NBER WORKING PAPER SERIES

GLOBAL VALUE CHAINS AND EFFECTIVE EXCHANGE RATES AT THE COUNTRY-SECTOR LEVEL

Nikhil Patel
Zhi Wang
Shang-Jin Wei

Working Paper 20236
http://www.nber.org/papers/w20236

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
June 2014

We are grateful to Charles Engel, Stephanie Schmitt-Grohe, Martin Uribe and our discussants Martin Berka and Giorgio Valente for helpful comments and discussions. We would also like to thank seminar participants at Columbia and the ABFER annual conference in Singapore for useful comments and Joy Glazener for editorial assistance. All remaining errors are our own. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

Global Value Chains and Effective Exchange Rates at the Country-Sector Level
Nikhil Patel, Zhi Wang, and Shang-Jin Wei
NBER Working Paper No. 20236
June 2014
JEL No. F1,F3

ABSTRACT

The real effective exchange rate (REER) is one of the most cited statistical constructs in international macroeconomics. With the rising importance of offshoring and outsourcing, the standard measures are increasingly flawed. In addition, because different sectors within a country may participate in international production sharing at different stages, sector level variations are also important. We develop a theoretical framework to compute REER at both the sector and country levels. It nests the existing measures in the literature and addresses their shortcomings. As an application, we exploit the recently available World Input-Output Database (WIOD) to study the properties of the new measures of the REER for 40 countries, 35 sectors, over 1995-2011.

Nikhil Patel
Department of Economics
Columbia University
New York, NY 10027
np2333@columbia.edu

Zhi Wang
Research Division
Office of Economics
US International Trade Commission
500 E Street SW
Washington, DC 20436
zhi.wang@usitc.gov

Shang-Jin Wei
Graduate School of Business
Columbia University
Uris Hall 619
3022 Broadway
New York, NY 10027-6902
and NBER
shangjin.wei@columbia.edu

1 Introduction

The Real Effective Exchange Rate (REER) is one of the the most quoted indices in international economics among academics, policy makers and financial market participants. A google search on "effective exchange rates" yields close to 58 million hits, higher than terms like "inflation targeting "(~5 million) and "comparative advantage"(~9 million). The underlying purpose behind the construction of REER is to have a statistic that measures competitiveness by quantifying the sensitivity of demand for output originating from a particular country as a function of changes in world prices. It is a partial equilibrium construct which takes observed price changes as given and does not require modeling of primitive shocks that lead to those price changes, or any other general equilibrium constraints like balanced trade.[1] Among the common applications of REER in international macroeconomics is its use as an indicator to draw inferences regarding currency manipulation, currency misalignment and vulnerability to crises–see for instance Chinn (2000) , Goldfajn and Valdés (1999) and Gagnon (2012).

The importance of REER is further evident from the time, effort and resources devoted to computing REER indices by leading organizations like the International Monetary Fund (IMF), Bank of International Settlements (BIS), OECD as well as various central banks around the world.[2]

Standard REER measures make a number of simplifying assumptions which we evaluate in this paper. For instance, they assume that every country exports only final goods which are produced without using imported intermediate goods. The first point we wish to emphasize is that this assumption is not innocuous, as the following example shows.

Consider a stylized world with three countries involved in a global value chain–China, Japan and the US. Suppose Japan manufactures raw materials for the production of a mobile phone and ships it to China which acts as an assembly point. China in turn exports the finished product to the US which is then consumed by US consumers. According to the traditional REER measures like the one used at the IMF, the phone would be classified as China's "product" and China would be assumed to be competing with other providers of phones. Consequently these models would conclude that an increase in the price of a mobile phone in Japan would lead to an increase in the demand for China's mobile phone and hence increase its competitiveness. In reality however, China is not producing the entire phone but is the producer of assembly services, which accounts for only a small fraction of the total value of the product. It therefore competes with other providers of such processing services and not phone manufacturers. Once we recognize this, it becomes clear

[1] See Chinn (2006) for a primer on the concept of REER and Rogoff (2005) for an application and discussion.

[2] For the IMF REER computation see McGuirk (1986) and Bayoumi et al. (2005). For the Federal reserve's REER measure see Loretan (2005). The BIS's REER methodology is summarized in Klau et al. (2008)

that an increase in Japanese prices of mobile phone components could very well lead to a decline in demand for China's services and hence a decline in its competitiveness.

This example shows that REER computed by organizations like the IMF is not only inaccurate in terms of magnitude, but may also have the wrong sign. Given that trade in intermediate goods can potentially have a major influence on the REER, it is important to incorporate it in the model used to compute REER, especially given the prominence and rising trend in intermediate goods trade in the last two decades.[3]

Another limitation common to all previous work in this literature is that REER is defined at the level of individual countries.[4] With increasing specialization and trade in intermediate inputs, inter-sectoral linkages between countries differ substantially from aggregate country level relationships. Wang et al. (2013) (henceforth WWZ) have documented this heterogeneity substantially. In particular, they find that total foreign content (VS) sourced from manufacturing and services sectors used in world manufacturing goods production has increased by 8.3 percentage points from 1995 to 2011 (from 22. 5% in 1995 to 30.8% in 2011). Moreover, they show that this increase is primarily accounted for by the increase in foreign double counted terms (FDC) which are a result of back and fourth trading between countries. Once we take the concept of global value chains seriously, we have to recognize the fact that different sectors in a country tend to participate in cross-border production sharing by different extents and in different ways. For example, according to WWZ (2013), some sectors mostly engage in regional value chains (i.e., buying or selling intermediate inputs with neighboring countries), whereas others engage in truly global value chains (i.e., sourcing and selling a significant amount of inputs to countries on different continents.) This implies substantial heterogeneity in changes in competitiveness across sectors to a given change in foreign price vector. An aggregate country level measure is incapable of capturing these. Indeed, in section 10 we document several instances where the REERs move in opposite directions for different sectors within a country. In these cases where the country level REER moves in the opposite direction to certain key sectors within the country, lack of information on the latter can lead to false conclusions and inefficient or even counter-productive policy measures. For

[3]For OECD countries Miroudot et al. (2009) find the share of trade in intermediate goods and services to be 56% and 73% respectively. As emphasized in Baldwin and Lopez-Gonzalez (2012), intermediate goods trade and vertical specialization have grown many fold in developing countries starting in the 1980s (see also Wang et al., 2013) Also important is the import content of exports, epitomized by the prevalence of processing trade involving Asian economies, especially China. Koopman et al. (2014) find that the import content of exports is as high as 90 percent for some sectors in China.

[4]Although Goldberg (2004) develops sectoral effective exchange rates for the US and Kiyotaka et al. (2012) and Kiyotaka et al. (2013) do so for Japan, Korea and China, these papers do not take into account trade in intermediate inputs or differences in elasticity of substitution. In fact they proceed in a reduced form manner and do not allow for even third country competition as is done in our framework as well as in the models by the IMF and BIS.

example, we document in figure 2 that Mexico as a whole has experienced a loss in competitiveness (appreciation) starting in 1995. One policy measure this may instigate is (implicit or explicit) subsidies, especially to exporters. However applying these measures uniformly across all sectors would be erroneous since there are sectors like the financial intermediation sector that are already experiencing a gain in competitiveness. This illustrates the usefulness of sector level exchange rates in enabling countries to better target and manage their producers and exporters. Although lack of data and absence of the global value chain phenomenon had prevented the feasibility and need for such measures in the past, in this paper we emphasize that this is no longer the case.

Recognizing all these shortcomings, this paper proposes a concept of REER that improves upon the existing REER measures in the literature along four dimensions. Firstly, by explicitly allowing for trade in intermediate inputs and distinguishing trade by end use category, our model recognizes that value added and gross output are not the same. We therefore compute different REER indices for value added (GVC-REER) and gross output (Q-REER). Secondly, we start at the level of sectors within countries and build our way up, allowing us to define and compute not just country REERs but also REERs for sectors within countries. Given our data source (to be discussed in detail later on in the paper) we can compute REER indices for 35 sectors within each of the 40 countries in the sample. We find substantial heterogeneity in the REER across sectors within countries and are in a position to talk about competitiveness of individual sectors within the same countries. Thirdly, we estimate and explicitly incorporate different elasticities of substitution across different groups of goods in our REER measure. This is a significant improvement relative to other attempts to take GVCs into account which continue to work with a simplified Cobb-Douglas case in which all elasticities are identically equal to one. Lastly, we compute sector level price indices and use these in our REER measure instead of the more coarse country level price indices (GDP deflator or CPI) which have been used in the literature so far.

We show that our model can also be used to construct better measures of bilateral real exchange rates (RERs). Bilateral RERs are typically used for either price level comparisons or relative movements in competitiveness (see for example Chinn, 2006). If the goal is the latter, then we show that the insights gained from our discussion of REERs can be used to construct measures of bilateral RERs using sector level trade flows that better reflect movements in relative competitiveness. We find in the data that our measure of bilateral RER paints a substantially different picture compared to the standard GDP deflator based RER for certain key country pairs. Section 11 for instance shows that for US-China, we find that although the standard RER displays a non monotonic pattern for the RER including substantial depreciation of the Chinese exchange rate prior to the abandonment of the peg in 2005 (which was widely reported and discussed among academics, financial participants as well as in the media) our RER measure which exploits sector level linkages shows a secular appreciation during the sample period.

It is worth emphasizing that our objective is to model relatively short-term and small-scale movements in competitiveness. We therefore take the nature of GVC and trade patterns across countries and sectors as given and do not consider the issue of endogenous off-shoring and production sharing decisions.[5] Moreover, due to the complex nature of the model we solve it using log linearization techniques. This further reinforces the view that our REER indices are best suited for short-term movements resulting from shocks that are not too large so as to affect organization of GVCs.

This paper is related to two different strands of the literature. First and foremost the paper is a contribution to the literature on international trade and finance. The most prominent and commonly cited REER measure today is the IMF's REER measure. This along with other commonly used REER indices like the ones by the Federal Reserve and Bank of International Settlements (BIS) do not distinguish between trade in intermediate and final goods and consider all trade flows to be in the latter category. The consequence of making this assumption could be quite detrimental, as we have discussed above.

A few recent papers have recognized this drawback and have made attempts to address them. Bems and Johnson (2012) (henceforth BJ) allow for trade in intermediates and compute the REER weighting matrix at the country level. Bayoumi et al. (2013) propose a measure of competitiveness in which they borrow the weighting matrix from the IMF but adjust the price indices to acknowledge the presence of imported inputs. But these papers work with the constant elasticity (Cobb Douglas) assumption and country level (instead of more detailed sector level) price indices. Our attempt to incorporate sector level price indices and build sector level exchange rates has a precedent in the work of Bennett and Zarnic (2009). But their work does not incorporate trade in intermediate goods and uses an IMF-like weighting matrix. Moreover, they use unit labor costs to proxy for price of value added, whereas we have a more comprehensive measure of value added price index at the sector level which includes not only labor but also capital.

In Table 1 we summarize our contribution to this literature by drawing comparisons across the most influential as well as the most advanced REER computation frameworks available. A check mark ($\surd$) indicates that the paper attempts to address the particular attribute, irrespective of whether they achieve the desired objective in our opinion. In fact, in most cases where there are check marks in the table, we show drawbacks that are addressed by our framework. As will be shown in the sections to follow, our measure is not only more comprehensive, but nests most of the common measures in the literature.

More broadly, the paper is motivated by and is linked to new but rapidly expanding literature

[5] There is a growing literature on organization of global value chains that looks into these questions. See for instance Antràs and Chor (2013), Antràs (2014), Costinot et al. (2013) and Johnson and Moxnes (2012).

Table 1 – Comparison with the literature on computing REER

	IMF	FED	BIS	BJ	BST	BZ	This paper
Value added competitiveness				$\surd$			$\surd$
Sector level exchange rates/prices						$\surd$	$\surd$
Trade in intermediate goods				$\surd$	$\surd$		$\surd$
Heterogenous elasticities							$\surd$

Key: BJ: Bems and Johnson (2012); BST: Bayoumi et al. (2013); BZ: Bennett and Zarnic (2009); A check mark ($\surd$) indicates that the paper attempts to address the particular attribute, irrespective of whether they achieve the desired objective in our opinion

on global value chains and vertical specialization in trade (see Hummels et al., 2001 and Baldwin and Lopez-Gonzalez, 2012 among many others) as well as the literature on trade statistics and export accounting in the presence of intermediate goods trade (Koopman et al., 2012 and Wang et al., 2013).

After briefly reviewing the concept of REER in section 2 we start by presenting a simplified version of our framework in section 3 to illustrate some of the features of our exchange rate measure before moving on to the general model in section 4 which is discussed in greater detail and applied to the data.

2 The Concept of REER as a Measure of Competitiveness

The real effective exchange rate measures change in competitiveness by quantifying changes in the demand for goods produced by a country as a function of changes in relative prices.[6] To be more precise, if V_J is the demand for the goods produced (or alternatively, value added) by country J, then the effective exchange rate of country J is defined as:

$$\triangle REER_J = \triangle V_J = G_J\left(\{\triangle p\}_{i=1}^{n}\right) \tag{2.1}$$

where $\{\triangle p\}_{i=1}^{n}$ is a vector of price changes in all countries including the home country. Note that no other variables except the prices explicitly enter the function $G(.)$. Hence by construction REER is a partial equilibrium construct where the primitive shocks that lead to the observed price changes are not modeled. Moreover the demand side of the economy is assumed to be exogenous and the

[6] In this literature the use of the word "competitiveness" is appropriate only in conjunction with the perfect competition assumption. With imperfect competition, an increase in demand for value added may coincide with a decrease in profits for the producers (as would be the case for instance with a monetary policy shock with sticky prices). It seems misleading to label this an increase in competitiveness of the producers. We thank Charles Engel for pointing this out.

aggregate final demand is assumed to be constant (although relative demands are allowed to change when prices change).

The function $G_J(.)$ is homogenous of degree zero, so that the model satisfies neutrality in the sense that if all prices (including the home price) double, then the relative demands remain unchanged (and since by construction aggregate demand is held fixed, the absolute demand for each good also remains unaffected).

It is worth pointing out that REER models like ours do not assume balanced trade or any restrictions on the trade balance. Trade balances are allowed to be non-zero in the steady state and are calibrated to their observed counterparts in the data. This is in line with the partial equilibrium setup in which the demand side is exogenous.

Throughout the paper we work with a static partial equilibrium model. Before describing the model it is pertinent to address concerns regarding its ostensibly restrictive nature.

Competitiveness is a supply side concept and measures how changes in cost structure of a producer makes its product more competitive by enabling it to capture demand from other producers. In order to isolate the role of competitiveness it is therefore critical to shut out other effects that might be operational, most notably aggregate demand.

Consider the case of a favorable supply shock like a productivity shock or a tariff reduction that affects a single producer. In the partial equilibrium setting of this paper, the shock manifests itself only in the form of a lower price, which leads to an increase in demand for the good (at the expense of other goods). This is interpreted as an increase in competitiveness. In a general equilibrium framework, a supply shock of this form will have additional effects. In particular, it will affect the real incomes of agents which in turn will affect aggregate demand (and also its distribution in a heterogenous agent economy). This latter effect, however, is not a direct consequence of the cost advantage gained by the producer and hence must not be included in the competitiveness measure.

To make the point more precise, consider a two country (A and B), two sector (1 and 2) model and suppose (A, 1) is hit by a positive productivity shock. The direct impact of this shock would be to lower the price of (A, 1). This in turn would lead to a shift in demand towards (A, 1) at the expense of demand going to all the three remaining sectors. This demand is captured by (A, 1) due to its cost advantage triggered by the favorable productivity shock. This is the effect that a competitiveness measure should be designed to capture and our REER measure does exactly this. In addition, the favorable productivity shock would most likely also lead to an increase overall demand (i.e quantity demanded) for all goods. This effect may imply an increase in the quantity demanded for either or all of the three remaining sectors. This latter increase however comes despite an unfavorable movement in relative costs (i.e a decline in competitiveness), and hence must not be interpreted as an increase in competitiveness as would be the case if we measure competitiveness solely by increase in quantity demanded in a general equilibrium setting. For instance, with home bias in consumption, the transfer effect leads to an improvement in home terms of trade which in turn leads to an increase in the absolute demand for foreign goods as well, but this happens despite an increase in costs, not due to a favorable movement in costs which is what competitiveness in meant to capture.

It is often relevant for researchers working with partial equilibrium models to justify that the ignored general equilibrium effects are not likely to overturn their main partial equilibrium conclusions. The goal in this paper, however, is the construction of statistic as opposed to forecasting, comparative statics or explaining stylized facts in the data. We therefore do not need to show robustness with respect to generalization of the model beyond the partial equilibrium setup. On the contrary we argue that the question at hand requires us to look precisely at partial equilibrium effects and although general equilibrium effects might overturn partial equilibrium ones as hinted above, the latter are not the object of attention as far as measurement of competitiveness is concerned.

Regarding the static nature of the model, although the model does not have any inter-temporal features like capital accumulation and investment, it nevertheless imports the dynamics from the persistence of shocks in the data, which are the observed price movements. In fact, the model is calibrated to the exact time series of price changes so that we let the data completely govern the dynamic aspects of our REER measures.

Aside from effective exchange rates, unit labor costs (both at the country and sector level) are often used as indicators of competitiveness. Although these are widely available and easy to comprehend, REER, especially if constructed taking into account trade in intermediate goods and sector level trade flows as in this paper, is superior as a measure of competitiveness on several counts. Firstly the REER incorporates movements in relative prices as opposed to looking at individual or a pair of prices in isolation. It combines information from prices in all countries by weighting them in proportion to how important they are in affecting the competitiveness of the country or sector in question based on past trade flows. Looking at individual prices cannot deliver such information. For instance consider two countries A and B. If prices in A rise by more than prices in B, this does not automatically make B more competitive than A since they may be trading with different partners whose prices themselves may have changed differently. Moreover, B might be using A's output as an intermediate input in production, in which case B's competitiveness might actually fall when A's price rises. Secondly, a measure like the unit labor cost considers only one component of value added (namely labor) and ignores the effect of prices of other inputs (most notably capital) which account for one third of the total value added on average (and much higher in certain sectors). This is also a drawback of all effective exchange rate measures that use unit labor costs as deflators. In this paper we compute price indices for value added which incorporates all components of the latter (including labor and capital) to overcome this shortcoming.

3 A Stylized Three Country Global Value Chain

We start with a simplified 3 country, 2 sector model capturing the basic Global Value Chain (GVC) features. There are three countries (J, C and U for Japan, China and USA respectively) and two sectors indexed by {1, 2}. Table 2 displays the input output table. All boxes with "X" denote non-zero entries while the remaining entries are zero. Note that the IO matrix is sparse and contains only one (out of a possible 36) non-zero entries. The Global value chain is modeled across two sectors in three different countries. The upstream sector J1 (sector 1 in country J) produces raw materials that are exported to country C. Sector 2 in country C combines these intermediate inputs from J along with its own value added to produce final goods that are then exported to countries J and U in addition to being consumed internally by country C. All other sectors (i. e J2, C1, U1 and U2) only produce goods using own value added (i. e no intermediate inputs) and sell them as final demand in the home country. Sector 2 can be interpreted as the electronics sector and sector 1 can be interpreted as a (raw) materials sector.

Table 2 *– A stylized 3 country 2 sector global value chain set up*

		J		C		U		JFinal	CFinal	Ufinal	total output
		J1	J2	C1	C2	U1	U2				
J	J1	0	0	0	X	0	0	X	0	0	X
	J2	0	0	0	0	0	0	X	0	0	X
C	C1	0	0	0	0	0	0	0	X	0	X
	C2	0	0	0	0	0	0	X	X	X	X
U	U1	0	0	0	0	0	0	0	0	X	X
	U2	0	0	0	0	0	0	0	0	X	X
VA		X	X	X	X	X	X				
total output		X	X	X	X	X	X				

Specifying the model:

We use Q_l^h to denote the gross output of sector l is country h. [7] X_{sl}^{fh} denotes the intermediate input from country f sector s used in production by (h, l). V_l^h denotes the value added by (h, l). We assume a constant elasticity of substitution which is allowed to differ across consumption and production aggregators as clarified below.

Production:

The production function of $(C, 2)$ is given by

$$Q_2^C = \left[(w^V)^{\frac{1}{\sigma}} (V_2^C)^{\frac{\sigma-1}{\sigma}} + (w^X)^{\frac{1}{\sigma}} (X_{12}^{JC})^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{3.1}$$

[7] Throughout this paper, superscripts will be used for countries and subscripts for sectors.

where σ is the elasticity of substitution between the two inputs and w^V and w^X are weights that can be mapped to the shares of the two inputs.

Since all other production comprises entirely of own value added, the remaining production functions are of the form:

$$Q_l^h = V_l^h \forall (h,l) \neq (C,2) \tag{3.2}$$

Consumption:

We use F_l^{hf} to denote output of (h,l) that is absorbed in country f as final demand. Based on table 2, the consumption aggregators for the three countries are given as follows:

$$\begin{aligned}
F^J &= [(\kappa_1^{JJ})^{\frac{1}{\theta}}(F_1^{JJ})^{\frac{\theta-1}{\theta}} + (\kappa_2^{JJ})^{\frac{1}{\theta}}(F_2^{JJ})^{\frac{\theta-1}{\theta}} + (\kappa_2^{CJ})^{\frac{1}{\theta}}(F_2^{CJ})^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}} && (3.3)\\
F^C &= [(\kappa_1^{CC})^{\frac{1}{\theta}}(F_1^{CC})^{\frac{\theta-1}{\theta}} + (\kappa_2^{CC})^{\frac{1}{\theta}}(F_2^{CC})^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}} && (3.4)\\
F^U &= [(\kappa_1^{UU})^{\frac{1}{\theta}}(F_1^{UU})^{\frac{\theta-1}{\theta}} + (\kappa_2^{UU})^{\frac{1}{\theta}}(F_2^{UU})^{\frac{\theta-1}{\theta}} + (\kappa_2^{CU})^{\frac{1}{\theta}}(F_2^{CU})^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}} && (3.5)
\end{aligned}$$

We use large case P to denote CPI, small case p^Q is used to denote price of gross output; p^V will be used to denote price of value added and p^X for price of intermediate goods. θ is the elasticity of substitution between different goods in the final consumption good bundle and is assumed to be the same across all countries[8]. κ s are weights that denote the shares of the different components in the aggregators.

Appendix A shows how the linearized version of the model can be solved to write the demand for value added for each entity as a function of all the value added prices assuming that total final demand remains constant (i.e $\hat{F}^h = 0 \forall h$), or what we define as the real exchange rate (GVC-REER) following this partial equilibrium literature[9]:

$$\begin{aligned}
\triangle GVC - REER_s^h &= \hat{V}_s^h = w(v)_{s1}^{hJ}\hat{p}(V)_1^J + w(v)_{s2}^{hJ}\hat{p}(V)_2^J + w(v)_{s1}^{hC}\hat{p}(V)_1^C \\
&\quad + w(v)_{s2}^{hC}\hat{p}(V)_2^C + w(v)_{s1}^{hU}\hat{p}(V)_1^U + w(v)_{s2}^{hU}\hat{p}(V)_2^U && (3.6)
\end{aligned}$$

where $h \in \{J, C, U\}, s \in \{1, 2\}$

To illustrate the properties of our REER weighting scheme, we focus on the weight assignment between the two sectors that are involved in the the GVC. We use the notation $p(V)_l^a V_l^{ab}$ to denote the total value added created in (a,l) that is ultimately absorbed as country b's final demand. Our

8 This and many other restrictions imposed so far will be relaxed in the general model presented later.

9 In addition to the first order conditions, the only equilibrium conditions we need are market clearing conditions. These are listed in the appendix.

model yields the following expression for the weight assigned by C2 o J1:

$$\begin{aligned} w(v)_{21}^{CJ} &= \theta\left[\left(\frac{p(V)_2^C V_2^{CJ}}{p(V)_2^C V_2^C}\right)\left(\frac{p(V)_1^J V_1^{JJ}}{P^J F^J}\right)+\left(\frac{p(V)_2^C V_2^{CC}}{p(V)_2^C V_2^C}\right)\left(\frac{p(V)_1^J V_1^{JC}}{P^C F^C}\right)+\left(\frac{p(V)_2^C V_2^{CU}}{p(V)_2^C V_2^C}\right)\left(\frac{p(V)_1^J V_1^{JU}}{P^U F^U}\right)\right] \\ &+\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)(\sigma-\theta) \end{aligned} \tag{3.7}$$

Several aspects of the weighting schemes now become evident from (3.7). Firstly, under the constant elasticity assumption ($\theta = \sigma = 1$)the weight reduces to:

$$w(v)_{21}^{CJ} = \sum_{k=J,C,U}\left[\frac{\left(p(V)_2^C V_2^{Ck}\right)\left(p(V)_1^J V_1^{Jk}\right)}{\left(p(V)_2^C V_2^C\right)\left(P^K F^k\right)}\right] \geq 0 \tag{3.8}$$

This is the same as the sum of the three terms inside the bracket in (3.7)[10]. Each of these terms can be interpreted as capturing the intensity of competition between $(C,2)$ and $(J,1)$ in the three final goods markets, namely C, J and U. The higher is the intensity of competition as measured by these terms, the higher would be the weight assigned by $(C,2)$ to $(J,1)$, since in that case a fall in the price of $(J,1)$'s value added would hurt $(C,2)$ more. As an example, if the last term in (3.7) $\left[\left(\frac{p(V)_2^C V_2^{CU}}{p(V)_2^C V_2^C}\right)\left(\frac{p(V)_1^J V_1^{JU}}{P^U F^U}\right)\right]$ is high, it conveys that value added by $(C,2)$ and $(J,1)$ are competing intensively in order to satisfy country U's final demand, so a fall in $(J,1)$'s price will hurt $(C,2)$ to a greater extent.

Secondly, note that $w(v)_{21}^{CJ}$ is strictly increasing in σ. Intuitively, the lower is the elasticity of substitution between own value added and Japanese imports in $(C,2)$'s production, the higher will be the co-movement between the value added by $(C,2)$ and $(J,1)$. In this case, the weight assigned by $(C,2)$ to $(J,1)$ will be lower, since a fall in the price of $(J,1)$'s value added, which increases demand for output of $(J,1)$, will also end up exerting a positive effect on the demand for value added by $(C,2)$.

Thirdly, unlike σ, the effect of an increase in θ is ambiguous. On the one hand an increase in θ exerts a positive effect on $w(v)_{21}^{CJ}$ via the standard expenditure switching effect–if consumers are more willing to substitute between different goods in their consumption bundle, a fall in the price of a substitute will decrease demand for the own good to a greater extent. On the other hand, to the extent that the value added by $(C,2)$ and $(J,1)$ are complementary, expenditure switching towards $(J,1)$'s value added indirectly also implies a shift in expenditure towards $(C,2)$'s output

[10] In fact, under the assumption of constant elasticity, all weights can be represented by a formula mimicking (3.8), i. e.

$$w(v)_{ls}^{hk} \;=\; \sum_{k=J,C,U}\left[\frac{\left(p(V)_l^h V_l^{hk}\right)\left(p(V)_s^c V_s^{ck}\right)}{\left(p(V)_l^h V_l^h\right)\left(P^K F^k\right)}\right] \forall h,k,s,l$$

This is the same as 5.2 in the case of the general model.

and hence towards $(C, 2)$'s value added which is embodied in its own output. In this example the two effects run in opposite directions.

Lastly, (3.7) and (3.8)also illustrate the restrictive nature of the weighting scheme in the constant elasticity case. In particular the complementarity effect discussed above is not present, and as is evident from (3.8) the weighting scheme is not flexible enough to accommodate negative weights.

The appendix shows how we can derive REERs for gross output competitiveness (as opposed to competitiveness for value added which is what GVC-REER measures) and how it compares to our GVC-REER weighting scheme in this 3 by 2 world.

Defining Aggregate Real Effective Exchange Rates for Countries

We compute aggregate country level REERs by exploiting information on sector level trade flows. For comparison, in the appendix we also compute an alternate measure that aggregates all trade flows within a country as is commonly done in the literature. Appendix A shows that the demand for value added produced by country h can be written as:

$$\hat{V}^h = \sum_{f \in (J,C,U)} \sum_{k \in \{1,2\}} \left[\left(\frac{p(V)_1^h V_1^h}{p(V)^h V^h} \right) w(v)_{1k}^{hf} + \left(\frac{p(V)_2^h V_2^h}{p(V)^h V^h} \right) w(v)_{2k}^{hf} \right] \hat{p}(V)_k^f \tag{3.9}$$

If sector level price indices are available, then (3.9) provides the most accurate measure of competitiveness for a country as a whole. Typically however, country level REER is computed using an aggregate country-wide price index (like the GDP deflator, CPI or some measure of unit labor cost). To define an analogous measure in our framework we need to make an assumption regarding the link between sector level prices and the aggregate GDP deflator of a country. We choose to make the simplest possible assumption (which is also made implicitly throughout this literature), namely that all sector level prices change by the same proportion as the change in the aggregate GDP deflator. In particular,

$$\hat{p}(V)^h = \hat{p}(V)_1^h = \hat{p}(V)_2^h \tag{3.10}$$

With this assumption we can simplify (3.9) to write:

$$\hat{V}^h = \sum_{f \in (J,C,U)} \underbrace{\left[\left(\frac{p(V)_1^h V_1^h}{p(V)^h V^h} \right) \left(w(v)_{11}^{hf} + w(v)_{12}^{hf} \right) + \left(\frac{p(V)_2^h V_2^h}{p(V)^h V^h} \right) \left(w(v)_{21}^{hf} + w(v)_{21}^{hf} \right) \right]}_{w_A(v)^{hf}} \hat{p}(V)^f \tag{3.11}$$

Here $w_A(v)^{hf}$ denotes the aggregate (i.e country level) weight assigned by country h to country f in the real exchange rate of country h. As is evident from (3.11), this weight is itself a weighted sum of the weights assigned by each sector in h to each sector in f, with the weights given by the value added shares of the sectors in h.

In the appendix we compare our country level REER measure against the common approach of aggregating across sectors within each country and show that these two approaches yield different results except in knife-edge cases.[11]

4 The General Model

Consider a world economy comprising of n countries. There are m sectors within each country. Each country-sector is called an "entity" so that there are a total of nm production entities in the world economy. Each entity uses a production function with its own value added and a composite intermediate input which can contain intermediate inputs from all mn entities including itself. The output of each entity can be used either as a final good (consumed in any of the n countries) or as an input by another entity. Thus there are a total of nm producers and $nm + n$ consumers (nm entities plus n final goods consumers) in the economy. Both the production function and final goods consumption aggregators are nested CES (constant elasticity of substitution) aggregators which are described in detail next.

4.1 Production

Consider the production process for entity (c, l). The production process is assumed to follow the following three stage hierarchy:

First, for each sector, inputs from all foreign countries from that sector are aggregated (with a constant elasticity of substitution) to form sectoral intermediate inputs $\{X(f)^c_{sl}\}^m_{s=1}$. In other words, $X(f)^c_{sl}$ is the aggregate sector s foreign intermediate input used in production by country c sector l

$$X(f)^c_{sl} = \left[\sum_{i=1, i\neq c}^{n} (w^{ic}_{sl})^{1/\sigma^1_s(c,l)} (X^{ic}_{sl})^{\frac{\sigma^1_s(c,l)-1}{\sigma^1_s(c,l)}}\right]^{\frac{\sigma^1_s(c,l)}{\sigma^1_s(c,l)-1}}, s = 1, 2, ..m \tag{4.1}$$

Here X^{ic}_{sl} denotes inputs from country i sector s used in production by country c sector l, the w's are aggregation weights and $\sigma^1_s(c, l)$ is the (constant) elasticity of substitution between different foreign varieties of the sector s output in the production function of entity (c, l)

The sector s import bundle is then combined with the domestic sector s input to form the aggregate sector s input. The elasticity of substitution between these two inputs is $\sigma^{1h}_s(c, l)$.

[11]The issue of aggregation is explored in more detail after we present the general model.

$$X_{sl}^c = \left[(w_{sl}^{cc})^{1/\sigma_s^{1h}(c,l)} (X_{sl}^{cc})^{\frac{\sigma_s^{1h}(c,l)-1}{\sigma_s^{1h}(c,l)}} + (w(f)_{sl}^c)^{1/\sigma_s^{1h}(c,l)} (X(f)_{sl}^c)^{\frac{\sigma_s^{1h}(c,l)-1}{\sigma_s^{1h}(c,l)}} \right]^{\frac{\sigma_s^{1h}(c,l)}{\sigma_s^{1h}(c,l)-1}} \tag{4.2}$$

With this two step framework we are allowing for a distinction between "macro" ($\sigma_s^{1h}(c,l)$) and "micro" ($\sigma_s^1(c,l)$) elasticities for each sector, which is a feature of the data documented in the literature–see Feenstra et al. (2010) .

Next, these m sectoral aggregates are combined to form the aggregate intermediate input X_l^c

$$X_l^c = \left[\sum_{s=1}^{m} (w_{sl}^c)^{1/\sigma^2(c,l)} (X_{sl}^c)^{\frac{\sigma^2(c,l)-1}{\sigma^2(c,l)}} \right]^{\frac{\sigma^2(c,l)}{\sigma^2(c,l)-1}} \tag{4.3}$$

Finally, the aggregate intermediate input is combined with the entity's own value added to form the gross output for entity (c,l) which is used both as intermediate and final good. This is denoted by Q_l^c. The elasticity of substitution between value added and aggregate intermediate is $\sigma^3(c,l)$

$$Q_{.l}^c = \left[(w_l^{vc})^{1/\sigma^3(c,l)} (V_l^c)^{\frac{\sigma^3(c,l)-1}{\sigma^3(c,l)}} + (w_l^{Xc})^{1/\sigma^3(c,l)} (X_l^c)^{\frac{\sigma^3(c,l)-1}{\sigma^3(c,l)}} \right]^{\frac{\sigma^3(c,l)}{\sigma^3(c,l)-1}} \tag{4.4}$$

4.2 Preferences

A country specific final good is obtained by aggregating goods from all nm production entities in two stages.

Firstly, for each sector s, goods from all foreign countries are aggregated to form an aggregate sector s final imported good for consuming country c. The elasticity of substitution for each aggregate is $\theta_s^1(c)$

$$F_s^c(f) = \left[\sum_{i=1, i\neq c}^{n} (\kappa_s^{ic})^{1/\theta_s^1(c)} (F_s^{ic})^{\frac{\theta_s^1(c)-1}{\theta_s^1(c)}} \right]^{\frac{\theta_s^1(c)}{\theta_s^1(c)-1}} \tag{4.5}$$

Each of these imported goods is then combined with the domestic goods from its respective sector to form an aggregate sector s consumption good for country c.

$$F_s{}^c = \left[(\kappa_s^{cc})^{1/\theta_s^{1h}(c)} (F_s^{cc})^{\frac{\theta_s^1(c)-1}{\theta_s^1(c)}} + (\kappa(f)_s^c)^{1/\theta_s^{1h}(c)} (F(f)_s^c)^{\frac{\theta_s^{1h}(c)-1}{\theta_s^{1h}(c)}} \right]^{\frac{\theta_s^{1h}(c)}{\theta_s^{1h}(c)-1}} \tag{4.6}$$

Finally, these s sectoral aggregates are combined (with constant elasticity $\theta^2(c)$) to form the aggregate consumption good for country c

$$F^c = \left[\sum_{s=1}^{m} (\kappa_s^c)^{1/\rho^2(c)} (F_s^c)^{\frac{\theta^2(c)-1}{\theta^2(c)}} \right]^{\frac{\theta^2(c)}{\theta^2(c)-1}} \tag{4.7}$$

4.3 Market clearing:

Gross output from an entity is absorbed either as an intermediate input or a final good (we do not allow for inventory accumulation or any inter-temporal effects). Thus the following market clearing condition holds $\forall(c,l)$

$$Q_l^c = \sum_{i=1}^{n} F_l^{ci} + \sum_{j=1}^{m} \sum_{k=1}^{n} X_{lj}^{ck} \tag{4.8}$$

5 Computation of Effective Exchange Rate Weighting Matrices

In order to define the exchange rates we take prices and final demands in all countries as exogenous and compute the demand for value added and gross output of different entities as functions of prices. This partial equilibrium setup is common in the literature and requires only one market clearing condition along with the different optimality conditions for production and consumption.

5.1 Demand for value added as a function of price of value added:(GVC-REER)

The appendix shows that the demand for value added can be written as

$$vec\left(\hat{V}_l^c\right) = W_V vec\left(\hat{p}(V)_l^c\right) + W_{FV} vec\left(\hat{F}^c\right) \tag{5.1}$$

Here $\left(vec\left(\hat{V}_l^c\right)\right)_{nmX1}$ is the vector of changes in value added stacked across all countries and sectors, and W_V and W_F are nm by nm matrices derived in the appendix. Putting the change in final demand $vec\left(\hat{F}^c\right)$ to zero, the nm by nm matrix premultiplying $vec\left(\hat{p}(V)_l^c\right)$ can be interpreted as a matrix of weights for the real effective exchange rate, as it measures how the demand for value added originating in a country-sector changes when price of value added changes in any other entity.

Interpretation in the case with constant elasticity:

Appendix D shows that under the constant elasticity assumption the weight assignment by country sector (h,l) to country-sector (c,s) where $(h,l) \neq (c,s)$ can be written as follows:

$$w_{ls}^{hc} = \sum_{k=1}^{n} \left[\frac{\left(p(V)_l^h V_l^{hk}\right)\left(p(V)_s^c V_s^{ck}\right)}{\left(p(V)_l^h V_l^h\right)\left(P^K F^k\right)} \right], (h,l) \neq (c,s) \tag{5.2}$$

where we use lower case w to denote constant elasticity weights. This is a generalized form of equation 3.8 which was derived in the context of a simplified model and the intuition is similar. In particular, the weight assigned by country sector (h,l) to country-sector (c,s) where $(h,l) \neq (c,s)$ is a weighted sum of the value added created by country-sector (c,s) and absorbed by each of the countries $k(=1,..,n)$, where the weights are given by the value added created by (h,l) that is absorbed in the same country k. This captures both mutual and third country competition, because the weight is high if both $\left(p(V)_l^h V_l^{hk}\right)$ and $\left(p(V)_s^c V_s^{ck}\right)$ are high, which happens when both (h,l) and (c,s) have a high share of value added exports to country k.

Relaxing the Uniform Elasticity Assumption

Since a full analytical characterization of the role played by different elasticities is infeasible given the complex nature of the model, we will illustrate the role played by the different elasticities in a series of examples in section 8. However, in order to provide some intuition the following proposition shows the effect of a small change in elasticity on the REER weights in the neighborhood of the constant elasticity equilibrium.

Proposition 5.1. *Suppose all production and consumption elasticities are constant and equal to σ and θ respectively*[12]*. Then starting at the uniform elasticity equilibrium, the effect of a change in elasticity on the weight assigned by entity (h,l) to entity (c,s) is given by:*

$$\frac{\partial w_{ls}^{hc}}{\partial \theta} = w_{ls}^{hc} - \frac{v_l^h v_s^c \sum_{c_1=1}^{n} \sum_{c_2=1}^{n} \sum_{k=1}^{m} b_{lk}^{hc_1} b_{sk}^{cc_1} (p(Q)_k^{c_1} F_k^{c_1 c_2})}{p(V)_l^h V_l^h}, (h,l) \neq (c,s) \tag{5.3}$$

Proof(sketch): See appendix D.

Here b is used to denote elements of the global Leontief inverse matrix and $p(Q)$ is used to denote price of gross output.(5.3) shows that an increase in elasticity of substitution of consumption holding everything else constant(including the production elasticity) has two opposing effects on the weight assigned by home entity (h,l) to the foreign entity (c,s). The two terms correspond to the expenditure switching and complementarity effect illustrated earlier with the stylized model. In particular, the first effect (expenditure switching) is positive and is given by the constant elasticity weight w_{ls}^{hc}, which, it should be recalled, is always positive in the constant elasticity case. In addition, there is the countervailing complementarity effect which comes from the second term on the right hand side. This term is high when the products $b_{lk}^{hc_1} b_{sk}^{cc_1}$ are high for various entities

[12] This is equivalent to assumption (A2) in section 6

indexed by (c_1, k), which in turn happens if the outputs of the two entities are used together in production (i, e entities such as $(c_{1,}k)$ which use the output of (c, s) as an input, also uses the output of (h, l) as an input).

Intuitively, when the price of (c, s) decreases, its quantity demanded increases. This effect is greater the greater is the elasticity of substitution between goods(θ). Moreover, an increase in demand for (c, s) will end up increasing the output of (h, l) if it is highly complementary with (c, s).

The corresponding expression for the two GVC sectors in section 3 is the following:

$$\frac{\partial w_{21}^{CJ}}{\partial \theta}|_{\theta=\sigma=1} = w_{21}^{CJ} - \left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right) \tag{5.4}$$

We will elaborate more on these mechanisms by the use of stylized examples.

5.2 Gross Output Competitiveness

We also derive the demand for aggregate output as a function of price of value added (this is analogous to the "goods" REER measure proposed in Bayoumi et al. (2013). See appendix for steps of proof)

$$vec\left(\hat{Q}_l^c\right) = W_Q vec\left(p(\hat{V})_l^c\right) + W_{FQ} vec\left(\hat{F}^c\right) \tag{5.5}$$

Here W_Q is an nm by nm weighting matrix derived in appendix C. Again putting the change in final demand $vec\left(\hat{F}^c\right)$ to be zero, the nm by nm matrix premultiplying $vec\left(\hat{p}_l^{vc}\right)$ can be interpreted as a matrix of weights for the real effective exchange rate with regard to gross competitiveness, i. e it measures how the demand for output of a country-sector changes with changes in prices of other country-sectors. This is in contrast to the first measure defined above, which looks at change in demand for value added. (As is shown in 6.1, the two are the same in the special case where gross output is the same as value added, as is assumed in most of the REER measures including IMF, FED and BIS).

Bayoumi et al. (2013) define the "Goods REER" as a measure of competitiveness of a country's gross output.

$$\triangle log(GoodsREER^i) = \sum_{j\neq i} W_{imf}^{ij}\left(\hat{p^i} - \hat{p^j}\right) \tag{5.6}$$

where W_{imf}^{ij} are the IMF weights. The analogous expression in the context of the framework proposed in this paper (and BJ) can be obtained by setting $m = 1$, $\hat{F}^c = 0 \forall c$ and taking the i^{th} row of the following expression (5.5)

However in general (5.5) and(5.6) are not the same since, as shown in the next section, the IMF weights coincide with the weights obtained in the present framework only under fairly restrictive conditions.

The idea behind the Integrated real exchange rate (IRER) measure proposed in Thorbecke (2011)[13] is also similar to Bayoumi et al. (2013) with the analogous expression in the present model given by (5.5) . However they too use the IMF weighting scheme, which means that their measure only coincides with the measure proposed in the present paper under fairly restrictive assumptions, which we discuss in detail below.

6 Relationship to other REER Weighting Matrices in the Literature

This section shows the link between the two REER measures proposed in the previous section and some common REER measures in the literature with particular emphasis on whether and under what conditions the different measures in the literature can be recovered from the more general measures proposed here. We start by listing the various REER measures that are compared in this section. With some abuse of terminology, we refer to the weighting matrix by the same name as the name given to the associated REER measure by the authors(irrespective of the price index used).

1. GVC-REER and Q-REER are as defined in the previous section

2. IOREER(BJ): Input-output real effective exchange rates as defined by BJ in their model

3. VAREER(BJ): Value added real effective exchange rates as defined by BJ in their model. It is a special case of IOREER with all elasticities (production and consumption) set equal to each other.

4. GOODS-REER: as defined by Bayoumi et al. (2013) .

5. IRER: Integrated real exchange rate as proposed by Thorbecke (2011), but without lagged dependence

6. IMF-REER

   As shown in Bayoumi et al. (2005) the weight assignment by country i to country j in the IMF's REER measure is given by:

$$W^{ij}_{imf} = (\alpha_m + \alpha_s)W^{ij}_{imfm} + (\alpha_c)W^{ij}_{imfc} + (\alpha_T)W^{ij}_{imfT} \tag{6.1}$$

   where α_m, $\alpha_s, \alpha_c, and \alpha_T$ are shares of manufactures, (non-tourism) services, commodities, and tourism in overall trade.

[13] Thorbecke (2011) defines the IRER measure with lag dependence. However, I ignore this feature and refer to IRER as the Thorbecke (2011) measure without lag dependence in order to make the measure comparable to the other measures discussed here and in the following sections

Assumptions:

(A1)$m = 1$. i. e, each country has only one sector

(A2) Elasticities are the same across consumption and production entities

1. $\sigma_s^1(c,l) = \sigma_1 \forall s,c,l$, $\sigma_s^{1h}(c,l) = \sigma_1 \forall s,c,l$, $\sigma^2(c,l) = \sigma_2 \forall c,l$, $\sigma^3(c,l) = \sigma_3 \forall c,l$
2. $\theta_s^1(c) = \theta_1 \forall s,c$, $\theta_s^{1h}(c) = \theta_1 \forall s,c$, $\theta^2(c) = \theta_2 \forall c$

(A3) All elasticities(in both consumption and production) are the same

- $\sigma_1 = \sigma_2 = \sigma_3 = \theta_1 = \theta_2$=1(wlog)

(A4) No intermediates in production and only final goods are traded.

(A5) All trade flows comprise of trade in manufacturers and non tourism services, i. e$\alpha_c = \alpha_T = 0$

Proposition 6.1.

1. *Under (A1) and (A2):*

   GVC-REER =IOREER

2. *Under (A1) , (A2) and (A3):*

   GVC-REER=IOREER=VAREER

3. *Under (A1), (A2), (A3) and (A4):*

   GVC-REER=Q-REER=IOREER=VAREER

4. *Under (A1), (A2), (A3), (A4) and (A5)*

   IMF-REER = GVC-REER=Q-REER=IOREER=VAREER=GOODSREER=IRER[14]

proof: see appendix.

In general, the GVC-REER measure does not reduce to the common measures currently is use, such as those of the FED (Loretan, 2005), BIS (Klau et al., 2008) or IMF (Bayoumi et al., 2005). Certain parallels can, however, be drawn between the IMF measure and the GVC-REER (and BJ) measure as shown in part 4 of proposition 6.1.

[14] *As mentioned before, note that the IMF uses CPI to compute REER, but in this section we will use IMF-REER to denote total effective exchange rates computed with IMF weights but using the GDP deflator, to make the measure comparable with other measures proposed here and in BJ*

7 Building Country-level REER From Ground Up:

Value added weights at the country level

This section provides a method to aggregate the country-sector level weights derived above and defines country level weights analogous to the ones commonly discussed in the literature. We show that the aggregated weights so derived in general do not correspond to any of the ones proposed in the literature except in knife-edge cases. This is attributable to the fact that our measure exploits inter-sectoral linkages between countries to provide a more comprehensive measure of competitiveness than what can be obtained by using just country level data.

To derive the expression for country-level value added weights, we start with the following decomposition of the nominal GDP of country c into its different sectoral components:

$$p(V)^c V^c = \sum_{l=1}^{m} p(V)_l^c V_l^c \tag{7.1}$$

log linearizing this equation we get:

$$\hat{p}(V)^c + \hat{V}^c = \sum_{l=1}^{m} \left(\frac{p(V)_l^c V_l^c}{p(V)^c V^c} \right) \left[\hat{p}(V)_l^c + \hat{V}_l^c \right] \tag{7.2}$$

Stacking the n equations in (7.2) we can write the system in matrix notation as:

$$vec\left(\hat{p}(V)^c\right)_{nX1} + vec\left(\hat{V}^c\right)_{nX1} = R_V \left[vec\left(\hat{p}(V)_l^c\right)_{nmX1} + vec\left(\hat{V}_l^c\right)_{nmX1} \right] \tag{7.3}$$

where

$$(R_V)_{nXnm} = \begin{pmatrix} S_1^V & 0_m' & .. & 0_m' \\ 0_m' & S_2^V & & : \\ : & & .. & : \\ 0_m' & 0_m' & .. & S_n^V \end{pmatrix} \tag{7.4}$$

and $\left(S_i^V\right)_{1Xm} = \left(\frac{p(V)_1^i V_1^i}{p(V)^i V^i}, \frac{p(V)_2^i V_2^i}{p(V)^i V^i}, .., \frac{p(V)_m^i V_m^i}{p(V)^i V^i} \right)$ and 0_m is an m by 1 matrix of zeros. By definition the change in the GDP deflator is the weighted sum of change in its components and hence (7.3) reduces to

$$vec\left(\hat{V}^c\right)_{nX1} = R_V \left[vec\left(\hat{V}_l^c\right)_{nmX1} \right] \tag{7.5}$$

using (5.1) in (7.5) and imposing $vec\left(\hat{F}^c\right) = 0$ as before we get:

$$vec\left(\hat{V}_l^c\right) = R_V W_V vec\left(\hat{p}(V)_l^c\right) \tag{7.6}$$

Defining the two measures of country level value added exchange rates:

When sector level price indices are available, (7.6) defines the change in the country level GVC-REER, i. e

$$\triangle log(GVC - REER) = W_V(C) vec\left(\hat{p}(V)_l^c\right) \tag{7.7}$$

where the n by nm matrix $W_V = R_V W_V$ is the weighting matrix which can be interpreted as follows: the weight assigned by country i to country j sector l is itself a weighted sum of the weights assigned by each sector of country i to (j, l), with the weights being proportional to the country i sector's share of value added as a fraction of total value added by country i

$$W_{V}{}_l^{ij} = \sum_{s=1}^{m} \left(\frac{p(V)_s^i V_s^i}{p(V)^i V^i}\right) (W_V)_{sl}^{ij} \tag{7.8}$$

Sector level prices are often not available for many countries. In such cases we need a further approximation. In particular, we need to assume a mapping between sector level prices and GDP deflator, i.e between $\hat{p}^{vc}$ and $\{\hat{p}_l^{vc}\}_{l=1}^M$. We make the relatively uninformed assumption that all sectoral level prices change in the same proportion as the aggregate GDP, i. e we make the following assumption[15].

Assumption (AP):

$$p(\hat{V})^j = \hat{p}(V)_l^j \forall l \forall j \tag{7.9}$$

Using this assumption we can define our second measure of country level value added exchange rate, GVC-REER(GDPdef) as follows:

$$\triangle log(GVC - REER(GDPdef)) = W_V(CG) vec\left(\hat{p}(V)^c\right) \tag{7.10}$$

where $W_V(GDPdef) = R_V W_V R_g$ is an n by n matrix of weights and $R_g = I_n \otimes 1_m$

[15]Note that in a world with price rigidity and producer currency pricing this assumption is satisfied automatically.

Link to other measures in the literature:

Our second measure of country level exchange rates which uses only the GDP deflator (GVC-REER(GDPdef)) has an n by n weighting matrix as all other measures in the literature and we can hence make a comparison with them.

Given the country-sector level weights(W_V), the country level weights ($W_V(CG)$) have an intuitive interpretation. The weight assigned by country i to country j is a weighted sum of the weights assigned by each sector of country i to each sector of country j, with the weights being proportional to the home sector's share of value added as a fraction of total home value added.

$$W_V(GDPdef)^{ij} = \sum_{s=1}^{m} \left(\frac{p(V)_s^i V_s^i}{p(V)^i V^i}\right) \left(\sum_{k=1}^{m} (W_V)_{sk}^{ij}\right) \tag{7.11}$$

These country level weights defined here are different from others proposed in the literature in several respects as will be discussed in the following sections. The closest to our measure is the one by Bems and Johnson (2012) who also take into account the input-output linkages in their measure and define weights in terms of value added, but do not exploit sector level linkages across countries. Because of the greater information used in our measure, it is in general different from their VAREER and IOREER, even under the assumption of all elasticities being the same. The following proposition shows that even under the uniform elasticity assumption, GVC-REER and VAREER differ from each other except in special cases. [16]

Condition 7.1

$$v^i \sum_{c=1}^{n} b^{ic} F^{cj} = \sum_{l=1}^{m} v_s^i \sum_{c=1}^{n} \sum_{s=1}^{m} b_{ls}^{ic} F_s^{cj} \forall i, j \tag{7.12}$$

where $v_l^i = \frac{p(V)_l^i V_l^i}{p(Q)_l^i Q_l^i}$ is the value added share for entity (i, l) and b denotes a generic element of the global inter-country Leontief inverse matrix.

Proposition 7.1.

The country level weights ($W_V(GDPdef)$) defined above reduces to VAREER (and IOREER) weights defined in Bems and Johnson (2012) if either of the two conditions below are satisfied.

1. (A2), (A3) and condition 5. 1

[16] We make the comparison with Bems and Johnson (2012) because it is the closest to our framework. Although they do not allow for sector level linkages in their theoretical model, in the empirical implementation of their model they do use sector level linkages from Johnson and Noguera (2012) by making some simplifications. (However their simplification only works in the constant elasticity case).

2. (A3), (A4) and $\theta_1 = \theta_1^h = \theta_2$

The proof is given in appendix D. The first part of the proposition shows that outside of the knife-edge case in which the above condition is satisfied, the GVC-REER(GDPdef) weights which exploit inter-sectoral linkages between countries will dominate the VAREER measure even under the uniform elasticity assumption (they would of course differ if elasticities are different even if the first condition in the proposition is satisfied). Intuitively, condition (7.12) is satisfied if different sectors within a country are "symmetric" with regard to their input-output linkages with the rest of the world, for in that case aggregation across sectors within a country will be a closer approximation to the behavior of each individual sector. The next section will provide an example to illustrate the role played by the condition in aggregating weights at the country level.

The second part of the proposition shows that differences between GVC-REER and VAREER vanish when there is no trade in intermediates. This shows that if there is no trade in intermediates, then aggregating trade flows across sectors within a country does not lead to any loss of information as far as computation of real effective exchange rate is concerned. Intuitively, if all production by all entities involves only own value added and no intermediates, then there is no asymmetry between sectors within a country with regard to the foreign value added embodied in their output (which is zero in all cases) and hence aggregation does not lead to any loss of relevant information.

Gross output exchange rate at country level

Following a similar procedure to the one used for GVC-REER, we can define weights and exchange rates for gross output at the country level:

$$\triangle log(QREER) = W_Q vec\left(\hat{p}(V)^c_l\right) \tag{7.13}$$

$$\triangle log(QREER(GDPdef)) = W_Q(GDPdef) vec\left(\hat{p}(V)^c\right) \tag{7.14}$$

8 Illustrative Examples

This section presents some examples to illustrate the different aspects of the weighting matrices derived above and compare them to other measures proposed in the literature.

Example 8.1. Three country world with limited trade in intermediate inputs:

Consider the following 3 country one sector example where the input output linkages are restricted to just one non-zero entry. Country C imports intermediates from country J, puts in own value added and sells the output to all the three countries as final output. Table 3 displays the associated input-output table.

Table 3 – *Input output table for 8.1*

	J	C	U	J final	C final	U final	Total output
J	0	1	0	1	0	0	2
C	0	0	0	0.1	0.1	1	1. 2
U	0	0	0	0	0	1	1
Value added	2	0.2	1				
Total output	2	1. 2	1				

In this simplified example only two elasticities are relevant, namely σ_3 (elasticity of substitution between C's value added and intermediate input from J in C's gross output) and θ_1 (elasticity of substitution between final goods in the final consumption basket of all countries. For simplicity, this elasticity is assumed to be common across countries).

Consider the weight assigned by country C to country J, W_{CJ}, which measures the change in demand for value added by C when price of value added by J changes. A decrease in $p(V)^J$ affects the demand for C's value added via two channels. Firstly, with regard to final goods consumption, a decrease in $p(V)^J$ leads to a shift towards J's value added (and goods containing value added by J, namely the gross output of C) in the final goods consumption bundle of all countries. The strength of this effect depends on θ_1. A higher θ_1 means that goods are more substitutable in the final goods consumption bundle of countries and hence the shift towards J's value added will be more pronounced when its price decreases.

Secondly, with regard to intermediate goods and production mix, a decrease in the price of J's value added leads to a shift towards J's value added and a shift away from C's value added in the production function of C. The strength of this effect depends on σ_3. The higher is this elasticity, the higher is the shift towards J's value added in C's production (at the expense of C's own value added) and hence higher is the fall in demand for C's value added.

As a result of these two effects W_{CJ} is an increasing function of σ_3 and a decreasing function of θ_1, as was pointed out in proposition (5.1). Interestingly, when θ_1 is sufficiently high and σ_3 is sufficiently low, W_{CJ} may indeed be negative, something that the IMF weights or the value added weights in BJ do not allow for.

Table 4 presents weights based on different schemes for this example when $\sigma_3 = 1.5$, $\theta_1 = 5$. (as is done by the IMF and others, weights are normalized so that own weight is -1 and is not reported). Several aspects of the differences in the weighting schemes are noteworthy. Firstly, note that there are no negative weights in the IMF and the VAREER weighting matrix. In fact it can be easily shown that these weighting schemes are not flexible enough to accommodate negative weights under any circumstances. Next, note from column 1 that W_{JC} and W_{CJ} are negative in the GVC-REER measure. As discussed above, this is a consequence of the input output structure and a combination of a relatively high θ_1 (=5) and low σ_3 (=1. 5). Column 3 illustrates that as

Table 4 – Comparison of weights under different measures for example 8.1

	GVC-REER (PWW)	VAREER (BJ)	Q-REER (PWW)	GOODS-REER (BST)	IMF Weights (BLS)
W_{JC}	**-0.04**	**0.19**	**-0.04**	**1.0**	**1.00**
W_{JU}	1. 04	0.80	1. 04	0	0
W_{CJ}	**-0.25**	**0.54**	**-4.07**	**-3.40**	**0.26**
W_{CU}	1.25	0.45	5.07	4.40	0.73
W_{UJ}	0.83	0.83	0.83	0.83	0
W_{UC}	0.16	0.16	0.16	0.16	1

key	
PWW	Patel, Wang, Wei(2014)
BJ	Bems and Johnson (2012)
BST	Bayoumi et al. (2013)
BLS	Bayoumi et al. (2005)

far as gross output is concerned, the magnitude of the negative weight assigned by country C to country J is much larger. This is because only the first effect discussed above (i. e shift in final demand) affects gross output, whereas the second effect (shift towards intermediate composition) does not affect the gross output measure.

The uniform elasticity assumption is overly restrictive can also be noted from the observation that the VAREER(BJ) weight which does take into account trade in intermediates, does worse than the IMF weight which ignores it, although both have the wrong sign.

Column 4 shows that the Goods-REER measure of Bayoumi et al. (2013) falls somewhere in between the GVC-REER and the Q-REER measures (columns 1 and 3) so that it measures neither gross output competitiveness nor value added competitiveness but some arbitrary combination of the two. Although the aim in Bayoumi et al. (2013) is to capture gross competitiveness, they fall short of doing so because their measure uses the IMF weighting scheme which does not account for trade in intermediates. This aspect is further illustrated by the fact that the GOODS-REER measure(which in turn inherits this property from the IMF measure) assigns a value of 0 to W_{JU} because there is no direct trade between J and U. However, J's value added does reach U via C and so the correct weighting matrix must have $W_{JU} \neq 0$.

Lastly, note from the last two rows of table 4 that the weights assigned by country U to the remaining two countries are the same in all the measures except IMF. This is a consequence of the fact that the US trades in only final goods and all its production comprises entirely of its own value added.

Figure 7 in the appendix shows how the weight assigned by C to J changes with the elasticities. The top left figure plots W_{CJ} for three measure(GVC-REER, VAREER(BJ) and IMF) for different values of σ_3 with θ_1 fixed at 1. 5. The top right picture plots the same weights for different values of θ_1 with σ_3 fixed at 5. The bottom left figure shows a 3D plot of W_{CJ} for the GVC-REER measure

Table 5 – IO table and elasticities for example 8.2

		J1	J2	C1	C2	U1	U2	JFinal	CFinal	Ufinal	total output
		J		C		U					
J	J1	0	0	0	2	0	0	1	0	0	3
	J2	0	0	0	0	0	0	1	0	0	1
C	C1	0	0	0	0	0	0	0	2	0	2
	C2	0	0	0	0	0	0	0.5	0.5	2.5	3.5
U	U1	0	0	0	0	0	0	0	0	2	2
	U2	0	0	0	0	0	0	0	0	1	1
VA		3	1	2	1.5	2	1				
total output		3	1	2	3.5	2	1				

Elasticities:
$\sigma_1 = 2$, $\sigma_2 = 2$, $\sigma_3 = 2$, $\theta_1 = 5$, $\theta_2 = 5$

for different values of σ_3 and θ_1 while the bottom right augments this graph by adding a surface each for VAREER and IMF weights.

Example 8.2. A three country 2 sector world:

This example is an extension of example 8.1 which will be used to illustrate the role of aggregation and comment on the practice of normalization of weights. In each country we now have two distinct production sectors. The main object of attention will be country C which is assumed to have two sectors that are different with regard to their production function. Sector 1(C1) uses its own value added and produces only final goods absorbed at home. Sector 2 (C2) operates downstream and uses intermediates from a different country and produces only a final good. The elasticities and input-output table is given in table 5 below.

Table 5 displays the full 6X6 country-sector level weighting matrix (which is not normalized to illustrate the mechanics of aggregation based on equation (7.11) later on). In line with the observations made in example 8.1, C2 and J1 are found to attach negative weights to each other. Table 7 shows the weights at the country level. As can be seen from the country level weighting matrix, the negative weights disappear when sectors are aggregated by country. To understand the intuition behind this, table 8 shows a 2 by 2 sub matrix from table 7 which contains weights assigned by sectors in C and J to each other, along with the value added and gross output shares of the respective sectors, derived from table 5.

Note that the aggregate country weight W_{CJ} is a combination of the 4 sector level weights (in line with (7.11)). Since in value added terms the size of C_1 and J_1 is higher compared to C_2 and J_2 respectively, the country level GVC-REER weight is likely to be dominated by $W_{J_1C_1}$, which is positive (0.57)[17]

[17]the number 0.3 can be recovered from (7.11) as follows
$W^{CJ} = 0.57(W^{CJ}_{11} + W^{CJ}_{12}) + 0.43(W^{CJ}_{21} + W^{CJ}_{22}){=}0.57(0.571 + 0) + 0.43(-0.338 + 0.28) \simeq 0.3$

Table 6 – *GVC-REER weights at country-sector level(raw) for example8.2*

	J_1	J_2	C_1	C_2	U_1	U_2
J_1	-2.36	0.86	0.38	-0.169	0.86	0.43
J_2	2.57	-3	0	0.43	0	0
C_1	0.571	0	-1	0.4	0	0
C_2	-0.338	0.28	0.571	-2.4	1. 3	0.65
U_1	1.30	0	0	0.97	-3.18	0.90
U_2	1.30	0	0	0.97	1. 81	-4.0

Table 7 – *GVC-REER weights at country level for example8.2*

	J	C	U
J	-1.24	0.26	0.97
C	0.30	-1.14	0.83
U	1.30	0.97	-2.27

Table 9 provides a summary of country-level weights assigned by different measures in the literature alongside the weighting matrices proposed in the preceding sections (now the weights are normalized to make the comparison easier). Unlike the case with GVC-REER weights, note that Q-REER, which focuses on gross output, does end up assigning negative weights (see row 3 containing W_{CJ}) even at the country level, the intuition for which is again clear from noting that in terms of gross output shares the dominant sectors are J_1 and C_2 so the country level weight W_{CJ} is likely to be dominated by the weight assigned by C_2 to J_1, which is negative(-0.338). However as in the previous example since the GOODS-REER measure uses IMF weights, it does not completely account for the input-output linkages and hence offers a different weight from our Q-REER measure.

More on aggregation:

To focus exclusively on the role of aggregation we now set all elasticities equal to one. Table 10 shows the country-level input output table derived from the general country- sector level IO table. The country level IO table is what is used to compute weights when inter-sectoral flows are ignored, as is common in the literature. Table 11 gives the weights under the two difference schemes, GVC-REER and the corresponding measure derived from a country level IO table, which we call A-REER (for aggregate). Note that in theory the A-REER measure is equivalent to VAREER in

Table 8 – *2X2 weighting matrix for example 8.2*

	J_1	J_2	value added share	gross output share
C_1	0.571	0	**0.57**	0.36
C_2	-0.338	0.28	0.43	**0.64**
value added share	**0.75**	0.25		
gross output share	**0.75**	0.25		

Table 9 – Summary of different weights at country-country (normalized) example 8.2

	GVC-REER (PWW)	IO-REER (BJ)	VAREER (BJ)	Q-REER (PWW)	GOODS-REER (BST)	IMF Weights (BLS)
W_{JC}	0.21	0.41	0.69	0.21	1	1
W_{JU}	0.78	0.58	0.30	0.79	0	0
W_{CJ}	**0.26**	**0.29**	**0.56**	**-0.80**	**0.25**	**0.53**
W_{CU}	0.73	0.71	0.44	1.80	0.75	0.47
W_{UJ}	0.57	0.36	0.36	0.57	0.36	0
W_{UC}	**0.42**	**0.63**	0.63	-0.42	0.63	1

Table 10 – Country level IO table for example 8.2

table 6. 2. 6	J	C	U	J final	C final	U final	Total output
J	0	2	0	2	0	0	4
C	0	0	0	0.5	2.5	2.5	5.5
U	0	0	0	0	0	3	3
Value added	4	3.5	3				
Total output	4	5.5	3				

Bems and Johnson (2012).

The difference between the 2 weighting matrices can be illustrated using W_{CU} as an example. W_{CU} is an increasing function of value added by C that is ultimately absorbed in country U. By exploiting sector level information the GVC-REER measure recognizes that all the exports from C to U are associated with sector C2, which uses foreign (J) value added (and therefore less of its own value added) and hence it tends to reduce the weight assigned by C to U. The aggregate measure on the other hand looks at aggregate country level data and as a result attributes a higher amount of value added by C in its exports to U (because C2 has a higher fraction of own value added). As a result, W_{CU} is higher under the aggregate measure compared to GVC-REER. Table 12 shows a comparison between normalized and raw weights under the two measures. Note that now the ordering in W_{CU} is reversed and the GVC-REER measure assigned a higher weight than A-REER. This undesirable feature is a consequence of the arbitrariness involved in normalization and highlights its drawbacks.

Table 11 – GVC-REER and VAREER weights (raw, constant elasticity)

	GVC-REERC	A-REER
W_{JC}	0.18	0.27
W_{JU}	0.19	0.12
W_{CJ}	0.20	0.32
W_{CU}	**0.16**	**0.25**
W_{UJ}	0.25	0.16
W_{UC}	0.19	0.28

Table 12 – *Normalization and the role of aggregation*

	Raw Weights		Normalized weights	
	GVC-REER	A-REER(raw)	GVC-REER	A-REER
W_{JC}	0.18	0.27	0.48	0.69
W_{JU}	0.19	0.12	0.51	0.37
W_{CJ}	0.20	0.32	0.55	0.56
W_{CU}	**0.16**	**0.25**	**0.44**	**0.43**
W_{UJ}	0.25	0.16	0.57	0.36
W_{UC}	0.19	0.28	0.43	0.63

9 Data

We use recently released data from the World Input-Output Database(WIOD) which was developed by a consortium of eleven European research institutions with funding from the European Commission. The database consists of a time series of input-output tables covering 40 countries and 35 sectors from 1995-2011[18]. The data is available in both current and previous year prices which enables us to compute price indices for different entries in the input-output table. A detailed description of this database can be found in Timmer and Erumban (2012). As documented by these authors, the database is more precise than previous attempts in the literature (for instance Johnson and Noguera, 2012) as it uses less approximations and more detailed trade data[19].

Estimation of Elasticities

The examples in the previous section have shown that the different elasticities of substitution are key parameters in computing the REER weights. In the next section we take up the task of estimating these elasticities. The availability of input output tables at previous year prices in the World Input Output Database (WIOD) allows us to estimate elasticities used in the computation of weights instead of assuming all elasticities to be unity as is done in the literature. We use the framework pioneered in Feenstra (1994) and subsequently used in Broda and Weinstein (2006) and Soderbery (2013) to estimate the different elasticities used in the CES aggregators for production and consumption which also enter the expression for the real exchange rate. Appendix F provides a brief overview of the framework followed by a discussion of how the estimation is carried out in the context of our model.

[18]The full set of countries and sectors is listed in appendix J

[19]For instance, unlike Johnson and Noguera (2012) who use a proportionality assumption to split intermediate and final goods imports, the WIOD uses detailed data from com-trade to distinguish trade flow into the different categories.

10 Results

10.1 Elasticity Estimation

We generalize the framework used in Feenstra (1994) and subsequently in Broda and Weinstein (2006) and Soderbery (2013) to allow for elasticities less than unity. To minimize the effect of outliers we winsorize the data at the 10th and 90th percentiles. Further, we propose to use the bootstrap median as our point estimate since we find this to be more stable than the MLE in our simulations. Table 18 in the appendix reports the moments of the sample bootstrap distribution obtained using 50 draws. In table 13 we report estimates across different country and sector groups.

10.1.1 Baseline Calibration:

As shown in table 13 we find substantial heterogeneity in the elasticities across country and sector groups. The p-values for the null of equality of medians across samples is often insignificantly different from zero. For our baseline calibration of consumption elasticities we pool across sectors and split countries into two groups, OECD and non-OECD. For production elasticities we pool across countries and split sectors into primary, secondary and tertiary. Thus we use 16 different elasticities in the baseline calibration. These are highlighted in bold in table 13.

Table 13 – *Comparison of median elasticities across different country and sector groups*

	Consumption Elasticities				Production Elasticities			
	θ^1	θ^{1h}	θ^2		σ^1	σ^{1h}	σ^2	σ^3
OECD(28)	**7.75**	**3.38**	**1.35**	primary(2)	**12.25**	**8.19**	**4.76**	**1.015**
Non-OECD(13)	**17.80**	**9.45**	**1.925**	secondary(15)	**5.81**	**8.02**	**4.36**	**1.015**
p(OECD,Non-OECD)	0.00***	0.074*	0.266	tertiary(18)	**9.14**	**7.29**	**3.22**	**1.015**
observations	1435	41	41	p(secondary, tertiary)	0.00***	0.34	0.00***	

Notes: p(a, b) denotes the p-value for the null hypothesis that the medians are constant for a and b using Moods chi squared test. Tests for σ^1 are based only on 2 countries(i. e 2450 (=2*35*35) observations). Agriculture and mining are classified as primary sectors while the rest are split into manufacturing (secondary, 15 sectors) and services (tertiary, 18 sectors).
For p values, "***", "**" and "*" denote significance at 1%, 5% and 10% level respectively

10.2 Multilateral Exchange rates

In this section we illustrate the properties of our different REER indices. Figure 1 illustrates the different kinds of REER indices that we generate using our framework taking the example of one country, the United States. The first plot displays eight different country REER indices that can be constructed using the framework developed in this paper. Four of these (solid lines) are value added exchange rate indices and the other four are indices for gross output. The eight indices can also be seen as two groups of four indices each corresponding to the uniform elasticity elasticity (UE) or our baseline calibration. Here we use normalized versions of the weighting matrices, the rationale for which we will discuss below. All indices are in logs and normalized to zero at the start of the sample period so the value on the y axis can be read as the percentage deviation from the start of the sample. The second plot illustrates how each of the 8 indices in the first plot can be split into 35 sectoral components using the baseline GVC-REER as an example.

For reference, figures 9-14 in appendix L show the 8 indices for all 40 countries in the sample. Several interesting observations are worthy of mention. Firstly, note that there is substantial heterogeneity across the indices which speaks to the importance of incorporating trade in intermediate goods. Secondly, we note that the 8 indices, although different, show high comovement for most countries. The notable exception is China. Here a comparison of the red dotted line and the pink solid line for instance shows that although there was an appreciation in China's value added exchange rate (GVC-REER) in the initial part of the sample, if we impose the constant elasticity assumption and look at China's gross output instead of value added, the conclusion seems to be the opposite. China is a country that shows the most disparity across REER indices and we will discuss and illustrate them in the remainder of this section.

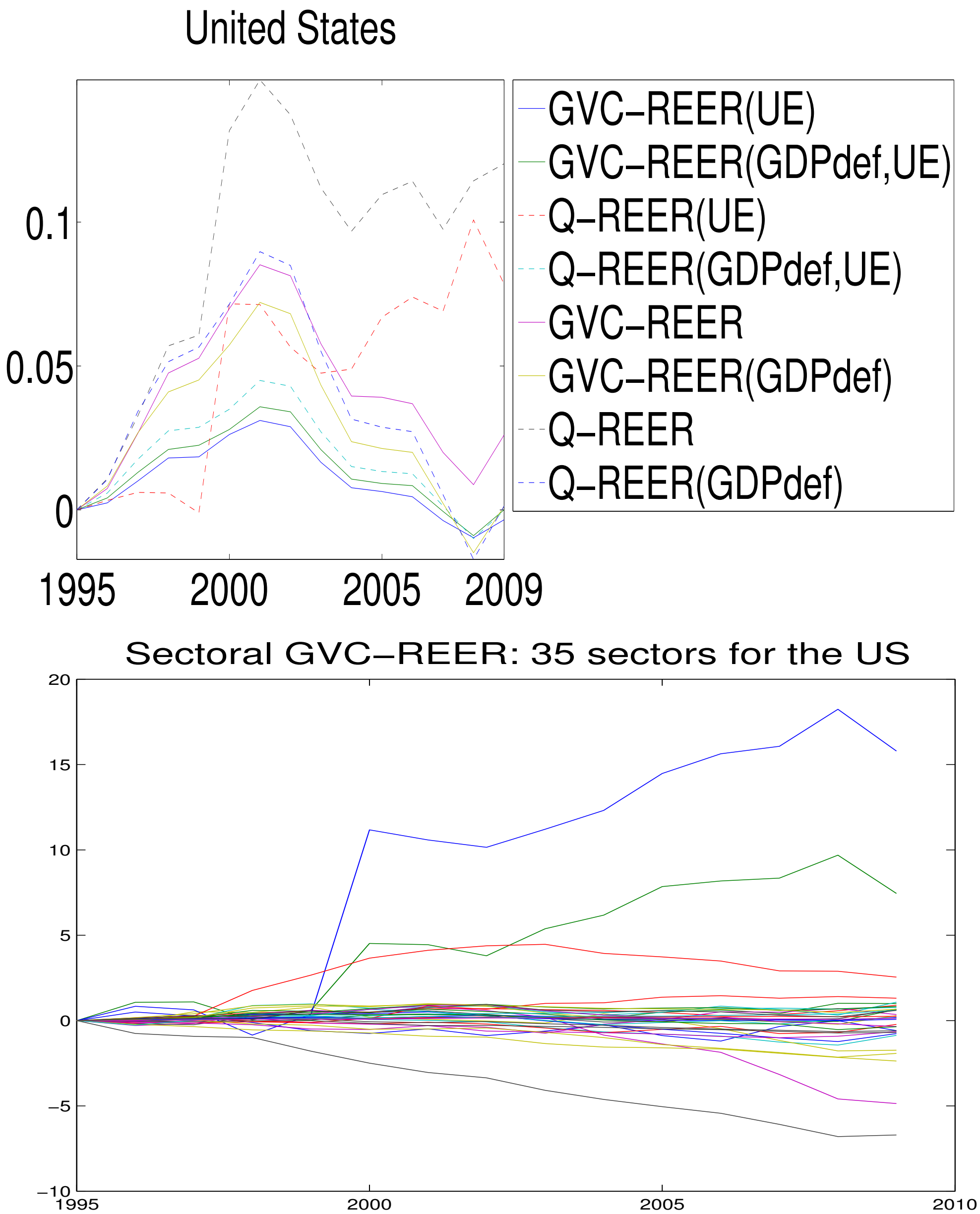
As mentioned before, our framework also allows us to compute exchange rates at the sector level to gauge competitiveness of individual sectors within a country. Figure 2 shows some sector level exchange rates for select countries. As can be seen in the figure, we find evidence of substantial heterogeneity across movements in competitiveness for sectors within countries. For Mexico for instance we find that although the aggregate exchange rate appreciates through the sample period, the REER for the financial intermediation sector indicates depreciation, implying an increase in its competitiveness even as the overall competitiveness of the economy falls.

We next illustrate the role of different aspects of our REER indices in isolation. Figure 3 shows a comparison of uniform elasticity and baseline elasticity GVC-REER indices for select countries. The figure clearly shows the dramatic increase in the volatility of REER when moving from Cobb-Douglas case(constant elasticity) to the case where more realistic elasticities estimated from the data are incorporated [20]. This is the reason we chose to display indices based on normalized weights in figure 1.

Due to the high volatility of the REER with heterogenous elasticities, a mere visual comparison between the two indices is not informative. Moreover, our focus in this paper is not on second moments of REER. Therefore, in order to illustrate the role played by heterogenous elasticities we define a statistic to qualitatively capture the differences in REER based on uniform and heterogenous elasticity. For each entity (e) and for each year, we create a variable d_t^e which takes the value one if

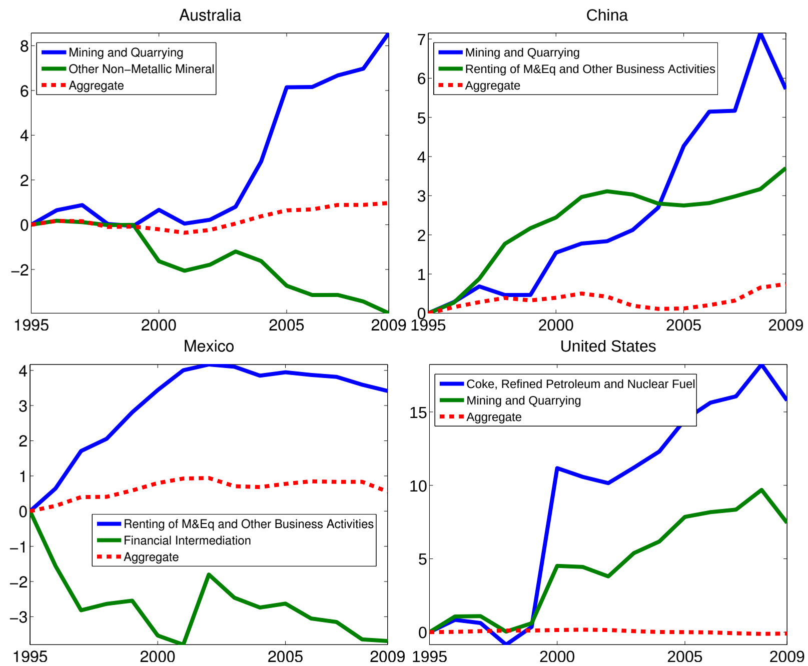
[20]The same pattern holds in a comparison of constant and heterogenous elasticities indices for gross output competitiveness as well (These results are not reported)

Figure 1 *– REER indices for USA*



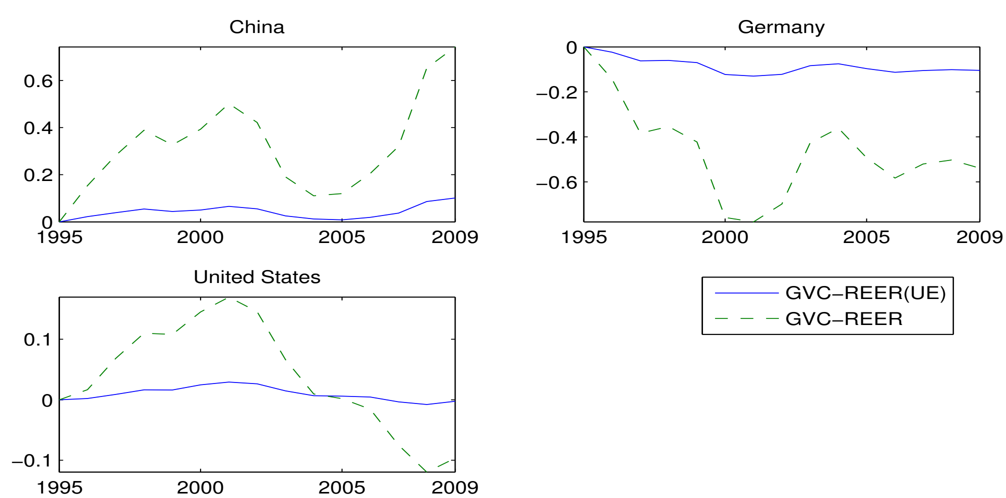
Notes: This figure illustrates all the different REER indices that we compute using our framework taking the case of USA as an example. The first plot shows the 8 indices at the aggregate (country) level and the second plot shows how each of the 8 can be further split into 35 sectoral components. All indices are in logs and normalized to zero at the start of the sample period so the value on the y axis can be read as the percentage deviation from the start of the sample. Further, all own weights are normalized (as done by the IMF) to make a visual comparison feasible. In this figure the IMF convention is adopted so that an increase corresponds to an appreciation.

Figure 2 – *Sector level Exchange rates along with Aggregate country REER for select countries*



Notes: All indices are in logs and normalized to zero at the start of the sample period so the reading on the value on the y axis can be read as the percentage deviation from the start of the sample. In this figure the IMF convention is adopted so that an increase corresponds to an appreciation.

Figure 3 – *The role of heterogenous elasticities*



Notes: All indices are in logs and normalized to zero at the start of the sample period so the reading on the value on the y axis can be read as the percentage deviation from the start of the sample. In this figure the IMF convention is adopted so that an increase corresponds to an appreciation.

Table 14 – Divergence index for Countries

d^e	number of countries
0.21	1
0.14	4
0.07	15
0	21
	total=41

the GVC-REER uniform elasticity and heterogenous elasticity (with baseline calibration) indices move in opposite directions and zero otherwise.

$$d_t^e = 1\left(sign(\triangle GVC - REER(BM)_t) \neq sign(\triangle GVC - REER(CE)_t)\right) \tag{10.1}$$

We then compute the mean of d_t^e for each e across all time periods and to define the "Divergence index" for entity e as follows:

$$d^e = \frac{\sum_{t=2}^{T} d_t^e}{T-1} \tag{10.2}$$

Note that d^e takes the value zero if the two REER measures always agree in their direction of movement and takes the value of 1 if they never agree, i. e always move in opposite directions.

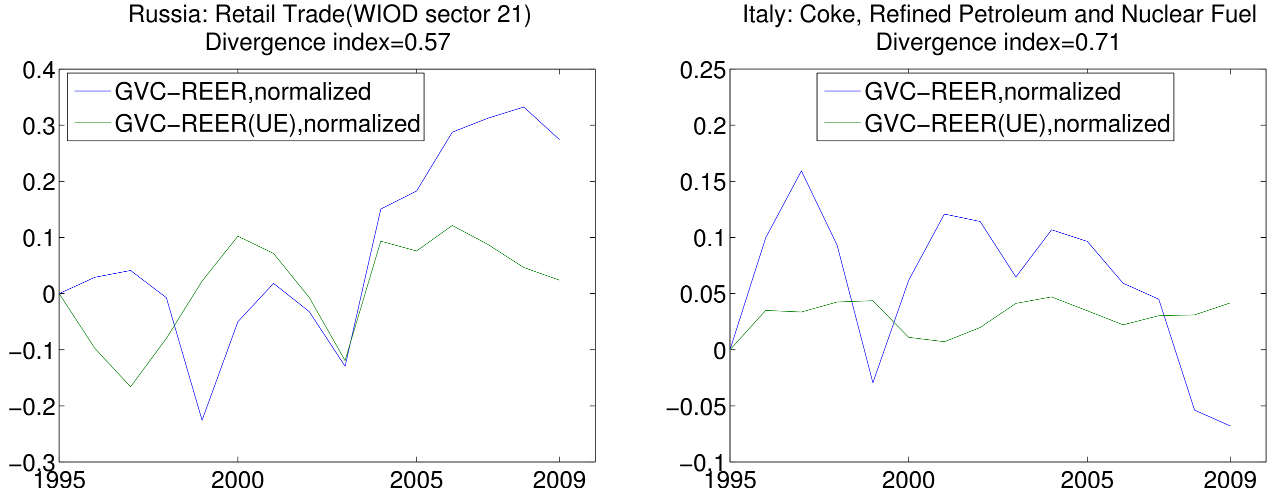
Table 14 summarizes the distribution of the divergence index for country level GVC-REER[21]. A large fraction of countries (21) never see a divergence between the two measures. The maximum number of times the measures move in opposite directions is 3, which is still a small fraction (20%) of the total number of years. This happens for Slovenia. Our main takeaway from these statistics is that incorporation of heterogenous elasticities does not significantly alter the REER indices at the country level, at least qualitatively.

The story however is different when we go to more disaggregated level of sectors within each country. Here we find examples where the two measures disagree on sign in as many as 10 (67%) of the time periods. Figure 8 in the appendix displays a histogram plot of the divergence index for 1435 country-sector pairs, and figure 4 shows some examples where the two measures have the highest disparity.

Next, we consider the role of using sector level price indices in isolation. Figure 5 plots the GVC-REER indices based on the baseline calibration using sector level prices alongside the same indices constructed using an aggregate price index (namely the GDP deflator) for select countries. While there is very little difference between the two indices for some countries (Germany) the divergence is substantial for countries like China and Turkey. The difference is most stark when

[21]The results for other REERs including Q-REER are qualitatively similar

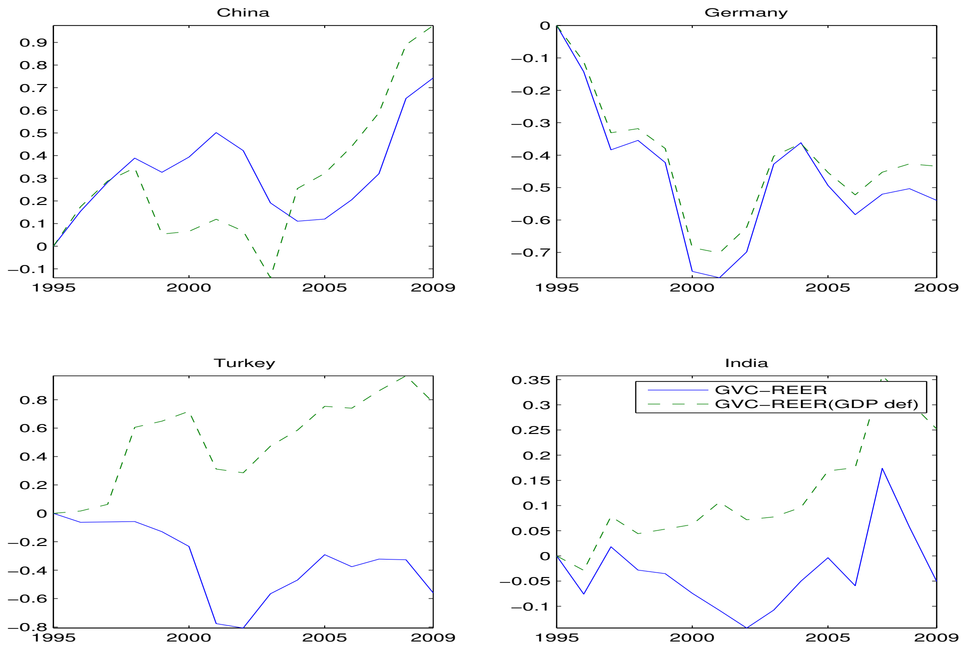
Figure 4 *– Examples with high divergence between uniform elasticity and heterogenous elasticity GVC-REER*



Notes: All indices are in logs and normalized to zero at the start of the sample period so the reading on the value on the y axis can be read as the percentage deviation from the start of the sample. In this figure the IMF convention is adopted so that an increase corresponds to an appreciation.

the indices move in opposite directions (as is the case for these countries at various points in the sample), as it shows that ignoring sector level information can lead to an error in computing not only the magnitude but also the direction of exchange rate movement. For instance, in 2003, while the GDP deflator based REER indicates a depreciation, the more comprehensive REER based on sector level prices actually indicates an appreciation of the Chinese GVC-REER. Similar instances are also observed for other countries, most notably for India and Turkey as shown in figure 5.

Figure 5 – *The role of sector level price indices*



Notes: All indices are in logs and normalized to zero at the start of the sample period so the reading on the value on the y axis can be read as the percentage deviation from the start of the sample. In this figure the IMF convention is adopted so that an increase corresponds to an appreciation.

11 Application: Bilateral Real Exchange Rates:

The bilateral real exchange rate (RER) is commonly used to gauge competitiveness as well as cost of living differentials between countries. In particular, the bilateral real exchange rate between countries h and f is defined as follows:

$$RER^{hf} = \hat{p}(V)^f - \hat{p}(V)^h \tag{11.1}$$

where $\hat{p}^f$ and $\hat{p}^h$ are changes in aggregate (country wide) price indices measured in a common currency.

Based on the the insights gained from the previous sections we argue that if the goal is to measure competitiveness of one country against the other then the standard RER measures computed using an aggregate price index(such as those in Chinn, 2006) like the GDP deflator and ignoring trade in intermediates can be misleading since they ignore sector level linkages between the countries.

Consider a two country world where each country has two sectors. There is no trade in intermediate goods and production comprises entirely of own value added. Table 15 shows how the final demand is distributed across sectors.

Table 15 – *IO table for bilateral RER*

		C		U		CFinal	Ufinal	total output
		C1	C2	U1	U2			
C	C1	0	0	0	0	1	1	2
	C2	0	0	0	0	3	0	3
U	U1	0	0	0	0	0	1	1
	U2	0	0	0	0	0	1	1
VA		2	3	1	1			
total output		2	3	1	1			

Suppose in addition, $\hat{p}(V)_1^C = -0.01, \hat{p}(V)_2^C = 0.02, \hat{p}(V)_1^U = 0, \hat{p}(V)_2^U = 0$ (all prices are in a common currency, so nominal exchange rate is already incorporated)

Based on the conventional RER definition using an aggregate country level price index,

$$\hat{RER}^{US-CH} = \hat{p}(V)^C - \hat{p}(V)^U = 0.008 \tag{11.2}$$

and hence the conventional RER measure would indicate an increase in competitiveness of the US. This however is misleading since the entire price increase comes from China's sector 2 which does not compete with any of the US sectors. Moreover, the Chinese sector which does compete with the US is C1, which actually experiences a decrease in its price, so the correct measure of competitiveness must signal an appreciation of the US exchange rate against China, not a depreciation as measured by the standard RER in 11.1.

Our framework to compute effective exchange rates can be easily modified to adjust these bilateral RERs to better reflect movements in competitiveness. We define our bilateral RER measure, the "GVC-RER" as follows

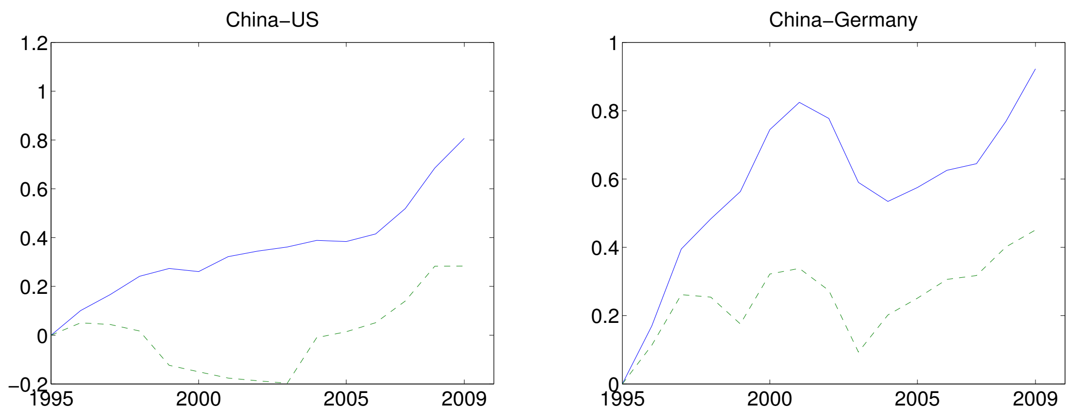
$$GVC - RER^{hf} = \sum_{i=1}^{m} v_i^h \left[\sum_{j=1}^{m} w_{ij}^{hh} \hat{p}(V)_j^h + \sum_{j=1}^{m} w_{ij}^{hf} \hat{p}(V)_j^f \right] \tag{11.3}$$

Here,$v_i^h = \frac{p(V)_i^h V_i^h}{\sum_{j=1}^m p(V)_j^h V_j^h}$ is the share of sector i in country h's total value added, so that $\sum_{i=1}^m v_i^h = 1$. The ws are weights that are analogous to the GVC-REER weights. Based on this measure, we get $GVC - \hat{RER}^{US-CH} = -0.01$. Hence the two measures differ not just in magnitude but also the sign. The conventional measure shows a depreciation whereas the new measure shows an appreciation that is consistent with intuition.

Figure 6 Shows the comparison of the two RER measures for China against two of its major trading partners–The United States and Germany. In computing these indices, the weights are normalized so that the sum of the home country and foreign country weights are equal in magnitude, as is the case with the standard RER measure. Unlike in the GVC-REER effective exchange rate computation, here the normalization of weights cannot be avoided, since otherwise the GVC-RER measure would be dominated by home prices because home sectors (especially the own sector) on average carry much higher GVC-REER weights.

It can be seen that there are substantial differences between the two measures for some country pairs. For China's bilateral exchange rate against the US for instance, whereas the standard RER shows a U shaped pattern, the GVC-REER shows a secular appreciation during the sample period, indicating that price movements during this period have meant that China has lost competitiveness against the US steadily.

Figure 6 – *Comparison of GVC-RER and standard RER bilateral exchange rates for China*



Notes: All indices are in logs and normalized to zero at the start of the sample period so the reading on the value on the y axis can be read as the percentage deviation from the start of the sample. In this figure the IMF convention is adopted so that an increase corresponds to an appreciation.

12 Conclusion

This paper proposes a theoretical framework to compute real effective exchange rates (REER) as a measure of competitiveness by incorporating four features that have been typically overlooked in the literature and that we show are likely to lead to mis-measurement in competitiveness. Firstly, we distinguish between trade by end use category (i. e intermediate vs final). Recognizing that with trade in intermediate inputs, value added and gross output become delinked, we define and compute REER indices to quantify competitiveness both in terms of gross output (Q-REER) and value added (GVC-REER). Secondly, we go beyond aggregate REERs for countries and compute REERs for individual sectors within countries. We are able to do so by exploiting detailed sector level trade flows in the data and by specifying a general multi-country multi-sector model on the theoretical side. Thirdly, we construct sector level price indices and use these in our REER indices instead of relying on the more coarse country level price indices like CPI, GDP deflator or some measure of unit labor cost. Fourthly, we explicitly estimate and incorporate different elasticities of substitution in production functions and final demand aggregators in our REER indices, which is a significant improvement from the typical practice of assuming all elasticities to be unity as is done in the literature. We illustrate the importance of each of these additions using illustrative examples as well as actual REER indices computed using data from the World Input-Output Database (WIOD) and outline the conditions under which our general framework nests the other measures in the literature.

We take our framework to the data by utilizing detailed input-output tables from the World Input-Output Database (WIOD). We compute REER indices for 40 countries and 1435 country-sector pairs for the period 1995-2009 and display various aspects of our REER measures and contrast them with other measures in the literature.

In addition to addressing the issue of competitiveness in a comprehensive manner, we see two other important auxiliary contributions of the paper. Firstly, our modeling of the production and consumption aggregators is the most comprehensive in the literature and allows for features like intermediate inputs and several elasticities of substitution including a distinction between macro and micro elasticities that has been shown to be a feature of the data (see Feenstra et al., 2010). Although we worked with a static partial equilibrium model in this paper in order to best address the primary question of interest, the model can be extended to a dynamic general equilibrium setting to study other important issues in international macroeconomics including international transmission of shocks. Secondly, this is the first paper to our knowledge that has taken up the task of estimating elasticities of substitution comprehensively by making a distinction between consumption and production elasticities on the one hand and micro and macro elasticities on the other. Since even the most advanced empirical estimates of elasticities available in the literature to date do not distinguish between production and consumption elasticities (see for instance Broda and Weinstein, 2006, Soderbery, 2013 or Feenstra et al., 2010), DSGE models aiming to study the role of production sharing are often missing a key component in their calibration[22]. Our elasticity estimates provide a first step toward filling this void.

Appendix

A Stylized 3 by 2 GVC <Not for publication>

The associated price indices for final goods consumption (CPIs) can be computed as follows:

[22]The list of papers that would benefit from such estimates is too large to summarize comprehensively. But several recent examples are Burstein et al. (2008), Johnson (2012) and Di Giovanni and Levchenko (2010).

$$\begin{aligned} P^J &= [(\kappa_1^{JJ})(p(Q)_1^J)^{1-\theta} + (\kappa_2^{JJ})(p(Q)_2^J)^{1-\theta} + (\kappa_2^{CJ})(p(Q)_2^C)^{1-\theta}]^{\frac{1}{1-\theta}} && (A.1)\\ P^C &= [(\kappa_1^{CC})(p(Q)_1^C)^{1-\theta} + (\kappa_2^{CC})(p(Q)_2^C)^{1-\theta}]^{\frac{1}{1-\theta}} && (A.2)\\ P^U &= [(\kappa_1^{UU})(p(Q)_1^U)^{1-\theta} + (\kappa_2^{UU})(p(Q)_2^U)^{1-\theta} + (\kappa_2^{CU})(p(Q)_2^C)^{1-\theta}]^{\frac{1}{1-\theta}} && (A.3)\end{aligned}$$

Market clearing conditions:

Output of all entities except $(C, 2)$ is sold only as final good. Table 2 implies the following market clearing conditions:

$$\begin{aligned} Q_1^J &= X_{12}^{JC} + F_1^{JJ} && (A.4)\\ Q_2^J &= F_2^{JJ} && (A.5)\\ Q_1^C &= F_1^{CC} && (A.6)\\ Q_2^C &= F_2^{CC} + F_2^{CJ} + F_2^{CU} && (A.7)\\ Q_1^U &= F_1^{UU} && (A.8)\\ Q_2^U &= F_2^{UU} && (A.9)\end{aligned}$$

Solving the model:

We solve the model by combining the log linearized first order conditions with the market clearing conditions. The first order condition for final good can be written as:

$$F_s^{fh} = \kappa_s^{fh}\left(\frac{p(Q)_s^f}{P^h}\right)^{-\theta} F^h, h, f \in \{J, C, U), s \in \{1, 2\} \tag{A.10}$$

(note that only 8 out of the 18 values of F_s^{fh} are positive, as denoted in table 2).
We will work with the following log linearized version:

$$\hat{F}_s^{fh} = -\theta\hat{p}_s^f + \theta\hat{P}^h + \hat{F}^h \tag{A.11}$$

Linearizing the expressions for the CPIs in the three countries we get:

$$\hat{P}^U = \left(\frac{p_2^C F_2^{CU}}{P^U F^U}\right) \hat{p}(Q)_2^C + \left(\frac{p(Q)_1^U F_1^{UU}}{P^U F^U}\right) \hat{p}(V)_1^U + \left(\frac{p(Q)_2^U F_2^{UU}}{P^U F^U}\right) \hat{p}(V)_2^U \tag{A.12}$$

$$\hat{P}^C = \left(\frac{p(Q)_1^C F_1^{CC}}{P^C F^C}\right) \hat{p}(Q)_1^C + \left(\frac{p(Q)_2^C F_2^{CC}}{P^C F^C}\right) \hat{p}(Q)_2^C \tag{A.13}$$

$$\hat{P}^J = \left(\frac{p(Q)_1^J F_1^{JJ}}{P^J F^J}\right) \hat{p}(Q)_1^J + \left(\frac{p(Q)_2^J F_2^{JJ}}{P^J F^J}\right) \hat{p}(Q)_2^J + \left(\frac{p(Q)_2^C F_2^{CJ}}{P^J F^J}\right) \hat{p}(Q)_2^C \tag{A.14}$$

The first order conditions for production are as follows:

$$X_{12}^{JC} = w^X \left(\frac{p(Q)_1^J}{p(Q)_2^C}\right)^{-\sigma} Q_2^C$$

$$V_2^C = = w_2^V \left(\frac{p(V)_2^C}{p(Q)_2^C}\right)^{-\sigma} Q_2^C$$

These along with the production function (3.1) and its associated price index can be linearized as follows:

$$\hat{X}_{12}^{JC} = -\sigma \hat{p}(Q)_1^J + \sigma \hat{p}(Q)_2^C + \hat{Q}_2^C \tag{A.15}$$

$$\hat{V}_2^C = -\sigma \hat{p}(V)_2^C + \sigma \hat{p}(Q)_2^C + \hat{Q}_2^C \tag{A.16}$$

$$\hat{Q}_2^C = \left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right) \hat{V}_2^C + \left(\frac{p(V)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right) \hat{X}_{12}^{JC} \tag{A.17}$$

$$\hat{p}_2^C = \left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right) \hat{p}(V)_2^C + \left(\frac{p(V)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right) \hat{p}(V)_1^J \tag{A.18}$$

Next, the non trivial market clearing conditions (A.7) and (A.4) can be linearized as follows:

$$\hat{Q}_2^C = \left(\frac{p(Q)_2^C F_2^{CC}}{p(Q)_2^C Q_2^C}\right) \hat{F}_2^{CC} + \left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right) \hat{F}_2^{CJ} + \left(\frac{p(Q)_2^C F_2^{CU}}{p(Q)_2^C Q_2^C}\right) \hat{F}_2^{CU} \tag{A.19}$$

$$\hat{Q}_1^J = \left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_1^J Q_1^J}\right) \hat{X}_{12}^{JC} + \left(\frac{p(Q)_1^J F_1^{JJ}}{p(Q)_1^J Q_1^J}\right) \hat{F}_1^{JJ} \tag{A.20}$$

$$\tag{A.21}$$

Computation of linearized expression for $\hat{V}_2^C$ and Q_2^C in section 3.

From (B.15) and (7.6) we get:

$$\hat{Q}_2^C = \hat{V}_2^C + \left(\frac{p(Q)_1^J X_{12}^{JC}}{p(V)_2^C V_2^C}\right)\left(-\sigma \hat{p}(V)_1^J + \sigma \hat{p}(Q)_2^C\right) \tag{A.22}$$

Using the linearized first order conditions for final goods consumption (A.23) in the market clearing condition (B.21) we get:

$$\begin{aligned}
\hat{Q}_2^C &= \left(\frac{p(Q)_2^C F_2^{CC}}{p(Q)_2^C Q_2^C}\right)\hat{F}_2^{CC} + \left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\hat{F}_2^{CJ} + \left(\frac{p(Q)_2^C F_2^{CU}}{p(Q)_2^C Q_2^C}\right)\hat{F}_2^{CU} \\
&= \left(\frac{p(Q)_2^C F_2^{CC}}{p(Q)_2^C Q_2^C}\right)\left(-\theta \hat{p}(Q)_2^C + \theta \hat{P}^C + \hat{F}^C\right) + \left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(-\theta \hat{p}(Q)_2^C + \theta \hat{P}^J + \hat{F}^J\right) && \text{(A.23)} \\
&\quad + \left(\frac{p(Q)_2^C F_2^{CU}}{p(Q)_2^C Q_2^C}\right)\left(-\theta \hat{p}(Q)_2^C + \theta \hat{P}^U + \hat{F}^U\right) && \text{(A.24)}
\end{aligned}$$

Using the expressions for the linearized CPIs ((B.26) (B.27) and (B.42))) as well as (B.35) we can write (A.23) as follows:

$$\hat{Q}_2^c = w(Q)_{21}^{CJ}\hat{p}(V)_1^J + w(Q)_{22}^{CJ}\hat{p}(V)_2^J + w(Q)_{21}^{CC}\hat{p}(V)_1^C + w(Q)_{22}^{CC}\hat{p}(V)_2^C + w(Q)_{21}^{CU}\hat{p}(V)_1^U + w(Q)_{22}^{CU}\hat{p}(V)_2^U \tag{A.25}$$

where

$$\begin{aligned}
w(Q)_{21}^{CJ} &= -\theta\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right) + \theta\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{P^J F^J}\right) + \theta\left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CC}}{P^C F^C}\right) \\
&\quad + \theta\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_1^J F_1^{JJ}}{P^J F^J}\right) + \theta\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CU}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CU}}{P^U F^U}\right) && \text{(A.26)}
\end{aligned}$$

$$\begin{aligned}
w(Q)_{22}^{CJ} &= \theta\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^J F_2^{JJ}}{P^J F^J}\right) && \text{(A.27)} \\
w(Q)_{21}^{CC} &= \theta\left(\frac{p(Q)_2^C F_2^{CC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_1^C F_1^{CC}}{P^C F^C}\right) && \text{(A.28)}
\end{aligned}$$

$$\begin{aligned}
w(Q)_{22}^{CC} &= -\theta + \theta\left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CC}}{P^C F^C}\right) + \theta\left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{P^J F^J}\right) \\
&\quad + \theta\left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(V)_2^C F_2^{CU}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CU}}{P^U F^U}\right)
\end{aligned}$$

$$
\begin{aligned}
w(Q)^{CU}_{21} &= \theta\left(\frac{p(Q)^C_2 F^{CU}_2}{p(Q)^C_2 Q^C_2}\right)\left(\frac{p(Q)^U_1 F^{UU}_1}{P^U F^U}\right)\\
w(Q)^{CU}_{22} &= \theta\left(\frac{p(Q)^C_2 F^{CU}_2}{p(Q)^C_2 Q^C_2}\right)\left(\frac{p(Q)^U_2 F^{UU}_2}{P^U F^U}\right)
\end{aligned}
$$

From (A.22) and (A.25) we can write the demand for value added by $(C, 2)$ as a function of prices as follows:

$$
\hat{V}^c_2 = w(v)^{CJ}_{21}\hat{p}(V)^J_1 + w(v)^{CJ}_{22}\hat{p}(V)^J_2 + w(v)^{CC}_{21}\hat{p}(V)^C_1 + w(v)^{CC}_{22}\hat{p}(V)^C_2 + w(v)^{CU}_{21}\hat{p}(V)^U_1 + w(v)^{CU}_{22}\hat{p}(V)^U_2 \tag{A.29}
$$

where

$$
\begin{aligned}
w(V)^{CJ}_{21} &= w(Q)^{CJ}_{21} + \sigma\left(\frac{p(Q)^J_1 X^{JC}_{12}}{p(Q)^C_2 Q^C_2}\right)\\
w(V)^{CJ}_{22} &= w(Q)^{CJ}_{22}\\
w(V)^{CC}_{21} &= w(Q)^{CC}_{21}\\
w(V)^{CC}_{22} &= w(Q)^{CC}_{22} + \sigma\left(\frac{p(Q)^J_1 X^{JC}_{12}}{p(Q)^C_2 Q^C_2}\right)\\
w(V)^{CU}_{21} &= w(Q)^{CU}_{21}\\
w(V)^{CU}_{22} &= w(Q)^{CU}_{22}
\end{aligned}
$$

$$
\hat{V}^c_1 = w(v)^{CJ}_{11}\hat{p}(V)^J_1 + w(v)^{CJ}_{12}\hat{p}(V)^J_2 + w(v)^{CC}_{11}\hat{p}(V)^C_1 + w(v)^{CC}_{12}\hat{p}(V)^C_2 + w(v)^{CU}_{11}\hat{p}(V)^U_1 + w(v)^{CU}_{12}\hat{p}(V)^U_2 \tag{A.30}
$$

where

$$
\begin{aligned}
w(V)^{CC}_{11} &= -\theta\left(1 - \frac{p(Q)^C_1 F^{CC}_1}{P^C F^C}\right)\\
w(V)^{CC}_{12} &= \theta\left(\frac{p(Q)^C_2 F^{CC}_2}{P^C F^C}\right)\left(\frac{p(V)^C_2 V^C_2}{p^C_2 Q^C_2}\right)\\
w(V)^{CJ}_{11} &= \theta\left(\frac{p(Q)^C_2 F^{CC}_2}{P^C F^C}\right)\left(\frac{p(Q)^J_1 X^{JC}_{12}}{p(Q)^C_2 Q^C_2}\right)\\
w(V)^{CJ}_{12} &= w(V)^{CU}_{11} = w(V)^{CU}_{12} = 0
\end{aligned} \tag{A.31}
$$

The appendix shows that the weight assigned by sector 2 in country C to sector 1 in country J (its input supplier) is given by

$$w(v)_{21}^{CJ} = \left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)(\sigma-\theta) + \underbrace{\theta\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{P^J F^J}\right)}_{term2}$$

$$+\underbrace{\theta\left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CC}}{P^C F^C}\right)}_{term3} \tag{A.32}$$

$$+\underbrace{\theta\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_1^J F_1^{JJ}}{P^J F^J}\right)}_{term4} + \underbrace{\theta\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CU}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CU}}{P^U F^U}\right)}_{term5} \tag{A.33}$$

We can interpret the different terms on the right hand side of the above equation as follows:

$$\begin{aligned} term2 + term4 &= \theta\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{P^J F^J}\right) + \theta\left(\frac{p(Q)_2^C F_2^{CJ}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_1^J F_1^{JJ}}{P^J F^J}\right) \\ &= \theta\left[\left(\frac{p(Q)_2^{VC} V_2^C}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{p(V)_2^C V_2^C}\right)\right]\left[\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right)\left(\frac{p(Q)_2^C F_2^{CJ}}{P^J F^J}\right) + \left(\frac{p_{(Q)1}^J F_1^{JJ}}{P^J F^J}\right)\right] \end{aligned} \tag{A.34}$$

$\left(\frac{p(V)_2^C V_2^C}{p(Q)_2^C Q_2^C}\right) p(Q)_2^C F_2^{CJ}$ is the value added created by $(C,2)$ that is ultimately absorbed in country J. Similarly $\left(\frac{p(Q)_1^J X_{12}^{JC}}{p(Q)_2^C Q_2^C}\right) p(Q)_2^C F_2^{CJ} + p(Q)_1^J F_1^{JJ}$ is the value added created in $(J,1)$ that is ultimately absorbed in country J. Therefore we can simplify (A.34) to write:[23]

$$term2 + term4 = \theta\left(\frac{p(V)_2^C V_2^{CJ}}{p(V)_2^C V_2^C}\right)\left(\frac{p(V)_1^J V_1^{JJ}}{P^J F^J}\right) \tag{A.35}$$

Comparison of GVC-REER and Q-REER

With trade in intermediate inputs, competitiveness in gross output and value added can be delinked. The expression for gross output competitiveness in our model is as follows (We label this measure of gross output competitiveness "Q-REER").

$$\triangle Q - REER_2^c = \hat{Q}_2^c = w(Q)_{21}^{CJ}\hat{p}(V)_1^J + w(Q)_{22}^{CJ}\hat{p}(V)_2^J + w(Q)_{21}^{CC}\hat{p}(V)_1^C + w(Q)_{22}^{CC}\hat{p}(V)_2^C + w(Q)_{21}^{CU}\hat{p}(V)_1^U + w(Q)_{22}^{CU}\hat{p}(V)_2^U \tag{A.36}$$

where

[23] The derivation follows by multiplying and dividing both terms by $\left(\frac{p_2^{VC} V_2^C}{p_2^C Q_2^C}\right)$ and rearranging.

$$
\begin{aligned}
w(Q)^{CJ}_{21} &= \theta\left[\left(\frac{p(V)^C_2V^{CJ}_2}{p(V)^C_2V^C_2}\right)\left(\frac{p(V)^J_1V^{JJ}_1}{P^JF^J}\right)+\left(\frac{p(V)^C_2V^{CC}_2}{p(V)^C_2V^C_2}\right)\left(\frac{p(V)^J_1V^{JC}_1}{P^CF^C}\right)+\left(\frac{p(V)^C_2V^{CU}_2}{p(V)^C_2V^C_2}\right)\left(\frac{p(V)^J_1V^{JU}_1}{P^UF^U}\right)\right] \\
&\quad -\theta\left(\frac{p(Q)^J_1X^{JC}_{12}}{p(Q)^C_2Q^C_2}\right) \qquad \text{(A.37)} \\
&= w(v)^{CJ}_{21}-\sigma\left(\frac{p(Q)^J_1X^{JC}_{12}}{p(Q)^C_2Q^C_2}\right) \qquad \text{(A.38)}
\end{aligned}
$$

The idea behind the "Goods-REER" measure of Bayoumi et al. (2013) is to measure competitiveness of gross output as opposed to value added. The analogous expression in our framework is given by (A.36), but as will be shown later, the two measures do not coincide except in very restrictive and/or knife-edge cases.

Two differences between the value added weight $w(v)^{CJ}_{21}$ and gross output weight $w(Q)^{CJ}_{21}$ are worth highlighting. First, note that as long as the production function is not Leontief (i. e $\sigma \neq 0$) , the gross output competitiveness weight is always lower then the value added weight ($w(Q)^{CJ}_{21} < w(V)^{CJ}_{21}$). This is a consequence of the fact that substitutability in the production function which causes the weight $w(V)^{CJ}_{21}$ to increase because of the possibility of a shift occurring from V^C_2 to $(J,1)$'s value added(embodied in X^{JC}_{12}) does not affect the gross output weight $w(Q)^{CJ}_{21}$, for as far as gross output is concerned the substitution between different inputs in production is irrelevant as long as the final demand for the good increases.

Secondly note that when $X^{JC}_{12} = 0$, the two weights are equivalent. As will be shown in the paper later on, this is a general result–that in the absence of intermediate inputs the gross output and value added weighting matrices are identical.

Computing Aggregate Real Effective Exchange Rates for Countries

To derive the expression for country-level value added weights, we start with the following decomposition of the nominal GDP of country h into its different sectoral components:

$$
p(V)^hV^h = p(V)^h_1V^h_1 + p(V)^h_2V^h_2 \qquad \text{(A.39)}
$$

where p^{vh} is the GDP deflator of country h. Log linearizing this equation we get:

$$
\hat{p}(V)^h + \hat{V}^h = \left(\frac{p(V)^h_1V^h_1}{p(V)^hV^h}\right)\left[\hat{p}(V)^h_1+\hat{V}^h_1\right] + \left(\frac{p(V)^h_2V^h_2}{p(V)^hV^h}\right)\left[\hat{p}(V)^h_2+\hat{V}^h_2\right] \qquad \text{(A.40)}
$$

Since (up to a first order approximation) the change in GDP deflator is a weighted sum of changes in the different sector level prices, the above equation reduces to

$$
\begin{aligned}
\hat{V}^h &= \left(\frac{p(V)_1^h V_1^h}{p(V)^h V^h}\right)\left[\hat{V}_1^h\right] + \left(\frac{p(V)_2^h V_2^h}{p(V)^h V^h}\right)\left[\hat{V}_2^h\right] \\
&= \left(\frac{p(V)_1^h V_1^h}{p(V)^h V^h}\right)\left[\sum_{f\in\{J,C,F\}}\sum_{k\in\{1,2\}} w(v)_{1k}^{hf}\hat{p}(V)_k^f\right] + \left(\frac{p(V)_2^h V_2^h}{p(V)^h V^h}\right)\left[\sum_{f\in\{J,C,F\}}\sum_{k\in\{1,2\}} w(v)_{2k}^{hf}\hat{p}(V)_k^f\right] \\
&= \sum_{f\in(J,C,U)}\sum_{k\in\{1,2\}}\left[\left(\frac{p(V)_1^h V_1^h}{p(V)^h V^h}\right)w(v)_{1k}^{hf} + \left(\frac{p(V)_2^h V_2^h}{p(V)^h V^h}\right)w(v)_{2k}^{hf}\right]\hat{p}(V)_k^f \qquad \text{(A.41)}
\end{aligned}
$$

Solution of the single sector version of the model:

Final demand:

$$
\hat{F}^{fh} = -\theta\hat{p}(Q)_s^f + \theta\hat{P}^h + \hat{F}^h, (h,f) \in \{(C,C),(J,J),(U,U),(C,J),(C,U)\} \qquad \text{(A.42)}
$$

Linearizing the expressions for the CPIs in the three countries((B.22)-(B.23)) we get:

$$
\begin{aligned}
\hat{P}^U &= \left(\frac{p(Q)^C F^{CU}}{P^U F^U}\right)\hat{p}(V)^C + \left(\frac{p(Q)^U F^{UU}}{P^U F^U}\right)\hat{p}(V)^U \qquad \text{(A.43)} \\
\hat{P}^C &= \hat{p}(V)^C \qquad \text{(A.44)} \\
\hat{P}^J &= \left(\frac{p(Q)^J F^{JJ}}{P^J F^J}\right)\hat{p}(Q)^J + \left(\frac{p(Q)^C F^{CJ}}{P^J F^J}\right)\hat{p}(Q)^C \qquad \text{(A.45)}
\end{aligned}
$$

The first order conditions for production are as follows:

$$
\begin{aligned}
X_{12}^{JC} &= w^X\left(\frac{p(Q)_1^J}{p(Q)_2^C}\right)^{-\sigma} Q_2^c \\
V_2^C = &= w_2^V\left(\frac{p(Q)_2^C}{p(Q)_2^C}\right)^{-\sigma} Q_2^c
\end{aligned}
$$

These along with the production function (7.5) and its associated price index can be linearized as follows:

Table 16 – Single sector version of the 3 by 2 model

	J	C	U	JFinal	CFinal	Ufinal	total output
J	0	X	0	X	0	0	X
C	0	0	0	X	X	X	X
U	0	0	0	0	0	X	X
VA	X	X	X				
total output	X	X	X				

$$\begin{aligned}\hat{X}^{JC} &= -\sigma\hat{p}(Q)^J + \sigma\hat{p}(Q)^C + \hat{Q}^C && \text{(A.46)}\\ \hat{V}^C &= -\sigma\hat{p}(Q)_2^C + \sigma\hat{p}(Q)_2^C + \hat{Q}_2^C && \text{(A.47)}\\ \hat{Q}^C &= \left(\frac{p(Q)^C V^C}{p(Q)^C Q^C}\right)\hat{V}_2^C + \left(\frac{p(V)^J X^{JC}}{p(Q)^C Q^C}\right)\hat{X}^{JC} && \text{(A.48)}\\ \hat{p}^C &= \left(\frac{p(V)^C V^C}{p(Q)^C Q^C}\right)\hat{p}(V)^C + \left(\frac{p(V)^J X^{JC}}{p(Q)^C Q^C}\right)\hat{p}(V)^J && \text{(A.49)}\end{aligned}$$

Next, the market clearing conditions (4.8) and (B.14) can be linearlized as follows:

$$\begin{aligned}\hat{Q}^C &= \left(\frac{p(Q)^C F^{CC}}{p(Q)^C Q^C}\right)\hat{F}^{CC} + \left(\frac{p(Q)^C F^{CJ}}{p(Q)^C Q^C}\right)\hat{F}^{CJ} + \left(\frac{p(Q)_2^C F_2^{CU}}{p(Q)^C Q^C}\right)\hat{F}^{CU} && \text{(A.50)}\\ \hat{Q}^J &= \left(\frac{p(Q)_1^J X^{JC}}{p(Q)^J Q^J}\right)\hat{X}^{JC} + \left(\frac{p(Q)^J F^{JJ}}{p(Q)^J Q^J}\right)\hat{F}^{JJ} && \text{(A.51)}\end{aligned}$$

Using these linearized first order and market clearing conditions we can derive an expression for change in demand for value added by country C (equation (D.13) and (D.1))

Country Level exchange rate without exploiting sector level heterogeneity

The production functions, price indices and final demands and market clearing conditions are now given as follows:

$$Q^c = \left[(w^{CV})^{\frac{1}{\sigma}}(V^c)^{\frac{\sigma-1}{\sigma}} + (w^{CX})^{\frac{1}{\sigma}}(X^{Jc})^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{A.52}$$

$$p(Q)^c = \left[(w^{CV})(p(V)^c)^{1-\sigma} + (w^{CX})(p(Q)^J)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{A.53}$$

$$Q^h = V^h, h \in \{J, U\} \tag{A.54}$$

Consumption

$$
\begin{aligned}
F^J &= [(\kappa^{JJ})^{\frac{1}{\theta}}(F^{JJ})^{\frac{\theta-1}{\theta}} + (\kappa^{CJ})^{\frac{1}{\theta}}(F^{CJ})^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}} && \text{(A.55)}\\
F^C &= F^{CC} && \text{(A.56)}\\
F^U &= [(\kappa^{UU})^{\frac{1}{\theta}}(F^{UU})^{\frac{\theta-1}{\theta}} + (\kappa^{CU})^{\frac{1}{\theta}}(F^{CU})^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}} && \text{(A.57)}
\end{aligned}
$$

$$
\begin{aligned}
P^J &= [(\kappa^{JJ})(p(Q)^J)^{1-\theta} + (\kappa^{CJ})(p(Q)^C)^{1-\theta}]^{\frac{1}{1-\theta}} && \text{(A.58)}\\
P^C &= p(Q)^C && \text{(A.59)}\\
P^U &= [(\kappa^{UU})(p(Q)^U)^{1-\theta} + (\kappa^{CU})(p(Q)^C)^{1-\theta}]^{\frac{1}{1-\theta}} && \text{(A.60)}
\end{aligned}
$$

$$
\begin{aligned}
Q^J &= X^{JC} + F^{JJ}\\
Q^C &= F^{CC} + F^{CJ} + F^{CU}\\
Q^U &= F^{UU}
\end{aligned}
$$

The appendix shows that the weight assigned by country C to country J in this case is given by:

$$
\hat{V}^C = w(v)^{CJ}\hat{p}(V)^J + w(v)^{CC}\hat{p}(V)^C + w(v)^{CU}\hat{p}(V)^U \tag{A.61}
$$

$$
\begin{aligned}
w(v)^{CJ} &= \sigma\left(\frac{p(V)^J X^{JC}}{p(Q)^C Q^C}\right) - \theta\left(\frac{p(V)^J X^{JC}}{p(Q)^C Q^C}\right) + \theta\left(\frac{p(V)^C F^{CC}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CC}}{P^C F^C}\right) && \text{(A.62)}\\
&+\theta\left(\frac{p(V)^J X^{JC}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CJ}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CJ}}{P^J F^J}\right) + \theta\left(\frac{p(V)^C F^{CJ}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CJ}}{P^J F^J}\right)\\
&+\theta\left(\frac{p(V)^J X^{JC}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CU}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CU}}{P^U F^U}\right) && \text{(A.63)}\\
w(v)^{CC} &= -\sigma\left(\frac{p(V)^J X^{JC}}{p(Q)^C Q^C}\right) - \theta\left(\frac{p(V)^C V^C}{p(Q)^C Q^C}\right) + \theta\left(\frac{p(V)^C V^C}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CC}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CC}}{P^C F^C}\right)\\
&+\theta\left(\frac{p(V)^C V^C}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CJ}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CJ}}{P^J F^J}\right)\\
&+\theta\left(\frac{p(V)^C V^C}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CU}}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CU}}{P^U F^U}\right) && \text{(A.64)}\\
w(v)^{CU} &= \theta\left(\frac{p(V)^C V^C}{p(Q)^C Q^C}\right)\left(\frac{p(V)^C F^{CU}}{p(V)^C V^C}\right)\left(\frac{p(V)^U F^{UU}}{P^U F^U}\right) && \text{(A.65)}
\end{aligned}
$$

It is evident from A.62 and 3.11 that $w(v)^{CJ}$ and $w_A(v)^{CJ}$ are not equal. The issue of the non-equivalence of the two weighting matrices will be discussed after we have specified the general model. For now we just want to emphasize that our weighting matrix which exploits sector level information is unique in the literature, and so a starting point to begin a comparison with other measures in the literature is to consider the case where there is only one sector within each country.

We next move on to our general model which builds on the intuition developed from this 3 by 2 setting. After discussing the relationship to other measures in the literature based on the general model we come back to the 3 by 2 model to show some illustrative examples.

B Solution of the general model <Not for publication>

B.1 First order conditions

B.1.1 first order conditions for production:

$$V_l^c = w_l^{vc} \left(\frac{p(V)_l^c}{p(Q)_l^c} \right)^{-\sigma^3(c,l)} Q_l^c \tag{B.1}$$

$$X_l^c = w_l^{Xc} \left(\frac{p(X)_l^c}{p(Q)_l^c} \right)^{-\sigma^3(c,l)} Q_l^c \tag{B.2}$$

$$X_{sl}^c = w_{sl}^c \left(\frac{p(X)_{sl}^c}{p(X)_l^c} \right)^{-\sigma^2(c,l)} X_l^c \tag{B.3}$$

$$X_{sl}^{ic} = w_{sl}^{ic} \left(\frac{p(Q)_s^i}{p(X)_{sl}^{(f)c}} \right)^{-\sigma_s^1(c,l)} X(f)_{sl}^c \tag{B.4}$$

$$X_{sl}^{cc} = w_{sl}^{cc} \left(\frac{p(Q)_s^c}{p(X)_{sl}^{(f)c}} \right)^{-\sigma_s^{1h}(c,l)} X_{sl}^c \tag{B.5}$$

$$X_{sl}^c(f) = w(f)_{sl}^c \left(\frac{p(X)_{sl}^{(f)c}}{p(X)_{sl}^c} \right)^{-\sigma_s^{1h}(c,l)} X_{sl}^c \tag{B.6}$$

Here q_l^c and q_{sl}^c are price indices corresponding to X_l^c and X_{sl}^c respectively and are given by:

$$p(X)_l^c = \left[\sum_{s=1}^{m}(w_{sl}^c)(p(X)_{sl}^c)^{1-\sigma^2(c,l)}\right]^{\frac{1}{1-\sigma^2(c,l)}} \tag{B.7}$$

$$p(X)_{sl}^{(f)c} = \left[\sum_{i=1,i\neq c}^{n}(w_{sl}^{ic})(p(Q)_s^i)^{1-\sigma_s^1(c,l)}\right]^{\frac{1}{1-\sigma_s^1(c,l)}} \tag{B.8}$$

$$p(X)_{sl}^c = \left[(w_{sl}^{cc})(p(Q)_s^c)^{1-\sigma^{1h}(c,l)} + (w_l^{Xc})(p(X)_{sl}^{(f)c})^{1-\sigma^{1h}(c,l)}\right]^{\frac{1}{1-\sigma^{1h}(c,l)}} \tag{B.9}$$

and price of gross output is given by:

$$p(Q)_l^c = \left[(w_l^{vc})(p(V)_l^c)^{1-\sigma^3(c,l)} + (w_l^{Xc})(p(X)_l^c)^{1-\sigma^3(c,l)}\right]^{\frac{1}{1-\sigma^3(c,l)}} \tag{B.10}$$

where $p(V)_l^c$ is the price of value added(i. e price of factor of production) of country c sector l

B.1.2 First order conditions for final consumption:

$$F_s^{ic} = \kappa_s^{ic}\left(\frac{p(Q)_s^i}{P(f)_s^c}\right)^{-\theta_s^1(c)} F(f)_s^c \tag{B.11}$$

$$F_s^{cc} = \kappa_s^{cc}\left(\frac{p(Q)_s^c}{P_s^c}\right)^{-\theta_s^{1h}(c)} F_s^c \tag{B.12}$$

$$F(f)_s^c = \kappa(f)_s^c\left(\frac{P(f)_s^c}{P_s^c}\right)^{-\theta_s^{1h}(c)} F_s^c \tag{B.13}$$

$$F_s^c = \kappa_s^c\left(\frac{P_s^c}{P^c}\right)^{-\theta^2(c)} F^c \tag{B.14}$$

Here P_s^c and P^c are price indices for sector s good and aggregate good consumed by country c, respectively and are given by

$$P_s^c(f) = \left[\sum_{i=1,i\neq c}^{n}(\kappa_s^{ic})(p(Q)_s^i)^{1-\theta_s^1(c)}\right]^{\frac{1}{1-\theta_s^1(c)}} \tag{B.15}$$

$$P_s^c = \left[(\kappa_s^{cc})(p(Q)_s^c)^{1-\theta_s^{1h}(c)} + (\kappa(f)_l^c)(P(f)_s^c)^{1-\theta_s^{1h}(c)}\right]^{\frac{1}{1-\theta_s^{1h}(c)}} \tag{B.16}$$

$$P^c = \left[\sum_{s=1}^{m}(\kappa_s^c)(P_s^c)^{1-\theta^2(c)}\right]^{\frac{1}{1-\theta^2(c)}} \tag{B.17}$$

Let $[A]_{nmXnm}$ be the input-output coefficient matrix at the country-sector level, i. e the $(i,j)^{th}$ block which has dimension mXm is given by

$$[A]_{mXm}^{ij} = \begin{pmatrix} a_{11}^{ij} & a_{12}^{ij} & .. & a_{1m}^{ij} \\ a_{21}^{ij} & a_{22}^{ij} & .. & a_{2m}^{ij} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}^{ij} & a_{m2}^{ij} & .. & a_{mm}^{ij} \end{pmatrix} \tag{B.18}$$

where a_{sl}^{ij} denotes the output of (i,s) used in the production of one unit of (j,l), i. e

$$a_{sl}^{ij} = \frac{p(Q)_s^i X_{sl}^{ij}}{p(Q)_l^j Q_l^j} \tag{B.19}$$

Let $[B]_{nmXnm}$ be the corresponding total requirement matrix given by

$$[B]_{nmXnm} \quad = \quad (I_{nm} - [A])^{-1} \tag{B.20}$$

Also, define the matrix $[D_Q]_{nmxnm}$ to be a diagonal matrix with the $(cl)^{th}$ diagonal entry given by $\frac{1}{p_l^c Q_l^c}$

Log Linearization:

A note on notation:

- for any variable Y_{cd}^{ab}, $vec\left(Y_{cd}^{ab}\right)$ denotes a vector with all components of Y_{cd}^{ab} stacked together
- The stacking order is as follows: d, c, b, d. i, e first the home sector index changes, followed by foreign sector, followed by home country and finally foreign country
  - $vec\left(Y_{cd}^{b}\right), vec\left(Y_{c}^{ab}\right)$ etc are defined accordingly.
- Examples in a 2 by 2 case $(m = n = 2)$
  - $vec\left(Y_{cd}^{ab}\right) = (Y_{11}^{11}, Y_{12}^{11}, Y_{21}^{11}, Y_{22}^{11}, Y_{11}^{12}, Y_{12}^{12}, Y_{21}^{12}, Y_{22}^{12}, Y_{11}^{21}, Y_{12}^{21}, Y_{21}^{21}, Y_{22}^{21}, Y_{11}^{22}, Y_{12}^{22}, Y_{21}^{22}, Y_{22}^{22})'$
  - $vec\left(Y_{cd}^{b}\right) = (Y_{11}^{1}, Y_{12}^{1}, Y_{21}^{1}, Y_{22}^{1}, Y_{11}^{2}, Y_{12}^{2}, Y_{21}^{2}, Y_{22}^{2})'$

This appendix contains the log linearized first order and market clearing conditions and organizes them in stacked matrix notation which will be useful in deriving the results that follow. A variable with a "^" denotes log deviation from steady state.

Log linearizing and stacking components of production function and price indices:(to simplify notation further, we omit the parenthesis for gross output prices, i.e the parenthesis containing "Q" is omitted)

Raw expression $X(f)^c_{sl} = \left[\sum_{i=1,i\neq c}^{n}(w^{ic}_{sl})^{1/\sigma^1_s(c,l)}(X^{ic}_{sl})^{\frac{\sigma^1_s(c,l)-1}{\sigma^1_s(c,l)}}\right]^{\frac{\sigma^1_s(c,l)}{\sigma^1_s(c,l)-1}}$

Log linearized expression $\hat{X(f)}^c_{sl} = \sum_{i=1,i\neq c}^{n}\left(\frac{p^i_s X^{ic}_{sl}}{P(X)^{(f)}{}^c_{sl}X(f)^c_{sl}}\right)\hat{X}^{ic}_{sl}$

Stacked vector:

$$\left(vec(\hat{X}(f)^c_{sl})\right) = \underbrace{W_{1XXH}}_{nm^2Xn^2m^2}\, vec(\hat{X}^{ic}_{sl}) \tag{B.21}$$

Raw expression $X^c_{sl} = \left[(w^c_{sl})^{1/\sigma^{1h}_s(c,l)}(X^{cc}_{sl})^{\frac{\sigma^{1h}_s(c,l)-1}{\sigma^{1h}_s(c,l)}} + (w(f)^c_{sl})^{1/\sigma^{1h}_s(c,l)}(X(f)^c_{sl})^{\frac{\sigma^{1h}_s(c,l)-1}{\sigma^{1h}_s(c,l)}}\right]^{\frac{\sigma^{1h}_s(c,l)}{\sigma^{1h}_s(c,l)-1}}$

Log linearized expression $\hat{X}^c_{sl} = \sum_{i=1}^{n}\left(\frac{p^i_s X^{ic}_{sl}}{p(X)^c_{sl}X^c_{sl}}\right)\hat{X}^{ic}_{sl}$

Stacked Vector:

$$\left(vec(\hat{X}^c_{sl})\right) = \underbrace{W_{1XX}}_{nm^2Xn^2m^2}\, vec(\hat{X}^{ic}_{sl}) \tag{B.22}$$

Raw expression $X^c_l = \left[\sum_{s=1}^{m}(w^c_{sl})^{1/\sigma^2(c,l)}(X^c_{sl})^{\frac{\sigma^2(c,l)-1}{\sigma^2(c,l)}}\right]^{\frac{\sigma^2(c,l)}{\sigma^2(c,l)-1}}$

Log linearized expression $\hat{X}^c_l = \sum_{s=1}^{m}\left(\frac{p(X)q^c_{sl}X^c_{sl}}{p(X)^c_l X^c_l}\right)\hat{X}^c_{sl}$

Stacked vector:

$$vec(\hat{X}^c_l) = (W_{2XX})_{nmXnm^2}\, vec(\hat{X}^c_{sl}) \tag{B.23}$$

Raw expression $q(f)^c_{sl} = \left[\sum_{i=1,i\neq c}^{n}(w^{ic}_{sl})(p^i_s)^{1-\sigma^1_s(c,l)}\right]^{\frac{1}{1-\sigma^1_s(c,l)}}$

Log linearized expression $\hat{q}^c_{sl}(f) = \sum_{i=1,i\neq c}^{n}\left(\frac{p^i_s X^{ic}_{sl}}{p(X)^c_{sl}X^c_{sl}}\right)\hat{p}^i_s$

Stacked vector:

$$vec(\hat{q}^c_{sl}(f)) = (W_{1XPH})_{nm^2Xnm}\, vec(\hat{p}^i_s) \tag{B.24}$$

Raw expression $q_{sl}^{c}=\left[(w_{sl}^{cc})(p^{c}{}_{s})^{1-\sigma^{1h}(c,l)}+(w_{l}^{Xc})(p(X)^{(f)}{}_{sl}^{c})^{1-\sigma^{1h}(c,l)}\right]^{\frac{1}{1-\sigma^{1h}(c,l)}}$

Log linearized expression $\hat{p(X)}_{sl}^{c}=\sum_{i=1}^{n}\left(\frac{p_{s}^{i}X_{sl}^{ic}}{p(X)_{sl}^{c}X_{sl}^{c}}\right)\hat{p_{s}^{i}}$

Stacked vector:

$$vec(\hat{p(X)}_{sl}^{c})=(W_{1XP})_{nm^{2}Xnm}\,vec(\hat{p}_{s}^{i}) \tag{B.25}$$

Raw expression $p(X)_{l}^{c}=\left[\sum_{s=1}^{m}(w_{sl}^{c})(p(X)_{sl}^{c})^{1-\sigma^{2}(c,l)}\right]^{\frac{1}{1-\sigma^{2}(c,l)}}$

Log linearized expression $\hat{q}_{l}^{c}=\sum_{s=1}^{m}\left(\frac{p(X)_{sl}^{c}X_{sl}^{c}}{p(X)_{l}^{c}X_{l}^{c}}\right)\hat{q_{sl}^{c}}$

Stacked vector:

$$vec(\hat{q}_{l}^{c})=(W_{2Xp})_{nmXnm^{2}}\,vec(\hat{p(X)}_{sl}^{c}) \tag{B.26}$$

Raw expression $P_{s}^{c}(f)=\left[\sum_{i=1,i\neq c}^{n}(\kappa_{s}^{ic})(p_{s}^{i})^{1-\theta_{s}^{1}(c)}\right]^{\frac{1}{1-\theta_{s}^{1}(c)}}$

Log linearized expression $\hat{P(f)}_{s}^{c}=\sum_{i=1,i\neq c}^{n}\left(\frac{p_{s}^{i}F_{s}^{ic}}{P(f)_{s}^{c}F(f)_{s}^{c}}\right)\hat{p}_{s}^{i}$

Stacked vector:

$$vec\left(\hat{P}_{s}^{c}\right)_{nmX1}=(W_{1FPH})_{nmXnm}vec\left(p_{s}^{i}\right)_{nmX1} \tag{B.27}$$

Raw expression $P_{s}^{c}=\left[(\kappa_{s}^{cc})(p_{s}^{cc})^{1-\theta_{s}^{1h}(c)}+(\kappa(f)_{l}^{c})(P(f)_{s}^{c})^{1-\theta_{s}^{1h}(c)}\right]^{\frac{1}{1-\theta_{s}^{1h}(c)}}$

Log linearized expression $\hat{P}_{s}^{c}=\sum_{i=1}^{n}\left(\frac{p_{s}^{i}F_{s}^{ic}}{P_{s}^{c}F_{s}^{c}}\right)\hat{p}_{s}^{i}$

Stacked vector:

$$vec\left(\hat{P}_{s}^{c}\right)_{nmX1}=(W_{1FP})_{nmXnm}vec\left(p_{s}^{i}\right)_{nmX1} \tag{B.28}$$

Raw expression $P^{c}=\left[\sum_{s=1}^{m}(\kappa_{s}^{c})(P_{s}^{c})^{1-\theta^{2}(c)}\right]^{\frac{1}{1-\theta^{2}(c)}}$

Log linearized expression $\hat{P}^{c}=\sum_{s=1}^{m}\left(\frac{P_{s}^{c}F_{s}^{c}}{P^{c}F^{c}}\right)\hat{P}_{s}^{c}$

Stacked vector:

$$vec\left(\hat{P}^{c}\right)_{nX1}=(W_{2FP})_{nXnm}Vec\left(\hat{P}_{s}^{c}\right)_{nmX1} \tag{B.29}$$

Final goods consumption first order conditions:

$$\hat{F_s^{ic}} = -\theta_s^1(c)(\hat{p}_s^i - P(\hat{f})_s{}^c) + F(\hat{f})_s{}^c \tag{B.30}$$
$$\hat{F_s^{cc}} = -\theta_s^{1h}(c)(\hat{p}_s^c - \hat{P_s}{}^c) + \hat{F_s}{}^c \tag{B.31}$$
$$F(\hat{f})_s^c = -\theta_s^{1h}(c)(\hat{P}(f)_s^c - \hat{P_s}{}^c) + \hat{F_s}{}^c \tag{B.32}$$
$$\hat{F_s^c} = -\theta^2(c)(\hat{P}_s^c - \hat{P}^c) + \hat{F}^c \tag{B.33}$$

We can combine these 4 conditions to write:

$$\begin{aligned}
\hat{F_s}{}^{ic} &= -\theta_s^1(c)\hat{p}_s^i + \left(\theta_s^1(c) - \theta_s^{1h}(c)\right) P(\hat{f})_s^c + \left(\theta_s^{1h}(c) - \theta^2(c)\right) \hat{P}_s^c + \theta^2(c)\hat{P}^c + \hat{F}^c \\
\hat{F_s}{}^{cc} &= -\theta_s^{1h}(c)\hat{p}_s^c + \left(\theta_s^{1h}(c) - \theta^2(c)\right) \hat{P}_s^c + \theta^2(c)\hat{P}^c + \hat{F}^c
\end{aligned}$$

We can now stack the above n^2m equations to write a single matrix equation as follows:

$$\begin{aligned}
vec\left(\hat{F_s^{ic}}\right)_{n^2mX1} &= J_F(i \neq c)\left[(Y_1)_{n^2mXnm} vec(\theta_s^1(c))_{nmX1}\right] \odot \left[(Y_2)_{n^2mXnm} vec(\hat{p}_s^i)_{nmX1}\right] \\
&- -J_F(i = c)\left[(Y_1)_{n^2mXnm} vec(\theta_s^{1h}(c))_{nmX1}\right] \odot \left[(Y_2)_{n^2mXnm} vec(\hat{p}_s^i)_{nmX1}\right] \\
&+ J_F(i \neq c)\left[Y_1\left(vec(\theta_s^1(c))_{nmX1} - vec(\theta_s^{1h}(c))_{nmX1}\right)\right] \odot \left[Y_1 vec(\hat{P}(f)_s^c)_{nmX1}\right] \\
&+ \left(Y_1 vec(\theta_s^{1h}(c))_{nmX1} - (Y_3)_{n^2mXn} vec(\theta^2(c))_{nmX1}\right) \odot \left(Y_1 vec(\hat{P}_s^c)_{nmX1}\right) \\
&+ \left[Y_3 vec(\theta^2(c))_{nmX1}\right] \odot \left[Y_3 vec(\hat{P}^c)_{nX1}\right] + Y_3\hat{F}^c
\end{aligned}$$

where $Y_1 = 1_n \otimes I_{nm}$, $Y_2 = I_n \otimes 1_n \otimes I_m$, $Y_3 = 1_n \otimes I_n \otimes 1_m$, $\odot$ is the element by element multiplication operator for two vectors and $J_F(x)$ is an n^2m by 1 vector with ones in all indices that satisfy the condition x and zero elsewhere.

Combining this with (B.29) and (B.28),

$$vec\left(\hat{F_s^{ic}}\right)_{n^2mX1} = Z_F vec(\hat{p}_s^i)_{nmX1} + Y_3\hat{F}^c \tag{B.34}$$

where

$$\begin{aligned}
(Z_F)_{n^2mXnm} &= J_F(i \neq c)\left[(Y_1)_{n^2mXnm} vec(\theta_s^1(c))_{nmX1}\right] \odot \left[(Y_2)_{n^2mXnm}\right] \\
&- -J_F(i = c)\left[(Y_1)_{n^2mXnm} vec(\theta_s^{1h}(c))_{nmX1}\right] \odot \left[(Y_2)_{n^2mXnm}\right] \\
&+ J_F(i \neq c)\left[Y_1\left(vec(\theta_s^1(c))_{nmX1} - vec(\theta_s^{1h}(c))_{nmX1}\right)\right] \odot \left[Y_1 W_{FH}\right] \\
&+ \left(Y_1 vec(\theta_s^{1h}(c))_{nmX1} - (Y_3)_{n^2mXn} vec(\theta^2(c))_{nmX1}\right) \odot \left(Y_1 W_{1FP}\right) \\
&+ \left[Y_3 vec(\theta^2(c))_{nmX1}\right] \odot \left[Y_3 W_{2FP} W_{1FP}\right]
\end{aligned} \tag{B.35}$$

Log linearizing Production first order conditions:

$$V_l^c = w_l^{vc}\left(\frac{p_l^{vc}}{p_l^c}\right)^{-\sigma^3(c,l)} Q_l^c \tag{B.36}$$

$$X_l^c = w_l^{Xc}\left(\frac{q_l^c}{p_l^c}\right)^{-\sigma^3(c,l)} Q_l^c \tag{B.37}$$

$$X_{sl}^c = w_{sl}^c\left(\frac{q_{sl}^c}{q_l^c}\right)^{-\sigma^2(c,l)} X_l^c \tag{B.38}$$

$$X_{sl}^{ic} = w_{sl}^{ic}\left(\frac{p_s^i}{q(f)_{sl}^c}\right)^{-\sigma_s^1(c,l)} X(f)_{sl}^c \tag{B.39}$$

$$X_{sl}^{cc} = w_{sl}^{cc}\left(\frac{p_s^c}{q_{sl}^c}\right)^{-\sigma_s^{1h}(c,l)} X_{sl}^c \tag{B.40}$$

$$X_{sl}^c(f) = w(f)_{sl}^c\left(\frac{q(f)_{sl}^c}{q_{sl}^c}\right)^{-\sigma_s^{1h}(c,l)} X_{sl}^c \tag{B.41}$$

$$\begin{aligned}
\hat{X_{sl}^{ic}} &= -\sigma_s^1(c,l)\hat{p}_s^i + \sigma_s^1(c,l)p(\hat{X})_{sl}^{(f)c} + \hat{X}(f)_{sl}^c \\
\hat{X_{sl}^{cc}} &= -\sigma_s^{1h}(c,l)\hat{p}_s^c + \sigma_s^{1h}(c,l)p(\hat{X})_{sl}^c + \hat{X}_{sl}^c \\
X_{sl}^{\hat{c}}(f) &= -\sigma_s^{1h}(c,l)p(\hat{X})_{sl}^{(f)c} + \sigma_s^{1h}(c,l)p(\hat{X})_{sl}^c + \hat{X}_{sl}^c \\
\hat{X_{sl}^c} &= -\sigma^2(c,l)p(\hat{X})_{sl}^c + \sigma^{2h}(c,l)p(\hat{X})_l^c + \hat{X}_l^c
\end{aligned}$$

$$\begin{aligned}
\hat{X}_{sl}^{ic} &= -\sigma_s^1(c,l)\hat{p}_s^i + \left(\sigma_s^1(c,l) - \sigma_s^{1h}(c,l)\right)p(\hat{X}) + \left(\sigma_s^{1h}(c,l) - \sigma^2(c,l)\right)p(\hat{X})_{sl}^c \\
&+ \left(\sigma^2(c,l) - \sigma^3(c,l)\right)p(\hat{X})_l^c + \sigma^3(c,l)\hat{p}_l^c + \hat{Q}_l^c \\
\hat{X}_{sl}^{cc} &= -\sigma_s^{1h}(c,l)\hat{p}_s^c + \left(\sigma_s^{1h}(c,l) - \sigma^2(c,l)\right)p(\hat{X})_{sl}^c \\
&+ \left(\sigma^2(c,l) - \sigma^3(c,l)\right)\hat{q}_l^c + \sigma^3(c,l)\hat{p}_l^c + \hat{Q}_l^c
\end{aligned}$$

These n^2m^2 equations can be stacked to write

$$
\begin{aligned}
vec\left(\hat{X}^{ic}_{sl}\right)_{n^2m^2} &= -J_X(i\neq c)\left[C_1 vec\left(\sigma^1_s(c,l)\right)_{nm^2X1}\right]\odot\left[C_3 vec(\hat{p}^i_s)_{nmX1}\right] \\
&- J_X(i=c)\left[C_1 vec\left(\sigma^{1h}_s(c,l)\right)_{nm^2X1}\right]\odot\left[C_3 vec(\hat{p}^i_s)_{nmX1}\right] \\
&+ J_X(i\neq c)\left[C_1\left(vec\left(\sigma^1_s(c,l)\right)_{nm^2X1} - vec\left(\sigma^{1h}_s(c,l)\right)_{nm^2X1}\right)\right]\odot\left[C_1 p(\hat{X})^{(f)c}_{sl}\right] \\
&+ \left[C_2\left(vec\left(\sigma^2(c,l)\right)_{nmX1} - vec\left(\sigma^3(c,l)\right)_{nmX1}\right)\right]\odot\left[C_2 p(\hat{X})^c_l\right] \\
&+ \left[C_1 vec\left(\sigma^{1h}_s(c,l)\right)_{nm^2X1} - C_2 vec\left(\sigma^2(c,l)\right)_{nmX1}\right]\odot\left[C_1 p(\hat{X})^c_{sl}\right] \\
&+ \left[C_2 vec\left(\sigma^3(c,l)\right)_{nmX1}\right]\odot\left[C_2 vec(\hat{p}^i_s)_{nmX1}\right] + C_2\hat{Q}^c_l
\end{aligned}
$$

where $C_1 = 1_n \otimes I_{nm^2}$, $C_2 = 1_n \otimes I_n \otimes 1_m \otimes I_m$, $C_3 = I_n \otimes 1_n \otimes I_m \otimes 1_m$. $J_X(y)$ is an n^2m by 1 vector with ones in all indices that satisfy the condition y and zero elsewhere.

Combining this with (B.22) - (B.26) we get:

$$
vec\left(\hat{X}^{ic}_{sl}\right)_{n^2m^2} = Z_X vec(\hat{p}^i_s)_{nmX1} + C_2\hat{Q_l}^c \tag{B.42}
$$

where

$$
\begin{aligned}
Z_X &= -J_X(i\neq c)\left[C_1 vec\left(\sigma^1_s(c,l)\right)_{nm^2X1}\right]\odot[C_3] \\
&- J_X(i=c)\left[C_1 vec\left(\sigma^{1h}_s(c,l)\right)_{nm^2X1}\right]\odot[C_3] \\
&+ J_X(i\neq c)\left[C_1\left(vec\left(\sigma^1_s(c,l)\right)_{nm^2X1} - vec\left(\sigma^{1h}_s(c,l)\right)_{nm^2X1}\right)\right]\odot[C_1 W_{XH}] \\
&+ \left[C_2\left(vec\left(\sigma^2(c,l)\right)_{nmX1} - vec\left(\sigma^3(c,l)\right)_{nmX1}\right)\right]\odot[C_2 W_{2XP} W_{1XP}] \\
&+ \left[C_1 vec\left(\sigma^{1h}_s(c,l)\right)_{nm^2X1} - C_2 vec\left(\sigma^2(c,l)\right)_{nmX1}\right]\odot[C_1 W_{1XP}] \\
&+ \left[C_2 vec\left(\sigma^3(c,l)\right)_{nmX1}\right]\odot[C_2]
\end{aligned} \tag{B.43}
$$

Next, linearizing the production function we have:

$$
vec\left(\hat{Q}^c_l\right) = (D_v)_{nmXnm}\left(vec\left(\hat{V}^c_l\right)\right)_{nmX1} + (D_X)_{nmXnm}\, vec\left(\hat{X}^c_l\right) \tag{B.44}
$$

$$
vec\left(\hat{p^c_l}\right) = D_v vec\left(\hat{p}(V)^c_l\right) + D_X vec\left(p(\hat{X})^c_l\right) \tag{B.45}
$$

(here D_v and D_X are $nmXnm$ diagonal matrices denoting the shares of value added and intermediate inputs in the production of different goods , i. e the lc^{th} diagonal entry of D_v is $\frac{p(V)^c_l V^c_l}{p(Q)^c_l Q^c_l}$ and that of D_X is $\frac{p(X)^c_l X^c_l}{p(Q)^c_l Q^c_l}$. We can use (B.25) and (B.26) in (B.45)to obtain the following expression linking price of gross output and price of value added:

$$
vec\left(\hat{p^c_l}\right) = \left(I - D_X W_{2XP} W_{1XP}\right)^{-1} D_V vec\left(p(\hat{V})^c_l\right) \tag{B.46}
$$

The market clearing conditions (4.8) can be linearized as:

$$\hat{Q}_j^i = \sum_{h=1}^{n}\sum_{l=1}^{m}\frac{X_{jl}^{ih}}{Q_j^i}\hat{X}_{jl}^{ih} + \sum_{h=1}^{n}\frac{F_j^{ih}}{Q_j^i}\hat{F}_j^{ih} \tag{B.47}$$

As before, these can be written in stacked form by creating matrices S_X and S_F from the above equations to yield:

$$vec\left(\hat{Q}_l^c\right) = (S_F)_{nmXn^2m}\, vec\left(\hat{F_s^{fc}}\right) + (S_X)_{nmXn^2m^2}\, vec\left(\hat{X}_{sl}^{fc}\right) \tag{B.48}$$

C Derivations of the expressions for GVC-REER and Q-TREER((5.1) and (5.5))<Not for publication>

From (B.48) and (B.42) we get

$$vec\left(\hat{Q}_l^c\right)[I_{nm} - S_X C_2] = (S_X Z_X + S_F Z_F)\, vec\left(\hat{p}_l^c\right) + S_F Y_3 vec\left(\hat{F}^c\right) \tag{C.1}$$

Using (B.46) in (C.1)and rearranging we get:

$$\begin{aligned} vec\left(\hat{Q}_l^c\right) \;&=\; [I_{nm} - S_X C_2]^{-1}(S_X Z_X + S_F Z_F)(I - D_X W_{2XP} W_{1XP})^{-1} D_V vec\left(p(\hat{V})_l^i\right) \\ &\quad + [I_{nm} - S_X C_2]^{-1} S_F Y_3 vec\left(\hat{F}^i\right) \end{aligned} \tag{C.2}$$

This is equation (5.5) in the main text.

Next, starting from the linearized production function $vec\left(\hat{Q}_l^c\right) = D_v vec\left(\hat{V}_l^c\right) + D_X vec\left(\hat{X}_l^c\right)$ we first use (B.23) and (B.22) to get:

$$vec\left(\hat{Q}_l^c\right) = D_v vec\left(\hat{V}_l^c\right) + D_X W_{2XX} W_{1XX} vec\left(\hat{X}_{sl}^{ic}\right) \tag{C.3}$$

substituting (B.42) in (C.3) and rearranging we get:

$$vec\left(\hat{Q}_l^c\right)[I - D_X W_{2XX} W_{1XX} C_2] = D_v vec\left(\hat{V}_l^c\right) + D_X W_{2XX} W_{1XX} Z_X vec\left(\hat{p}_l^c\right) \tag{C.4}$$

It can be shown that $W_{2XX}W_{1XX}C_2 = I$ and hence $[I - D_X W_{2XX} W_{1XX} Z_4 Z_6] = D_v$ so that the above expression simplifies to:

$$vec\left(\hat{Q}_l^c\right) = vec\left(\hat{V}_l^c\right) + D_V^{-1} D_X W_{2XX} W_{1XX} Z_X (I - D_X W_{2XP} W_{1XP})^{-1} D_V vec\left(p(\hat{V})_l^c\right) \tag{C.5}$$

eliminating $vec\left(\hat{Q}_l^c\right)$ from (C.2) and (C.5) we get:

$$\begin{aligned} vec\left(\hat{V}_l^c\right) &= \left\{(I_{nm} - S_X C_2)^{-1}(S_F Z_F + S_X Z_X) - D_v^{-1} D_X W_{2XX} W_{1XX} Z_X\right\}(I - D_X W_{2XP} W_{1XP})^{-1} D_V vec\left(\hat{p}_l^{vc}\right) \\ &+ (I - S_X Z_4 Z_6)^{-1} S_F Y_3 vec\left(\hat{F}^c\right) \end{aligned} \tag{C.6}$$

It is easy to show the following identities:

$$(I_{nm} - S_X C_2)^{-1} = D_Q^{-1} B D_Q \tag{C.7}$$

$$(I - D_X W_{2XP} W_{1XP})^{-1} = B' \tag{C.8}$$

Substituting (C.7) and (C.8) in (C.6) we get (5.1) in the main text, with:

$$W_V = \left[D_Q^{-1} B D_Q (S_F Z_F + S_X Z_X) - D_V^{-1} D_X W_{2XX} W_{1XX} Z_X\right] B' D_V \tag{C.9}$$

D Proofs of Propositions<Not for publication>

D.1 Sketch of Proof of Proposition 5.1

In this appendix we sketch the proof of proposition 5.1. Since the underlying intuition is preserved in the case with $m = 1$, we will sketch the proof for this simplified case.

The expression for the weighting matrix is given by:

$$w = \left\{D_Q^{-1} B D_Q (S_F Z_F + S_X Z_X) - D_v^{-1} D_X W_{2XX} W_{1XX} Z_X\right\} B' D_V \tag{D.1}$$

As shown in proposition (6.1), under the constant elasticity assumption and $m = 1$, the GVC-REER weighting matrix reduces to VAREER weighting matrix defined in Bems and Johnson (2012), which according to equation (18) in that paper is given by

$$w = -I + D_Q^{-1} B D_Q S_F M_2 B' D_v \tag{D.2}$$

define the matrices

$Z_1 = Z_4 = 1_n \otimes I_n \equiv M_2$

$Z_2 = Z_5 = I_n \otimes 1_n \equiv M_1$

Under the constant elasticity assumption, from (B.43) and (B.35) we have:

$$\begin{aligned} Z_X &= \sigma(M_2 - M_1) && \text{(D.3)} \\ Z_F &= \theta(M_2 W_{FP} - M_1) && \text{(D.4)} \end{aligned}$$

Taking the partial derivative of (D.1) wrt θ

$$\frac{\partial w}{\partial \theta} = D_Q^{-1} B D_Q S_F (M_2 W_{FP} - M_1) B' D_V \tag{D.5}$$

using (D.2) in (D.5) , the following relationship holds for the off diagonal elements of w

$$\frac{\partial w^{ij}}{\partial \theta} = w^{ij} - \left[D_Q^{-1} B D_Q S_F M_1 B' D_V \right]_{ij}, i \neq j \tag{D.6}$$

Simplifying the last term in the above expression gives (5.3) in the main text.

D.2 Proof of Proposition (6.1):

Part 1.

the GVC-REER weighting matrix under (A2) is given by:

$$W_V = \left\{ (I - S_X Z_4 Z_6)^{-1} - D_v^{-1} D_X W_{2XX} W_{1XX} Z_X \right\} (I - D_X W_{2Xp} W_{1Xp})^{-1} D_V \tag{D.7}$$

where

$Z_X = \sigma_1(Z_4 W_{1XP} - Z_5) + \sigma_2(Z_4 Z_6 W_{2XP} W_{1XP} - Z_4 W_{1XP}) + \sigma_3(Z_4 Z_6 - Z_4 Z_6 W_{2XP} W_{1XP})$

with

$Z_1 = 1_n \otimes I_{nm}$, $Z_2 = I_n \otimes (1_n \otimes I_m)$, $Z_3 = I_n \otimes 1_m$, $(Z_4)_{n^2m^2Xnm^2} = 1_n \otimes I_{nm^2}$, $(Z_5)_{n^2m^2Xnm} = I_n \otimes 1_n \otimes I_m \otimes 1_m$ and $Z_6 = (I_n \otimes 1_m) \otimes I_m$

for $m = 1$, the different matrices in the above equation simplify as:

$Z_1 = Z_4 = 1_n \otimes I_n \equiv M_2$[24]

$Z_2 = Z_5 = I_n \otimes 1_n \equiv M_1$

$Z_3 = Z_6 = I_n$

$W_{2FP} = W_{2XX} = W_{2XP} = I_n$

$D_X W_{1Xp} = \Omega'$, where Ω is the country level input output matrix with $\Omega_{ij} = \frac{p^i X_{ij}}{p^j Q_j}$

$$\begin{aligned}
Z_X &= \sigma_1(Z_4 W_{1XP} - Z_5) + \sigma_2(Z_4 Z_6 W_{2XP} W_{1XP} - Z_4 W_{1XP}) + \sigma_3(Z_4 Z_6 - Z_4 Z_6 W_{2XP} W_{1XP}) \\
&= \sigma_1(M_2 W_{1XP} - M_1) + \sigma_2(M_2 W_{1XP} - M_2 W_{1XP}) + \sigma_3(M_2 - M_2 W_{1XP}) \\
&= \sigma_1(M_2 W_{1XP} - M_1) + \sigma_3(M_2 - M_2 W_{1XP}) \qquad \text{(D.8)}
\end{aligned}$$

[24]In this section the matrices M_1 and M_2 are as defined in Bems and Johnson (2012) and are different from the ones defined earlier in this paper.

$$
\begin{aligned}
Z_F &= \theta_1(Z_1 W_{1FP} - Z_2) + \theta_2\left(Z_1 Z_3 W_{2FP} W_{1FP} - Z_1 W_{1FP}\right) \\
&= \theta_1(M_2 W_{FP} - M_1)
\end{aligned}
$$

Substituting all these in the expression for Z_{Vclp} we get

$$
\begin{aligned}
W_V &= -\theta_1\left(I - S_X M_2\right)^{-1} S_F(M_1 - M_2 W_{FP})(I-\Omega')^{-1} D_V \\
&+ \left(I - S_X M_2\right)^{-1} S_X\left[\sigma_1(M_2 W_{1XP} - M_1) + \sigma_3(M_2 - M_2 W_{1XP})\right](I-\Omega')^{-1} D_V \\
&- D_V^{-1} D_X W_X\left[\sigma_1(M_2 W_{1XP} - M_1) + \sigma_3(M_2 - M_2 W_{1XP})\right](I-\Omega')^{-1} D_V
\end{aligned}
$$

This is the same as equation (33) in section 5 of Bems and Johnson (2012) IOREER-BJ. Part 2 and 3 follow directly from Bems and Johnson (2012).

Part 4:

The IMF manufacturing weights are given by (Bayoumi et al. (2005))

$$
W^{ij}_{imfm} = \frac{\sum_k w^{ik} s^{jk}}{\sum_k w^{ik}(1 - s^{ik})} \tag{D.9}
$$

where $s^{jk} = \frac{sales^{jk}}{\sum_l sales^{lk}}$ and $w^{ik} = \frac{sales^{ik}}{\sum_n sales^{in}}$($sales^{ij}$ denotes gross sales from country i to country j)

Substituting the expressions for s^{jk} and w^{ik} in W^{ij} and simplifying we get:

$$
W^{ij}_{imfm} = \frac{1}{T_i^{imfm}} \sum_k \left(\frac{sales^{ik}}{\sum_n sales^{in}}\right)\left(\frac{sales^{jk}}{\sum_l sales^{lk}}\right) \tag{D.10}
$$

where

$$
T_i^{imfm} = 1 - \sum_k \left(\frac{sales^{ik}}{\sum_n sales^{in}}\right)\left(\frac{sales^{ik}}{\sum_l sales^{lk}}\right) \tag{D.11}
$$

From parts 1-3 we know that under (A1), (A2) TEER and VAREER-BJ are equivalent and given by equation (24) in BJ which is reproduced below.

$$
W^{ij}_{BJ} = \frac{1}{T_i^{BJ}} \sum_k \left(\frac{p^{iv} V^{ik}}{P^{iv} V^i}\right)\left(\frac{p^{jv} V^{jk}}{P^k F^k}\right) \tag{D.12}
$$

with $T_i^{BJ} = \sum_k \left(\frac{p^{iv} V^{ik}}{P^{iv} V^i}\right)\left(\frac{p^{iv} V^{ik}}{P^k F^k}\right)$

Under the assumption of no intermediates (A3)we have:

- $p^{iv} = p^i$, $Q^i = V^i$, $V^{ik} = F^{ik}$

- $sales^{ik} = p^{iv}V^{ik} = p^{i}V^{ik}$
- $\sum_n sales^{in} = \sum_n p^{iv}V^{in} = p^{iv}V^{i}$
- $\sum_l sales^{lk} = \sum_l p^{lv}V^{lk} = P^{k}F^{k}$

Substituting these in (D.10) and (D.11)

$W^{ij}_{imfm} = W^{ij}_{BJ}$

Finally, using $\alpha_c = \alpha_T = 0$ we have

$W^{ij}_{imf} = W^{ij}_{BJ}$

The equivalence of IMF-REER to GOOD-SREER and IRER follows in a straightforward manner from the respective papers (Bayoumi et al. (2013) and Thorbecke (2011))

D.3 Proof of Proposition 7.1

Part 1

We start with the following expression for GVC-REER weights at the country-sector level (D.7). under the constant elasticity assumption:

$$\begin{aligned} Z_X &= -Z_5 + Z_4 Z_6 && \text{(D.13)} \\ Z_F &= -Z_2 + Z_1 Z_3 W_{2FP} W_{1FP} && \text{(D.14)} \end{aligned}$$

Here, without loss of generality we can assume that the elasticity is 1.

$$\begin{aligned} (I - S_X Z_4 Z_6)^{-1} &= D_Q^{-1} B D_Q \equiv \lambda && \text{(D.15)} \\ (I - D_X W_{2XP} W_{1XP})^{-1} &= B' && \text{(D.16)} \end{aligned}$$

Substituting (D.13), (D.14), (D.15) and (D.16) in (D.7)

$$\begin{aligned} W_V &= \left[(\lambda(S_F Z_F + S_X Z_X) - D_V^{-1} D_X W_{2XX} W_{1XX} Z_X \right] B' D_V \\ &= \lambda S_F Z_1 Z_3 W_{2FP} W_{1FP} B' D_v + \left[\lambda S_X Z_4 Z_6 - \lambda (S_F Z_2 + S_X Z_5) - D_V^{-1} D_X W_{2XX} W_{1XX} Z_X \right] B' D_V && \text{(D.17)} \end{aligned}$$

Using the identities $S_F Z_2 + S_X Z_5 = I$ and $D_V - D_X W_{2XX} W_{1XX} Z_X = (I - A)' = B'^{-1}$, we can show that the second term in (D.17) is the identity matrix, so that (D.17) reduces to:

$$\begin{aligned} W_V &= -I_{nm} + \lambda S_F Z_1 Z_3 W_{2FP} W_{1FP} [B_l^c]' D_v && \text{(D.18)} \\ &= -I_{nm} + M_{1m} M_{2m} \end{aligned}$$

where

$$\begin{aligned} M_{1m} &= \lambda S_F Z_1 Z_3 \\ M_{2m} &= W_{2FP} W_{1FP} [B_l^c]' D_v \end{aligned}$$

Next, the country level weights (which correspond to VAREER in Bems and Johnson (2012)) are given by:

$$W_V^1 = \left\{ \left(I - S_X^1 Z_4^1 Z_6^1\right)^{-1} \left(S_F^1 Z_F^1 + S_X^1 Z_X^1\right) - (D_v^1)^{-1} D_X^1 W_{1XX} Z_X^1 \right\} \left(I - D_X^1 W_{1Xp}\right)^{-1} D_V^1 \quad \text{(D.19)}$$

(where the superscript 1 on the matrices on the RHS of (D.19) indicates that the matrix corresponds to the case where $m = 1$)

Following steps similar to those used to derive (D.18) we can get an analogous expression:

$$\begin{aligned} W_V^1 &= -I_n + \lambda^1 S_F^1 Z_1^1 Z_3^1 W_{1FP}^1 [B^c]' D_v \quad \text{(D.20)} \\ &= -I_n + M_1 M_2 \end{aligned}$$

where

$$\begin{aligned} M_1 &= \lambda^1 S_F^1 Z_1^1 Z_3^1 \quad \text{(D.21)} \\ M_2 &= W_{1FP}^1 [B^c]' D_v^1 \end{aligned}$$

The 2 country level weights are equal iff

$$R_V W_V R_g = W_V(CG) \quad \text{(D.22)}$$

'

Since $R_V R_g = I_n$, a necessary and sufficient condition for (D.22) to hold is :

$$(R_V M_{1m})(M_{2m} R_g) = M_1 M_2 \quad \text{(D.23)}$$

$$
\begin{aligned}
(R_V M_{1m})_{ij} &= \sum_{c=1}^{n}\sum_{l=1}^{m}\sum_{s=1}^{m}\left(\frac{v_s^i b_{sl}^{ic} F_l^{cj}}{p^{vi}V^i}\right)\\
(M_{2m}R_g)_{ij} &= \sum_{c=1}^{n}\sum_{l=1}^{m}\sum_{s=1}^{m}\left(\frac{v_s^j b_{sl}^{jc} F_l^{ci}}{P^i F^i}\right)\\
(M_1)_{ij} &= \sum_{c=1}^{n}\left(\frac{v^i b^{ic} F^{cj}}{p^{vi}V^i}\right)\\
(M_2)_{ij} &= \sum_{c=1}^{n}\left(\frac{v^j b^{jc} F^{ci}}{P^i F^i}\right)
\end{aligned}
$$

here $v_s^i = \left(\frac{p_s^{vi}V_s^i}{p_s^i Q_s^i}\right)$.
From these expressions it is clear that the condition (D.23) is satisfied for all values if and only if

$$
v^i \sum_{c=1}^{n} b^{ic} f^{cj} = \sum_{l=1}^{m} v_s^i \sum_{c=1}^{n}\sum_{s=1}^{m} b_{ls}^{ic} F_s^{cj} \forall i, j \tag{D.24}
$$

or stacking these conditions in matrix notation:

$$
diag[v^c]_{nXn}[B^c]_{nXn}[F^C]_{nXn} = (M_V)_{nXnm} diag[v_l^c]_{nmXnm}[B_l^c]_{nmXnm}[F_l^c]_{nmXn} \tag{D.25}
$$

which is the same as (7.12) in the main text.

Interpretation in the case of constant elasticity

Under the assumption that all elasticities (both in production and consumption) are the same, we can interpret the country-sector level weights purely in terms of value added trade flows. Suppose the common elasticity is η . Without loss of generality we can assume η to be unity since it factors out. Then the weighting matrix W can be written as above:

$$
W_V = -I_{nm} + M_1 M_2 \tag{D.26}
$$

The matrix M_1 is an nm by n matrix with each row corresponding to a unique production entity. Along this row, the n columns give the value added created by the production entity that is finally absorbed by each country. As an example, the entry corresponding to row (i, l) and column j gives the value added created by production entity (i, l) that is eventually absorbed in country j as a fraction of total value added created by the production entity (i, l). Entries in this matrix can thus be interpreted as export shares in value added terms. The corresponding mathematical expression is[25]

$$M_1((i,l),j) = \frac{v_l^i \sum_{c=1}^{n} \sum_{s=1}^{m} b_{ls}^{ic} \left(p(Q)_s^c F_s^{cj}\right)}{p(V)_l^i V_l^i} \tag{D.27}$$

where $v_l^i = \frac{p_l^{vi} V_l^i}{p_l^i Q_l^i}$. For later, it is convenient to write this expression compactly as:

$$M_1((i,l),j) = \frac{p(V)_l^i V_l^{ij}}{p(V)_l^i V_l^i} \tag{D.28}$$

where $p_l^{vi} V_l^{ij}$ is the value added created by production entity (i,l) that is finally absorbed in country j.

Matrix M_2 is an n by nm matrix with each column corresponding to a unique production entity and each row containing the value added created by the entity corresponding to the column that is absorbed in each country, as a fraction of the total final demand in that country. As an example, the entry corresponding to column (i,l) and row j gives the value added created by production entity (i,l) that is ultimately absorbed in country j as a fraction of total final demand of country j. The corresponding mathematical expression is :

$$M_2(j,(i,l)) = \frac{v_l^i \sum_{c=1}^{n} \sum_{s=1}^{m} b_{ls}^{ic} \left(p(Q)_s^c F_s^{cj}\right)}{P^j F^j} \tag{D.29}$$

As above, it turns out to be more convenient to rewrite the above expression in short-hand notation as follows:

$$M_2(j,(i,l)) = \frac{p(V)_l^i V_l^{ij}}{P^j F^j} \tag{D.30}$$

Using the generic terms from (D.28) and (D.30)we can write the weight assignment by country sector (h,l) to country-sector (c,s) where $(h,l) \neq (c,s)$ as follows:

$$w_{ls}^{hc} = \sum_{k=1}^{n} \left[\frac{\left(p(V)_l^h V_l^{hk}\right) \left(p(V)_s^c V_s^{ck}\right)}{\left(p(V)_l^h V_l^h\right) \left(P^K F^k\right)} \right], (h,l) \neq (c,s) \tag{D.31}$$

where we use lower case w to denote constant elasticity weights. This is a generalized form of equation 3.8 which was derived in the context of a simplified model and the intuition is similar. In particular, the weight assigned by country sector (h,l) to country-sector (c,s) where $(h,l) \neq (c,s)$ is a weighted sum of the value added created by country-sector (c,s) and absorbed by each of the countries $k(=1,..,n)$, where the weights are given by the value added created by (h,l) that

[25]The raw expression of the matrix M_1 is $\frac{\sum_{c=1}^{n} \sum_{s=1}^{m} b_{ls}^{ic} p(Q)_s^c F_s^{cj}}{p(Q)_l^i Q_l^i}$. Multiplying and dividing by $v_l^i = \frac{p(V)_l^i V_l^i}{p(Q)_l^i Q_l^i}$ yields the expression below.

is absorbed in the same country k. This captures both mutual and third country competition, because the weight is high if both $\left(p(V)_l^h V_l^{hk}\right)$ and $\left(p(V)_s^c V_s^{ck}\right)$ are high, which happens when both (h, l) and (c, s) have a high share of value added exports to country k.

Part 2:

Under (A3), (A4) and $\theta_1 = \theta_2$ (=1(wlog)) we have,
$diag[v^c]_{nXn} = [B^c]_{nXn} = I_n$,
$diag[v_l^c]_{nmXnm} = [B_l^c]_{nmXnm} = I_{nm}$
$(M_V)_{nXnm}[F_l^c]_{nmXn} = [F^C]_{nXn}$
With these simplifications condition (D.25) is automatically satisfied and hence GVC-REER(CG) is equivalent to VAREER.

(D.32)

E Illustration of Role of Elasticities: Example 8.1

F Estimation of elasticities <Not for publication>

F.1 Framework

The approach used here will be based on recent work by Soderbery (2013) which outlines certain drawbacks in the preceding two papers and proposes an estimator which outperforms them. Consider a generic CES Armington aggregator defined as follows:

$$D_t = \left[\sum_{k \in K} (w_k)^{1/\eta} (D_{kt})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{F.1}$$

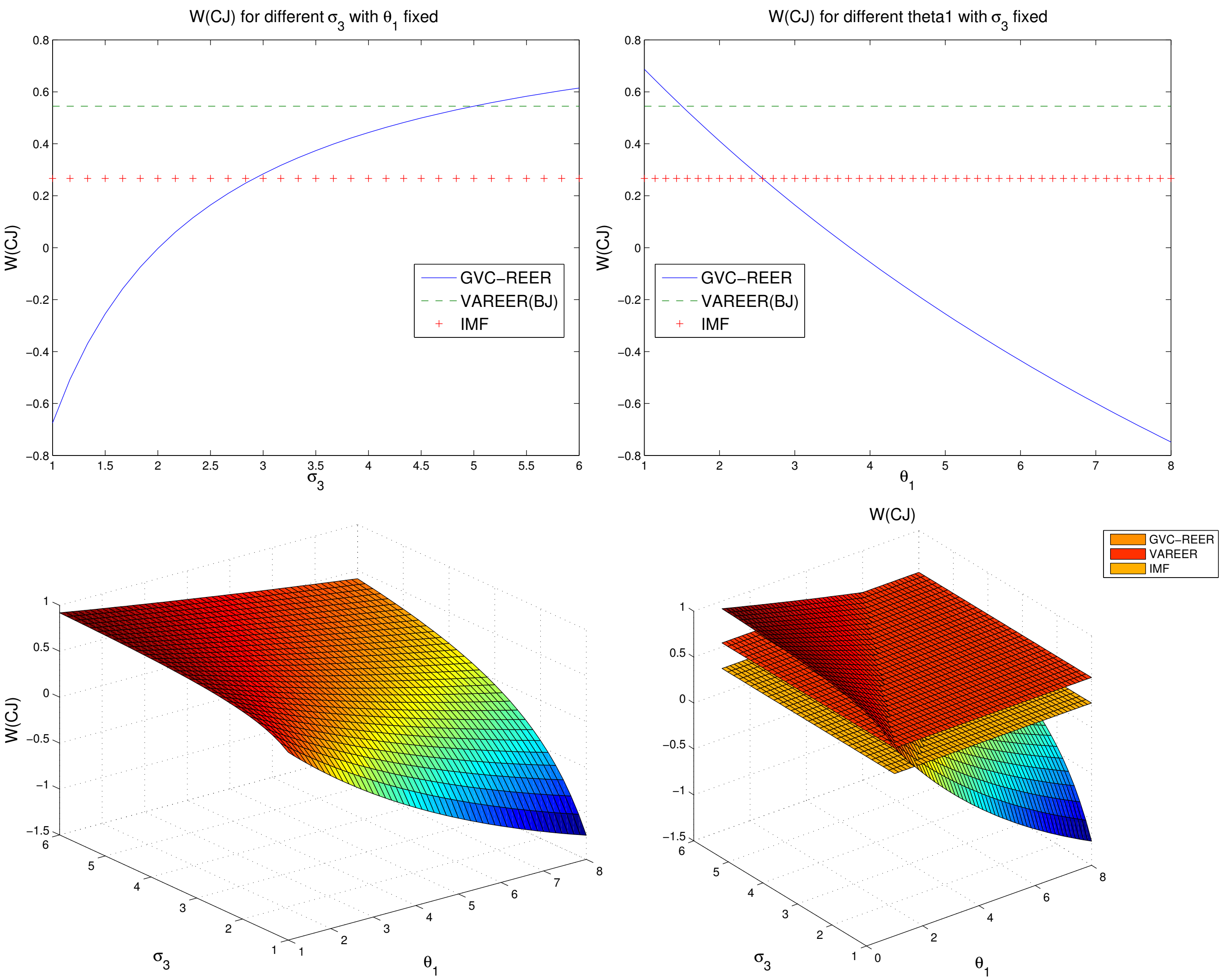
The objective is to estimate the demand elasticity η. The double differenced demand equation in terms of expenditure shares is given by[26]:

$$\triangle^r ln(s_{kt}) = -(\eta - 1)\triangle^r ln(p_{kt}) + \epsilon_{kt}^r \tag{F.2}$$

where $\triangle^r ln(x_{kt}) = \triangle ln(x_{kt}) - \triangle ln(x_{rt})$ and $\triangle ln(x_{jt}) = ln(x_{jt}) - ln(x_{j(t-1)}), x = s, p$ r is called a reference variety and is typically chosen to be the one with the largest share . s_{kt} is the expenditure

[26]See Soderbery (2013) , Broda and Weinstein (2006) or Feenstra (1994) for further details including the actual derivation

Figure 7 *– Illustration of Role of Elasticities: Example 8.1*



share of the k^{th} variety and is given by:

$$s_{kt} = \frac{p_{kt} D_{kt}}{\sum_{k \in K} p_{kt} D_{kt}} \tag{F.3}$$

Next, given a supply curve with elasticity ρ, the supply curve in terms of differenced shares and prices can be written as:

$$\triangle^r ln(p_{kt}) = \left(\frac{\rho}{1+\rho} \right) \triangle^r ln(s_{kt}) + \delta^r_{kt} \tag{F.4}$$

If the demand and supply disturbances are independent across time, then the 2 equations can be multiplied and scaled to yield:

$$Y_{kt} = \theta_1 Z_{1kt} + \theta_2 Z_{2kt} + u_{kt} \tag{F.5}$$

where $Y_{kt} = (\triangle^r ln(p_{kt}))^2$, $Z_{1kt} = (\triangle^r ln(s_{kt}))^2$, $Z_{2kt} = (\triangle^r ln(p_{kt}))(\triangle^r ln(s_{kt}))$, and $u_{kt} = \frac{\epsilon^r_{kt} \delta^r_{kt}}{1-\phi}$.

Further, the parameters of this regression model can be mapped to the primitive parameters of the demand and supply system as follows:

$\phi = \frac{\rho(\eta-1)}{1+\rho\eta} \in [0, \frac{\sigma-1}{\sigma})$

$\theta_1 = \frac{\phi}{(\eta-1)^2(1-\phi)}$ $\theta_2 = \frac{2\phi-1}{(\eta-1)(1-\phi)}$

Consistent estimates of θ_1 can be obtained by using the moment condition $E(u_{kt}) = 0$, where consistency relies on $T \to \infty$. [27] If standard procedures (2SLS or LIML) yield a value of θ_1 that gives imaginary values for η and ρ or values with the wrong sign, then the grid search or the non-linear search method of Soderbery (2013) can be used.

F.2 Implementation:

We construct sectoral price indices for all cells in the WIOD input output table using the tables in previous year prices. For a fixed production entity(identified by the country-sector pair (c, l)) and a fixed sector s, table 17 shows how the estimation of the different elasticities in the model maps onto the procedure outlined above.

[27] Given the nature of the data, the value of T is typically very small. For example Soderbery (2013) uses an unbalanced panel with 15 years of data

Table 17 – Elasticity Estimation

	D	D_k	p_k
Production elasticities			
$\sigma^1_s(c,l)$	$(X(f)^c_{sl})$	(X^{kc}_{sl})	$p(Q)^k_s$
$\sigma^{1h}_s(c,l)$	(X^c_{sl})	$(X^{cc}_{sl}),(X(f)^c_{sl})$	$p(Q)^k_s$
$\sigma^2(c,l)$	(X^c_l)	(X^c_{kl})	$p(X)^c_{kl}$
$\sigma^3(c,l)$	(Q^c_l)	(X^c_l, V^c_l)	$(p(Q)^c_l, p(V)^c_l)$
Consumption elasticities			
$\theta^1_s(c)$	$(F(f)^c_s)$	(F^{kc}_s)	$p(Q)^k_s$
$\theta^{1h}_s(c)$	(F^c_s)	$(F^c_s),(F(f)^c_s)$	$p(Q)^k_s$
$\theta^1(c)$	(F^c)	(F^c_k)	P^c_k

This table shows how the model in section 4 maps into the general framework for estimation of elasticities discussed in section F

G Bootstrap moments of elasticities <Not for publication>

Table 18 – Summary of Bootstrap moments

	Consumption Elasticities			Production Elasticities			
	θ^1	θ^{1h}	θ^2	σ^1	σ^{1h}	σ^2	σ^3
15th percentile	1. 774	1. 174	1. 04	1. 65	1. 782	1. 008	0.867
median	9. 876	7. 438	1. 527	7. 88	7. 7	3. 816	**1. 015**
85th percentile	82. 535	65. 550	4. 73	67. 22	37. 607	14. 553	1. 501
sample size	1435(=35*41)	41	41	2450	1435	1435	41

Note: the table reports the percentiles of bootstrap medians, for example, for θ^1, 1. 774 is the 15th percentile of the distribution of medians of the 1435 θ^1 bootstrap distributions. We propose this quantity as our point estimate as they are more stable than the LIML point estimate. The moments reported above are based on 50 iterations.
Statistics for σ^1 in this and the next table are based only on observations for 2 countries (China and the US), i. e $<$ 5percent of the total number of possible observations.

H Description of matrices defined in the main text with examples for the case with $n = m = 2$ <Not for publication>

This appendix includes details of each of the matrices as they appear in the text, including dimension and content. As always, n stands for the number of countries and m for the number of sectors within each country. An example with $n = m = 2$ is provided in each case.

H.1 Matrix $(W_{1XX})_{nm^2Xn^2m^2}$

$$(W_{1XX})_{nm^2Xn^2m^2} = [\left(W^1_{1XX}\right)_{nm^2Xnm^2}, \left(W^2_{1XX}\right)_{nm^2Xnm^2}, .., (W^n_{1XX})_{nm^2Xnm^2}] \tag{H.1}$$

$$(W^i_{1XX})_{nm^2Xnm^2} = \begin{pmatrix} (diag(W^{i1}_{1XX}))_{m^2Xm^2} & 0_{m^2Xm^2} & .. & 0_{m^2Xm^2} \\ 0_{m^2Xm^2} & (diag(W^{i2}_{1XX}))_{m^2Xm^2} & .. & 0_{m^2Xm^2} \\ : & : & : & : \\ 0_{m^2Xm^2} & 0_{m^2Xm^2} & .. & (diag(W^{in}_{1XX}))_{m^2Xm^2} \end{pmatrix} \tag{H.2}$$

$$(W^{ij}_{1XX})_{m^2X1} = \left[\left\{\left(\frac{p^1_1 X^{ij}_{11}}{q^j_{11}X^j_{11}}\right), \left(\frac{p^1_1 X^{ij}_{12}}{q^j_{12}X^j_{12}}\right), .., \left(\frac{p^1_1 X^{ij}_{1m}}{q^j_{1m}X^j_{1m}}\right)\right\}, .., \left\{\left(\frac{p^1_m X^{ij}_{m1}}{q^j_{m1}X^j_{m1}}\right), \left(\frac{p^1_m X^{ij}_{m2}}{q^j_{m2}X^j_{m2}}\right), .., \left(\frac{p^1_m X^{ij}_{mm}}{q^j_{mm}X^j_{mm}}\right)\right\}\right]$$

Example with $n = m = 2$

$$\begin{aligned}
W^{11}_{1XX} &= \left[\left(\frac{p^1_1 X^{11}_{11}}{q^1_{11}X^1_{11}}\right), \left(\frac{p^1_1 X^{11}_{12}}{q^1_{12}X^1_{12}}\right), \left(\frac{p^1_2 X^{11}_{21}}{q^1_{21}X^1_{21}}\right), \left(\frac{p^1_2 X^{11}_{22}}{q^1_{22}X^1_{22}}\right)\right] \\
W^{12}_{1XX} &= \left[\left(\frac{p^1_1 X^{12}_{11}}{q^2_{11}X^2_{11}}\right), \left(\frac{p^1_1 X^{12}_{12}}{q^2_{12}X^2_{12}}\right), \left(\frac{p^1_2 X^{12}_{21}}{q^2_{21}X^2_{21}}\right), \left(\frac{p^1_2 X^{12}_{22}}{q^2_{22}X^2_{22}}\right)\right] \\
W^{21}_{1XX} &= \left[\left(\frac{p^2_1 X^{21}_{11}}{q^1_{11}X^1_{11}}\right), \left(\frac{p^2_1 X^{21}_{12}}{q^1_{12}X^1_{12}}\right), \left(\frac{p^2_2 X^{21}_{21}}{q^1_{21}X^1_{21}}\right), \left(\frac{p^2_2 X^{21}_{22}}{q^1_{22}X^1_{22}}\right)\right] \\
W^{22}_{1XX} &= \left(\frac{p^2_1 X^{22}_{11}}{q^2_{11}X^2_{11}}\right), \left(\frac{p^2_1 X^{22}_{12}}{q^2_{12}X^2_{12}}\right), \left(\frac{p^2_2 X^{22}_{21}}{q^2_{21}X^2_{21}}\right), \left(\frac{p^2_2 X^{22}_{22}}{q^2_{22}X^2_{22}}\right)
\end{aligned}$$

$$W^1_{1XX} = \begin{pmatrix}
\left(\frac{p^1_1 X^{11}_{11}}{q^1_{11}X^1_{11}}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \left(\frac{p^1_1 X^{11}_{12}}{q^1_{12}X^1_{12}}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \left(\frac{p^1_2 X^{11}_{21}}{q^1_{21}X^1_{21}}\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \left(\frac{p^1_2 X^{11}_{22}}{q^1_{22}X^1_{22}}\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \left(\frac{p^1_1 X^{12}_{11}}{q^2_{11}X^2_{11}}\right) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \left(\frac{p^1_1 X^{12}_{12}}{q^2_{12}X^2_{12}}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{p^1_2 X^{12}_{21}}{q^2_{21}X^2_{21}}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{p^1_2 X^{12}_{22}}{q^2_{22}X^2_{22}}\right)
\end{pmatrix}$$

$$W_{1XX}^{2} = \begin{pmatrix} \left(\frac{p_1^2 X_{11}^{21}}{q_{11}^1 X_{11}^1}\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{p_1^2 X_{12}^{21}}{q_{12}^1 X_{12}^1}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{p_2^2 X^{21}}{q_{21}^1 X_{21}^1}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{p_2^2 X_{22}^{21}}{q_{22}^1 X_{22}^1}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{p_1^2 X_{11}^{22}}{q_{11}^2 X_{11}^2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{p_1^2 X_{12}^{22}}{q_{12}^2 X_{12}^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{p_2^2 X_{21}^{22}}{q_{21}^2 X_{21}^2}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{p_2^2 X_{22}^{22}}{q_{22}^2 X_{22}^2}\right) \end{pmatrix}$$

$W_{XX} = [W_{1XX}^1, W_{1XX}^2]$

H.2 Matrices $(W_{2XX})_{nmXnm^2}$ and $(W_{2Xp})_{nmXnm^2}$ $((W_{2XP})_{nmXnm^2}{=}(W_{2XX})_{nmXnm^2})$

$$(W_{2Xp})_{nmXnm^2} = \begin{pmatrix} \left(W_{2Xp}^1\right)_{mXm^2} & 0_{mXm^2} & .. & 0_{mXm^2} \\ 0_{mXm^2} & \left(W_{2Xp}^2\right)_{mXm^2} & .. & 0_{mXm^2} \\ : & : & & : \\ 0_{mXm^2} & 0_{mXm^2} & .. & \left(W_{2Xp}^n\right)_{mXm^2} \end{pmatrix} \tag{H.3}$$

$$\left(W_{2Xp}^i\right)_{mXm^2} = \left(\left(W_{2Xp}^{i,1}\right)_{mXm}, \left(W_{2Xp}^{i,2}\right)_{mXm}, .., \left(W_{2Xp}^{i,m}\right)_{mXm}\right) \tag{H.4}$$

$$\left(W_{2Xp}^{i,k}\right)_{mXm} = diag\left(\frac{q_{k1}^i X_{k1}^i}{q_1^i X_1^i}, \frac{q_{k2}^i X_{k2}^i}{q_2^i X_2^i}, .., \frac{q_{km}^i X_{km}^i}{q_m^i X_m^i}\right) \tag{H.5}$$

Example with $n = m = 2$

$$\begin{aligned} \left(W_{2Xp}^{1,1}\right)_{mXm} &= \begin{pmatrix} \frac{q_{11}^1 X_{11}^1}{q_1^1 X_1^1} & 0 \\ 0 & \frac{q_{12}^1 X_{12}^1}{q_2^1 X_2^1} \end{pmatrix} \\ \left(W_{2Xp}^{1,2}\right)_{mXm} &= \begin{pmatrix} \frac{q_{21}^1 X_{21}^1}{q_1^1 X_1^1} & 0 \\ 0 & \frac{q_{22}^1 X_{22}^1}{q_2^1 X_2^1} \end{pmatrix} \\ \left(W_{2Xp}^{2,1}\right)_{mXm} &= \begin{pmatrix} \frac{q_{11}^2 X_{11}^2}{q_1^2 X_1^2} & 0 \\ 0 & \frac{q_{12}^2 X_{12}^2}{q_2^2 X_2^2} \end{pmatrix} \\ \left(W_{2Xp}^{2,2}\right)_{mXm} &= \begin{pmatrix} \frac{q_{21}^2 X_{21}^2}{q_1^2 X_1^2} & 0 \\ 0 & \frac{q_{22}^2 X_{22}^2}{q_2^2 X_2^2} \end{pmatrix} \end{aligned}$$

$$\left(W_{2Xp}^1\right)_{mXm^2} = \begin{pmatrix} \frac{q_{11}^1 X_{11}^1}{q_1^1 X_1^1} & 0 & \frac{q_{21}^1 X_{21}^1}{q_1^1 X_1^1} & 0 \\ 0 & \frac{q_{12}^1 X_{12}^1}{q_2^1 X_2^1} & 0 & \frac{q_{22}^1 X_{22}^1}{q_2^1 X_2^1} \end{pmatrix}$$

$$\left(W_{2Xp}^1\right)_{mXm^2} = \begin{pmatrix} \frac{q_{11}^2 X_{11}^2}{q_1^2 X_1^2} & 0 & \frac{q_{21}^2 X_{21}^2}{q_1^2 X_1^2} & 0 \\ 0 & \frac{q_{12}^2 X_{12}^2}{q_2^2 X_2^2} & 0 & \frac{q_{22}^2 X_{22}^2}{q_2^2 X_2^2} \end{pmatrix}$$

$$(W_{2Xp})_{nmXnm^2} = \begin{pmatrix} \frac{q_{11}^1 X_{11}^1}{q_1^1 X_1^1} & 0 & \frac{q_{21}^1 X_{21}^1}{q_1^1 X_1^1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q_{12}^1 X_{12}^1}{q_2^1 X_2^1} & 0 & \frac{q_{22}^1 X_{22}^1}{q_2^1 X_2^1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q_{11}^2 X_{11}^2}{q_1^2 X_1^2} & 0 & \frac{q_{21}^2 X_{21}^2}{q_1^2 X_1^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q_{12}^2 X_{12}^2}{q_2^2 X_2^2} & 0 & \frac{q_{22}^2 X_{22}^2}{q_2^2 X_2^2} \end{pmatrix}$$

Matrix $(W_{1XP})_{nm^2Xnm}$

$$(W_{1XP})_{nm^2Xnm} = \begin{pmatrix} \left(W_{1Xp}^{11}\right)_{m^2Xm} & \left(W_{1Xp}^{21}\right)_{m^2Xm} & .. & \left(W_{1Xp}^{n1}\right)_{m^2Xm} \\ \left(W_{1Xp}^{12}\right)_{m^2Xm} & \left(W_{1Xp}^{22}\right)_{m^2Xm} & .. & \left(W_{1Xp}^{n2}\right)_{m^2Xm} \\ : & : & : & : \\ \left(W_{1Xp}^{1n}\right)_{m^2Xm} & \left(W_{1Xp}^{2n}\right)_{m^2Xm} & & \left(W_{1Xp}^{nn}\right)_{m^2Xm} \end{pmatrix} \tag{H.6}$$

$$\left(W_{1Xp}^{ij}\right)_{m^2Xm} = \begin{pmatrix} \left(W_{1Xp}^{ij}\right)_{mX1}^1 & 0_{mX1} & .. & 0_{mX1} \\ 0_{mX1} & \left(W_{1Xp}^{ij}\right)_{mX1}^2 & .. & 0_{mX1} \\ : & : & : & : \\ 0_{mX1} & 0_{mX1} & .. & \left(W_{1Xp}^{ij}\right)_{mX1}^m \end{pmatrix} \tag{H.7}$$

$$\left(W_{1Xp}^{ij}\right)_{mX1}^k = \left(\frac{p_k^i X_{k1}^{ij}}{q_{k1}^j X_{k1}^j}, \frac{p_k^i X_{k2}^{ij}}{q_{k2}^j X_{k2}^j}, .., \frac{p_k^i X_{km}^{ij}}{q_{km}^j X_{km}^j}\right)' \tag{H.8}$$

Example with $n = m = 2$

$$
\begin{aligned}
\left(W^{11}_{1Xp}\right)^1 &= \left(\frac{p^1_1 X^{11}_{11}}{q^1_{11} X^1_{11}}, \frac{p^1_1 X^{11}_{12}}{q^1_{12} X^1_{12}}\right)' \\
\left(W^{11}_{1Xp}\right)^2 &= \left(\frac{p^1_2 X^{11}_{21}}{q^1_{21} X^1_{21}}, \frac{p^1_2 X^{11}_{22}}{q^1_{22} X^1_{22}}\right)' \\
\left(W^{21}_{1Xp}\right)^1 &= \left(\frac{p^2_1 X^{21}_{11}}{q^1_{11} X^1_{11}}, \frac{p^2_1 X^{21}_{12}}{q^1_{12} X^1_{12}}\right)' \\
\left(W^{21}_{1Xp}\right)^2 &= \left(\frac{p^2_2 X^{21}_{21}}{q^1_{21} X^1_{21}}, \frac{p^2_2 X^{21}_{22}}{q^1_{22} X^1_{22}}\right)' \\
\left(W^{12}_{1Xp}\right)^1 &= \left(\frac{p^1_1 X^{12}_{11}}{q^2_{11} X^2_{11}}, \frac{p^1_1 X^{12}_{12}}{q^2_{12} X^2_{12}}\right)' \\
\left(W^{12}_{1Xp}\right)^2 &= \left(\frac{p^1_2 X^{12}_{21}}{q^2_{21} X^2_{21}}, \frac{p^1_2 X^{12}_{22}}{q^2_{22} X^2_{22}}\right)' \\
\left(W^{22}_{1Xp}\right)^1 &= \left(\frac{p^2_1 X^{22}_{11}}{q^2_{11} X^2_{11}}, \frac{p^2_1 X^{22}_{12}}{q^2_{12} X^2_{12}}\right)' \\
\left(W^{22}_{1Xp}\right)^2 &= \left(\frac{p^2_2 X^{22}_{21}}{q^2_{21} X^2_{21}}, \frac{p^2_2 X^{22}_{22}}{q^2_{22} X^2_{22}}\right)'
\end{aligned}
$$

$$\left(W^{11}_{1XP}\right)_{4X2} = \begin{pmatrix} \frac{p_1^1 X_{11}^{11}}{q_{11}^1 X_{11}^1} & 0 \\ \frac{p_1^1 X_{12}^{11}}{q_{12}^1 X_{12}^1} & 0 \\ 0 & \frac{p_2^1 X_{21}^{11}}{q_{21}^1 X_{21}^1} \\ 0 & \frac{p_2^1 X_{22}^{11}}{q_{22}^1 X_{22}^1} \end{pmatrix}$$

$$\left(W^{21}_{1XP}\right) = \begin{pmatrix} \frac{p_1^2 X_{11}^{21}}{q_{11}^1 X_{11}^1} & 0 \\ \frac{p_1^2 X_{12}^{21}}{q_{12}^1 X_{12}^1} & 0 \\ 0 & \frac{p_2^2 X_{21}^{21}}{q_{21}^1 X_{21}^1} \\ 0 & \frac{p_2^2 X_{22}^{21}}{q_{22}^1 X_{22}^1} \end{pmatrix}$$

$$\left(W^{12}_{1XP}\right) = \begin{pmatrix} \frac{p_1^1 X_{11}^{12}}{q_{11}^2 X_{11}^2} & 0 \\ \frac{p_1^1 X_{12}^{12}}{q_{12}^2 X_{12}^2} & 0 \\ 0 & \frac{p_2^1 X_{21}^{12}}{q_{21}^2 X_{21}^2} \\ 0 & \frac{p_2^1 X_{22}^{12}}{q_{22}^2 X_{22}^2} \end{pmatrix}$$

$$\left(W^{22}_{1XP}\right) = \begin{pmatrix} \frac{p_1^2 X_{11}^{22}}{q_{11}^2 X_{11}^2} & 0 \\ \frac{p_1^2 X_{12}^{22}}{q_{12}^2 X_{12}^2} & 0 \\ 0 & \frac{p_2^2 X_{21}^{22}}{q_{21}^2 X_{21}^2} \\ 0 & \frac{p_2^2 X_{22}^{22}}{q_{22}^2 X_{22}^2} \end{pmatrix}$$

$$W_{1XP} = \begin{pmatrix} \frac{p_1^1 X_{11}^{11}}{q_{11}^1 X_{11}^1} & 0 & \frac{p_1^2 X_{11}^{21}}{q_{11}^1 X_{11}^1} & 0 \\ \frac{p_1^1 X_{12}^{11}}{q_{12}^1 X_{12}^1} & 0 & \frac{p_1^2 X_{12}^{21}}{q_{12}^1 X_{12}^1} & 0 \\ 0 & \frac{p_2^1 X_{21}^{11}}{q_{21}^1 X_{21}^1} & 0 & \frac{p_2^2 X_{21}^{21}}{q_{21}^1 X_{21}^1} \\ 0 & \frac{p_2^1 X_{22}^{11}}{q_{22}^1 X_{22}^1} & 0 & \frac{p_2^2 X_{22}^{21}}{q_{22}^1 X_{22}^1} \\ \frac{p_1^1 X_{11}^{12}}{q_{11}^2 X_{11}^2} & 0 & \frac{p_1^2 X_{11}^{22}}{q_{11}^2 X_{11}^2} & 0 \\ \frac{p_1^1 X_{12}^{12}}{q_{12}^2 X_{12}^2} & 0 & \frac{p_1^2 X_{12}^{22}}{q_{12}^2 X_{12}^2} & 0 \\ 0 & \frac{p_2^1 X_{21}^{12}}{q_{21}^2 X_{21}^2} & 0 & \frac{p_2^2 X_{21}^{22}}{q_{21}^2 X_{21}^2} \\ 0 & \frac{p_2^1 X_{22}^{12}}{q_{22}^2 X_{22}^2} & 0 & \frac{p_2^2 X_{22}^{22}}{q_{22}^2 X_{22}^2} \end{pmatrix} \tag{H.9}$$

Matrix $(S_X)_{nmXn^2m^2}$

$$(S_X)_{nmXn^2m^2} = \begin{pmatrix} (S_X^1)_{mXnm^2} & 0_{mXnm^2} & .. & 0_{mXnm^2} \\ 0_{mXnm^2} & (S_X^2)_{mXnm^2} & .. & 0_{mXnm^2} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{mXnm^2} & 0_{mXnm^2} & .. & (S_X^n)_{mXnm^2} \end{pmatrix} \tag{H.10}$$

$$(S_X^i)_{mXnm^2} = \left(\left(S_X^{i1}\right)_{mXm^2}, \left(S_X^{i2}\right)_{mXm^2}, ..., \left(S_X^{in}\right)_{mXm^2} \right) \tag{H.11}$$

$$(S_X^{ij})_{mXm^2} = \begin{pmatrix} \left(S_{X1}^{ij}\right)_{1Xm} & 0_{1Xm} & .. & 0_{1Xm} \\ 0_{1Xm} & \left(S_{X2}^{ij}\right)_{1Xm} & .. & 0_{1Xm} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{1Xm} & 0_{1Xm} & .. & \left(S_{Xm}^{ij}\right)_{1Xm} \end{pmatrix} \tag{H.12}$$

$$\left(S_{Xk}^{ij}\right)_{1Xm} = \left(\frac{X_{k1}^{ij}}{Q_k^i}, \frac{X_{k2}^{ij}}{Q_k^i}, ..., \frac{X_{km}^{ij}}{Q_k^i} \right) \tag{H.13}$$

Example with $n = m = 2$:

$(S_{X1}^{11})_{1X2} = \left(\frac{X_{11}^{11}}{Q_1^1}, \frac{X_{12}^{11}}{Q_1^1} \right)$

$(S_{X2}^{11}) = \left(\frac{X_{21}^{11}}{Q_2^1}, \frac{X_{22}^{11}}{Q_2^1} \right)$

$(S_{X1}^{12}) = \left(\frac{X_{11}^{12}}{Q_1^1}, \frac{X_{12}^{12}}{Q_1^1} \right)$

$(S_{X2}^{12}) = \left(\frac{X_{21}^{12}}{Q_2^1}, \frac{X_{22}^{12}}{Q_2^1} \right)$

$(S_{X1}^{21}) = (\frac{X_{11}^{21}}{Q_1^2}, \frac{X_{12}^{21}}{Q_1^2})$

$(S_{X2}^{21}) = \left(\frac{X_{21}^{21}}{Q_2^2}, \frac{X_{22}^{21}}{Q_2^2} \right)$

$(S_{X1}^{22}) = (\frac{X_{11}^{22}}{Q_1^2}, \frac{X_{12}^{22}}{Q_1^2})$

$(S_{X2}^{22}) = (\frac{X_{21}^{22}}{Q_2^2}, \frac{X_{22}^{22}}{Q_2^2})$

$S_X^{11} = \begin{pmatrix} \frac{X_{11}^{11}}{Q_1^1} & \frac{X_{12}^{11}}{Q_1^1} & 0 & 0 \\ 0 & 0 & \frac{X_{21}^{11}}{Q_2^1} & \frac{X_{22}^{11}}{Q_2^1} \end{pmatrix}$

$S_X^{12} = \begin{pmatrix} \frac{X_{11}^{12}}{Q_1^1} & \frac{X_{12}^{12}}{Q_1^1} & 0 & 0 \\ 0 & 0 & \frac{X_{21}^{12}}{Q_2^1} & \frac{X_{22}^{12}}{Q_2^1} \end{pmatrix}$

$S_X^{21} = \begin{pmatrix} \frac{X_{11}^{21}}{Q_1^2} & \frac{X_{12}^{21}}{Q_1^2} & 0 & 0 \\ 0 & 0 & \frac{X_{21}^{21}}{Q_2^2} & \frac{X_{22}^{21}}{Q_2^2} \end{pmatrix}$

$$S_X^{22} = \begin{pmatrix} \frac{X_{11}^{22}}{Q_1^2} & \frac{X_{12}^{22}}{Q_1^2} & 0 & 0 \\ 0 & 0 & \frac{X_{21}^{22}}{Q_2^2} & \frac{X_{22}^{22}}{Q_2^2} \end{pmatrix}$$

$$S_X^{1} = \begin{pmatrix} \frac{X_{11}^{11}}{Q_1^1} & \frac{X_{12}^{11}}{Q_1^1} & 0 & 0 & \frac{X_{11}^{12}}{Q_1^1} & \frac{X_{12}^{12}}{Q_1^1} & 0 & 0 \\ 0 & 0 & \frac{X_{21}^{11}}{Q_2^1} & \frac{X_{22}^{11}}{Q_2^1} & 0 & 0 & \frac{X_{21}^{12}}{Q_2^1} & \frac{X_{22}^{12}}{Q_2^1} \end{pmatrix}$$

$$S_X^{2} = \begin{pmatrix} \frac{X_{11}^{21}}{Q_1^2} & \frac{X_{12}^{21}}{Q_1^2} & 0 & 0 & \frac{X_{11}^{22}}{Q_1^2} & \frac{X_{12}^{22}}{Q_1^2} & 0 & 0 \\ 0 & 0 & \frac{X_{21}^{21}}{Q_2^2} & \frac{X_{22}^{21}}{Q_2^2} & 0 & 0 & \frac{X_{21}^{22}}{Q_2^2} & \frac{X_{22}^{22}}{Q_2^2} \end{pmatrix}$$

$$S_X = \begin{pmatrix} \frac{X_{11}^{11}}{Q_1^1} & \frac{X_{12}^{11}}{Q_1^1} & 0 & 0 & \frac{X_{11}^{12}}{Q_1^1} & \frac{X_{12}^{12}}{Q_1^1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{X_{21}^{11}}{Q_2^1} & \frac{X_{22}^{11}}{Q_2^1} & 0 & 0 & \frac{X_{21}^{12}}{Q_2^1} & \frac{X_{22}^{12}}{Q_2^1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{X_{11}^{21}}{Q_1^2} & \frac{X_{12}^{21}}{Q_1^2} & 0 & 0 & \frac{X_{11}^{22}}{Q_1^2} & \frac{X_{12}^{22}}{Q_1^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{X_{21}^{21}}{Q_2^2} & \frac{X_{22}^{21}}{Q_2^2} & 0 & 0 & \frac{X_{21}^{22}}{Q_2^2} & \frac{X_{22}^{22}}{Q_2^2} \end{pmatrix}$$

Matrix $(S_F)_{nmXn^2m}$

$$(S_F)_{nmXn^2m} = \begin{pmatrix} (S_F^1)_{mXnm} & 0_{mXnm} & .. & 0_{mXnm} \\ 0_{mXnm} & (S_F^2)_{mXnm} & .. & 0_{mXnm} \\ : & : & : & : \\ 0_{mXnm} & 0_{mXnm} & .. & (S_F^n)_{mXnm} \end{pmatrix}$$

$$(S_F^i)_{mXnm} = ((S_F^{i1})_{mXm}, (S_F^{i2})_{mXm}, ..., (S_F^{in})_{mXm})$$

$$(S_F^{ij})_{mXm} = \begin{pmatrix} S_{F1}^{ij} & 0 & .. & 0 \\ 0 & S_{F2}^{ij} & .. & 0 \\ : & : & : & : \\ 0 & 0 & .. & S_{Fm}^{ij} \end{pmatrix}$$

$$\left(S_{Fk}^{ij}\right)_{1X1} = \left(\frac{F_k^{ij}}{Q_k^i}\right)$$

Example with $n = m = 2$:

$S_{F1}^{11} = \frac{F_1^{11}}{Q_1^1}$, $S_{F2}^{11} = \frac{F_2^{11}}{Q_2^1}$, $S_{F1}^{12} = \frac{F_1^{12}}{Q_1^1}$, $S_{F2}^{12} = \frac{F_2^{12}}{Q_2^1}$

$S_{F1}^{21} = \frac{F_1^{21}}{Q_1^2}$, $S_{F2}^{21} = \frac{F_2^{21}}{Q_2^2}$, $S_{F1}^{22} = \frac{F_1^{22}}{Q_1^2}$, $S_{F2}^{22} = \frac{F_2^{22}}{Q_2^2}$

$$S_F^{11} = \begin{pmatrix} \frac{F_1^{11}}{Q_1^1} & 0 \\ 0 & \frac{F_2^{11}}{Q_2^1} \end{pmatrix}, \; S_F^{12} = \begin{pmatrix} \frac{F_1^{12}}{Q_1^1} & 0 \\ 0 & \frac{F_2^{12}}{Q_2^1} \end{pmatrix}$$

$$S_F^{21} = \begin{pmatrix} \frac{F_1^{21}}{Q_1^2} & 0 \\ 0 & \frac{F_2^{21}}{Q_2^2} \end{pmatrix}, \; S_F^{22} = \begin{pmatrix} \frac{F_1^{22}}{Q_1^2} & \\ & \frac{F_2^{22}}{Q_2^2} \end{pmatrix}$$

$$S_F^{1} = \begin{pmatrix} \frac{F_1^{11}}{Q_1^1} & 0 & \frac{F_1^{12}}{Q_1^1} & 0 \\ 0 & \frac{F_2^{11}}{Q_2^1} & 0 & \frac{F_2^{12}}{Q_2^1} \end{pmatrix}, \; S_F^{2} = \begin{pmatrix} \frac{F_1^{21}}{Q_1^2} & 0 & \frac{F_1^{22}}{Q_1^2} & 0 \\ 0 & \frac{F_2^{21}}{Q_2^2} & & \frac{F_2^{22}}{Q_2^2} \end{pmatrix}$$

$$S_F = \begin{pmatrix} \frac{F_1^{11}}{Q_1^1} & 0 & \frac{F_1^{12}}{Q_1^1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{F_2^{11}}{Q_2^1} & 0 & \frac{F_2^{12}}{Q_2^1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{F_1^{21}}{Q_1^2} & 0 & \frac{F_1^{22}}{Q_1^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{F_2^{21}}{Q_2^2} & & \frac{F_2^{22}}{Q_2^2} \end{pmatrix}$$

Matrix $(W_{1FP})_{nmXnm}$

$$W_{1FP} = \begin{pmatrix} (W_{1FP}^1)\, mXnm \\ (W_{1FP}^2)\, mXnm \\ \vdots \\ (W_{1FP}^n)\, mXnm \end{pmatrix}$$

where $\left(W_{1FP}^k\right) = \left[\left(W_{1FP}^{1k}\right)_{mXm} \left(W_{1FP}^{2k}\right)_{mXm} \cdots \left(W_{1FP}^{nk}\right)_{mXm}\right]$

$\left(W_{1FP}^{jk}\right)_{mXm} = diag\left(\frac{p_1^j F_1^{jk}}{P_1^k F_1^k}, \frac{p_2^j F_2^{jk}}{P_2^k F_2^k}, ..., \frac{p_m^j F_m^{jk}}{P_m^k F_m^k}\right)$

Example with $n = m = 2$

$$W_{1FP} = \begin{pmatrix} \frac{p_1^1 F_1^{11}}{P_1^1 F_1^1} & 0 & \frac{p_1^2 F_1^{21}}{P_1^1 F_1^1} & 0 \\ 0 & \frac{p_2^1 F_2^{11}}{P_2^1 F_2^1} & 0 & \frac{p_2^2 F_2^{21}}{P_2^1 F_2^1} \\ \frac{p_1^1 F_1^{12}}{P_1^2 F_1^2} & 0 & \frac{p_1^2 F_1^{22}}{P_1^2 F_1^2} & 0 \\ 0 & \frac{p_2^1 F_2^{12}}{P_2^2 F_2^2} & 0 & \frac{p_2^2 F_2^{22}}{P_2^2 F_2^2} \end{pmatrix}$$

Matrix $(W_{2FP})_{nXnm}$

$W_{2FP} = [(W_{2FP}^1)_{nXm}\,(W_{2FP}^2)_{nXm}\,,..,(W_{2FP}^n)_{nXm}]$

$(W_{2FP}^i)_{nXm}$ is a matrix with the i^{th} row given by $\left(\frac{P_1^i F_1^i}{P^i F^i}, \frac{P_2^i F_2^i}{P^i F^i}, ..., \frac{P_m^i F_m^i}{P^i F^i}\right)$

Example with $n = m = 2$

$$W_{2FP} = \begin{pmatrix} \frac{P_1^1 F_1^1}{P^1 F^1} & \frac{P_2^1 F_2^1}{P^1 F^1} & 0 & 0 \\ 0 & 0 & \frac{P_1^2 F_1^2}{P^2 F^2} & \frac{P_2^2 F_2^2}{P^2 F^2} \end{pmatrix}$$

Matrix $(D_V)_{nmXnm}$

$$(D_V)_{nmXnm} = diag\left(\underbrace{\frac{p_1^{v1} V_1^1}{p_1^1 Q_1^1}, \frac{p_2^{v1} V_2^1}{p_2^1 Q_2^1}, ..., \frac{p_m^{v1} V_m^1}{p_m^1 Q_m^1}}_{mX1}, \underbrace{\frac{p_1^{v2} V_1^2}{p_1^2 Q_1^2}, \frac{p_2^{v2} V_2^2}{p_2^2 Q_2^2}, ..., \frac{p_m^{v2} V_m^2}{p_m^2 Q_m^2}}_{mX1}, .., \underbrace{\frac{p_1^{vn} V_1^n}{p_1^n Q_1^n}, \frac{p_2^{vn} V_2^n}{p_2^n Q_2^n}, ..., \frac{p_m^{vn} V_m^n}{p_m^n Q_m^n}}_{mX1}\right)$$

Example with $m = n = 2$

$$(D_V)_{4X4} = diag\left(\underbrace{\frac{p_1^{v1} V_1^1}{p_1^1 Q_1^1}, \frac{p_2^{v1} V_2^1}{p_2^1 Q_2^1}}_{2X1}, \underbrace{\frac{p_1^{v2} V_1^2}{p_1^2 Q_1^2}, \frac{p_2^{v2} V_2^2}{p_2^2 Q_2^2}}_{2X1}\right)$$

Matrix $(D_X)_{nmXnm}$

$$(D_X)_{nmXnm} = diag\left(\underbrace{\frac{q_1^1X_1^1}{p_1^1Q_1^1}, \frac{q_2^1X_2^1}{p_2^1Q_2^1}, ..., \frac{q_m^1X_m^1}{p_m^1Q_m^1}}_{mX1}, \underbrace{\frac{q_1^2X_1^2}{p_1^2Q_1^2}, \frac{q_2^2X_2^2}{p_2^2Q_2^2}, ..., \frac{q_m^2X_m^2}{p_m^2Q_m^2}}_{mX1}, .., \underbrace{\frac{q_1^nX_1^n}{p_1^nQ_1^n}, \frac{q_2^nX_2^n}{p_2^nQ_2^n}, ..., \frac{q_m^nX_m^n}{p_m^nQ_m^n}}_{mX1}\right)$$

Example with $n = m = 2$

$$(D_X)_{4X4} = diag\left(\underbrace{\frac{q_1^1X_1^1}{p_1^1Q_1^1}, \frac{q_2^1X_2^1}{p_2^1Q_2^1}}_{2X1}, \underbrace{\frac{q_1^2X_1^2}{p_1^2Q_1^2}, \frac{q_2^2X_2^2}{p_2^2Q_2^2}}_{2X1}\right)$$

I General algebra and results with Armington aggregators<Not for publication>

$$M = \left[\sum_g (w_g)^{1/\delta}(m_g)^{\frac{\delta-1}{\delta}}\right]^{\frac{\delta}{\delta-1}} \tag{I.1}$$

foc:

$$m_g = w_g \left(\frac{p_g}{p_M}\right)^{-\delta} M \tag{I.2}$$

price index:

$$P_M = \left[\sum_g (w_g)(p_g)^{1-\delta}\right]^{\frac{1}{1-\delta}} \tag{I.3}$$

linearized version:

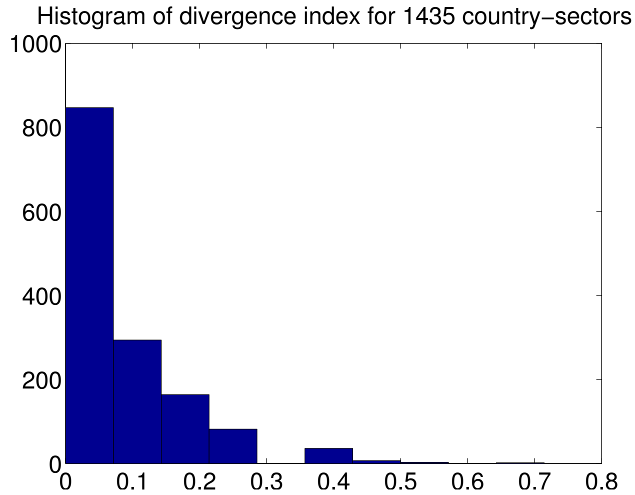
$$\hat{M} = \sum_g \left(\frac{p_g m_g}{P_M M}\right) \hat{m_g} \tag{I.4}$$

$$\hat{P_M} = \sum_g \left(\frac{p_g m_g}{P_M M}\right) \hat{p_g} \tag{I.5}$$

J List of countries and sectors <Not for publication>

	Countries		Sectors
1	'Australia'	1	'Agriculture, Hunting, Forestry and Fishing'
2	'Austria'	2	'Mining and Quarrying'
3	'Belgium'	3	'Food, Beverages and Tobacco'
4	'Bulgaria'	4	'Textiles and Textile Products'
5	'Brazil'	5	'Leather, Leather and Footwear'
6	'Canada'	6	'Wood and Products of Wood and Cork'
7	'China'	7	'Pulp, Paper, Paper , Printing and Publishing'
8	'Cyprus'	8	'Coke, Refined Petroleum and Nuclear Fuel'
9	'Czech Republic'	9	'Chemicals and Chemical Products'
10	'Germany'	10	'Rubber and Plastics'
11	'Denmark'	11	'Other Non-Metallic Mineral'
12	'Spain'	12	'Basic Metals and Fabricated Metal'
13	'Estonia'	13	'Machinery, Nec'
14	'Finland'	14	'Electrical and Optical Equipment'
15	'France'	15	'Transport Equipment'
16	'United Kingdom'	16	'Manufacturing, Nec; Recycling'
17	'Greece'	17	'Electricity, Gas and Water Supply'
18	'Hungary'	18	'Construction'
19	'Indonesia'	19	'Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel'
20	'India'	20	'Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles'
21	'Ireland'	21	'Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods'
22	'Italy'	22	'Hotels and Restaurants'
23	'Japan'	23	'Inland Transport'
24	'Korea'	24	'Water Transport'
25	'Lithuania'	25	'Air Transport'
26	'Luxembourg'	26	'Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies'
27	'Latvia'	27	'Post and Telecommunications'
28	'Mexico'	28	'Financial Intermediation'
29	'Malta'	29	'Real Estate Activities'
30	'Netherlands'	30	'Renting of M&Eq and Other Business Activities'
31	'Poland'	31	'Public Admin and Defense; Compulsory Social Security'
32	'Portugal'	32	'Education'
33	'Romania'	33	'Health and Social Work'
34	'Russia'	34	'Other Community, Social and Personal Services'
35	'Slovak Republic'	35	'Private Households with Employed Persons'
36	'Slovenia'		
37	'Sweden'		
38	'Turkey'		
39	'Taiwan'		
40	'United States'		

Figure 8 – *Divergence index at the country-sector level*



K Divergence Index for 1435 country- sectors pairs

L REER indices plots <Not for publication>

Figure 9 – *GVC-REER and Q-REER indices for all countries*

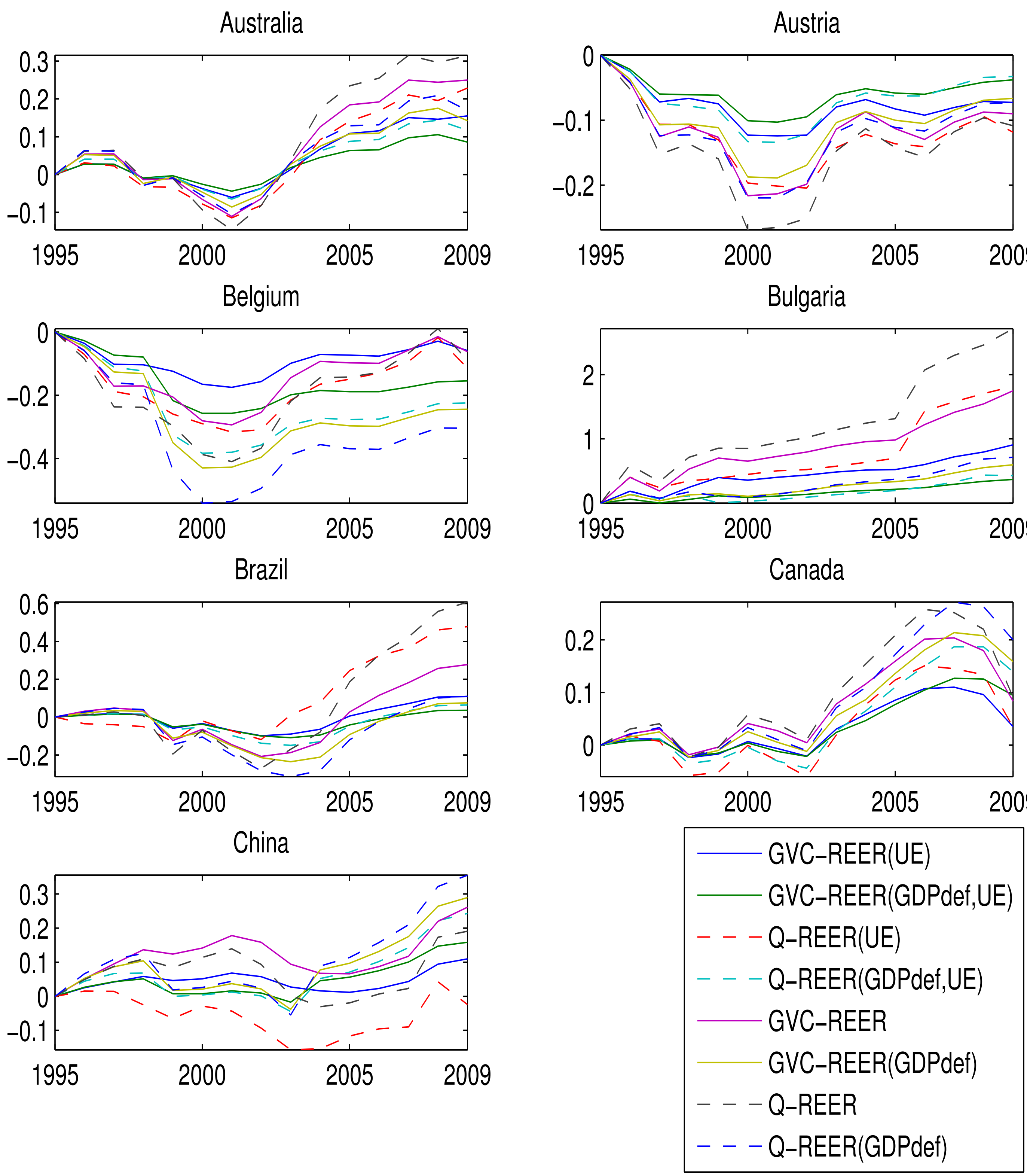


Figure 10 – *GVC-REER and Q-REER indices for all countries cont.*

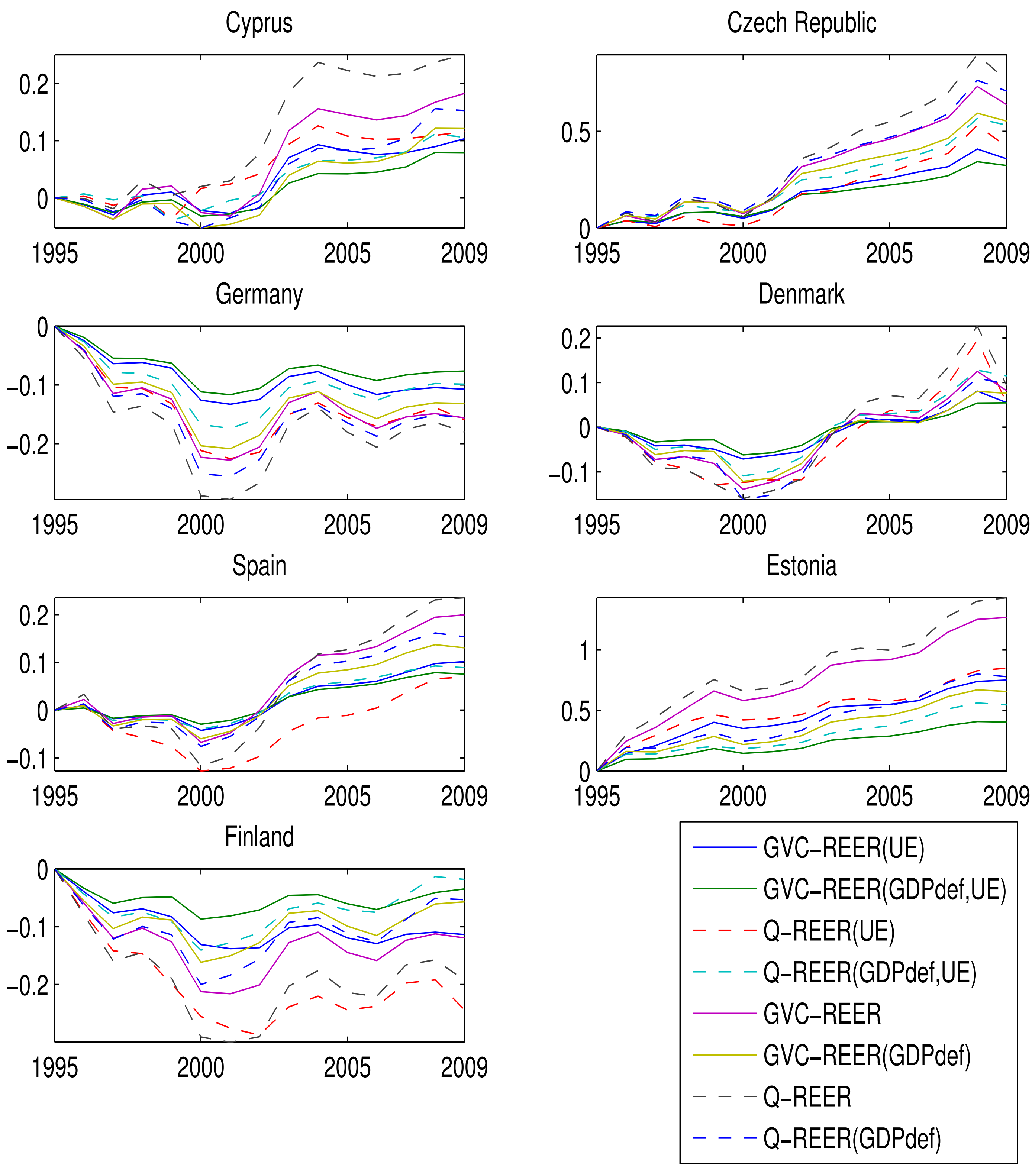


Figure 11 – *GVC-REER and Q-REER indices for all countries cont.*

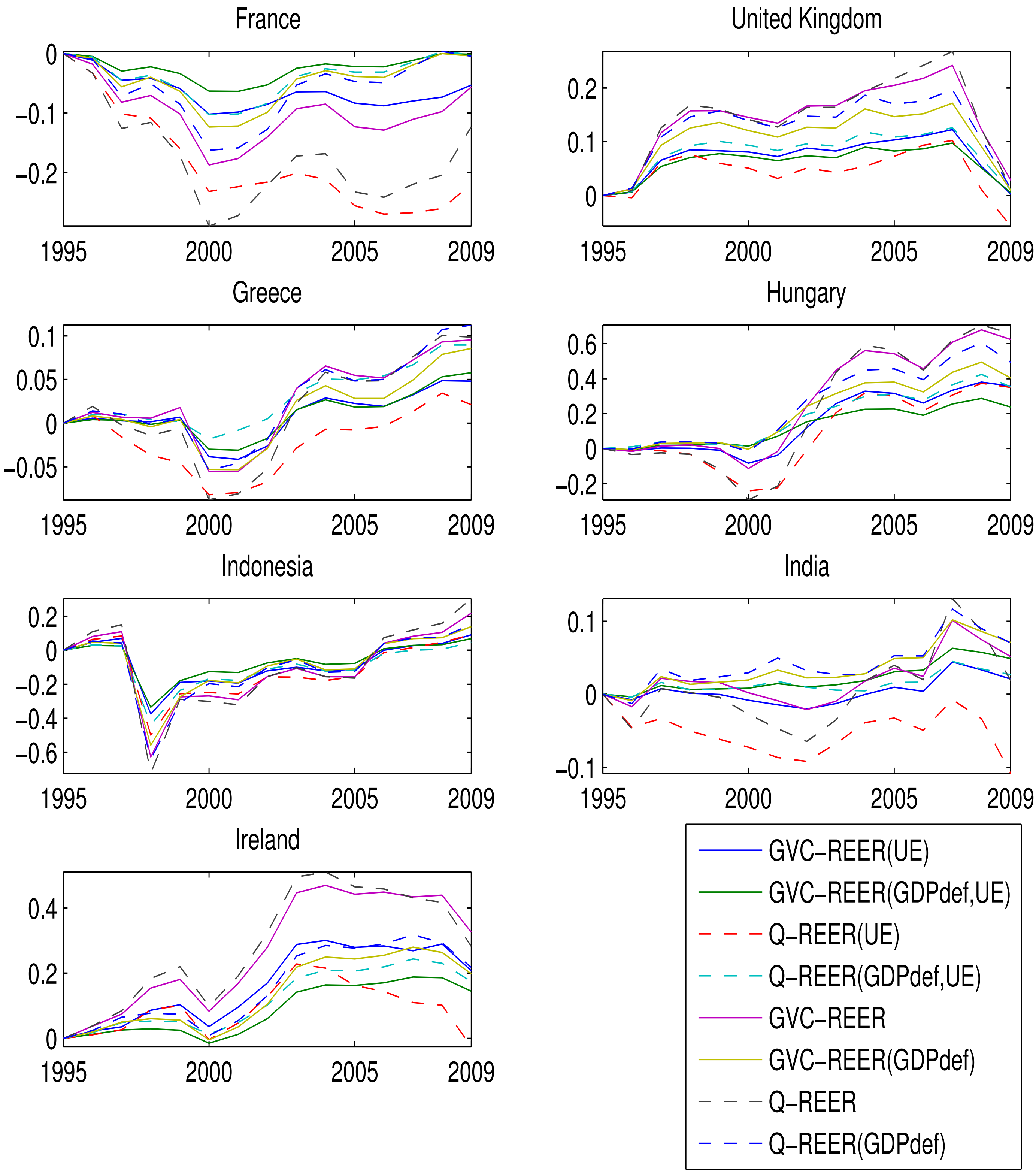


Figure 12 – *GVC-REER and Q-REER indices for all countries cont.*

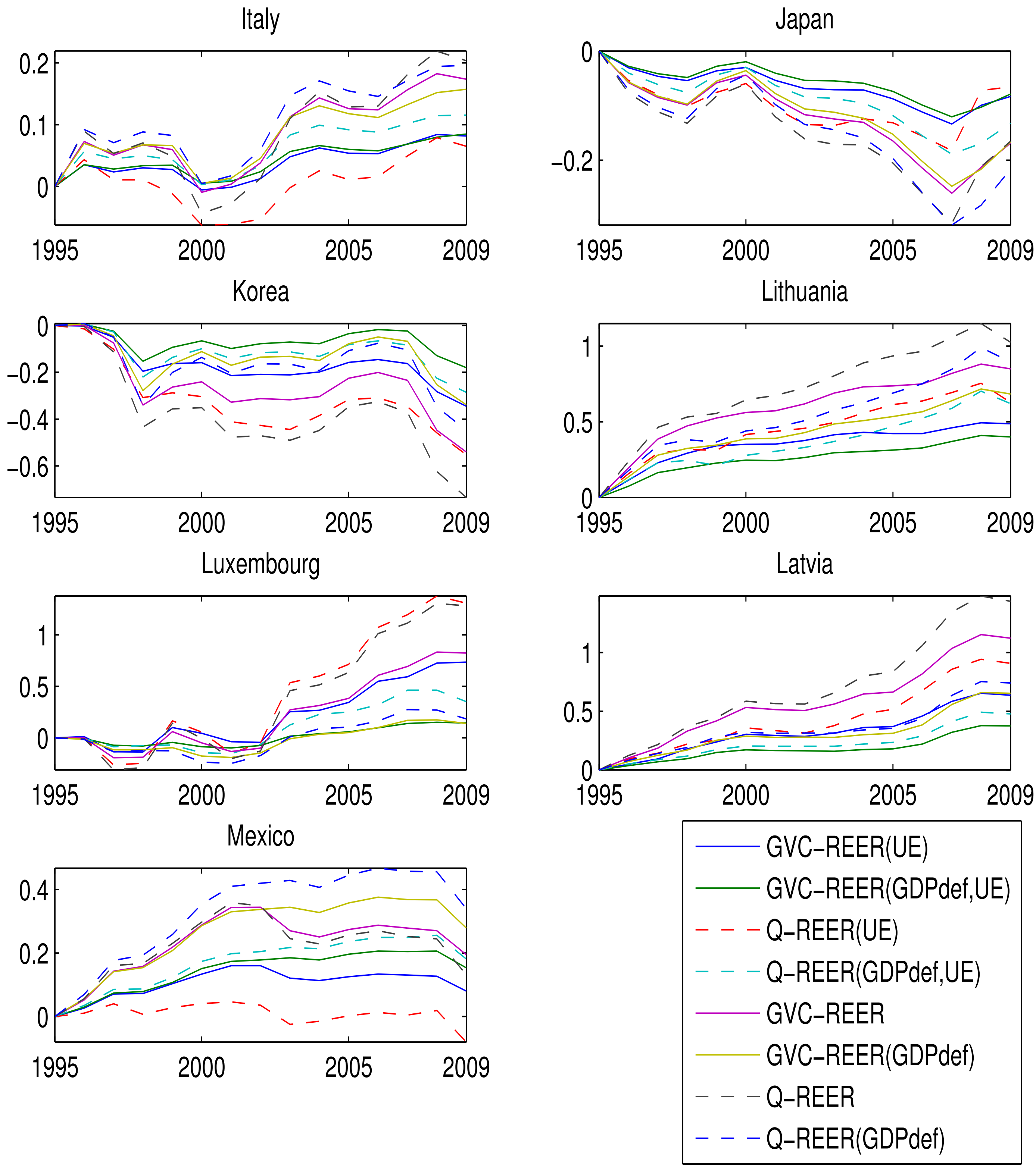


Figure 13 – *GVC-REER and Q-REER indices for all countries cont.*

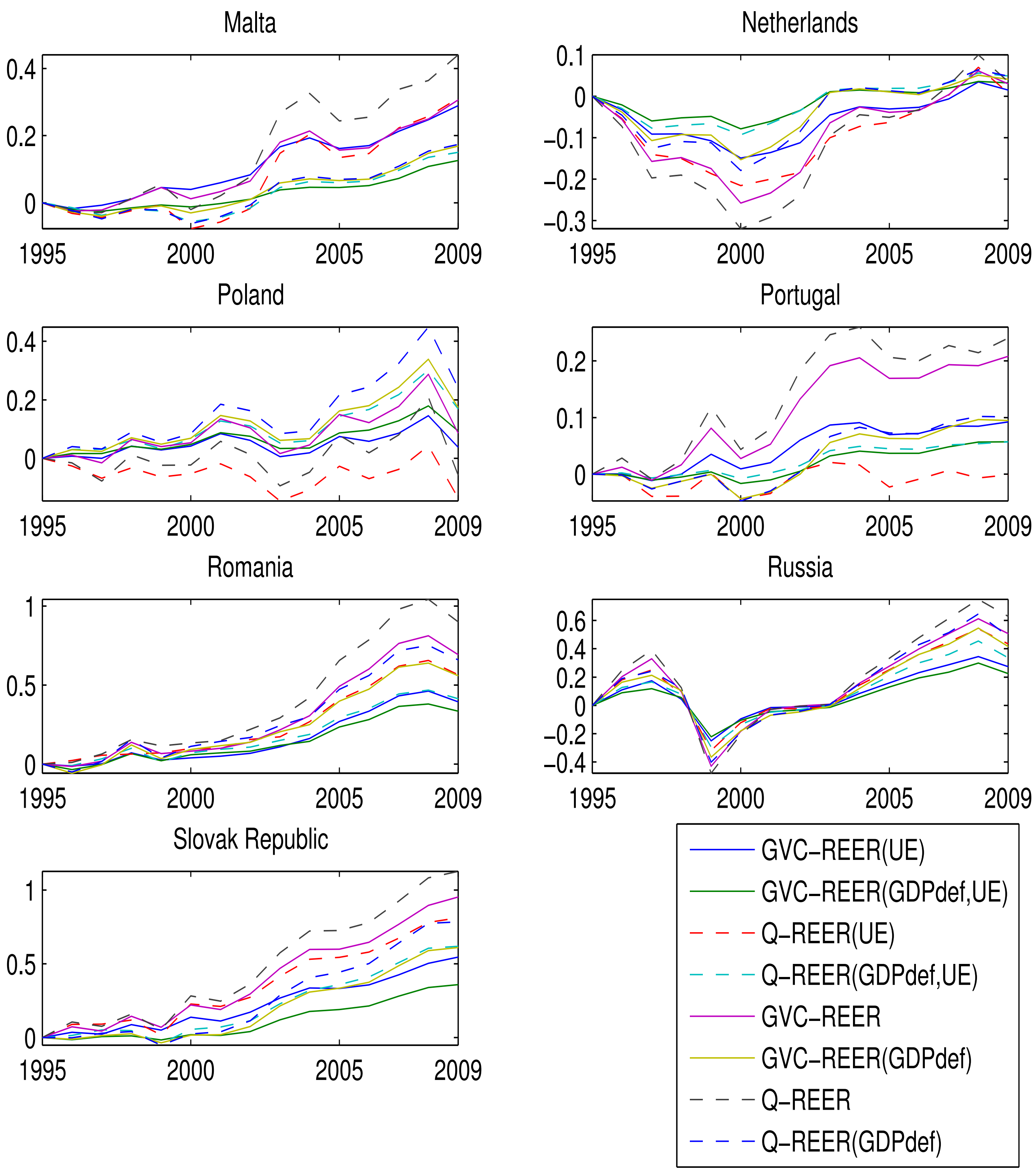
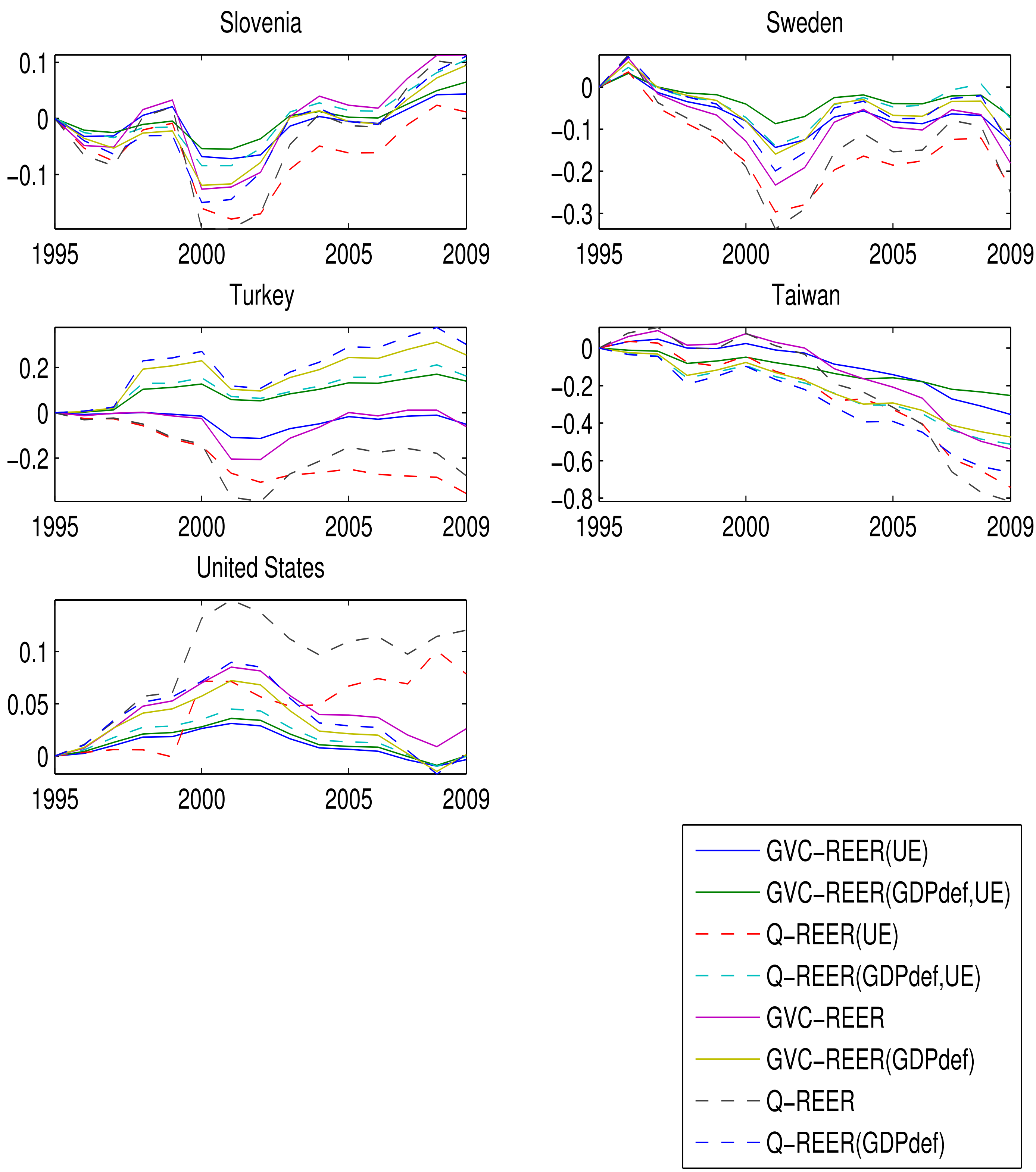


Figure 14 – *GVC-REER and Q-REER indices for all countries cont.*



References

Antràs, P. (2014). Global production: A contracting perspective.

Antràs, P. and Chor, D. (2013). Organizing the global value chain. *Econometrica*, 81(6):2127–2204.

Baldwin, R. and Lopez-Gonzalez, J. (2012). Supply-chain trade: A portrait of global patterns and several testable hypotheses. *HEI manuscript.*

Bayoumi, M. T., Saito, M., and Turunen, M. J. (2013). *Measuring Competitiveness: Trade in Goods or Tasks?* Number 13-100. International Monetary Fund.

Bayoumi, T., Lee, J., and Jayanthi, S. (2005). New rates from new weights.

Bems, R. and Johnson, R. C. (2012). Value-added exchange rates. Technical report, National Bureau of Economic Research.

Bennett, H. Z. and Zarnic, Z. (2009). International competitiveness of the Mediterranean quartet: A heterogeneous-product approach. *IMF Staff Papers*, 56(4):919–957.

Broda, C. and Weinstein, D. E. (2006). Globalization and the gains from variety. *The Quarterly Journal of Economics*, 121(2):541–585.

Burstein, A., Kurz, C., and Tesar, L. (2008). Trade, production sharing, and the international transmission of business cycles. *Journal of Monetary Economics*, 55(4):775–795.

Chinn, M. D. (2000). Before the fall: were East Asian currencies overvalued? *Emerging Markets Review*, 1(2):101–126.

Chinn, M. D. (2006). A primer on real effective exchange rates: determinants, overvaluation, trade flows and competitive devaluation. *Open economies review*, 17(1):115–143.

Costinot, A., Vogel, J., and Wang, S. (2013). An elementary theory of global supply chains. *The Review of Economic Studies*, 80(1):109–144.

Di Giovanni, J. and Levchenko, A. A. (2010). Putting the parts together: Trade, vertical linkages, and business cycle comovement. *American Economic Journal: Macroeconomics*, 2(2):95–124.

Feenstra, R. C. (1994). New product varieties and the measurement of international prices. *The American Economic Review*, pages 157–177.

Feenstra, R. C., Obstfeld, M., and Russ, K. N. (2010). In search of the Armington elasticity. *University of California-Davis, unpublished mimeo.*

Gagnon, J. E. (2012). Combating widespread currency manipulation. *Policy Brief in International Economics*, pages 12–19.

Goldberg, L. S. (2004). Industry-specific exchange rates for the United States. *Federal Reserve Bank of New York Economic Policy Review*, 10(1).

Goldfajn, I. and Valdés, R. O. (1999). The aftermath of appreciations. *The Quarterly Journal of Economics*, 114(1):229–262.

Hummels, D., Ishii, J., and Yi, K.-M. (2001). The nature and growth of vertical specialization in world trade. *Journal of international Economics*, 54(1):75–96.

Johnson, R. C. (2012). Trade in intermediate inputs and business cycle comovement. Technical report, National Bureau of Economic Research.

Johnson, R. C. and Moxnes, A. (2012). Technology, trade costs, and the pattern of trade with multi-stage production.

Johnson, R. C. and Noguera, G. (2012). Accounting for intermediates: Production sharing and trade in value added. *Journal of International Economics*, 86(2):224–236.

Kiyotaka, S., Junko, S., Shrestha, Nagendra, N., and Zhang, Shajuan, S. (2012). The construction and analysis of industry-specific effective exchange rates in japan. Technical report.

Kiyotaka, S., Junko, S., Shrestha, N. N., and ZHANG, S. (2013). Exchange rate appreciation and export price competitiveness: Industry-specific real effective exchange rates of japan, korea, and china. Technical report.

Klau, M. et al. (2008). The new BIS effective exchange rate indices. *BIS Quarterly Review*.

Koopman, R., Wang, Z., and Wei, S.-J. (2012). Tracing value-added and double counting in gross exports. Technical report, National Bureau of Economic Research.

Koopman, R., Wang, Z., and Wei, S.-J. (2014). Tracing value-added and double counting in gross exports. *The American Economic Review*, pages 459–494.

Loretan, M. (2005). Indexes of the foreign exchange value of the dollar. *Fed. Res. Bull.*, 91:1.

McGuirk, A. (1986). Measuring price competitiveness for industrial country trade in manufactures.

Miroudot, S., Lanz, R., and Ragoussis, A. (2009). Trade in intermediate goods and services. Technical report, OECD Publishing.

Rogoff, K. (2005). Rethinking exchange rate competitiveness. *The Global Competitiveness Report*, 2006:99–105.

Soderbery, A. (2013). Estimating import supply and demand elasticities: Analysis and implications. *Available at SSRN 2272749*.

Thorbecke, W. (2011). Investigating the effect of exchange rate changes on China's processed exports. *Journal of the Japanese and International Economies*, 25(2):33–46.

Timmer, M. and Erumban, A. (2012). The world input-output database (wiod): Contents, sources and methods. *WIOD Background document available at www. wiod. org*.

Wang, Z., Wei, S.-J., and Zhu, K. (2013). Quantifying international production sharing at the bilateral and sector level. *working paper*.