

NBER WORKING PAPER SERIES

THE WEALTH DISTRIBUTION IN BEWLEY ECONOMIES WITH INVESTMENT
RISK

Jess Benhabib
Alberto Bisin
Shenghao Zhu

Working Paper 20157
<http://www.nber.org/papers/w20157>

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
May 2014

Thanks to Ben Moll for kindly providing us with very useful detailed comments. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

NBER working papers are circulated for discussion and comment purposes. They have not been peer-reviewed or been subject to the review by the NBER Board of Directors that accompanies official NBER publications.

© 2014 by Jess Benhabib, Alberto Bisin, and Shenghao Zhu. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

The Wealth Distribution in Bewley Economies with Investment Risk
Jess Benhabib, Alberto Bisin, and Shenghao Zhu
NBER Working Paper No. 20157
May 2014
JEL No. E13,E21

ABSTRACT

We study the wealth distribution in Bewley economies with idiosyncratic capital income risk (investment risk). We find, under rather general conditions, a unique ergodic distribution of wealth which displays fat tails (a Pareto distribution in the right tail).

Jess Benhabib
Department of Economics
New York University
19 West 4th Street, 6th Floor
New York, NY 10012
and NBER
jess.benhabib@nyu.edu

Shenghao Zhu
Department of Economics
National University of Singapore
AS2 Level 4, 1 Arts Link
Singapore 117570
ecszhus@nus.edu.sg

Alberto Bisin
Department of Economics
New York University
19 West 4th Street, 5th Floor
New York, NY 10012
and NBER
alberto.bisin@nyu.edu

1 Introduction

Bewley economies, as e.g. in Bewley (1977, 1983) and Aiyagari (1994)¹ represent one of the fundamental workhorses of modern macroeconomics, its main tool when moving away from the study of efficient economies with a representative agent by allowing e.g., for incomplete markets.² In these economies the evolution of aggregate variables does not generally constitute a sufficient representation of equilibrium, which instead requires the characterization of the dynamics of the distributions across heterogeneous agents.

In Bewley economies each agent faces a stochastic process for labor earnings and solves an infinite horizon consumption-saving problem with incomplete markets. Typically, agents are restricted to save by investing in a risk-free bond and are not allowed to borrow. The postulated process for labor earnings determines the dynamics of the equilibrium distributions for consumption, savings, and wealth. More recent specifications of the model allow for aggregate risks and an equilibrium determination of labor earnings and interest rates.³

Bewley models have been successful in the study of several macroeconomic phenomena of interest. Calibrated versions of this class of models have been used to study welfare costs of inflation (Imrohoroglu, 1992), asset pricing (Mankiw, 1986 and Huggett, 1993), unemployment benefits (Hansen and Imrohoroglu, 1992), fiscal policy (Aiyagari, 1995 and Heathcote, 2005), labor productivity (Heathcote, Storesletten, and Violante, 2008a, 2008b; Storesletten, Telmer, and Yaron, 2001; and Krueger and Perri, 2008); see Heathcote-Storesletten-Violante (2010) for a recent survey of the quantitative implications of Bewley models.

Stochastic labor endowments can in principle generate some skewness in the distribution of wealth, especially if the labor endowment process is itself skewed and persistent. On the other hand, Bewley models have generally found it difficult to reproduce the observed distribution of wealth in many countries; see e.g., Aiyagari (1994) and Huggett (1993).⁴ More specifically, they have found it difficult to reproduce the high inequality (as measured, e.g., by Gini coefficients) and the fat tails (as e.g., in Pareto distributions) that empirical distributions of wealth tend to display.⁵ This is because at high wealth

¹The *Bewley economy* terminology is rather generally adopted and has been introduced by Sargent-Ljungqvist (2004).

²The assumption of complete markets is generally rejected in the data; see e.g., Attanasio and Davis (1996), Fisher and Johnson (2006) and Jappelli and Pistaferri (2006).

³These extensions have been first introduced by Huggett (1993) and Aiyagari (1994). See Ljungqvist and Sargent (2004), Ch. 17, for a review of results. See also Rios-Rull (1995) and Krusell-Smith (2006, 2008).

⁴Most empirical studies of labor earnings find some form of stationarity of the earning process; see Guvenen (2007) and e.g., the discussion of Primiceri and van Rens (2006) by Heathcote (2008). Persistent income shocks are often postulated to explain the cross-sectional distribution of consumption but seem hardly enough to produce fat tailed distributions of wealth; see e.g., Storesletten, Telmer, Yaron (2004).

⁵Large top wealth shares in the U.S. since the 60's are documented e.g., by Wolff (1987, 2004). Fat

levels, the incentives for further precautionary savings tapers off and the tails of wealth distribution remain thin; see Carroll (1997) and Quadrini (1999) for a discussion of these issues.⁶

In the present paper we study the wealth distribution in the context of Bewley economies extended to allow for idiosyncratic capital income risk. Capital income risk is naturally interpreted as entrepreneurial risk.⁷ To this end we provide first an analysis of the standard *income fluctuation problem*, as e.g., in Chamberlain-Wilson (2000), extended to account for capital income risk; see Ljungqvist and Sargent (2004), Ch. 16, as well as Rios-Rull (1995) and Krusell-Smith (2006), for a review of results regarding the standard *income fluctuation problem*.⁸ We restrict ourselves to idiosyncratic labor earnings and capital income for simplicity. We finally embed the economy into general equilibrium, through a neoclassical production function along the lines of Aiyagari (1994) and Angeletos (2007), where capital risk and labor earnings are endogenously determined at equilibrium.

Complementing our previous papers (Benhabib, Bisin, and Zhu, 2011 and 2013), which focus on overlapping generation economies,⁹ we show that Bewley economies with idiosyncratic capital income risk display under rather general assumptions a stationary wealth distribution which is fat tailed, more precisely it is a Pareto distribution in the right tail. We also show that it is capital income risk, rather than labor earnings, that drives the properties of the right tail of the wealth distribution.¹⁰

tails for the distributions of wealth are also well documented, for example by Nirei-Souma (2004) for the U.S. and Japan from 1960 to 1999, by Clementi-Gallegati (2004) for Italy from 1977 to 2002, and by Dagsvik-Vatne (1999) for Norway in 1998. Using the richest sample of the U.S., the Forbes 400, during 1988-2003 Klass et al. (2007) find e.g., that the top end of the wealth distribution obeys a Pareto law.

⁶See also Cagetti and De Nardi (2008) for a survey.

⁷Capital income risk has been introduced by Angeletos and Calvet (2005) and Angeletos (2007) and further studied by Panousi (2008) and by ourselves (Benhabib, Bisin, and Zhu, 2011 and 2013). Quadrini (1999, 2000) and Cagetti and De Nardi (2006) study entrepreneurial risk explicitly. We refer to these papers and our previous papers, as well as to Benhabib and Bisin (2006) and Benhabib and Zhu (2008), for more general evidence on the macroeconomic relevance of capital income risk.

⁸The work by Levhari and Srinivasan (1969), Schectman (1976), Schectman and Escudero (1977), Chamberlain-Wilson (2000), Huggett (1993), Rabault (2002), Carroll and Kimball (2005) has been instrumental to provide several incremental pieces to the characterization of the solution of (various specifications of) the income fluctuation problem. On the other hand, sidestepping the income fluctuation problem by assuming a constant savings rate, Nirei (2013) shows that thick tails are a direct consequence of the linearity of the wealth equation.

⁹Other life-cycle models of the distribution of wealth include Huggett (1996) and Rios-Rull (1995).

¹⁰An alternative approach to generate fat tails without stochastic returns or discounting is to introduce a model with bequests, where the probability of death (and/or retirement) is independent of age. In these models, the stochastic component is not stochastic returns but the length of life. For models that embody such features see Wold and Whittle (1957), Castaneda, Gimenez and Rios-Rull (2003) and Benhabib and Bisin (2006). Relatedly, Krusell and Smith (1998) introduce heterogeneous discount rates to produce some skewness in the distribution of wealth.

2 The income fluctuation problem with idiosyncratic capital income risk

Consider an infinite horizon agent at time $t = 0$ choosing a consumption process $\{c_t\}_{t=0}^{\infty}$ and a wealth process $\{a_{t+1}\}_{t=0}^{\infty}$ to maximize his utility, discounted at a rate $\beta < 1$ subject to the accumulation equation for wealth,

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1},$$

where $\{y_t\}_{t=0}^{\infty}$ is the earning process and $\{R_{t+1}\}_{t=0}^{\infty}$ the rate of return process. Suppose the agent also faces a no-borrowing constraint at each time t :

$$c_t \leq a_t.$$

In this paper we consider the following specification of this *income fluctuation problem*:

The utility function is Constant Relative Risk Aversion (CRRA):

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

with $\gamma \geq 1$.

R_t and y_t are stochastic processes, identically and independently distributed (i.i.d.) over time; furthermore,

y_t has probability density function $f(y)$ on bounded support $[\underline{y}, \bar{y}]$, with $\underline{y} > 0$;

R_t has probability density function $g(R)$ on support $[\underline{R}, \infty)$, with $\underline{R} > 0$.

We also impose the following assumptions.

Assumption 1 $\beta ER_t^{1-\gamma} < 1$.

Assumption 2 $(\beta ER_t^{1-\gamma})^{\frac{1}{\gamma}} ER_t < 1$.

Note that Assumption 2 implies that $\beta ER_t < 1$.

Assumption 3 $\Pr(\beta R_t > 1) > 0$ and any finite moment of R_t exists.

Assumption 4 $(\bar{y})^{-\gamma} < \beta E [R_t (y_t)^{-\gamma}]$.

We shall in part relax these assumptions in Section 4.

In summary, the *income fluctuation problem with idiosyncratic capital income risk* (IF) that we study in this section is the following:

$$\begin{aligned} & \max_{\{c_t\}_{t=0}^{\infty}, \{a_{t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} & (1) \\ & s.t. \ a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1} \\ & \quad c_t \leq a_t \\ & \quad a_0 \text{ given.} \end{aligned}$$

It is useful to briefly outline at the outset our strategy to show that the Bewley economy, extended to include idiosyncratic capital income risk, can generate a wealth distribution with fat tails, that is a distribution that does not have all moments. As in the Aiyagari (1994) model, the borrowing constraint together with stochastic incomes assures a lower bound to assets which acts as a reflecting barrier (See Lemma 7 in the Appendix). Since discounted expected returns, $\beta E(R_t)$, are less than 1 on average, the economy contracts, giving rise to a stationary distribution of assets. However, since we cannot obtain explicit solutions for consumption or savings policies, we have to explicitly show that under suitable assumptions there are no disjoint invariant sets or cyclic sets in assets, so that agents do not get trapped in subsets of the support of the asset distribution. In other words we have to show that the stochastic process for assets is ergodic, and that a unique stationary distribution exists. We show this in Lemmas 6 and 8 in the Appendix. We then have to show that, unlike in the basic Aiyagari (1994) model with stochastic earnings and deterministic returns on wealth, introducing idiosyncratic capital income risk can generate a fat-tailed asset distribution. Since explicit linear solutions are not available even under CRRA preferences, we cannot use the results of Kesten (1973) for linear recursions. Instead we use a generalization of the Kesten results for non-linear stochastic processes that are asymptotically linear, due to Mirek (2011). We show the asymptotic linearity of the consumption and savings policies which, under appropriate assumptions, allow us to use the results of Mirek (2011) (see Propositions 3, 4 and 5) and we characterize the fat tail of the stationary distribution in Theorem 3. Finally, using a "span of control" approach with idiosyncratic productivity shocks and a competitive labor market as in Angeletos (2007), we show in Section 5 that our results for a fat tailed distribution of wealth can be embedded in a general equilibrium setting.

In the remainder of this section we show several technical results about the consumption function $c(a)$ which solves this problem, as a build-up for its characterization in the next section. All proofs are in the Appendix.

Theorem 1 *A consumption function $c(a)$ which satisfies the constraints of the IF problem (1) and furthermore satisfies*

i the Euler equation

$$u'(c(a)) \geq \beta ER_{t+1}u'(c[R(a - c(a)) + y]) \text{ with equality if } c(a) < a; \quad (2)$$

and

ii the transversality condition

$$\lim_{t \rightarrow \infty} E\beta^t u'(c_t)a_t = 0. \quad (3)$$

represents a solution of the IF problem given by 1.

By strict concavity of $u(c)$, there exists a unique $c(a)$ which solves 1, the IF problem.

The study of $c(a)$ requires studying two auxiliary problems. The first is a version the IF problem 1, where the stochastic process for earnings $\{y_t\}_0^\infty$ is turned off, that is, $y_t = 0$, for any $t \geq 0$. The second is a finite horizon version of the IF problem 1. In both cases we naturally maintain the relevant specification and assumptions imposed on 1, our main IF problem.

2.1 The IF problem with no earnings

The formal problem is:

$$\begin{aligned} \max_{\{c_t\}_{t=0}^\infty, \{a_{t+1}\}_{t=0}^\infty} E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \quad (4) \\ \text{s.t. } a_{t+1} = R_{t+1}(a_t - c_t) \\ c_t \leq a_t \\ a_0 \text{ given.} \end{aligned}$$

This problem can indeed be solved in closed form, following Levhari and Srinivasan (1969). Note that for this problem the borrowing constraint is never binding because Inada conditions are satisfied for CRRA utility.

Proposition 1 *The unique solution of (4), the IF problem with no earnings, is*

$$c^{no}(a) = \phi a, \text{ for some } 0 < \phi < 1.$$

2.2 The finite IF problem

For any $T > 0$, let the finite IF problem be:

$$\begin{aligned} & \max_{\{c_t\}_{t=0}^T, \{a_{t+1}\}_{t=0}^{T-1}} E \sum_{t=0}^T \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} & (5) \\ \text{s.t. } & a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}, \quad \text{for } 0 \leq t \leq T-1 \\ & c_t \leq a_t, \quad \text{for } 0 \leq t \leq T \\ & a_0 \text{ given.} \end{aligned}$$

With some notational abuse, let c^t denote consumption t periods from the end-period T , that is, at time $T - t$.

Proposition 2 *The unique solution of (5), the finite IF problem, is a consumption function $c^t(a)$ which is continuous and increasing in a . Furthermore, let s^t denote the induced savings function,*

$$s^t(a) = a - c^t(a).$$

Then $s^t(a)$ is also continuous and increasing in a .

2.3 The IF problem

We can now derive a relation between $c^t(a)$, $c^{no}(a)$ and $c(a)$. This result is a straightforward extension of Proposition 2.3 and Proposition 2.4 in Rabault (2002).

Lemma 1 *$\lim_{t \rightarrow \infty} c^t(a)$ exists, it is continuous and satisfies the Euler equation. Furthermore, $\lim_{t \rightarrow \infty} c^t(a)$ is higher than the optimal consumption function of the no labor-income problem,*

$$\lim_{t \rightarrow \infty} c^t(a) \geq c^{no}(a).$$

The main result of this section follows:

Theorem 2 *The solution of (1), the IF problem, is the consumption function $c(a)$ which is obtained as $\lim_{t \rightarrow \infty} c^t(a)$.*

2.4 Characterization of $c(a)$

Let the induced savings function $s(a)$ be

$$s(a) = a - c(a).$$

Proposition 3 *The consumption and savings functions $c(a)$ and $s(a)$ are continuous and increasing in a .*

Carroll and Kimball (2005) show that $c^t(a)$ is concave.¹¹ But Lemma 2 guarantees that $c(a) = \lim_{t \rightarrow \infty} c^t(a)$ and thus $c(a)$ is also a concave function of a .

Proposition 4 *The consumption function $c(a)$ is a concave function of a .*

The most important result of this section is that the optimal consumption function $c(a)$, in the limit for $a \rightarrow \infty$, is linear and has the same slope as the optimal consumption function of the income fluctuation problem with no earnings, ϕ .

Proposition 5 *The consumption function $c(a)$ satisfies $\lim_{a \rightarrow \infty} \frac{c(a)}{a} = \phi$.*

The proof is non-trivial; see the Appendix.

3 The stationary distribution

In this section we study the distribution of wealth in an economy populated by a continuum of measure 1 agents who solve the income fluctuation problem IF, given by 1, when their earnings and investment risk are uncorrelated, that is, i.i.d. in the cross-section. The wealth accumulation equation of the IF problem in 1 is

$$a_{t+1} = R_{t+1}(a_t - c(a_t)) + y_{t+1}. \quad (6)$$

It is useful to compare it with the IF problem given by 4 that has no earnings, $y_t = 0$. Using Lemma 1 we have:

$$\begin{aligned} a_{t+1} &= R_{t+1}(a_t - c(a_t)) + y_{t+1} \\ &\leq R_{t+1}(a_t - c^{no}(a_t)) + y_{t+1} \\ &= R_{t+1}(1 - \phi)a_t + y_{t+1}. \end{aligned}$$

Let

$$\mu = 1 - \phi = (\beta ER^{1-\gamma})^{\frac{1}{\gamma}}.$$

Thus $\mu < 1$. We have

$$a_{t+1} \leq \mu R_{t+1} a_t + y_{t+1}.$$

The main results in this section are the following.

Theorem 3 *There exists a unique stationary distribution for a_{t+1} which satisfies the stochastic wealth accumulation equation 6.*

¹¹See also Carroll, Slacalek, and Tokuoka (2014).

The proof, in the Appendix, requires several steps. First we show that the wealth accumulation process $\{a_{t+1}\}_{t=0}^{\infty}$ induced by equation 6 above is φ -irreducible, *i.e.* there exists a non trivial measure φ on $[\underline{y}, \infty)$ such that if $\varphi(A) > 0$, the probability that the process enters the set A in finite time is strictly positive for any initial condition (see Chapter 4 of Meyn and Tweedie (2009)). We also show that $a = y$ represents a reflecting barrier for the process. To show that there exists a unique stationary wealth distribution we exploit the results in Meyn and Tweedie (2009) and show that the process $\{a_{t+1}\}_{t=0}^{\infty}$ is ergodic.

Finally, in the next theorem we show that the wealth accumulation process $\{a_{t+1}\}_{t=0}^{\infty}$ has a fat tail. We use the characterization of $c(a)$ and $s(a)$ in Section 2.4, and in particular the fact that $\frac{s(a)}{a}$ is increasing in a and $\frac{s(a)}{a}$ approaches μ as a goes to infinity; this allows us to apply some results by Mirek (2011) regarding conditions for asymptotically Pareto stationary distributions for processes induced by non-linear stochastic difference equations.

Theorem 4 *The unique stationary distribution for a_{t+1} which satisfies the stochastic wealth accumulation equation 6 has a fat tail; *i.e.*, there exist $1 < \alpha < \infty$ and an $\epsilon > 0$ arbitrarily small such that*

$$E(M^\epsilon)^\alpha = 1, \quad M^\epsilon = \mu^\epsilon R_t \quad \text{and} \quad \mu - \mu^\epsilon < \epsilon.$$

and

$$\liminf_{a \rightarrow \infty} \frac{\Pr(a_{t+1} > a)}{a^{-\alpha}} \geq C,$$

where C is a positive constant.

Proof. Since $\frac{s(a)}{a}$ is increasing in a and $\frac{s(a)}{a}$ approaches μ as a goes to infinity, we can pick a large a^ϵ such that

$$\mu - \frac{s(a^\epsilon)}{a^\epsilon} < \epsilon.$$

Let

$$\mu^\epsilon = \frac{s(a^\epsilon)}{a^\epsilon}.$$

Thus $\mu - \epsilon < \mu^\epsilon \leq \mu$.

Let

$$l(a) = \begin{cases} s(a), & a \leq a^\epsilon \\ \mu^\epsilon a, & a \geq a^\epsilon \end{cases}. \quad (7)$$

Note that $l(a) \leq s(a)$ for $\forall a \in [\underline{y}, \infty)$, since $\frac{s(a)}{a}$ is increasing in a ; furthermore, the function $l(a)$ in (7) is Lipschitz continuous, since $s(a)$ is Lipschitz continuous.

Let

$$\psi(a) = R_t l(a) + y.$$

Now we apply Theorem 1.8 of Mirek (2011), to show that the stochastic process $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$, induced by $\tilde{a}_{t+1} = \psi(\tilde{a}_t)$, has a unique stationary distribution and that the tail of the stationary distribution for \tilde{a}_{t+1} is asymptotic to a Pareto law, *i.e.*

$$\lim_{a \rightarrow \infty} \frac{\Pr(\tilde{a}_{t+1} > a)}{a^{-\alpha}} = C,$$

where C is a positive constant.

In order to apply Theorem 1.8 of Mirek (2011), we need to verify Assumption 1.6 and Assumption 1.7 of Mirek (2011).

By the definition of $\psi(\cdot)$ we have

$$\lim_{\tau \rightarrow 0} \left[\tau \psi \left(\frac{1}{\tau} a \right) \right] = M^\epsilon a \quad \text{for } \forall a \in [\underline{y}, \infty).$$

Let

$$N_t = \Omega R_t + y_t$$

where

$$\Omega = \max_{a \in [\underline{y}, a^\epsilon]} |s(a) - \mu^\epsilon a|.$$

It is easy to verify that

$$|\psi(a) - M^\epsilon a| < N_t \quad \text{for } \forall a \in [\underline{y}, \infty).$$

Thus $\psi(\cdot)$ satisfies Assumption 1.6 (Shape of the mappings) of Mirek (2011).

Obviously, the conditional law of $\log M^\epsilon$ is non arithmetic. Let $h(d) = \log E(M^\epsilon)^d$. By Assumption 2 we have $E(\mu R_t) < 1$. Thus $h(1) = \log E(M^\epsilon) \leq \log E(\mu R) < 0$. We now show that Assumption 2 and Assumption 3 imply that there exists $\kappa > 1$ such that $\mu^\kappa E(R_t)^\kappa > 1$. By Jensen's inequality we have $E(R_t)^{1-\gamma} \geq (ER_t)^{1-\gamma}$. And Assumption 2 implies that $\beta ER_t < 1$. Thus

$$\mu = (\beta E(R_t)^{1-\gamma})^{\frac{1}{\gamma}} \geq \beta.$$

Thus

$$E(\mu R_t)^\kappa \geq E(\beta R_t)^\kappa \geq \int_{\{\beta R_t > 1\}} (\beta R_t)^\kappa.$$

By Assumption 3, $\Pr(\beta R_t > 1) > 0$. Thus there exists $\kappa > 1$ such that $\mu^\kappa E(R_t)^\kappa > 1$. We could pick μ^ϵ such that $(\mu^\epsilon)^\kappa E(R_t)^\kappa > 1$. Thus $h(\kappa) = \log E(M^\epsilon)^\kappa > 0$. By Assumption 3, any finite moment of R_t exists. Thus $h(d)$ is a continuous function of d . Thus there exists $\alpha > 1$ such that $h(\alpha) = 0$, *i.e.* $E(M^\epsilon)^\alpha = 1$. Also we know that $h(d)$ is a convex function of d . Thus there is a unique $\alpha > 0$, such that $E(M^\epsilon)^\alpha = 1$.

Moreover, $E[(M^\epsilon)^\alpha | \log M^\epsilon] < \infty$, since M^ϵ has a lower bound, and, by Assumption 3, any finite moment of R_t exists.

We also know that $E(N_t)^\alpha < \infty$ since y_t has bounded support and, by Assumption 3, any finite moment of R_t exists.

Thus M^ϵ and N_t satisfy Assumption 1.7 (Moments condition for the heavy tail) of Mirek (2011).

By Lemma 7, $a = \underline{y}$ is a reflecting barrier of the process $\{a_{t+1}\}_{t=0}^\infty$. Also we assume that the support of R_t is unbounded. Thus the support of the stationary distribution for \tilde{a}_{t+1} is unbounded.

Applying Theorem 1.8 of Mirek (2011), we find that the stationary distribution \tilde{a}_{t+1} has a Pareto tail. Finally, we show that the stationary wealth distribution a_{t+1} , has a fat tail.

Pick $a_0 = \tilde{a}_0$. The stochastic process $\{a_{t+1}\}_{t=0}^\infty$ is induced by

$$a_{t+1} = R_{t+1}s(a_t) + y_{t+1}.$$

And the stochastic process $\{\tilde{a}_{t+1}\}_{t=0}^\infty$ is induced by

$$\tilde{a}_{t+1} = R_{t+1}l(\tilde{a}_t) + y_{t+1}.$$

For a path of (R_t, y_t) , we have $a_t \geq \tilde{a}_t$. Thus for $\forall a > \underline{y}$, we have

$$\Pr(a_t > a) \geq \Pr(\tilde{a}_t > a).$$

This implies that

$$\Pr(a_{t+1} > a) \geq \Pr(\tilde{a}_{t+1} > a),$$

since the stochastic processes $\{a_{t+1}\}_{t=0}^\infty$ and $\{\tilde{a}_{t+1}\}_{t=0}^\infty$ are ergodic. Thus

$$\liminf_{a \rightarrow \infty} \frac{\Pr(a_{t+1} > a)}{a^{-\alpha}} \geq \liminf_{a \rightarrow \infty} \frac{\Pr(\tilde{a}_{t+1} > a)}{a^{-\alpha}} = \lim_{a \rightarrow \infty} \frac{\Pr(\tilde{a}_{t+1} > a)}{a^{-\alpha}} = C. \quad \blacksquare$$

4 Extensions

We discuss how to relax the specification of the IF problem 1 along two main relevant directions.

Earning's growth. We can allow for exogenous growth $g > 1$ in earnings y_t as in Aiyagari-McGrattan (1998). To this end, we need to deflate the variables by the growth rate. In principle the borrowing constraint should be assumed to grow at the economy's growth rate in this case. In our context, since we allow for no borrowing, no modification of the constraint is needed. However, Assumption 3 would have to be modified so that $\Pr(\frac{\beta R_t}{g^\gamma} > 1) > 0$.

Bounded returns. We can also allow for an upper bound in the support of R_t . Consider a distribution of R on a bounded support $[\underline{R}, \bar{R}]$ which satisfies all assumptions except

that of unbounded support. Let its density be denoted f . Evaluate $\frac{c(a)}{a}$ at $a = \bar{y}$ and compute $\bar{R} \left(1 - \frac{c(\bar{y})}{\bar{y}}\right)$. If $\bar{R} \left(1 - \frac{c(\bar{y})}{\bar{y}}\right) > 1$, then the economy's stationary distribution of wealth has fat tails. But suppose instead that $\bar{R} \left(1 - \frac{c(\bar{y})}{\bar{y}}\right) < 1$. In this case pick an $\hat{R} > \bar{R}$ and such that $\hat{R} \left(1 - \frac{c(\bar{y})}{\bar{y}}\right) > 1$ and perturb f , the distribution of R , as follows: $f(R; \varepsilon) = (1 - \varepsilon)f(R)$ for any $R \in [\underline{R}, \bar{R}]$ and $f(\hat{R}; \varepsilon)$ has mass ε . Note that $f(\cdot, 0) = f$, so that we effectively produced a continuous parametrization of the distribution f . The parametrization is continuous in the sense that $\int g(r)f(R; \varepsilon)dR$ is continuous in ε for any continuous function g . Now this construction guarantees that wealth a can escape to the expanding region with positive probability ε . Indeed by Berge's maximum theorem $\frac{c(\bar{y})}{\bar{y}}$ is continuous in ε and \hat{R} can be chosen large enough to compensate any local variation in $\frac{c(\bar{y})}{\bar{y}}$. As a consequence, this construction produces an economy whose stationary distribution of wealth has fat tails even with a distribution of R which is bounded above. What is really needed is that the distribution of R has any positive density above the R^* such that $R^* \left(1 - \frac{c(\bar{y})}{\bar{y}}\right) = 1$, even if the support is not connected.

5 General equilibrium

In this section we embed the analysis of the distribution of wealth induced by the IF problem 1 in general equilibrium. Following Angeletos (2007) we assume that each agent acts as entrepreneur of his own individual firm. Each firm has a constant returns to scale production function

$$F(k, n, A)$$

where k, n are, respectively, capital and labor, and A is an idiosyncratic productivity shock. Note that for notational economy we suppress the superscript i that denotes the i th firm. The agent can only use his own savings as capital in his own firm. In each period $t + 1$, the agent first observes his firm's productivity shock A_{t+1} and then decides how much labor to hire in a competitive labor market, n_{t+1} . Therefore, each firm faces the same market wage rate w_{t+1} . The agent can decide not to engage in production, in which case $n_{t+1} = 0$ and he can carry over the firm's capital to next period. The agent's earnings in period $t + 1$ are $w_{t+1}e_{t+1}$, where e_{t+1} is his idiosyncratic (exogenous) labor supply. The firm's profits in period $t + 1$ are denoted π_{t+1} :

$$\pi_{t+1} = \max_{n_{t+1}} (\max\{F(k_{t+1}, n_{t+1}, A_{t+1}) - wn_{t+1}\} + (1 - \delta)k_{t+1}, k_{t+1})$$

Furthermore,

$$k_{t+1} = a_t - c_t$$

and

$$a_{t+1} = \pi_{t+1} + w_{t+1}e_{t+1}$$

Definition 5 *A stationary general equilibrium consists of policy functions, c_t , n_t , and k_{t+1} , a (constant) wage rate w , and a distribution $v(a_{t+1})$, such that the following conditions hold:*

- (i) c_t , n_t , and k_{t+1} are optimal policy functions given w .
- (ii) $\int n_t di = \int e_t di = 1$.
- (iii) v is a stationary distribution of a_{t+1} .

We can now construct such a stationary distribution. The first order conditions of each agent firm's labor choice requires

$$F_2(k_{t+1}, n_{t+1}, A_{t+1}) = w_{t+1}.$$

which, under constant returns to scale, implies

$$F_2\left(1, \frac{n_{t+1}}{k_{t+1}}, A_{t+1}\right) = w. \quad (8)$$

From constant returns to scale, we then obtain

$$\pi_{t+1} = R_{t+1}k_{t+1}.^{12}$$

The dynamic equation for wealth is then

$$a_{t+1} = R_{t+1}(a_t - c_t) + w_{t+1}e_{t+1}.$$

From equation (8) we can solve

$$\frac{n_{t+1}}{k_{t+1}} = g(w_{t+1}, A_{t+1}).$$

Thus

$$n_{t+1} = g(w_{t+1}, A_{t+1}) k_{t+1}.$$

and hence we obtain the labor market clearing condition:

$$1 = \int n_{t+1} di = \int g(w_{t+1}, A_{t+1}) \int k_{t+1} di = \int g(w_{t+1}, A_{t+1}) K_{t+1}, \quad (9)$$

where

$$\int k_{t+1} di = K_{t+1}.$$

¹²More specifically, $R_{t+1} = \max(\omega_{t+1}, 1)$ and $\omega_{t+1} = \left[F_1\left(1, \frac{n_{t+1}}{k_{t+1}}, A_{t+1}\right) + 1 - \delta\right]$.

Given the wealth a_t , the agent chooses its consumption c_t and savings k_{t+1} . Given the wage rate w_{t+1} , k_{t+1} , and its realization of A_{t+1} , the firm chooses employment n_{t+1} . Then each agent receives its own firm's profits and labor earnings $w_{t+1}e_{t+1}$, which form a_{t+1} . The stochastic process $\{a_{t+1}\}_{t=0}^{\infty}$ permits a unique stationary distribution wealth distribution induced by the IF problem given by 1. The stationary distribution of a_{t+1} induces then a stationary distribution of k_{t+1} . The aggregate capital K_{t+1} is the first moment of the stationary distribution of k_{t+1} and is therefore constant. As a consequence, from equation (9), the wage clearing the labor market, w_{t+1} , can be solved as a constant wage w .

6 Simulation

In this section we carry out a simulation of the Bewley economy we studied in the paper.¹³ The objective is simply to illustrate that a reasonable (though not calibrated) parameter set produces a wealth distribution which is Pareto in the tail with an exponent in the same order of magnitude as the one estimated for various developed economies, that is, about 2.¹⁴ We shall defer a careful calibration to future work.

Consider then a Bewley economy, as introduced in Section 2, with log preferences ($\gamma = 1$) and discount rate $\beta = .95$. The earning process is uniformly distributed over support $[1, 10]$, while the process for the rate of return R_t has a uniform component over support $[.8, 1.2]$ and a .01 mass on $R^* = 5.5$ (to approximate an unbounded support as discussed in Section 4). The consumption function rapidly converges to a concave function, linear in the tail, which is shown in Figure 1.

The stationary distribution of wealth, for 10 thousand households over 5 thousands periods, is shown in Figure 2.

It has a Pareto tail with exponent 2.6505.¹⁵ The top 1 percent of the population owns 13.19 percent of aggregate wealth, while the top 10 percent owns 32.22 percent of aggregate wealth. The top wealth shares in the simulation are lower than in the data, e.g., for the U.S. Larger shares (and a lower tail exponent) can be obtained, however, by skewing the distribution of returns or by introducing even a small serial autocorrelation to capture frictions in social mobility.

7 Conclusions

In this paper we construct a general equilibrium model with idiosyncratic capital income risks in a Bewley economy and show that the resulting wealth distribution can have fat

¹³See Nirei (2013) for simulations of a related model with similar properties.

¹⁴See the references in footnote 5.

¹⁵The tail exponent is computed using a Matlab package, *plfit*, based on Clauset, Shalizi, and Newman (2009).

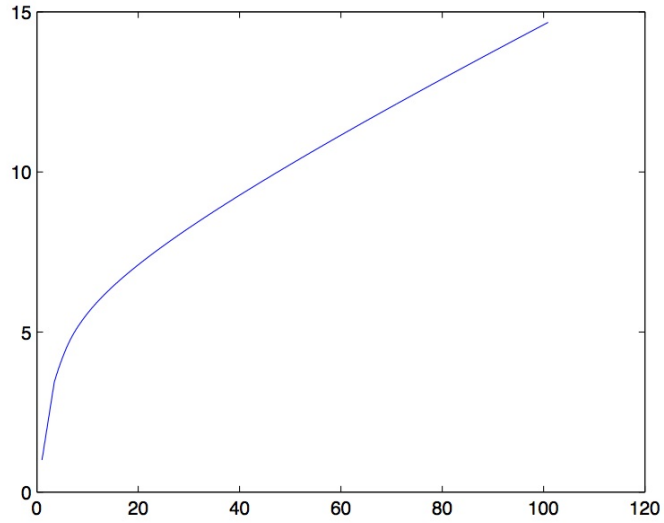


Figure 1: The Consumption Function

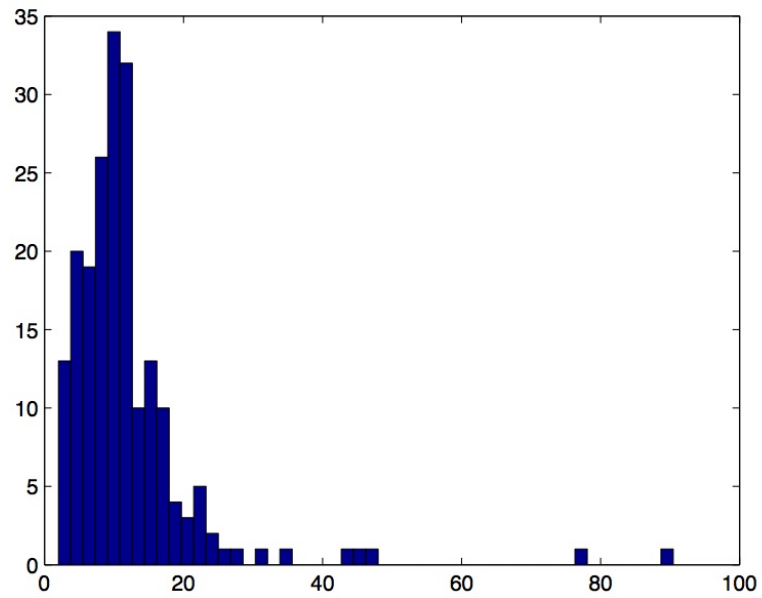


Figure 2: The Wealth Distribution

tails. Simulations of the economy suggest that, once idiosyncratic capital income risk is taken into account, Bewley models can reproduce fundamental stylized properties of the

wealth distribution observed in the data for the U.S. and other developed economies.

References

- [1] Aiyagari, S.R. (1994): "Uninsured idiosyncratic risk and aggregate saving," *Quarterly Journal of Economics*, 109(3), 659-84.
- [2] Aiyagari, S.R. (1995): "Optimal capital income taxation with incomplete markets and borrowing constraints," *Journal of Political Economy*, 103(6), 1158-75.
- [3] Angeletos, G. (2007): "Uninsured Idiosyncratic Investment Risk and Aggregate Saving," *Review of Economic Dynamics*, 10, 1-30.
- [4] Angeletos, G. and L.E. Calvet (2005): "Incomplete-Market Dynamics in a Neoclassical Production Economy," *Journal of Mathematical Economics*, 41, 407-438.
- [5] Attanasio, O.P. and S.J. Davis (1996): "Relative wage movements and the distribution of consumption," *Journal of Political Economy*, 104(6), 1227-62.
- [6] Benhabib, J. and A. Bisin (2006): "The Distribution of Wealth and Redistributive Policies," mimeo, New York University.
- [7] Benhabib, J. and S. Zhu (2008): "Age, Luck and Inheritance, NBER W.P. 14128.
- [8] Benhabib, J., A. Bisin, and S. Zhu (2011): "The Distribution of Wealth and Fiscal Policy in Economies with Finitely Lived Agents," *Econometrica*, 79(1), 122-57.
- [9] Benhabib, J., A. Bisin, and S. Zhu (2013): "The distribution of wealth in the Blanchard-Yaari model," forthcoming in *Macroeconomic Dynamics*, special issue on Complexity in Economic Systems.
- [10] Bewley, T. (1977): "The permanent income hypothesis: A theoretical formulation," *Journal of Economic Theory*, 16(2), 252-92.
- [11] Bewley, T. (1983): "A difficulty with the optimum quantity of money," *Econometrica*, 51(5), 1485-504.
- [12] Cagetti, M. and M. De Nardi (2006): "Entrepreneurship, frictions, and wealth," *Journal of Political Economy*, 114(5), 835-70.
- [13] Cagetti, M. and M. De Nardi (2008): "Wealth inequality: Data and models," *Macroeconomic Dynamics*, 12(52), 285-313.
- [14] Carroll, C. D. (1997): "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics*, 112, 1-56.
- [15] Carroll, C. D. and M.S. Kimball (2005): "Liquidity constraints and precautionary saving," mimeo, Johns Hopkins University.

- [16] Carroll, C. D., J. Slacalek, and K. Tokuoka (2014). "The Distribution of Wealth and the Marginal Propensity to Consume," mimeo, Johns Hopkins University.
- [17] Castaneda, A., J. Diaz-Gimenez, and J.V. Rios-Rull (2003): "Accounting for the US earnings and wealth inequality," *Journal of Political Economy*, 111(4), 818-57.
- [18] Chamberlain G. and C.A. Wilson (2000): "Optimal intertemporal consumption under uncertainty," *Review of Economic Dynamics*, 3(3), 365-95.
- [19] Clauset, A., Shalizi, C.R., and M. E. J. Newman (2009): Power-law Distributions in Empirical Data, *SIAM Review*, 51(4), 661-703.
- [20] Clementi, F. and M. Gallegati (2005): "Power Law Tails in the Italian Personal Income Distribution," *Physica A: Statistical Mechanics and Theoretical Physics*, 350, 427-438.
- [21] Dagsvik, J.K., and B.H. Vatne (1999): "Is the Distribution of Income Compatible With a Stable Distribution?" Discussion Paper 246, Research Department, Statistics Norway.
- [22] Fisher, J. and D. Johnson (2006): "Consumption mobility in the United States: Evidence from two panel data sets," *B.E. Journal of Economic Analysis and Policy*, 6(1), art. 16.
- [23] Guvenen, F. (2007): "Learning your earning: Are labor income shocks really very persistent?" *American Economic Review* 97(3), 687-712.
- [24] Hansen, G.D. and A. Imrohorglu (1992): "The role of unemployment insurance in an economy with liquidity constraints and moral hazard," *Journal of Political Economy*, 100(1), 118-42.
- [25] Heathcote, J. (2005): "Fiscal policy with heterogeneous agents and incomplete markets," *Review of Economic Studies*, 72(1), 161-88.
- [26] Heathcote, J. (2008): "Discussion of Heterogeneous Life-Cycle Profiles, Income Risk, and Consumption Inequality, by G. Primiceri and T. van Rens," mimeo, Federal Bank of Minneapolis.
- [27] Heathcote, J., K. Storesletten, and G.L. Violante (2005): "Two views of inequality over the life cycle," *Journal of the European Economic Association*. 32(3), 765-75.
- [28] Heathcote, J., K. Storesletten, and G.L. Violante (2007): "Consumption and labor supply with partial insurance: An analytical framework," mimeo, New York University.

- [29] Heathcote, J., K. Storesletten, and G.L. Violante (2008a): "Insurance and opportunities: a welfare analysis of labor market risk," *Journal of Monetary Economics*, 55(3), 501-25.
- [30] Heathcote, J., K. Storesletten, and G.L. Violante (2008b): "The macroeconomic implications of rising wage inequality in the United States," mimeo, New York University.
- [31] Huggett, M. (1993): "The risk-free rate in heterogeneous-agent incomplete-insurance economies," *Journal of Economic Dynamics and Control*, 175(6), 953-69.
- [32] Huggett, M. (1996): "Wealth distribution in life-cycle economies," *Journal of Monetary Economics*, 38(3), 469-94.
- [33] Imrohroglu, A. (1992): "The welfare cost of inflation under imperfect insurance," *Journal of Economic Dynamics and Control*, 16(1), 79-91.
- [34] Krueger, D. and F. Perri (2003): "On the welfare consequences of the increase in inequality in the United States," *NBER Macroeconomics Annual 2003*, edited by M. Gertler and K. Rogoff, 83-121, Cambridge, MA: MIT Press.
- [35] Jappelli, T. and L. Pistaferri (2006): "Intertemporal choice and consumption mobility. Journal of the European Economic Association," 4(1), 75-115.
- [36] Kesten, H. (1973): "Random Difference Equations and Renewal Theory for Products of Random Matrices," *Acta Mathematica*, 131, 207-248.
- [37] Klass, O.S., O. Biham, M. Levy, O. Malcai, and S. Solomon (2007): "The Forbes 400, the Pareto Power-Law and Efficient Markets," *The European Physical Journal B - Condensed Matter and Complex Systems*, 55, 143-147.
- [38] Krusell, P. and A.A. Smith (1998): "Income and wealth heterogeneity in the macroeconomy. Journal of Political Economy," 106(5), 867-96.
- [39] Krusell, P. and A.A. Smith (2006): "Quantitative macroeconomic models with heterogeneous agents," in *Advances in Economics and Econometrics: Theory and Applications*, edited by R. Blundell, W. Newey, and T. Persson, Cambridge: Cambridge University Press.
- [40] Levhari, D. and T. N. Srinivasan (1969): "Optimal savings under uncertainty," *Review of Economic Studies*, 36, 153-163.
- [41] Ljungqvist, L. and T.J. Sargent (2004): *Recursive Macroeconomic Theory*, Cambridge, MA: MIT Press.

- [42] Mankiw, N.G. (1986): "The equity premium and the concentration of aggregate shocks," *Journal of Financial Economics*, 17(1), 211-19.
- [43] Meyn, S.P. and Tweedie, R.L. (2009): *Markov chains and stochastic stability*, Cambridge University Press.
- [44] Mirek, M. (2011): "Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps," *Probability Theory and Related Fields*, 151, 705-34.
- [45] Nirei, M. (2013): "Pareto Distribution of Income in Neoclassical Growth Models," mimeo, Hitotsubashi University.
- [46] Nirei, M. and W. Souma (2004): "Two Factor Model of Income Distribution Dynamics," mimeo, Utah State University.
- [47] Pistaferri, L. (2003): "Anticipated and unanticipated wage changes, wage risk, and intertemporal labor supply," *Journal of Labor Economics*, 21(3), 729-82.
- [48] Primiceri, G. and T. van Rens (2013): "Heterogeneous life-cycle profiles," forthcoming in *Journal of Monetary Economics*.
- [49] Quadrini, V. (1999): "The Importance of Entrepreneurship for Wealth Concentration and Mobility," *Review of Income and Wealth*, 45, 1-19.
- [50] Quadrini, V. (2000): "Entrepreneurship, saving and social mobility," *Review of Economic Dynamics*, 3(1), 1-40.
- [51] Rabault, G. (2002): "When do borrowing constraints bind? Some new results on the income fluctuation problem," *Journal of Economic Dynamics and Control*, 26, 217-45.
- [52] Rios-Rull, J.V. (1995): "Models with heterogeneous agents," In *Frontiers of Business Cycle Research*, edited by T.F. Cooley, Princeton, NJ: Princeton University Press.
- [53] Royden, H. L.(1988): *Real Analysis*, (third ed.), Upper Saddle River, New Jersey: Prentice-Hall, Inc.
- [54] Schechtman, J. (1976): "An income fluctuation problem," *Journal of Economic Theory*, 12(2), 218-41.
- [55] Schechtman, J. and V.L.S. Escudero (1977): "Some results on an income fluctuation problem." *Journal of Economic Theory*, 16(2), 151-66.

- [56] Storesletten, K., C.L. Telmer, and A. Yaron (2001): "The welfare cost of business cycles revisited: Finite lives and cyclical variation in idiosyncratic risk," *European Economic Review*, 45(7), 1311-39.
- [57] Storesletten, K., C.L. Telmer, and A. Yaron (2004): "Consumption and risk sharing over the life cycle," *Journal of Monetary Economics*, 51(3), 609-33.
- [58] Wold, H.O.A. and P. Whittle (1957): "A Model Explaining the Pareto Distribution of Wealth," *Econometrica*, 25, 591-595.
- [59] Wolff, E. (1987): "Estimates of Household Wealth Inequality in the U.S., 1962-1983," *Review of Income and Wealth*, 33, 231-256.
- [60] Wolff, E. (2004): "Changes in Household Wealth in the 1980s and 1990s in the U.S.," mimeo, NYU.

Appendix

Proof of Theorem 1. A feasible policy $c(a)$ is said to overtake another feasible policy $\hat{c}(a)$ if starting from the same initial wealth a_0 , the policies $c(a)$ and $\hat{c}(a)$ yield stochastic consumption processes (c_t) and (\hat{c}_t) that satisfy

$$E \left[\sum_{t=0}^T \beta^t (u(c_t) - u(\hat{c}_t)) \right] > 0 \quad \text{for all } T > \text{some } T_0.$$

Also, a feasible policy is said to be optimal if it overtakes all other feasible policies.

Proof: For an a_0 , the stochastic consumption process (c_t) is induced by the policy $c(a)$. Let (\hat{c}_t) be an alternative stochastic consumption process, starting from the same initial wealth a_0 . By the strict concavity of $u(\cdot)$, we have

$$E \left[\sum_{t=0}^T \beta^t (u(c_t) - u(\hat{c}_t)) \right] \geq E \left[\sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \right].$$

From the budget constraint we have

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}$$

and

$$\hat{a}_{t+1} = R_{t+1}(\hat{a}_t - \hat{c}_t) + y_{t+1}.$$

For a path of (R_t, y_t) , we have

$$\frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} = a_t - c_t - (\hat{a}_t - \hat{c}_t) \tag{10}$$

and

$$c_t - \hat{c}_t = a_t - \hat{a}_t - \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}}.$$

Therefore we have

$$\sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) = \sum_{t=0}^T \beta^t u'(c_t) \left(a_t - \hat{a}_t - \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} \right).$$

Using $a_0 = \hat{a}_0$ and rearranging terms, we have

$$\sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) = - \sum_{t=0}^T \beta^t [u'(c_t) - \beta R_{t+1} u'(c_{t+1})] \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} - \beta^T u'(c_T) \frac{a_{T+1} - \hat{a}_{T+1}}{R_{T+1}}.$$

Using equation (10) we have

$$\begin{aligned}
\sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) &= - \sum_{t=0}^T \beta^t [u'(c_t) - \beta R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} \\
&\quad - \beta^T u'(c_T) [a_T - c_T - (\hat{a}_T - \hat{c}_T)] \\
&\geq - \sum_{t=0}^T \beta^t [u'(c_t) - \beta R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} - \beta^T u'(c_T) a_T.
\end{aligned}$$

Thus we have

$$\begin{aligned}
E \left[\sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \right] &\geq -E \left(\sum_{t=0}^T \beta^t [u'(c_t) - \beta E R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} \right) \\
&\quad - E \beta^T u'(c_T) a_T. \tag{11}
\end{aligned}$$

By the Euler equation (2) we have $u'(c_t) - \beta E R_{t+1} u'(c_{t+1}) \geq 0$. If $c_t < a_t$, then $u'(c_t) = \beta E R_{t+1} u'(c_{t+1})$. If $c_t = a_t$, then $a_t - c_t - (\hat{a}_t - \hat{c}_t) = -(\hat{a}_t - \hat{c}_t) \leq 0$. Thus

$$-E \left(\sum_{t=0}^T \beta^t [u'(c_t) - \beta E R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} \right) \geq 0. \tag{12}$$

Combining equations (11) and (12) we have

$$E \left[\sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \right] \geq -E \beta^T u'(c_T) a_T.$$

By the transversality condition (3) we know that for large T ,

$$E \left[\sum_{t=0}^T \beta^t (u(c_t) - u(\hat{c}_t)) \right] \geq 0. \quad \blacksquare$$

Proof of Proposition 1. The Euler equation of this problem is

$$c_t^{-\gamma} = \beta E R_{t+1} c_{t+1}^{-\gamma}. \tag{13}$$

Guess $c_t = \phi a_t$. From the Euler equation (13) we have

$$\phi = 1 - (\beta E R^{1-\gamma})^{\frac{1}{\gamma}} > 0.$$

It is easy to verify the transversality condition,

$$\lim_{t \rightarrow \infty} E (\beta^t c_t^{-\gamma} a_t) = 0. \quad \blacksquare$$

Let $V^t(a)$ be the optimal value function of an agent who has wealth a and has t periods to the end T . Thus we have

$$V^t(a) = \max_{c \leq a} \{u(c) + \beta EV^{t-1}(R(a-c) + y)\} \quad \text{for } t > 1$$

and

$$V^1(a) = \max_{c \leq a} u(c).$$

We have the Euler equation of this problem, for $t > 1$

$$u'(c^t(a)) \geq \beta E[Ru'(c^{t-1}(R(a-c^t(a)) + y))] \text{ with equality if } c^t(a) < a.$$

Proof of Proposition 2. Continuity is a consequence of the Theorem of the Maximum and mathematical induction. The proof that $c^t(a)$ and $s^t(a)$ are increasing can be easily adapted from the proof of Theorem 1.5 of Schechtman (1976); it makes use of the fact that $c^t(a) > 0$, a consequence of Inada conditions which hold for CRRA utility functions. ■

Proof of Theorem 2. By Lemma 1 we know that $c(a)$ satisfies the Euler equation. Now we verify that $c(a)$ satisfies the transversality condition (3).

By theorems 1 and 1 we have

$$c_t \geq \phi a_t.$$

Note that $a_t \geq \underline{y}$ for $t \geq 1$. We have

$$u'(c_t)a_t \leq \phi^{-\gamma} (\underline{y})^{1-\gamma} \quad \text{for } t \geq 1.$$

Thus

$$\lim_{t \rightarrow \infty} E\beta^t u'(c_t)a_t = 0. \quad \blacksquare$$

Proof of Proposition 3. By Lemma 1, $c(a)$ is continuous. Thus $s(a)$ is continuous since $s(a) = a - c(a)$.

Also, by Lemma 1, $\lim_{t \rightarrow \infty} s^t(a) = s(a)$, since $\lim_{t \rightarrow \infty} c^t(a) = c(a)$, $s^t(a) = a - c^t(a)$, and $s(a) = a - c(a)$. The conclusion that $c(a)$ and $s(a)$ are increasing in a follows from part (ii) of Proposition 2. ■

Note that Proposition 3 implies that $c(a)$ and $s(a)$ are Lipschitz continuous. For $\tilde{a}, \hat{a} > 0$, without loss of generality, we assume that $\tilde{a} < \hat{a}$. We have $c(\tilde{a}) \leq c(\hat{a})$ and $s(\tilde{a}) \leq s(\hat{a})$. Also $c(\tilde{a}) + s(\tilde{a}) = \tilde{a}$ and $c(\hat{a}) + s(\hat{a}) = \hat{a}$. Thus

$$c(\hat{a}) - c(\tilde{a}) + s(\hat{a}) - s(\tilde{a}) = \hat{a} - \tilde{a}.$$

Thus we have

$$0 \leq c(\hat{a}) - c(\tilde{a}) \leq \hat{a} - \tilde{a}$$

and

$$0 \leq s(\hat{a}) - s(\tilde{a}) \leq \hat{a} - \tilde{a}.$$

Thus

$$|c(\hat{a}) - c(\tilde{a})| \leq |\hat{a} - \tilde{a}|$$

and

$$|s(\hat{a}) - s(\tilde{a})| \leq |\hat{a} - \tilde{a}|.$$

Proof of Proposition 5. The proof involves several steps, producing a characterization of $\frac{c(a)}{a}$.

Lemma 2 $\exists \zeta > \underline{y}$, such that $s(a) = 0, \forall a \in (0, \zeta]$.

Proof. Suppose that $s(a) > 0$ for $a > \underline{y}$. Pick $a_0 > \underline{y}$. For any finite $t \geq 0$, we have $a_t > \underline{y}$ and $u'(c_t) = \beta ER_{t+1}u'(c_{t+1})$. Thus

$$u'(c_0) = \beta^t ER_1 R_2 \cdots R_{t-1} R_t u'(c_t). \quad (14)$$

By theorems 1 and 1 we have

$$c_t \geq \phi a_t > \phi \underline{y}.$$

Thus equation (14) implies that

$$u'(c_0) \leq (\phi \underline{y})^{-\gamma} (\beta ER)^t. \quad (15)$$

Thus the right hand side of equation (15) approaches 0 as t goes to infinity. A contradiction. Thus $s(\zeta) = 0$ for some $\zeta > \underline{y}$. By the monotonicity of $s(a)$, we know that $s(a) = 0, \forall a \in (0, \zeta]$. ■

We can now show the following:

Lemma 3 $\frac{c(a)}{a}$ is decreasing in a .

Proof. By Lemma 2 we know that $c(\underline{y}) = \underline{y}$. For $\forall a > \underline{y}$, $\frac{c(a)}{a} \leq 1 = \frac{c(\underline{y})}{\underline{y}}$. Note that $-c(a)$ is a convex function of a , since $c(a)$ is a concave function of a . For $\hat{a} > \tilde{a} > \underline{y}$, we have¹⁶

$$\frac{c(\hat{a}) - c(\underline{y})}{\hat{a} - \underline{y}} \leq \frac{c(\tilde{a}) - c(\underline{y})}{\tilde{a} - \underline{y}}.$$

¹⁶See Lemma 16 on page 113 of Royden (1988).

This implies that

$$c(\hat{a})\tilde{a} \leq c(\tilde{a})\hat{a} - [\hat{a} - \tilde{a} - (c(\hat{a}) - c(\tilde{a}))]\underline{y}. \quad (16)$$

Since $c(a)$ is Lipschitz continuous, we have

$$c(\hat{a}) - c(\tilde{a}) \leq \hat{a} - \tilde{a}. \quad (17)$$

Combining inequalities (16) and (17) we have

$$c(\hat{a})\tilde{a} \leq c(\tilde{a})\hat{a},$$

i.e.

$$\frac{c(\hat{a})}{\hat{a}} \leq \frac{c(\tilde{a})}{\tilde{a}}.$$

■

By Theorems 1 and Proposition 1 we know that $\frac{c(a)}{a} \geq \phi$. Thus we have

$$\lim_{a \rightarrow \infty} \frac{c(a)}{a} \text{ exists.}$$

Let

$$\lambda = \lim_{a \rightarrow \infty} \frac{c(a)}{a}. \quad (18)$$

Note that $\lambda \leq 1$ since $c(a) \leq a$.

The Euler equation of this problem is

$$c_t^{-\gamma} \geq \beta ER_{t+1}c_{t+1}^{-\gamma} \text{ with equality if } c_t < a_t. \quad (19)$$

Lemma 4 $\lambda \in [\phi, 1)$.

Proof. Suppose that $\lambda = 1$. Thus

$$\liminf_{a_t \rightarrow \infty} \frac{c(a_t)}{a_t} = \lim_{a_t \rightarrow \infty} \frac{c(a_t)}{a_t} = 1.$$

From the Euler equation (19) we have

$$c_t^{-\gamma} \geq \beta ER_{t+1}c_{t+1}^{-\gamma} \geq \beta ER_{t+1}a_{t+1}^{-\gamma}$$

since $c_{t+1} \leq a_{t+1}$ and $\gamma \geq 1$.

Thus

$$\left(\frac{c(a_t)}{a_t}\right)^{-\gamma} \geq \beta ER_{t+1} \left(R_{t+1} \left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma}.$$

By Fatou's lemma we have

$$\begin{aligned} & \liminf_{a_t \rightarrow \infty} ER_{t+1} \left(R_{t+1} \left(1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \\ & \geq E \liminf_{a_t \rightarrow \infty} \left[R_{t+1} \left(R_{t+1} \left(1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \right]. \end{aligned}$$

Thus

$$\begin{aligned} 1 & = \lim_{a_t \rightarrow \infty} \left(\frac{c(a_t)}{a_t} \right)^{-\gamma} \\ & \geq \beta \lim_{a_t \rightarrow \infty} ER_{t+1} \left(R_{t+1} \left(1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \\ & = \beta \liminf_{a_t \rightarrow \infty} ER_{t+1} \left(R_{t+1} \left(1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \\ & \geq \beta E \liminf_{a_t \rightarrow \infty} \left[R_{t+1} \left(R_{t+1} \left(1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \right] \\ & = \beta E \lim_{a_t \rightarrow \infty} \left[R_{t+1} \left(R_{t+1} \left(1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \right] \\ & = \infty. \end{aligned}$$

A contradiction. ■

From Lemma 4 we know that $c_t < a_t$ when a_t is large enough. Thus the equality of the Euler equation holds

$$c_t^{-\gamma} = \beta ER_{t+1} c_{t+1}^{-\gamma}.$$

Thus

$$\left(\frac{c_t}{a_t} \right)^{-\gamma} = \beta ER_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma}. \quad (20)$$

Taking limits on both sides of equation (20) we have

$$\lim_{a_t \rightarrow \infty} \left(\frac{c_t}{a_t} \right)^{-\gamma} = \beta \lim_{a_t \rightarrow \infty} ER_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma}.$$

Thus

$$\lambda^{-\gamma} = \beta \lim_{a_t \rightarrow \infty} ER_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma}. \quad (21)$$

We turn to the computation of $\lim_{a_t \rightarrow \infty} ER_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma}$.

In order to compute $\lim_{a_t \rightarrow \infty} ER_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma}$, we first show a lemma.

Lemma 5 For $\forall H > 0, \exists J > 0$, such that $a_{t+1} > H$ for $a_t > J$. Here J does not depend on realizations of R_{t+1} and y_{t+1} .

Proof. Note that

$$\frac{a_{t+1}}{a_t} = \frac{R_{t+1}(a_t - c_t) + y_{t+1}}{a_t} \geq R_{t+1} \left(1 - \frac{c_t}{a_t} \right).$$

From equation (18) we know that for some $\varepsilon > 0, \exists J_1 > 0$, such that

$$\frac{c_t}{a_t} < \lambda + \varepsilon$$

for $a_t > J_1$. Thus

$$\frac{a_{t+1}}{a_t} \geq R_{t+1} \left(1 - \frac{c_t}{a_t} \right) \geq R_{t+1}(1 - \lambda - \varepsilon). \quad (22)$$

And

$$\frac{a_{t+1}}{a_t} \geq R_{t+1}(1 - \lambda - \varepsilon) \geq \underline{R}(1 - \lambda - \varepsilon).$$

We pick $J > J_1$ such that $\underline{R}(1 - \lambda - \varepsilon) \geq \frac{H}{J}$. Thus for $a_t > J$, we have

$$\frac{a_{t+1}}{a_t} \geq \frac{H}{J}.$$

This implies that

$$a_{t+1} \geq \frac{H}{J} a_t > H.$$

■

From equation (18) we know that for some $\eta > 0, \exists H > 0$, such that

$$\frac{c_{t+1}}{a_{t+1}} > \lambda - \eta \quad (23)$$

for $a_{t+1} > H$.

From Lemma 5 and equations (22) and (23) we have

$$R_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma} = R_{t+1} \left(\frac{c_{t+1}}{a_{t+1}} \frac{a_{t+1}}{a_t} \right)^{-\gamma} \leq (\lambda - \eta)^{-\gamma} (1 - \lambda - \varepsilon)^{-\gamma} R_{t+1}^{1-\gamma}$$

for $a_t > J$. And

$$(\lambda - \eta)^{-\gamma} (1 - \lambda - \varepsilon)^{-\gamma} E R_{t+1}^{1-\gamma} < \infty$$

since $\gamma \geq 1$. Thus when a_t is large enough, $(\lambda - \eta)^{-\gamma} (1 - \lambda - \varepsilon)^{-\gamma} R_{t+1}^{1-\gamma}$ is a dominant function of $R_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma}$.

Note that

$$\lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_{t+1}} = \lim_{a_t \rightarrow \infty} \frac{c(a_{t+1})}{a_{t+1}} = \lambda \quad a.s.$$

by Lemma 5 and equation (18). And

$$\lim_{a_t \rightarrow \infty} \frac{a_{t+1}}{a_t} = \lim_{a_t \rightarrow \infty} \left(\frac{R_{t+1}(a_t - c_t) + y_{t+1}}{a_t} \right) = R_{t+1}(1 - \lambda) \quad a.s.$$

since $y_{t+1} \in [\underline{y}, \bar{y}]$. Thus

$$\lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_t} = \lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_{t+1}} \frac{a_{t+1}}{a_t} = \lambda(1 - \lambda)R_{t+1} \quad a.s.$$

Thus by the Dominated Convergence Theorem, we have

$$\lim_{a_t \rightarrow \infty} ER_{t+1} \left(\frac{c_{t+1}}{a_t} \right)^{-\gamma} = ER_{t+1} \left(\lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_t} \right)^{-\gamma} = \lambda^{-\gamma}(1 - \lambda)^{-\gamma} ER_{t+1}^{1-\gamma}. \quad (24)$$

Combining equations (21) and (24) we have

$$\lambda^{-\gamma} = \beta \lambda^{-\gamma} (1 - \lambda)^{-\gamma} ER_{t+1}^{1-\gamma}. \quad (25)$$

By Lemma 4 we know that $\lambda \geq \phi > 0$. Thus we find λ from equation (25)

$$\lambda = 1 - (\beta ER^{1-\gamma})^{\frac{1}{\gamma}}.$$

Thus $\lambda = \phi$.

Proof of Theorem 3. The proof requires several steps.

Lemma 6 *The wealth accumulation process (a_t) is ψ -irreducible.*

Proof. First we show that the process (a_t) is φ -irreducible. We construct a measure φ on $[\underline{y}, \infty)$ such that

$$\varphi(A) = \int_A f(y) dy.$$

where $f(y)$ is the density of labor earnings $\{y\}$. Note that the borrowing constraint binds in finite time with a positive probability for $\forall a_0 \in [\underline{y}, \infty)$. Suppose not. For any finite $t \geq 0$, we have $a_t > \underline{y}$ and $u'(c_t) = \beta ER_{t+1} u'(c_{t+1})$. Following the same procedure as in the proof of lemma 2, we obtain a contradiction. If the borrowing constraint binds at period t , then $a_{t+1} = y_{t+1}$. Thus any set A such that $\int_A f(y) dy > 0$ can be reached in finite time with a positive probability. The process (a_t) is φ -irreducible.

By Proposition 4.2.2 in Meyn and Tweedie (2009), there exists a probability measure ψ on $[\underline{y}, \infty)$ such that the process $\{a_{t+1}\}_{t=0}^{\infty}$ is ψ -irreducible, since it is φ -irreducible. ■

Lemma 7 *Under Assumption 4, $a = \underline{y}$ is a reflecting barrier of the process (a_t) .*

Proof. If $a_t = \underline{y}$, then there exists \hat{y} close to \bar{y} such that $\Pr(a_{t+1} \in [\hat{y}, \bar{y}] | a_t = \underline{y}) = \Pr(y_{t+1} \in [\hat{y}, \bar{y}]) > 0$, since $s(\underline{y}) = 0$. To show that a_{t+2} can be greater than \bar{y} with a positive probability, it is sufficient to show that $s(\bar{y}) > 0$. Suppose that $s(\bar{y}) = 0$. Thus $s(a) = 0$ for $a \in [\underline{y}, \bar{y}]$. Thus by the Euler equation we have

$$(\bar{y})^{-\gamma} \geq \beta E [R_t (y_t)^{-\gamma}].$$

This is impossible under Assumption 4. Thus $s(\bar{y}) > 0$ and $a = \underline{y}$ is a reflecting barrier of the process $\{a_{t+1}\}_{t=0}^{\infty}$. ■

To show that there exists a unique stationary wealth distribution, we have to show that the process (a_t) is ergodic. Actually, we can show that it is geometrically ergodic.

Lemma 8 *The process $\{a_{t+1}\}_{t=0}^{\infty}$ is geometrically ergodic.*

Proof. To show that the process (a_t) is geometrically ergodic, we use part (iii) of Theorem 15.0.1 of Meyn and Tweedie (2009). We need to verify that

- a the process $\{a_{t+1}\}_{t=0}^{\infty}$ is ψ -irreducible;
- b the process $\{a_{t+1}\}_{t=0}^{\infty}$ is aperiodic;¹⁷ and
- c there exists a petite set C ,¹⁸ constants $b < \infty$, $\rho > 0$ and a function $V \geq 1$ finite at some point in $[\underline{y}, \infty)$ satisfying

$$EV(a_{t+1}) - V(a_t) \leq -\rho V(a_t) + bI_C(a_t), \quad \forall a_t \in [\underline{y}, \infty).$$

By Lemma 6, the process $\{a_{t+1}\}_{t=0}^{\infty}$ is ψ -irreducible.

For a φ -irreducible Markov process, when there exists a v_1 -small set A with $v_1(A) > 0$,¹⁹ then the stochastic process is called strongly aperiodic; see Meyn and Tweedie (2009, p. 114). We construct a measure v_1 on $[\underline{y}, \infty)$ such that

$$v_1(A) = \int_A f(y) dy.$$

By lemma 2, we know that $s(a) = 0, \forall a \in [\underline{y}, \zeta]$. Thus $[\underline{y}, \zeta]$ is v_1 -small and $v_1([\underline{y}, \zeta]) = \int_{\underline{y}}^{\zeta} f(y) dy > 0$. The process (a_t) is strongly aperiodic.

¹⁷For the definition of aperiodic, see page 114 of Meyn and Tweedie (2009).

¹⁸For the definition of petite sets, see page 117 of Meyn and Tweedie (2009).

¹⁹For the definition of small sets, see page 102 of Meyn and Tweedie (2009).

We now show that an interval $[\underline{y}, B]$ is a petite set for $\forall B > \underline{y}$. To show this, we first show that $\underline{R}s(a) + \underline{y} < a$ for $a \in (\underline{y}, \infty)$. For $s(a) = 0$, this is obviously true. For $s(a) > 0$, suppose that $\underline{R}s(a) + \underline{y} \geq a$, we have

$$u'(c(a)) = \beta ER_t u'(c(R_t s(a) + y)) \leq \beta ER_t u'(c(a)).$$

We obtain a contradiction since Assumption 2 implies that $\beta ER_t < 1$. Also by Lemma 2, there exists an interval $[\underline{y}, \zeta]$, such that $s(a) = 0, \forall a \in [\underline{y}, \zeta]$. For an interval $[\underline{y}, B], \forall a_0 \in [\underline{y}, B]$, there exists a common t such that the borrowing constraint binds at period t with a positive probability. Then for any set $A \subset [\underline{y}, \bar{y}]$, $\Pr(a_{t+1} \in A | s(a_t) = 0) = \int_A f(y) dy$. Note that a t -step probability transition kernel is the probability transition kernel of a specific sampled chain. Thus we construct a measure v_a on $[\underline{y}, \infty)$ such that v_a has a positive measure on $[\underline{y}, \bar{y}]$ and $v_a((\bar{y}, \infty)) = 0$. The t -step probability transition kernel of a process starting from $\forall a_0 \in [\underline{y}, B]$ is greater than the measure v_a . An interval $[\underline{y}, B]$ is a petite set for $\forall B > \underline{y}$.

We pick a function $V(a) = a + 1, \forall a \in [\underline{y}, \infty)$. Thus $V(a) > 1$ for $a \in [\underline{y}, \infty)$. Pick $0 < q < 1 - \mu ER_t$. Let $\rho = 1 - \mu ER_t - q > 0$ and $b = 1 - \mu ER_t + Ey$. Pick $B > \underline{y}$, such that $B + 1 \geq \frac{b}{q}$. Let $C = [\underline{y}, B]$. Thus C is a petite set. Therefore, for $\forall a_t \in [\underline{y}, \infty)$, we have

$$\begin{aligned} EV(a_{t+1}) - V(a_t) &= E(a_{t+1}) - a_t \\ &\leq -(1 - \mu ER_t)V(a_t) + 1 - \mu ER_t + Ey \\ &\leq -\rho V(a_t) + bI_C(a_t) \end{aligned}$$

where $I_C(\cdot)$ is an indicator function.

By Theorem 15.0.1 of Meyn and Tweedie (2009) the process (a_t) is geometrically ergodic. ■