. I also adopt $\alpha(\omega) = \alpha(0) + [1 - \alpha(0)]\omega$. Higher $\alpha(0)$, higher θ , and lower c all have similar qualitative effects on the key equations of the moral hazard model so I will focus on θ , η , and μ .

As a benchmark, set $\theta = 3$, $\eta = 4$, and $\mu = \frac{1}{5}$ with $\alpha(0) = \frac{3}{10}$ and $c = \frac{1}{4}$.¹⁵ Figure 2 shows how the percent welfare loss in the moral hazard equilibrium varies with parameters. I measure welfare as the total present discounted value of net output. Given R, the net output of a (matched) type ω is $e(\omega, R) \alpha(\omega) \theta + c \ln(1 - e(\omega, R))$. Red lines calculate losses relative to the welfare achieved by the capital-maximizing planner discussed above (K-MAX) while blue lines calculate losses relative to a planner who maximizes the net output metric subject to aggregate feasibility (W-MAX). The loss relative to W-MAX is always larger because, as discussed in Subsection 4.4, the W-MAX problem implicitly assumes everyone can store. For the benchmark parameters, the loss imparted by only the resource allocation externalities (i.e., the loss relative to K-MAX) is non-trivial: it amounts to 2.10% of net output.

All else constant, Figure 2 suggests that the percent loss is bigger for low θ , low μ , and moderate η . Low θ scales down the welfare function without changing the nature of the externalities while low μ allows resource allocations to have a more persistent effect on the distribution of available borrowers. To understand the result on η , consider the bindingness of

¹⁵Interpret a model period as a year. The minimum maturity of a loan is thus one year and, with $\mu = 1/5$, the average maturity is 5 years. This is consistent with FRED data: conditional on lasting at least one year, the weighted-average maturity of business loans made by domestic banks averaged 4.7 years prior to the crisis. The choice of $\eta = 4$ implies p(1) = 0.98 and p(0.5) = 0.86. In other words, η is high enough that one activity is extremely likely to succeed if all resources are devoted to it but not so high that both activities are extremely likely to succeed if resources are split between them. The remaining parameters are then set to get a reasonable productivity distribution among financed borrowers. One unit of capital to type ω at loan rate R yields expected gross output of $e(\omega, R) \alpha(\omega) \theta$ so I use $\ln(e(\omega, R) \alpha(\omega) \theta)$ as the productivity metric. Under the benchmark parameters, the 90-10 percentile range is 0.99, the interquartile range is 0.38, and the standard deviation is 0.43. Using $\ln(TFP)$, Syverson (2004) finds an average 90-10 percentile range of 0.99, an average interquartile range of 0.45, and a standard deviation of industry medians of 0.37.

the lender resource constraint. Low η (relative to z) implies a tight constraint: an unmatched lender does not have enough resources to make even one intermediation activity succeed with high probability. The fraction of matches that are informed is thus small, making the distributional externality small as well. The inability to rematch at a high rate then helps disincentivize informed lenders from breaking more matches in response to the smaller distributional deterioration. Low values of η thus dampen the externalities discussed in Subsection 2.5 and result in a smaller welfare loss. Increasing η relaxes the bindingness of the resource constraint and initially increases the welfare loss. However, as η continues to increase, it becomes possible to make both matching and screening succeed with probabilities very close to one. In other words, the constraint all but disappears, resource allocation becomes moot, and the welfare loss relative to K-MAX tends to zero.

Figure 2 also plots the welfare losses predicted by the baseline model when $q(\omega) = \frac{g(\omega)}{\theta(1-\delta)}$. Recall that the moral hazard model distinguishes between an expected payoff of $g(\omega)$ for informed lenders and an expected payoff of $h(\omega, \overline{R}(\pi, \underline{\omega}))$ for uninformed lenders. Under the posited $q(\omega)$, the baseline model uses $g(\omega)$ for both. Figure 2 shows that the welfare loss is almost always smaller in the baseline: not having the externalities feed through uninformed loan rates attenuates the loss.

As a final exercise, Figure 3 explores whether using W-MAX rather than K-MAX as the efficiency benchmark for the moral hazard model changes any of the qualitative results. Proposition 10 established $(\pi^*, \underline{\omega}^*) \gg (\widehat{\pi}, \widehat{\underline{\omega}})$ when $(\widehat{\pi}, \widehat{\underline{\omega}})$ is chosen by a capital-maximizing planner. The first column of Figure 3 reveals that the same result prevails under a welfaremaximizing planner provided μ and η are not too high. The externalities imparted by individual resource allocation decisions die out as μ and η get very large, leaving only the failure of lenders to internalize their monopoly on storage. We can thus interpret Figure 3 as saying that W-MAX also predicts $(\pi^*, \underline{\omega}^*) \gg (\widehat{\pi}, \widehat{\underline{\omega}})$ for parameters where the externalities I am concerned with are strongest. Turn next to the size of the overall credit market. As discussed in Subsection 4.4, the decentralized market is too small relative to K-MAX. Since a welfare-maximizing planner cares about both capital and consumption, he should also settle on less capital than the capital-maximizing planner. However, for parameters where my externalities are sufficiently strong, the second column of Figure 3 reveals that the decentralized market is too small even when judged against W-MAX. The third column then shows that the W-MAX comparison yields an unambiguous prediction about the average default rate: it is always too high in the decentralized market. Relative to K-MAX though, the decentralized rate could be either too high or too low. The orders of magnitude are small but it is generally too high for parameters where the externalities I am concerned with are strongest.

5 Conclusion

This paper has examined the allocation of bank resources across two important intermediation activities: creating credit market matches and screening the borrowers in those matches. I began by constructing a model to disentangle the implications of this allocation decision in an environment with private information, many lenders, and a Walrasian interbank market. I then showed that the privately optimal allocation of resources is constrained inefficient when the cutoff between desirable and undesirable borrowers is endogenous. More precisely, too much emphasis is placed on getting rather than vetting borrowers but, once properly vetted, not enough matches are actually retained. The direction of inefficiency reflects both a distributional externality and an outside option externality. From a policy perspective, the results contribute to the current debate on bank taxes. In particular, the inefficiencies identified by my model suggest that taxing and/or regulating matching activities would be more effective than imposing a general profit tax. Extending the model to evaluate different implementations of this matching tax is therefore an interesting direction for future research.

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Figure 2: Welfare Loss (%)





Figure 3: Moral Hazard Model

Appendix - Proofs

Proof of Proposition 1

 ℓ

Define $S \equiv \{r, \psi\}$ and write U(S) to make explicit that S is taken as given. Objects of the form $max\{x_1, x_2\}$ can be expressed as $\max_{a \in [0,1]} \{ax_1 + (1-a)x_2\}$ so equations (1) and (2) define the following mapping:

$$\mathcal{T}U\left(S\right) = \max_{\pi \in [0,z], \ a(\cdot) \in [0,1]} \left\{ \ell\left(\pi, a\left(\cdot\right), S\right) + y\left(\pi, a\left(\cdot\right), S\right) \beta U\left(S\right) \right\} \\ where$$
$$(\pi, a\left(\cdot\right), S) \equiv \int_{0}^{1} p\left(\pi\right) \left[1 - \left[1 - a\left(\omega\right)\right] p\left(z - \pi\right)\right] \left[\frac{\mu \theta(1 - \delta)q(\omega)}{1 - \beta(1 - \mu)} - (1 + r)\right] \psi\left(\omega\right) d\omega$$
$$y\left(\pi, a\left(\cdot\right), S\right) \equiv 1 - p\left(\pi\right) \left[1 - \left[1 - \int_{0}^{1} a\left(\omega\right) \psi\left(\omega\right) d\omega\right] p\left(z - \pi\right)\right] \frac{(1 - \beta)(1 - \mu)}{1 - \beta(1 - \mu)}$$

If U exists in the set of bounded and continuous functions (C), then the Theorem of the Maximum implies $\mathcal{T}U(S) \in C$. Therefore, $\mathcal{T} : C \to C$. The following shows that \mathcal{T} satisfies Blackwell's sufficient conditions for a contraction:

• (discounting) Consider $U_1 \in \mathcal{C}$ and a constant k > 0:

$$\mathcal{T} (U_{1} + k) = \max_{\pi \in [0,z], a(\cdot) \in [0,1]} \{ \ell (\pi, a (\cdot), S) + y (\pi, a (\cdot), S) \beta (U_{1} + k) \}$$

$$= \max_{\pi \in [0,z], a(\cdot) \in [0,1]} \{ \ell (\pi, a (\cdot), S) + y (\pi, a (\cdot), S) \beta U_{1} + y (\pi, a (\cdot), S) \beta k \}$$

$$\le \max_{\pi \in [0,z], a(\cdot) \in [0,1]} \{ \ell (\pi, a (\cdot), S) + y (\pi, a (\cdot), S) \beta U_{1} \} + \beta k = \mathcal{T} U_{1} + \beta k$$

The inequality stems from $y(\pi, a(\cdot), S) \in (0, 1]$ for all $\pi \in [0, z]$ and $a(\cdot) \in [0, 1]$.

• (monotonicity) Consider $U_1, U_2 \in \mathcal{C}$ where $U_1(\cdot) \geq U_2(\cdot)$ and let $\{\pi_i^*, a_i^*(\cdot)\}$ denote the optimal choices under U_i :

$$\begin{aligned} \mathcal{T}U_{1}\left(S\right) &= \ell\left(\pi_{1}^{*}, a_{1}^{*}\left(\cdot\right), S\right) + y\left(\pi_{1}^{*}, a_{1}^{*}\left(\cdot\right), S\right) \beta U_{1}\left(S\right) \\ &\geq \ell\left(\pi_{2}^{*}, a_{2}^{*}\left(\cdot\right), S\right) + y\left(\pi_{2}^{*}, a_{2}^{*}\left(\cdot\right), S\right) \beta U_{1}\left(S\right) \\ &\geq \ell\left(\pi_{2}^{*}, a_{2}^{*}\left(\cdot\right), S\right) + y\left(\pi_{2}^{*}, a_{2}^{*}\left(\cdot\right), S\right) \beta U_{2}\left(S\right) = \mathcal{T}U_{2}\left(S\right) \end{aligned}$$

The first inequality follows from the fact that $\{\pi_2^*, a_2^*(\cdot)\}$ is feasible under U_1 . The second inequality follows from $y(\pi, a(\cdot), S) > 0$ for all $\pi \in [0, z]$ and $a(\cdot) \in [0, 1]$ along with the assumption that $U_1(\cdot) \geq U_2(\cdot)$.

We can now invoke the Contraction Mapping Theorem to establish existence of a unique $U \in \mathcal{C}$ such that $U = \mathcal{T}U$.

Proof of Proposition 2

Let $\pi_k(\underline{\omega})$ and $\pi_l(\underline{\omega})$ denote the functions implicitly defined by equations (9) and (10) respectively. Proving existence of equilibrium thus requires proving existence of a pair $(\pi^*, \underline{\omega}^*)$ such that $\pi^* = \pi_l(\underline{\omega}^*) = \pi_k(\underline{\omega}^*)$.

Begin with equation (9). Define ξ such that $q(\xi) \equiv \frac{1}{\theta(1-\delta)}$. Assumption 1 and $q'(\cdot) > 0$ imply $q(1) > \frac{1}{\theta(1-\delta)} > q(0)$ so ξ exists uniquely and is interior. Differentiating (9) reveals $\pi'_k(\xi) = 0$ and $\pi'_k(\underline{\omega}) [q(\underline{\omega}) - q(\xi)] < 0$ for $\underline{\omega} \neq \xi$. A necessary condition for $\pi^* > 0$ is $\pi_k(\xi) > 0$. To ensure $\pi_k(\xi) > 0$, we need $p(z) > \frac{\frac{1}{\theta(1-\delta)} - \int_0^1 q(\underline{\omega})d\omega}{\int_0^{\xi} [\frac{1}{\theta(1-\delta)} - q(\underline{\omega})]d\omega}$ which is the essence of Assumption 2. We can now conclude existence of unique points $\xi_{k,1} \in (0,\xi)$ and $\xi_{k,2} \in (\xi,1)$ satisfying $\pi_k(\xi_{k,1}) \equiv 0$ and $\pi_k(\xi_{k,2}) \equiv 0$. The restriction to $\xi_{k,1} > 0$ and $\xi_{k,2} < 1$ reflects the fact that $\pi_k(\cdot)$ is not defined at $\underline{\omega} = 0$ or $\underline{\omega} = 1$ under Assumption 1 and p(z) < 1.

Turn to equation (10). When evaluated at $\underline{\omega} = \xi$, the right-hand sides of (9) and (10) are the same so Assumption 2 also ensures $\pi_l(\xi) > 0$. We can then show $\pi_l(\xi) < \pi_k(\xi)$ in two steps. First, the left-hand side of (9) is increasing in π . Second, the left-hand side of (10) equals the left-hand side of (9) plus a function of π . This function is zero if $\pi = 0$ and positive otherwise. Therefore, $\pi_l(\xi) \ge \pi_k(\xi)$ is impossible. The following lemma completes the existence proof by finding a point $\underline{\omega} \in [\xi_{k,1}, \xi)$ such that $\pi_l(\underline{\omega}) > \pi_k(\underline{\omega})$:

Lemma 1 If $\pi_l(\xi_{k,1})$ exists, then $\pi_l(\xi_{k,1}) > \pi_k(\xi_{k,1})$. If $\pi_l(\xi_{k,1})$ does not exist, then there is a point $\underline{\omega}_z \in (\xi_{k,1}, \xi)$ such that $\pi_l(\underline{\omega}_z) = z > \pi_k(\underline{\omega}_z)$.

Proof. Equation (9) and the definition of $\xi_{k,1}$ yield:

$$1 - p\left(z\right) = \frac{\int_{\xi_{k,1}}^{1} \left[q(\omega) - \frac{1}{\theta\left(1 - \delta\right)}\right] d\omega}{\int_{0}^{\xi_{k,1}} \left[\frac{1}{\theta\left(1 - \delta\right)} - q(\omega)\right] d\omega} < \frac{\int_{\xi_{k,1}}^{1} \left[q(\omega) - q\left(\xi_{k,1}\right)\right] d\omega}{\int_{0}^{\xi_{k,1}} \left[q\left(\xi_{k,1}\right) - q(\omega)\right] d\omega}$$

where the inequality follows from $q(\xi_{k,1}) < q(\xi) \equiv \frac{1}{\theta(1-\delta)}$. Return to equation (10). If $\pi_l(\xi_{k,1})$ exists, then the above inequality implies $\pi_l(\xi_{k,1}) > 0 \equiv \pi_k(\xi_{k,1})$. If $\pi_l(\xi_{k,1})$ does not exist, then it must be the case that:

$$\frac{\int_{\xi_{k,1}}^{1} \left[q(\omega) - q(\xi_{k,1}) \right] d\omega}{\int_{0}^{\xi_{k,1}} \left[q(\xi_{k,1}) - q(\omega) \right] d\omega} > 1 + \frac{p(z)p'(0)}{p'(z)}$$

With $\frac{d}{dx} \left(\frac{\int_x^1 [q(\omega) - q(x)] d\omega}{\int_0^x [q(x) - q(\omega)] d\omega} \right) < 0$, we can thus look for a point $\underline{\omega}_z > \xi_{k,1}$ satisfying $\pi_l(\underline{\omega}_z) = z$. Substituting $\pi = z$ into equation (9) returns $\int_0^1 q(\omega) d\omega = \frac{1}{\theta(1-\delta)}$. This violates Assumption 1 so we can conclude $\pi_k(\cdot) < z$ and thus $\pi_l(\xi) < z$. Therefore, if $\pi_l(\xi_{k,1})$ does not exist, there is a point $\underline{\omega}_z \in (\xi_{k,1}, \xi)$ such that $\pi_l(\underline{\omega}_z) = z > \pi_k(\underline{\omega}_z)$. \Box We have now shown existence of an equilibrium $(\pi^*, \underline{\omega}^*)$ with $\underline{\omega}^* \in (\xi_{k,1}, \xi) \subset (0, 1)$ and $\pi^* = \pi_k(\underline{\omega}^*) \in (0, z)$. Consider next uniqueness. The left-hand side of equation (10) can be written as $\frac{A(\pi)}{B(\pi)}$, where $A(\pi) \equiv 1 - p(z - \pi) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}$ and $B(\pi) \equiv 1 - \frac{(1 - \mu)p(\pi)p(z - \pi)}{\mu + (1 - \mu)p(\pi)}$. Assumption 3 is sufficient for $\frac{d}{d\pi} \frac{A(\pi)}{B(\pi)} > 0$. More precisely, $\frac{d}{d\pi} \frac{A(\pi)}{B(\pi)} > 0$ amounts to:

$$\left[2 - \frac{p(\pi)}{p'(\pi)} \left(\frac{p''(z-\pi)}{p'(z-\pi)} + \frac{p''(\pi)}{p'(\pi)}\right)\right] \frac{p'(z-\pi)}{p(z-\pi)} > A(\pi) \left[\frac{p'(z-\pi)}{p(z-\pi)} - \frac{\mu}{\mu + (1-\mu)p(\pi)} \frac{p'(\pi)}{p(\pi)}\right] \frac{(1-\mu)p(\pi)}{\mu + (1-\mu)p(\pi)[1-p(z-\pi)]}$$

The right-hand side of the above inequality is maximized at $\mu = 0$ so a sufficient condition can be found by imposing $\mu = 0$ and rearranging to get Assumption 3. It is straightforward to see that the right-hand side of (10) is decreasing in $\underline{\omega}$ so Assumption 3 guarantees $\pi'_l(\cdot) < 0$. We already know $\pi'_k(\underline{\omega}) > 0$ for any $\underline{\omega} \in (\xi_{k,1}, \xi)$ so, to conclude uniqueness, we just need to show that all equilibria satisfy $\underline{\omega}^* \in (\xi_{k,1}, \xi)$. We can do this by rearranging equations (9) and (10) to isolate $\int_{\omega^*}^{1} q(\omega) d\omega$ in each then equating to get:

$$\frac{1}{\theta(1-\delta)} - q\left(\underline{\omega}^*\right) = \frac{p(\pi^*)p'(z-\pi^*)}{p'(\pi^*)} \frac{[\mu+(1-\mu)p(\pi^*)]\int_0^{\underline{\omega}^*} [q(\underline{\omega}^*)-q(\omega)]d\omega}{\mu[1-\underline{\omega}^*p(z-\pi^*)]+(1-\mu)p(\pi^*)[1-p(z-\pi^*)]} > 0$$
(23)

Invoking $q(\xi) \equiv \frac{1}{\theta(1-\delta)}$ and $q'(\cdot) > 0$ establishes the desired result.

Proof of Proposition 3

To show uniqueness of the constrained efficient allocation $(\widehat{\pi}, \underline{\widehat{\omega}})$, use the planner's objective function $\mathcal{W} \equiv \frac{\mu\theta}{1-\beta} \int_0^1 q(\omega) [\lambda(\omega) + \phi(\omega)] d\omega$. Any $(\widehat{\pi}, \underline{\widehat{\omega}})$ must also satisfy equation (9) so consider \mathcal{W} when π is evaluated at $\pi_k(\underline{\omega})$. This yields:

$$\mathcal{W}(\underline{\omega}) = \frac{\mu}{(1-\beta)(1-\delta)} \frac{p(\pi_k(\underline{\omega}))}{\mu + (1-\mu)p(\pi_k(\underline{\omega}))} \left[1 - \frac{\frac{1}{\theta(1-\delta)} - \int_0^1 q(\omega)d\omega}{\frac{1}{\theta(1-\delta)} - \frac{1}{\underline{\omega}}\int_0^{\omega} q(\omega)d\omega} \right]$$

By definition, $\mathcal{W}'(\widehat{\omega}) = 0$. It will suffice to establish $\mathcal{W}''(\widehat{\omega}) < 0$. After some algebra, we find that we can write:

$$\mathcal{W}''(\widehat{\underline{\omega}}) \propto \Upsilon_1(\pi_k(\widehat{\underline{\omega}})) \pi'_k(\widehat{\underline{\omega}}) - \Upsilon_2(\widehat{\underline{\omega}})$$

The function $\Upsilon_1(\cdot)$ is such that $\mu > \frac{p(z/2)^2}{5-p(z)+p(z/2)^2}$ ensures $\Upsilon_1(\cdot) < 0$. The function $\Upsilon_2(\cdot)$ is such that $\underline{\omega}q(\underline{\omega}) \leq \int_0^{\underline{\omega}} q(\omega) d\omega + q'(\underline{\omega}) \underline{\omega}^2$ ensures $\Upsilon_2(\cdot) > 0$. Using integration by parts, the condition for $\Upsilon_2(\cdot) > 0$ reduces to $q'(\underline{\omega}) \underline{\omega} \geq \frac{1}{\underline{\omega}} \int_0^{\underline{\omega}} q'(\omega) \omega d\omega$. This is certainly true if $q'(\omega) \omega$ is non-decreasing in ω or, equivalently, if $q(\cdot)$ is not too concave. Therefore, μ and $q''(\cdot)$ not too low are sufficient for $\Upsilon_1(\cdot) < 0$ and $\Upsilon_2(\cdot) > 0$. The last step is to show $\pi'_k(\underline{\hat{\omega}}) > 0$. The planner's first order condition for $\underline{\omega}$ is $q(\underline{\hat{\omega}}) = \frac{\gamma}{\frac{\mu\theta}{1-\theta}+\gamma\theta(1-\delta)}$, where $\gamma > 0$ is the Lagrange

multiplier on equation (9). With $\frac{\mu\theta}{1-\beta} > 0$, this implies $q(\widehat{\underline{\omega}}) < \frac{1}{\theta(1-\delta)}$ and thus $\pi'_k(\widehat{\underline{\omega}}) > 0$.

Proof of Proposition 4

The decentralized equilibrium is defined by equations (9) and (10). The constrained efficient allocation is defined by equations (9) and (11). With $\mu = 1$, equations (10) and (11) are identical so the decentralized equilibrium is constrained efficient.

Turn now to $\mu < 1$. As in the proof of Proposition 2, use $\pi_k(\underline{\omega})$ and $\pi_l(\underline{\omega})$ to denote the functions implicitly defined by equations (9) and (10) respectively. The decentralized market achieves $(\pi^*, \underline{\omega}^*)$ such that $\pi^* = \pi_l(\underline{\omega}^*) = \pi_k(\underline{\omega}^*)$. Letting $\pi_e(\underline{\omega})$ denote the function implicitly defined by equation (11), the planner achieves $(\widehat{\pi}, \widehat{\underline{\omega}})$ such that $\widehat{\pi} = \pi_e(\widehat{\underline{\omega}}) = \pi_k(\widehat{\underline{\omega}})$. We already know $\pi'_k(\widehat{\underline{\omega}}) > 0$ and $\pi'_k(\underline{\omega}^*) > 0$ from the proofs of Propositions 2 and 3. Note that $\pi'_k(\underline{\omega}^*) > 0$ was established without reference to Assumption 3 and is thus true for any equilibrium. The direction of inefficiency can now be established in three steps.

First, there is no $\underline{\omega} \in (0,1)$ such that $\pi_l(\underline{\omega}) = \pi_e(\underline{\omega}) \in (0,z)$. This can be seen by comparing equations (10) and (11): the right-hand sides are equal when evaluated at the same $\underline{\omega}$ but, unless $\pi = 0$ or $\pi = z$, the left-hand side of (11) is always bigger when evaluated at the same π . Second, there is a unique $\underline{\omega}_l \in (\xi, \xi_{k,2})$ such that $\pi_l(\underline{\omega}_l) = \pi_e(\underline{\omega}_l) = 0$. This can be seen as follows. The right-hand sides of (10) and (11) are decreasing in $\underline{\omega}$ so, with $p(z) \in (0, 1)$, there are unique values $\underline{\omega}_l \in (0, 1)$ and $\underline{\omega}_e \in (0, 1)$ satisfying $\pi_l(\underline{\omega}_l) =$ $\pi_e(\underline{\omega}_e) = 0$. However, at $\pi = 0$, the left-hand sides of (10) and (11) are equal so it must be the case that $\underline{\omega}_l = \underline{\omega}_e$. To complete the second step, recall $0 < \pi_l(\xi) < \pi_k(\xi)$ from the proof of Proposition 2. Also recall from equation (23) that $\pi_l(\underline{\omega})$ and $\pi_k(\underline{\omega})$ can only intersect for $\underline{\omega} < \xi$. Therefore, $\pi_l(\underline{\omega}) < \pi_k(\underline{\omega})$ for all $\underline{\omega} \in (\xi, \xi_{k,2})$ and, with $\pi_k(\xi_{k,2}) = 0$, it follows that $\underline{\omega}_l \in (\xi, \xi_{k,2})$. The third step is $\pi'_l(\underline{\omega}_l) < \pi'_e(\underline{\omega}_l) < 0$. This follows from differentiating equations (10) and (11) then evaluating at $\pi = 0$. Putting everything together, we can now conclude $\pi_l(\underline{\omega}) > \pi_e(\underline{\omega})$ for $\underline{\omega} < \underline{\omega}_l$. Uniqueness of $(\hat{\pi}, \hat{\underline{\omega}})$ then implies $(\pi^*, \underline{\omega}^*) \gg (\hat{\pi}, \hat{\underline{\omega}})$.

Proof of Proposition 5

Total lending is $K \equiv \int_0^1 [\lambda(\omega) + \phi(\omega)] d\omega$. Using equation (8) and the definition of welfare in Subsection 2.4 yields $K = \frac{(1-\beta)(1-\delta)}{\mu} \mathcal{W}$. The interbank clearing equation is the same as the planner's aggregate feasibility constraint so the decentralized equilibrium is a feasible allocation. That the planner does not choose it (see Proposition 4) implies $\widehat{\mathcal{W}} > \mathcal{W}^*$ and thus $\widehat{K} > K^*$. Consider next uninformed lending: $K_N \equiv \int_0^1 \phi(\omega) d\omega$. Using equations (5) and (6) to solve for $\phi(\omega)$ yields $K_N = k_1(\pi) [1 + k_2(\pi) \underline{\omega}]$ where $k_1(\pi) \equiv \frac{p(\pi)[1-p(z-\pi)]}{\mu+(1-\mu)p(\pi)}$ and $k_2(\pi) \equiv$ $\frac{(1-\mu)p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)[1-p(z-\pi)]}. \text{ Notice } \frac{\partial K_N}{\partial \omega} > 0 \text{ and } \frac{\partial K_N}{\partial \pi} = k'_1(\pi) + [k'_1(\pi) k_2(\pi) + k_1(\pi) k'_2(\pi)] \underline{\omega} \text{ with } k'_1(\pi) > 0. \text{ If } k'_1(\pi) k_2(\pi) + k_1(\pi) k'_2(\pi) > 0, \text{ then } \frac{\partial K_N}{\partial \pi} > 0 \text{ follows trivially. Otherwise, a sufficient condition for } \frac{\partial K_N}{\partial \pi} > 0 \text{ is } k'_1(\pi) [1 + k_2(\pi)] + k_1(\pi) k'_2(\pi) > 0. \text{ It is straightforward to verify that the sufficient condition is true so } (\pi^*, \underline{\omega}^*) \gg (\widehat{\pi}, \widehat{\omega}) \text{ implies } K_N^* > \widehat{K}_N. \text{ Noting that informed lending must satisfy } K_I \equiv K - K_N \text{ then yields } K_I^* < \widehat{K}_I. \text{ Turn now to the average default rate. Recall that a maturing project defaults with probability <math>1 - q(\omega)$ so the average default rate is $\mu \left(1 - \int_0^1 q(\omega) \frac{\lambda(\omega) + \phi(\omega)}{\int_0^1 [\lambda(x) + \phi(x)] dx} d\omega\right).$ This simplifies to $\mu \left(1 - \frac{1}{\theta(1-\delta)}\right)$ under equation (8) and is thus the same for the market and the planner.

Proof of Proposition 6

Under the proposed tax, an unmatched lender chooses $\pi \in [0, z]$ to maximize:

$$p(\pi) \int_{\underline{\omega}}^{1} \left[q(\omega) - q(\underline{\omega}) \right] \psi(\omega) \, d\omega - p(\pi) \left[1 - p(z-\pi) \right] \int_{0}^{\underline{\omega}} \left[q(\underline{\omega}) - q(\omega) \right] \psi(\omega) \, d\omega - \frac{1 - \beta(1-\mu)}{\mu \theta(1-\delta)} \tau \pi$$

Combining the first order condition from this problem with the equilibrium expression for $\psi(\cdot)$ yields the following lender optimality condition:

$$\frac{1 - p\left(z - \pi\right) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}}{1 - \frac{(1 - \mu)p(\pi)p(z - \pi)}{\mu + (1 - \mu)p(\pi)}} = \frac{\int_{\underline{\omega}}^{1} \left[q\left(\omega\right) - q\left(\underline{\omega}\right)\right] d\omega - \frac{\left[1 - \beta(1 - \mu)\right]\left(1 + \frac{(1 - \mu)p(\pi)p(z - \pi)\underline{\omega}}{\mu + (1 - \mu)p(\pi)\left[1 - p(z - \pi)\right]}\right)\tau}{\mu\theta(1 - \delta)}}{\int_{0}^{\underline{\omega}} \left[q\left(\underline{\omega}\right) - q\left(\omega\right)\right] d\omega}$$
(24)

The interbank clearing condition is unchanged since tax revenues are added back to the capital supply. Therefore, the decentralized equilibrium must now solve equations (9) and (24) while the constrained efficient allocation $(\widehat{\pi}, \widehat{\omega})$ still solves equations (9) and (11). If $\tau = \frac{\mu\theta(1-\delta)}{1-\beta(1-\mu)} \frac{(1-\mu)p(\widehat{\pi})p(z-\widehat{\pi})}{\mu+(1-\mu)p(\widehat{\pi})} \frac{\int_{\widehat{\omega}}^{1}[q(\omega)-q(\widehat{\omega})]d\omega}{1+\frac{(1-\mu)p(\widehat{\pi})p(z-\widehat{\pi})\widehat{\omega}}{1+\frac{(1-\mu)p(\widehat{\pi})p(z-\widehat{\pi})\widehat{\omega}}}}$, then $(\widehat{\pi}, \widehat{\omega})$ is a solution to (9) and (24).

Proof of Proposition 7

I start by showing that the value functions defined by equations (12) and (13) do indeed generate a reservation strategy for informed retention. Define $S \equiv \left\{ \tilde{r}, \tilde{\psi} \right\}$ and write $\tilde{U}(S)$ to make explicit that S is taken as given. Also rewrite equation (12) as:

$$\widetilde{J}_{I}(\omega,S) = \max_{a \in [0,1]} \left\{ a \left[\theta \left(1-\delta\right) q \left(\omega\right) - \left(1+\widetilde{r}\right) + \beta \left(1-\mu\right) \widetilde{J}_{I}(\omega,S) + \beta \mu \widetilde{U}\left(S\right) \right] + \left(1-a\right) \beta \widetilde{U}\left(S\right) \right\}$$

Next define indicator i and value function \mathcal{V} such that:

$$\mathcal{TV}(S,\omega,i) = i \times \mathcal{V}(S,\omega,1) + (1-i) \times \mathcal{V}(S,0,0)$$

where $\mathcal{V}(S, \omega, 1) \equiv \tilde{J}_I(\omega, S)$ and $\mathcal{V}(S, 0, 0) \equiv \tilde{U}(S)$. Now suppose \mathcal{V} exists in the set of bounded and continuous functions (\mathcal{C}). By the Theorem of the Maximum, $\mathcal{V}(\cdot, 0) \in \mathcal{C}$ and $\mathcal{V}(\cdot, 1) \in \mathcal{C}$ so $\mathcal{TV} \in \mathcal{C}$. We can also show that Blackwell's sufficient conditions for a contraction are satisfied so, by the Contraction Mapping Theorem, there is indeed a unique $\mathcal{V} \in \mathcal{C}$. By implication, \tilde{J}_I and \tilde{U} exist and are unique, bounded, and continuous. A similar contraction mapping argument can then be used to show that \tilde{J}_I lies in the set of increasingin- ω functions. The optimal retention strategy is thus defined by the reservation strategy described in the main text and the equilibrium can indeed be characterized by a pair $(\pi, \underline{\omega})$.

Turn now to the inefficiency results. Let $\tilde{\pi}_k(\underline{\omega}), \tilde{\pi}_l(\underline{\omega})$, and $\tilde{\pi}_e(\underline{\omega})$ denote the functions implicitly defined by equations (16), (17), and (18) respectively. The decentralized equilibrium solves (16) and (17) while the constrained efficient allocation solves (16) and (18). In other words, $(\tilde{\pi}^*, \underline{\widetilde{\omega}}^*)$ satisfies $\tilde{\pi}^* = \tilde{\pi}_k(\underline{\widetilde{\omega}}^*) = \tilde{\pi}_l(\underline{\widetilde{\omega}}^*)$ and $(\tilde{\widetilde{\pi}}, \underline{\widetilde{\omega}})$ satisfies $\tilde{\widetilde{\pi}} = \tilde{\pi}_k(\underline{\widetilde{\omega}}) = \tilde{\pi}_e(\underline{\widetilde{\omega}})$. Existence of equilibrium and the sufficiency of Assumption 3 for uniqueness of this equilibrium is proven similarly to Proposition 2.

With $\mu = 1$, equations (17) and (18) are identical so the decentralized equilibrium is constrained efficient. Next, notice that equations (16) and (18) are independent of β so $\tilde{\pi}$ is also independent of β and \mathcal{B} is explicitly defined. With $\beta = \mathcal{B}$ in equation (17), the planner's allocation solves the system of equations that defines the decentralized equilibrium for any $\mu \in (0, 1]$. Invoking Assumption 3, the equilibrium is unique so we can conclude $(\tilde{\pi}^*, \underline{\tilde{\omega}}^*) = (\tilde{\tilde{\pi}}, \underline{\tilde{\tilde{\omega}}})$. Turning now to $\mu < 1$ and $\beta \neq \mathcal{B}$, the following lemma will be useful:

Lemma 2 $\widetilde{\pi}'_{l}(\cdot) < 0, \ \widetilde{\pi}'_{k}(\underline{\widetilde{\omega}}^{*}) > 0, \ and \ \widetilde{\pi}'_{k}\left(\underline{\widetilde{\widetilde{\omega}}}\right) > 0$

Proof. Define $C(\pi) \equiv 1 + \frac{(1-\mu)[1-p(\pi)p(z-\pi)]}{\mu+(1-\mu)p(\pi)}$ and $H(\underline{\omega}) \equiv \frac{\int_{\underline{\omega}}^{1}[q(\underline{\omega})-q(\underline{\omega})]d\underline{\omega}}{\int_{0}^{0}[q(\underline{\omega})-q(\underline{\omega})]d\underline{\omega}}$. With $A(\pi)$ as per the proof of Proposition 2, $\tilde{\pi}_{l}(\cdot)$ solves $\frac{A(\tilde{\pi}_{l}(\underline{\omega}))}{C(\tilde{\pi}_{l}(\underline{\omega}))} = \frac{\mu H(\underline{\omega})}{1-\beta(1-\mu)}$. Some algebra reveals that Assumption 3 is sufficient for $\frac{d}{d\pi}\frac{A(\pi)}{C(\pi)} > 0$ so $H'(\underline{\omega}) < 0$ implies $\tilde{\pi}'_{l}(\cdot) < 0$. Establishing $\tilde{\pi}'_{k}(\underline{\widetilde{\omega}}^{*}) > 0$ proceeds as in the proof of Proposition 2. In particular, rewrite equations (16) and (17) to isolate $\int_{\underline{\widetilde{\omega}}^{*}}^{1}q(\omega) d\omega$ then equate. Rearrange the equated expression to isolate $\frac{1}{\theta(1-\delta)} - q(\underline{\widetilde{\omega}}^{*}) = 0$ which, by differentiation of equation (16) and Assumption 1, means $\tilde{\pi}'_{k}(\underline{\widetilde{\omega}}^{*}) > 0$. Finally, the planner's first order condition for $\underline{\omega}$ implies $q(\underline{\widetilde{\omega}}) < \frac{1}{\theta(1-\delta)}$ so $\tilde{\pi}'_{k}(\underline{\widetilde{\omega}}) > 0$ is also true. \Box

Given Lemma 2, showing $(\tilde{\pi}^*, \underline{\widetilde{\omega}}^*) \ll (\tilde{\widetilde{\pi}}, \underline{\widetilde{\widetilde{\omega}}})$ amounts to showing $\tilde{\pi}_l(\underline{\widetilde{\omega}}) < \tilde{\pi}_e(\underline{\widetilde{\omega}})$. Similarly, showing $(\tilde{\pi}^*, \underline{\widetilde{\omega}}^*) \gg (\tilde{\widetilde{\pi}}, \underline{\widetilde{\widetilde{\omega}}})$ amounts to showing $\tilde{\pi}_l(\underline{\widetilde{\omega}}) > \tilde{\pi}_e(\underline{\widetilde{\omega}})$. With $A(\pi), C(\pi)$, and $H(\underline{\omega})$ as defined in the proof of Lemma 2, $\tilde{\pi}_l(\cdot)$ and $\tilde{\pi}_e(\cdot)$ solve $\frac{A(\tilde{\pi}_l(\underline{\omega}))}{C(\tilde{\pi}_l(\underline{\omega}))} = \frac{\mu H(\underline{\omega})}{1-\beta(1-\mu)}$ and $\frac{A(\tilde{\pi}_e(\underline{\omega}))}{C(\tilde{\pi}_e(\underline{\omega}))} = \mu C(\tilde{\pi}_e(\underline{\omega})) H(\underline{\omega})$ respectively. If $\beta < \mathcal{B}$, then $\frac{1}{1-\beta(1-\mu)} < C(\tilde{\widetilde{\pi}}) = C(\tilde{\pi}_e(\underline{\widetilde{\omega}}))$

and, therefore, $\frac{A(\tilde{\pi}_{l}(\underline{\widetilde{\omega}}))}{C(\tilde{\pi}_{l}(\underline{\widetilde{\omega}}))} < \mu C\left(\widetilde{\pi}_{e}\left(\underline{\widetilde{\omega}}\right)\right) H\left(\underline{\widetilde{\omega}}\right) = \frac{A(\tilde{\pi}_{e}(\underline{\widetilde{\omega}}))}{C(\tilde{\pi}_{e}(\underline{\widetilde{\omega}}))}$. We know $\frac{d}{d\pi} \frac{A(\pi)}{C(\pi)} > 0$ from the proof of Lemma 2 so $\frac{A(\tilde{\pi}_{l}(\underline{\widetilde{\omega}}))}{C(\tilde{\pi}_{l}(\underline{\widetilde{\omega}}))} < \frac{A(\tilde{\pi}_{e}(\underline{\widetilde{\omega}}))}{C(\tilde{\pi}_{e}(\underline{\widetilde{\omega}}))}$ implies $\tilde{\pi}_{l}\left(\underline{\widetilde{\omega}}\right) < \tilde{\pi}_{e}\left(\underline{\widetilde{\omega}}\right)$. In other words, $(\tilde{\pi}^{*}, \underline{\widetilde{\omega}}^{*}) \ll \left(\tilde{\tilde{\pi}}, \underline{\widetilde{\omega}}\right)$ if $\beta < \mathcal{B}$. In an analogous manner, $\beta > \mathcal{B}$ yields $\frac{A(\tilde{\pi}_{l}(\underline{\widetilde{\omega}}))}{C(\tilde{\pi}_{l}(\underline{\widetilde{\omega}}))} > \frac{A(\tilde{\pi}_{e}(\underline{\widetilde{\omega}}))}{C(\tilde{\pi}_{e}(\underline{\widetilde{\omega}}))}$ so $\tilde{\pi}_{l}\left(\underline{\widetilde{\omega}}\right) > \tilde{\pi}_{e}\left(\underline{\widetilde{\omega}}\right)$ and thus $(\tilde{\pi}^{*}, \underline{\widetilde{\omega}}^{*}) \gg \left(\overline{\tilde{\pi}}, \underline{\widetilde{\omega}}\right)$.

The last step is to prove the relationship lending analog of Proposition 6. A remitted matching tax, now denoted by $\tilde{\tau}$, yields the following in lieu of equation (17):

$$\frac{1 - p\left(z - \pi\right) + \frac{p(\pi)p'(z - \pi)}{p'(\pi)}}{\mu\left(1 + \frac{(1 - \mu)[1 - p(\pi)p(z - \pi)]}{\mu + (1 - \mu)p(\pi)}\right)} = \frac{\frac{\int_{\omega}^{1} [q(\omega) - q(\underline{\omega})]d\omega}{1 - \beta(1 - \mu)} - \frac{\left(1 + \frac{(1 - \mu)p(\pi)}{\mu} \frac{1 - \mu[1 - p(z - \pi)]}{1 + (1 - \mu)p(\pi)[1 - p(z - \pi)]}\underline{\omega}\right)\tilde{\tau}}{\int_{0}^{\omega} [q\left(\underline{\omega}\right) - q\left(\omega\right)]d\omega}$$

The tax is defined so that equation (16) is unchanged. Implementing $\left(\widetilde{\widetilde{\pi}}, \widetilde{\underline{\widetilde{\omega}}}\right)$ thus requires:

$$\widetilde{\tau} = \left[\frac{\beta(1-\mu)}{1-\beta(1-\mu)} - \frac{(1-\mu)\left[1-p\left(\tilde{\tilde{\pi}}\right)p\left(z-\tilde{\tilde{\pi}}\right)\right]}{\mu+(1-\mu)p\left(\tilde{\tilde{\pi}}\right)}\right] \frac{\theta\left(1-\delta\right)p'\left(\tilde{\tilde{\pi}}\right)\int_{\widetilde{\omega}}^{1}\left[q\left(\omega\right)-q\left(\tilde{\omega}\right)\right]d\omega}{1+\frac{(1-\mu)p\left(\tilde{\tilde{\pi}}\right)}{\mu}\frac{1-\mu\left[1-p\left(z-\tilde{\tilde{\pi}}\right)\right]}{1+(1-\mu)p\left(\tilde{\tilde{\pi}}\right)\left[1-p\left(z-\tilde{\tilde{\pi}}\right)\right]}\widetilde{\omega}}$$

If $\beta \geq \mathcal{B}$, then $\frac{\beta(1-\mu)}{1-\beta(1-\mu)} \geq \frac{(1-\mu)\left[1-p(\widehat{\pi})p(z-\widehat{\pi})\right]}{\mu+(1-\mu)p(\widehat{\pi})}$ and $\widetilde{\tau} \geq 0$.

Proof of Proposition 8

The first step is to show that the retention decision of an informed lender still follows a reservation strategy, $\omega \geq \underline{\omega}$. This involves showing that the value function of an informed lender is increasing in ω . Given the focus on non-binding borrower participation constraints, the proof proceeds by a standard contraction mapping argument. Turn now to the loan rates. A type ω borrower succeeds with probability $e(\omega, R) \alpha(\omega)$, in which case his lender gets R. Given the borrower strategy in equation (19), an informed lender chooses $R \in \left(0, \theta - \frac{c}{\alpha(\omega)}\right)$ to maximize $\left(1 - \frac{c}{\alpha(\omega)(\theta - R)}\right) \alpha(\omega) R$. This yields $R(\omega) = \theta - \sqrt{\frac{c\theta}{\alpha(\omega)}}$, where $c \leq \frac{\alpha(0)}{\alpha(1)}\theta\alpha(0)$ ensures $c < \theta\alpha(\omega)$ and thus $e(\omega, R(\omega)) > 0$. Since an uninformed lender does not know ω exactly, he instead offers a pooled rate which solves:

$$\max_{\overline{R}} \int_{\underline{\omega}_l(\overline{R})}^1 \left(1 - \frac{c}{\alpha(\omega)(\theta - \overline{R})} \right) \alpha(\omega) \,\overline{R}\psi(\omega) \, d\omega \quad s.t. \quad \underline{\omega}_l(\overline{R}) \equiv \arg\min_{\omega \in [0,1]} \left| \theta - \frac{c}{\alpha(\omega)} - \overline{R} \right|$$

The problem yields $\overline{R} = \theta - \sqrt{\frac{c\theta}{\widetilde{\alpha}(\overline{R},\psi(\cdot))}}$ with $\widetilde{\alpha}(\overline{R},\psi(\cdot)) \equiv \frac{\int_{\underline{\omega}_l}^1(\overline{R})^{\alpha(\omega)\psi(\omega)d\omega}}{\int_{\underline{\omega}_l}^1(\overline{R})^{\psi(\omega)d\omega}} \in (\alpha(0),\alpha(1)).$ If

interior, $\underline{\omega}_l(\overline{R})$ is the highest type that exerts no effort when charged \overline{R} so $\underline{\omega}_l(\overline{R}) > 0$ requires $e(0,\overline{R}) = 0$. The latter amounts to $c \geq \frac{\theta \alpha(0)^2}{\tilde{\alpha}(\overline{R},\psi(\cdot))}$ which violates $c \leq \frac{\theta \alpha(0)^2}{\alpha(1)}$ and thus implies $\overline{R} = \theta - \sqrt{\frac{c\theta}{\int_0^1 \alpha(\omega)\psi(\omega)d\omega}}$. Using equations (5) to (7) yields $\psi(\omega) = \frac{\mu + (1-\mu)p(\pi)}{\mu + (1-\mu)p(\pi)[1-p(z-\pi)]} \frac{1}{1+\sigma(\pi)\omega}$ for $\omega \in [0,\underline{\omega})$ and $\psi(\omega) = \frac{1}{1+\sigma(\pi)\omega}$ for $\omega \in [\underline{\omega},1]$. Substituting into $\int_0^1 \alpha(\omega)\psi(\omega)d\omega$ and simplifying then produces $\overline{\alpha}(\pi,\underline{\omega})$ as defined in the proposition.

Proof of Proposition 9

The proof proceeds in three parts. Abstracting from borrower participation constraints, Part 1 establishes the existence of a stationary equilibrium while Part 2 provides sufficient conditions for uniqueness of this equilibrium. Part 3 then shows that, under similar sufficient conditions as Part 2, the equilibrium does indeed satisfy the participation constraints.

Part 1: Existence of Equilibrium (Conditional on Non-Binding P/C)

Let $\pi_l(\underline{\omega})$ and $\pi_k(\underline{\omega})$ denote the functions implicitly defined by equations (21) and (20) respectively. Also define ξ such that $g(\xi) \equiv 1$. I will assume $g(1) > 1 > \int_0^1 g(\omega) d\omega$ in all of what follows. This assumption and $g'(\cdot) > 0$ ensure that ξ exists uniquely and is interior. Finally, it will be convenient to define the following function:

$$f(\pi,\underline{\omega}) \equiv 1 - g(\underline{\omega}) - \frac{(1-\mu)p(\pi)[1-p(z-\pi)]}{\mu} \left[g(\underline{\omega}) - h\left(\underline{\omega},\overline{R}(\pi,\underline{\omega})\right)\right]$$

Differentiating equation (20) and invoking the envelope theorem on $\overline{R}(\cdot)$ reveals that $\pi'_k(\underline{\omega})$ and $f(\pi_k(\underline{\omega}), \underline{\omega})$ have the same sign. Additional properties of $\pi_k(\cdot)$ are summarized next:

Lemma 3 There exists a $\underline{p}_1 \in (0,1)$ such that $p(z) \in \left(\underline{p}_1,1\right)$ and $c \in \left[0,\frac{\theta\alpha(0)^2}{\alpha(1)}\right]$ imply: (i) a unique $\xi_{k,1} \in (0,\xi)$ satisfying $\pi_k\left(\xi_{k,1}\right) = 0$, (ii) a unique $\xi_{k,2} \in (\xi,1)$ satisfying $\pi_k\left(\xi_{k,2}\right) = 0$, and (iii) at least one $\underline{\omega}_0 \in (\xi_{k,1},\xi_{k,2})$ satisfying $\pi'_k(\underline{\omega}_0) = 0$.

Proof. Use equation (20) to write $\pi_k(\underline{\omega}) = 0$ as $s(\underline{\omega}) = H$, where $s(\underline{\omega}) \equiv \int_{\underline{\omega}}^1 [g(\omega) - 1] d\omega$ and $H \equiv \frac{1-p(z)}{p(z)} \left[1 - \left(\sqrt{\theta \int_0^1 \alpha(\omega) d\omega} - \sqrt{c} \right)^2 \right]$. Notice that $p(z) \in (0,1)$, $\int_0^1 g(\omega) d\omega < 1$, and Jensen's inequality imply H > 0. Also notice s(0) < 0 = s(1) and $s'(\underline{\omega}) = 1 - g(\underline{\omega})$ so showing $s(\xi) > H$ will show results (i) and (ii) of the lemma. Result (iii) will then follow because $g(\xi_{k,1}) < 1 < g(\xi_{k,2})$ implies $f(\pi_k(\xi_{k,1}), \xi_{k,1}) > 0 > f(\pi_k(\xi_{k,2}), \xi_{k,2})$ and thus
$$\begin{split} &\lim_{\underline{\omega}\to\xi_{k,1}^+}\pi_k'\left(\underline{\omega}\right)>0>\lim_{\underline{\omega}\to\xi_{k,2}^-}\pi_k'\left(\underline{\omega}\right). \ \text{ To get } s\left(\xi\right)>H, \ define \ \underline{p}_1 \ \text{ so that } s\left(\xi\right)=H \ \text{for } c=\frac{\theta\alpha(0)^2}{\alpha(1)}\\ &\text{ and } p\left(z\right)=\underline{p}_1. \ \text{ Notice } \underline{p}_1\in(0,1) \ \text{since } p\left(z\right)=0 \ \text{yields } H=\infty>s\left(\cdot\right) \ \text{and } p\left(z\right)=1 \ \text{yields } H=0< s\left(\xi\right) \ \text{given } g\left(\xi\right)\equiv 1 \ \text{ and } \xi<1. \ \text{It is then straightforward to show } \frac{ds(\xi)}{dp(z)}=0>\frac{dH}{dp(z)}\\ &\text{ and } \frac{ds(\xi)}{dc}<0<\frac{dH}{dc} \ \text{when } c\leq\frac{\theta\alpha(0)^2}{\alpha(1)} \ \text{so } s\left(\xi\right)>H \ \text{follows for } c\leq\frac{\theta\alpha(0)^2}{\alpha(1)} \ \text{and } p\left(z\right)>\underline{p}_1. \ \Box \end{split}$$

Differentiating equation (21) and invoking Assumption 3 reveals $\pi'_l(\cdot) < 0$ so there is a unique ξ_l such that $\pi_l(\xi_l) = 0$. Equations (21) and (20) imply that ξ_l and any ξ_k satisfying $\pi_k(\xi_k) = 0$ are related by: $\int_0^{\xi_l} g(\omega) d\omega + \frac{1-p(z)\xi_l}{p(z)}g(\xi_l) = \int_0^{\xi_k} g(\omega) d\omega + \frac{1-p(z)\xi_k}{p(z)}$. Using $g(\xi_{k,1}) < 1 < g(\xi_{k,2})$ and $\frac{d}{d\omega} \left[\int_0^{\omega} g(\omega) d\omega + \frac{1-p(z)\omega}{p(z)}g(\omega) \right] > 0$ then implies $\xi_l \in (\xi_{k,1}, \xi_{k,2})$. Therefore, $\pi_k(\xi_l) > 0 = \pi_l(\xi_l)$ and either $\pi_l(\xi_{k,1}) > 0 = \pi_k(\xi_{k,1})$ or $\pi_l(\omega_z) = z > \pi_k(\omega_z)$ for a $\omega_z \in (\xi_{k,1}, \xi_l)$. There thus exists at least one $\omega^* \in (\xi_{k,1}, \xi_l)$ where $\pi_l(\omega^*) = \pi_k(\omega^*)$.

Part 2: Uniqueness of Equilibrium (Conditional on Non-Binding P/C)

The proof so far has established existence of at least one pair $(\pi^*, \underline{\omega}^*)$ satisfying equations (20) and (21). Rearranging (20) and (21) to get two expressions for $\Delta(\pi, \underline{\omega})$, equating these expressions, and rearranging again yields:

$$T\left(\pi,\underline{\omega}\right) \equiv \left(1 - \underline{\omega} + \frac{A(\pi)}{B(\pi)}\underline{\omega}\right) \left[1 - g\left(\underline{\omega}\right)\right] - \frac{p(\pi)p'(z-\pi)}{[1 - p(z-\pi)]p'(\pi)} \int_{\underline{\omega}}^{1} \left[g\left(\omega\right) - 1\right] d\omega = 0$$
(25)

where $A(\cdot)$ and $B(\cdot)$ are as defined in the proof of Proposition 2. Any equilibrium $(\pi^*, \underline{\omega}^*)$ can now also be defined as the solution to equations (20) and (25). In what follows, I use $\pi_i(\underline{\omega})$ to denote the function implicitly defined by equation (25). Differentiating yields:

$$\pi'_{i}(\underline{\omega}) = -\frac{\left(1 - \underline{\omega} + \frac{A(\pi)}{B(\pi)}\underline{\omega}\right)\frac{g'(\underline{\omega})}{1 - g(\underline{\omega})} + \frac{A(\pi)}{B(\pi)}\frac{\mu p(z - \pi)}{[\mu + (1 - \mu)p(\pi)][1 - p(z - \pi)]}}{\left(1 - \underline{\omega} + \frac{A(\pi)}{B(\pi)}\underline{\omega}\right)\left(\frac{p'(\pi)}{p(\pi)} - \frac{p''(\pi)}{p'(\pi)} - \frac{p''(z - \pi)}{p'(z - \pi)} - \frac{p'(z - \pi)}{1 - p(z - \pi)}\right) - \frac{A(\pi)}{B(\pi)}\underline{\omega}\left(\frac{A'(\pi)}{A(\pi)} - \frac{B'(\pi)}{B(\pi)}\right)}$$
(26)

with π evaluated at $\pi_i(\underline{\omega})$. Additional properties of $\pi_i(\cdot)$ are summarized in the next lemma. As in Part 1, $\xi \in (0, 1)$ is uniquely defined by $g(\xi) \equiv 1$.

Lemma 4 $\pi_i(\xi) = 0$; $\pi_i(\xi_z) = z$ for a unique $\xi_z \in (0,\xi)$; and $\pi'_i(\underline{\omega}) < 0$ for $\underline{\omega} \in (\xi_z,\xi)$. Moreover, $\pi_i(\underline{\omega})$ is not defined for $\underline{\omega} \notin [\xi_z,\xi]$.

Proof. Recall $s(\underline{\omega})$ from the proof of Lemma 3. Since $\int_0^1 g(\omega) d\omega < 1$ and $g'(\cdot) > 0$, we have $s(\underline{\omega}) > 0$ if and only if $\underline{\omega} \in (\xi_0, 1)$ with ξ_0 uniquely defined by $s(\xi_0) \equiv 0$. Therefore, the second term in equation (25) is non-negative if and only if $\underline{\omega} \in [\xi_0, 1]$ while the first term is non-negative if and only if $\underline{\omega} \in [0, \xi]$. The definitions of ξ_0 and ξ imply $\xi_0 < \xi$ so $\underline{\omega} \in [\xi_0, \xi]$ is necessary for $\pi_i(\underline{\omega})$ to exist. It is clear from equation (25) that $\pi_i(\xi) = 0$. The equation

also yields $\pi_i(\xi_z) = z$ for ξ_z defined by $\zeta(\xi_z) \equiv 1 + \frac{p'(z)}{p(z)p'(0)}$, where $\zeta(x) \equiv \frac{\int_x^1 [g(\omega) - g(x)] d\omega}{1 - g(x)}$. Notice $\zeta(\xi_0) = 1 - \xi_0$, $\zeta(\xi) = \infty$, and $\zeta'(x) > 0$ for $x \in (\xi_0, \xi)$. Therefore, $\xi_z \in (\xi_0, \xi)$ is uniquely defined. If $\pi'_i(\cdot) < 0$ is false, then there are points $\underline{\omega}_1 \in (\xi_0, \xi)$ and $\underline{\omega}_2 \in (\underline{\omega}_1, \xi)$ such that $\pi_i(\underline{\omega}_1) = \pi_i(\underline{\omega}_2)$ and $\pi'_i(\underline{\omega}_1) > 0 > \pi'_i(\underline{\omega}_2)$. Equation (26) would then imply $\underline{\omega}_1 > \underline{\omega}_2$ which contradicts $\underline{\omega}_2 \in (\underline{\omega}_1, \xi)$. Therefore, $\pi'_i(\cdot) < 0$. Combined with $\pi_i(\xi_z) = z$ for a unique $\xi_z \in (\xi_0, \xi)$, this also implies that $\pi_i(\underline{\omega})$ cannot exist for $\underline{\omega} \in [\xi_0, \xi_z)$. \Box

With $\pi_i(\cdot)$ not defined above ξ and $\pi_k(\xi) > 0 = \pi_i(\xi)$, we can restrict attention to $\underline{\omega}^* < \xi$. The following lemma provides sufficient conditions for a unique equilibrium:

Lemma 5 Define $\Omega_0 \equiv \min \{\underline{\omega}_0 | \pi'_k(\underline{\omega}_0) = 0\}$. The following are sufficient for uniqueness:

- Condition 1: $\pi_i(\Omega_0) < \pi_k(\Omega_0)$
- Condition 2: $\frac{\partial f(\pi,\Omega_0)}{\partial \pi} < 0$ for any $\pi \in [\pi_i(\Omega_0), \pi_k(\Omega_0)]$
- Condition 3: $\frac{df(\pi_i(\underline{\omega}),\underline{\omega})}{d\omega} > 0$ for any $\underline{\omega} > \Omega_0$ satisfying $f(\pi_i(\underline{\omega}),\underline{\omega}) = 0$

Proof. The proof of Lemma 3 established $f(\pi_k(\xi_{k,1}), \xi_{k,1}) > 0$ and the definition of ξ establishes $f(\pi_k(\xi), \xi) < 0$. The fact that $\pi'_k(\cdot)$ and $f(\pi_k(\cdot), \cdot)$ have the same sign then implies $\Omega_0 \in (\xi_{k,1}, \xi)$. It also implies $\pi'_k(\underline{\omega}) > 0$ for all $\underline{\omega} \in (\xi_{k,1}, \Omega_0)$ which, taken with $\pi'_i(\cdot) < 0$, means there is at most one equilibrium with $\underline{\omega}^* \in (\xi_{k,1}, \Omega_0)$. A necessary condition for multiplicity is thus existence of an equilibrium with $\underline{\omega}^* \in (\Omega_0, \xi)$. Suppose such an $\underline{\omega}^*_a$ exists. Then $\pi'_i(\cdot) < 0$ and Condition 1 imply existence of a point $\underline{\omega}^*_b \in (\Omega_0, \underline{\omega}^*_a]$ satisfying $\pi_i(\underline{\omega}^*_b) = \pi_k(\underline{\omega}^*_b)$ and $\pi'_k(\underline{\omega}^*_b) < 0$. Invoking again that $\pi'_k(\cdot)$ and $f(\pi_k(\cdot), \cdot)$ have the same sign further implies $f(\pi_i(\underline{\omega}^*_b), \underline{\omega}^*_b) = f(\pi_k(\underline{\omega}^*_b), \underline{\omega}^*_b) < 0$. Notice that Conditions 1 and 2 yield $f(\pi_i(\Omega_0), \Omega_0) > f(\pi_k(\Omega_0), \Omega_0) = 0$. Therefore, existence of $\underline{\omega}^*_b \in (\Omega_0, \underline{\omega}^*_a]$ such that $f(\pi_i(\underline{\omega}^*_b), \underline{\omega}^*_b) < 0$ implies existence of a point $\underline{\omega}_c \in (\Omega_0, \underline{\omega}^*_b)$ satisfying $f(\pi_i(\underline{\omega}_c), \underline{\omega}_c) = 0$ and $\frac{df(\pi_i(\underline{\omega})}{d\underline{\omega}}\Big|_{\underline{\omega}=\underline{\omega}_c}$ < 0. This violates Condition 3 so there cannot exist an equilibrium with $\underline{\omega}^*_a \in (\Omega_0, \xi)$. Existence of at most one $\underline{\omega}^* \in (\xi_{k,1}, \Omega_0)$ and existence of at least one equilibrium (as per Part 1) then imply existence of a unique equilibrium.

The rest of this section derives restrictions so that the conditions in Lemma 5 hold. I first show that the desired conditions can be achieved with certain restrictions on the critical point(s) of $\pi_k(\cdot)$. I then show how the critical point restrictions can be achieved with restrictions on the borrower disutility parameter c. Lemma 6 completes the first step: **Lemma 6** Define $\Pi_0 \equiv \max \{\pi_k(\underline{\omega}_0) | \pi'_k(\underline{\omega}_0) = 0\}$. If $p(z) \in [\underline{p}_2, 1)$ for some $\underline{p}_2 \in (0, 1)$, then there exist scalars $\underline{\underline{\omega}} \in (0, 1)$ and $\overline{\pi} \in (0, z)$ such that the conditions in Lemma 5 hold when $\Omega_0 > \underline{\underline{\omega}}$ and $\Pi_0 < \overline{\pi}$.

Proof. Take each condition in turn:

• Condition 1: Recall that $\pi_i(\underline{\omega})$ solves $T(\pi,\underline{\omega}) = 0$, with $T(\cdot)$ as defined in equation (25). Also notice $\frac{dT(\pi,\underline{\omega})}{d\underline{\omega}} < 0$. If $T(\pi_k(\Omega_0),\Omega_0) < 0$, then there exists an $\varepsilon \in (0,\Omega_0)$ such that $T(\pi_k(\Omega_0),\Omega_0-\varepsilon) = 0$. In other words, there exists an $\varepsilon \in (0,\Omega_0)$ such that $\pi_i(\Omega_0-\varepsilon) = \pi_k(\Omega_0)$. With $\pi'_i(\cdot) < 0$, it then follows that $\pi_i(\Omega_0) < \pi_i(\Omega_0-\varepsilon) = \pi_k(\Omega_0)$ so $T(\pi_k(\Omega_0),\Omega_0) < 0$ will be enough to establish Condition 1. Turn now to $\pi_k(\Omega_0)$ which solves equation (20) when $\underline{\omega} = \Omega_0$. Rewrite (20) as $\frac{1-p(z-\pi)}{p(z-\pi)} = \frac{1}{1-\Gamma(\pi,\underline{\omega})} \frac{s(\underline{\omega})}{1+\sigma(\pi)\underline{\omega}}$, where:

$$\Gamma\left(\pi,\underline{\omega}\right) \equiv \frac{1}{1+\sigma(\pi)\underline{\omega}} \left[\frac{1}{B(\pi)} \int_{0}^{\underline{\omega}} h\left(\omega,\overline{R}\left(\pi,\underline{\omega}\right)\right) d\omega + \int_{\underline{\omega}}^{1} h\left(\omega,\overline{R}\left(\pi,\underline{\omega}\right)\right) d\omega \right]$$

By definition, $\pi'_k(\Omega_0) = 0$ so we cannot have $\frac{\partial}{\partial \underline{\omega}} \left(\frac{s(\underline{\omega})}{1 + \sigma(\pi)\underline{\omega}} \right) \times \frac{\partial \Gamma(\pi,\underline{\omega})}{\partial \underline{\omega}} > 0$ at $\underline{\omega} = \Omega_0$. Taking derivatives, $\frac{\partial \Gamma(\pi,\underline{\omega})}{\partial \underline{\omega}} = \frac{\sigma(\pi) \left[h\left(\underline{\omega}, \overline{R}(\pi,\underline{\omega})\right) - \Gamma(\pi,\underline{\omega}) \right]}{1 + \sigma(\pi)\underline{\omega}}$. With $h\left(\omega, \overline{R}\left(\cdot\right)\right)$ and $\overline{\alpha}\left(\pi,\underline{\omega}\right)$ as per the text, $h\left(\underline{\omega}, \overline{R}\left(\pi,\underline{\omega}\right)\right) \ge \Gamma\left(\pi,\underline{\omega}\right)$ amounts to $\alpha\left(\underline{\omega}\right) \ge \overline{\alpha}\left(\pi,\underline{\omega}\right)$. Notice $\alpha'\left(\cdot\right) > 0$ implies $\underline{\omega} \int_0^1 \alpha\left(\omega\right) d\omega > \int_0^{\underline{\omega}} \alpha\left(\omega\right) d\omega$ and thus $\int_0^1 \alpha\left(\omega\right) d\omega > \overline{\alpha}\left(\pi,\underline{\omega}\right)$. Defining $\underline{\omega}_1 \equiv \alpha^{-1} \left(\int_0^1 \alpha\left(\omega\right) d\omega\right)$ then yields $\frac{\partial \Gamma(\pi,\underline{\omega})}{\partial \underline{\omega}} > 0$ for $\underline{\omega} \in \left(\underline{\omega}_1, 1\right)$. Consider now $\frac{s(\underline{\omega})}{1 + \sigma(\pi)\underline{\omega}}$. If $T\left(\pi,\underline{\omega}\right) \ge 0$, then:

$$\frac{\partial}{\partial \underline{\omega}} \left(\frac{s(\underline{\omega})}{1 + \sigma(\pi)\underline{\omega}} \right) \ge \frac{1 - g(\underline{\omega})}{1 + \sigma(\pi)\underline{\omega}} \left[1 - \frac{\sigma(\pi)}{1 + \sigma(\pi)\underline{\omega}} \left(1 - \underline{\omega} + \frac{A(\pi)}{B(\pi)}\underline{\omega} \right) \frac{[1 - p(z - \pi)]p'(\pi)}{p(\pi)p'(z - \pi)} \right]$$

Defining $n(\pi) \equiv \frac{\mu + (1-\mu)p(\pi)[1-p(z-\pi)]}{\mu p(z-\pi)} \left(1 - \frac{\mu + (1-\mu)p(\pi)}{(1-\mu)p(z-\pi)} \frac{p'(z-\pi)}{A(\pi)p'(\pi)}\right)$, the above derivative is positive if $\underline{\omega} \in (n(\pi), \xi)$. A sufficient condition for $\underline{\omega} > n(\pi)$ is $\frac{\mu + (1-\mu)p(\pi)}{(1-\mu)p(z-\pi)} \frac{p'(z-\pi)}{A(\pi)p'(\pi)} \ge 1$. Given the upper bound on $\frac{p'(z-\pi)}{1-p(z-\pi)}$ from Assumption 4, the left-hand side of the sufficient condition is increasing in π so it will be enough to have it hold at $\pi = 0$. That is, $p(z) [1-p(z)] \le \frac{\mu p'(z)}{(1-\mu)p'(0)}$ will be enough for $\underline{\omega} > n(\pi)$. Notice that $p(z) [1-p(z)] \le \frac{\mu p'(z)}{(1-\mu)p'(0)}$ can be translated into an upper bound on p(z) which I will call \underline{p}_2 . Now recall $\Omega_0 < \xi$ and thus $g(\Omega_0) < 1$ from the proof of Lemma 5. Putting everything together: If $\Omega_0 > \underline{\omega}_1$, then $T(\pi_k(\Omega_0), \Omega_0) \ge 0$ violates $\pi'_k(\Omega_0) = 0$ so it must be the case that $T(\pi_k(\Omega_0), \Omega_0) < 0$ and, therefore, $\pi_i(\Omega_0) < \pi_k(\Omega_0)$.

• Condition 2: The partial of $f(\pi, \underline{\omega})$ with respect to π is:

$$\frac{\partial f(\pi,\underline{\omega})}{\partial \pi} = -\frac{(1-\mu)p(\pi)[1-p(z-\pi)]}{\mu} \left(\frac{p'(\pi)A(\pi)\left[g(\underline{\omega})-h\left(\underline{\omega},\overline{R}(\cdot)\right)\right]}{p(\pi)[1-p(z-\pi)]} - \frac{\partial h\left(\underline{\omega},\overline{R}(\cdot)\right)}{\partial\overline{R}} \frac{\partial\overline{R}(\cdot)}{\partial\pi} \right) \right)$$

Recall $g(\omega) \geq h\left(\omega, \overline{R}\left(\cdot\right)\right)$ for all ω so establishing $\frac{\partial h\left(\omega, \overline{R}(\cdot)\right)}{\partial \overline{R}} \frac{\partial \overline{R}(\cdot)}{\partial \pi} < 0$ will be enough to establish $\frac{\partial f(\pi, \omega)}{\partial \pi} < 0$. Using the expressions for $h(\cdot)$ and $\overline{R}(\cdot)$ yields $\frac{\partial h\left(\omega, \overline{R}(\cdot)\right)}{\partial \overline{R}} > 0$ if and only if $\alpha(\underline{\omega}) > \overline{\alpha}(\pi, \underline{\omega})$. Therefore, $\frac{\partial h\left(\omega, \overline{R}(\cdot)\right)}{\partial \overline{R}} > 0$ for $\underline{\omega} \in \left(\underline{\omega}_1, 1\right)$ with $\underline{\omega}_1$ as defined in the proof of Condition 1. Turn next to $\frac{\partial \overline{R}(\cdot)}{\partial \pi} = -\frac{\sigma'(\pi)}{2\overline{\alpha}(\pi, \underline{\omega})} \sqrt{\frac{c\theta}{\overline{\alpha}(\pi, \underline{\omega})}} \frac{\omega \int_0^1 \alpha(\omega) d\omega - \int_0^{\omega} \alpha(\omega) d\omega}{[1+\sigma(\pi)\underline{\omega}]^2}$. Given $\alpha'(\cdot) > 0$, it follows that $\frac{\partial \overline{R}(\cdot)}{\partial \pi}$ and $\sigma'(\pi)$ have opposite signs. Taking derivatives then reveals $\sigma'(\pi)$ and $m(\pi) \equiv \frac{\mu p(z-\pi)p'(\pi)}{\mu+(1-\mu)p(\pi)} - p(\pi)p'(z-\pi)$ have the same sign. Notice that $p''(\cdot) < 0$ implies $m'(\pi) < 0$ so defining $\overline{\pi}_1 \equiv \arg\min_{\pi\in(0,z)} |m(\pi)|$ yields $\frac{\partial \overline{R}(\cdot)}{\partial \pi} < 0$ for $\pi \in (0, \overline{\pi}_1)$. We have now established $\frac{\partial f(\pi, \omega)}{\partial \pi} < 0$ for $\underline{\omega} \in \left(\underline{\omega}_1, 1\right)$ and $\pi \in (0, \overline{\pi}_1)$. Condition 2 requires $\frac{\partial f(\pi, \Omega_0)}{\partial \pi} < 0$ for any $\pi \in [\pi_i(\Omega_0), \pi_k(\Omega_0)]$ so it will be sufficient to have $\Omega_0 > \underline{\omega}_1$ and $\Pi_0 < \overline{\pi}_1$.

• Condition 3: If $f(\pi_i(\underline{\omega}), \underline{\omega}) = 0$, then:

$$\frac{df(\pi_{i}(\underline{\omega}),\underline{\omega})}{d\underline{\omega}} = -\left[1 - g\left(\underline{\omega}\right)\right] \left[\left(1 + \frac{(1-\mu)p(\pi)[1-p(z-\pi)]}{\mu}\right) \frac{g'(\underline{\omega})}{1-g(\underline{\omega})} + \frac{p'(\pi)A(\pi)}{p(\pi)[1-p(z-\pi)]}\pi'_{i}\left(\underline{\omega}\right) \right] \\ + \frac{(1-\mu)p(\pi)[1-p(z-\pi)]}{\mu} \left[\frac{\partial h(\underline{\omega},\overline{R}(\cdot))}{\partial\underline{\omega}} + \frac{\partial h(\underline{\omega},\overline{R}(\cdot))}{\partial\overline{R}} \frac{\partial\overline{R}(\cdot)}{\partial\underline{\omega}} + \frac{\partial h(\underline{\omega},\overline{R}(\cdot))}{\partial\overline{R}} \frac{\partial\overline{R}(\cdot)}{\partial\underline{\omega}} \pi'_{i}\left(\underline{\omega}\right) \right]$$

with π evaluated at $\pi_i(\underline{\omega})$. Recall $\pi'_i(\cdot) < 0$. Also recall $\frac{\partial h\left(\underline{\omega},\overline{R}(\cdot)\right)}{\partial \overline{R}} > 0$ and $\frac{\partial h\left(\underline{\omega},\overline{R}(\cdot)\right)}{\partial \overline{R}} \frac{\partial \overline{R}(\cdot)}{\partial \pi} < 0$ for $\underline{\omega} \in \left(\underline{\omega}_1, 1\right)$ and $\pi \in (0, \overline{\pi}_1)$. It is now straightforward to show $\frac{\partial h\left(\underline{\omega},\overline{R}(\cdot)\right)}{\partial \underline{\omega}} = \alpha'(\underline{\omega}) \overline{R}(\cdot) > 0$ and $\frac{\partial \overline{R}(\cdot)}{\partial \underline{\omega}} = \frac{1}{2\overline{\alpha}(\pi,\underline{\omega})} \sqrt{\frac{c\theta}{\overline{\alpha}(\pi,\underline{\omega})}} \frac{\sigma(\pi)[\alpha(\underline{\omega})-\overline{\alpha}(\pi,\underline{\omega})]}{1+\sigma(\pi)\underline{\omega}} > 0$ for $\underline{\omega} \in \left(\underline{\omega}_1, 1\right)$. A sufficient condition for $\frac{df(\pi_i(\underline{\omega}),\underline{\omega})}{d\underline{\omega}} > 0$ is thus $\frac{p'(\pi)A(\pi)}{p(\pi)[1-p(z-\pi)]}\pi'_i(\underline{\omega}) < -\left(1+\frac{(1-\mu)p(\pi)[1-p(z-\pi)]}{\mu}\right)\frac{g'(\underline{\omega})}{1-g(\underline{\omega})}$. Using (26) to substitute out $\pi'_i(\underline{\omega})$ and invoking the lower bound on $\frac{p'(z-\pi)}{1-p(z-\pi)}$ from Assumption 4, the sufficient condition is sure to hold if $\underline{\omega} \ge \frac{1}{2\left(1-\frac{1-p(z-\pi)}{B(\pi)}\right)}$. Notice $\frac{\partial}{\partial\pi}\left(\frac{1-p(z-\pi)}{B(\pi)}\right) > 0$ so define $\underline{\omega}_2(\pi) \equiv \frac{1}{2\left(1-\frac{1-p(z-\pi)}{B(\pi)}\right)}$ and $\overline{\pi}_2 \equiv \arg\min_{\pi\in(0,z)}\left|\frac{1-p(z-\pi)}{B(\pi)} - \frac{1}{2}\right|$. Having $\Omega_0 > \max\left\{\underline{\omega}_1,\underline{\omega}_2(\Pi_0)\right\}$ and $\Pi_0 < \min\left\{\overline{\pi}_1,\overline{\pi}_2\right\}$ will thus suffice for Condition 3. \Box

We have now seen that a sufficiently high value of Ω_0 and a sufficiently low value of Π_0 will ensure uniqueness of equilibrium. The last step is to translate these restrictions into restrictions on the parameter c. The next lemma will be useful in this regard:

Lemma 7 $\frac{d\overline{\pi}}{dc} = 0$ and $\frac{d\omega}{d\overline{c}} \leq 0$ and $\frac{d\Pi_0}{dc} < 0$ and $\frac{d\xi_{k,1}}{dc} > 0$

Proof. From Lemma 6, $\overline{\pi} = \min\{\overline{\pi}_1, \overline{\pi}_2\}$ and $\underline{\underline{\omega}} = \max\{\underline{\underline{\omega}}_1, \underline{\underline{\omega}}_2(\Pi_0)\}$. Recall that $\overline{\pi}_1$ and $\overline{\pi}_2$ are defined by $\frac{\mu p(z-\overline{\pi}_1)p'(\overline{\pi}_1)}{\mu+(1-\mu)p(\overline{\pi}_1)} \equiv p(\overline{\pi}_1)p'(z-\overline{\pi}_1)$ and $\frac{1-p(z-\overline{\pi}_2)}{B(\overline{\pi}_2)} \equiv \frac{1}{2}$. Also recall that $\underline{\underline{\omega}}_1$ is defined by $\alpha\left(\underline{\underline{\omega}}_1\right) \equiv \int_0^1 \alpha(\omega) d\omega$. None of these definitions depend on c so $\frac{d\overline{\pi}}{dc} = \frac{d\underline{\underline{\omega}}_1}{dc} = 0$.

Turn now to $\frac{d\underline{\omega}_2(\Pi_0)}{dc} = \underline{\omega}'_2(\Pi_0) \frac{d\Pi_0}{dc}$. The proof of Lemma 6 showed $\underline{\omega}'_2(\cdot) > 0$ so $\frac{d\Pi_0}{dc} < 0$ will establish $\frac{d\underline{\omega}}{dc} \leq 0$. To show $\frac{d\Pi_0}{dc} < 0$, treat c as a variable in equation (20) and differentiate to find that $\frac{d\pi}{dc}$ has the opposite sign as $\int_{\underline{\omega}}^1 \sqrt{\frac{g(\omega)}{c}} d\omega + \frac{[1-p(z-\pi)][1+\sigma(\pi)\underline{\omega}]\overline{R}(\pi,\underline{\omega})}{p(z-\pi)[\theta-\overline{R}(\pi,\underline{\omega})]} - \frac{\mu\sigma(\pi)f(\pi,\underline{\omega})}{(1-\mu)p(\pi)p(z-\pi)} \frac{d\underline{\omega}}{dc}$. Since Π_0 solves $\Pi_0 = \pi_k(\underline{\omega}_0)$ and $f(\Pi_0,\underline{\omega}_0) = 0$ for some $\underline{\omega}_0 \in (\xi_{k,1},\xi)$, it follows that $\frac{d\Pi_0}{dc} < 0$. What remains now is to show $\frac{d\xi_{k,1}}{dc} > 0$. Recall that $\xi_{k,1} \in (0,\xi)$ solves $s(\xi_{k,1}) = H$ with $s(\cdot)$ and H as defined in the Lemma 3 proof. Differentiating the equation for $\xi_{k,1}$ yields $\frac{d\xi_{k,1}}{1-g(\xi_{k,1})} = \frac{\int_{\xi_{k,1}}^1 \left(\sqrt{\frac{\theta\alpha(\omega)}{c}} - 1\right) d\omega + \frac{1-p(z)}{p(z)} \left(\sqrt{\frac{\theta}{c}} \int_0^1 \alpha(\omega) d\omega} - 1\right)}{1-g(\xi_{k,1})}$. Both numerator terms are positive under the assumptions of Proposition 8 so $\xi_{k,1} \in (0,\xi)$ implies $g(\xi_{k,1}) < 1$ and thus $\frac{d\xi_{k,1}}{dc} > 0$. \square

Taken together, Lemmas 5 and 6 say that $\Omega_0 > \underline{\omega}$ and $\Pi_0 < \overline{\pi}$ are sufficient for a unique equilibrium. Recalling from the proof of Lemma 5 that $\xi_{k,1}$ is always a lower bound on Ω_0 , Lemma 7 then says that higher values of c can be used to increase Ω_0 relative to $\underline{\omega}$ and decrease Π_0 relative to $\overline{\pi}$. I must now show that $c \leq \frac{\theta \alpha(0)^2}{\alpha(1)}$ as per Proposition 8 does not preclude existence of a sufficiently high c. In proving Proposition 9 up to this point, I have assumed $g(1) > 1 > \int_0^1 g(\omega) d\omega$. Given $g(\omega) \equiv \left(\sqrt{\theta \alpha(\omega)} - \sqrt{c}\right)^2$, this imposes implicit bounds $c \in (\underline{c}_q, \overline{c}_g)$. I have also made two assumptions about p(z). The first, $p(z) > \underline{p}_{1} = \left(1 + \frac{\int_{\xi_{x}}^{1} \left[\left(\sqrt{\theta\alpha(\omega)} - \sqrt{\frac{\theta}{\alpha(1)}}\alpha(0)\right)^{2} - 1\right] d\omega}{1 - \left(\sqrt{\theta}\int_{0}^{1} \alpha(\omega) d\omega - \sqrt{\frac{\theta}{\alpha(1)}}\alpha(0)\right)^{2}}\right)^{-1} \text{ where } \alpha\left(\xi_{x}\right) \equiv \frac{1}{\theta} \left(1 + \sqrt{\frac{\theta}{\alpha(1)}}\alpha\left(0\right)\right)^{2}, \text{ was } \beta\left(1 + \sqrt{\frac{\theta}{\alpha(1)}}\alpha\left(0\right)\right)^{2}\right)^{-1}$ in Lemma 3. The second, $p(z) \ge \underline{p}_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\mu p'(z)}{(1-\mu)p'(0)}}$, was in Lemma 6. Notice that $\alpha(0) = \overline{a} \equiv \alpha(1) - \sqrt{\frac{\alpha(1)}{\theta}}$ would yield $\xi_x = 1$ and thus $\underline{p}_1 = 1 > \underline{p}_2$. Also notice $\frac{d\underline{p}_1}{d\alpha(0)} > 0$ and $\frac{d\underline{p}_2}{d\alpha(0)} = 0$. Therefore, there is an $\underline{a}_1 \in (0, \overline{a})$ such that $\alpha(0) \in [\underline{a}_1, \overline{a})$ yields $\underline{p}_1 \in (\underline{p}_2, 1)$ and collapses the two assumptions on p(z) into $p(z) > \underline{p}_1$. Return now to the issue of $c \leq \frac{\theta \alpha(0)^2}{\alpha(1)}$. Define $\underline{a}_2 \equiv \sqrt{\alpha(1)} \left[\int_0^1 \sqrt{\alpha(\omega)} d\omega - \sqrt{\frac{1}{\theta} + \left(\int_0^1 \sqrt{\alpha(\omega)} d\omega \right)^2 - \int_0^1 \alpha(\omega) d\omega} \right] < \overline{a}$. Imposing $\alpha(0) \in (\underline{a}_2, \overline{a})$ ensures $\frac{\theta \alpha(0)^2}{\alpha(1)} \in (\underline{c}_g, \overline{c}_g)$. If $c = \frac{\theta \alpha(0)^2}{\alpha(1)}$ and $p(z) = \underline{p}_1$, then $\xi_{k,1} = \xi_x = \xi_{k,2}$. This means that $\pi_k(\underline{\omega})$ is only defined for $\underline{\omega} = \xi_x$ and, at that point, $\pi_k(\xi_x) = 0$. Trivially then, $\Pi_0 = 0$ so $\Pi_0 < \overline{\pi}$ is satisfied. Moreover, $\underline{\underline{\omega}}_2(0) = \frac{1}{2p(z)}$ so, given that $\alpha''(\cdot) \leq 0$ implies $\underline{\underline{\omega}}_1 \leq \frac{1}{2}$, we are left with $\underline{\underline{\omega}} = \frac{1}{2p(z)}$. Satisfying $\Omega_0 > \underline{\underline{\omega}}$ will thus require $\xi_x > \frac{1}{2p(z)}$ or, equivalently, $p(z) > \frac{1}{2\xi_x}$. A sufficient condition is $\underline{p}_1 \ge \frac{1}{2\xi_x}$. We have already seen that $\alpha(0) = \overline{a}$ would yield $\xi_x = 1$ and $\underline{p}_1 = 1$, satisfying $\underline{p}_1 \ge \frac{1}{2\xi_x}$ with strict inequality. The derivatives $\frac{d\underline{p}_1}{d\alpha(0)} > 0$ and $\frac{d\xi_x}{d\alpha(0)} > 0$ then ensure existence of an $\underline{a}_3 \in (0, \overline{a})$ such that $\alpha(0) \in [\underline{a}_3, \overline{a})$ yields $\underline{p}_1 \in [\frac{1}{2\xi_r}, 1]$. In short, imposing $\alpha(0) \in (\max\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}, \overline{a})$ prevents $c \leq \frac{\theta \alpha(0)^2}{\alpha(1)}$ from precluding existence of a sufficiently high c when p(z) is close enough to \underline{p}_1 . We can then impose $p(z) \in \left(\underline{p}_1, \underline{p}_1 + \varepsilon\right)$ for some $\varepsilon > 0$ and find $[\underline{c}, \overline{c}] \subseteq \left[\underline{c}_g, \frac{\theta \alpha(0)^2}{\alpha(1)}\right)$.

Part 3: Validation of Non-Binding P/C

Begin with the participation constraints for a type $\omega \in [\underline{\omega}, 1]$ borrower. Let $Y(\omega)$ denote ω 's value if unmatched and let $V(\omega, R)$ denote his value if newly matched at loan rate R. Recall from the main text that the informed and uninformed loan rates $(R(\omega) \text{ and } \overline{R} \text{ respectively})$ both induce $e(\omega, R) = 1 - \frac{c}{\alpha(\omega)(\theta - R)} > 0$ so we can write:

$$V(\omega, R) = \frac{\mu}{1-\beta(1-\mu)} \left[\alpha(\omega)(\theta - R) + c \ln\left(\frac{c}{\alpha(\omega)(\theta - R)}\right) - c + \beta Y(\omega) \right]$$
$$Y(\omega) = p(\pi) p(z - \pi) V(\omega, R(\omega)) + p(\pi) \left[1 - p(z - \pi)\right] V(\omega, \overline{R}) + \left[1 - p(\pi)\right] \beta Y(\omega)$$

There are two participation constraints: $V(\omega, R(\omega)) \ge \beta Y(\omega)$ and $V(\omega, \overline{R}) \ge \beta Y(\omega)$. The first, $V(\omega, R(\omega)) \ge \beta Y(\omega)$, ensures that ω prefers an informed match now over waiting for an uninformed match. The second, $V(\omega, \overline{R}) \ge \beta Y(\omega)$, ensures that he prefers an uninformed match now over waiting for an informed match. Using the expressions for $V(\omega, R)$ and $Y(\omega)$ along with the definitions of $R(\omega)$ and \overline{R} , the constraints reduce to:

$$\Theta_{1}(\omega) \geq \max\left\{-\frac{1-\beta(1-\mu)[1-p(\pi)p(z-\pi)]}{1-\beta(1-\mu)}\Theta_{2}\left(\omega,\overline{\alpha}\left(\pi,\underline{\omega}\right)\right), \frac{\beta(1-\mu)p(\pi)[1-p(z-\pi)]}{1-\beta(1-\mu)}\Theta_{2}\left(\omega,\overline{\alpha}\left(\pi,\underline{\omega}\right)\right)\right\}$$

$$where$$

$$\Theta_{1}(\omega) \equiv \sqrt{c\theta\alpha\left(\omega\right)} + \frac{c}{2}\ln\left(\frac{c}{\theta\alpha(\omega)}\right) - c$$

$$\Theta_{2}\left(\omega,\overline{\alpha}\left(\pi,\underline{\omega}\right)\right) \equiv \left(\sqrt{\frac{\alpha(\omega)}{\overline{\alpha}(\pi,\underline{\omega})}} - 1\right)\sqrt{c\theta\alpha\left(\omega\right)} + \frac{c}{2}\ln\left(\frac{\overline{\alpha}(\pi,\underline{\omega})}{\alpha(\omega)}\right)$$

If $\Theta_2(\omega, \overline{\alpha}(\pi, \underline{\omega})) \geq 0$, then the constraint is just $\Theta_1(\omega) \geq \frac{\beta(1-\mu)p(\pi)[1-p(z-\pi)]}{1-\beta(1-\mu)}\Theta_2(\omega, \overline{\alpha}(\pi, \underline{\omega}))$. Since $e(\omega, R(\omega)) > 0$ implies $\Theta_1(\omega) > 0$, the constraint is satisfied with strict inequality in the limiting case of $\pi = 0$. Consider now $\pi > 0$. With $\overline{\alpha}(\pi, \underline{\omega})$ as defined in the main text, $\frac{d\Theta_2(\omega, \overline{\alpha}(\pi, \underline{\omega}))}{d\pi} = \frac{c\sigma'(\pi)}{2\overline{\alpha}(\pi, \underline{\omega})} \left(\sqrt{\frac{\theta\alpha(\omega)^2}{c\overline{\alpha}(\pi, \underline{\omega})}} - 1\right) \frac{\omega \int_0^1 \alpha(\omega) d\omega - \int_0^{\underline{\omega}} \alpha(\omega) d\omega}{[1+\sigma(\pi)\underline{\omega}]^2}$. The assumptions of Proposition 8 guarantee $\sqrt{\frac{\theta\alpha(\omega)^2}{c\overline{\alpha}(\pi, \underline{\omega})}} > 1$ and the proof of Lemma 6 established $\underline{\omega} \int_0^1 \alpha(\omega) d\omega > \int_0^{\underline{\omega}} \alpha(\omega) d\omega$. The proof also established $\sigma'(\pi) > 0$ for $\pi \in (0, \overline{\pi}_1)$. Therefore, $\frac{d\Theta_2(\omega, \overline{\alpha}(\pi, \underline{\omega}))}{d\pi} > 0$ for $\pi \in (0, \overline{\pi}_1)$. Combined with $\frac{d\Theta_1(\omega)}{d\pi} = 0$ and $\frac{dp(\pi)[1-p(z-\pi)]}{d\pi} > 0$, we can now conclude that there exists a $\overline{\pi} \in (0, \overline{\pi}_1)$ such that the participation constraint is satisfied for all $\pi \in (0, \overline{\pi})$. If $\Theta_2(\omega, \overline{\alpha}(\pi, \underline{\omega})) < 0$, the constraint becomes $\Theta_1(\omega) \geq \frac{1-\beta(1-\mu)[1-p(\pi)p(z-\pi)]}{1-\beta(1-\mu)} |\Theta_2(\omega, \overline{\alpha}(\pi, \underline{\omega}))|$.

A sufficient condition for this is
$$\frac{\beta(1-\mu)p(\pi)p(z-\pi)}{1-\beta(1-\mu)[1-p(\pi)p(z-\pi)]}\sqrt{\frac{\theta\alpha(\omega)}{c}} \le \sqrt{\frac{\theta\alpha(\omega)^2}{c\overline{\alpha}(\pi,\underline{\omega})}} + \frac{1}{2}\ln\left(\frac{c\overline{\alpha}(\pi,\underline{\omega})}{\theta\alpha(\omega)^2}\right) - 1.$$

The right-hand side of the sufficient condition is decreasing in $\overline{\alpha}(\cdot)$ so $\overline{\alpha}(\cdot) \leq \int_0^1 \alpha(\omega) d\omega$ yields an even stricter condition, namely $\frac{\beta(1-\mu)p(\pi)p(z-\pi)}{1-\beta(1-\mu)[1-p(\pi)p(z-\pi)]}\sqrt{\frac{\theta\alpha(\omega)}{c}} \leq y(\omega) - \ln(y(\omega)) - 1$ with $y(\omega) \equiv \sqrt{\frac{\theta\alpha(\omega)^2}{c\int_0^1 \alpha(x)dx}} > 1$. By properties of the $\ln(\cdot)$ function, the right-hand side of the stricter condition is strictly positive while the left-hand side is zero at $\pi = 0$. Increasing π initially increases the left-hand side while leaving the right-hand side unchanged. So, once again, the participation constraint will be satisfied for any π below some positive bound. Sufficient parameter restrictions will then go in the same direction as the end of Part 2.

Derivation of Equation (22)

Conditional on the allocations $\kappa(\omega)$ and $\overline{\kappa}$, the expected capital from putting type ω in an informed match is $e(\omega, \kappa(\omega)) \alpha(\omega) \kappa(\omega)$ while the expected capital from putting him in an uninformed match is $e(\omega, \overline{\kappa}) \alpha(\omega) \overline{\kappa}$. Conditional on π and $\underline{\omega}$, the proportion of type ω s in informed matches is $\lambda(\omega)$ while the proportion in uninformed matches is $\phi(\omega)$ as per equations (5) and (6). The capital-maximizing planner thus chooses $\pi, \underline{\omega}, \kappa(\cdot)$, and $\overline{\kappa}$ to maximize $\frac{\mu}{1-\beta} \int_0^1 [e(\omega, \kappa(\omega)) \alpha(\omega) \kappa(\omega) \lambda(\omega) + e(\omega, \overline{\kappa}) \alpha(\omega) \overline{\kappa} \phi(\omega)] d\omega$ subject to an aggregate feasibility constraint. This constraint requires that the amount of capital allocated to firms equals the amount of capital available to the planner each period. In other words, $\mu \int_0^1 [\lambda(\omega) + \phi(\omega)] d\omega = \mu \int_0^1 [e(\omega, \kappa(\omega)) \alpha(\omega) \kappa(\omega) \lambda(\omega) + e(\omega, \overline{\kappa}) \alpha(\omega) \overline{\kappa} \phi(\omega)] d\omega$. Define $\widetilde{\gamma} \equiv \gamma (1-\beta)$ where γ is the multiplier on the feasibility constraint. The Lagrangian is:

$$\mathcal{L} = (1+\widetilde{\gamma}) \left(\frac{p(\pi)[1-p(z-\pi)]}{\mu+(1-\mu)p(\pi)[1-p(z-\pi)]} \int_0^{\omega} e(\omega,\overline{\kappa}) \alpha(\omega) d\omega + \frac{p(\pi)[1-p(z-\pi)]}{\mu+(1-\mu)p(\pi)} \int_{\omega}^1 e(\omega,\overline{\kappa}) \alpha(\omega) d\omega \right) \overline{\kappa} + \frac{(1+\widetilde{\gamma})p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)} \int_{\omega}^1 e(\omega,\kappa(\omega)) \alpha(\omega) \kappa(\omega) d\omega - \widetilde{\gamma} \left(\frac{p(\pi)[1-p(z-\pi)]\omega}{\mu+(1-\mu)p(\pi)[1-p(z-\pi)]} + \frac{p(\pi)(1-\omega)}{\mu+(1-\mu)p(\pi)} \right)$$

Optimizing with respect to $\kappa(\cdot)$ and $\overline{\kappa}$ yields $\kappa(\omega) = R(\omega)$ and $\overline{\kappa}(\pi,\underline{\omega}) = \overline{R}(\pi,\underline{\omega})$, with $R(\omega)$ and $\overline{R}(\pi,\underline{\omega})$ as defined in Proposition 8. Therefore and as desired, there are no direct inefficiencies stemming from $R(\cdot)$ and \overline{R} . Optimizing with respect to π and $\underline{\omega}$ then combining the first order conditions yields equation (22) in the main text.

Proof of Proposition 10

Compare equations (21) and (22) to get:

$$\frac{A(\pi)}{B(\pi)} \stackrel{\mathrm{Eq}(22)}{=} \widehat{X}\left(\pi,\underline{\omega}\right) \quad \equiv \quad \frac{\int_{\underline{\omega}}^{1} [g(\omega) - g(\underline{\omega})] d\omega - \frac{(1-\mu)p(\pi)[1-p(z-\pi)]}{\mu} Q(\pi,\underline{\omega})}{\frac{1}{B(\pi)} \int_{0}^{\underline{\omega}} [g(\underline{\omega}) - h\left(\omega,\overline{R}(\cdot)\right)] d\omega + B(\pi) \int_{\underline{\omega}}^{1} [g(\omega) - h\left(\omega,\overline{R}(\cdot)\right)] d\omega}$$

$$< \frac{\int_{\underline{\omega}}^{1} [g(\omega) - g(\underline{\omega})] d\omega}{\int_{0}^{\underline{\omega}} [g(\underline{\omega}) - h\left(\omega, \overline{R}(\cdot)\right)] d\omega + B(\pi) \int_{\underline{\omega}}^{1} [g(\omega) - h\left(\omega, \overline{R}(\cdot)\right)] d\omega} \equiv X(\pi, \underline{\omega}) \stackrel{\text{Eq}(21)}{=} \frac{A(\pi)}{B(\pi)}$$

where the inequality reflects $Q(\pi,\underline{\omega}) < 0$ and $\frac{1}{B(\pi)} > 1$ for any $\pi \in (0, z)$ and $\underline{\omega} \in (0, 1)$. Note that $A(\cdot)$ and $B(\cdot)$ are again as defined in the proof of Proposition 2. Recall $\frac{d}{d\pi} \frac{A(\pi)}{B(\pi)} > 0$ and use $\pi_e(\underline{\omega})$ to denote the function implicitly defined by equation (22). For a given value of π , we thus have $X(\pi, \pi_l^{-1}(\pi)) = \hat{X}(\pi, \pi_e^{-1}(\pi)) < X(\pi, \pi_e^{-1}(\pi))$. It is straightforward to show $\frac{\partial X(\pi,\underline{\omega})}{\partial \underline{\omega}} < 0$ so $\pi_l^{-1}(\pi) > \pi_e^{-1}(\pi)$ follows. In other words, $\pi_l(\cdot)$ lies to the right of $\pi_e(\cdot)$ in $\underline{\omega}$ - π space. From Part 2 of the Proposition 9 proof, recall $\underline{\omega}^* \in (\xi_{k,1}, \Omega_0)$ and $\pi'_k(\underline{\omega}) > 0$ for all $\underline{\omega} \in (\xi_{k,1}, \Omega_0)$. Combining with the fact that $(\hat{\pi}, \underline{\widehat{\omega}})$ also lies on $\pi_k(\cdot)$ completes the proof of $(\pi^*, \underline{\omega}^*) \gg (\hat{\pi}, \underline{\widehat{\omega}})$. Turn now to the moral moral hazard analog of Proposition 6. The remitted matching tax yields the following in lieu of equation (21):

$$\frac{1-p\left(z-\pi\right)+\frac{p(\pi)p'(z-\pi)}{p'(\pi)}}{1-\frac{(1-\mu)p(\pi)p(z-\pi)}{\mu+(1-\mu)p(\pi)}} = \frac{\int_{\underline{\omega}}^{1}\left[g\left(\omega\right)-g\left(\underline{\omega}\right)\right]d\omega - \left(1+\frac{(1-\mu)p(\pi)p(z-\pi)\underline{\omega}}{\mu+(1-\mu)p(\pi)[1-p(z-\pi)]}\right)\frac{1-\beta(1-\mu)}{\mu p'(\pi)}\tau}{\int_{0}^{\underline{\omega}}\left[g\left(\underline{\omega}\right)-g\left(\omega\right)\right]d\omega + \Delta\left(\pi,\underline{\omega}\right)}$$

The tax is defined so that equation (20) is unchanged. Implementing $(\hat{\pi}, \hat{\omega})$ thus requires:

$$\tau = \frac{\frac{(1-\mu)p(\widehat{\pi})p(z-\widehat{\pi})}{\mu+(1-\mu)p(\widehat{\pi})} \left[\int_{\underline{\widehat{\omega}}}^{1} [g(\omega) - g(\underline{\widehat{\omega}})] d\omega - A(\widehat{\pi}) \int_{\underline{\widehat{\omega}}}^{1} D\left(\omega, \overline{R}(\widehat{\pi}, \underline{\widehat{\omega}})\right) d\omega \right] + \frac{(1-\mu)p(\widehat{\pi})[1-p(z-\widehat{\pi})]}{\mu} B(\widehat{\pi})Q(\widehat{\pi}, \underline{\widehat{\omega}})}{\frac{1-\beta(1-\mu)}{\mu p'(\widehat{\pi})} \left(1 + \frac{(1-\mu)p(\widehat{\pi})p(z-\widehat{\pi})\underline{\widehat{\omega}}}{\mu+(1-\mu)p(\widehat{\pi})[1-p(z-\widehat{\pi})]} \right)}$$

Substituting in for $Q(\widehat{\pi}, \underline{\widehat{\omega}})$ and rearranging confirms $\tau > 0$.