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Daniel Andrei
Bruce Carlin
Michael Hasler

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ABSTRACT

We study the impact of model disagreement on the dynamics of asset prices, return volatility, and trade in the market. In our continuous-time framework, two investors have homogeneous preferences and equal access to information, but disagree about the length of the business cycle. We show that model disagreement amplifies return volatility and trading volume by inducing agents to have different economic outlooks, which generates a term structure of disagreement. Different economic outlooks imply that investors will trade even if they do not disagree about the current value of fundamentals. Also, we find that while the absolute level of return volatility is driven by long-run risk, the variation and persistence of volatility (i.e., volatility clustering) is driven by disagreement. Compared to previous studies that consider model uncertainty with a representative agent or those that study heterogeneous beliefs with no model disagreement, our paper offers a theoretical foundation for the GARCH-like behavior of stock returns.

Daniel Andrei
UCLA
ddandrei@gmail.com

Bruce Carlin
Anderson Graduate School of Management
UCLA
110 Westwood Plaza, Suite C413
Los Angeles, CA 90095-1481
and NBER
bruce.carlin@anderson.ucla.edu

Michael Hasler
105 St. George Street
Toronto, ON
M5S 1E6
Michael.Hasler@rotman.utoronto.ca

1 Introduction

The field of finance is currently grappling with the fact that there are limits to applying the standard Bayesian paradigm to asset pricing. Specifically, in a standard Bayesian framework, beliefs are updated with a particular model in mind. However, as noted by Hansen and Sargent (2007), many economic models cannot be trusted completely, thereby introducing the notion of model uncertainty. Theoretically, though, as long as the potential set of models that all agents in an economy consider is the same ex ante, the Bayesian framework can still apply because agents can update their beliefs about which model explains the economy. However, if the agents consider different sets of models or they adhere to different paradigms, then disagreement will persist regarding which model is best to describe the world or predict the future. It is this notion of *model disagreement* that we focus on in this paper and characterize its effects on asset prices, return volatility, and trade in the market.

Empirically, model disagreement appears to be important. For example, in a recent paper by Carlin, Longstaff, and Matoba (2013), the authors study the effects of disagreement about prepayment speed forecasts in the mortgage-backed securities market on risk premia, volatility, and trading volume. Indeed, the prepayment models that traders use are often proprietary and differ from each other, while the inputs to these models are publicly observable (e.g., unemployment, interest rates, inflation). In that paper, the authors show that disagreement is associated with a positive risk premium and is the primary channel through which return volatility impacts trading volume.

In this paper, we analyze a continuous-time framework in which investors exhibit model disagreement and study how this affects the dynamics of asset prices and trading volume. In our setup, two investors have homogenous preferences and equal access to information, but disagree about the length of the business cycle. Each investor knows that the expected dividend growth rate mean-reverts, but uses a different parameter that governs the rate at which this fundamental returns to its long-term mean. The disagreement is commonly known, but each agent adheres to his own model when deciding whether to trade.[1]

Using disagreement about the length of the business cycle is natural and plausible. For example, Massa and Simonov (2005) show that forecasters strongly disagree on recession probabilities, which implies that they have different beliefs regarding the duration of recessionary and expansionary phases. The origin of this disagreement may arise from many sources. Indeed, there still remains much debate regarding the validity of long-run risk models (e.g., Beeler and Campbell (2012); Bansal, Kiku, and Yaron (2012)). Additionally, in practice agents might use different time-series to estimate the mean-reversion parameter

[1]This form of disagreement arises if agents are uncertain about the interpretation of public information, even after observing infinitely many signals (Acemoglu, Bimpikis, and Ozdaglar, 2010).

(e.g., use consumption versus production data). Likewise, their estimation methods may differ (e.g., fitting the model to past analyst forecast data versus a moving-average of output growth versus performing maximum-likelihood Kalman filter estimation). Finally, as Yu (2012) documents, least-squares and maximum-likelihood estimators of the mean-reversion speed of a continuous-time process are significantly biased. Some investors might be aware of the existence of this bias and would adjust their estimation accordingly, whereas other investors might ignore it.[2]

In our equilibrium model, two distinct quantities turn out to be important determinants of asset prices and trade in the market. The first is the *disagreement over fundamentals*, which is the instantaneous difference in beliefs about the expected growth rate in the economy. The second is the *difference in economic outlooks*, which affects expectations of future economic variables and takes into account how both agents will disagree over fundamentals in the future. The differences in economic outlooks at all horizons into the future are dictated by the term structure of disagreement. Interestingly, different outlooks amplify return volatility and trading volume, even when the agents agree about the current fundamentals. To show this, we perform a numerical analysis that compares an economy populated by a representative agent to that populated by two agents with model disagreement. Both settings are otherwise observationally equivalent in terms of their average expected growth rate and average uncertainty. Additionally, we set the disagreement over fundamentals to zero. We show that the volatility with model disagreement is higher than what an observationally equivalent representative agent economy generates. Also, while there is no trade in the representative agent economy, as the difference in economic outlook increases, trading volume follows suit. These results imply that model disagreement not only amplifies volatility, but also provides an important mechanism by which uncertainty affects trade.

Also in the equilibrium of our model, we show that while the absolute level of volatility is driven primarily by long-run risk, the variation and persistence of volatility (i.e., volatility clustering) is driven by disagreement. Disagreement is persistent and increases the volatility of the risk-adjusted discount factor and consequently also the volatility of stock returns.[3] Persistent transmission from investors beliefs to stock market volatility via disagreement causes excess volatility, which is time-varying and persistent. We disentangle the impact of disagreement from the impact generated by the other driving forces by decomposing stock return volatility. We show that, indeed, disagreement is the main driving force of persistent

[2] We further justify the assumption of different parameters in Appendix A.1 by performing a simulation exercise in which we let the agents estimate the mean-reversion parameter with different methods. We show that the difference between the estimated parameters is typically substantial, even though we perform 1,000 simulations of economies of length of 50 years at quarterly frequency.

[3] Persistent disagreement is consistent with empirical findings by Patton and Timmermann (2010) and Andrade, Crump, Eusepi, and Moench (2014).

fluctuations in stock market volatility, whereas the level of the volatility is mainly driven by long-run risk, as the long-run risk literature (Bansal and Yaron, 2004) suggests.[4]

Our results help to explain three well-known characteristics about financial market volatility. First, volatility systematically exceeds that justified by fundamentals (Shiller, 1981; LeRoy and Porter, 1981). Indeed, we show that model disagreement amplifies volatility, over and above the usual effect of uncertainty. Second, volatility is time-varying (Schwert, 1989; Mele, 2008). This arises naturally out of our model because disagreement is mean-reverting. Last, volatility is persistent (Engle, 1982; Bollerslev, 1986; Nelson, 1991), occurring in clusters. This persistence (or predictability) has been described extensively in the empirical literature, but there is a paucity of theoretical explanations. We show that model disagreement generates a new channel of persistence transmission from investors beliefs to stock market volatility and we fit a GARCH model on simulated stock returns to show that volatility is indeed persistent. Our paper proposes therefore a theoretical foundation for the GARCH-like behavior of stock returns.

Finally, we conclude the paper with a survival analysis. Indeed, in any model with heterogeneous agents, whether all types survive in the long-run is a reasonable concern. To address this, we perform simulations and show that all agents in our economy with model disagreement survive for long periods of time, consistent with previous findings in the literature (Yan, 2008). Based on this, we posit that model disagreement can have long-lasting effects on asset prices without eliminating any players from the marketplace, which likely makes our analysis economically important.

Our approach contrasts with previous work and thus adds to the previous finance literature. As already mentioned, Hansen and Sargent (2007) studies model misspecification and model uncertainty, but does so for a single investor.[5] In contrast, our study investigates the consequence of disagreement about models in an economy with different investors. We assume that investors disagree about the model governing the economy. Certainly, there are many other forms of disagreement; in particular, several papers feature a setting in which investors agree on the model governing the economy but disagree on the information that they receive (see, e.g., Scheinkman and Xiong 2003, Dumas, Kurshev, and Uppal 2009, or Xiong and Yan 2010). These models are able to generate excess volatility but they do not identify the cause of persistent fluctuations in volatility.

[4]We define long-run risk here as the risk associated to a persistent expected dividend growth rate only. In Bansal and Yaron (2004) long-run risk captures the risk associated to a persistent expected dividend growth rate, a persistent dividend growth volatility, and a persistent expected dividend growth volatility. In contrast, we do not assume that any fundamental variable features stochastic volatility. Instead, stock return volatility becomes stochastic in equilibrium exactly because agents disagree about the magnitude of long-run risk. We thus argue that long-run risk *per se* is not a cause of fluctuations in volatility, whereas *disagreement about long-run risk* endogenously gives rise to such fluctuations.

[5]See also Uppal and Wang (2003), Maenhout (2004), Liu, Pan, and Wang (2005), and Drechsler (2013).

The remainder of the paper is organized as follows. Section 2 describes the model and its solution. Section 3 explores how model disagreement affects volatility and trading volume. Section 4 addresses the survival of investors. Section 5 concludes. All derivations and computational details are in Appendix A.

2 Model Disagreement

Consider a pure exchange economy defined over a continuous time horizon $[0, \infty)$, in which a single consumption good serves as the numéraire. The underlying uncertainty of the economy is characterized by a 2-dimensional Brownian motion $W = \{W_t : t > 0\}$, defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The aggregate endowment of consumption is assumed to be positive and to follow the process:

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dW_t^\delta \tag{1}$$

$$df_t = \lambda(\overline{f} - f_t)dt + \sigma_f dW_t^f, \tag{2}$$

where W^δ and W^f are two independent Brownian motions under the objective probability measure $\mathbb{P}$. The expected consumption growth rate f, henceforth called *the fundamental*, is unobservable and mean-reverts to its long-term mean $\bar{f}$ at the speed λ. The parameters σ_δ and σ_f are the volatilities of the consumption growth and of the fundamental.

There is a single risky asset (the stock), defined as the claim to the aggregate consumption stream over time. The total number of outstanding shares is unity. In addition, there is also a risk-free bond, available in zero-net supply.

The economy is populated by two agents, A and B. Each agent is initially endowed with equal shares of the stock and zero bonds, can invest in these two assets, and derives utility from consumption over his or her lifetime. Each agent chooses a consumption-trading policy to maximize his or her expected lifetime utility:

$$U_i = \mathbb{E}^i \left[\int_0^\infty e^{-\rho t} \frac{c_{it}^{1-\alpha}}{1-\alpha} dt \right], \tag{3}$$

where $\rho > 0$ is the time discount rate, $\alpha > 0$ is the relative risk aversion coefficient, and c_{it} denotes the consumption of agent $i \in \{A, B\}$ at time t. The expectation in (3) depends on agent i's perception of future economic conditions. Agents value consumption streams using the same preferences with identical risk aversion and time discount rate but, as we will describe below, have heterogeneous beliefs.

2.1 Learning and Disagreement

The agents commonly observe the process δ, but have incomplete information and heterogeneous beliefs about the dynamics of the fundamental f. Specifically, the agents agree that the fundamental mean-reverts but disagree on the value of the mean-reversion parameter λ. As such, they have different perceptions about the length of the business cycle.

Agent A's perception of the aggregate endowment and the fundamental is

$$\frac{d\delta_t}{\delta_t} = f_{At}dt + \sigma_\delta dW^\delta_{At} \tag{4}$$

$$df_{At} = \lambda_A \left(\bar{f} - f_{At}\right) dt + \sigma_f dW^f_{At}, \tag{5}$$

where W^δ_A and W^f_A are two independent Brownian motions under agent A's probability measure $\mathbb{P}^A$. On the other hand, agent B believes that

$$\frac{d\delta_t}{\delta_t} = f_{Bt}dt + \sigma_\delta dW^\delta_{Bt} \tag{6}$$

$$df_{Bt} = \lambda_B \left(\bar{f} - f_{Bt}\right) dt + \sigma_f dW^f_{Bt}, \tag{7}$$

where W^δ_B and W^f_B are two independent Brownian motions under agent B's probability measure $\mathbb{P}^B$. Both agents agree on the long-term mean of the fundamental $\bar{f}$ and on the volatility of the fundamental σ_f.[6]

Neither agent uses the right parameter λ. Instead, the true parameter λ is assumed to lie somewhere in between the parameters perceived by the agents. As such, there are 3 probability measures: the objective probability measure $\mathbb{P}$ and the two probability measures $\mathbb{P}^A$ and $\mathbb{P}^B$ as perceived by agents A and B.

The agents both observe the aggregate endowment process δ and use it to estimate the fundamental f under their respective probability measures.[7] Since they use different models, they have different estimates of f. Define $\widehat{f}_A$ and $\widehat{f}_B$ as each agent's estimate of the unobservable fundamental f:

$$\widehat{f}_{it} \equiv \mathbb{E}^i_t \left[f_{it}\right], \quad \text{for } i \in \{A, B\}, \tag{8}$$

[6]We have considered extensions of the model where agents have heterogeneous parameters $\bar{f}$ and σ_f, with similar results. The parameter bearing the main implications is the mean-reversion speed λ and thus we choose to focus on heterogeneity about it and to isolate our results from other sources of belief heterogeneity.

[7]We assume that the only public information available is the history of the aggregate endowment process δ. The model can also accommodate public news informative about the fundamental, but here we chose not to obscure the model's implications and we abstract away from additional public news. The effects of heterogeneous beliefs about public news are well-understood (see, e.g., Scheinkman and Xiong (2003) or Dumas et al. (2009) among others).

which are computed using standard Bayesian updating techniques. Learning is implemented via Kalman filtering and yields[8]

$$d\widehat{f}_{it} = \lambda_i \left(\bar{f} - \widehat{f}_{it} \right) dt + \frac{\gamma_i}{\sigma_\delta} d\widehat{W}^\delta_{it}, \quad \text{for } i \in \{A, B\}, \tag{9}$$

where γ_i denotes the posterior variance perceived by agent i and $\widehat{W}^\delta_i$ represents the normalized innovation process of the dividend under agent i's probability measure

$$d\widehat{W}^\delta_{it} = \frac{1}{\sigma_\delta} \left(\frac{d\delta_t}{\delta_t} - \widehat{f}_{it} dt \right). \tag{10}$$

The process in Equation (10) has a simple interpretation. Agent i observes a realized growth of $d\delta_t/\delta_t$ and has an expected growth of $\widehat{f}_{it}dt$. The difference between the realized and the expected growth, normalized by the standard deviation σ_δ, represents the surprise or the innovation perceived by agent i.

The posterior variance γ_i (i.e., *Bayesian uncertainty*) reflects incomplete knowledge of the true expected growth rate. It is defined by[9]

$$\gamma_i \equiv \mathrm{Var}^i_t \left[f_{it} \right] = \sigma^2_\delta \left(\sqrt{\lambda^2_i + \frac{\sigma^2_f}{\sigma^2_\delta}} - \lambda_i \right) > 0, \quad \text{for } i \in \{A, B\}. \tag{11}$$

Equation (11) shows how γ_i depends on the initial parameters. The posterior variance increases with the volatility of the fundamental σ_f and with the volatility of the aggregate endowment σ_δ, and decreases with the mean-reversion parameter λ_i. Intuitively, if λ_i is small then agent i believes the process f to be persistent and thus the perceived uncertainty in the estimation is large. Since agents A and B use different mean-reversion parameters, it follows that their individual posterior variances are different, that is, one of the agents will perceive a more precise estimate of the expected growth rate. Therefore, one of the agents appears "overconfident" with respect to the other agent, although overconfidence here does not arise from misinterpretation of public signals as in Scheinkman and Xiong (2003) or Dumas et al. (2009), but from different underlying models.

The innovation processes $\widehat{W}^\delta_A$ and $\widehat{W}^\delta_B$ are Brownian motions under $\mathbb{P}^A$ and $\mathbb{P}^B$, respec-

[8] See Theorem 12.7 in Liptser and Shiryaev (2001) and Appendix A.2 for computational details.

[9] As in Scheinkman and Xiong (2003) or Dumas et al. (2009), we assume that the posterior variance has already converged to a constant. The convergence arises because investors have Gaussian priors and all variables are normally distributed. This generates a deterministic path for the posterior variance and a quick convergence (at an exponential rate) to a steady-state value.

tively. They are such that agent i has the following system in mind

$$\frac{d\delta_t}{\delta_t} = \widehat{f}_{it} dt + \sigma_\delta d\widehat{W}_{it}^\delta \tag{12}$$

$$d\widehat{f}_{it} = \lambda_i(\bar{f} - \widehat{f}_{it})dt + \frac{\gamma_i}{\sigma_\delta} d\widehat{W}_{it}^\delta, \quad i \in \{A, B\}. \tag{13}$$

A few points are worth mentioning. First, although the economy is governed by two Brownian motions under the objective probability measure $\mathbb{P}$ (as shown in (1)-(2)), there is only one Brownian motion under each agent's probability measure $\mathbb{P}^i$. This arises because there is only one *observable* state variable, the aggregate endowment δ. Second, the instantaneous variance of the observable process δ is the same for both agents, which is not the case for the instantaneous variance of the filter $\widehat{f}_i$. Because of the "overconfidence" effect induced by different parameters λ, one of the agents will perceive a more volatile filter than the other.

Furthermore, agreeing to disagree implies that each agent knows how the other agent perceives the economy and that they are aware that their different perceptions will generate disagreement—*although they observe the same process* δ. This important feature (that the aggregate endowment process is observable and thus it should be the same for both agents) provides the link between the two probability measures $\mathbb{P}^A$ and $\mathbb{P}^B$. Writing the aggregate consumption process (12) for both agents and restricting the dynamics to be equal provides a relationship between the innovation processes $\widehat{W}_A^\delta$ and $\widehat{W}_B^\delta$ (technically, a change of measure from $\mathbb{P}^A$ to $\mathbb{P}^B$):

$$d\widehat{W}_{At}^\delta = d\widehat{W}_{Bt}^\delta + \frac{1}{\sigma_\delta}\left(\widehat{f}_{Bt} - \widehat{f}_{At}\right) dt. \tag{14}$$

Equation (14) shows how one can convert agent A's perception of the innovation process $\widehat{W}_A^\delta$ to agent B's perception $\widehat{W}_B^\delta$. The change of measure consists of adding the drift term on the right hand side of (14). For example, suppose that agent A has an estimate of the expected growth rate of $\widehat{f}_{At} = 1\%$, whereas agent B's estimate is $\widehat{f}_{Bt} = 3\%$. Assume that the realized growth rate (observed by both agents) turns out to be $d\delta_t/\delta_t = 2\%$. It follows that agent B was optimistic and $d\widehat{W}_{Bt}^\delta = -0.01/\sigma_\delta$, whereas agent A was pessimistic and $d\widehat{W}_{At}^\delta = 0.01/\sigma_\delta$.

The extra drift term in Equation (14) comprises the difference between each agent's estimates of the growth rate $(\widehat{f}_{Bt} - \widehat{f}_{At})$ or the *disagreement*, which we denote hereafter by $\widehat{g}_t$. We can now use this relationship to compute the dynamics of $\widehat{g}_t$, under one of the agent's probability measure, say $\mathbb{P}^B$.

Proposition 1. *(Evolution of Disagreement) Under the probability measure $\mathbb{P}^B$, the dynam-*

ics of disagreement are given by

$$d\widehat{g}_t = d\widehat{f}_{Bt} - d\widehat{f}_{At} = \left[(\lambda_A - \lambda_B)(\widehat{f}_{Bt} - \bar{f}) - \left(\frac{\gamma_A}{\sigma_\delta^2} + \lambda_A\right)\widehat{g}_t\right] dt + \frac{\gamma_B - \gamma_A}{\sigma_\delta} d\widehat{W}_{Bt}^{\delta}. \tag{15}$$

Proof. See Appendix A.3 □

Proposition 1 characterizes the dynamics of disagreement[10], which yields several properties that make it different from previous models of overconfidence that have been studied in the literature. First, if one of the agents believes in long-run risk, disagreement is persistent. Second, if agents have different degrees of precision in their estimates (which happens to be the case when they use different parameters λ), disagreement is stochastic. Third, because its long-term drift is stochastic, disagreement will never converge to a constant but will always be regenerated—even without a stochastic term.

To see this, observe that Equation (15) shows that disagreement is mean-reverting around a stochastic mean, driven by $\widehat{f}_B$. This arises because $\lambda_A \neq \lambda_B$. If agents adhered to the same models, disagreement would revert to zero, as in Scheinkman and Xiong (2003) and Dumas et al. (2009). In contrast, in our setup, the mean is driven by $\widehat{f}_B$ because the agents use different models. In addition, if one of the agents, say agent B, believes in long-run risk, disagreement becomes persistent because it mean reverts to a persistent $\widehat{f}_B$.[11]

To appreciate the relationship between the agents' precision and the stochastic nature of disagreement, let us focus on the stochastic term in the dynamics of disagreement expressed in Equation (15). This term arises because $\gamma_A \neq \gamma_B$. As previously observed in Equation (11), different posterior variances are a result of different mean-reversion parameters. This generates stochastic shocks in disagreement. Although models of overconfidence (Scheinkman and Xiong, 2003; Dumas et al., 2009) generate a similar stochastic term, a key difference arises in our setup. To see this, suppose we shut down this stochastic term. This can be done by properly adjusting the initial learning problem of the agents.[12] Equation (15) shows that, even though the stochastic term disappears, disagreement will still be time-varying—and persistent—precisely due to the first term in its drift. In contrast, shutting down the stochastic term in models of overconfidence eliminates disagreement through prompt convergence toward its long-term mean, zero. This highlights the "structural" form of disagreement generated by different economic models.

[10]The dynamics of disagreement in (15) comprise only $\widehat{W}_B$ but not $\widehat{W}_A$. Without loss of generality, we choose to work under agent B's probability measure $\mathbb{P}^B$; however, by using (14), we could easily switch to agent A's probability measure and all the results would still hold.

[11]Alternatively, if agent A believes the fundamental is persistent, then we can write the dynamics of disagreement under $\mathbb{P}^A$ and the same intuition holds.

[12]Precisely, we can consider that agents have different parameters σ_f chosen in such a way that $\gamma_A = \gamma_B$. This will shut down the stochastic term in Equation (15).

2.2 Economic Outlook

Now, let us consider how model disagreement affects each agent's relative economic *outlook*. Since each agent perceives the economy under a different probability measure, any random economic variable X, measurable and adapted to the observation filtration $\mathscr{O}$, now has two expectations: one under the probability measure $\mathbb{P}^A$, and the other under the probability measure $\mathbb{P}^B$. Naturally, they are related to each other by the formula

$$\mathbb{E}^A[X] = \mathbb{E}^B[\eta X], \tag{16}$$

where η measures the relative difference in outlook from one agent to the other.

Proposition 2. *(Economic Outlook) Under the probability measure $\mathbb{P}^B$, the relative difference in economic outlook satisfies*

$$\eta_t \equiv \left.\frac{d\mathbb{P}^A}{d\mathbb{P}^B}\right|_{\mathscr{O}_t} = e^{-\frac{1}{2}\int_0^t \left(\frac{1}{\sigma_\delta}\widehat{g}_s\right)^2 ds - \int_0^t \frac{1}{\sigma_\delta}\widehat{g}_s d\widehat{W}^\delta_{Bs}}, \tag{17}$$

where $\mathscr{O}_t$ is the observation filtration at time t and η obeys the dynamics

$$\frac{d\eta_t}{\eta_t} = -\frac{1}{\sigma_\delta}\widehat{g}_t d\widehat{W}^\delta_{Bt}. \tag{18}$$

Proof. See Girsanov's Theorem. □

On the surface, the expression in (17) is simply the Radon-Nikodym derivative for the change of measure between the agents' beliefs. But this has a natural economic interpretation here as the difference in economic outlook between the agents, since it captures the difference in expectations that each agent has for the future. This contrasts with previous papers that use η_t to express differences in the *sentiment* between agents (Dumas et al., 2009). In our setting, agents do not have behavioral biases like overconfidence or optimism. Rather, because they adhere to different models of the world, they rationally have different economic outlooks, which are not a function of how they are feeling per se (i.e., sentiment).

One important implication of Proposition 2 is that disagreement over fundamentals and economic outlook are different entities. In fact, relative outlook is a function of disagreement, and there may be differences in the agents' outlook even though they agree today on the underlying fundamentals of the economy. This is because disagreement in our setup expresses the difference in beliefs about the expected growth rate *today*, while outlook enters into the expectations of future economic variables and thus captures the way in which agents' beliefs will differ into the *future*.

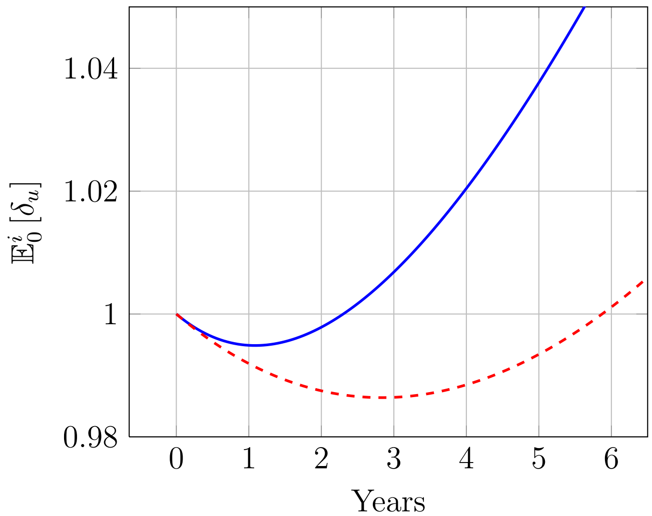

Figure 1: Different economic outlooks

Expectation of future aggregate consumption computed by agent A (solid blue line) and agent B (dashed red line). The state variables at time $t = 0$ are $\delta_0 = 1$, $\widehat{f}_{B0} = -1\%$, $\widehat{g}_0 = 0$, and $\eta_0 = 1$. Other parameters for this example are listed in Table 1.

This is best appreciated by observing that the relative difference in outlook in (17) is a function of the integral of disagreement that is realized over a particular horizon, not just the disagreement that takes place at one particular instant. This implies that two agents may have very different outlooks, even though they currently agree on the fundamentals in the market. That is, even though their models currently yield the same fundamentals, because they use different models, they will have different outlooks for the future. This will drive the results in future sections where we show that trading volume may be substantial even when there is currently no disagreement about fundamentals: trade will still take place because the agents take into account that they will disagree in the future (i.e., they have different economic outlooks).

To see this more clearly, consider the following example. Suppose that, at t=0, $\widehat{f}_{A0} = \widehat{f}_{B0} = -1\%$. Because $\widehat{g}_0 = 0$, both agents agree that the economy is going through a recession. Furthermore, assume that agent B believes the economic cycles are longer than agent A, that is, $\lambda_B = 0.1$ whereas $\lambda_A = 0.3$. Figure 1 shows the different economic outlooks that agents hold, *even though they are in agreement today.* It calculates the expectation of future dividends, $\mathbb{E}_0^i[\delta_u]$. Agent A (solid blue line) believes that the economy will recover quickly, in about two years, whereas agent B (dashed red line) believes that it will take six years for the economy to get back to its initial level of aggregate consumption.

It is also instructive to observe in (18) that disagreement affects the evolution of relative outlook. It is the primary driver of fluctuations in η_t. When $\widehat{g}_t$ is large, η_t will also have large fluctuations. Note, however, that even though $d\eta_t$ is zero when $\widehat{g}_t = 0$, η_t itself can take any positive value and thus it still bears implications for the pricing of assets in the economy.

2.3 Equilibrium Pricing

To compute the equilibrium, we first write the optimization problem of each agent under agent B's probability measure $\mathbb{P}^B$. Since we have decided to work (without loss of generality) under $\mathbb{P}^B$, let us write from now on and for notational ease the following conditional expectations operator

$$\mathbb{E}_t[\cdot] \equiv \mathbb{E}^B[\cdot \mid \mathscr{O}_t]. \tag{19}$$

The market is complete in equilibrium since under the observation filtration of both agents there is a single source of risk. Consequently, we can solve the problem using the martingale approach of Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989).[13]

Proposition 3. *(Equilibrium) Assume that the coefficient of relative risk aversion α is an integer.*[14] *The equilibrium price of the risky asset at time t is*

$$S_t = \int_t^{\infty} S_t^u du, \tag{20}$$

where S_t^u is

$$S_t^u = \mathbb{E}_t\left[\frac{\xi_u^B}{\xi_t^B}\delta_u\right] = e^{-\rho(u-t)}\delta_t^{\alpha}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\omega(\eta_t)^j\left[1-\omega(\eta_t)\right]^{\alpha-j}\mathbb{E}_t\left[\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\delta_u^{1-\alpha}\right], \tag{21}$$

where ξ^B denotes the state-price density perceived by agent B

$$\xi_t^B = e^{-\rho t}\delta_t^{-\alpha}\left[\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} + \left(\frac{1}{\kappa_B}\right)^{1/\alpha}\right]^{\alpha}, \tag{22}$$

and $\omega(\eta)$ denotes agent A's share of consumption

$$\omega(\eta_t) = \frac{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}}{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} + \left(\frac{1}{\kappa_B}\right)^{1/\alpha}}. \tag{23}$$

[13] The martingale approach transforms the dynamic consumption and portfolio choice problem into a consumption choice problem subject to a static, lifetime budget constraint.

[14] This assumption greatly simplifies the calculus. To the best of our knowledge, it has been first pointed out in Yan (2008) and Dumas et al. (2009). If the coefficient of relative risk aversion is real, the computations can still be performed using Newton's generalized binomial theorem.

The risk free rate r and the market price of risk θ are

$$r_t = \rho + \alpha \widehat{f}_{Bt} - \alpha\omega(\eta_t)\widehat{g}_t + \frac{1}{2}\left[\frac{\alpha-1}{\alpha\sigma_\delta^2}\omega(\eta_t)(1-\omega(\eta_t))\widehat{g}_t^2 - \alpha(\alpha+1)\sigma_\delta^2\right] \tag{24}$$

$$\theta_t = \alpha\sigma_\delta + \omega(\eta_t)\frac{\widehat{g}_t}{\sigma_\delta}. \tag{25}$$

Proof. The proof mainly follows Dumas et al. (2009) and is provided in Appendix A.4. The moment-generating function in Equation (21) is solved in Appendix A.5. □

Equation (22) shows how the state-price density ξ^B depends on the outlook variable η. Since disagreement $\widehat{g}$ directly drives the volatility of the state-price density (as shown in Equation 18), it follows from (22) that persistence in disagreement generates persistence in the volatility of the state-price density. Therefore, even though in our model agents disagree about a drift component, it directly impacts the diffusion of the state price density and consequently all the equilibrium quantities.

The optimal share of consumption, stated in Equation (23), is exclusively driven by the outlook variable η. If η tends to infinity, which means that agent A's perception of the economy is more likely than agent B's perception[15], then agent A's share of consumption tends to one. Conversely, if η tends to zero, then $\omega(\eta)$ converges to zero. Unsurprisingly, agent A's consumption share increases with the likelihood of agent A's probability measure being true.

The single-dividend paying stock, expressed in Equation (21), consists in a weighted sum of expectations, with weights characterized by the consumption share $\omega(\cdot)$, which itself is driven by the economic outlook η. It is instructive to study first the case $\alpha = 1$ (log-utility case), when the price of the single-dividend paying stock becomes

$$S_t^u = \omega(\eta_t)S_{At}^u + [1-\omega(\eta_t)]\, S_{Bt}^u, \tag{26}$$

where S_{it}^u is the price of the asset in a hypothetical economy populated by only group i agents. A similar aggregation result is provided by Xiong and Yan (2010). In contrast, when the coefficient of relative risk aversion is greater than one, the aggregation must be adapted to accomodate the additional intermediary terms (for $j = 1, ..., \alpha - 1$) in the summation (21). In fact, the summation has now $\alpha + 1$ terms and the price of the single dividend paying

[15]This can be seen from Equation (18): high η can arise either if (i) agent B is optimistic ($\widehat{g} > 0$) and $\widehat{W}_B$ shocks are negative or if (ii) agent B is pessimistic ($\widehat{g} < 0$) and $\widehat{W}_B$ shocks are positive.

stock becomes

$$S_t^u = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega(\eta_t)^j \left[1 - \omega(\eta_t)\right]^{\alpha - j} S_{jt}^u, \tag{27}$$

where S_{jt}^u is the price of the asset in a hypothetical economy populated by agents with relative economic outlook $\eta^{j/\alpha}$ ($j = 0$ corresponds to agent B and $j = \alpha$ corresponds to agent A). Since the binomial coefficients in (27) sum up to one, the price is therefore a weighted average of $\alpha + 1$ prices arising in representative agent economies populated by agents with relative economic outlook $\eta^{j/\alpha}$. Hence, the outlook variable η not only affects the price valuation through the expectations in (21), but also through the weights in the summation (27).

The weighted average form (27) highlights the origin of fluctuations in stock price volatility and the key role played by disagreement and the relative outlook η. The intuition is as follows. The relative outlook η fluctuates in the presence of disagreement and causes investors to speculate against each other. This speculative activity generates fluctuations in consumption shares: if the hypothetical investor j's model is confirmed by the data, he or she will consume more and thus his or her weight in the pricing formula (27) increases. The price S_t^u will therefore approach S_{jt}^u not only through the expectation but also through changes in the relative weights. These fluctuations in relative weights further amplify the impact of disagreement on the stock price.

3 Return Volatility and Trading Volume

We analyze the impact of model disagreement on the stock return volatility and trading volume. We show that economic outlook plays a pivotal role in generating excess volatility and trading volume in financial markets. We then turn to the implications of model disagreement and different economic outlook for the level, fluctuations, and persistence of volatility.

Proposition 4. *(Stock Return Volatility) The time t stock return volatility satisfies*

$$|\sigma_t| = \left| \frac{\sigma(X_t)^\top \frac{\partial S_t}{\partial X_t}}{S_t} \right| = \left| \frac{\sigma(X_t)^\top \int_0^\infty \frac{\partial S_t^u}{\partial X_t} du}{\int_0^\infty S_t^u du} \right|, \tag{28}$$

where $\sigma(x_t)$ denotes the diffusion of the state vector $x = (\zeta,\ \widehat{f}_B,\ \widehat{g},\ \mu)$. The stock return diffusion, σ_t, can be written

$$\sigma_t = \sigma_\delta + \underbrace{\frac{S_f}{S} \frac{\gamma_B}{\sigma_\delta}}_{\equiv \sigma_{f,lr}} + \underbrace{\frac{S_g}{S} \left(\frac{\gamma_B - \gamma_A}{\sigma_\delta} \right)}_{\equiv \sigma_{g,lr}} + \underbrace{\frac{-S_\mu}{S \sigma_\delta} \widehat{g}_t}_{\equiv \sigma_{g,i}}, \tag{29}$$

where S_f, S_g, and S_μ represent partial derivatives of stock price with respect to $\widehat{f}_B$, $\widehat{g}$, and $\mu \equiv \ln \eta$ respectively.

Proof. The diffusion of the state vector $(\zeta,\ \widehat{f}_B,\ \widehat{g},\ \mu)$ is obtained from Equations (12), (13), (15), and (18). Multiply these with $S_\zeta/S = 1$, S_f/S, S_g/S, and S_μ/S to obtain (29). □

Equation (29) shows that the stock return diffusion σ consists in the standard Lucas (1978) volatility σ_δ and three terms representing the *long-run* impact of changes in the estimated fundamental $\widehat{f}_B$ (denoted by $\sigma_{f,lr}$), the *long-run* impact of changes in the disagreement $\widehat{g}$ (denoted by $\sigma_{g,lr}$), and the *instantaneous* impact of changes in the disagreement $\widehat{g}$ (denoted by $\sigma_{g,i}$). Since we assume the volatility of the dividend σ_δ to be constant, the volatility of the price-dividend ratio is exclusively driven by these last 3 terms. Therefore, all the following interpretations apply to both the stock return volatility and the volatility of the price-dividend ratio.

3.1 Economic Outlook, Excess Volatility, and Trading Volume

We use a numerical example to show how model disagreement leads to excess volatility, even when $\widehat{g}$ is currently zero. Specifically, we compare an economy populated by a representative agent to one populated by two agents with model disagreement, when the two settings are observationally equivalent in terms of their average expected growth rate and average uncertainty. By shutting down the direct effect of disagreement, this exercise highlights the key role played by differences in economic outlook in generating excess volatility.

The calibration is provided in Table 1. These parameters are adapted from Brennan and Xia (2001) and Dumas et al. (2009), with a few differences. We choose lower values for the volatility of the fundamental and the dividend growth volatility. For the preference parameters, we choose a smaller coefficient of relative risk aversion and a positive subjective discount rate.

Parameter	**Symbol**	**Value**
Relative Risk Aversion	α	3
Subjective Discount Rate	ρ	0.015
Agent A's Initial Share of Consumption	ω_0	0.5
Consumption Growth Volatility	σ_δ	0.03
Mean-Reversion Speed of the Fundamental	λ_A	0.3
	λ_B	0.1
Long-Term Mean of the Fundamental	$\bar{f}$	0.025
Volatility of the Fundamental	σ_f	0.015

Table 1: Calibration

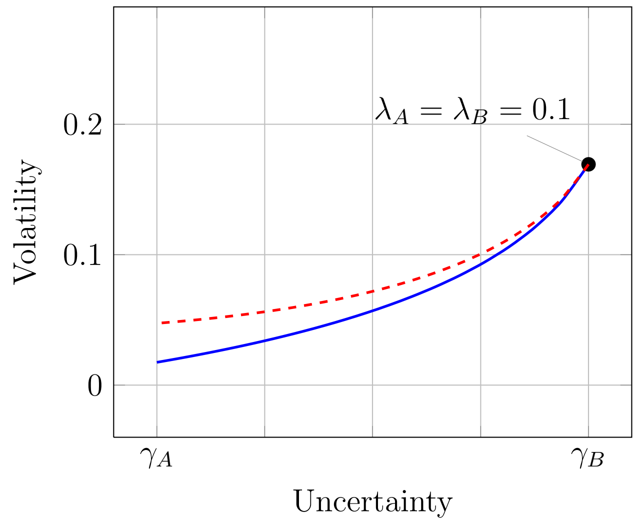

Figure 2: Amplification of volatility through model disagreement
The graph compares stock volatility in two economies that are observationally equivalent with respect to (i) uncertainty, (ii) average views of the agents on the growth rate, and (iii) disagreement. The only difference between the economies is the *existence* of model disagreement in one of them, case represented by the red dashed line. The blue solid line thus depicts an observationally equivalent representative agent economy. Parameters are provided in Table 1.

The mean-reversion speed chosen by agent B is 0.1, corresponding to a business cycle half-life of approximately seven years. Agent B consequently believes in long-run risk. On the other hand, agent A, who choses $\lambda_A = 0.3$, believes that the length of the business is shorter with a perceived half-life of approximately two years. We assume that the true λ lies somewhere in between λ_B and λ_A, and thus neither agent has a superior learning model.

Suppose now that in one economy a representative agent uses a mean-reversion parameter $\lambda_{\text{rep}} \in [0.1, 0.3]$. Different levels of λ_{rep} result in different levels of uncertainty, denoted hereafter γ_{rep}. As such, if the agent believes the growth rate of the economy to be persistent, uncertainty is higher—due to the long-run risk effect—and thus volatility is higher. This is reflected by the blue solid line in Figure 2. If the representative agent believes λ_{rep} to be 0.3, then uncertainty takes the value γ_A. As λ_{rep} decreases, uncertainty rises up to γ_B, which is attained for $\lambda_{\text{rep}} = 0.1$. To keep it simple, we assume that the filtered growth rate of the representative agent is $\widehat{f}_{\text{rep}} = \bar{f}$. Figure 2 thus confirms the direct, positive, effect of uncertainty on volatility in a representative agent economy.

Now, compare this to a second economy where there is model disagreement and assume that there are equal consumption weights for the agents (i.e., $\omega_A = \omega_B = 1/2$) and equal expected growth rates (i.e., $\widehat{f}_{\text{rep}} = \widehat{f}_A = \widehat{f}_B = \bar{f}$). To make the comparison meaningful, we keep the underlying uncertainty equal to the previous case, so that with $\lambda_B = 0.1$ and $\lambda_A \geq 0.1$, uncertainty in the representative agent economy equals the weighted average uncertainty in the heterogeneous agent economy; that is, λ_A solves

$$\gamma_{\text{rep}} \equiv \omega_A \gamma_A + \omega_B \gamma_B. \tag{30}$$

We then compute the volatility that arises for all values of λ_A which solve (30) and with $\lambda_B = 0.1$ fixed.

The red dashed line in Figure 2 shows that model disagreement amplifies volatility with respect to a representative agent economy. This is meaningful because the two economies are observationally equivalent. Indeed, (i) uncertainty is the same and equals γ_{rep}, (ii) the average views of agents on the growth rate are the same and equal $\bar{f}$, and (iii) disagreement is the same and equals $\widehat{g} = 0$. The only difference between these economies is the existence of model disagreement, which presumably is not observable by the econometrician. But this difference generate excess volatility through different economic outlooks—even if agents agree today, they hold different economic outlooks about the future.[16] Given this, it appears that, through the different economic outlooks that it generates, model disagreement amplifies volatility beyond that observed in an observationally equivalent representative agent economy.

Now, we consider how model disagreement affects trading volume in the economy. Trading volume represents the absolute value of the change in agents' risky position. Measuring trading volume is straightforward in discrete time. In continuous time, however, diffusion processes have infinite variation. We therefore follow Xiong and Yan (2010) and proxy trading volume with the volatility of agents' risky position changes.[17] For this matter, picking agent A or agent B gives the same measure of trading volume. In order to be consistent with what has been done so far, we choose to focus on agent B.

The number of assets held by agent B, is given by the martingale representation theorem:

$$\pi_{B,t} S_t \sigma_t = \frac{\partial V_{Bt}}{\partial x_t} \sigma(x_t) \tag{31}$$

where V_{Bt} is the wealth of agent B at time t (provided in Appendix A.4) and $x_t = (\zeta \ \widehat{f}_B \ \widehat{g} \ \mu)$ is the state vector of the economy. Equation (31) states that fluctuations in the price of the risky asset, scaled by the number of assets held by agent B, are perfectly matched to fluctuations in agent B's wealth. In other words, the agent's position in the risky asset is set in such a way to replicate wealth fluctuations. Naming the term on the right hand side

[16] Separate calculations show that volatility is further amplified with respect to the dashed red line when $\widehat{g}_t < 0$ (i.e., when the long-term agent B is pessimistic) and remains almost unchanged with respect to the dashed red line when $\widehat{g}_t > 0$ (i.e., when the long-term agent B is optimistic).

[17] Trading actually occurs in discrete time and it is thus reasonable to measure changes in position across small intervals (but finite). On average these changes increase with the volatility of investors' risky position changes.

Economy	Parameters	Trading volume
(1) No model disagreement	$\lambda_A = 0.3,\ \lambda_B = 0.3$	0
(2) Moderate model disagreement	$\lambda_A = 0.3,\ \lambda_B = 0.2$	0.085
(3) Severe model disagreement	$\lambda_A = 0.3,\ \lambda_B = 0.1$	0.200
(4) Moderate model disagreement	$\lambda_A = 0.2,\ \lambda_B = 0.1$	0.155
(5) No model disagreement	$\lambda_A = 0.1,\ \lambda_B = 0.1$	0

Table 2: Model disagreement and trading volume

$\sigma_{V_B,t}$, the position in the risky asset is

$$\pi_{B,t} = \frac{\sigma_{V_B,t}}{S_t \sigma_t} \tag{32}$$

We are interested in measuring fluctuations in this position. These fluctuations can be gauged either by simulations, or by simply computing the absolute value of the position's diffusion:

$$\sigma\left(\pi_{B,t}\right) = \left| \frac{\partial \pi_{B,t}}{\partial \zeta_t} \sigma_\delta + \frac{\partial \pi_{B,t}}{\partial \widehat{f}_B} \frac{\gamma_B}{\sigma_\delta} + \frac{\partial \pi_{B,t}}{\partial \widehat{g}} \frac{\gamma_B - \gamma_A}{\sigma_\delta} - \frac{\partial \pi_{B,t}}{\partial \mu_t} \frac{1}{\sigma_\delta} \widehat{g}_t \right| \tag{33}$$

Inspecting (33), the last term shows how disagreement *directly* moves trading volume. Of course, it does enter indirectly as well through the partial derivatives, as do the other state variables.

By the same train of thought as before, we assume that $\widehat{g} = 0$ so we can compare economies populated by two agents having model disagreement with those populated by a representative agent. Table 2 describes the five distinct economies we analyze, which are different with respect to the set of parameters (λ_A, λ_B) considered. Severity of model disagreement is measured by the distance between the parameters λ_A and λ_B. As such, two of the economies feature moderate model disagreement (economies 2 and 4), one economy features more severe model disagreement (economy 3), and the last two economies have a representative agent (economies 1 and 5).

The last column of Table 2 shows that the level of trading volume increases with the severity of model disagreement. Clearly, trading volume is zero in the representative agent cases (economies 1 and 5). In between, agents take on speculative positions against each other, which increases trading volume. These results also show that investors change their positions even though disagreement today is zero, i.e., $\widehat{g} = 0$. They do so because they know that their underlying models are different and thus they have different economic outlooks. Once again, the key variable in generating trading volume is not disagreement *per se*, but the relative economic outlook.

3.2 Characteristics of Volatility

We turn now to the analysis of the characteristics of stock return volatility. We highlight the key role played by the relative outlook variable η in the propagation of disagreement shocks to volatility shocks. We then show that, while the level of volatility is mostly driven by long-run risk as in Bansal and Yaron (2004), both the variation and persistence of volatility are driven by disagreement.

3.2.1 Level and Variation of Volatility

Coming back to Proposition 4, a direct analysis of the stock diffusion formula (29) is obscured by the presence of the partial derivatives S_f, S_g, and S_μ. These derivatives depend on the state variables themselves and thus are time-varying. In order to gain more intuition and to understand which terms drive the level of volatility and which ones drive its fluctuations, we simulate the last three terms in Equation (29). Simulations are done at weekly frequency for 100 years. Figure 3 illustrates one simulated path of the stock return diffusion and its components. The significant driver of changes in stock market volatility is the fourth term in Equation (29), $\sigma_{g,i}$, whereas terms representing long-run changes in disagreement, $\sigma_{g,lr}$, and long-run changes in the estimated fundamental, $\sigma_{f,lr}$, are slightly time-varying but have less significant impact on the dynamics of volatility.

The fourth term in Equation (29) is therefore key to understanding the impact of disagreement on stock return volatility. This term consists in the partial derivative of the stock price with respect to the relative outlook variable η multiplied by the volatility of η, which itself is directly driven by disagreement (according to 18). Both disagreement and economic outlook therefore play a role in driving volatility, by the following mechanism. When agents are in disagreement, they hold different economic outlooks and thus the stock price fluctuates in order to accommodate speculative trading by both agents. Higher disagreement generates large fluctuations in economic outlook (according to 18) and thus large changes in the stock price. One can therefore say that disagreement drives the volatility of stock returns *through changes in the relative economic outlook*.

To disentangle the role played by disagreement from the role played by the relative economic outlook, we plot in the left panel of Figure 4 the fourth diffusion component $\sigma_{g,i}$ and the disagreement $\widehat{g}$. The correlation coefficient between the two lines in this particular example yields a value of 0.95. In the right pannel of Figure 4 we plot the distribution of the correlation between the diffusion term and disagreement for 1,000 simulations and we find that the coefficient stays mainly between 0.8 and 1. It is therefore disagreement which drives the fluctuations in $\sigma_{g,i}$, whereas the relative economic outlook is the primary channel through which these fluctuations are transmitted to stock market volatility.

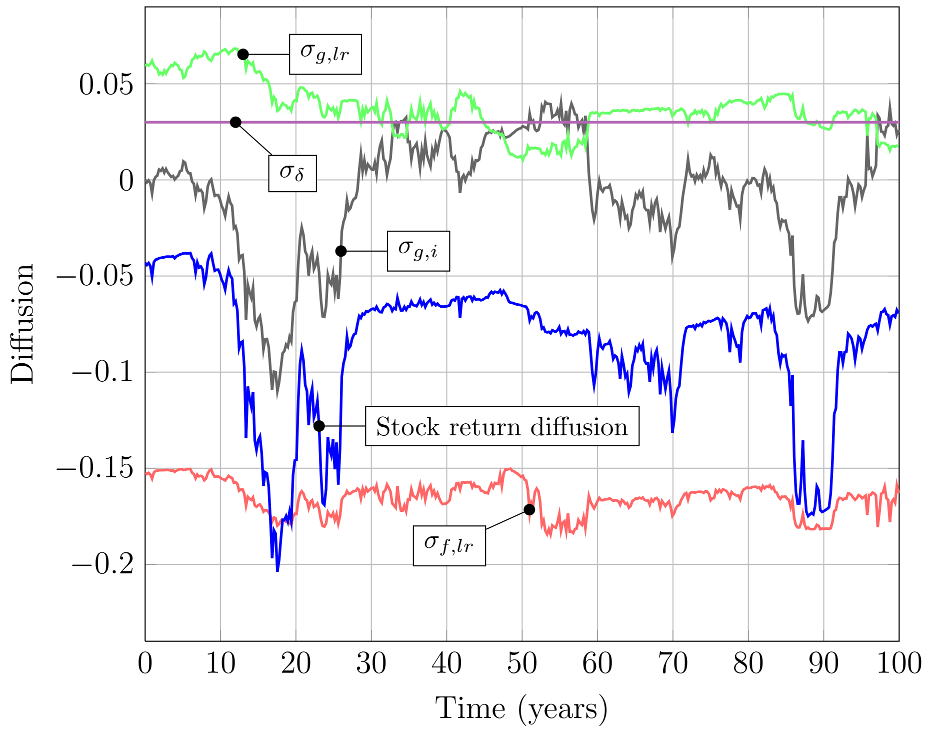

Figure 3: Stock return diffusion and its components

One simulated path (100 years) of the stock return diffusion and its components. Simulations are performed at weekly frequency, but lines are plotted at quarterly frequency to avoid graph cluttering. The diffusion components $\sigma_{f,lr}$, $\sigma_{g,lr}$, and $\sigma_{g,i}$ are defined in Equation (29). The calibration is provided in Table 1.

We examine whether the dynamics illustrated on Figure 3 are particular to one simulation. To this end, we plot in Figures 5 and 6 the distributions of the averages and variances of $\sigma_{f,lr}$, $\sigma_{g,lr}$, and $\sigma_{g,i}$. Averages and variances are computed over the length of each simulation which is chosen to be 100 years at weekly frequency.

Figure 5 shows that the diffusion components $\sigma_{g,lr}$ and $\sigma_{g,i}$ do not have a significant impact on the level of volatility. The level of volatility is primarily determined by the $\widehat{f}_B$-term defined by $\sigma_{f,lr}$. It is worth mentioning that the $\widehat{f}_B$-term is negative because in our model the precautionary savings effect dominates the substitution effect. Indeed, a positive shock in the fundamental increases future consumption. Because agents want to smooth consumption over time, they increase their current consumption and so reduce their current investment. This tendency to disinvest outweighs the substitution effect (which pushes investors to invest more) and implies a drop in prices as long as agents are sufficiently risk averse ($\alpha > 1$). Hence the stock return diffusion component determined by changes in the fundamental, $\sigma_{f,lr}$, is negative. The smaller the mean-reversion speed λ_B, the more negative the $\sigma_{f,lr}$ component, and consequently the larger stock return volatility becomes. The reason is that a small mean-reversion speed implies a significant amount of long-run

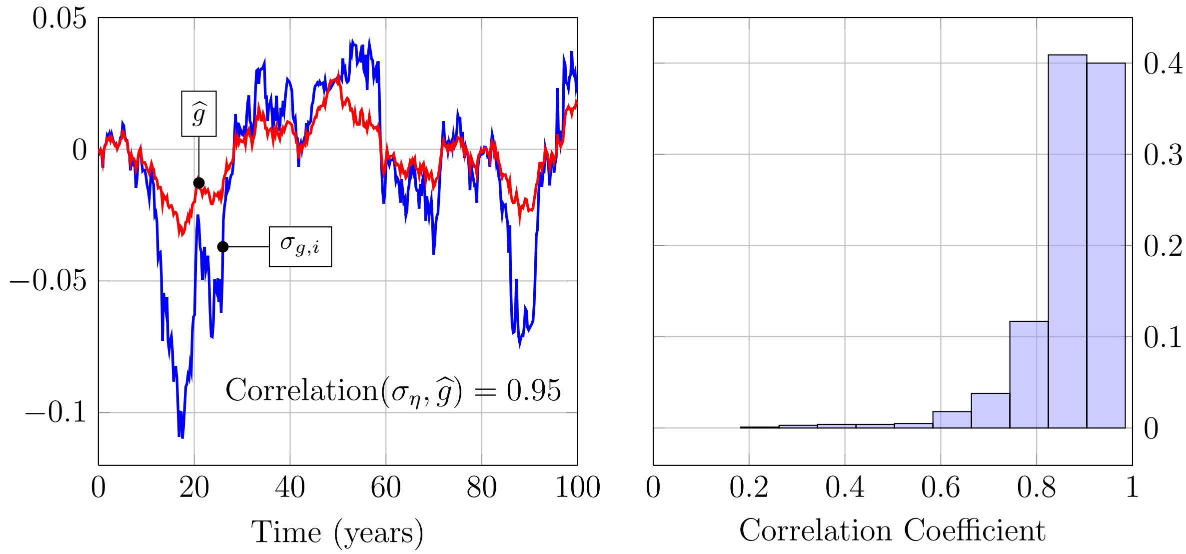

Figure 4: Volatility component $\sigma_{g,i}$ and disagreement $\widehat{g}$
The left panel depicts one 100 years simulation of the volatility component $\sigma_{g,i}$ and the associated disagreement $\widehat{g}$. Simulations are performed at weekly frequency, but lines are plotted at quarterly frequency to avoid graph cluttering. The volatility term $\sigma_{g,i}$ is defined in Equation (29). The calibration is provided in Table 1. The right panel shows the distribution of the correlation between $\sigma_{g,i}$ and $\widehat{g}$. This correlation is computed over an horizon of 100 years (simulated at weekly frequency), for 1,000 simulations.

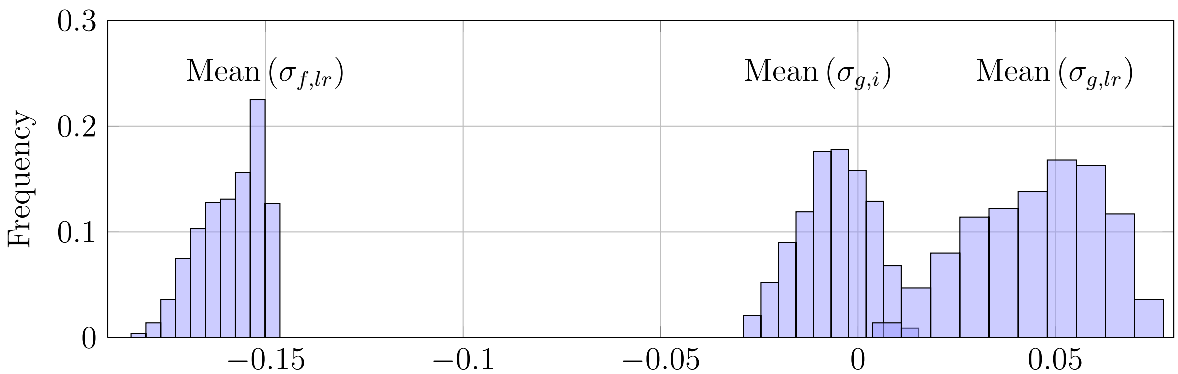

Figure 5: Distribution of the average of the diffusion components
The average over 1,000 simulations of each of the last three diffusion components in Equation (29) is computed over a 100 years horizon, at weekly frequency. The calibration is provided in Table 1.

risk and therefore the stock price is very sensitive to movements in the fundamental, as in Bansal and Yaron (2004).

We try now to understand which components drive the variability of stock return diffusion. This is shown in Figure 6, which depicts the variances of the diffusion components and confirms the conclusions drawn from the example depicted in Figure 3. Variations incurred by the stock return diffusion are almost exclusively generated by variations in the third and

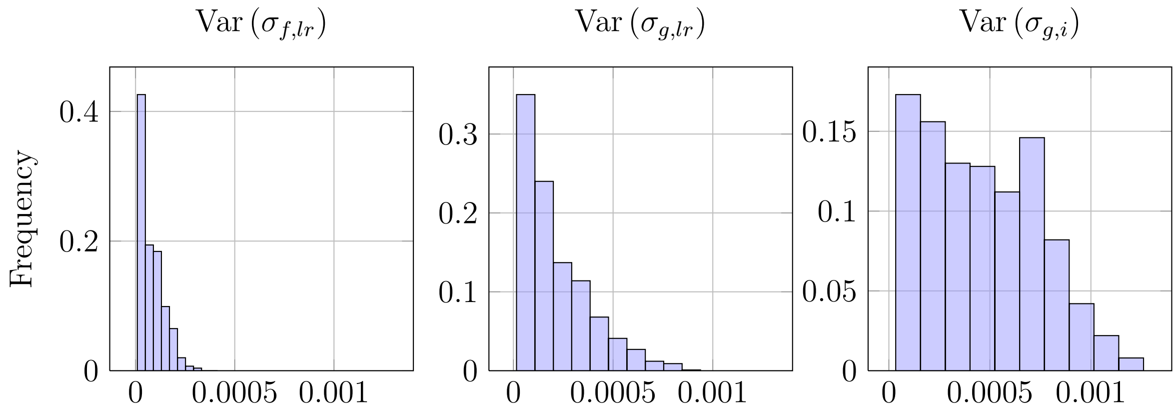

Figure 6: Distribution of the variance of the diffusion components

The variance over 1,000 simulations of each of the last three diffusion components in Equation (29) is computed over a 100 years horizon, at weekly frequency. The calibration is provided in Table 1.

fourth diffusion terms, $\sigma_{g,lr}$ and $\sigma_{g,i}$, which are both driven by disagreement. Indeed, variations in $\sigma_{f,lr}$ are relatively small. We can therefore conclude that the level of the volatility is mainly driven by the persistence of the expected consumption growth, whereas fluctuations in volatility are driven by differences of beliefs regarding the persistence of the expected consumption growth.

3.2.2 Persistence of Volatility

We turn now to the time-varying properties of volatility and address the question whether the fluctuations in volatility generated by disagreement are persistent. Indeed, stock return volatility clusters in our model because of the following mechanism. As shown in Proposition 1, disagreement $\widehat{g}$ mean-reverts to a stochastic mean driven by $\widehat{f}_B$. Because one of the agents (in this case agent B) believes the fundamental is persistent, agent B's estimation of the fundamental $\widehat{f}_B$ is persistent and so becomes the disagreement. Given that the disagreement enters the diffusion of state-price density through the outlook variable η (see Proposition 2) and then enters volatility through the last component in Equation (29), stock return volatility clusters. This mechanism, new to our knowledge, shows how persistence in the fundamental (a component of the *drift*) can transmute into the *diffusion* of stock return and generate volatility clustering.

To provide evidence that persistent disagreement indeed implies GARCH-type dynamics, we simulate 1,000 paths of stock returns over a 100 years horizon at weekly frequency. For each simulated path we compute the demeaned returns, ϵ, by extracting the residuals of the

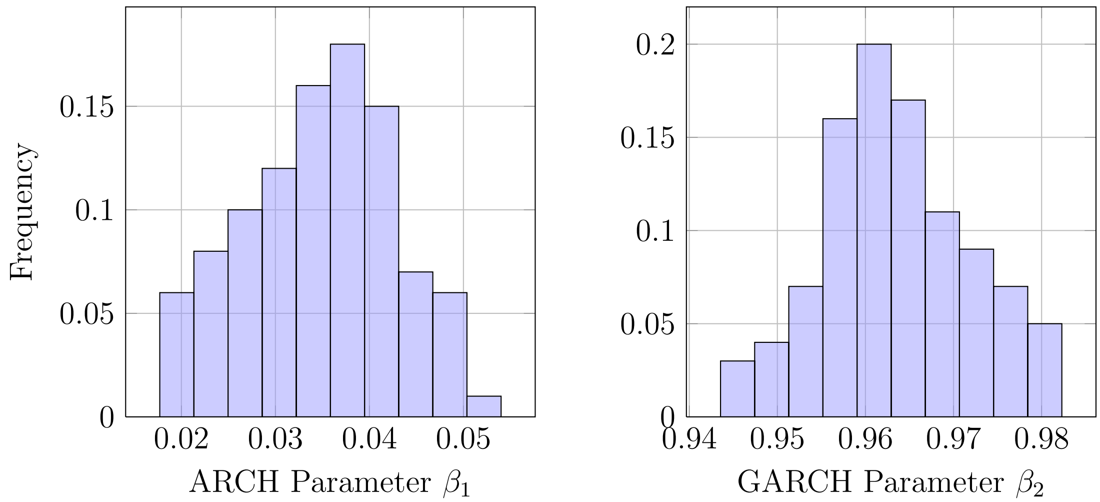

Figure 7: Model implied ARCH and GARCH parameters
Distribution of the ARCH and GARCH parameters, resulted from 1,000 simulations over 100 years, at weekly frequency. The calibration is provided in Table 1.

$AR(1)$ regression

$$r_{t,t+1} = \alpha_0 + \alpha_1 r_{t-1,t} + \epsilon_{t+1}, \tag{34}$$

where $r_{t,t+1}$ stands for the stock return between time t and $t+1$. The demeaned returns ϵ is then fitted to a GARCH(1,1) process defined by

$$\epsilon_t = \sigma_t z_t, \text{ where } z_t \sim \mathrm{N}(0,1) \tag{35}$$

$$\sigma_{t+1}^2 = \beta_0 + \beta_1 \epsilon_t^2 + \beta_2 \sigma_t^2. \tag{36}$$

Figure 7 illustrates the distribution of the ARCH parameter β_1 and the GARCH parameter β_2. Their associated t-stats range between 6 and 11 for the ARCH parameter and between 150 and 350 for the GARCH parameter. First, the large t-stats suggest that the estimated parameters are significant. Second, the values of β_1, β_2, and in particular their sum show that stock return volatility clusters and is close to be integrated. That is, the model implied volatility clusters because its main driver—the disagreement among agents—is persistent.

3.2.3 Volatility Clustering in Alternative Theoretical Models

We conclude this section with two questions. First, can a single agent framework generate volatility clustering? Second, if agents's difference of beliefs is generated by overconfidence instead of disagreement (Scheinkman and Xiong, 2003; Dumas et al., 2009), can we also observe volatility clustering?

To address the first question, we observe that in a single agent model the last two terms in Equation (29) disappear and volatility depends only on σ_δ and $\sigma_{f,lr}$. The analysis above indicates that this second term does not move significantly. Therefore, without disagreement there are no significant fluctuations in volatility and thus single agent models have difficulties in generating volatility clustering.

Turning now to the second question, Proposition 1 shows that disagreement mean-reverts around a persistent $\widehat{f}_B$ and thus itself becomes persistent. In contrast, disagreement generated by overconfidence, as in Dumas et al. (2009), is not easily persistent, even though both agents would be long-term believers. The reason is that disagreement mean-reverts around zero with a parameter equal to $\lambda + \gamma/\sigma_\delta^2$ (see Lemma 2 in Dumas et al. 2009). Because of the second term, this parameter is large under usual calibrations, which is not enough to generate persistent dynamics.

On a final note, in Appendix A.6, Table 6, we perform a robustness analysis which further confirms the role played by disagreement for generating persistence and fluctuations in volatility. The analysis consists in comparing the properties of the model implied volatility for different calibrations. Consistent with our theoretical results, we find that strong long-run risk increases the average level of volatility, while severe disagreement increases both the variation and the persistence of volatility.

4 Survival

In our model we make the assumption that the fundamental is unobservable. It is consequently reasonable to assume that both investors have different beliefs regarding the dynamics of an unobservable process. Furthermore, it would be arbitrary and non-realistic to assume that one of the two agents has the correct beliefs i.e. the right model in mind. This raises two questions: what is the true data-generating process and how long do agents survive given this true data-generating process? This section is devoted to a discussion of these two questions.

In order to investigate how long each agent survives, we have to assume a realistic data-generating process in the sense that it has to be consistent with agents beliefs. Indeed, although both agents might realistically have wrong beliefs, they agents are not too far from the truth. Therefore, we assume that the true data-generating process is

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma dW_t^\delta \tag{37}$$

$$df_t = \lambda(\bar{f} - f_t)dt + \sigma_f dW_t^f, \tag{38}$$

where W^δ and W^f are two independent Brownian motions under the true probability measure

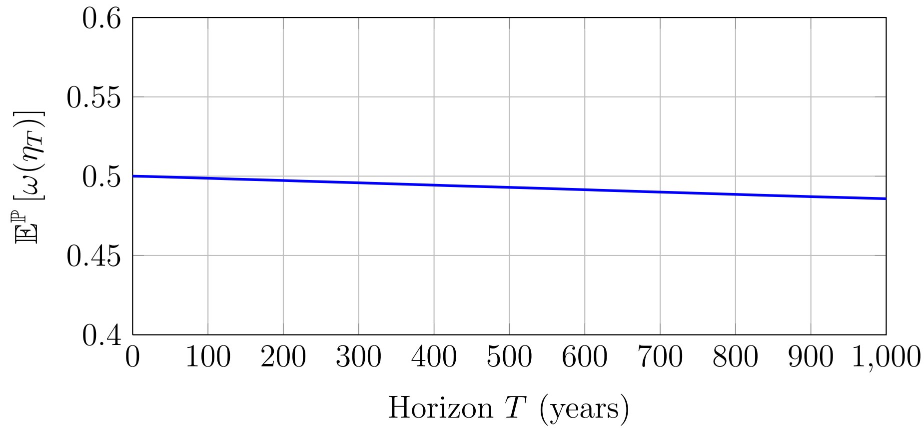

Figure 8: Expected share of consumption of agent A

Expectation under the true probability measure $\mathbb{P}$ of the consumption share of Agent A, $\omega(\eta_T)$. 10,000 simulations over 1,000 years are performed. The calibration is provided in Table 1.

$\mathbb{P}$. The true mean-reversion speed λ is assumed to be the average of agents A and B estimated mean-reversion speeds. Table 3 provides the values of the true and perceived mean-reversion speeds as well as their corresponding half-lives. Both agents misperceive the true length of the business cycle, but one overestimates it whereas the other underestimates it.

Belief	λ	**Half-Life**
Agent A	0.3	2.31
True value	0.2	3.46
Agent B	0.1	6.93

Table 3: Mean-reversion speed and length of the business cycle

To investigate the speed at which agent A or agent B disappears from the economy, we follow Yan (2008) and compute the $\mathbb{P}$-expectation of the consumption share of agent A. To this end we first simulate the dividend process using the true data-generating process provided in (37) and (38). Then, we perform each agent's learning exercise and we compute the expectation of the consumption share of agent A, $\omega(\eta_T)$, for T ranging from 0 to $1,000$ years.

Figure 8 depicts the $\mathbb{P}$-expectation (i.e., under the true probability measure) of the consumption share of agent A over 1,000 years. This expected consumption share slightly decreases on average. Investors who believe in long-run risk are expected to save more and thus have a lower survival index (Yan, 2008). Nevertheless, Figure 8 shows that both shares of consumption remain very close to each other and that both agents survive for more than 1,000 years. This is also consistent with Yan (2008). We conclude therefore that the type of disagreement considered here is economically important over long horizons.

5 Conclusion

We consider a theoretical framework in which two agents interpret information using different economic models of the economy. Specifically, in our setup agents disagree on the length of the business cycle. We analyze the asset pricing implications of such disagreement.

We first show that the disagreement is strongly persistent, affects the volatility of the stochastic discount factor and consequently impacts the stock return volatility. Disagreement generates different expectations of future economic variables, or different economic outlooks, which further amplify the return volatility and the trading volume. We decompose the dynamics of volatility and show that disagreement is the main driver of volatility fluctuations, while the absolute level of volatility is driven primarily by long-run risk. We thus provide a theoretical foundation of the GARCH-like behavior of stock returns.

Several questions are the subject of our ongoing research. First, we assume that investors do not change their economic models. It is important to understand how our results would change if agents were to perform the full learning exercise. Our expectation is that investors' estimates should end up close to the true model only after a very long time. Indeed, an accurate estimator of the mean-reversion speed of a relatively persistent process requires a large sample of data. In addition, the large set of plausible models governing the real economy makes it virtually impossible for the agents to end up in agreement.

The learning uncertainty is by construction constant in our setup. This is because we do not consider here any additional news (newspapers, quarterly reports, economic data, and so on). In a setting with additional news and in which investors' attention to news is fluctuating, uncertainty will fluctuate. Our conjecture is that spikes in attention will exacerbate the disagreement among agents, further amplifying the effects on volatility described in this paper. It is therefore important to study the synergistic relationships between attention, uncertainty, and disagreement and their impact on asset prices.

Finally, our model generates a term structure of disagreement whose shape is governed by the difference between the mean-reversion parameters. Empirically observed term structures of disagreement (as in Patton and Timmermann (2010) or Andrade et al. (2014)) can therefore help estimating the magnitude of the difference between these parameters. The term structure of disagreement should also have implications on the pricing of firms with different characteristics and probably explain well-known anomalies such as the value premium.

A Appendix

A.1 Kalman/Maximum-Likelihood vs. Particle Filtering

Let us assume that the true data-generating process satisfies

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dW_t^\delta \tag{39}$$

$$df_t = \lambda(\bar{f} - f_t)dt + \sigma_f dW_t^f, \tag{40}$$

where W^δ and W^f are independent Brownian motions. The true parameters defining the dynamics of the dividend δ and the fundamental f are provided in Table 4.

Parameter	Symbol	Value
Dividend Growth Volatility	σ_δ	0.03
Mean-Reversion Speed of the Fundamental	λ	0.2
Long-Term Mean of the Fundamental	$\bar{f}$	0.025
Volatility of the Fundamental	σ_f	0.015

Table 4: True parameters

We simulate dividend data at quarterly frequency over a 50-year horizon[18] using the true data-generating process defined in Equations (39) and (40) and the parameters provided in Table 4. Each agent uses the quarterly dividend data to estimate the following discrete-time model

$$\log\left(\frac{\delta_{t+\Delta}}{\delta_t}\right) = \left(f_t - \frac{1}{2}\sigma_\delta^2\right)\Delta + \sigma_\delta\sqrt{\Delta}\epsilon_{1,t+\Delta} \tag{41}$$

$$f_{t+\Delta} = A_f f_t + B_f + C_f \epsilon_{2,t+\Delta}, \tag{42}$$

where $A_f = e^{-\lambda\Delta}$, $B_f = \bar{f}\left(1 - e^{-\lambda\Delta}\right)$, $C_f = \frac{\sigma_f}{\sqrt{2\lambda}}\sqrt{1 - e^{-2\lambda\Delta}}$, and ϵ_1, ϵ_2 are independent Gaussian random variables with mean 0 and variance 1.

Although agents have the same information at hand, we assume that they use different econometrics techniques to perform their estimation exercise. Agent A estimates the unobservable fundamental and the parameters by applying the Kalman filter together with Maximum-Likelihood (Hamilton, 1994), while agent B applies the particle filtering algorithm presented in Liu and West (2001).[19]

Table 5 shows that agent A and B obtain parameter estimates of the dividend volatility σ_δ, the long-term mean of the fundamental $\bar{f}$, and the volatility of the fundamental σ_f that are relatively close to each other. Their estimation of the mean-reversion speed λ, however, differs significantly from one another. Indeed, the absolute difference between the mean-reversion speed estimated by agent A and that estimated by agent B is worth 0.1742. Relative to the true value of the parameter, the difference in the estimated mean-reversion speeds is about 87%, whereas it is less than 25% for all other parameters. Therefore, this calibration exercise motivates, first, our assumption to consider heterogeneity in mean-reversion speeds only and, second, our choice to consider mean-reversion speeds such that $|\lambda_A - \lambda_B| = 0.2$.

[18] The frequency and horizon considered match those of the Real GDP growth time-series available on the Federal Reserve Bank of Philadelphia's website.

[19] We would like to thank Arthur Korteweg and Michael Rockinger for providing us with various particle filtering codes. The particle filtering algorithm of Liu and West (2001) estimates, at each point in time, the unobservable fundamental and the parameters of the model.

Definition	Symbol	Value
Absolute Difference in Dividend Growth Volatility	$\lvert\sigma_{A\delta} - \sigma_{B\delta}\rvert$	0.0022
Absolute Difference in Mean-Reversion Speed of the Fundamental	$\lvert\lambda_A - \lambda_B\rvert$	0.1742
Absolute Difference in Long-Term Mean of the Fundamental	$\lvert\bar{f}_A - \bar{f}_B\rvert$	0.0056
Absolute Difference in Volatility of the Fundamental	$\lvert\sigma_{Af} - \sigma_{Bf}\rvert$	0.0032
Relative Difference in Dividend Growth Volatility	$\frac{\lvert\sigma_{A\delta}-\sigma_{B\delta}\rvert}{\sigma_\delta}$	0.0749
Relative Difference in Mean-Reversion Speed of the Fundamental	$\frac{\lvert\lambda_A-\lambda_B\rvert}{\lambda}$	0.8683
Relative Difference in Long-Term Mean of the Fundamental	$\frac{\lvert\bar{f}_A-\bar{f}_B\rvert}{\bar{f}}$	0.2254
Relative Difference in Volatility of the Fundamental	$\frac{\lvert\sigma_{Af}-\sigma_{Bf}\rvert}{\sigma_f}$	0.2105

Table 5: Estimated parameters: maximum-likelihood vs. particle filter
Agent A applies the Kalman-filter together with Maximum-Likelihood, while agent B applies the particle filter algorithm of Liu and West (2001). The parameter values $\sigma_{B\delta}$, λ_B, $\bar{f}_B$, and σ_{Bf} are those obtained at the terminal time. Numbers reported above are medians computed over 1,000 simulations.

A.2 Filtering Problem

Agent A's learning problem

Following the notations of Liptser and Shiryaev (2001), the observable process is

$$\frac{d\delta_t}{\delta_t} = (A_0 + A_1 f_{At})dt + B_1 dW^f_{At} + B_2 dW^\delta_{At} \tag{43}$$

$$= (0 + 1 \cdot f_{At})dt + 0 \cdot dW^f_{At} + \sigma_\delta dW^\delta_{At}. \tag{44}$$

The unobservable process f_A satisfies

$$df_{At} = (a_0 + a_1 f_{At})dt + b_1 dW^f_{At} + b_2 dW^\delta_{At} \tag{45}$$

$$= (\lambda_A \bar{f} + (-\lambda_A) f_{At})dt + \sigma_f dW^f_{At} + 0 \cdot dW^\delta_{At}. \tag{46}$$

Thus,

$$bob = b_1 b_1' + b_2 b_2' = \sigma_f^2 \tag{47}$$

$$BoB = B_1 B_1' + B_2 B_2' = \sigma_\delta^2 \tag{48}$$

$$boB = b_1 B_1' + b_2 B_2' = 0. \tag{49}$$

The estimated process defined by $\widehat{f}_{At} = \mathbb{E}^{\mathbb{P}^A}(f_{At}|\mathscr{O}_t)$ has dynamics

$$d\widehat{f}_{At} = (a_0 + a_1\widehat{f}_{At})dt + (boB + \gamma_{At}A_1')(BoB)^{-1}(\frac{d\delta_t}{\delta_t} - (A_0 + A_1\widehat{f}_{At})dt), \tag{50}$$

where the posterior variance γ_{At} solves the ODE

$$\dot{\gamma}_{At} = a_1\gamma_{At} + \gamma_{At}a_1' + bob - (boB + \gamma_{At}A_1')(BoB)^{-1}(boB + \gamma_{At}A_1')'. \tag{51}$$

Assuming that we are at the steady-state yields

$$a_1\gamma_{At} + \gamma_{At}a_1' + bob - (boB + \gamma_{At}A_1')(BoB)^{-1}(boB + \gamma_{At}A_1')' = 0. \tag{52}$$

Consequently,

$$d\widehat{f}_{At} = \lambda_A(\bar{f} - \widehat{f}_{At})dt + \frac{\gamma_A}{\sigma_\delta}d\widehat{W}^\delta_{At} \tag{53}$$

where

$$\gamma_A = \sqrt{\sigma_\delta^2(\sigma_\delta^2\lambda_A^2 + \sigma_f^2)} - \lambda_A\sigma_\delta^2 \tag{54}$$

$$d\widehat{W}^\delta_{At} = \frac{1}{\sigma_\delta}\left(\frac{d\delta_t}{\delta_t} - \widehat{f}_{At}dt\right). \tag{55}$$

Agent B's learning problem

The estimated process is defined by $\widehat{f}_{Bt} = \mathbb{E}^B(f_{Bt}|\mathscr{O}_t)$. Doing the same computations as before yields

$$d\widehat{f}_{Bt} = \lambda_B(\bar{f} - \widehat{f}_{Bt})dt + \frac{\gamma_B}{\sigma_\delta}d\widehat{W}^\delta_{Bt}, \tag{56}$$

where

$$\gamma_B = \sqrt{\sigma_\delta^2(\sigma_\delta^2\lambda_B^2 + \sigma_f^2)} - \lambda_B\sigma_\delta^2 \tag{57}$$

$$d\widehat{W}^\delta_{Bt} = \frac{1}{\sigma_\delta}\left(\frac{d\delta_t}{\delta_t} - \widehat{f}_{Bt}dt\right). \tag{58}$$

A.3 Proof of Proposition 1

The dynamics of $\widehat{f}_A$ under the measure $\mathbb{P}^B$ are written

$$\begin{aligned} d\widehat{f}_{At} &= \lambda_A(\bar{f} - \widehat{f}_{At})dt + \frac{\gamma_A}{\sigma_\delta^2}(\widehat{f}_{Bt} - \widehat{f}_{At})dt + \frac{\gamma_A}{\sigma_\delta}d\widehat{W}^\delta_{Bt} \\ &= \lambda_A\bar{f}dt + \lambda_A\widehat{g}_tdt - \lambda_A\widehat{f}_{Bt}dt + \frac{\gamma_A}{\sigma_\delta^2}\widehat{g}_tdt + \frac{\gamma_A}{\sigma_\delta}d\widehat{W}^\delta_{Bt} \end{aligned} \tag{59}$$

because by Girsanov's Theorem

$$d\widehat{W}^\delta_{At} = d\widehat{W}^\delta_{Bt} + \frac{1}{\sigma_\delta}\widehat{g}_tdt. \tag{60}$$

Consequently, the dynamics of $\widehat{g}$ satisfy

$$
\begin{aligned}
d\widehat{g}_t &\equiv d\widehat{f}_{Bt} - d\widehat{f}_{At} && (61)\\
&= \left[(\lambda_A - \lambda_B)(\widehat{f}_{Bt} - \bar{f}) - \left(\lambda_A + \frac{\gamma_A}{\sigma_\delta^2}\right)\widehat{g}_t\right] dt + \frac{\gamma_B - \gamma_A}{\sigma_\delta} d\widehat{W}^\delta_{Bt}. && (62)
\end{aligned}
$$

A.4 Proof of Proposition 3

The optimization problem for agent B is

$$
\begin{aligned}
&\max_{c_{Bt}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_{Bt}^{1-\alpha}}{1-\alpha}dt\right] && (63)\\
&\text{s.t. } \mathbb{E}\left[\int_0^\infty \xi_t^B c_{Bt} dt\right] \le x_{B0}, && (64)
\end{aligned}
$$

where ξ^B denotes the state-price density perceived by agent B and x_{B0} is his or her initial wealth. The problem for agent A (under the probability measure $\mathbb{P}^B$) is

$$
\begin{aligned}
&\max_{c_{At}} \mathbb{E}\left[\int_0^\infty \eta_t e^{-\rho t}\frac{c_{At}^{1-\alpha}}{1-\alpha}dt\right] && (65)\\
&\text{s.t. } \mathbb{E}\left[\int_0^\infty \xi_t^B c_{At} dt\right] \le x_{A0}. && (66)
\end{aligned}
$$

Note how the change of measure enters the objective function of agent A, but that the expectation in the budget constraint (66) does not need to be adjusted. This is because the state-price density inside the expectation, ξ^B, is the one perceived by agent B.[20]

The first-order conditions are

$$
\begin{aligned}
c_{Bt} &= \left(\kappa_B e^{\rho t}\xi^B\right)^{-\frac{1}{\alpha}} && (67)\\
c_{At} &= \left(\frac{\kappa_A}{\eta_t}e^{\rho t}\xi^B\right)^{-\frac{1}{\alpha}}, && (68)
\end{aligned}
$$

where κ_A and κ_B are the Lagrange multipliers associated with the budget constraints of agents A and B. Summing up the agents' optimal consumption policies and imposing market clearing, i.e., $c_{At} + c_{Bt} = \delta_t$, yields the state-price density perceived by agent B:

$$
\xi_t^B = e^{-\rho t}\delta_t^{-\alpha}\left[\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} + \left(\frac{1}{\kappa_B}\right)^{1/\alpha}\right]^\alpha \tag{69}
$$

Substituting the state-price density ξ^B in the optimal consumption policies yields the following consumption sharing rules

$$
\begin{aligned}
c_{At} &= \omega(\eta_t)\,\delta_t && (70)\\
c_{Bt} &= \left[1 - \omega(\eta_t)\right]\delta_t, && (71)
\end{aligned}
$$

[20] Alternatively, we could have defined ξ^A, the state-price density under agent A's probability measure. Then, we would have $\mathbb{E}^A\left[\xi^A \mathbf{1}_x\right] = \mathbb{E}^B\left[\eta\xi^A \mathbf{1}_x\right] = \mathbb{E}^B\left[\xi^B \mathbf{1}_x\right]$ for any event x. This implies that $\xi^B = \eta\xi^A$.

where $\omega(\eta)$ denotes agent A's share of consumption, which satisfies

$$\omega\left(\eta_t\right)=\frac{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}}{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}+\left(\frac{1}{\kappa_B}\right)^{1/\alpha}}. \tag{72}$$

We assume, as in Yan (2008) and Dumas et al. (2009), that the relative risk aversion α is an integer. The state-price density at time T satisfies

$$\xi_T^B = e^{-\rho T}\delta_T^{-\alpha}\left(\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_T}{\kappa_A}\right)^{1/\alpha}\right)^{\alpha} \tag{73}$$

$$= e^{-\rho T}\delta_T^{-\alpha}\frac{1}{\kappa_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{\eta_T\kappa_B}{\kappa_A}\right)^{\frac{j}{\alpha}} \tag{74}$$

$$= e^{-\rho T}\delta_T^{-\alpha}\frac{1}{\kappa_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\eta_t\kappa_B}{\kappa_A}\right)^{\frac{j}{\alpha}}\eta_T^{\frac{j}{\alpha}} \tag{75}$$

$$= e^{-\rho T}\delta_T^{-\alpha}\frac{1}{\kappa_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)}\right)^{j}\eta_T^{\frac{j}{\alpha}}, \tag{76}$$

where the last equality comes from the fact that

$$\omega(\eta_t)=\frac{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}}{\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}} \tag{77}$$

$$1-\omega(\eta_t)=\frac{\left(\frac{1}{\kappa_B}\right)^{1/\alpha}}{\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}} \tag{78}$$

and consequently

$$\left(\frac{\eta_t\kappa_B}{\kappa_A}\right)^{\frac{1}{\alpha}}=\frac{\omega(\eta_t)}{1-\omega(\eta_t)}. \tag{79}$$

Rewriting Equation (78) yields

$$\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}=\left(\frac{1}{1-\omega(\eta_t)}\right)^{\alpha}\frac{1}{\kappa_B}. \tag{80}$$

Thus the single-dividend paying stock price satisfies

$$S_t^T = \mathbb{E}_t\left(\frac{\xi_T^B}{\xi_t^B}\delta_T\right) \tag{81}$$

$$\overset{(76)\text{ and }(80)}{=} \mathbb{E}_t\left(\frac{e^{-\rho T}\delta_T^{-\alpha}\frac{1}{\kappa_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)}\right)^j\eta_T^{\frac{j}{\alpha}}}{e^{-\rho t}\left(\frac{1}{1-\omega(\eta_t)}\right)^{\alpha}\frac{1}{\kappa_B}\delta_t^{-\alpha}}\delta_T\right) \tag{82}$$

$$= \mathbb{E}_t\left(e^{-\rho(T-t)}\frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)}\right)^j\eta_T^{\frac{j}{\alpha}}}{\left(\frac{1}{1-\omega(\eta_t)}\right)^{\alpha}\delta_t^{-\alpha}}\delta_T^{1-\alpha}\right) \tag{83}$$

$$= e^{-\rho(T-t)}(1-\omega(\eta_t))^{\alpha}\delta_t^{\alpha}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)}\right)^j\mathbb{E}_t\left(\eta_T^{\frac{j}{\alpha}}\delta_T^{1-\alpha}\right). \tag{84}$$

Finally the stock price is given by

$$S_t = \int_t^{\infty} S_t^u du. \tag{85}$$

The wealth of agent B at time t satisfies

$$V_{Bt} = \mathbb{E}_t\left(\int_t^{\infty}\frac{\xi_u^B}{\xi_t^B}c_{Bu}du\right). \tag{86}$$

The definitions of agent B's consumption, c_B, the state-price density, ξ^B, and the share of consumption, $\omega(\eta)$, imply that

$$\mathbb{E}_t\left(\frac{\xi_u^B}{\xi_t^B}c_{Bu}\right) = \mathbb{E}_t\left(e^{-\rho(u-t)}\frac{\left[\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_u}{\kappa_A}\right)^{1/\alpha}\right]^{\alpha}}{\left[\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}\right]^{\alpha}\delta_t^{-\alpha}}(1-\omega(\eta_u))\delta_u^{1-\alpha}\right) \tag{87}$$

$$= \mathbb{E}_t\left(e^{-\rho(u-t)}\kappa_B(1-\omega(\eta_t))^{\alpha}\delta_t^{\alpha}\left(\frac{1}{\kappa_B}\right)^{1/\alpha}\left[\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_u}{\kappa_A}\right)^{1/\alpha}\right]^{\alpha-1}\delta_u^{1-\alpha}\right). \tag{88}$$

Since the relative risk aversion α is an integer we have

$$\begin{aligned}
\left[\left(\frac{1}{\kappa_B}\right)^{1/\alpha}+\left(\frac{\eta_u}{\kappa_A}\right)^{1/\alpha}\right]^{\alpha-1}\left(\frac{1}{\kappa_B}\right)^{1/\alpha} &= \left[\left(\frac{\eta_u\kappa_B}{\kappa_A}\right)^{1/\alpha}+1\right]^{\alpha-1}\frac{1}{\kappa_B} \\
&= \frac{1}{\kappa_B}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\left(\frac{\eta_u\kappa_B}{\kappa_A}\right)^{j/\alpha} \\
&= \frac{1}{\kappa_B}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\left(\frac{1}{\eta_t}\right)^{j/\alpha}\left(\frac{\eta_t\kappa_B}{\kappa_A}\right)^{j/\alpha}\eta_u^{j/\alpha} \\
&\overset{(79)}{=} \frac{1}{\kappa_B}\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\left(\frac{1}{\eta_t}\right)^{j/\alpha}\left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)}\right)^{j}\eta_u^{j/\alpha} \qquad (89)
\end{aligned}$$

Substituting Equation (89) in Equation (88) yields the desired result

$$\mathbb{E}_t\left(\frac{\xi_u^B}{\xi_t^B}c_{Bu}\right) = e^{-\rho(u-t)}\delta_t^\alpha\sum_{j=0}^{\alpha-1}\binom{\alpha-1}{j}\omega(\eta_t)^j(1-\omega(\eta_t))^{\alpha-j}\mathbb{E}_t\left(\left(\frac{\eta_u}{\eta_t}\right)^{j/\alpha}\delta_u^{1-\alpha}\right). \qquad (90)$$

A.5 State Vector and Transform Analysis

Finding the equilibrium price boils down to computing the following expectation:

$$\mathbb{E}_t\left[\eta_u^{\frac{j}{\alpha}}\delta_u^{1-\alpha}\right] = \mathbb{E}_t\left[e^{(1-\alpha\ 0\ 0\ j/\alpha\ 0\ 0\ 0)X}\right], \qquad (91)$$

where we define the augmented vector of state variables X by

$$X = \begin{pmatrix}\zeta & \widehat{f}_B & \widehat{g} & \mu & \widehat{g}^2 & \widehat{g}\widehat{f}_B & \widehat{f}_B^2\end{pmatrix}^\top. \qquad (92)$$

In Equation (92), ζ represents the log aggregate consumption ($\zeta \equiv \ln\delta$), whereas μ represents the log relative outlook ($\mu \equiv \ln\eta$). Observe that the vector of state variables (initially four) has been augmented by adding three quadratic and cross-product terms. By doing so, the initially *affine-quadratic* vector $(\zeta, \widehat{f}_B, \widehat{g}, \mu)^\top$ is transformed into the *affine* vector X (see Cheng and Scaillet, 2007). It follows that the expectation in Equation (91) is the moment-generating function of an affine vector and thus we can apply the theory of affine processes (Duffie, Pan, and Singleton, 2000) to compute this quantity, which becomes

$$\mathbb{E}_t\left[\eta_u^{\frac{j}{\alpha}}\delta_u^{1-\alpha}\right] = \mathbb{E}_t\left[e^{(1-\alpha\ 0\ 0\ j/\alpha\ 0\ 0\ 0)X}\right] = e^{\widetilde{\alpha}(u-t)+\widetilde{\beta}(u-t)X_t}. \qquad (93)$$

In Equation (93), $\widetilde{\alpha}$ is a 1-dimensional function of the maturity u, with boundary condition $\widetilde{\alpha}(0) = 0$, whereas $\widetilde{\beta}$ is a 7-dimensional function of the maturity u, with boundary condition $\widetilde{\beta}(0) = (1-\alpha\ 0\ 0\ j/\alpha\ 0\ 0\ 0)$.

In order to solve Equation (93), let us write the dynamics of the affine state-vector X as follows:

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\widehat{W}^{\delta}_{Bt} \tag{94}$$

$$\mu(X) = K_0 + K_1 X \tag{95}$$

$$\left(\sigma(X)\sigma(X)^{\top}\right)_{ij} = H_{0ij} + H_{1ij} \cdot X. \tag{96}$$

From Duffie (2010) we know that

$$\mathbb{E}_t\left(\delta_u^{\epsilon}\eta_u^{\chi}\right) = \mathbb{E}_t\left(e^{\epsilon\zeta_u + \chi\mu_u}\right) = e^{\widetilde{\alpha}(\tau)+\widetilde{\beta}(\tau)X_t}, \tag{97}$$

where $\tau = u - t$ and ϵ and χ are arbitrary constants. $\widetilde{\alpha}$ and $\widetilde{\beta}$ solve the following system of 8 Ricatti ODEs

$$\widetilde{\beta}'(\tau) = K_1^{\top}\widetilde{\beta}(\tau) + \frac{1}{2}\widetilde{\beta}^{\top}(\tau)H_1\widetilde{\beta}(\tau) \tag{98}$$

$$\widetilde{\alpha}'(\tau) = K_0^{\top}\widetilde{\alpha}(\tau) + \frac{1}{2}\widetilde{\beta}^{\top}(\tau)H_0\widetilde{\beta}(\tau) \tag{99}$$

with boundary conditions $\widetilde{\beta}_1(0) = \epsilon$, $\widetilde{\beta}_2(0) = 0$, $\widetilde{\beta}_3(0) = 0$, $\widetilde{\beta}_4(0) = \chi$, $\widetilde{\beta}_5(0) = 0$, $\widetilde{\beta}_6(0) = 0$, $\widetilde{\beta}_7(0) = 0$, and $\alpha(0) = 0$. This system cannot be directly solved in closed form. However, we know that $\widetilde{\beta}_1(\tau) = \epsilon$ and $\widetilde{\beta}_4(\tau) = \chi$. Thus, the system can be written in a matrix Riccati form as follows[21]

$$Z' = J + B^{\top}Z + ZB + ZQZ, \tag{100}$$

where

$$Z = \begin{pmatrix} \Gamma & \widetilde{\beta}_3/2 & \widetilde{\beta}_2/2 \\ \widetilde{\beta}_3/2 & \widetilde{\beta}_5 & \widetilde{\beta}_6/2 \\ \widetilde{\beta}_2/2 & \widetilde{\beta}_6/2 & \widetilde{\beta}_7 \end{pmatrix} \tag{101}$$

and Γ is a function of τ. The matrices J, B, and Q satisfy

$$J = \begin{pmatrix} 0 & -\frac{\epsilon\chi}{2} & \frac{\epsilon}{2} \\ -\frac{\epsilon\chi}{2} & \frac{(\chi-1)\chi}{2\sigma\delta^2} & 0 \\ \frac{\epsilon}{2} & 0 & 0 \end{pmatrix} \tag{102}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -\gamma_A\epsilon + \gamma_B\epsilon - (\lambda_A - \lambda_B)\bar{f} & -\frac{\lambda_A\sigma\delta^2+\gamma_A-\gamma_A\chi+\gamma_B\chi}{\sigma\delta^2} & \lambda_A - \lambda_B \\ \gamma_B\epsilon + \bar{f}\lambda_B & -\frac{\gamma_B\chi}{\sigma\delta^2} & -\lambda_B \end{pmatrix} \tag{103}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2(\gamma_A-\gamma_B)^2}{\sigma\delta^2} & \frac{2\gamma_B(\gamma_B-\gamma_A)}{\sigma\delta^2} \\ 0 & \frac{2\gamma_B(\gamma_B-\gamma_A)}{\sigma\delta^2} & \frac{2\gamma_B^2}{\sigma\delta^2} \end{pmatrix}. \tag{104}$$

Note that we set J_{11} and J_{23} to zero since they can be any real numbers. Using Radon's lemma,

[21]See Andrei and Cujean (2010) for detailed explanations related to this methodology.

we get

$$
\begin{aligned}
Z(\tau) &= Y^{-1}(\tau)X(\tau) \;\text{ where } X \text{ and } Y \text{ satisfy} && (105)\\
X' &= BX + JY, \;\; X(0) = [0]_{3\times 3} && (106)\\
Y' &= -QX - B^{\top}Y, \;\; Y(0) = I_{3\times 3}. && (107)
\end{aligned}
$$

The solution of this system is

$$
\begin{aligned}
(X(\tau)\;\; Y(\tau)) &= (X(0)\;\; Y(0))\, M(\tau), \;\text{ where } M(\tau) \text{ is the matrix exponential} && (108)\\
M(\tau) &= exp\left(\begin{pmatrix} B & -Q \\ J & -B^{\top}\end{pmatrix}\tau\right). && (109)
\end{aligned}
$$

Note that the matrix exponential $M(\tau)$ has to be computed using a Jordan decomposition. Indeed, we have

$$
M(\tau) = Sexp(J_o\tau)S^{-1}, \tag{110}
$$

where J_o and S are, respectively, the Jordan and the similarity matrix extracted from the Jordan decomposition. The Betas are consequently given by

$$
\begin{aligned}
\widetilde{\beta}_1(\tau) &= \epsilon && (111)\\
\widetilde{\beta}_2(\tau) &= \frac{n_{01} + \sum_{i=1}^{8} n_{i1}e^{j_i\tau}}{b_{01} + \sum_{i=1}^{8} b_{i1}e^{j_i\tau}} && (112)\\
\widetilde{\beta}_3(\tau) &= \frac{n_{02} + \sum_{i=1}^{8} n_{i2}e^{j_i\tau}}{b_{02} + \sum_{i=1}^{8} b_{i2}e^{j_i\tau}} && (113)\\
\widetilde{\beta}_4(\tau) &= \chi && (114)\\
\widetilde{\beta}_5(\tau) &= \frac{n_{03} + \sum_{i=1}^{8} n_{i3}e^{j_i\tau}}{b_{03} + \sum_{i=1}^{8} b_{i3}e^{j_i\tau}} && (115)\\
\widetilde{\beta}_6(\tau) &= \frac{n_{04} + \sum_{i=1}^{8} n_{i4}e^{j_i\tau}}{b_{04} + \sum_{i=1}^{8} b_{i4}e^{j_i\tau}} && (116)\\
\widetilde{\beta}_7(\tau) &= \frac{n_{05} + \sum_{i=1}^{8} n_{i5}e^{j_i\tau}}{b_{05} + \sum_{i=1}^{8} b_{i5}e^{j_i\tau}}. && (117)
\end{aligned}
$$

Notice that the function $\widetilde{\alpha}(\tau)$ is obtained through a numerical integration. Thus, this function is not obtained in closed form. Since in our setup $\chi = \frac{j}{\alpha}$ and $\epsilon = 1 - \alpha$, the stock price simplifies to

$$
\begin{aligned}
S_t &= \int_0^{\infty} S_t^{\tau}\, d\tau && (118)\\
&= \delta_t \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega(\eta_t)^j (1-\omega(\eta_t))^{\alpha-j} \times && (119)\\
&\times \int_0^{\infty} e^{-\rho\tau} e^{\widetilde{\alpha}_j(\tau) + \widetilde{\beta}_{2j}(\tau)\widehat{f}_{Bt} + \widetilde{\beta}_{3j}(\tau)\widehat{g}_t + \widetilde{\beta}_{5j}(\tau)\widehat{g}_t^2 + \widetilde{\beta}_{6j}(\tau)\widehat{f}_{Bt}\widehat{g}_t + \widetilde{\beta}_{7j}(\tau)\widehat{f}_{Bt}^2} d\tau. && (120)
\end{aligned}
$$

Even though the above integral is computed numerically, the price process can be simulated very efficiently.

A.6 Robustness Analysis

Table 6 confirms that disagreement and long-run risk have different impacts on stock-return volatility. Indeed, an increase in long-run risk increases the average level of volatility (Bansal and Yaron, 2004), while an increase in disagreement increases both the variation of volatility and the persistence of volatility.

Economy	Parameter	Mean	Min.	Max.	Vol.	Persist.
(1) No model disagreement and severe long-run risk	$\lambda_A = 0.1$ $\lambda_B = 0.1$	0.169	0.164	0.181	0.002	0.463
(2) Moderate model disagreement and strong long-run risk	$\lambda_A = 0.2$ $\lambda_B = 0.1$	0.101	0.081	0.147	0.012	0.913
(3) Severe model disagreement and moderate long-run risk	$\boldsymbol{\lambda_A = 0.3}$ $\boldsymbol{\lambda_B = 0.1}$	**0.085**	**0.046**	**0.199**	**0.028**	**0.997**
(4) Severe model disagreement and weak long-run risk	$\lambda_A = 0.4$ $\lambda_B = 0.2$	0.023	0.013	0.043	0.005	0.996
(5) Severe model disagreement and no long-run risk	$\lambda_A = 0.5$ $\lambda_B = 0.3$	0.005	0.002	0.017	0.003	0.999

Table 6: Properties of volatility for various calibrations

This table presents the mean, minimum, maximum, volatility, and persistence of volatility in five different models. Persistence is calculated as the sum of the parameters β_1 and β_2 in the GARCH(1,1) estimation. In bold is the benchmark model considered throughout the paper. Numbers reported above are (annualized) averages computed over 1,000 simulations of weekly data over a 100-year horizon.

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