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INFLATING AWAY THE PUBLIC DEBT? AN EMPIRICAL ASSESSMENT

Jens Hilscher
Alon Raviv
Ricardo Reis

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ABSTRACT

We propose and implement a method that provides quantitative estimates of the extent to which higher-than-expected inflation can lower the real value of outstanding government debt. Looking forward, we derive a formula for the debt burden that relies on detailed information about debt maturity and claimholders, and that uses option prices to construct risk-adjusted probability distributions for inflation at different horizons. The estimates suggest that it is unlikely that inflation will lower the US fiscal burden significantly, and that the effect of higher inflation is modest for plausible counterfactuals. If instead inflation is combined with financial repression that ex post extends the maturity of the debt, then the reduction in value can be significant.

Jens Hilscher
Brandeis University
International Business School
Lemberg Academic Center 252
415 South Street
Waltham, MA 02454-9110
hilscher@brandeis.edu

Ricardo Reis
Department of Economics, MC 3308
Columbia University
420 West 118th Street, Rm. 1022 IAB
New York NY 10027
and NBER
rreis@columbia.edu

Alon Raviv
Brandeis University
International Business School
Mailstop 32
415 South Street
Waltham, MA 02454-9110
araviv@brandeis.edu

1 Introduction

A higher inflation target has some benefits, and one of its most celebrated is to erode the real value of outstanding debt. While across centuries and countries, a common way that sovereigns have paid for high public debt is by having higher, and sometimes even hyper, inflation, this rarely came without some or all of fiscal consolidation, financial repression, and partial default (Reinhart and Rogoff, 2009). As a result, the effectiveness of higher inflation in alleviating the fiscal burden of a country both in the past and in the future is an open empirical question. A pressing application of this question is: with U.S. total public debt at its highest ratio of GDP since 1947, would higher inflation be an effective way to pay for it?

Providing an answer requires tackling two separate issues. The first is to calculate by how much would 1% unanticipated and permanently higher inflation lower the debt burden. If all of the U.S. public debt outstanding in 2012 (101% of GDP) were held in private hands, if it were all nominal, and if it all had a maturity equal to the average (5.4 years), then a quick back-of-the-envelope answer is 5.5%.¹ However, we will show that this approximation is misleading. In fact, we estimate that the probability that the reduction in U.S. debt is as large as 5.5% of GDP is below 0.05%. The approximation is inaccurate since the underlying assumptions are inaccurate. The debt number is exaggerated because large shares of the debt are either held by other branches of the government or have payments indexed to inflation and the maturity number is inaccurate because it does not take into account the maturity composition of privately-held nominal debt.

The second issue is that assuming a sudden and permanent increase in inflation by an arbitrary amount (1% in the above example) is empirically not helpful. After all, if

¹To justify this calculation, assume the government owes X , all due in m periods. Its current market value is $V = Xe^{-(r+\pi)m}$ with continuous-time discounting and where r is the real interest rate and π is the rate of inflation, both assumed to be constant. Then $\partial V/\partial\pi = -mV$. An alternative justification assumes that the government owes an amount $Xe^{-t/m}$ at every future date t , so the distribution of outstanding debt is exponential. Its market value today then is: $X/(r + \pi + 1/m)$. Since the nominal interest was approximately zero in 2012, we get approximately the same answer.

the price level could suddenly jump to infinity, the entire nominal debt burden would be trivially eliminated. It is important first to recognize that inflation is stochastic, and that investors will take this risk into account when pricing and choosing to hold government debt—if investors anticipated sudden infinite inflation, they would not be willing to hold government debt at a positive price. Second, the central bank does not perfectly control inflation, so that even if it wanted to raise inflation by 1% it might not be able to. Moreover, there are many possible paths to achieving higher inflation, either doing so gradually or suddenly, permanently or transitorily, in an expected or unexpected way, and we would like to know how they vary in effectiveness. Therefore, it is important to consider counterfactual experiments that economic agents believe are possible.

This paper addresses both issues, by proposing and implementing a method to accurately estimate the effect of plausible levels of inflation on the real value of debt. Section 2 starts by deriving a simple formula for the fiscal burden of outstanding debt. By taking a forward-looking approach to the intertemporal government budget constraint, we show that the real value of debt is equal to a weighted average of the payments due at different horizons, with the weights given by the expected inverse of compounded risk-adjusted inflation. The formula makes transparent how high inflation can affect the fiscal burden via these weights.

Section 3 collects data on the maturity structure of payments associated with Treasury securities made by the federal government to various entities. We focus first on private holdings, then separate private holdings between domestic and foreign, and finally single out the Federal Reserve in the public holdings.² We find that the private sector holds a disproportionate share of the debt of shorter maturities and very few longer maturities, whereas the Federal Reserve's portfolio has the opposite composition. We show that this difference in composition results in private holdings having a lower exposure to inflation than the Federal Reserve.

²We refer to marketable debt held by the private sector as private holdings.

We next construct risk-adjusted distributions for inflation. Section 4 introduces data on inflation contracts in the form of inflation caps and floors with payoffs that depend on the realizations of CPI inflation. Exploring the variety of contracts that are traded with different strike prices, we extract the implied stochastic discount factors for inflation used by market participants at different maturities. We build on, and go beyond, the calculations in contemporaneous work by Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2013), providing maturity structures of risk-adjusted marginal density functions for both cumulative inflation and one-year forward inflation.

Combing both sets of marginal densities, section 5 proposes a new method to estimate the joint risk-adjusted distribution of inflation at different future horizons. Our method has the intriguing feature that it uses no time-series data on the realizations of inflation. It is based entirely on the information provided by the different options contracts at a given date, and relies on the theory of copulas. The joint distributions allow us to calculate novel value-at-risk measures of the debt debasement due to inflation, and to consider a rich set of counterfactual inflation distributions to investigate what drives the results.

Using all these inputs, section 6 calculates the probability that the present value of debt debasement due to inflation is larger than any given threshold. The 5th percentile of this value at risk calculation is a mere 3.1% of GDP, and any loss above 4.2% has less than 1% probability. Interestingly, much of the effect of inflation would fall on foreign holders of the government debt, who hold the longer maturities. The Federal Reserve, which also holds longer maturities, would also suffer larger capital losses.

Section 7 explores what is behind the modest effect of inflation on the fiscal burden by analyzing a series of counterfactuals. The result is driven by two complementary factors. First, the maturity of U.S government debt that is privately held is quite low. By comparison, we show that with the outstanding debt composition of 2000 the effects would be more than one third higher. Second and related, over short horizons of a few years, market participants

place a very low probability on U.S. inflation being significantly high. In the near term, there is much debt but little extra inflation, and for longer horizons, there can be significant inflation but there is little debt. The total resulting effect is small.

Section 8 explores the role of an active policy tool that interacts with inflation and is often used in developing countries: financial repression. It drives a wedge between market interest rates and the interest rate on government bonds, and acts as a tax on the existing holders of the government debt. We show that extreme financial repression, where bondholders are paid with reserves at the central bank which they must hold for a fixed number of periods, is equivalent to ex post extending the maturity of the debt. Under such circumstances inflation has a much larger impact, such that if repression lasts for a decade, permanently higher inflation that previously lowered the real value of debt by 3.7% now lowers it by 23% of GDP.

Section 9 concludes with suggestions for further research.

In terms of its place in the literature, our paper is one in a long list that studies the link between fiscal policy and inflation. More recently, Cochrane (2011*b*) and Davig, Leeper and Walker (2011) have argued that high levels of U.S. debt may lead to higher inflation through the fiscal theory of the price level, while Aizenman and Marion (2011) argue that policymakers have a strong incentive to inflate this debt. Our goal is more applied as we quantify the amount by which inflation can actually lower the public debt burden. Our estimates might be useful to calibrate this class of models in the future.

Closer to our question, Hall and Sargent (2011) provide an accounting decomposition of the evolution of public debt applied to U.S. historical data, while Reinhart and Sbrancia (2011) emphasize how inflation coupled with financial repression helped developed countries pay their debts after World War II. Our methods are instead forward looking, so we can consider different counterfactual scenarios. We confirm some of their key results, showing that it is the interaction between financial repression and long maturities of debt that allows

for significant effects of inflation. At the same time, we exploit the risk-adjusted joint distribution of inflation to derive a set of richer predictions that depend on the dynamics of inflation, and we use a simpler and forward looking formula for the debt burden.

Berndt, Lustig and Yeltekin (2012) and Chung and Leeper (2007) use vector autoregressions to estimate the impact of fiscal spending shocks on different terms in the intertemporal budget constraint. We focus on inflation shocks instead, and we directly measure the impact on future discount rates using inflation options data. Giannitsarou and Scott (2008) show that fiscal imbalances do not help to forecast future inflation in six advanced economies. We instead use options market data to make forecasts, and we do not ask whether in the past governments have used inflation to pay for debts, but rather what would be the impact of doing so today. Our goal is to understand what the limits are to using the option to inflate, rather than to ask whether or not that option has been chosen in the past.

Aizenman and Marion (2011) and Bohn (2011) also consider counterfactuals about the future, but they make rough approximations of the maturity of the privately-held debt and treat inflation as deterministic. We tackle these two issues directly and as precisely as possible. Our data should make these approximations no longer necessary in the future.

Krause and Moyen (2013) use a DSGE model that makes many structural and behavioral assumptions in order to investigate several links through which inflation may affect debt, fiscal surpluses, and seignorage. We only assume no arbitrage in the government debt market, and focus exclusively on the debasement of debt. At the same time, they use several approximations in treating the data, whereas we go into more detail. Faraglia et al. (2013) and Leeper and Zhou (2013) also write DSGE models to study how optimal inflation depends on the maturity of government debt partly through its effect on the real value of debt. In the other direction, Missale and Blanchard (1994) study how the maturity of debt depends on the government's credibility to keep inflation low. Our goal is positive, not normative, and again our estimates should allow researchers to calibrate their models.

Higher inflation also redistributes wealth between different domestic households, or between them and foreigners (Doepke and Schneider (2006); Berriel (2013)). We focus on the redistribution from private holders of debt, domestic or foreign, to the fiscal authority. Moreover, higher inflation may also raise fiscal surpluses, for instance by raising seignorage (Hilscher, Raviv and Reis, 2014), but here we consider only its effects on the real value of debt outstanding. Our estimates provide one valuable piece to assess the redistributive effects of inflation, but there are more pieces that future research can investigate.

Finally, while there are many ways to extract objective and subjective probability forecasts for inflation, including financial prices, surveys, and economic and statistical models, these methods tend to forecast the mean while being silent on higher moments. Crucially, they are not appropriate for pricing. Our goal is to measure the market value of different policies, so we need the risk-neutral probabilities that are relevant for pricing the government debt. Together with Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2013), we are one of the first papers to extract the risk-neutral density for inflation from option contracts and use it to ask macroeconomic questions. Kitsul and Wright (2013) look at the response of the density around monetary policy announcements, while Fleckenstein, Longstaff and Lustig (2013) assess the risk of deflation in the United States. We ask a different question. Moreover, whereas they only use data on cumulative inflation, we are the first to construct distributions of one year forward inflation as well, and to integrate cumulative and forward inflation data to estimate a risk-adjusted joint distribution for inflation at different horizons.³

³One-year forward distributions are constructed from year-on-year inflation options, whose payoff depends on the realization of inflation over one-year periods.

2 Theory: the debt burden and risk-neutral densities

Our goal is to measure the fall in the debt burden due to higher inflation. This requires coming up with a workable definition of the debt burden, seeing the effect of inflation on it, and estimating its size. This section derives a simple formula that accomplishes these three goals.

2.1 The public debt

Letting W_t denote the real market value of government debt at date t :

$$W_t = \sum_{j=0}^{\infty} \frac{H_t^j B_t^j}{P_t} + \sum_{j=0}^{\infty} Q_t^j K_t^j. \quad (1)$$

Going over each of the terms on the right-hand side: B_t^j is the par value of zero-coupon nominal debt held at date t that has a maturity of j years, so that at date t the government expects to pay B_t^j dollars at date $t + j$. K_t^j is the par value of real debt held at date t that has a maturity of j years, referring mostly to Treasury inflation protected securities (TIPS). H_t^j is the market price (or inverse-yield) at which nominal debt with a maturity of j years trades at date t . Likewise, Q_t^j is the price (or inverse-yield) of TIPS with a maturity of j years at date t . Finally, P_t is the price level, and we will use the notation $\pi_{t,t+j} = P_{t+j}/P_t$ to denote gross cumulative inflation between two dates. The following normalizations apply: $H_t^0 = Q_t^0 = 1$ and $P_0 = 1$.

Modeling the government debt this way involves some simplifications. First, the government often has a wide variety of non-market outstanding debt. The implicit assumption above is that their price is the same as that of marketable debt, which should be the case through the forces of arbitrage between these different securities. Second, it assumes that coupon-paying bonds can be priced as portfolios of zero-coupon bonds. In this way, we

limit the huge variety of debt instruments issued by the government and simply consider promised payments (either principal or coupon payments) at each point in time. Again, arbitrage should imply that this assumption is reasonable. Third, unfunded nominal liabilities of the government like Social Security could be included in B_t^j , and the real assets (and real liabilities) of the government could be included in K_t^j . Theoretically, they pose no problem. In practice, measuring any of these precisely, or taking into account their lower liquidity, is a challenge that goes beyond this paper, so we will leave them out.

If all debt were short-term, then the expression in equation (1) would reduce to $B_t^0/P_t + K_t^0$. The simple rule of thumb that an increase in P_t lowers the debt burden proportionately to the privately-held nominal debt is accurate. However, with longer maturities, future inflation and higher future price levels affect yields and so also the value of debt. Without knowing how yields of different government liabilities depend on inflation, this equation cannot answer our question.

2.2 The law of motion for debt

To pay for the debt, the government must either collect a real fiscal primary surplus of s_t , or borrow more from the private sector:

$$W_t = s_t + \sum_{j=0}^{\infty} \frac{H_t^{j+1} B_{t+1}^j}{P_t} + \sum_{j=0}^{\infty} Q_t^{j+1} K_{t+1}^j. \quad (2)$$

Combining the previous two equations provides a law of motion for debt. Looking forward from date 0 for t periods, we can write it as:

$$\begin{aligned} W_0 = & W_{t+1} \prod_{\tau=0}^t Q_{\tau}^1 + \sum_{i=0}^t \prod_{\tau=0}^i Q_{\tau}^1 s_i + \sum_{i=0}^t \prod_{\tau=0}^i Q_{\tau}^1 \sum_{j=0}^{\infty} (H_i^{j+1} - H_i^1 H_{i+1}^j) \frac{B_{i+1}^j}{P_i} \\ & + \sum_{i=0}^t \prod_{\tau=0}^i Q_{\tau}^1 \sum_{j=0}^{\infty} (Q_i^{j+1} - Q_i^1 Q_{i+1}^j) K_{i+1}^j + \sum_{i=0}^t \prod_{\tau=0}^i Q_{\tau}^1 \left(\frac{H_i^1 P_{i+1}}{P_i} - Q_i^1 \right) \sum_{j=0}^{\infty} \frac{H_i^{j+1} B_{i+1}^j}{P_{i+1}}. \quad (3) \end{aligned}$$

This equation makes apparent why it is difficult to answer our question. Inflation can affect almost every term on the right-hand side without a clear way to decompose them. Worse, in order to judge how a particular path for inflation $\{\pi_{0,i}\}_{i=0}^t$ affects the fiscal burden, we would need to know how inflation will change the slope of the yield curve at every maturity (the $H_i^{j+1} - H_i^1 H_{i+1}^j$ term) or the maturity composition and the share of nominal versus real bonds that future governments will issue and the private sector will choose to hold (the B_{i+1}^j term). Likewise, we would need to know the link between inflation and the real yield curve (the $Q_i^{j+1} - Q_i^1 Q_{i+1}^j$ term) as well as the ex post differences between nominal and real returns (the $H_i^1 P_{i+1}/P_i - Q_i^1$ term). Finally, recall that this expression holds for every possible path of inflation as well as for realization of uncertainty in the economy. Therefore, there is an unwieldy large number of possible measures of how much the fiscal burden will change in the future.

Hall and Sargent (2011), and many that preceded them, partially overcome these problems by using a version of this equation that looks backwards, instead of forward, in time. Given debt in the present (W_{t+1}) and in the past (W_0), there are historical data on most of the terms above. However, this decomposition of the factors affecting the evolution of the debt is only able to measure the effect of inflation while keeping fixed every other interest rate, fiscal surplus, and outstanding bonds, so it is not possible to isolate the impact of inflation alone on the public debt. Moreover, our question requires us to look forward to figure out how debt depends on future, not past, inflation.

2.3 Looking forward: the intertemporal budget constraint

Our approach relies on one assumption: that there is a stochastic discount factor to price all of these government liabilities. It is well understood that this is equivalent to requiring the absence of arbitrage. While there is evidence against no-arbitrage across all assets (Cochrane, 2011a), note that all we require is that there is no arbitrage between Treasuries

of different maturities. U.S. government bond markets are one of the most liquid in the world, have fewer restrictions on short-selling, and serve as the fundamental asset for many traded derivatives. Moreover, note that we do not require the stochastic discount factor to be unique, so we are not assuming complete markets. We are also not excluding the possibility that there are varying risk premia or excess profits across different maturities, as in models of segmented markets or preferred habitats, so we are not ruling out possible liquidity premia or the effectiveness of quantitative easing policies. We have the much weaker assumption that there is no risk-free arbitrage across maturities of Treasuries.⁴

The stochastic discount factor at date t for a real payoff at date $t+j$ is denoted by $m_{t,t+j}$, and the conventional pricing equations for j -period bonds are:

$$1 = \mathbb{E}_t \left(\frac{m_{t,t+j}}{Q_t^j} \right) = \mathbb{E}_t \left(\frac{m_{t,t+j} P_t}{H_t^j P_{t+j}} \right). \quad (4)$$

A nominal discount bond costs H_t^j/P_t in real units at date t , and pays off $1/P_{t+j}$ real units in j periods; its return times the stochastic discount factor has to have an expectation of 1. Likewise for a real bond. The absence of arbitrage over time implies that the stochastic discount factors across any two maturities, n and j are linked by: $m_{t,t+j} = m_{t,t+n} m_{t+n,t+j}$ for $1 \leq n \leq j$.

Multiplying by stochastic discount factors at different dates and taking expectations of equation (3), while taking the limit as time goes to infinity and imposing that the government cannot run a Ponzi scheme, we get the following result:⁵

$$W_0 = \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} \left(\frac{B_0^t}{P_t} \right) \right] + \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} K_0^t \right] = \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} s_t \right]. \quad (5)$$

⁴Our construction of risk-neutral distributions in section 4 also requires no arbitrage in the inflation derivatives market.

⁵We could allow for bubbles or Ponzi schemes by the government. As long as their value does not depend on inflation, then proposition 1 below is unchanged.

The first equality provides a workable measure of the debt burden. The government liabilities were fixed in the past, so this expression depends only on real discount factors looking into the future and the distribution of all future price levels. Moreover, it shows that because the real payments do not depend on inflation, we can focus on the nominal debt to assess the effect of inflation.

The second equality shows that we can interpret our measure as saying how much fewer taxes the government can collect by lowering the debt burden. Higher inflation may not only lower the real payments on the outstanding nominal debt, but also change primary fiscal surpluses. In companion work (Hilscher, Raviv and Reis, 2014), we measure one of these effects through the seignorage revenues that higher inflation generates. But the question in this paper is what is the effect of inflation on the outstanding public debt, and expression 5 shows that this depends on outstanding nominal debt alone.

2.4 A formula for the debt burden as a weighted average

The only uncertainty in how much will pay the outstanding nominal debts that mature in t periods is on the realization of the price level. Therefore, even though the stochastic discount factor depends, in principle, on all sources of uncertainty in the economy, only its marginal distribution with respect to inflation will lead to non-zero terms once we multiply by inverse inflation and take expectations. Therefore:

$$\mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} \left(\frac{B_0^t}{P_t} \right) \right] = \sum_{t=0}^{\infty} B_0^t \mathbb{E} \left(\frac{m(\pi_{0,t})}{\pi_{0,t}} \right) = \sum_{t=0}^{\infty} R_t^{-1} B_0^t \int \left(\frac{f(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t}. \quad (6)$$

The second equality uses the standard definition of a risk-neutral density $f(\cdot)$, where R_t is the real risk-free return between 0 and t from the perspective of date 0. By definition, it does not depend on future realizations of inflation.

Combining all the results gives our formula for the debt burden as a function of inflation.

Proposition 1. *The debt burden is a weighted average of the nominal payments that the government must make at all present and future dates:*

$$\sum_{t=0}^{\infty} \omega_t B_0^t \tag{7}$$

with weights given by:

$$\omega_t = R_t^{-1} \int \left(\frac{f(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t} \tag{8}$$

This formula makes clear how future inflation affects the debt burden today. It explicitly takes into account that inflation is stochastic and not perfectly controlled by policy. It is forward looking, and it delivers a single number, in spite of all the possible future realizations for inflation. It depends on inflation only, as all of its relevant effects on prices are captured in the inflation densities. This includes the possible effect of inflation on real interest rates through Fisher effects, or changes over time in the compensation for inflation risk, or in liquidity premia, since the relevant densities are for risk-adjusted inflation. Finally, it allows for a discussion of counterfactuals that is somewhat disciplined, in terms of either different realizations from these densities or shifts in the densities themselves. Therefore, it satisfies all of the requirements that we laid out to answer our question.

Using this formula requires two key inputs: the payments due to private investors, and the risk-adjusted density for inflation at each future maturity. The next two sections explain how we measure them for the United States.

3 Data: U.S. Treasuries by holder and maturity

The total U.S. federal debt at the end of 2012, reported by the Bureau of the Public Debt, was \$16.4 trillion, or 101% of GDP. Yet, this number does not distinguish between nominal and indexed bonds, or include information about the maturity structure. Moreover, it includes

debt held by different branches of the U.S. government, even though we want to measure net government liabilities since any gains of the Treasury at the expense of Social Security, or any other government account, will sooner or later have to be covered by the Treasury.

3.1 Data on holdings of U.S. Treasuries at different maturities

The appendix describes our multiple data sources. The main source of data is the Center for Research on Security Prices (CRSP) that reports the private holdings of all outstanding marketable government notes and bonds at the end of 2012.⁶ We measure monthly total nominal payments, using both face value and coupons at each maturity. We also adjust the size of payments to exclude holdings by state and local governments. Then, we use the nominal yield curve to evaluate the formula in equation (1). The total market value of privately-held marketable nominal Treasury securities at the end of 2012 was \$8.3 trillion, or 51.2% of GDP.⁷

It is instructive to compare the headline debt number of \$16.4 trillion reported by the Treasury to our baseline number. For this purpose we focus on face values, as reported by the Monthly Statement of the Public Debt. The CRSP data does not include non-marketable securities (\$5.4 trillion), the vast majority of which are part of intragovernment holdings (\$4.8 trillion). The largest single such holding (\$2.6 trillion) is the Federal Old-Age and Survivors Insurance Trust Fund (Social Security). Since we focus on nominal debt, we exclude the amount of outstanding TIPS (\$0.8 trillion), which are almost exclusively held by the private sector. We next adjust the CRSP amounts to take account of the share of debt held by state and local governments, especially in state and local pension funds.⁸

⁶We build on Hall and Sargent (2011) and significantly extend their work, to a finer distribution of maturities, to consider different claimholders, and to use more sources of data.

⁷We can calculate this number according to equation (7) instead. The discrepancy is negligible, equal to about \$12 billion.

⁸We do not include debt issued by state and local government, estimated by the Census to amount to \$2.9 trillion in 2011. There is a large variety of these debt instruments, and no good source that reports their private holdings. We therefore restrict ourselves to federal debt.

Holdings are approximately 7.5% of total privately held debt. Finally, while the number so far is misleadingly called “debt held by the public” by the Treasury Bulletin, it includes the holdings by the Federal Reserve. Any losses on the portfolio of the central bank will map directly into smaller seignorage payments to the Treasury (Hall and Reis, 2013), so the same argument that excludes Social Security holdings applies to Federal Reserve holdings (\$1.6 trillion). Combining all of these numbers, we arrive at a face value of \$8 trillion, which corresponds closely to the face value of debt constructed from CRSP.

Next, we break private holdings into domestic and foreign by maturity. Foreigners held \$5.3 trillion in Treasury debt, or 33% of GDP. We also use data on holdings by Chinese investors, the largest holders of debt, which account for 6.9% of GDP.⁹

Finally, the Federal Reserve Bank of New York keeps the information on each bond held by the Federal Reserve in its System Open Market Account (SOMA). We use this information to also obtain detailed holdings for the Federal Reserve at each maturity. Valued at \$1.9 trillion these are not included in our baseline estimates, but we will consider them separately in the analysis.

3.2 The maturity distribution of holdings

Figure 1 shows B_0^t as a function of t using monthly data. A noticeable feature of the distribution is how concentrated it is on the short end. The average maturity of the U.S. government debt, weighted by private holdings, is 4.8 years according to our calculations. This is well below the 5.4 years reported by the Treasury for all outstanding debt.¹⁰

Moreover, it is salient from figure 1 that a simple approximation, like assuming a single bond with a maturity equal to the average, or an exponential distribution, is not appropriate. The maturity distribution decays very quickly in the first few years, then flattens between

⁹These amounts probably over-state how much foreigners ultimately hold, since they include holdings in offshore financial centers, some of which may be by American citizens.

¹⁰The Treasury estimate comes from the Quarterly Refunding Documents.

5 and 10 years, becomes close to zero between 10 to 25 years partly because of the lack of issuance of 30-year bonds between 2001 and 2006, and then picks up between 25 and 30 years.

Figure 2 aggregates to the annual frequency, and plots also the foreign holdings (so domestic is the difference between private and foreigners), together with the Fed's holdings. Private domestic holdings of U.S. Treasuries are concentrated in low maturities, with little holdings above 5 years. Foreigners account for a larger share of the 5-10 year holdings, and the Federal Reserve holds a disproportionate share of maturities above 5 years partly due to quantitative easing policies.

4 The marginal densities for inflation

The weights in equation (8) require knowing the term structure of risk-free real rates and the risk-adjusted density of inflation at different horizons. For the former, we use standard estimates of the real yield curve from Gürkaynak, Sack and Wright (2010). For the densities, we use new data on option contracts.¹¹

4.1 The data on zero-coupon and year-on-year inflation options

The market for over-the-counter USD inflation options emerged in 2002 and it has grown at a very fast rate, especially after 2008. Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2013) use data from these markets as well, and argue that since 2009 the market has been liquid enough to reliably reflect market expectations of inflation.¹² We use daily data of caps and floors on CPI inflation on December 31 of 2012.¹³

¹¹The appendix contains more details on the data, the method to estimate the distributions, and several robustness checks.

¹²Kitsul and Wright (2013) write that by 2011, trading in the inter-dealer market was close to \$ 22 billion, while J.P. Morgan (2013) estimates the annual trading volume in inflation derivatives at the start of 2013 was \$50 billion.

¹³The appendix looks at dates nearby to ensure there was nothing special about this particular date.

Figure 1: Government payments due to the private sector by maturity (B_0^t)

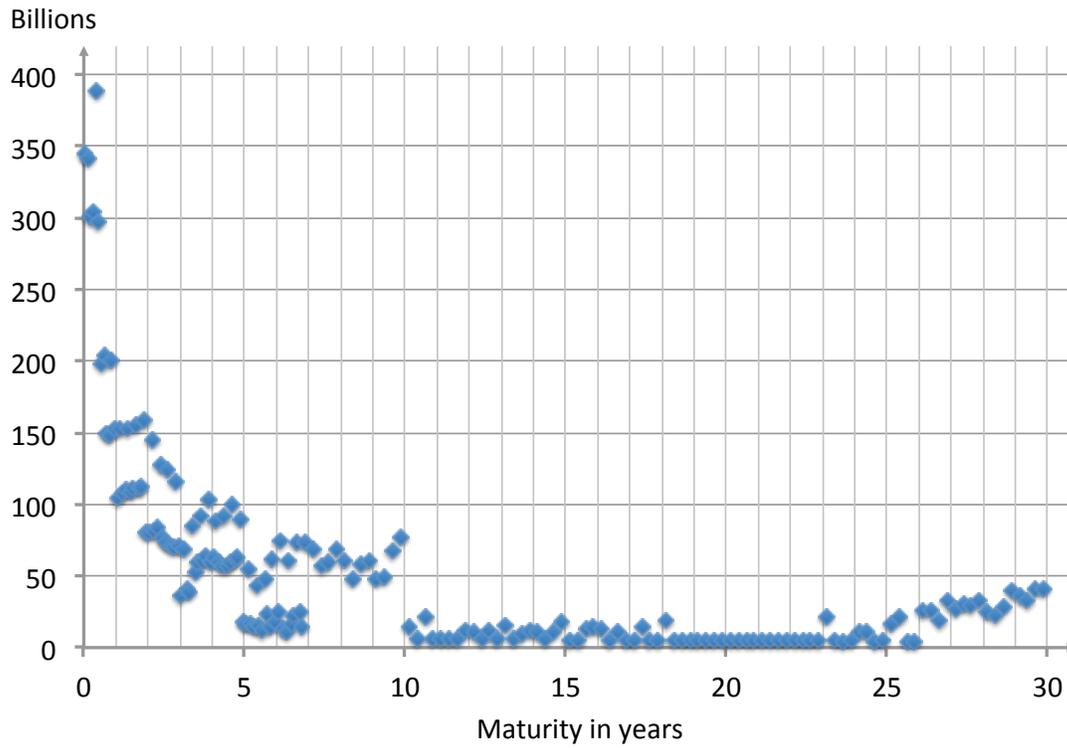
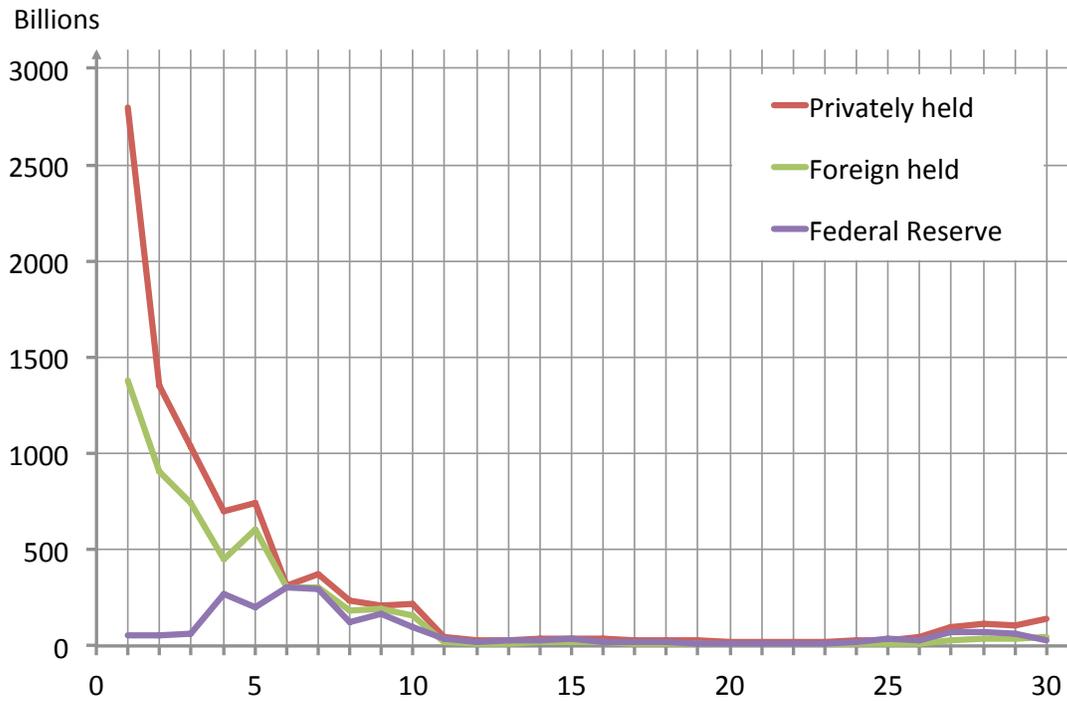


Figure 2: Breakdown of annual payments by holder of debt, by maturity



Like Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2013), we use data on zero-coupon caps and floors that pay off if cumulative inflation between the start of the contract and its maturity lies above or below the annually compounded strike price. The strike price ranges from -2% to 3% (floors) and 1% to 6% (caps), in 0.5% increments. We have data for all maturities between 1 and 10 years, together with data for 12 and 15 years, which we will use to check our estimates.

Unlike Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2013), we also use data on year-on-year inflation caps and floors. These contracts are portfolios of caplets and floorlets that pay off at the end of each year if inflation during that year is above or below the strike price. Strike prices range from -3% to 6% . There seems to be no market demand for instruments with strike prices outside of this range and indeed we find that risk-adjusted probabilities outside this range are very low. The horizons of the year-on-year options range from 1 to 10 years.

4.2 Estimating the risk-adjusted probability density of inflation

As the classic work of Breeden and Litzenberger (1978) noted, given a rich enough set of option contracts with observable prices, it is possible to recover non-parametrically the risk-neutral distribution of inflation without making any specific distributional assumption about inflation or its link to other asset prices. The classic formula linking the risk-adjusted distribution of inflation and the price (a_0) and strike (k) of an option with maturity of t years is:

$$f(\pi_{0,t}) = R_t \left(\frac{\partial^2 a_0}{\partial k^2} \right). \quad (9)$$

Intuitively, if the price of a call declines quickly with the strike price, then the outcome at that point is more likely. Because our data give us many option prices for different strike prices and at different intervals, we can estimate this partial derivative by using the differences in

these prices.

In practice, the data require a considerable amount of cleaning. Aside from measurement error, we face the difficulty that the options in general are not traded simultaneously resulting in call option pricing functions that are not always well behaved. To screen out such data, first we drop option prices from the data if they contain simple arbitrage opportunities: (i) if the call (put) premium does not monotonically decrease (increase) in the strike price, (ii) if the option premium does not increase monotonically with maturity, and (iii) if butterfly spreads that correspond to Arrow-Debreu prices do not have positive prices. Next, before we take differences of the data, we smooth it in implied volatility space: for each maturity, we calculate Black and Scholes (1973) implied volatilities for all strike prices, smooth them with a natural spline, and convert back to option prices. We then use equation (9) and the delta method to calculate two finite differences that approximate the second partial derivative.

Through this procedure, the data do not reveal point expectations of future inflation but rather the risk-adjusted distributions for inflation in the future. That is, the distributions reflect the likelihood of different values of inflation, the risk associated with them, and the market price of this risk. This distinguishes our measures from many of the common measures of inflation expectations. Unlike opinion surveys, we are extracting risk-neutral rather than subjective expectations, and we do so from observing profit-making behavior. Unlike the break-even rate of inflation from comparing real and nominal yields, we have a whole distribution for inflation instead of a single number. Moreover, we do not need to worry about the liquidity differences between nominal bond and TIPS markets, or the price of the embedded floor which ensures that TIPS always pay back at least par value. Finally, unlike models of the term structure that use the yield curve to extract market-based inflation expectations, our measure does not rely on the associated (often strong) identifying assumptions in these models.

4.3 The estimates

Using our data on zero-coupon floors and caps, we extract the density $f(\pi_{t,t+j})$, with $j = 1, \dots, 10$. This gives a term structure of the cumulative risk-adjusted inflation distributions. In turn, using data on year-on-year contracts, we construct forward risk-adjusted distributions for year-on-year inflation $f(\pi_{t+j-1,t+j})$. Figures 3 and 4 plot the distributions at the end of 2012.

Noticeably, the risk-adjusted mode of inflation in 2013 was only 1.25%.¹⁴ Beyond 5 years, all of the distributions are bell-shaped and with similar median and mode, around 2.75%, which reflects partly the Federal Reserve target, and partly the compensation for inflation risk. All of the distributions have fat tails and are significantly non-normal. Depending on the maturity, excess kurtosis is between 0.69 and 1.07. Kitsul and Wright (2013) interpret the tails as reflecting investors' perception that both very high and very low inflation are the costly states of the world.

Another interesting feature of the distributions is that, as the horizon increases, the variance does not fall. Rather, the standard deviation for cumulative inflation rises from 1.3% at maturity 5 to 1.5% by maturity 10. The distribution becomes more spread out, either because extreme events far in the future are perceived as more costly, or because there is more uncertainty about inflation. Likewise, for year-on-year inflation, the standard deviations range between 1.3% and 2.2% in horizons 1 to 7, and are equal to 2.2% for horizons 8 to 10. Still, the probability that average annual inflation exceeds 5.75% is at most 5% across all horizons. Sustained high risk-adjusted inflation is perceived as being a very remote possibility.

¹⁴Because of the 0.5% granularity of our data, we report the probability of lying between 1% and 1.5% as 1.25%.

Figure 3: Marginal risk-adjusted distributions for cumulative annualized inflation

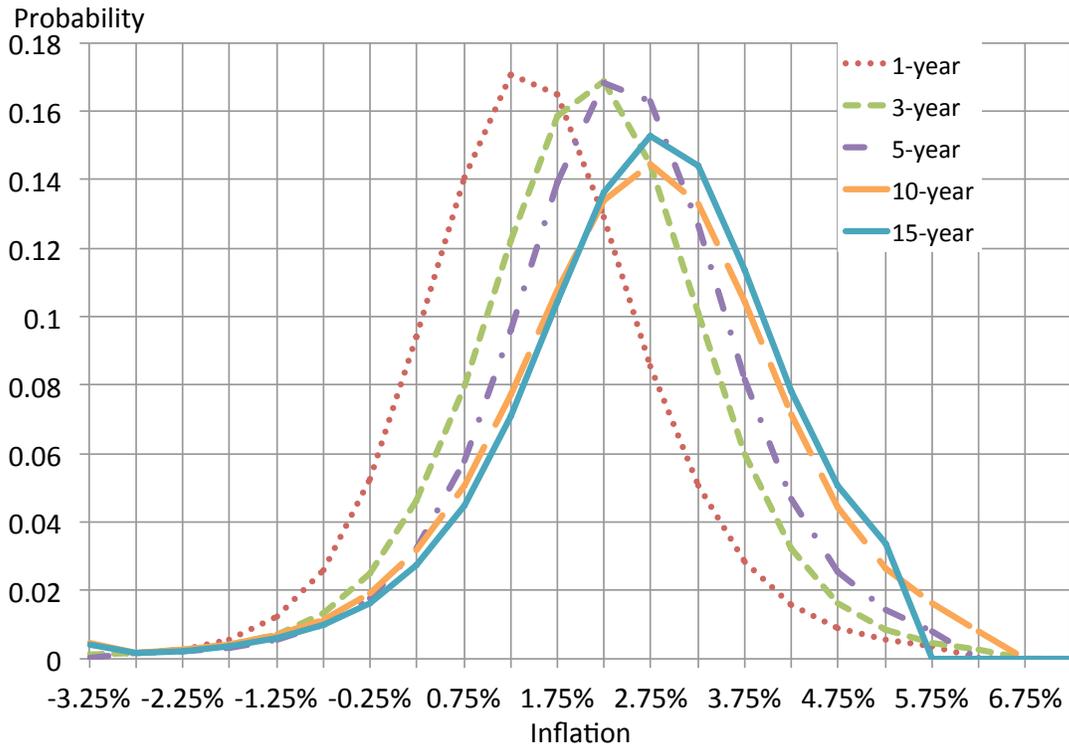
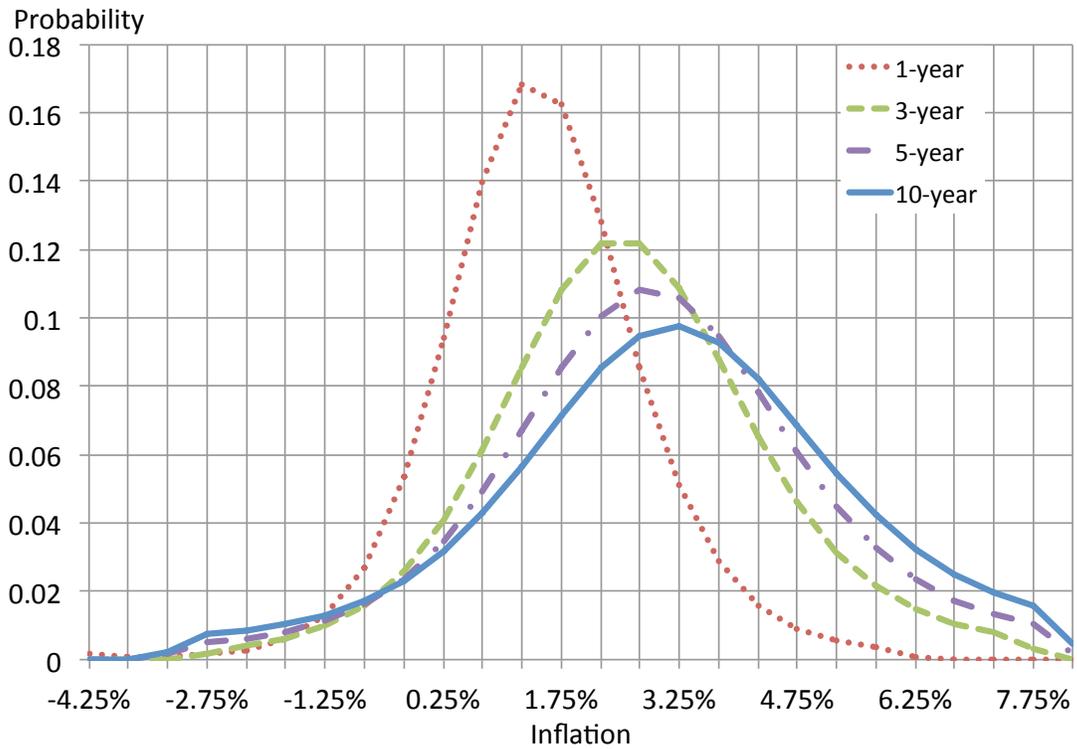


Figure 4: Marginal risk-adjusted distributions for year-on-year inflation



5 The joint distribution of inflation

To evaluate the weights in our formula, all that we need are the marginal distributions for cumulative inflation in the previous section. However, for our experiments, we would like to know the joint distribution of inflation across multiple years. With it, we can draw sequences of inflation to evaluate the probability of different scenarios and we can flexibly consider a series of different experiments.

5.1 A method of moments copula-based estimator

Understanding how the realizations of a random variable are related over time is, of course, the classic problem in time-series modeling. Our particular data on inflation contracts provides a novel way to approach this problem that has the intriguing feature of only using data at one date in time, by exploiting agents' expectations in financial markets to recover their beliefs about the time series of risk-adjusted inflation.

Consider the problem of obtaining the risk-adjusted joint density between annual inflation over the next two years: $f(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2})$. Sklar (1959) shows that there exists a copula function $C(\cdot) : [0, 1]^2 \rightarrow [0, 1]$ such that:

$$f(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2}) = C(f(\ln \pi_{t,t+1}), f(\ln \pi_{t+1,t+2})). \quad (10)$$

This function captures the co-dependence between the two random variables, so that we can obtain the joint density given information on the marginals. We use a parametric version of the copula function $\hat{C}(f(\ln \pi_{t,t+1}), f(\ln \pi_{t+1,t+2}), \rho)$, so the unknown copula function is fully characterized by a vector of parameters ρ in the known function $\hat{C}(\cdot)$, where ρ is of dimension M . The typical approach in the literature that estimates copulas would be to use the time series for past inflation to estimate both the marginal densities and the parameters

in ρ . Our unusual data allows us to approach the problem quite differently.

To start, we already have estimates of the marginal densities for year-on-year inflation. Moreover, from the zero-coupon options, we also have another marginal distribution: $f(\ln \pi_{t,t+2}) = f(\ln \pi_{t,t+1} + \ln \pi_{t+1,t+2})$. From the definition of the distribution:

$$f(\ln \pi_{t,t+2}) = \int_{\ln \pi_{t,t+1} + \ln \pi_{t+1,t+2} = \ln \pi_{t,t+2}} \hat{C}(f(\ln \pi_{t,t+1}), f(\ln \pi_{t+1,t+2}), \rho) d\pi_{t,t+1} d\pi_{t+1,t+2}. \quad (11)$$

Since we have N bins on the marginal distributions, this expression gives N moment conditions with which to estimate the M unknown parameters in ρ .

The appendix extends this logic to show that:

Proposition 2. *Given data for the marginal distributions of cumulative inflation $f(\ln \pi_{t,t+j})$ and year-on-year inflation $f(\ln \pi_{t+j-1,t+j})$ for $j = 1, \dots, J$, one can estimate the joint distribution $f(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2}, \dots, \ln \pi_{t+J-1,t+J})$ by estimating the M parameters in the ρ vector that satisfy the $(N - 1)(J - 1) \geq M$ conditions:*

$$f(\ln \pi_{t,t+j}) = \int_{\Pi} \hat{C}(f(\ln \pi_{t,t+1}), \dots, f(\ln \pi_{t+J-1,t+J}), \rho) d \ln \pi_{t,t+1} \dots d \ln \pi_{t+J-1,t+J}. \quad (12)$$

The integration set Π is such that: $\ln \pi_{t,t+1} + \dots + \ln \pi_{t+j-1,t+j} = \ln \pi_{t,t+j}$, for $j = 1, \dots, J$.

We can use these moments to estimate ρ , akin to GMM, although these are not moments of the distribution of the random variable, as is usual, but rather the distributions themselves.

Our data on options contracts only goes to 10 years, but the debt maturity goes all the way to 30 years. For inflation beyond 10 years, we extrapolate by assuming that the joint distribution is a stationary Markov process of order 9 with parameters given by the distribution from 1 to 10 years. The appendix discusses the details.

5.2 The estimates

We use a multivariate Gaussian copula, which has a single parameter to model the co-dependence between any two variables. Therefore $M = J(J - 1)/2 = 45$ since we have inflation over 10 years, which is well below the number of moments (180). This copula does not assume normality for inflation, it simply assumes that the joint dependence of inflation over time resembles a normal distribution in the sense that if the marginals were normal, then the multivariate would be normal, too.

We estimate two separate models of ρ . One of them searches for the correlations that minimize the equally-weighted squared deviations of the moments in proposition 2, subject only to the restriction that the correlation between inflation at two maturities is between -1 and 1. This model has the virtue of putting no restrictions and exploiting our proposition directly. Moreover, it fits the data quite well, in the sense that the discrepancies between the two sides of the moment conditions in proposition 2 are visually small. Yet, its estimates are hard to interpret. It is also difficult to understand intuitively what variation in the data is driving the estimates.

We focus instead on a restricted model for risk-adjusted inflation dynamics that assumes that the non-stationary component of inflation must be a random walk. This is a generalization of the common Beveridge-Nelson model for inflation that is frequently used. It has a few virtues relative to the unrestricted model. First, it has only 10 parameters, all of which are easy to interpret: the relative variance of the non-stationary and stationary components of inflation, and the 9 autocorrelations of the stationary part. Second, it fits the data almost as well as the unrestricted model, and it is easier to understand what in the data drives the estimates. Third, there is an out-of-sample test of our estimation procedure that uses only the maturities 1 to 10, which consists of comparing the copula model's prediction for cumulative inflation in years 12 and 15, for which we have data. In that test, the restricted model performs slightly better. Anyway, our results are robust to using one or the other

Table 1: Estimated correlation coefficients of year-on-year inflation in the joint distribution

<i>Maturity</i>	1	2	3	4	5	6	7	8	9	10
1	1.00	0.36	0.33	0.50	0.48	0.27	0.40	0.38	0.28	0.11
2		1.00	0.36	0.33	0.50	0.48	0.27	0.40	0.38	0.28
3			1.00	0.36	0.33	0.50	0.48	0.27	0.40	0.38
4				1.00	0.36	0.33	0.50	0.48	0.27	0.40
5					1.00	0.36	0.33	0.50	0.48	0.27
6						1.00	0.36	0.33	0.50	0.48
7							1.00	0.36	0.33	0.50
8								1.00	0.36	0.33
9									1.00	0.36
10										1.00

Notes: Estimated correlation coefficients for year-on-year inflation between date 2012+j and 2012+l, in column j, row l.

model, both for the two main findings from the estimates that we discuss below, as well as for the results on debt debasement.

The first finding is that, according to the options data, risk-adjusted inflation is a stationary process. In the restricted model, we estimate the variance of the random walk of risk-adjusted inflation to be zero. Even in the unrestricted model, the estimated correlation coefficients across maturities are only rarely above 0.6. This suggests that in risk-adjusted terms, long-run inflation expectations are well anchored. To understand what drives this result in the options data, note that if risk-adjusted inflation were non-stationary then the variance of year-on-year inflation should increase with the horizon. Yet, figure 4 shows that after 5 years the distributions do not spread out in a significant way.

Table 1 shows the estimated correlation parameters across maturities for the restricted model. So even though there is strong evidence of stationarity, the second interesting result is that the correlations fall only very slowly (if at all) with the horizon. Even as far as eight years into the future, the autocorrelation is still 0.28, which seems far from well-anchored

expectations. While stationary, risk-adjusted inflation is a process with a long memory, where shocks persist for many years. Intuitively, figure 3 shows that the distribution of cumulative average inflation does not shrink with the horizon, so agents must expect that inflation reverts slowly to its long-run mean. Curiously, this is consistent with the findings of Gürkaynak, Sack and Swanson (2005) who also use market expectations of the future in forward contracts to find that 10-year forward interest rates are highly correlated with 1-year interest rates, much more than what usual mean-reverting models of short term rates would suggest.

5.3 Interpreting the joint distributions

While we have been clear so far that all of these results are based on risk-adjusted distributions of inflation, it is useful to take a step back and understand what this implies for interpreting the results. When we refer to the distribution of inflation between any two dates after 2012, this is conditional on information that is available in 2012. There is no new information revealed by the joint distribution of 2016 and 2017 that affects the joint distributions between 2019 and 2020; both are measured given information in 2012.

Moreover, the relevant multi-period risk-adjusted expectation of inflation depends on the product of actual inflation and state-dependent risk compensation, which includes anything that changes marginal utility for different realizations of inflation. Our estimates have no way of distinguishing whether the risk-adjusted correlations between inflation at different horizons are due to serial correlation of actual inflation or of the stochastic discount factor.

A virtue of conjoining these two factors is that our estimates do not have to take a stand on the contentious issue of the slope of the Phillips curve, or on whether and how are inflation and real interest rates are correlated. This correlation is already taken into account within the risk-adjusted distributions. Moreover, one might argue that governments weight (or at least should weight) different inflation policies according to their effect on welfare, so

that because they adjust for the growth of marginal utility, risk-adjusted probabilities are at least as useful as actual probabilities for considering policy (Veronesi and Zingales, 2010; Kocherlakota, 2012).

The limitation of dealing with risk-adjusted inflation is that it makes it harder to compare the estimation results with the findings on actual inflation. To do so, one needs a model that provides identification restrictions to separate risk compensation from inflation, like that in e.g., Fleckenstein, Longstaff and Lustig (2013). Exploring such a model in the context of our estimates is high on the research agenda, but is best left for future work.

A concrete example perhaps makes these points clearer. When we evaluate the expectation of inflation between dates 0 and 2 using the estimated risk-neutral densities, we are computing:

$$\begin{aligned} \mathbb{E} \left[\frac{m_{0,2}}{\pi_{0,2}} \right] &= \mathbb{E} \left[\left(\frac{m_{0,1}}{\pi_{0,1}} \right) \left(\frac{m_{1,2}}{\pi_{1,2}} \right) \left(\frac{m_{0,2}}{m_{0,1}m_{1,2}} \right) \right] \\ &= \mathbb{E} \left(\frac{m_{0,1}}{\pi_{0,1}} \right) \mathbb{E} \left(\frac{m_{1,2}}{\pi_{1,2}} \right) + cov \left[\frac{m_{0,1}}{\pi_{0,1}}, \frac{m_{1,2}}{\pi_{1,2}} \right] \end{aligned} \quad (13)$$

Our first point is that all expectations are taken conditional on information at time 0. Therefore, the last term in brackets in the first line is exactly equal to 1, no matter what state gets realized at dates 1 and 2. Risk-adjustments (distributions of the stochastic discount factor across future states) or real interest rates (conditional means of the stochastic discount factor) may change over time, but that is taken into account by the date 0 expectation. Our second point is that the covariance can depend on the correlation between $\pi_{0,1}$ and $\pi_{1,2}$, as well as between $m_{0,1}$ and $m_{1,2}$, and between the two pairs. It takes a model for $m_{0,t}$ or for $\pi_{0,t}$ to separate them.

6 The distribution of debt debasement

Infinite sudden inflation would wipe out all of the nominal debt. However, market expectations of inflation should reflect this possibility. If in the inflation contracts that we observe, investors place less than 10% probability on inflation being above 4%, scenarios where inflation is suddenly infinity are of little relevance.

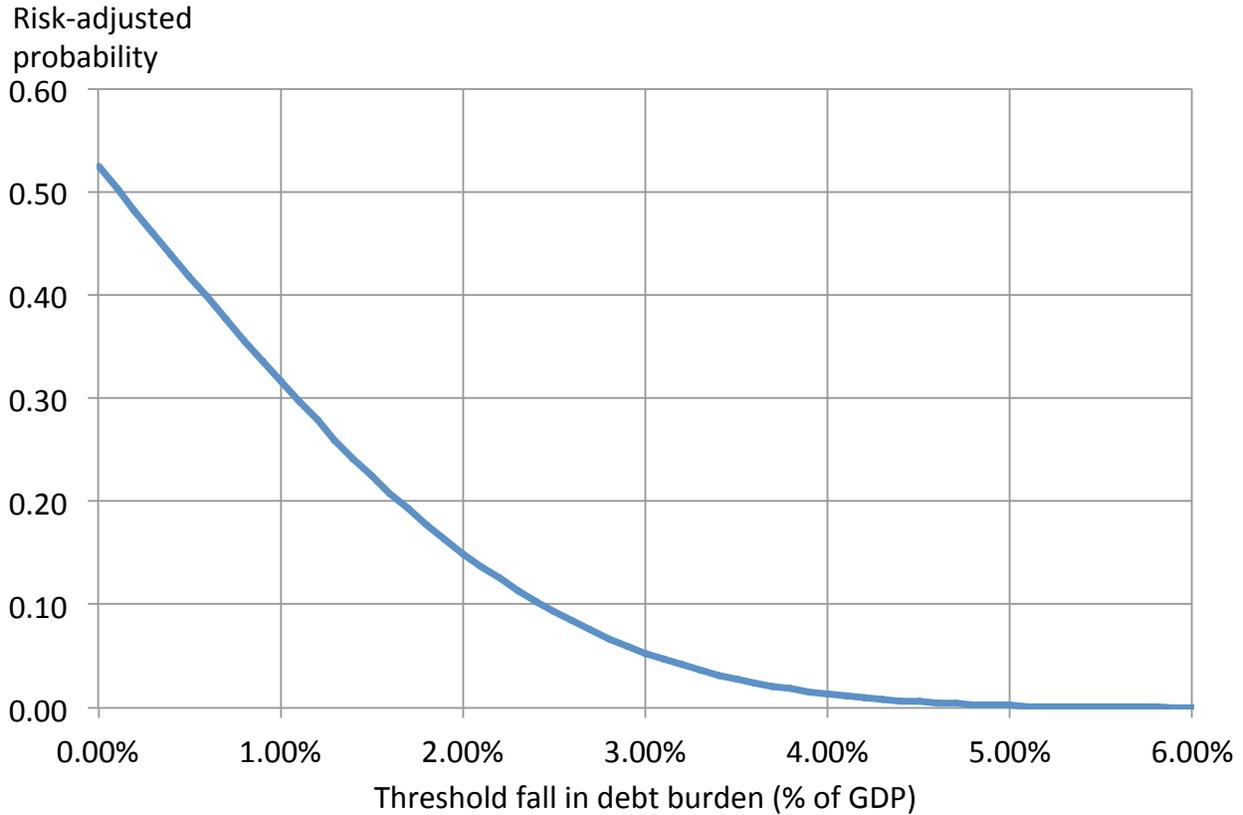
Using our densities for future inflation, we measure the probability that debt would fall by different amounts. We do this in the context of proposition 1, that is, we consider the present value of debt debasement. Using our estimated risk-adjusted joint distribution for inflation, we draw a large number of histories and, ordering them by their impact on the real value of debt, we estimate probabilities that debt will fall by some threshold. This is akin to measures of risk-adjusted Value-at-Risk (VaR) that are often used.¹⁵ These measures take into account all possible future scenarios for inflation as perceived today by market participants, including changes in the policy of the central bank, shocks to inflation beyond the control of monetary policy, and changes in risk-attitudes towards inflation.

Figure 5 shows the probability that the fiscal burden will fall by more than a few percentage points of GDP, according to the risk-adjusted distribution for inflation. Strikingly, the numbers are all quite small. The probability that debt falls by more than 5% of GDP is 0.2%. Having the real value of the debt fall by at least 1% of GDP due to inflation variation is likely, with a probability of 32% but anything more than only 3% of GDP has the quite small probability of 5.3%.

Table 2 presents VaR measures separately for each investor. Most of the gains are at the expense of foreigners, of which the largest nationality is Chinese. This happens not just because they hold more debt than domestics, but mostly because they hold longer maturity debt. Therefore, extreme situations where a succession of high realizations of annual inflation

¹⁵See, in particular, Ait-Sahalia and Lo (2000) as well as Jackwerth and Rubinstein (1996).

Figure 5: Probability that the fall in the value of debt exceeds a threshold



lead to large cumulative inflation affect foreigners more than domestics.¹⁶

The last column in the table shows the effect on a non-private holder of debt, the central bank. The Federal Reserve would potentially suffer large losses as a share of its portfolio. Even though it holds much less debt than domestic private investors, the central bank loses significantly more than they do. In 2012, the central bank held mostly long-term bonds, which are more exposed to inflation risk.

¹⁶From the perspective of foreigners, how much they ultimately lose also depends on how much the exchange rate devalues with the increase in inflation and on whether there are capital controls or not. We are not measuring the foreigners' losses. Rather, we are measuring how much the fiscal authority would gain at their expense, which is not the same as the country's gains. Our measure does not depend on whether the holder is domestic or foreign, or what happens to the exchange rate, even if these factors would be important for the impact of higher inflation on national welfare.

Table 2: Percentiles of the distribution of the fall in the value of debt for bondholders

<i>Percentile</i>	<u> Holders of the debt </u>				
	<i>Privately held</i> (51%)	<i>Domestic</i> (19%)	<i>Foreign</i> (32%)	<i>China</i> (7%)	<i>Central Bank</i> (12%)
<i>90th</i>	2.4%	0.8%	1.6%	0.4%	1.2%
<i>95th</i>	3.1%	1.1%	2.0%	0.5%	1.5%
<i>99th</i>	4.2%	1.4%	2.8%	0.7%	2.0%

Notes: Each cell shows the cutoff in the reduction of the real present value of debt as a share of GDP, so that the risk-adjusted probability of a larger loss (borne by the agent in the column) is equal to one minus the percentile in the row.

7 Counterfactual estimates of debt debasement

We next explore the effects of shifting the inflation distributions. This allows us to both interpret the VaR results as well as to consider hypothetical scenarios of interest. It is also similar to conducting stress tests. Importantly, note that these are shifts in the risk-adjusted distribution for inflation, not changes in inflation per se. They are useful to understand how different properties of inflation affect the debt debasement effect, but they do not cleanly correspond to actual policy changes.¹⁷

In practice, for each counterfactual, we propose a new distribution $\hat{f}(\cdot)$, recalculate the real value of the debt using equation (7), and subtract it from the market value of debt to obtain our estimate of the fall in real debt.¹⁸ This approach is the stochastic equivalent of asking what would happen if inflation was $x\%$ higher. As in the previous section, we discipline the experiments using the data. The shift x is pinned down to be consistent with the plausible set of scenarios in our original distribution $f(\cdot)$. Table 3 reports the results.

¹⁷Unlike the value at risk results in the previous section, but like any counterfactual exercise, these experiments are potentially exposed to the Lucas critique. However, because these are shifts to risk-adjusted distributions they do incorporate agents' shifting expectations about inflation and risk. It is not the shifts per se that are subject to criticism, but only their interpretation as policy changes.

¹⁸The appendix provides more details.

7.1 Counterfactual 1: The impact of higher inflation

The first experiment shifts the marginal distributions for annual inflation at every maturity so that the new median is at the old 90th percentile. We think of this experiment as capturing an announcement that the inflation target of the Fed is now expected to be higher, as suggested by Blanchard, Dell’Ariccia and Mauro (2010).¹⁹

The second experiment instead sets the density within the 90th percentile to zero, and scales the density outside of this range proportionately. This corresponds to a commitment that inflation will be higher for sure in the future. Only inflation realizations at the right tail of the current distribution become possible. At the same time, because there is no shift to the right as in the first case, very high levels of inflation are also not that likely.

The table shows that the first experiment lowers the debt burden by 3.7% while the second lowers it by 4.4%. Again, foreigners absorb a large share of the losses, because they both hold more debt and especially at longer maturities. Again, in spite of holding 37% less debt than domestic private bondholders, the Federal Reserve loses 37% more with inflation because it holds longer maturity debt.

7.2 Counterfactual: the role of uncertainty

Because we have the full probability distribution for inflation, we can inspect what is the effect that uncertainty about inflation has on the real value of the debt.

Our third experiment again shifts the marginal density so the new median is the old 90th percentile, but now this is accomplished by scaling inflation proportionately at every maturity. It is often said that higher average inflation comes with more variable inflation,

¹⁹Again to be clear, we are shifting the risk-neutral distribution of inflation, not the actual inflation target of the central bank. The link between the two depends both on the effectiveness of central bank policy as well as on changes in private assessments of risk. In the extreme case where the central bank controls the distribution of inflation, and where inflation is “pure” in the sense of Reis and Watson (2010), so that changes in inflation are independent of changes in relative prices, the two are the same.

Table 3: Counterfactual impact of higher inflation on the real present value of debt

<i>Inflation counterfactual</i>	<u>Holders of the debt</u>				
	<i>Privately held (51%)</i>	<i>Domestic (19%)</i>	<i>Foreign (32%)</i>	<i>China (7%)</i>	<i>Central Bank (12%)</i>
<i>1. Permanently higher</i>	3.7%	1.3%	2.3%	0.6%	1.8%
<i>2. Right tail only</i>	4.4%	1.5%	2.9%	0.7%	2.1%
<i>3. Higher and more variable</i>	3.4%	1.2%	2.2%	0.5%	1.7%
<i>4. Higher for sure</i>	3.8%	1.4%	2.4%	0.6%	1.9%
<i>5. Partially anticipated</i>	1.3%	0.4%	0.9%	0.2%	0.6%
<i>6. Temporary increase</i>	1.2%	0.4%	0.8%	0.2%	0.4%
<i>7. Gradual increase</i>	2.3%	0.9%	1.4%	0.3%	1.4%

Notes: Each cell shows the fall in the real real present value of debt as a ratio of GDP.

and this experiment tries to capture this possibility.

In the other direction, in the fourth counterfactual, we assume that year-on-year inflation is exactly equal to the average inflation in our estimates. This shows what would happen if inflation became deterministic, so we can understand better the effect of volatility.

The fifth counterfactual studies a shift in inflation that is partially expected. We assume that after an initial unexpected jump of inflation upwards, the distribution of inflation looking forward is equal to the conditional expectation that we have estimated. Therefore, whereas in the previous experiments all of the changes at all maturities were unexpected, now only the change in the first year catches agents by surprise, but they adjust their expectations right after.

From table 3, we see that more uncertainty lowers the effectiveness of inflation at debasing the debt. Intuitively, because the real value of future nominal payments are convex in inflation, uncertainty raises their value and so lowers the benefits of raising inflation.

Also as expected, in case 5, if agents adjust their expectations after one year of surprise inflation, the estimates are significantly smaller. In this case, in spite of the quite extreme

shift in the distribution for inflation that we considered, the fall in the real value of debt is quite far from even 2% of GDP.

7.3 Counterfactual: the time path for inflation

The five experiments so far assumed that the risk-adjusted inflation distribution would change immediately and permanently. The sixth case considers instead a temporary increase in inflation, with the distribution for year-on-year inflation shifting rightwards so the new median is at the 90th percentile the next year, but only at the 80th percentile the year after, and so on, so that for maturities of 5 or more years there is no change. The seventh case considers a gradual increase, with the one-year inflation distribution unchanged, while the two-year shifts horizontally so the new median is at the old 60th percentile, and so on until the fifth year, after which we have the same permanent shift as in the first case.

The last two rows in table 3 show that both of these reasonable deviations from the first counterfactual again cut significantly the effect of inflation on debt. If inflation only increases temporarily, the benefit for the Treasury is only 1.2%.

7.4 Why such low numbers? The role of inflation

All of the estimates in tables 2 and 3 are surprisingly small. Even when we considered unlikely and extreme scenarios, the debt never lost even one tenth of its real value. Why is this the case?

Table 4 shows the risk-adjusted harmonic mean of inflation for both the baseline and each of the counterfactuals at different maturities.²⁰ Even in the most extreme case, inflation between 2012 and 2013 never increases by more than 3%. From the perspective of actual market-based distributions, anything larger than this seems unreasonable. But from

²⁰We take the harmonic, instead of an arithmetic mean, since proposition 1 shows that it is the expectation of the inverse of inflation that matters for debt valuation.

Table 4: Expected adjusted average annual inflation for different counterfactuals

<i>Distribution for inflation</i>	<i>Horizon</i>			
	<i>1-year</i>	<i>3-year</i>	<i>5-year</i>	<i>30-year</i>
<i>Baseline</i>	1.5%	2.1%	2.4%	3.0%
<i>Counterfactuals</i>				
<i>1. Permanently higher</i>	3.2%	4.0%	4.5%	5.6%
<i>2. Right tail only</i>	4.4%	4.7%	5.2%	5.2%
<i>3. Higher and more variable</i>	3.1%	4.0%	4.5%	5.6%
<i>4. Higher for sure</i>	3.2%	4.0%	4.6%	5.7%
<i>5. Partially anticipated</i>	3.0%	2.9%	3.2%	3.3%
<i>6. Temporary increase</i>	3.2%	3.3%	3.2%	3.1%
<i>7. Gradual increase</i>	1.5%	2.5%	3.4%	5.5%

Notes: Each cell reports $1/E(n/\pi_{0,n})$ the harmonic mean of average inflation until horizon n .

the perspective of debt valuation, these are not large numbers. In order to raise the debt debasement effect to 10%, it would take shifting the distribution of inflation in experiment 1 so far to the right that the new median for annual inflation is almost 11%. Yet, in that case, the new and old distributions for inflation would have close to zero overlap, making this scenario literally incredible.

7.5 Why such low numbers? The role of maturity

Still, over many years, an additional percentage points of inflation can accumulate to a large effect on debt debasement. If all privately-held debt was at long maturities, inflation might significantly reduce its real value.

Table 5 investigates the effect of maturity on our estimates by considering the effect of the higher inflation only on the debt with maturity below 1 year, or only below 4.5 years. The numbers are significantly lower than when all the debt is included. Moreover, even

Table 5: Counterfactual impact of higher inflation with different maturity distributions

<i>Inflation counterfactual</i>	<i>Including only debt of maturity up to:</i>			<i>With the maturity distribution of debt in 2000</i>
	<i>1 year</i>	<i>4.5 years</i>	<i>All</i>	
<i>1. Permanently higher</i>	0.1%	1.1%	3.7%	5.1%
<i>2. Right tail only</i>	0.2%	1.5%	4.4%	5.8%
<i>3. Higher and more variable</i>	0.1%	1.1%	3.4%	4.7%
<i>4. Higher for sure</i>	0.1%	1.1%	3.8%	5.3%
<i>5. Partially anticipated</i>	0.1%	0.6%	1.3%	1.6%
<i>6. Temporary increase</i>	0.1%	0.7%	1.2%	1.2%
<i>7. Gradual increase</i>	0.0%	0.3%	2.3%	3.8%

Notes: Each cell shows the fall in the real present value of debt as a ratio of GDP.

though three quarters of the market value of debt has a maturity below 4.5 years, the debt debasement effect is well below 0.75 of our estimates when all the debt is included. This confirms that most of the benefits from higher inflation come from the longer maturity debt.²¹

The last column of the table confirms this conclusion. We estimated the distribution of debt held in private hands by maturity for the year 2000 using the same steps that we followed for our 2012 calculations. Whereas in 2012 only 6% of the market value of debt was in maturities above 10 years, in 2000 these long-term bonds accounted for 17% of the total debt. The duration of privately-held debt was 5.1 years in 2000, compared to 3.7 years in 2012.²² We then repeated our experiments using the 2000 distribution, scaled up proportionately so that the total market value of the debt is the same as in 2012. The question we are asking is whether inflation would be more effective if the public debt held in the private sector in 2012 had the 2000 maturity structure. The answer is a clear yes: the effects are significantly larger.

²¹The temporary increase now has a larger effect than the gradual, since the former has a greater impact on the short end of the maturity structure.

²²This is the Fisher and Weil (1971) duration.

7.6 Reassessing the rule of thumb

The conclusion so far is clear: over the short run, there is too little potential inflation but plenty of debt. Above 5 years, there is plenty of inflation but too little debt. Combined, the overall effect of inflation on the debt burden is small. In a way, this confirms the intuition in the popular rule of thumb that multiplies average maturity by expected higher inflation to calculate debt debasement, so long as the increase in inflation reflects current market expectations. At the same time, the introduction showed that this rule of thumb dramatically over-predicted the effect of higher inflation. Is there an alternative?

Starting with the formula that multiplies outstanding debt by average maturity and then by a counterfactual increase in inflation, all three terms need significant adjustments. The first is to realize that what matters is not outstanding debt, but privately-held debt. Thus, the right number is 51%, not 101%. The second is that weighted average maturity would only be useful if the distribution was approximately exponential, which it is very far from accurate. Rather, what is necessary is a measure of duration. Assuming, as we did in the introduction, that inflation increases by the same amount at all maturities, it is possible to approximate the impact on the value of the portfolio using the Fisher and Weil (1971) duration. For the portfolio of privately-held debt in 2012, it is 3.7. Third, and finally, this calculation can only estimate the impact of a fully expected permanent increase in inflation by a marginal amount. It is a local estimate, corresponding to the first term of a Taylor approximation. It cannot calculate the impact of uncertainty about inflation, whether the change is gradual or transitory, or provide any probabilities to assess what magnitudes of changes are likely.

Multiplying 51% by 3.7 gives an estimated impact of 0.1% permanently higher inflation of 0.189% of GDP. Using our accurate formula and our detailed data, in a counterfactual where we shift the risk-adjusted distribution for inflation by precisely 0.1% to the right at all frequencies and maturities, we end up with an estimate of 0.184%. The approximation

is quite good.

At the same time, even this adjusted rule of thumb has serious limitations. The fact that it ignores the stochastic process for inflation is one. Across experiments, increasing average inflation as reported in table 4 does not lead to proportional increases in the extent of debt debasement in table 3, so this can matter a great deal. Another serious limitation is that the rule of thumb is a local linear approximation. For 1% higher inflation, the rule of thumb implies an effect of 1.89% on the debt, while the actual number is 1.75%; as inflation gets even higher, the approximation error can become quite substantial. A third limitation is that to calculate the duration, one needs to know the entire distribution of bond holdings $\{B_0^t\}_t$ and associated prices. But if a researcher already has this information, then he/she is better off just using our formula in proposition 1 instead of this local approximation.

8 Financial repression

It may seem surprising that our calculations so far could be silent on the hotly disputed topic of whether inflation has real effects. One might think that if inflation lowers real interest rates, then because the government needs to pay less to roll over its debt, the fiscal burden will be smaller. Yet, if the interest rate is lower, this also means that investors discount the future debt by less. By our assumption of no arbitrage opportunities, the real interest paid on the government bonds and the real interest that private agents use to discount the future are the same. Therefore, in the present value of the fiscal burden, these two effects exactly offset, whatever is the impact of inflation on the real interest rate.

Financial repression is a way to drive a wedge between these two interest rates. This wedge works like a tax on the returns of government debt and as such provides a source of revenue that reduces the fiscal burden. The literature on financial repression, which dates back at least to McKinnon (1973), offers many examples of how this tax is collected and

enforced, through channels like caps on interest rates, direct lending to the government by captive domestic savers, or financial regulation, among others. In theory, this would show up as a factor $1 - \tau_t$ in each of the terms in our formula in proposition 1. But, at this general level, we cannot say more empirically about its size or how it varies with inflation.

8.1 Repression as financial regulation

Reinhart and Sbrancia (2011) discuss how many developed countries used a combination of caps on the interest rates on government bonds and inflation between 1945 and 1980 to pay for the World War II debt. One particular way in which this is achieved is by forcing the holders of outstanding debt to roll it over for “special” debt that sells for a higher price (or pays a lower return) than the market price for identical private securities. This is achieved for instance by forcing banks to accept this special debt and hold it under the guise of financial regulation and stability. An extreme case of this hidden financial repression is to require banks to hold zero-interest reserves at the central bank. Effectively, one type of government liability that pays market interest is replaced by another type that pays no interest.

To model this formally, assume for simplicity that all debt is nominal and has maturity of one period, and that the holders of maturing bonds B_0^t are forced to take special bonds as payment that promise to pay \tilde{B}_{t+1} next period and sell today for price \tilde{H}_t . The price of the bond is higher than the market price for nominal bonds, H_t , capturing repression, and below one, capturing the zero lower bound on interest rates. When it is equal to one, we have extreme financial repression with the special bonds being zero-interest reserves. The law of motion at any date after 0 for debt now becomes:

$$W_t = s_t + \left(\frac{\tilde{H}_t - H_t}{P_t} \right) \tilde{B}_{t+1} + W_{t+1} \frac{H_t P_{t+1}}{P_t}. \quad (14)$$

Because $\tilde{H}_t > H_t$, this expression makes clear that financial repression works like a

source of tax revenue. Because $B_0^t = \tilde{H}_t \tilde{B}_{t+1}$, this revenue subtracts from the real value of outstanding debt just like a tax on its holders would. Similar algebra as the one that led to proposition 1 shows that the debt burden now is:

$$E_0 \sum_{t=0}^{\infty} m_{0,t} \left(\frac{H_t}{\tilde{H}_t} \right) \left(\frac{B_0^t}{P_t} \right) \leq \sum_{t=0}^{\infty} \frac{B_0^t}{P_0} E_0 \left(\frac{m_{0,t+1}}{\pi_{0,t+1}} \right). \quad (15)$$

Since $\tilde{H}_t > H_t$, the debt burden is lower the higher is financial repression. The inequality binds in the case of extreme financial repression ($\tilde{H}_t = 1$), where the government rolls over its past debt through zero-interest required reserves. This expression shows that the effect of financial repression is essentially equivalent to delaying all payments on the debt for one year at a zero interest rate, or shifting the maturity structure by one year.

Generalizing the previous argument to have financial repression for N periods, we obtain a new version of proposition 1:

Proposition 3. *With financial repression for N periods, the debt burden is still equal to $\sum_{t=0}^{\infty} \omega_t^r B_0^t$ but now the weights are:*

$$\omega_t^r = R_t^{-1} \int \left(\frac{f(\pi_{0,t})}{\pi_{0,t}} \right) \left(\frac{H_t^N}{\tilde{H}_t^N} \right) d\pi_{0,t} \leq \omega_{t+N} \quad (16)$$

With extreme financial repression ($H_t^N = 1$), the inequality becomes an equality.

The weights in the formula for the debt burden are lower, as long as nominal interest rates are positive. Higher inflation is now more effective at lowering the real value of the debt. Inflation not only debases the debt but also lowers the real return on the zero-interest reserves.

Table 6: The effect of inflation with financial repression

<i>Years of repression</i>	<i>Repression</i>	<i>Higher inflation</i>	<i>Total</i>
<i>1 year</i>	0.7%	4.6%	5.3%
<i>5 year</i>	4.8%	8.1%	12.9%
<i>10 year</i>	12.2%	10.7%	22.9%

Notes: Each cell shows the fall in the real present value of debt as a ratio of GDP as a result first solely of repression, and then of higher inflation under experiment 1. The last column is the sum of the two previous ones.

8.2 Estimating the joint effectiveness of inflation and repression

Table 6 shows the effect of extreme financial repression. The first column of numbers shows that financial repressions alone, with no additional inflation, can significantly lower the real value of debt. Even with the low nominal interest rates in 2012, repression for 10 years would wipe out almost one quarter of the debt.

Column 2 then conducts our experiment 1 within the financially repressed economy, so it measures the effect solely of the inflation distribution shifting to the right. The effect of inflation is much higher than before. This confirms our conclusion of the previous section that a longer maturity of current debt is the key ingredient that makes inflation effective at lowering the real value of debt.

The total effect in column 3 shows the joint effect, at date 0, of both imposing financial repression and shifting the distribution of inflation to the right. The reduction of the real value of debt is substantial. Repression is the tool by which the government can roll over current debt to longer-term debt, without paying the price of higher nominal yields due to higher inflation expectations.

9 Conclusion

In this paper we present a method for evaluating the effectiveness of inflation at lowering the real value of debt. We find that, using data from 2012 for the United States, higher inflation is unlikely to lower the real value of debt by more than a few percentage points of GDP. The reason is two-fold. First, market participants expect that inflation over the next few years will be modest. Second, in 2012, the majority of the debt held in the private sector was of short maturity. Combining the two, inflation could do little.

To arrive at this conclusion we make five main contributions. (1) We derive a simple new formula for the debt burden, that shows how the maturity structure of debt holdings and inflation determine debasement. (2) We compile data measuring the maturity structure of debt held by different sets of investors for the U.S. in 2012, and the risk-adjusted marginal distributions of inflation at different horizons. (3) We propose a new estimator for the risk-adjusted joint distribution of inflation over time using copulas and inflation distributions. (4) We provide value-at-risk measures for the likelihood of debasement, as well as expected values for different scenarios. (5) We show how higher debt maturities, either ex ante as in the historical data, or ex post via financial repression, can increase the effectiveness of inflation at debt debasement.

One way to increase debasement would be to have significantly higher inflation that completely surprises the market. Yet informed, profit-maximizing agents view such outcomes as being virtually impossible. Another way would be to have the private sector hold more long-term public debt. However, perhaps it is precisely by holding short-term debt that private agents are reducing the incentive of the government to inflate away the debt, so that low maturity holdings and low inflation expectations are mutually consistent. If so, issuing more long-term debt might lead to a large fall in its price. Another way to achieve debasement is combining inflation with financial repression, which could be implemented

by forcing an ex-post extension of maturities. Of course, such a policy would likely impose significant costs on the economy by impairing financial intermediation.

If inflation may not pay for the U.S. public debt, then what will? Since market prices today put the probability of the United States defaulting at close to zero, the markets seem to be expecting budget surpluses, brought about either by increases in revenue or decreases in expenditures. Perhaps inflation itself, while not eroding the real value of government debt, will generate fiscal surpluses by decreasing the real value of nominally frozen public sector wages and pensions. Alternatively, perhaps market expectations are inconsistent or they are severely underestimating future inflation. What is sure and inescapable is that, one way or another, the budget constraint of the government will have to hold.

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Appendix to “Inflating Away the Public Debt? An Empirical Assessment”

Jens Hilscher

Alon Raviv

Ricardo Reis

Brandeis University

Brandeis University

Columbia University

July 2014

A Proof of proposition 1

While the bulk of the proof is already in section 2, here we fill in some missing steps. First, adding and subtracting $Q_t^1 W_{t+1}$ to the right-hand side of equation (2), and using equation (1) to replace out W_{t+1} , a few steps of algebra deliver the law of motion for the market value of government debt:

$$W_t = Q_t^1 W_{t+1} + s_t + x_{t+1} \tag{A1}$$

where the *revaluation* term x_{t+1} is equal to:

$$\sum_{j=0}^{\infty} (Q_t^{j+1} - Q_t^1 Q_{t+1}^j) K_{t+1}^j + \sum_{j=0}^{\infty} (H_t^{j+1} - H_t^1 H_{t+1}^j) \frac{B_{t+1}^j}{P_t} + \sum_{j=0}^{\infty} \frac{H_{t+1}^j B_{t+1}^j}{P_{t+1}} \left(\frac{P_{t+1} H_t^1}{P_t} - Q_t^1 \right). \tag{A2}$$

Iterating this equation forward, from date 0 to date $t + 1$, delivers equation (3) in the text.

Dividing both sides of the law of motion for W_t by Q_t^1 , multiplying by $m_{t,t+1}$ and taking

expectations gives:

$$W_t = \mathbb{E}_t(m_{t,t+1}W_{t+1}) + s_t + \mathbb{E}_t\left(\frac{m_{t,t+1}x_{t+1}}{Q_t^1}\right). \quad (\text{A3})$$

For now, assume that the last term on the right-hand side is zero. We will show it shortly. Multiply both sides of (A3) by $m_{0,t}$ and take expectations as of date 0, so that using the law of iterated expectations you get the recursion:

$$\mathbb{E}_0(m_{0,t}W_t) = \mathbb{E}_0(m_{0,t+1}W_{t+1}) + \mathbb{E}_0(m_{0,t}s_t). \quad (\text{A4})$$

Iterate this forward from date 0 to date T , and take the limit as T goes to infinity. With the no-Ponzi scheme condition $\lim_{T \rightarrow \infty} \mathbb{E}_0(m_{0,T}W_T) = 0$, we get the result in expression (5):

$$W_0 = \mathbb{E}_0\left[\sum_{t=0}^{\infty} m_{0,t}s_t\right]. \quad (\text{A5})$$

Finally, for the first equality in expression (5), replace out the bond prices from equation (1) using the equations in (4).

The missing step was to show that $\mathbb{E}_t(m_{t,t+1}x_{t+1}/Q_t^1) = 0$ for all t . Consider the first element of x_{t+1} and take the following steps:

$$\begin{aligned} & \mathbb{E}_t\left[m_{t,t+1} \sum_{j=0}^{\infty} \left(\frac{Q_t^{j+1}}{Q_t^1} - Q_{t+1}^j\right) K_{t+1}^j\right] = \\ & \sum_{j=0}^{\infty} K_{t+1}^j \left[Q_t^{j+1} \mathbb{E}_t\left(\frac{m_{t,t+1}}{Q_t^1}\right) - \mathbb{E}_t(m_{t,t+1}Q_{t+1}^j)\right] = \\ & \sum_{j=0}^{\infty} K_{t+1}^j [Q_t^{j+1} - \mathbb{E}_t(m_{t,t+1} \mathbb{E}_{t+1} m_{t+1,t+1+j})] = \\ & \sum_{j=0}^{\infty} K_{t+1}^j [Q_t^{j+1} - \mathbb{E}_t(m_{t,t+1+j})] = 0 \end{aligned} \quad (\text{A6})$$

The first equality comes from replacing the expectation of the sum by the sum of the expectations and from taking the prices known at date t out of the expectation ; the second from using the result in equation (4) twice; the third from using the law of iterated expectations; and the fourth from using equation (4). Identical steps show that the other terms in $\mathbb{E}_t(m_{t,t+1}x_{t+1}/Q_t^1)$ are also equal to zero.

The final part of the proof to clarify is expression (6). First note that in principle, $m_{0,t}$ can depend on many random variables. However, all we need to evaluate is $\mathbb{E}(m_{0,t}/\pi_{0,t})$. Therefore, only the dependence of $m_{0,t}$ on $\pi_{0,t}$ will lead to a non-zero term once the expectation of the product of the discount factor and inverse inflation is evaluated. Therefore, we can write $\mathbb{E}(m(\pi_{0,t})/\pi_{0,t})$.

Second, let $\hat{f}(\cdot)$ be the probability density function for inflation. Then, we can re-write the expression as: $\int (\hat{f}(\pi_{0,t})m(\pi_{0,t})/\pi_{0,t}) d\pi_{0,t}$. Define the inverse of the risk-free rate, from the perspective of date 0, as $R_t^{-1} = \int \hat{f}(\pi_{0,t})m(\pi_{0,t})d\pi_{0,t}$. From the definition of the stochastic discount factor, this is the price of a bond that would pay one dollar for sure in t periods, regardless of the realization of inflation. Then, define the risk-adjusted density for inflation as: $f(\pi_{0,t}) = \hat{f}(\pi_{0,t})m(\pi_{0,t})R_t$. Using these results it follows that:

$$\mathbb{E}(m(\pi_{0,t})/\pi_{0,t}) = \int \left(\frac{\hat{f}(\pi_{0,t})m(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t} = R_t^{-1} \int \left(\frac{f(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t}. \quad (\text{A7})$$

This confirms expression (6).

B Debt holdings

We construct monthly maturity structures, that is B_0^t , for five groups of investors: (1) private total, which is publicly held debt, excluding Federal Reserve and state and local government holdings (2) foreign, (3) domestic, which is private minus foreign, (4) China,

and (5) Central Bank (Federal Reserve). The data sources are the CRSP U.S. Treasury database, the Treasury Bulletin, the “Foreign Portfolio Holdings of U.S. Securities” report available from the U.S. Treasury, and the System Open Market Account (SOMA) holdings available from the FRBNY. The numbers for face value of total outstanding debt for different categories in section 3.1 come from the Monthly Statement of Public Debt, the Treasury Bulletin (OFS-2, FD-3), and the SOMA. Data are all for end of December 2012 for the United States.

We construct holdings of notes and bonds as follows. For (1) the data is available from CRSP, and we subtract out state and local government holdings assuming they have the same maturity structure. Detailed information for (2) is available for June 2012 and we assume proportionate growth of foreign holdings to construct December 2012 numbers. (3) is the difference of (1) and (2). For (4) we combine data on country level holdings of long term debt with information on the total foreign maturity structure. Note however that we do not have data on the holdings of China per maturity but only a scaling factor, so we assume that the Chinese maturity distribution is proportional to that of overall foreigners. For (5) we use the security level data available from the FRBNY. We assume that all coupon and principal payments mature in the middle of each month.

CRSP does not have data on Treasury bills. We use the issues of the Treasury bulletin to obtain information on bills and follow the same steps as we did above for notes and bonds. In particular, we construct holding of T-Bills as follows. For (1) We subtract Fed holdings from Treasury Bulletin Table FD-2 T-Bill holdings and then subtract out the proportionate amount of state and local government holdings; for (2) and (3) we do the same as for notes and bonds; for (4) we scale the June 2012 holdings; for (5) we use the SOMA holdings.

Aside from calculating debt holdings for more categories of investors, our method for constructing the maturity structure of (1) has the following differences relative to Hall and Sargent (2011). First, we construct a monthly term structure and assume that promised

payments in a month are paid in the middle of the month (instead of using an annual frequency). Second, we exclude state and local government holdings. Third, we base the T-Bill holdings on Table FD-2 of the Treasury bulletin and Fed holdings (rather than Table FD-5 of the Treasury bulletin).

Data on real interest rates is from Gürkaynak, Sack and Wright (2010), available from the Federal Reserve Board. The term structure extends to a maximum of 20 years. To construct real spot rates for longer maturities we assume that forward rates for years 21 to 30 are equal to the average forward rate for years 18 to 20.

C Estimating the marginal distributions for inflation

We estimate inflation distributions from data on zero coupon and year-on-year inflation caps and floors, which we collect from Bloomberg, as do Fleckenstein, Longstaff and Lustig (2013). Kitsul and Wright (2013) use data provided by an interdealer organization, whereas we use the raw reported numbers.

C.1 Zero-coupon inflation options

The basic methodology for construction of the distributions from zero coupon inflation options is fairly standard and similar to Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2013). Still, we take several steps in cleaning the data that this section of the appendix clarifies.

The zero coupon inflation call options are traded with strike prices between 1% and 6%, in 0.5% increments, and expiration dates ranging from 1 to 10 years as well as 12 and 15 years, although the data for the 2 and 9 year maturities are of generally lower quality. The zero coupon put options are available for strike prices between -2% and 3% and identical maturities as the caps. For the overlapping range of strike prices we use both option prices

to reduce measurement error.

A zero coupon inflation cap is the most traded contract among inflation derivatives. It pays, at expiry, the maximum between zero and the difference between the cumulate inflation during the period and the strike price so its payoff at maturity is $\max[0, (1 + \pi_{0,t}) - (1 + k)^t]$. Following Breeden and Litzenberger (1978) we non-parametrically construct risk neutral density functions using these option prices.

In particular, the price at date 0 a_0 of a simple European call option with maturity t , with a strike price k on inflation $\pi_{0,t}$ that has a risk-neutral density of $f(\pi_{0,t})$, and with the risk-free rate H_t^{-1} is:

$$a_0 = H_{0,t} \int_k^\infty (\pi_{0,t} - k) f(\pi_{0,t}) d\pi_{0,t}. \quad (\text{A8})$$

Taking the derivative of the pricing equation with respect to k gives the cumulative density function $F(k) = H_{0,t}^{-1} \partial a_0 / \partial k + 1$. A second derivative delivers the result:

$$f(\pi_{0,t}) = H_{0,t}^{-1} \left(\frac{\partial^2 a_0}{\partial k^2} \right). \quad (\text{A9})$$

Therefore, we can extract the risk-neutral density by observing how the price of the option varies with changes in the strike price.

Breeden and Litzenberger (1978) suggest using a butterfly trading strategy to construct Arrow-Debreu securities, claims that pay one unit of currency if at some specific time in the future the underlying asset price is equal to a specific value and zero otherwise. While this method provides a good first approximation to risk neutral probabilities, it does not adjust for irregular options prices, due to, for example, non-synchronous trading (Bahra, 1997). Therefore, while we check that all such prices are positive, we must smooth the data otherwise the results are very inaccurate.

To overcome the drawbacks of the unadjusted butterfly strategy, we do the following. First, we drop data that represent simple arbitrage opportunities (discussed in the text). We

next calculate Black-Scholes implied volatilities, and, following Shimko (1993) and Campa, Chang and Reider (1998), for each set of options at any expiry date, we fit a natural spline with two knot point. We constrain the estimated implied volatility function to ensure that the smoothing does not re-introduce arbitrage opportunities. This method reduces the weight of irregular data, while preserving its overall form. We convert back to option prices and construct risk neutral distributions.

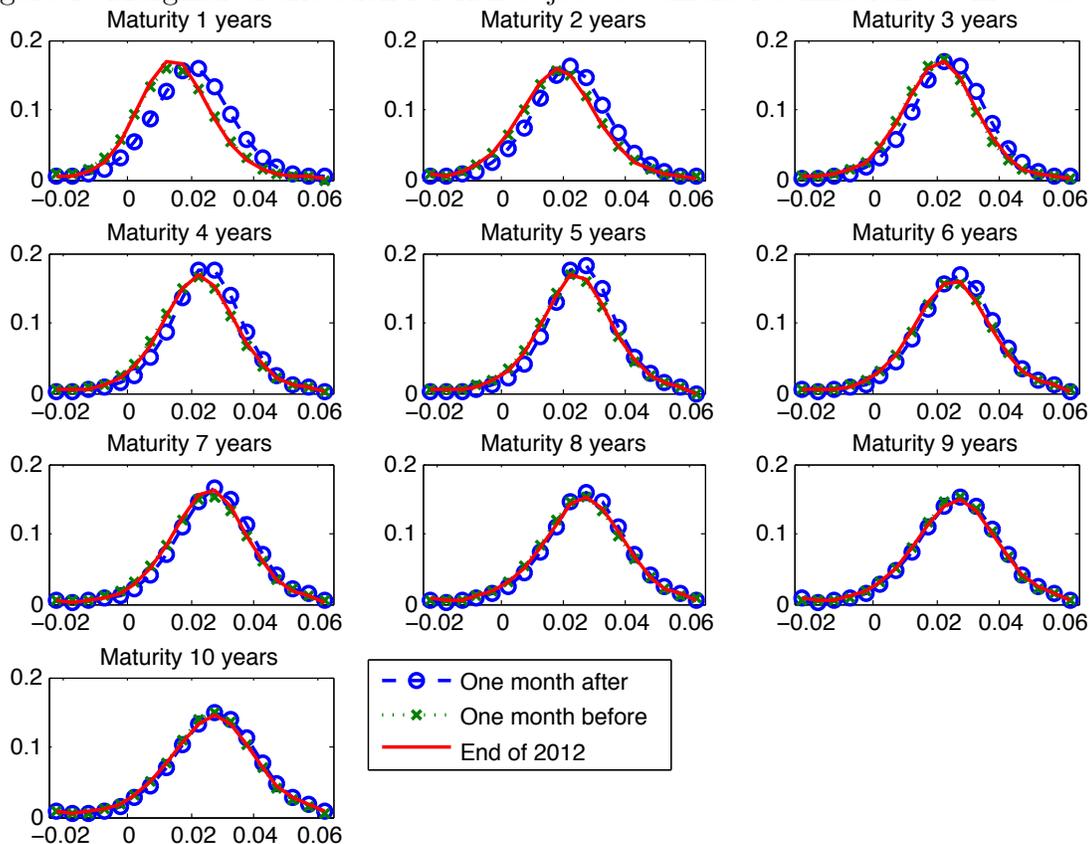
To be clear, this method does not assume that we can use the Black-Scholes formula for pricing. Instead, it is simply used as a nonlinear transformation on which smoothing is performed.

C.2 Year-on-year inflation options

The construction of distributions from year-on-year options requires a bootstrapping method where cap and floor contracts, which are portfolios of annual caplets and floorlets, are unbundled to recover prices of the underlying options. First, we use a bootstrapping procedure to extract the caplet and the floorlet prices from the cap and floor prices respectively. Second, when calculating the option's implied volatility we use the Rubinstein (1991) transformation which enables us to price the option as a plain vanilla option with a time to maturity equal to the option tenor between inflation reset times, discounting back using the real interest rate. For both types of options we use nominal and real interest rates from Gürkaynak, Sack and Swanson (2005).

For each horizon, for the smallest (largest) bin we report the risk-adjusted probability of inflation lying in or below (above) that bin. Because there was considerable mass in the last bin (5.5% to 6% inflation) for year-on-year inflation, we use our smoothed estimates to project four additional bins, so the figures show ranges up to 8%.

Figure 1: Marginal distribution for risk-adjusted cumulative inflation at three dates



C.3 Comparing option prices at different dates

We inspect option prices at the end of December 2012 (our data of interest) and select out of the last five trading days of the year the day that yields the maximum number of option prices that do not violate the threshold criteria, described above and in the text. This was December 31st, which is the benchmark for our calculations. We looked at distributions for the 5 days before and after. All of them looked almost identical to the ones we used.

More interesting, we also constructed inflation distributions one month before and after. Specifically, for November 30, 2012 and February 4, 2013. These days are again chosen as the trading days that are close to one month before and after the end of December and that have the maximum number of option prices that do not violate our threshold

criteria. Those could have changed because of the arrival of news about risk or inflation. Still, figure 1 shows that while there are some changes in the 1 and 2 year maturities, they are really small. More importantly, we repeated every calculation in the paper using these alternative distributions. The results on debt debasement for both the value at risk and the counterfactuals are essentially unchanged.

D Estimating the joint distributions for inflation

The marginal distributions for inflation are enough to evaluate the formula in proposition 1. Yet, to calculate inflation paths or counterfactuals, we need joint distributions.

D.1 Proof of proposition 2

Using $F(\cdot)$ to denote the cumulative density function, we have data for one-year inflation $F(\ln \pi_{t+j-1,t+j})$ for $j = 1 \dots J$ where $J = 10$ years, and for cumulative inflation $F(\ln \pi_{t,t+j})$. The data comes in N bins expressed as ranges for inflation.

Sklar's theorem states that there exists a function $c : [0, 1]^J \rightarrow [0, 1]$ such that:

$$F(\ln \pi_{t,t+1}, \dots, \ln \pi_{t+J-1,t+J}) = c(F(\ln \pi_{t,t+1}), F(\ln \pi_{t+1,t+2}), \dots, F(\ln \pi_{t+J-1,t+J})). \quad (\text{A10})$$

In turn, it follows from the link between marginal and joint distributions and the definition of cumulative inflation that:

$$F(\ln \pi_{t,t+j}) = \int_{\Pi} F(\ln \pi_{t,t+1}, \ln \pi_{t+1,t+2}, \dots, \ln \pi_{t+J-1,t+J}) d \ln \pi_{t,t+1} \dots d \ln \pi_{t+J-1,t+J} \quad (\text{A11})$$

where the set Π is defined as: $\left\{ \ln \pi_{t,t+j} : \ln \pi_{t,t+n} \sum_{i=1}^j \ln \pi_{t+i-1,t+i} \right\}$.

Combining these two results delivers the proposition in terms of cumulative distributions. We maintain the assumption throughout that all distributions are continuous. Therefore,

Sklar's theorem also applies to the marginal density functions, with $C(\cdot)$ replacing $c(\cdot)$. For numerical purposes, it was better to work with marginal, rather than cumulative, densities in the estimation.

A final point to note is that there are N bins and so N equalities in the distribution, of which one is redundant since probabilities must add up to one. There are J maturities, but for maturity one the equality is trivial. Therefore, in total there are $(N - 1)(J - 1)$ conditions.

D.2 Parametric copula and method of moments

We use the parametric normal copula, whose formula is:

$$\hat{C}(f(\ln \pi_{t,t+1}), \dots, f(\ln \pi_{t+J-1,t+J})) = \left(\frac{1}{\det R} \right) \exp \left(-\frac{1}{2} \begin{pmatrix} \Phi^{-1}(f(\ln \pi_{t,t+1})) \\ \vdots \\ \Phi^{-1}(f(\ln \pi_{t+J-1,t+J})) \end{pmatrix} (\rho^{-1} - \mathbf{I}_J) \begin{pmatrix} \Phi^{-1}(f(\ln \pi_{t,t+1})) \\ \vdots \\ \Phi^{-1}(f(\ln \pi_{t+J-1,t+J})) \end{pmatrix} \right) \quad (\text{A12})$$

where $\Phi(\cdot)^1$ is the inverse of the standard normal cdf, and ρ is a correlation matrix of dimension J .

The matrix ρ would only exactly equal the correlation matrix of the variables in the joint distribution if the marginal distributions happened to be normal. Yet, by drawing from the joint distribution using the formula above, and calculating correlation coefficients across many draws, we found that the difference between the actual correlation matrix and ρ was almost always less than 0.01.

To find the estimates, we minimize over the $J(J - 1)/2 = 45$ independent components of the correlation matrix ρ that lie between -1 and 1 . The objective function is the equally weighted average of the $(N - 1)(J - 1) = 20 \times 9 = 180$ squared deviations from the moments

in proposition 2. This is a difficult global minimization over a large parameter space, which we handle through a combination of global and local minimization algorithms and many repeated searches from randomly drawn starting points.

D.3 Inflation after 10 years

In drawing paths for inflation, for the first 10 years we use the joint distribution given by the multivariate copula. After that, we assume that inflation is a 9th order Markov process. Therefore, the distribution for inflation in year 11, conditional on inflation in years 2 to 10 is the same as the distribution for inflation in year 10, conditional on years 1 to 9. Since we have the joint distribution for inflation from year 1 to 10, it is easy to derive the conditional distribution for inflation in year 10, conditional on years 1 to 9. Thus, we have the conditional distribution for year 11, conditional on the draws so far. The same applies to year 12, and so on, all the way to 30. Note that, since we assumed that the joint distribution of maturities one to ten follows a Gaussian copula, then this procedure implicitly assumes that the joint distribution in maturities one to thirty is likewise a Gaussian copula. The key restriction is that the correlation matrix of $30 \times 29/2 = 435$ elements for the 1-30 copula only have 45 independent separate elements that we estimated for the 1-10 distribution.

D.4 Restricted distribution

We can represent inflation between two successive dates (or maturities) as:

$$\ln \pi_{t,t+1} = \mathbb{E}_t(\ln \pi_{t,t+1}) + p_{t+1} + s_{t+1}, \quad (\text{A13})$$

where p_{t+1} is a non-stationary part, and s_{t+1} a stationary one, with the two being independent and zero mean. From Wold's theorem, the stationary process is fully characterized by its covariance function $v_j = \mathbb{E}_t(s_{t+n}s_{t+n+j})$ for any arbitrary j . The definition of stationary is

that this is true for any positive n .

The restriction that we impose is that the non-stationary process is a random walk, so that $\mathbb{E}_t(p_{t+n}p_{t+n+j}) = \sigma n$. This constrains the way in which non-stationarity affects the correlation matrix over time. In particular, the correlation between inflation at date $t + n$ and date $t + j$, or the (n, j) element of the matrix ρ is given by the expression:

$$\rho_{n,n+j} = \frac{v_j + n\sigma}{\sqrt{(v_0 + n\sigma)(v_0 + (n + j)\sigma)}} \quad (\text{A14})$$

While the unrestricted ρ matrix has 45 parameters, with the random-walk constraint, there are only 10 parameters. Nine are the correlations of the stationary part $\{v_1/v_0, \dots, v_9/v_0\}$, and one more is the ratio of the relative variances of the permanent and transitory components σ/v_0 . We minimize the same objective function but over this small parameter space. The estimates when we set $\sigma = 0$ or estimate an unrestricted σ are identical up to 0.01, and the minimization algorithm always hits this bound for σ . Note that in this case, the ρ matrix has the easily identified form, $\rho_{n,n+j} = v_j/v_0$, which is the same whatever is n .

D.5 Goodness of fit and restricted versus unrestricted distribution

Table 1 shows the estimated ρ for the unrestricted model. Because the estimates vary so much, they are a little hard to interpret. The two key features that are in common with the restricted estimates are the ones emphasized in the text: the autocorrelation coefficients are not very high, and they do not seem to fall with maturity. But the estimates are quite noisy in that, across contiguous maturities, they jump up and down a bit.

We assess goodness of fit in multiple ways. First, figure 2 shows the data for the risk-neutral density of cumulative inflation at maturities 2 to 10 against the predicted densities, according to the restricted and unrestricted models. Our method of moments, following proposition 2, consisted of picking parameters ρ to minimize the difference between the data

Table 1: Estimated correlation coefficients of year-on-year inflation in the joint distribution

<i>Maturity</i>	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>	<i>7</i>	<i>8</i>	<i>9</i>	<i>10</i>
<i>1</i>	1.00	0.52	0.14	0.48	0.18	0.11	0.15	0.01	0.28	-0.02
<i>2</i>		1.00	0.35	0.33	0.12	0.22	-0.18	-0.04	0.41	-0.04
<i>3</i>			1.00	0.35	0.22	0.47	-0.08	0.54	0.44	0.25
<i>4</i>				1.00	0.61	0.59	0.69	0.53	0.39	0.31
<i>5</i>					1.00	0.47	0.61	0.68	0.53	0.55
<i>6</i>						1.00	0.40	0.51	0.36	0.42
<i>7</i>							1.00	0.57	0.43	0.44
<i>8</i>								1.00	0.71	0.45
<i>9</i>									1.00	0.52
<i>10</i>										1.00

Notes: Estimated correlation coefficients for year-on-year inflation between date 2012+j and 2012+l, in column j, row l.

in these 9 plots, and the models. Visually the fit is quite good, and the restriction seems to have almost no effect on the ability of the copula model to fit the data.

Second, we compare the models' prediction for inflation in horizons 12 and 15 with the data for those maturities. Note that this tests not only the normal copula, but also our 9th order Markov assumption to simulate beyond 10 years. Figure 3 shows the model-implied and data distributions. The model again seems to do well, with the stationary restriction doing better.

Third, we compare the model's predicted standard deviations of risk-adjusted inflation, with those in the marginal distributions. Figure 4 shows the two models. The restricted model again seems to do a slightly better job when applied to the horizon 12 and 15 distributions, although again it is clear that, in spite of the seemingly different estimates of ρ , the two models have quite similar fits.

Fourth, we repeated our calculations for one month after and one month before, to assess whether within this short time windows, the estimate of ρ was not too volatile. Table 2 shows

Figure 2: Marginal distribution for risk-adjusted cumulative inflation: data and models

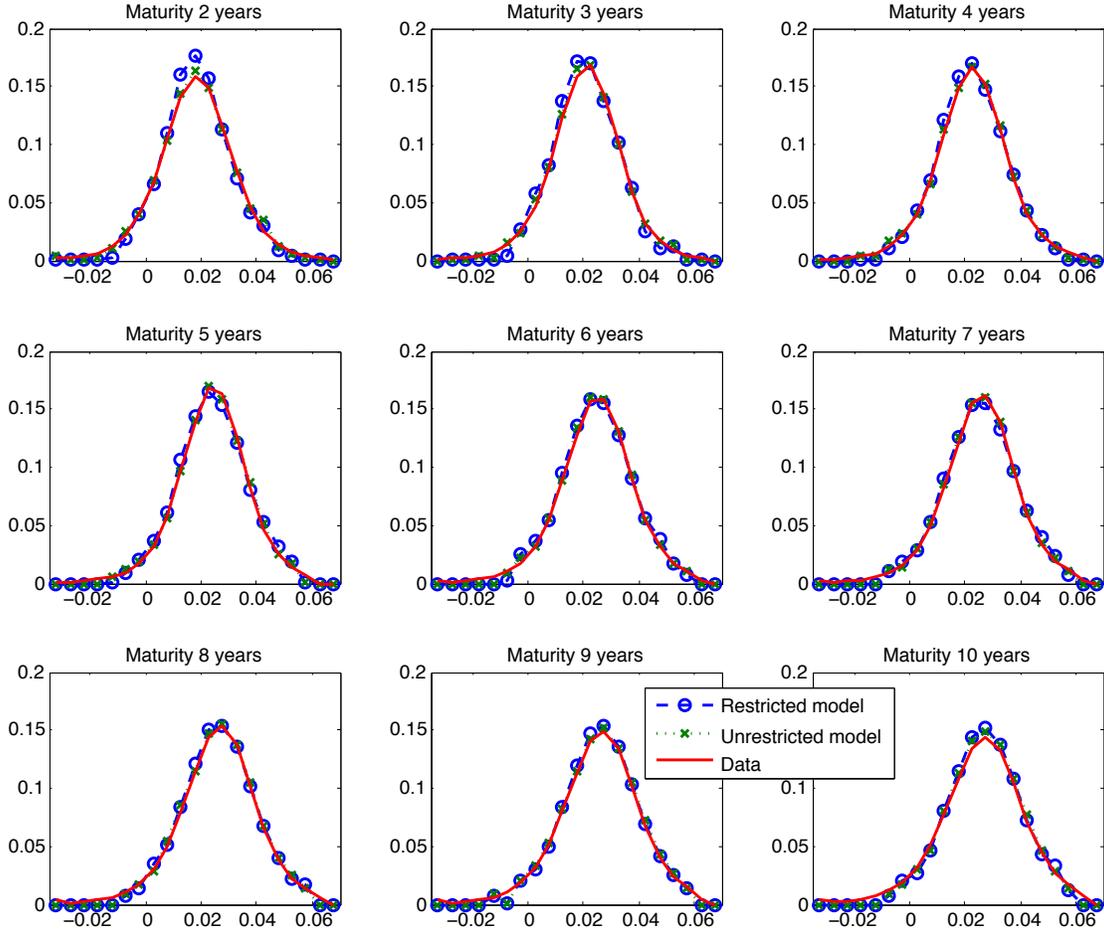


Figure 3: Distribution for $\ln \pi_t, t + 11$ and $\ln \pi_t, t + 15$, model and data

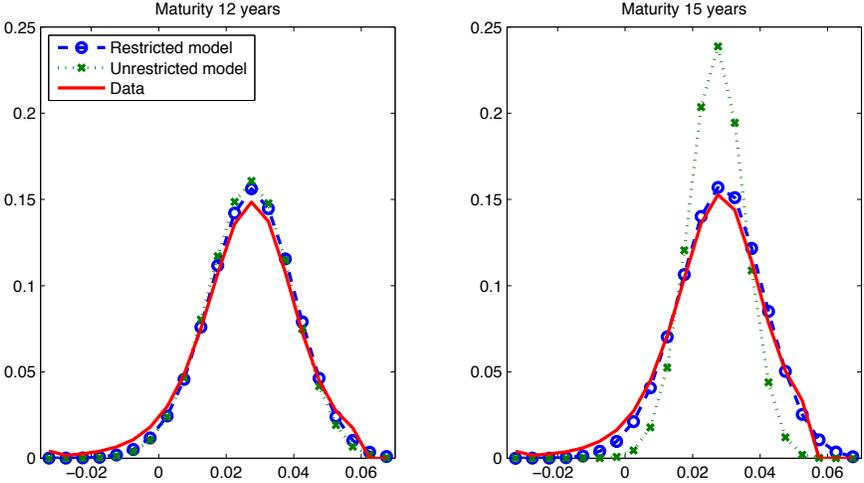


Figure 4: Standard deviation of risk-adjusted inflation at different horizons: data and models

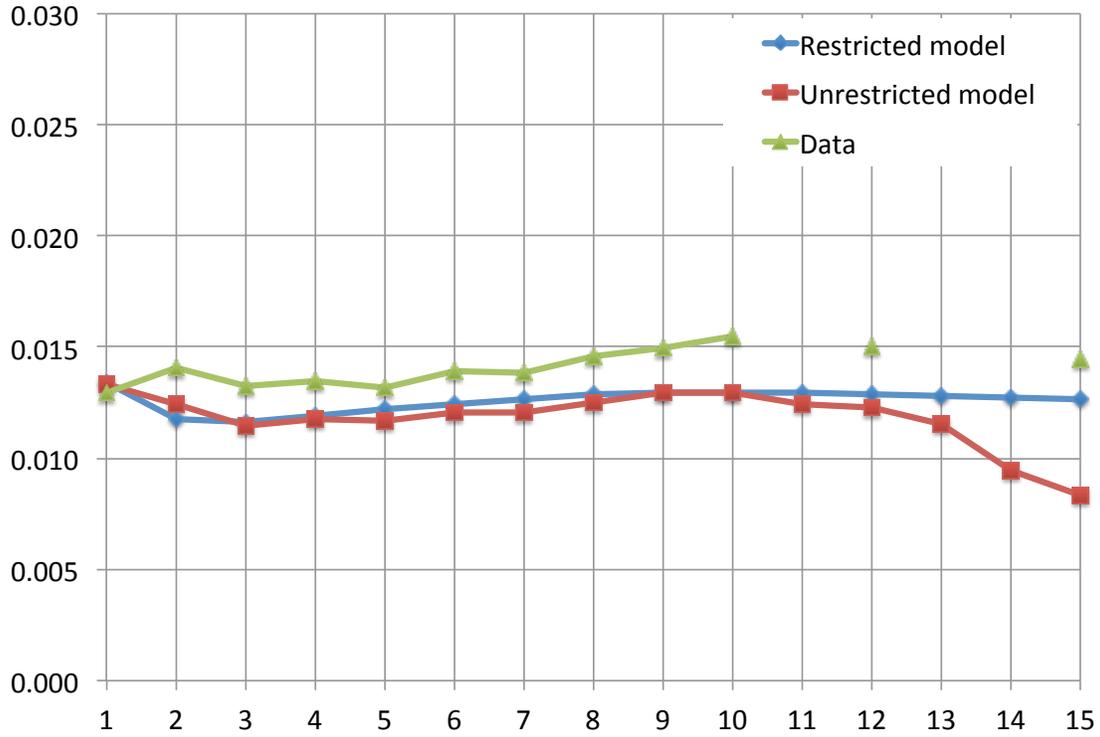


Table 2: Estimates of ρ one month before and after

<i>Horizon</i>	<i>Baseline (Dec 2012)</i>	<i>Nov 2012</i>	<i>Feb 2013</i>
1	0.36	0.31	0.23
2	0.33	0.36	0.27
3	0.50	0.47	0.37
4	0.48	0.50	0.37
5	0.27	0.25	0.44
6	0.40	0.39	0.30
7	0.38	0.32	0.48
8	0.28	0.24	0.43
9	0.11	0.01	-0.08

Notes: Estimated restricted correlation coefficients for year-on-year inflation

the resulting estimates for the restricted model only, to save space. While the estimates change somewhat, the overall pattern that we describe in the text is similar. Again, redoing all of our calculations using these estimates leads to almost no difference on the effects of inflation on the real debt burden.

E Estimating debt debasement

We draw 500,000 samples for 40 years of inflation using our joint distribution. We convert these into 480 month histories by assuming continuous compounding and a constant inflation rate within each year. We calculate the real value of the nominal payments for each of these draws, and order them, calculating their percentiles for table 2. We repeat the calculations using the B_0^t for each group of investors instead.

For the counterfactuals, we use alternative distributions for inflation from which to take draws, recalculate the value of the debt using the formula in proposition 1, and subtract it from the original number (51% of GDP). The alternative distributions are:

1. Permanently higher: Shift all the year-on-year distributions by the difference between the 90th and the 50th percentile at each maturity.
2. Right tail only: We draw from the baseline distribution but discard all histories in which average inflation is below its 90th percentile in any one of the years in the simulation.
3. Higher and more variable: For each maturity we multiply baseline inflation levels by a scaling factor so that the new mean is equal to the mean in case 1. This results in more variable inflation.
4. Higher for sure: This is the same as case 1 but we now assign all the weight to the mean at each maturity.

5. Partially expected: We set inflation equal to 3% in year 1. For the following histories we use marginal distributions conditional on the year 1 realization. After 10 years we assume that year-on-year inflation is 9th order Markov.
6. Temporary increase: We shift the year-on-year distributions in the same way as case 1 but we now shift them to the 90th percentile in year 1, 80th in year 2, 70th in year 3, and 60th in year 4. There is no change in the year-on-year distributions for maturities equal to and above five years. Note that this is different from the unexpected shock in the previous case since distributions in years 2 to 4 are shifted directly instead of changing only due to the new distribution in year 1.
7. Gradual increase: The one year distribution is unchanged, the 2 year median shifts to the previous 60th percentile, 70th for year 3, 80th in year 4, and 90th for 5 years and above.

F Proof of proposition 3

Financial repression consists of paying nominal bonds that are due at date t with new N -period special debt that sells for the price \tilde{H}_t^N :

$$B_t^0 = \tilde{H}_t^N \tilde{B}_{t+1}^{N-1}. \quad (\text{A15})$$

The value of outstanding debt at date t now is:

$$W_t = \sum_{j=0}^{\infty} \frac{H_t^j B_t^j}{P_t} + \sum_{j=0}^{\infty} Q_t^j K_t^j + \frac{\tilde{B}_{t-N+1}^{N-1}}{P_t}, \quad (\text{A16})$$

while the government budget constraint now is:

$$W_t = s_t + \sum_{j=0}^{\infty} \frac{H_t^{j+1} B_{t+1}^j}{P_t} + \sum_{j=0}^{\infty} Q_t^{j+1} K_{t+1}^j + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1}}{P_t}. \quad (\text{A17})$$

The text covered the special case where $N = 1$, there is no real debt, and all nominal debt had one period maturity. This appendix proves the general case.

Combining these two equations just as we did in the proof of proposition 1, we end up with a law of motion for debt:

$$W_t = Q_t^1 W_{t+1} + s_t + x_{t+1} + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1}}{P_t} - Q_t^1 \frac{\tilde{B}_{t-N+2}^{N-1}}{P_{t+1}} \quad (\text{A18})$$

By precisely the same steps as in the proof of proposition 1, it then follows that:

$$W_0 = \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} \left(\frac{B_0^t}{P_t} \right) \right] + \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} K_0^t \right] = \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} \left(s_t + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1} - Q_t^1 \tilde{B}_{t-N+2}^{N-1} P_t / P_{t+1}}{P_t} \right) \right] \quad (\text{A19})$$

Now, replacing the new debt with old debt using equation A15, we get that the nominal debt burden then becomes:

$$\mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} \left(\frac{B_0^t}{P_t} \right) \right] - \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} \left(\frac{B_0^t}{P_t} \right) \right] + \mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t} \frac{Q_t^1 B_{t-N+1}^0}{\tilde{H}_t^N P_{t+1}} \right] \quad (\text{A20})$$

Canceling terms and relabeling the limits of the sums (since financial repression started at date 0) we get the nominal debt burden as:

$$\mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t+N-1} \frac{Q_{t+N-1}^1 B_t^0}{\tilde{H}_t^N P_{t+N}} \right] \quad (\text{A21})$$

Finally, recall that $Q_{t+N-1}^1 = \mathbb{E}_{t+N-1}(m_{t+N-1,t+N})$. Using the law of iterated expectations

we end up with:

$$\mathbb{E} \left[\sum_{t=0}^{\infty} m_{0,t+N} \left(\frac{1}{\pi_{t,t+N} \tilde{H}_t^N} \right) \left(\frac{B_t^0}{P_t} \right) \right] \quad (\text{A22})$$

To get the equality in the proposition, simply use the upper bound $H_t^N = 1$. To get the equality written in terms of the price of nominal bonds, simply use the law of iterated expectations and the arbitrage condition: $H_t^N = \mathbb{E}(m_{t,t+N}/\pi_{t+N})$.

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