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UNCERTAINTY TRAPS

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ABSTRACT

We develop a theory of endogenous uncertainty and business cycles in which short-lived shocks can generate long-lasting recessions. In the model, higher uncertainty about fundamentals discourages investment. Since agents learn from the actions of others, information flows slowly in times of low activity and uncertainty remains high, further discouraging investment. The unique equilibrium of this economy displays uncertainty traps: self-reinforcing episodes of high uncertainty and low activity. While the economy recovers quickly after small shocks, large temporary shocks may have nearly permanent effects on the level of activity. The economy is subject to an information externality but uncertainty traps may remain even in the efficient allocation. We extend our framework to include additional features of standard business cycle models and show, in that context, that uncertainty traps can substantially worsen recessions and increase their duration.

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1 Introduction

We develop a theory of endogenous uncertainty and business cycles. The theory combines two forces: higher uncertainty about economic fundamentals deters investment, and uncertainty evolves endogenously because agents learn from the actions of others. The unique rational expectation equilibrium of the economy features *uncertainty traps*: self reinforcing episodes of high uncertainty and low economic activity that cause recessions to persist. Because of uncertainty traps, short-lived shocks can generate long-lasting recessions, and low activity may persist even after fundamentals have recovered. Thus, the theory rationalizes features of U.S. macroeconomic activity that are not easily explained by standard business cycle models, such as the slow recovery of output after recessions despite typically faster improvements in measured productivity.

We first build a model that only includes the essential features that give rise to uncertainty traps, and then embed them into a standard real business cycle model. In the model, firms decide whether to undertake an irreversible investment whose return depends on an imperfectly observed fundamental that evolves randomly according to a persistent process. Firms are heterogeneous in the cost of undertaking this investment and hold common beliefs about the fundamental. Beliefs are regularly updated with new information, and, in particular, firms learn by observing the return on the investment of other producers. We define uncertainty as the variance of these beliefs.

This environment naturally produces an interaction between beliefs and economic activity. Firms are more likely to invest if their beliefs about the fundamental have higher mean, but also if they have smaller variance (lower uncertainty). At the same time, the laws of motion for the mean and variance of beliefs depend on the investment rate. When few firms invest, little information is released, so uncertainty rises.

The key feature of the model is that this interaction between information and investment leads to uncertainty traps, formally defined as the coexistence of multiple stationary points in the dynamics of uncertainty and economic activity. The economy will converge to either a high regime (with high economic activity and low uncertainty) if the current level of uncertainty is sufficiently low, or to a low regime (with low activity and high uncertainty) if the current level of uncertainty is sufficiently high. As a result of this multiplicity, the economy exhibits strong non-linearities in its response to shocks: it quickly recovers after small temporary shocks, but it may shift into a low-activity regime after a large temporary shock. Once it has fallen in the low regime, only a large enough positive shock can push the economy back to the high-activity regime.

An important feature of the model is that, despite the presence of uncertainty traps, there is a unique recursive competitive equilibrium. That is, multiplicity of stationary points does not mean multiplicity of equilibria. Therefore, unlike in other macro models with complementarities, there is no room in our model for multiple equilibria or sunspots.[1]

The model features an inefficiently low level of investment because agents do not internalize the effect of their actions on common information. This inefficiency naturally creates room for welfare-

[1]For recent examples of business cycle models with multiple equilibria see Farmer (2013), Kaplan and Menzio (2013), Benhabib et al. (2015) and Schaal and Taschereau-Dumouchel (2015).

enhancing policy interventions. We study the problem of a constrained planner that is subject to the same informational constraints as private agents. The socially constrained-efficient allocation can be implemented with state-dependent subsidies. For example, it could be desirable to subsidize investment in times of high uncertainty and low activity. But, surprisingly, the optimal policy does not necessarily eliminate the uncertainty traps. Therefore, while policy interventions are desirable, they do not eliminate the non-linearities generated by the complementarity between uncertainty and economic activity.

After characterizing the baseline model, we embed the mechanism into a standard model of business cycles. We explore numerically the ability of the uncertainty trap mechanism to generate deep and persistent recessions for various parameter values. For that, we compare our baseline model, in which uncertainty fluctuates endogenously, with an economy in which the information flow is fixed. We find that the mechanism make recessions substantially deeper and longer relative to a framework with fixed uncertainty.

We also highlight how the model can explain certain features of the data that the standard RBC framework cannot replicate. First, uncertainty traps make output growth more persistent than TFP growth, as seen in the data. Second, evidence from a VAR shows that US output takes longer to recover after larger declines in productivity. This non-linearity is also present in impulse response functions generated from the model, but absent in the RBC model. Third, while the RBC model generates almost symmetric time series, the uncertainty trap mechanism generates negative skewness in output, as we observe in the data.

The theory is motivated by an empirical literature that investigates the impact of uncertainty on economic activity using VARs, as in Bloom (2009) and Bachmann et al. (2013), or using instrumental variables, as in Carlsson (2007), and finds that increases in uncertainty impedes economic activity. It also relates to the uncertainty-driven business cycle literature that analyzes the effect of uncertainty through real option effects as in Bloom (2009), Bloom et al. (2012), Bachmann and Bayer (2013), and Schaal (2015), and through financial frictions as in Arellano et al. (2012) and Gilchrist et al. (2014).[2]

Our theory adopts the concept of *Bayesian uncertainty*: in our model, agents use Bayes' rule to form beliefs about variables of interest (the *fundamentals* of the economy), and we define *uncertainty* as the variance of the probability distribution that describes these beliefs. The uncertainty-driven business cycle literature, instead, identifies uncertainty with time-varying volatility in various exogenous aggregate or idiosyncratic variables. These notions are related, but not identical. First, while time-varying volatility gives rise to Bayesian uncertainty, the latter is more general as it allows for additional channels, such as learning, to affect uncertainty. Second, measures of Bayesian uncertainty that capture subjective beliefs are countercyclical, like those used in the uncertainty-driven business cycle literature. Third, Bayesian uncertainty preserves the channel by which real option effects impact the economy, as they do with time-varying volatility. Thus, this more general

[2]Another literature studying time-varying risk is the literature on rare disasters (Barro, 2006) and time-varying disaster risk as in Gabaix (2012), Gourio (2012), and surveyed in Barro and Ursúa (2012).

and flexible concept, together with social learning, allows us to endogenize uncertainty, key to generating persistence in our model, while retaining the desirable properties of time-varying volatility.[3] Finally, as high-volatility events are short-lived in the data, models that focus on exogenous volatility shocks are hard to reconcile with the persistence of recessions. Bayesian uncertainty offers a promising alternative since some measures of subjective uncertainty display additional persistence.[4]

Our analysis also relates to a theoretical macroeconomic literature that studies environments characterized by learning from market outcomes such as Rob (1991), Caplin and Leahy (1993), Zeira (1994), Veldkamp (2005), Ordoñez (2009) and Amador and Weill (2010). Closely related to our paper is the analysis of Van Nieuwerburgh and Veldkamp (2006). They focus on explaining business-cycle asymmetries in an RBC model with incomplete information in which agents receive signals with procyclical precision about the economy's fundamental. During recessions, agents discount new information more heavily and the mean of their beliefs is slow to recover. Since the fundamental follows a two-state Markov process, beliefs are fully described by a single sufficient statistic, so that the mean and variance of beliefs are tied together. As a result, uncertainty does not provide an independent propagation mechanism and uncertainty traps do not arise. In contrast, our approach builds on a standard model of irreversible investment under uncertainty as in Dixit and Pindyck (1994) and Stokey (2008), and is able to disentangle the effects of mean vs. variance. The interaction between the option value of waiting due to irreversibilities and endogenous countercylical uncertainty is unique to our model, and essential to generate uncertainty traps.[5]

This paper is also related to the literature on fads and herding in the tradition of Banerjee (1992), Bikhchandani et al. (1992), Chamley and Gale (1994) and Chamley (2004). Articles in that tradition consider economies with an unknown fixed fundamental and study a one-shot evolution towards a stable state, whereas we study the full cyclical dynamics of an economy that fluctuates between regimes.

The dynamics generated by the model, with endogenous fluctuations between regimes, is reminiscent of the literature on static coordination games such as Morris and Shin (1998, 1999) and the dynamic coordination games literature as Angeletos et al. (2007) and Chamley (1999). These papers study games in which a complementarity in payoffs leads to multiple equilibria under complete information. The introduction of strategic uncertainty through noisy observation of the fundamental leads to a departure from common knowledge that eliminates the multiplicity. In contrast, the complete-information version of our model does not feature multiplicity, and complementarity only arises under incomplete information through social learning. Uniqueness does not obtain through strategic uncertainty, but by limiting the strength of the complementarities.

The paper is structured as follows. Section 2 presents the baseline model and the definition of the recursive equilibrium. Section 3 characterizes the partial-equilibrium investment decision of an

[3]Some recent papers discuss alternative channels that give rise to endogenous volatility over the business cycle. See Bachmann and Moscarini (2011) and D'Erasmo and Boedo (2011).

[4]See Appendix A for measures of uncertainty and additional discussion.

[5]Lang and Nakamura (1990) also consider environments in which economic activity increases the precision of information. Straub and Ulbricht (2014) propose a similar mechanism based on financial frictions.

individual firm and demonstrates the uniqueness of the equilibrium. Section 4 shows the existence of uncertainty traps, examines the non-linearities that they generate, and discusses the planner's problem. Section 5 describes the extended model and shows how uncertainty traps influence the response of the economy to various shocks. It also compares the dynamic properties of our model to an RBC model and the data. Section 6 concludes. Proofs can be found in the appendix.

2 Baseline Model

We begin by presenting a stylized model that only features the necessary ingredients to generate uncertainty traps. The intuitions from this simple model as well as the laws of motion governing the dynamics of uncertainty carry through to the extended model that we use for numerical analysis.

2.1 Population and Technology

Time is discrete. There is a fixed number of firms $\overline{N}$, chosen large enough that firms behave competitively. Each firm $j \in \{1, \ldots, \overline{N}\}$ holds a single investment opportunity that produces output x_j which is the sum of two components: a persistent common component θ, which denotes the economy's fundamental, as well as an idiosyncratic transitory component ε_j^x,

$$x_j = \theta + \varepsilon_j^x.$$

The common component follows an autoregressive process, so that the next period's fundamental is

$$\theta' = \rho_\theta \theta + \varepsilon^\theta, \tag{1}$$

where $0 < \rho_\theta < 1$. The innovations $\left(\varepsilon^\theta, \varepsilon_j^x\right)$ are normally distributed and independent over time and across firms,[6]

$$\varepsilon^\theta \sim \mathcal{N}\left(0, \left(1 - \rho_\theta^2\right) \sigma_\theta^2\right) \text{ and } \varepsilon_j^x \sim \mathcal{N}\left(0, \gamma_x^{-1}\right).$$

To produce, a firm must pay a fixed cost f, drawn each period from the continuous cumulative distribution F with mean $\bar{f}$ and standard deviation σ^f. Once production has taken place, the firm exits the economy and is immediately replaced by a new firm holding an investment opportunity. This assumption ensures that the mass of firms in the economy remains constant.[7]

Upon investment, the firm receives the payoff x_j. Firms have constant absolute risk-aversion,[8]

$$u\left(x_j\right) = \frac{1}{a}\left(1 - e^{-a x_j}\right),$$

[6] Idiosyncratic productivity ε^x is assumed to be i.i.d. for simplicity, but the theory could also accommodate persistence in this component.

[7] This assumption is made for tractability and is relaxed in the numerical exercise of Section 5.

[8] Agents can be thought of as entrepreneurs. Risk aversion simplifies the proof of Proposition 1 in the next section. It helps us establish, in particular, the convenient property that the equilibrium number of investing firms decreases with uncertainty everywhere, but is not crucial for the results. In the numerical section, we show that the mechanism carries through with risk neutrality.

where $a > 0$ is the coefficient of absolute risk aversion.

2.2 Timing and Information

At the beginning of each period, firms decide whether to invest or not without knowing their return on investment x_j. This decision therefore depends on their beliefs about the unobserved fundamental θ. As time unfolds, they learn about θ in various ways. First, they learn from a public signal Y observed at the end of each period,

$$Y = \theta + \varepsilon^y \tag{2}$$

where $\varepsilon^y \sim \text{iid } \mathcal{N}(0, \gamma_y^{-1})$. This signal captures the information released by statistical agencies or the media. Second, they learn by observing investment returns in the economy. Social learning takes place through this channel: when firm j invests, its return x_j is observed by all the other firms. [9] Since θ cannot be distinguished from the idiosyncratic term ε_j^x, production x_j acts as a noisy signal about the fundamental. Because of the normality assumption, a sufficient statistic for the information provided by all the firms' individual output is the public signal

$$X \equiv \frac{1}{N} \sum_{j \in I} x_j = \theta + \varepsilon_N^X, \tag{3}$$

where $N \in \left\{1, \ldots, \overline{N}\right\}$ is the endogenous number of firms that invest, I is the set of such firms, and

$$\varepsilon_N^X \equiv \frac{1}{N} \sum_{j \in I} \varepsilon_j^x \sim \mathcal{N}\left(0, (N\gamma_x)^{-1}\right).$$

Importantly, the precision $N\gamma_x$ of this signal increases with the number of investing firms N.

The timing of events is summarized in Figure 1.

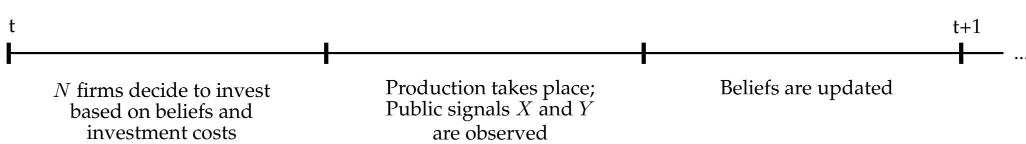


Figure 1: Timing of events

[9]In the context of our model, social learning aims at capturing the idea that firms learn from each other about various common components that affect their revenues — aggregate vs idiosyncratic productivities, but also demand, regulations, etc. Firms can also learn about demand conditions for broad product categories. For instance, observing that Firm A invests massively in a new phone might reveal to a competitor that Firm A has done a market study that reveals strong consumer demand for smartphones. Social learning has been found to influence economic decisions in various contexts. Foster and Rosenzweig (1995) estimate a model of the adoption of high-yielding seeds in India and find it consistent with social learning. Guiso and Schivardi (2007) find that peer-learning effects matter for the behavior of Italian industrial firms. Bikhchandani et al. (1998) survey the empirical social learning literature.

2.3 Beliefs

Under the assumption of a common initial prior, and because all information is public, beliefs are common across firms. In particular, there is no cross-sectional dispersion in beliefs. The normality assumptions about the signals and the fundamental imply that beliefs are also normally distributed

$$\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right),$$

where $\mathcal{I}$ is the information set at the beginning of the period. The mean of the distribution μ captures the optimism of agents about the state of the economy, while γ represents the precision of their beliefs about the fundamental. Precision γ is inversely related to the amount of uncertainty: as γ increases, the variance of beliefs decreases: uncertainty declines.

Firms start the period with beliefs (μ, γ) and use all the information available to update their beliefs according to Bayes' rule. By the end of the period, they have observed the public signals X and Y. Therefore, beliefs about next period's fundamental θ' are normally distributed with mean and precision equal to

$$\mu' = \rho_\theta \frac{\gamma\mu + \gamma_y Y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x}, \tag{4}$$

$$\gamma' = \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + N\gamma_x} + \left(1-\rho_\theta^2\right)\sigma_\theta^2\right)^{-1} \equiv \Gamma\left(N, \gamma\right). \tag{5}$$

These standard updating rules have straightforward interpretations: the mean of future beliefs μ' is a precision-weighted average of the present belief μ and the new signals, Y and X, whereas γ' depends on the precision of current beliefs, the precision of the signals and the variance of the shock to θ. Importantly, the precision of future beliefs does not depend on the realization of the public signals, but only on N and γ. The higher is N, the more precise is the public signal X, and the lower is uncertainty in the next period.[10] We use $\Gamma(N, \gamma)$ in (5) to denote the law of motion of the precision of information.

2.4 Firm Problem

We now describe the problem of a firm. In each period, given fixed cost f and beliefs about the fundamental, a firm can either wait or invest. It solves the Bellman equation

$$V\left(\mu, \gamma, f\right) = \max\left\{V^W\left(\mu, \gamma\right), V^I\left(\mu, \gamma\right) - f\right\}, \tag{6}$$

[10] In our current formulation, increasing the total number of firms would lead to a higher flow of information, suggesting that larger economies are at an informational advantage. In practice, however, a larger economy is more complex — combining many shocks, industries, products — and learning about its state may be more difficult. Additionally, other factors that influence firms' decisions are local (natural disasters, local demand conditions, changes in regional institutions, etc) and the social learning about these factors might be limited to regional firms. In this case, increasing the country's size would not necessarily convey more information about these local factors.

where $V^W(\mu,\gamma)$ is the value of waiting and $V^I(\mu,\gamma)$ is the value of investing after incurring the investment cost f. Specifically, they do not internalize the impact of their decisions on aggregate information.

If a firm waits, it starts the next period with updated beliefs (μ',γ') about the fundamental and a new draw of the fixed cost f'. Therefore, the value of waiting is

$$V^W(\mu,\gamma) = \beta \mathbb{E}_{\mu',\gamma'}\left[\int V(\mu',\gamma',f')\,dF(f') \mid \mu,\gamma\right]. \tag{7}$$

In turn, when a firm invests it receives output x and exits. Therefore,

$$V^I(\mu,\gamma) = \mathbb{E}\left[u(x) \mid \mu,\gamma\right] = \mathbb{E}\left[\frac{1}{a}\left(1-e^{-ax}\right) \mid \mu,\gamma\right]. \tag{8}$$

The firm's optimal investment decision takes the form of a cutoff rule $f_c(\mu,\gamma)$ such that a firm invests if and only if $f \leq f_c(\mu,\gamma)$. The cutoff is defined by the following indifference condition

$$f_c(\mu,\gamma) = V^I(\mu,\gamma) - V^W(\mu,\gamma). \tag{9}$$

2.5 Law of Motion for the Number of Investing Firms N

We now aggregate the individual decisions of the firms. As the investment decision follows the cutoff rule $f_c(\mu,\gamma)$, the process for the number of investing firms N satisfies

$$N\left(\mu,\gamma,\{f_j\}_{1\leq j\leq \overline{N}}\right) = \sum_{j=1}^{\overline{N}} \mathbb{I}\left(f_j \leq f_c(\mu,\gamma)\right). \tag{10}$$

Since investment depends on a random fixed cost, the number of investing firms is a random variable that depends on the realization of the shocks $\{f_j\}_{1\leq j\leq \overline{N}}$. As these costs are i.i.d., the ex-ante probability of investment is identical across firms. Therefore, the ex-ante distribution of N, as perceived by firms, is binomial,

$$N \mid \mu,\gamma \sim \text{Bin}\left(\overline{N}, p(\mu,\gamma)\right), \tag{11}$$

where $p(\mu,\gamma)$ captures the perceived probability of investment for other firms. In equilibrium, firms' expectations must be consistent with the actual probability of investing:

$$p(\mu,\gamma) = F\left(f_c(\mu,\gamma)\right). \tag{12}$$

Note that N is only a function of the beliefs (μ,γ) and the individual shocks $\{f_j\}_{1\leq j\leq \overline{N}}$. Since these shocks are independent from the fundamental θ and investment decisions are made before the observation of $\{x_j\}_{j\in I}$, there is nothing to learn from the non-investment of firms, nor from the realization of N itself.

2.6 Recursive Competitive Equilibrium

We define a recursive rational-expectation equilibrium as follows.

Definition 1. A recursive competitive equilibrium consists of a cutoff rule $f_c(\mu, \gamma)$, value functions $V(\mu, \gamma, f)$, $V^W(\mu, \gamma)$, $V^I(\mu, \gamma)$, a perceived ex-ante investment probability $p(\mu, \gamma)$, laws of motions for aggregate beliefs $\{\mu', \gamma'\}$, and a number of investing firms $N\left(\mu, \gamma, \{f_j\}_{1 \le j \le \overline{N}}\right)$, such that

1. The value function $V(\mu, \gamma, f)$ solves (6), with $V^W(\mu, \gamma)$ and $V^I(\mu, \gamma)$ defined according to (7) and (8), yielding the cutoff rule $f_c(\mu, \gamma)$ in (9);
2. The aggregate beliefs (μ, γ) evolve according to (4) and (5), where N is given by (11);
3. The ex-ante investment probability $p(\mu, \gamma)$ and the cutoff rule $f_c(\mu, \gamma)$ satisfy (12) and
4. The number $N\left(\mu, \gamma, \{f_j\}_{1 \le j \le \overline{N}}\right)$ of investing firms is given by (10).

3 Equilibrium Characterization

We first characterize the optimal investment decision of a firm. We provide conditions such that, due to the the irreversibility of investment, firms are less likely to invest when uncertainty is high. Then, we prove the existence and uniqueness of the recursive equilibrium and characterize its key properties.

3.1 Investment Rule Given the Evolution of Beliefs

The optimal investment rule $f_c(\mu, \gamma)$ depends on how beliefs evolve. We begin by establishing two simple lemmas about the dynamics of aggregate beliefs.

Evolution of the Mean of Beliefs

Using (4), we can characterize the stochastic process for the mean of beliefs as follows.

Lemma 1. *For a given N, mean beliefs μ follow an autoregressive process with time-varying volatility s,*

$$\mu' = \rho_\theta \mu + s(N, \gamma)\, \varepsilon,$$

where $s(N, \gamma) = \rho_\theta \left(\frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}\right)^{\frac{1}{2}}$ and $\varepsilon \sim \mathcal{N}(0, 1)$.

The mean of beliefs captures the optimism of agents about the fundamental and evolves stochastically due to the the arrival of new information. It inherits the autoregressive property of the fundamental, and its volatility $s(N, \gamma)$ is time-varying because the amount of information that firms collect over time is endogenous. The volatility is decreasing with γ and increasing with N. In times of low uncertainty (γ high) agents place more weight on their current information and less on new signals, making the mean of beliefs more stable. In contrast, in times of high activity (N high) more information is released, making beliefs more likely to fluctuate.

Evolution of Uncertainty

The precision of beliefs γ captures the (inverse of) uncertainty about the fundamental and its dynamics play a key role for the existence of uncertainty traps. Its law of motion satisfies the following properties.

Lemma 2. *The precision of next-period beliefs γ' increases with N and γ. For a given number of investing firms N, the law of motion for the precision of beliefs $\gamma' = \Gamma(N, \gamma)$ admits a unique stable stationary point in γ.*

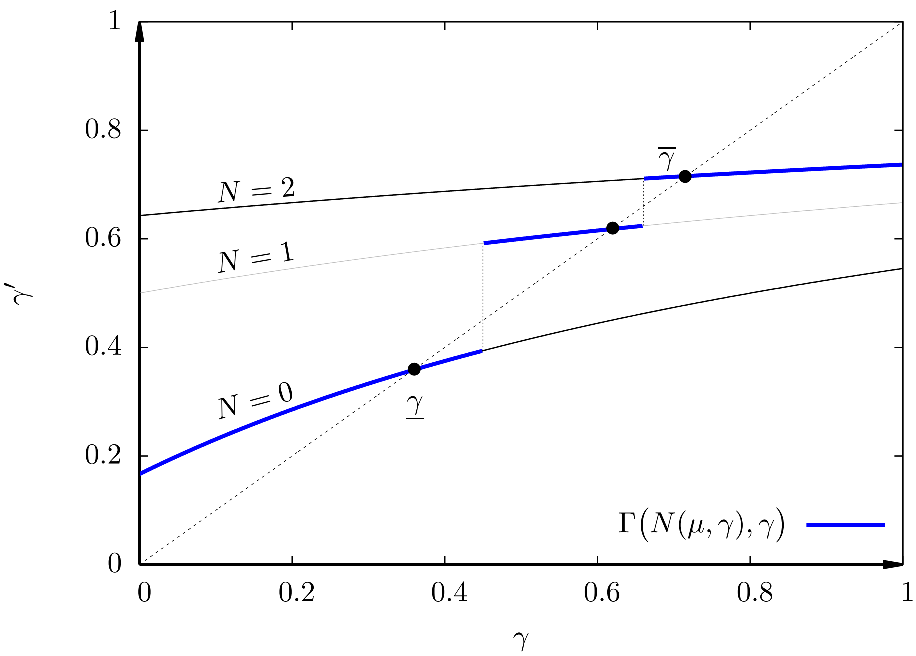


Figure 2: Example of dynamics for beliefs precision γ when $\overline{N} = 2$

The thin solid curves on Figure 2 depict $\Gamma(N, \gamma)$ for different constant values of N when $\overline{N} = 2$. An increase in the level of activity raises the next period precision of information γ' for each level of γ in the current period. Since N is between 0 and $\overline{N}$, the support of the ergodic distribution of γ must lie between the two bounds $\underline{\gamma}$ and $\overline{\gamma}$ defined by $\underline{\gamma} \equiv \Gamma(0, \underline{\gamma})$ and $\bar{\gamma} \equiv \Gamma(\overline{N}, \overline{\gamma})$. In other words, $\underline{\gamma}$ is the stationary level of precision when no firm invests, while $\overline{\gamma}$ is the one reached when all firms invest.

In equilibrium, N varies with μ and γ. Suppose, as an example, that N is a deterministic and increasing step function of γ, and let us keep μ fixed for the moment. Figure 2 illustrates how the feedback from uncertainty to investment opens up the possibility of multiple stationary points in the dynamics of the precision of beliefs, and therefore uncertainty. In this example, the function $\gamma' = \Gamma(N(\mu, \gamma), \gamma)$, depicted by the solid curve, has three fixed points. We formally establish, in part 4, that this type of multiplicity is a generic feature of the equilibrium.

Optimal Timing of Investment

With the laws of motion for aggregate beliefs at hand we can characterize the individual investment decision as a function of beliefs. Naturally, a more optimistic firm (higher μ) is more likely to invest. In turn, uncertainty (lower γ) may reduce the returns to investment for two reasons. First, risk averse firms dislike uncertain payoffs. Second, since investment is costly and irreversible, there is an option value of waiting: in the face of uncertainty, firms prefer to delay investment to gather additional information and avoid downside risk.

The next proposition formally establishes the validity of these intuitions. Specifically, it provides a partial-equilibrium characterization of the optimal investment behavior of a firm who, consistently with (11), perceives N as following a binomial distribution $\mathrm{Bin}\left(\overline{N}, p\left(\mu,\gamma\right)\right)$ for some sufficiently smooth function $p\left(\mu,\gamma\right) \in \mathcal{P}$, as defined in the Appendix.

Proposition 1. *Under Assumption 1 stated in Appendix G, given a random number of investing firms $N \sim Bin\left(\overline{N}, p\left(\mu,\gamma\right)\right)$ for some $p\left(\mu,\gamma\right) \in \mathcal{P}$, and for γ_x sufficiently low, there exists a unique solution $V\left(\mu,\gamma,f\right)$ to the firm's Bellman equation and the resulting cutoff $f_c\left(\mu,\gamma\right)$ is strictly increasing in μ and γ.*

The properties satisfied by the optimal investment rule are typical of optimal stopping time models of investment but, in our context, they are not straightforward to establish because of the endogeneity of information. As expected, investment is strictly increasing in μ. Establishing the last property, crucial to our mechanism, that the probability of investment decreases with uncertainty, is more challenging. On the one hand, uncertainty directly discourages investment through risk aversion and real option effects. The latter is guaranteed by Assumption 1, which is satisfied if the persistence of the fundamental is high enough and its volatility is sufficiently low. This ensures that the fundamental does not vary too much over time, so that firms may have an incentive to wait. On the other hand, these effects may be offset by an opposing effect through social learning: if the number of investing firms N declines with uncertainty, less information is released and the option to delay investment in order to collect more information becomes less attractive. With γ_x small, the amount of information that transits through the social learning channel remains small enough that the latter effect is negligible and the option value of waiting dominates. In particular, enough information can be gathered from the public signal Y despite the fluctuations in social learning, ensuring that the option to wait always remains attractive to firms.

3.2 Existence and Uniqueness

We have described in Lemmas 1 and 2 how beliefs depend on the number of investing firms, and, in Proposition 1, how firms' investment decisions are affected by beliefs. In the latter, firms make their decisions taking the aggregate investment probability p as given. We now close the equilibrium by requiring that the perceived investment behavior of firms, summarized by $N \sim \mathrm{Bin}\left(\overline{N}, p\left(\mu,\gamma\right)\right)$, is consistent with their actual investment decisions: $p\left(\mu,\gamma\right) = F\left(f_c\left(\mu,\gamma\right)\right)$. The next proposition shows that such a general equilibrium exists and is unique.

Proposition 2. *Under the same conditions as Proposition 1 and some regularity conditions stated in the Appendix, a recursive equilibrium exists and is unique. The equilibrium expected fraction of firms investing $p(\mu, \gamma)$ is increasing in the mean of beliefs μ and the precision γ.*

Showing uniqueness of the fixed point $p(\mu, \gamma) = F(f_c(\mu, \gamma))$ is challenging due to the ambiguous feedback from uncertainty to investment discussed in the previous section. Formally, this leads to a failure in the monotonicity of the mapping from the perceived investment probability $p(\mu, \gamma)$ to the investment probability of each firm, which prevents us from using Blackwell's sufficient conditions. Fortunately, we can explicitly show that the main fixed point problem is a contraction when γ_x is low. This assumption ensures that the complementarity between information and economic activity is not strong enough to support multiple equilibria. Uniqueness of equilibrium is an attractive feature as it leads to unambiguous predictions and makes the model amenable to quantitative work. Despite uniqueness of the equilibrium, the model features interesting non-linear dynamics and multiple regimes, as we show in part 4.

Figure 3 illustrates how the investment probability varies as a function of beliefs (μ, γ). The partial equilibrium results from Proposition 1 carry through to the general equilibrium: the number of investing firms increases as they become more optimistic about the fundamental (μ high) or less uncertain (γ high).

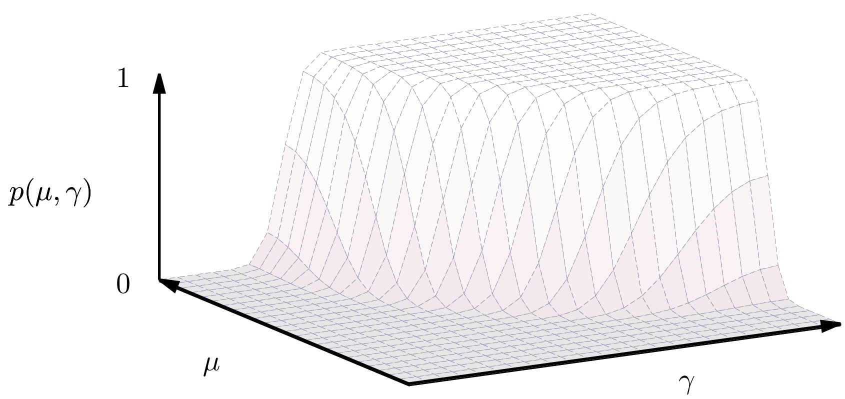


Figure 3: Example of aggregate investment probability

4 Uncertainty Traps

We now examine the interaction between firms' behavior in the face of uncertainty and social learning. This interaction leads to episodes of self-sustaining uncertainty and low activity, which we call uncertainty traps. We provide sufficient conditions on the parameters that guarantee the existence of such traps and discuss the type of aggregate dynamics that they imply. We find that the response of the economy to shocks is highly non-linear: it quickly recovers after small shocks, but large, short-lived shocks may plunge the economy into long-lasting recessions. We also characterize the constrained planner's problem and discuss its policy implications.

4.1 Definition and Existence

We assume at this point that the total number of firms $\overline{N}$ is large enough, so that

$$n(\mu,\gamma) \equiv \frac{N(\mu,\gamma)}{\overline{N}} \simeq p(\mu,\gamma). \tag{13}$$

With this assumption, we can treat the fraction of investing firms n as a deterministic function of beliefs, ignoring fluctuations due to the finiteness in the number of firms. The model's equations remain the same except that we must substitute $N(\mu,\gamma)$ with $n(\mu,\gamma)$.[11] We are now ready to define an uncertainty trap.

Definition 2. There is an *uncertainty trap* if there are at least two locally stable fixed points in the dynamics of beliefs precision $\gamma' = \Gamma(n(\mu,\gamma),\gamma)$ for some nonempty set $\mu \in M$.

The definition of an uncertainty trap captures the situation depicted in Figure 2: for a given mean of beliefs, the economy may find itself in distinct fixed points of the dynamics of uncertainty. We refer to these stationary points as *regimes.*

Note that multiplicity of regimes does not imply multiple equilibria. This distinction is important because it highlights that the model is not subject to indeterminacy. While multiple values of γ may satisfy the equation $\gamma = \Gamma(n(\mu,\gamma),\gamma)$ for a given μ, the regime that prevails at any given time is unambiguously determined by the history of past aggregate shocks, summarized by the current beliefs (μ,γ). The definition also emphasizes the notion of *stability,* which is required for the type of self-enforcing dynamics that we propose. Notice, however, that we only require local stability along the dimension γ while μ keeps evolving according to its law of motion.

The following proposition formally establishes that uncertainty traps exist for a range of mean of beliefs μ under some condition on the dispersion of investment costs.

Proposition 3. *Under the conditions of Proposition 2 and for σ^f small enough, there exists a non-empty interval $M = [\mu_l, \mu_h]$ such that, for all $\mu \in (\mu_l, \mu_h)$, the economy features an uncertainty trap with at least two regimes $\gamma_l(\mu) < \gamma_h(\mu)$. Regime γ_l is characterized by high uncertainty and low investment while regime γ_h is characterized by low uncertainty and high investment.*

Figure 4 presents examples for the law of motion of γ when the investment costs f are normally distributed. The solid curves represent the function $\gamma' = \Gamma(n(\mu,\gamma),\gamma)$ evaluated at five different values of μ, with the thick solid curve corresponding to an intermediate value of μ. In all

[11] To prevent uncertainty from vanishing completely as $\overline{N} \to \infty$, we assume that the precision of firms' individual signals decreases with $\overline{N}$: $\gamma_x\left(\overline{N}\right) = \gamma_x/\overline{N}$. The details of the limit and the corresponding economy are explained in Appendix F.

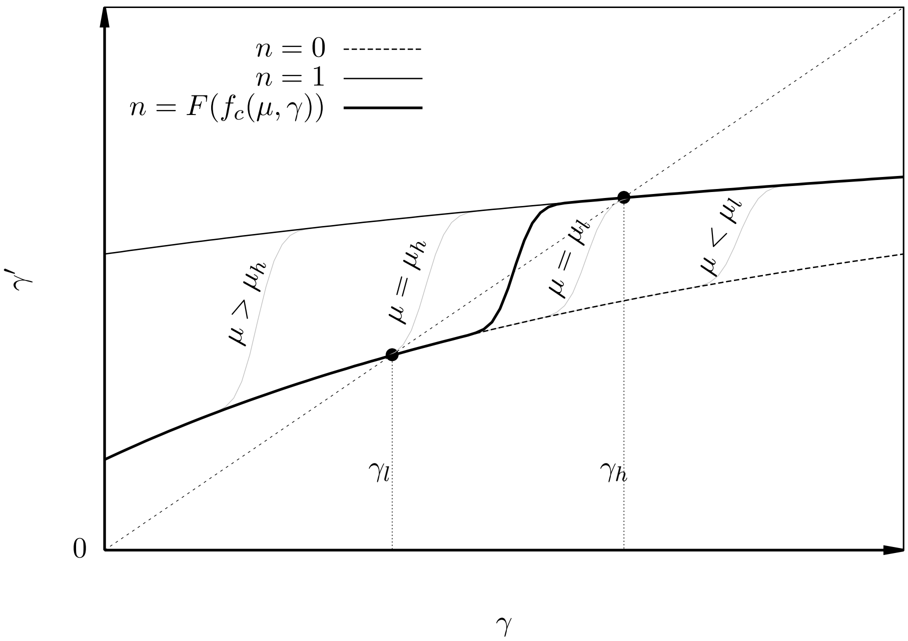


Figure 4: Dynamics of precision $\gamma' = \Gamma\left(n\left(\mu, \gamma\right), \gamma\right)$ for different values of μ

cases, for small precision γ, uncertainty is high and firms do not invest. As a result, they do not learn from observing aggregate activity and the precision of beliefs γ' remains low. As precision increases, uncertainty decreases and firms become sufficiently confident about the fundamental to start investing. As that happens, uncertainty decreases further.

In our example, the thick curve intersects the 45° line three times. The second intersection corresponds to an unstable regime, but the other two are locally stable. We denote these regimes by γ_l and γ_h. In regime γ_l, uncertainty is high and investment is low, while the opposite is true in regime γ_h.

Proposition 3 shows that this situation is a generic feature of the equilibrium when the dispersion of investment costs σ^f is small. This condition ensures that the feedback of investment on information is strong enough to sustain distinct stationary points.

4.2 Dynamics: Non-linearity and Persistence

We now describe the full dynamics of the economy by taking into account the evolution of μ in response to the arrival of new information. Figure 4 shows that, as long as μ stays between the values μ_l and μ_h, defined in Proposition 3, the two regimes $\gamma_l\left(\mu\right)$ and $\gamma_h\left(\mu\right)$ preserve their stability. As a result, uncertainty and the fraction of active firms n are relatively unaffected by changes in μ. In contrast, for values of μ above μ_h, a large enough fraction of firms invest, so the dynamics of beliefs only admits the high-activity regime as a stationary point. Similarly, for values below μ_l, the economy only admits the low-activity regime. Therefore, sufficiently large shocks to μ can make one regime unstable and trigger a regime switch.

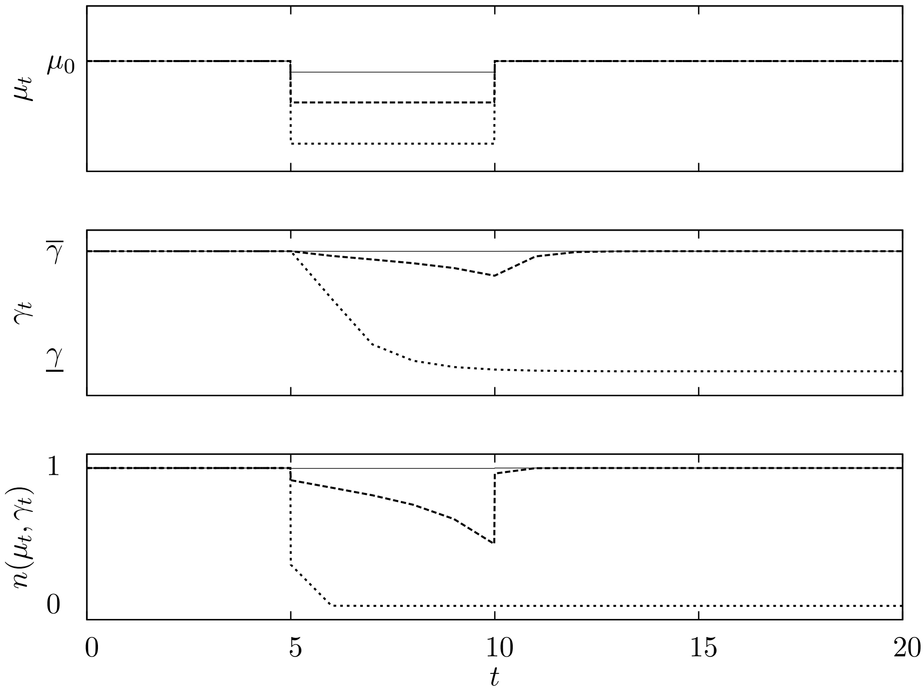


Figure 5: Persistent effects of temporary shocks

The economy displays non-linear dynamics: it reacts very differently to large shocks in comparison to small ones. Figure 5 shows various simulations to illustrate this feature using the example from Figure 4. The top panel presents three different series of shocks to the mean of beliefs μ. The three series start from the high-activity/low-uncertainty regime. At $t = 5$, the economy is hit by a negative shock to μ, due to a bad realization of either the public signals or the fundamental. The mean of beliefs then returns to its initial value at $t = 10$. Across the three series, the magnitude of the shock is different.

The middle and bottom panels show the response of beliefs precision γ and the fraction of investing firms n. The solid gray line represents a small temporary shock, such that μ remains within $[\mu_l, \mu_h]$. Despite the negative shocks to the mean of beliefs, all firms keep investing and the precision of beliefs is unaffected. When the economy is hit by a temporary shock of medium size (dashed line), some firms stop investing, leading to a gradual increase in uncertainty. As uncertainty rises, investment falls further and the economy starts to drift towards the low regime. However, when the mean of beliefs recovers, the precision of information and the number of active firms quickly return to the high-activity regime. In contrast, when the economy is hit by a large temporary shock (dotted line), the number of firms delaying investment is large enough to produce a self-sustaining increase in uncertainty. The economy quickly shifts to the low-activity regime and remains there even after the mean of beliefs recovers.

We now show how the economy escapes from the trap in which it fell in Figure 5. Figure 6 shows the effect of positive shocks when the economy starts from the low regime. The economy receives positive signals that lead to a temporary increase in mean beliefs between periods 20 and 25,

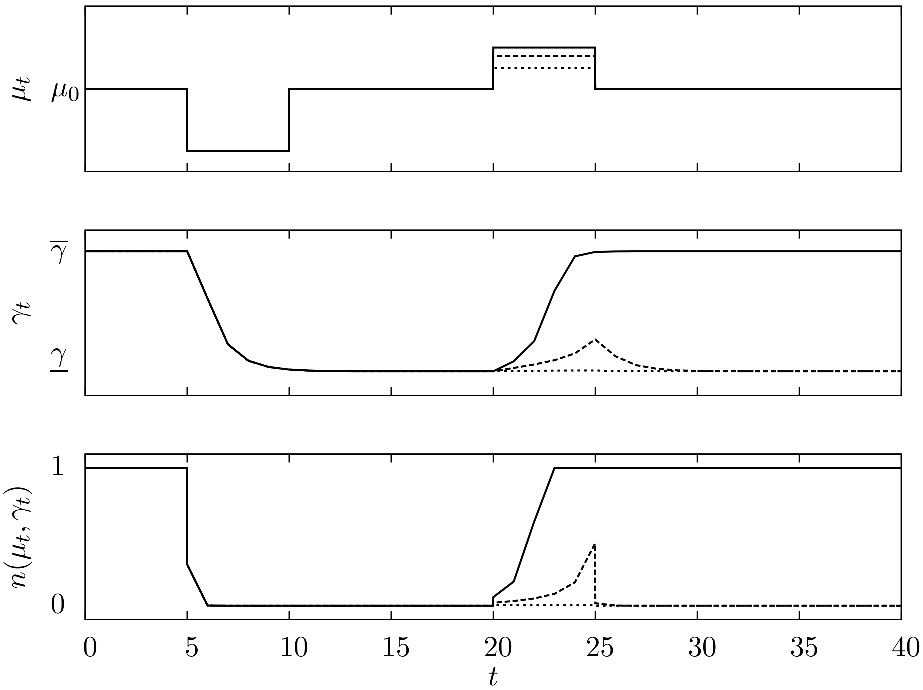


Figure 6: Escaping an uncertainty trap

possibly because of a recovery in the fundamental. When the temporary increase in average beliefs is not sufficiently strong, the recovery is interrupted as μ returns to its initial value. However, when the temporary increase is sufficiently large, the economy reverts back to the high-activity regime. Once again, temporary shocks of sufficient magnitude to the fundamental may lead to nearly permanent effects on the economy.

4.3 Additional Remarks

A number of additional lessons can be drawn from these simulations. First, in this framework, uncertainty is a by-product of recessions. This result echoes the empirical findings of Bachmann et al. (2013) who show that uncertainty is partly caused by recessions and conclude, by that, that it is of secondary importance for the business cycle. We show, however, that uncertainty may still have a large impact on the economy by affecting the persistence and depth of recessions, even if it is not what triggers them.

Second, as in models with learning in the spirit of Van Nieuwerburgh and Veldkamp (2006), this theory provides an explanation for asymmetries in business cycles. In good times, since agents receive a large flow of information, they react faster to shocks than in bad times.

Third, our economy may feature high uncertainty without volatility. For instance, in the low regime, agents are highly uncertain about the fundamental but the volatility of economic aggregates is low. Therefore, according to our theory, subjective uncertainty may affect economic fluctuations even if no volatility is observed in the data. This distinguishes our approach from the existing uncertainty-driven business cycle literature in the spirit of Bloom (2009). In particular, direct measures of subjective uncertainty rather than measures of volatility are important to capture the full amount of uncertainty in the economy.

Finally, a recent literature (Bachmann et al., 2013; Orlik and Veldkamp, 2013) uses survey data to derive measures of uncertainty based on ex-ante forecast errors. Our model highlights a potential difficulty about this approach, as uncertainty about *fundamentals* differs from uncertainty about *endogenous variables*, such as output or investment. For example, when the economy is trapped in the low activity regime, firms know that all firms are uncertain, and therefore that output and investment are likely to be low. As a result, their forecasts about economic aggregates are accurate even though their uncertainty about the fundamental is high. As implied by the model, forecast errors about variables like output may possibly be a bad proxy for uncertainty about fundamentals.

4.4 Policy Implications

The economy is subject to an information externality: in the decentralized equilibrium, firms invest less often than they should because they do not internalize the release of information to the rest of the economy caused by their investment. In Proposition 4, we solve the problem of a constrained planner subject to the same information technology as agents in the economy and show that the decentralized economy is constrained inefficient and that a simple policy instrument such as an investment subsidy that depends on current beliefs (μ, γ) is sufficient to restore constrained efficiency. Despite internalizing information flows, the constrained optimum still features uncertainty traps.

Proposition 4. *The recursive competitive equilibrium is constrained inefficient. The efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$ and a uniform tax. However, when γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

The subsidy that implements the optimal allocation takes a simple form to align social and private incentives. As shown in the proof of the proposition, it is simply the sum of the social value of releasing an additional signal to the economy and the private value of delaying investment.

The optimal policy being a subsidy, proposition 4 implies that firms are more likely to invest in the efficient allocation than in the laissez-faire economy. However, uncertainty traps can still arise in the efficient allocation. This may be surprising if one believes that the role of the planner is always to push the economy towards the high regime.[12] As it turns out, if the planner does not have more information than individual agents, it is still optimal to wait when uncertainty is high enough. Hence, there still exists a sufficiently strong complementarity between information and the level of activity in the constrained-efficient allocation to generate uncertainty traps. However, while uncertainty traps remain present in the efficient allocation, they are less persistent than in the laissez-faire economy because firms have stronger incentives to invest.

[12] In contrast to models with multiple equilibria, γ is a predetermined state variable that summarizes past information, not a forward-looking variable that the planner can pick to select equilibria.

5 Extended Model and Numerical Exercise

We now extend the model to incorporate standard features of business cycle models. The purpose of this exercise is to explore numerically the ability of the uncertainty trap mechanism to generate deep and persistent recessions in a more general context. To do so, we enrich the baseline theory along several dimensions. First, we introduce infinitely-lived firms that produce every period using a Cobb-Douglas production function that combines labor and capital as inputs. They also accumulate capital over time by investing through intensive and extensive margins. Second, a representative household maximizes utility over consumption streams and supplies labor inelastically.

5.1 Extended model

Preferences and Technology

A representative household maximizes utility over consumption with preferences

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}U\left(C_{t}\right),U'>0,U''\leqslant 0,$$

where C_t is aggregate consumption and $0<\beta<1$ is the discount factor. It is endowed with one unit of labor every period supplied inelastically.

There is a single consumption good produced by a unit measure of firms indexed by $j\in[0,1]$. Firm j operates a Cobb-Douglas technology and produces output

$$\left(A+Y\right)k_{j}^{\alpha}l_{j}^{1-\alpha},$$

using l_j units of labor and k_j units of capital, where

$$\begin{aligned} Y &= \theta+\varepsilon^{y} \\ \theta' &= \rho_{\theta}\theta+\varepsilon^{\theta} \end{aligned}$$

with $\varepsilon^{\theta}\sim$ iid $\mathcal{N}\left(0,\left(1-\rho_{\theta}^{2}\right)\sigma_{\theta}^{2}\right)$ and $\varepsilon^{y}\sim$ iid $\mathcal{N}\left(0,\gamma_{y}^{-1}\right)$, and where $A>0$ is the unconditional mean of total factor productivity.[13] As in the baseline model, the stochastic process θ is the fundamental of the economy. While Y corresponded to the public information provided by statistical agencies and the media in the baseline model, it now captures the information provided by economic aggregates like output. This specification is convenient as it guarantees that Y enters the laws of motion of beliefs in the same way as before and, at the same time, ensures that it contains all the information that diffuses through prices and other aggregates.

[13] Our specification of a multiplicative TFP ensures that the variance of Y does not affect expected output directly. We make sure that productivity never becomes negative in our numerical exercise.

Investment

In the baseline model, firms face a simple binary decision: *to invest* or *not*. To introduce irreversibilities in a more realistic way, we use a common device in the investment literature (Khan and Thomas, 2008) and model two different types of investment: *i)* normal investments, which corresponds to routine maintenance and small repairs of the current capital stock, *ii)* large investments, which we interpret as large purchases of plants and equipment, or the introduction of new products. All investments are subject to a convex variable cost $c(i_j)$, with $c' > 0$ and $c'' > 0$, and where i_j is the investment rate. In addition to this variable cost, large investments incur an i.i.d fixed cost $f_j > 0$, drawn from the continuous cumulative distribution F with mean $\overline{f}$ and standard deviation σ^f. Normal investments, on the other hand, do not require the payment of any additional cost, but are constrained to remain "small", i.e., within some bounds $i_j \in \left[\underline{i}, \overline{i}\right]$. To obtain aggregation, all costs are proportional to the stock of capital owned by the firm. Therefore, firm j must pay a total cost of $c(i_j)k_j$, with an additional cost of $f_j k_j$ if the investment is large, to increase its capital stock to

$$k'_j = (1 - \delta + i_j)\, k_j.$$

To introduce a significant option value of waiting, we assume that large investments can only be made if firms hold an investment opportunity. Investment opportunities arrive stochastically: a firm without an opportunity receives one with probability $\overline{q}$ at the end of the period and keeps it until exercised.[14] Only one investment opportunity can be held at a time. We use the dummy variable q_j to indicate whether a firm has an investment opportunity ($q_j = 1$) or not ($q_j = 0$).

Timing and Information

As in the baseline model, agents learn from two sources of information. First, observing aggregate output reveals the public signal Y, which summarizes all the information contained in prices and aggregate variables. Second, agents learn from a social learning channel. Normal investments, i.e., the replacement of light bulbs, do not bring in new information. However, because businesses often engage in substantial market research in preparation for large investments, we assume that if firm j undertakes a large investment, it releases a signal $x_j = \theta + \varepsilon_j^x$, where $\varepsilon_j^x \sim \mathcal{N}\left(0, (\gamma_x k_j)^{-1}\right)$. The individual signals $\{x_j\}$ are observed by everyone and their precision is proportional to the capital stock of the firm. We make this assumption for two reasons: first, it allows aggregation of the economy; second, it also seems realistic to assume that investment by large businesses reveal more information than investment by small mom-and-pop stores.

The timing of events is as follows:

1. All firms share the same prior distribution over the fundamental $\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right)$.

2. Firms that hold an investment opportunity draw their fixed cost f_j and decide whether to make a large investment or not.

[14] In Section 5.4, we allow for investment opportunities to be destroyed randomly.

3. All firms choose their investment rate i_j.

4. Firms choose labor l_j and production takes place.

5. Aggregate productivity Y and individual signals $\{x_j\}$ are publicly observed.

6. Firms that do not hold an investment opportunity receive one with probability $\overline{q}$.

7. Agents update their beliefs for the next period.

Value functions

We now define the problem faced by firms. Thanks to the linearity in individual capital stock k_j, the model admits aggregation: individual value functions are linear in k_j and the aggregate state space of the economy is (μ, γ, K, Q) where $K = \int k_j dj$ is the aggregate capital stock and $Q = \int q_j k_j dj$ is the stock of capital held by firms with an investment opportunity.

The value of a firm without an investment opportunity is

$$\begin{aligned} V_0(\mu, \gamma, K, Q, k) = \max_{i_0^c \in [\underline{i}, \overline{i}], l_0} \mathbb{E} \Big\{ & p(A+Y) k^\alpha l_0^{1-\alpha} - w l_0 - pc(i_0^c) k \\ & + \beta \left[\overline{q} \int V_1(\mu', \gamma', K', Q', k', f') dF(f') + (1-\overline{q}) V_0(\mu', \gamma', K', Q', k') \right] \Big| \mu, \gamma \Big\}, \end{aligned} \tag{14}$$

where $k' = (1 - \delta + i_0^c) k$, p denotes the price of the consumption good, and w is the wage. Firms without an opportunity simply produce, pay wages and choose their normal or "constrained" investment rate $i_0^c \in [\underline{i}, \overline{i}]$. In the following period, they receive an investment opportunity with probability $\overline{q}$ or remain at the value V_0. The value of a firm with an investment opportunity is

$$\begin{aligned} V_1(\mu, \gamma, K, Q, k, f) = \max_{l_1} \mathbb{E} & \left\{ p(A+Y) k^\alpha l_1^{1-\alpha} - w l_1 \Big| \mu, \gamma \right\} \\ & + \max \left\{ V_1^W(\mu, \gamma, K, Q, k), V_1^I(\mu, \gamma, K, Q, k) - \mathbb{E}[p] f k \right\}, \end{aligned} \tag{15}$$

where V_1^W is the value of waiting, i.e., undertaking a normal investment and holding on to the investment opportunity for the next period, and V_1^I is the value of acting, i.e., doing a large, unconstrained investment. The value of waiting is

$$V_1^W(\mu, \gamma, K, Q, k) = \max_{i_1^c \in [\underline{i}, \overline{i}]} \mathbb{E} \left\{ -pc(i_1^c) k + \beta \int V_1(\mu', \gamma', K', Q', k', f') dF(f') \Big| \mu, \gamma \right\}, \tag{16}$$

where $k' = (1 - \delta + i_1^c)\, k$. The value of acting is

$$V_1^I(\mu, \gamma, K, Q, k) = \max_{i_1} \mathbb{E}\Big\{ -pc(i_1)\, k \\ + \beta \left[\overline{q} \int V_1(\mu', \gamma', K', Q', k', f')\, dF(f') + (1 - \overline{q})\, V_0(\mu', \gamma', K', Q', k') \right] \Big| \mu, \gamma \Big\}, \tag{17}$$

where $k' = (1 - \delta + i_1)\, k$.

Taking the first-order condition on labor, we find that a firm's labor demand is proportional to its stock of capital and identical across firms. We define the labor demand per unit of capital $l(\mu, \gamma, K, Q)$. Substituting the optimal labor demand in the value functions and using the fact that the value functions (14)-(17) define contractions in the space of value functions, it is easy to show that the value functions are linear in k. In particular, the investment rates i_0^c, i_1^c and i_1 are independent of a firm's individual capital stock. The decision to undertake a large investment takes the form of a cutoff $f_c(\mu, \gamma, K, Q)$ in terms of the firm's individual fixed cost, which implies that firms with a fixed cost $f \leqslant f_c(\mu, \gamma, K, Q)$ choose a large investment while firm with $f > f_c(\mu, \gamma, K, Q)$ prefer to hold on to their investment opportunity until the next period. We define $n(\mu, \gamma, K, Q) = F(f_c(\mu, \gamma, K, Q))$ as the fraction of firms with investment opportunities that undertake large investments.

The aggregate capital stock follows the law of motion

$$K' = (K - Q)(1 - \delta + i_0^c) + Q(1 - \delta + n i_1 + (1 - n)\, i_1^c). \tag{18}$$

The first term corresponds to the evolution of the capital stock for firms without an investment opportunity (they hold $K - Q$ units of capital), and the second term captures the evolution of firms with an opportunity. A fraction n of them undertakes a large investment while the rest chooses a constrained normal investment i_1^c.

The stock of capital held by firms with an investment opportunity evolves according to the equation

$$Q' = (1 - \delta + i_0^c)\, \overline{q}\, (K - Q) + (1 - \delta + i_1^c)(1 - n)\, Q + (1 - \delta + i_1)\, \overline{q} n Q. \tag{19}$$

The first term captures the inflow of capital from firms without an opportunity (capital $K - Q$) that receive an investment opportunity (fraction $\overline{q}$), the second term is the capital of firms with opportunities that decide to wait and hold on to their opportunities (fraction $1 - n$ of the capital Q) and the third term denotes the inflow from firms with an investment opportunity that undertake a large investment (fraction n of capital Q) and are lucky enough to receive a new opportunity immediately (fraction $\overline{q}$).

As in the baseline model, the information that diffuses through social learning can be aggregated into a single aggregate signal X, which now is a capital-weighted average of all individual signals $\{x_j\}_{j \in [0,1]}$,

$$X = \frac{\int n_j q_j k_j x_j dj}{\int n_j q_j k_j dj} = \theta + \varepsilon^X, \varepsilon^X \sim \mathcal{N}\left(0, (nQ\gamma_x)^{-1}\right),$$

where $nQ\gamma_x$ is the precision of the social learning channel. The laws of motion for information become

$$\mu' = \rho_\theta \frac{\gamma\mu + \gamma_y Y + nQ\gamma_x X}{\gamma + \gamma_y + nQ\gamma_x}, \tag{20}$$

$$\gamma' = \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + nQ\gamma_x} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2 \right)^{-1}, \tag{21}$$

which correspond to the laws of motion governing beliefs in the baseline case, now adjusted for the new precision of X.

We are now ready to define a competitive equilibrium for this economy.

Definition 3. A recursive competitive equilibrium is a collection of value functions $V_0(\mu, \gamma, K, Q, k)$, $V_1(\mu, \gamma, K, Q, k, f)$, $V_1^W(\mu, \gamma, K, Q, k)$ and $V_1^I(\mu, \gamma, K, Q, k)$ for firms with individual policy functions $l(\mu, \gamma, K, Q)$, $i_0^c(\mu, \gamma, K, Q)$, $i_1^c(\mu, \gamma, K, Q)$, $i_1(\mu, \gamma, K, Q)$, $n(\mu, \gamma, K, Q)$ and $f_c(\mu, \gamma, K, Q)$; a policy function for the representative household $C(\mu, \gamma, K, Q, X, Y)$; and prices $p(\mu, \gamma, K, Q, X, Y)$ and $w(\mu, \gamma, K, Q)$ such that

1. The value functions and associated policy functions solve the Bellman equations (14)-(17) under laws of motion (18), (19), (20) and (21);
2. The household solves its problem: $p(\mu, \gamma, K, Q, X, Y) = U'(C(\mu, \gamma, K, Q, X, Y))$;
3. The labor market clears:
$$1 = l(\mu, \gamma, K, Q)\, K;$$
4. The goods market clears:
$$(A + Y)\, K^\alpha = C(\mu, \gamma, K, Q, X, Y) - c(i_0^c)(K - Q) - c(i_1^c)(1 - n)\, Q$$
$$- \left(nc(i_1) + \int^{f_c(\mu,\gamma,K,Q)} f dF(f) \right) Q.$$

Because of the irreversibility in investment, which arises when $\overline{q} < 1$, and the binarity of the investment decision, the extended model retains the key features which led to uncertainty traps in the baseline model. In particular, the precision of the information obtained through social learning, $nQ\gamma_x$, increases linearly with the fraction of investing firms, n. The option value of waiting is present because firms receive opportunities infrequently: when faced with an increase in uncertainty, they are reluctant to use a valuable investment opportunity and prefer to delay investment.

5.2 Simulations

Parameterization

We parameterize the model with the values shown in Table 1. Details on the data sources can be found in Appendix E. The time period is one month. The discount rate β is chosen to match an annual value of 0.95. The share α of capital in production is set to 0.4 to broadly match the average capital income share in postwar US. We calibrate the fundamental process to match key features of aggregate US TFP: an annual autocorrelation of 0.876 and a long-run standard deviation to 0.03.

In our benchmark calibration, we assume that the household has risk-neutral preferences. Under this assumption, we can evaluate the impact of real option effects alone. We relax this assumption in the sensitivity analysis of section 5.4.

We use a quadratic function $c(i) = i + \phi i^2$ for the variable cost of investment. To parameterize our investment-related parameters $\overline{q}$, ϕ, $\underline{i}$, $\overline{i}$, $\overline{f}$, and σ^f, we target firm-level moments from Compustat. First, we identify large investments with investment peaks in the data. We define an investment peak as a quarterly investment rate greater than 10%. Conditioning on having an investment peak, the median investment rate in Compustat is 18%. The capital-weighted fraction of firms in Compustat undergoing an investment peak (nQ/K in our model) is 2.8% on average in a quarter. We choose the parameters ϕ and $\overline{f}$ to match these two targets. The mean duration between investment peaks is 7 quarters in the data and the median is 14 quarters. We thus set the monthly probability $\overline{q}$ to $0.1 \times \frac{1}{3}$ to match an average duration of 10 quarters to receive an investment opportunity. We set $\underline{i}$ to 0 and choose $\overline{i}$ to match the average quarterly investment rate of 2.33% for firms not undergoing an investment peak.[15] In steady state, a yearly depreciation rate δ of 10.6% is consistent with our targets.[16] Finally, we study in our benchmark calibration a case with almost no heterogeneity in fixed costs and set $\sigma^f = 0.001 \times \overline{f}$. Section 5.4 provides sensitivity analysis on this parameter.

We are only left with the information parameters γ_y and γ_x to calibrate. Unfortunately, they lack obvious empirical counterparts, but we can use the Survey of Professional Forecasters (SPF) to obtain an order of magnitude. The SPF includes probability forecasts constructed from individual forecasters' expected distribution of output growth at different horizons, as well as "mean probability forecasts" which correspond to the average distribution of beliefs across forecasters. Focusing on a one-year horizon, the average standard deviation in the mean probability forecast is 1.3% for the period 1992-2015, with a maximum of 1.53% reached in the third quarter of 2009. Given ρ_θ and σ_θ, we set $\gamma_y = 100$ so that the maximum one-year-ahead standard deviation in beliefs about θ roughly corresponds to 1.5%.[17] We set γ_x so that the average precision from the social learning channel is a multiple of γ_y. In our benchmark calibration, we choose a multiple of 10, so that $nQ\gamma_x = 10 \times \gamma_y$, and perform sensitivity analysis on this parameter in section 5.4.

[15] In our simulations, all firms making a normal investment end up choosing $i = \overline{i}$ in our simulations.

[16] Using the law of motion for capital and using the fact that $i_0^c = i_1^c = \overline{i}$ in our simulations, the following relationship is satisfied in steady state, $\delta - \overline{i} = \frac{nQ}{K}(i_1 - \overline{i})$, which implies a value $\delta = 2.77\%$ at the quarterly level, or 10.6% annually for $\overline{i} = 0.0233$, $i_1 = 0.18$ and $nQ/K = 0.028$.

[17] At annual frequency, assuming that all the uncertainty is due to TFP ($= 1 + \theta$), the variance of TFP growth is

$$\mathrm{Var}_t\left(\log\left(1+\theta_{t+12}\right) - \log\left(1+\theta_t\right)\right) \simeq \mathrm{Var}_t\left(\theta_{t+12} - \theta_t\right) = \left(1 - \rho_\theta^{24}\right)\sigma_\theta^2 + \left(1 - \rho_\theta^{12}\right)^2 \frac{1}{\gamma_t}.$$

For $\gamma_y = 100$, the lower bound on γ is $\underline{\gamma} = 2817.9$ and the maximal one-year ahead standard deviation of TFP growth is 1.59%.

Parameter	Value
Time period	month
Total factor productivity	$A = 1$
Discount factor	$\beta = (0.95)^{1/12}$
Persistence of fundamental	$\rho_\theta = (0.876)^{1/12}$
Long-run standard deviation of fundamental	$\sigma_\theta = 0.03$
Share of capital in production	$\alpha = 0.4$
Probability of receiving an investment opportunity	$\overline{q} = 0.1/3$
Fixed cost of investment	$\overline{f} = 0.1$
Standard deviation of fixed costs	$\sigma^f = 0.001 \times \overline{f}$
Variable cost of investment	$\phi = 3.3$
Lower bound on constrained investments	$\underline{i} = 0$
Upper bound on constrained investments	$\overline{i} = 0.0233/3$
Depreciation rate	$1 - \delta = (1 - 0.0277)^{1/3}$
Precision of public signal	$\gamma_y = 100$
Precision of individual signals	$\gamma_x = 807$

Table 1: Parameter values for the numerical simulations

Benchmark Simulations

We first examine the properties of the policy function. Figure 7 presents the fraction of investing firms as a function of the mean of beliefs μ for three levels of uncertainty. As in the benchmark model, firms are more likely to invest when μ is high and uncertainty is low, which is where the real option effects appear.[18]

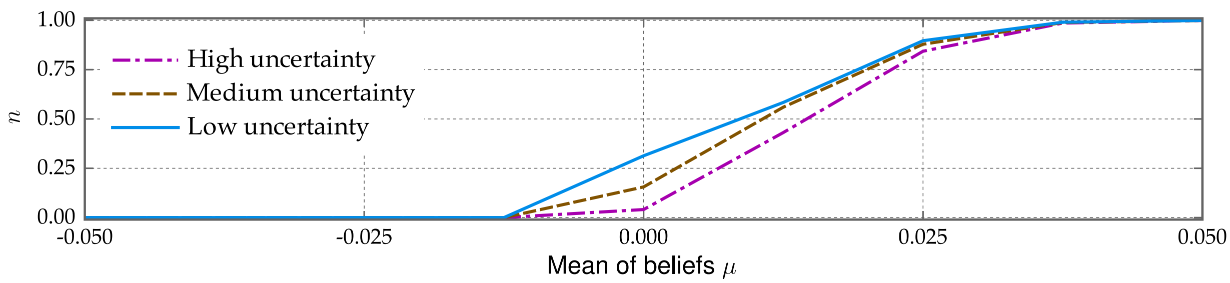
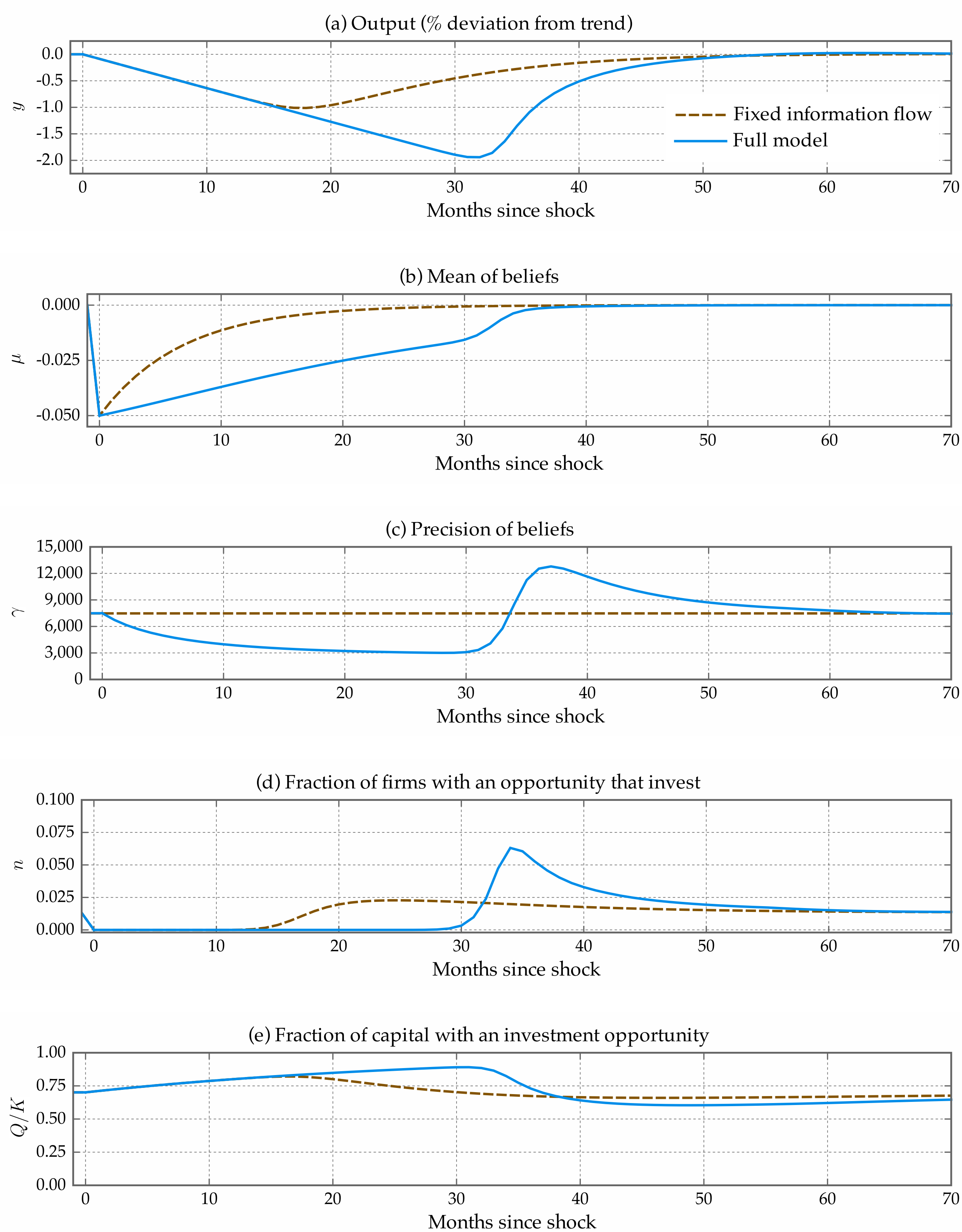


Figure 7: Investment decision $n(\mu, \gamma, K, Q)$ for K and Q constant at their steady-state level.

[18] As a result, the model is able to replicate the procyclicality of the cross-sectional dispersion of investment rates documented by Bachmann and Bayer (2014)



Notes: The solid curves show the evolution of the economy according to the full model, while the dashed curves show the evolution of a control economy in which the flow of public information is fixed at the steady-state level of the full model.

Figure 8: Evolution of the economy after a one-period 5% negative shock to μ.

We now investigate the strength of the uncertainty traps mechanism through a series of simulations. Since primitive shocks only affect the policy functions through their effects on beliefs, we examine directly the impact of shocks to beliefs. We consider the evolution of an economy hit by a negative 5% shock to the mean of beliefs μ resulting from bad realizations of the signals. The impulse response functions are represented by the solid curves in Figure 8. The dashed curves represent the response of a control economy with fixed information flow and no endogenous uncertainty.[19] Comparing the two economies allows us to quantify the impact of the endogenous uncertainty channel.

Let us consider the full model first. On impact, firms believe that productivity is low. The expected return from adding capital becomes lower than its cost, and firms cut back on large investments. As a result, fewer private signals are released and the precision of beliefs starts falling, as seen in Panel (c). Once the shock is over, agents start to receive signals suggesting that the fundamental is actually better than what they believed. Firms update their beliefs accordingly and, as shown in Panel (b), the mean of their beliefs starts to recover. The recovery in output is, however, delayed by the high uncertainty. Once the stock of capital has sufficiently declined and a large enough stock of opportunities has been accumulated, firms resume investing in large projects. This triggers an important release of information, μ recovers quickly, and uncertainty declines sharply, further raising investment.

In comparison, the recession is less severe in the fixed-information-flow economy. In this case, uncertainty does not rise after the initial shock. Thus, as the mean of beliefs μ recovers, firms resume investing earlier and the downturn is shorter. We see a drop in output of about 1% in the control economy, while production shrinks by 2% in the full model. The trough of the recession also happens 15 months later in the full model.

5.3 Comparison with the RBC Model and Data

This section compares the propagation properties of our model to a standard RBC model and provides empirical evidence supporting some of its key features.[20]

Persistence

We first evaluate the performance of the model regarding the persistence of output in relation to TFP. As has been noted before, the RBC model features weak internal propagation mechanisms and lacks persistence. Cogley and Nason (1995) make this point by comparing the autocorrelograms of output growth in the data and in an RBC model. We repeat their exercise in Figure 10 of Appendix B.1. Panel (a) shows the autocorrelograms of output and TFP growth in US data. The figure shows that output growth is more persistent than TFP growth. Panel (b) displays the same

[19] More precisely, in the control economy we keep the precision of the aggregate signal X constant at its steady-state value, so that the precisions of beliefs γ coincide in the steady states of the two economies.

[20] The RBC model that we use is the standard neoclassical growth model without adjustment costs, parametrized as in Appendix B.

autocorrelograms computed on data generated by a standard RBC model and by our full model with endogenous uncertainty. Along the findings of Cogley and Nason (1995), we find that the autocorrelogram of output growth in the RBC model mirrors that of TFP growth, implying that the RBC model adds little to no propagation to the exogenous shock process that is fed into the model. In our full model, however, the endogenous uncertainty generates substantial additional persistence in output growth, which is more aligned with the data.

Non-linearity

A key implication of the model is that the economy responds differently to shocks of different magnitudes. In particular, small TFP shocks are followed by mild recessions from which the economy recovers quickly, while large TFP shocks may trigger more protracted recessions.

To first illustrate how this prediction differs from a standard RBC model, Figure 11 in the Appendix displays the response of GDP in our model and in an RBC model to negative aggregate productivity shocks of different magnitudes. Interestingly, the duration of the downturn varies significantly with the size of the shock in our model, while it remains invariant in the RBC model. To provide a more accurate account of this feature, Table 2 reports the half-life, defined as the time that the economy spends with a level of GDP below 50% of its peak-to-trough fall, for each of the different shocks displayed in Figure 11. Due to its approximate log-linearity, the RBC model displays a constant duration for all the shocks. Our model, on the other hand, displays larger duration for larger shocks.

Shock $\Delta\theta$	Model	RBC
-0.1%	27	31
-0.5%	33	31
-2%	45	31

Notes: The duration displayed corresponds to the time spent by GDP below 50% of its peak-to-trough decline in quarters.

Table 2: Non-linearity in recession duration

We investigate the presence of this type of non-linearity in the data in the simplest possible way. Figure 12 in the Appendix shows the recovery paths for US GDP in the eight NBER recessions between 1960 and 2014. The blue curves are the 4 recessions with the larger peak-through falls in TFP; a quick look at the data suggests that these are the recessions where the recovery takes longer overall. To verify that this is the case, we estimate a bivariate VAR of TFP and GDP on 2 two subsamples of US data: one corresponding to the recessions and recoveries with *large* TFP declines, and another corresponding to *small* TFP declines. To identify a TFP shock, we order TFP first allowing for TFP only to have a contemporaneous impact on GDP. Figure 13 in Appendix B.2 reports the impulse responses to a minus one standard deviation shock to TFP in both VARs. GDP in the large-recession VAR falls more than the initial decline in TFP and takes longer to recover. In the case of the small-recession VAR, the response of GDP tracks closely the evolution of TFP, displaying little persistence. The half-life of GDP is more than twice as large for large than for small recessions (26 quarters for large recessions and only 11for small recessions). These results suggest that the response of the economy to TFP shocks is non-linear and that large TFP shocks generate deeper, more protracted recessions as our model predicts.

Asymmetry

Finally, the additional persistence and non-linearities generated by the uncertainty trap mechanism lead to substantial negative skewness in the ergodic distribution of log output. A long simulation of the full model yields a skewness of -0.30 for log output while, in the data, the corresponding number is -1.10.[21] In comparison, the RBC model generates a skewness of -0.02 in our simulations.

5.4 Sensitivity Analysis

Appendix C shows impulse response functions for different parameterizations of the information parameters. Figure 14 displays the response of output after a negative 5% shock to μ for three different values of γ_x: 250, 807 (benchmark) and 2500. Lower values of γ_x tend to make the downturn more protracted as beliefs take longer to catch up. The overall impact is, however, moderate. On the other hand, changes in γ_y have a large impact on the economy. Figure 15 presents the response of output for different values of γ_y: 100 (benchmark), 1000 and 5000. Larger values of γ_y limit the effect of endogenous uncertainty by reducing the overall level of uncertainty in the economy and therefore lowering the incentives to wait.[22]

Appendix C also provides a sensitivity analysis on the standard deviation of the distribution of fixed costs f. As can be seen in Figure 16, higher values of σ^f lead to shallower recessions and faster recoveries. When σ^f is large, f becomes the main determinant of investment and firms care less about θ: they simply wait for a good draw of f before making a large investment. Adding some autocorrelation to the f process would help make uncertainty matter even when σ^f is large.

For realism, we also consider an economy in which the firms endowed with an investment opportunity may loose it with probability $\underline{q}$. In this case, the option value is reduced as firms anticipate that unused opportunities might disappear. Appendix C.3 shows impulse response functions for shocks hitting economies with various $\underline{q}$. We find that if $\underline{q}$ is not too large the economy reacts similarly to our benchmark calibration.

Finally, we consider in Appendix C.4 the impact of the uncertainty traps mechanism in an economy with a risk-averse household and variable labor supply. The mechanism still generates

[21] We simulate the benchmark model for 100,000 periods before computing the skewness. The extended model with risk-aversion and variable labor, which we introduce in section 5.4, generates similar negative skewness.

[22] In our current setup, precision γ_y is directly related to the volatility in aggregate productivity. Since the standard deviation in the *level* of TFP in the US ranges from only 2% to 3%, the precision parameter γ_y cannot be too low. This may possibly limit the impact of the uncertainty trap mechanism. However, its importance can be restored if agents are uncertain about fundamentals that fluctuate more in the data. Hence, natural extensions for a full quantitative analysis would be to allow for uncertainty on the *growth rate* of TFP or on sector-specific productivity in a multi-sector economy, both of which display high volatility in the data.

persistence and amplification, but less so than in the risk-neutral case. Under risk aversion, households save more as uncertainty rises, pushing the interest rate down and increasing investment. This limits the impact of uncertainty on investment and, as a result, the uncertainty trap mechanism is weaker in that context.[23]

5.5 Further Applications

To allow for comparison with the uncertainty shock literature, we consider shocks to the precision of beliefs. Appendix D.1 provides the details of the exercise. Figure 19 in the appendix shows the impulse response functions of output, the fraction of firms undertaking large investments, and the precision of beliefs when the standard deviation of beliefs increase by 60% while keeping the mean constant.[24] Output drops by 0.75% in the full model and by 0.25% in the fixed-information economy. In both economies, γ recovers according to equation (21), but the recovery is faster in the control economy because the information flow is fixed. Thus, not only TFP shocks but also exogenous uncertainty shocks get propagated and amplified in our model as they slow down the learning process.

To evaluate the importance of the information externality, we solve the planner's problem and compare the efficient allocation to the competitive equilibrium. The response of the economy to a 5% shock to μ can be found in Appendix D.2. The economy recovers more quickly from this shock in the planner's allocation. In this case, as the recession worsens, the planner gathers information about the fundamental by letting some firms invest. Because of the decline in uncertainty, waiting becomes less attractive and the economy starts to recover. This suggests that policy interventions can have a sizable welfare impact in our environment.

6 Conclusion

We develop a theory of endogenous uncertainty and business cycles that combines two forces: higher uncertainty about economic fundamentals deters investment, and uncertainty evolves endogenously because agents learn from the actions of others. The interaction between investment and uncertainty leads to *uncertainty traps*: episodes in which high uncertainty leads firms to delay investment, further raising uncertainty. In the unique equilibrium of the model, the economy fluctuates between a high-activity/low-uncertainty regime and a low-activity/high-uncertainty regime and is subject to strong non-linear dynamics in which large shocks can have near permanent effects.

To explore the robustness of this mechanism, we embed it into a business cycle model. Uncertainty traps survive in that context and we find that recessions may become substantially deeper and longer relative to a framework with fixed exogenous uncertainty. The model improves on the RBC framework by generating *i)* increased persistence in output growth, *ii)* non-linear response

[23]We also use this extended model to perform a wedge decomposition and find that a negative shock to θ manifests itself mostly in a negative labor wedge on impact, followed by a mild deterioration in the investment wedge.

[24]This shock to γ corresponds to a movement from its steady-state to its lower bound.

to shocks, and *iii)* negative skewness in aggregates. Furthermore, our simulations suggest that optimal policy interventions could lead to faster recoveries.

We believe that the novel channel proposed in this paper is important for several reasons. First, the emphasis on subjective uncertainty — and beliefs about fundamentals in particular — implies that not only exogenous volatility shocks, but also other sources of uncertainty, matter for the economy. Thus, we view recent empirical work using survey data on forecasts or consumer and business expectations as an important step towards a more complete understanding of the role of uncertainty in business cycles. Second, we believe that our framework may be useful as a theoretical benchmark for empirical and quantitative studies seeking to estimate the direct and feedback effects of uncertainty on economic activity. Despite the multiplicity of regimes and strong non-linearities, the model features a single competitive equilibrium, which makes it amenable to applied work. Third, we have shown that allowing uncertainty to fluctuate endogenously may lead to a significant propagation and amplification mechanism. The type of non-linearities and the multiplicity in regimes that we obtain may be of broader interest for business cycle modeling in general and could also shed light on some particularly large historical downturns.

For the sake of clarity, we have exposited the mechanism in a purposely simple framework, but a number of generalizations may be worth investigating. In particular, it would be interesting to understand how uncertainty traps interact with frictions that could magnify their impact, such as financial frictions, demand externalities or belief heterogeneity. A full quantitative evaluation of the model is also needed. We leave these questions to future research.

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A Uncertainty and Business Cycles

As mentioned in the introduction, we adopt the general concept of Bayesian uncertainty, which, besides encompassing time-varying volatility, allows for other sources of uncertainty. Our model focuses, in particular, on a specific source of uncertainty, namely subjective uncertainty about aggregate TFP, which arises because of incomplete information and learning. The objective of this section is to provide measures that capture this subjective aspect of uncertainty and compare them to proxies already used in the literature.

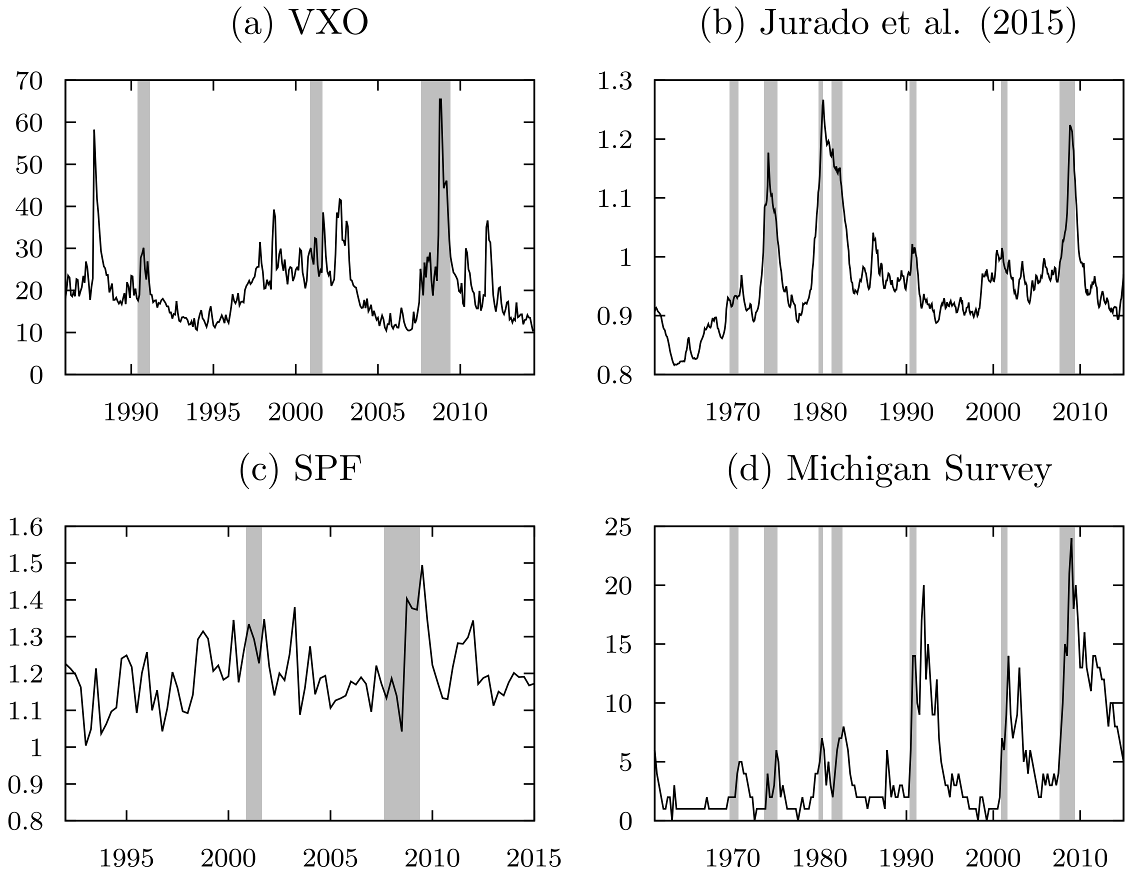
Measures of uncertainty

To capture the subjective aspect of uncertainty, Figure 9 proposes measures derived from asset prices or based on forecasting data and surveys.

We begin with two measures commonly used in the literature. Panel (a) presents the VXO from the Chicago Board Options Exchange.[25] It provides an index of the market expectations for stock market volatility over the next 30-day period and is constructed as the implied volatility from option prices. To the extent that aggregate TFP affects firm revenues, the type of uncertainty that we consider should be priced in options. The VXO thus captures part of our uncertainty, but is, however, an imperfect measure as it only includes uncertainty within the next 30 days, while investment decisions are likely to pay off further in the future. The VXO also reflects unrelated movements in volatility and risk premia. Panel (b) shows the uncertainty measure proposed by Jurado et al. (2015). To construct this measure, the authors estimate a factor-augmented forecasting model with time-varying volatility, and compute the volatility of the unforecastable component of the future value of a macroeconomic series. The data presented here is an average of uncertainty about the 12-month ahead value of a large number of macroeconomic series. Being a measure of ex-ante forecast error, this series captures the notion of Bayesian uncertainty. However, being derived from a factor model with time-varying volatility, it only reflects our notion of subjective uncertainty to the extent that this type of uncertainty may permeate to volatility in decisions.

Because of the imperfect nature of the series presented above, we now offer measures that attempt to isolate more accurately the uncertainty featured in our model. Panel (c) displays data from the Survey of Professional Forecasters (SPF). In this survey, each forecaster reports the distribution of output growth they anticipate at different horizons. The probability distribution is then averaged across forecasters under the label "mean probability forecasts". We compute the standard deviation of this average distribution over time as a measure of uncertainty about one-year ahead real output growth (in percentage). With the minor caveat that it represents uncertainty in aggregate output growth instead of TFP, this series arguably provides the best empirical counterpart to the uncertainty featured in our model, as it reflects uncertainty in subjective beliefs about aggregates shared across agents. Finally, the Michigan Survey of Consumers proposes a number

[25] The VXO is an index similar to the VIX, except that it is based on the S&P100 instead of the S&P500. Its main advantage is to cover the period 1986-2015 while the VIX only started in 1990.



Notes: (a) The VXO series are monthly averages of the CBOE series over 1986-2014. (b) The Jurado et al. (2015) series correspond to the H12 measure, i.e., an equal-weighted average of the 12-month ahead standard deviations over 132 macroeconomic series. (c) The SPF series is the standard deviation of the "mean probability forecast" of one-year ahead output growth in percentage terms. (d) The Michigan Survey series correspond to the percent fraction of all respondents that reply "uncertain future" to the question why people are not buying large household items. Shaded areas correspond to NBER recessions.

Figure 9: Various measures of uncertainty

of measures that reflect uncertainty as perceived by US households. Specifically, Panel (d) shows the fraction of respondents who answer "uncertain future" as a reason for why it is bad time to buy major household goods. Being based on a subjective assessment of uncertainty by individuals, this series is harder to interpret in a quantitative sense. It may, however, best capture the concept of uncertainty as perceived by individuals and might be the best predictor of consumption and investment decisions.

Shaded periods in Figure 9 are NBER recessions. All the presented series are countercyclical and increase during recessions. Interestingly, the two measures that best capture the notion of subjective uncertainty, in Panels (c) and (d), tend to display higher persistence. The measure from the Michigan Survey, in particular, has declined slowly since the 2007-2009 recession and, as of 2015:Q1, has not fully recovered to its pre-recession level.

Table 3 below presents various statistics about these series. The first row reports average uncertainty over recession quarters relative to average uncertainty during expansion quarters (i.e., non-recession) using various measures of uncertainty; the second and third rows report, respectively,

	VXO	Jurado et al. (2015)			SPF		Michigan Survey	
		H1	H3	H12	GNP	GDP	Goods	Cars
All Recessions	1.56	1.25	1.22	1.12	1.10	1.06	1.95	1.69
Mild Recessions	1.37	1.06	1.05	1.01	1.06	1.05	1.61	1.65
Deep Recession	1.75	1.38	1.34	1.20	1.11	1.06	2.18	1.72

Notes: This table reports the ratio of average uncertainty during recession quarters over non-recession quarters. Recession quarters follow the NBER definition. Recessions are classified as mild or deep according to whether the peak-through fall in TFP is above or below the the median peak-through fall in TFP across all recessions for which uncertainty data is available. According to this definition, mild recessions are those starting in 1960:Q2, 1969:Q4, 1990:Q3, and 2001:Q1, while deep recessions are those starting in 1973:Q4, 1980:Q1, 1981:Q3, and 2007:Q4. The VXO measure is available over 1986Q1-2014Q2. The H1 (1-month), H3 (3-month) and H12 (one-year) ahead forecast error measures are available from 1960:Q3-2014:Q4. The SPF-GNP measure is available from 1981:Q3-1991:Q4, and the SPF-GDP measure is available from 1992:Q1-2015:Q1. The Michigan Survey measures are available from 1960:Q1-2015:Q1 and include the responses to why it is not a good time to buy either cars or large household goods. See the explanation of Figure 9 for a description of each measure.

Table 3: Average uncertainty in recessions relative to expansions

average uncertainty during mild and deep recessions relative to expansion quarters. We see that these measures of subjective uncertainty are higher during recessions, and that uncertainty is on average larger during deep recessions than during mild recessions.

Comparison with cross-sectional measures

Since the uncertainty-driven business cycle literature often considers time-varying idiosyncratic volatility as a source of uncertainty, measures of cross-sectional dispersion are frequently used as a proxy. Among the most widely used measures is the cross-sectional dispersion of sales growth rates (Bloom, 2009), but other measures of cross-sectional dispersion have also been shown to be countercylical: output and productivity (Kehrig, 2011), prices (Vavra, 2014), employment growth (Bachmann and Bayer, 2014), and business forecasts (Bachmann et al., 2013).

Since we focus on aggregate uncertainty and because agents in our model share common beliefs, our model produces little variation in the cross-section, with the exception of the dispersion of investment rates which are procyclical in our model as in the data (Bachmann and Bayer, 2014). Consequently, these cross-sectional measures are in general inadequate to capture the type of uncertainty that we consider. In fact, this lack of variation in the cross-section is an important point of our argument, as we show that uncertainty may matter and impede economic recovery even when cross-sectional measures have returned to normal.

B Comparison with an RBC Model and Data

In this section, we compare some predictions of our extended framework to a standard RBC model and provide some empirical evidence for some key features of our mechanism. The tables and figures of this section are discussed in the body of the text.

The RBC model that we use is the standard neoclassical growth model without adjustment costs and with preferences given by

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[\log\left(C_t\right)-\frac{L_t^{1+\nu}}{1+\nu}\right],$$

with $\nu = 0.25$ for a Frisch elasticity of 4. Other parameters, including the stochastic process governing the dynamics of aggregate productivity θ, are the same as in our benchmark calibration. Table 4 lists the parameters.

Parameter	Value
Time period	month
Total factor productivity	$A = 1$
Discount factor	$\beta = (0.95)^{1/12}$
Persistence of fundamental	$\rho_\theta = (0.876)^{1/12}$
Long-run standard deviation of fundamental	$\sigma_\theta = 0.03$
Share of capital in production	$\alpha = 0.4$
Depreciation rate	$\delta = 1 - (0.894)^{1/12}$
Disutility of labor	$\nu = 0.25$

Table 4: Parameter values for the RBC model

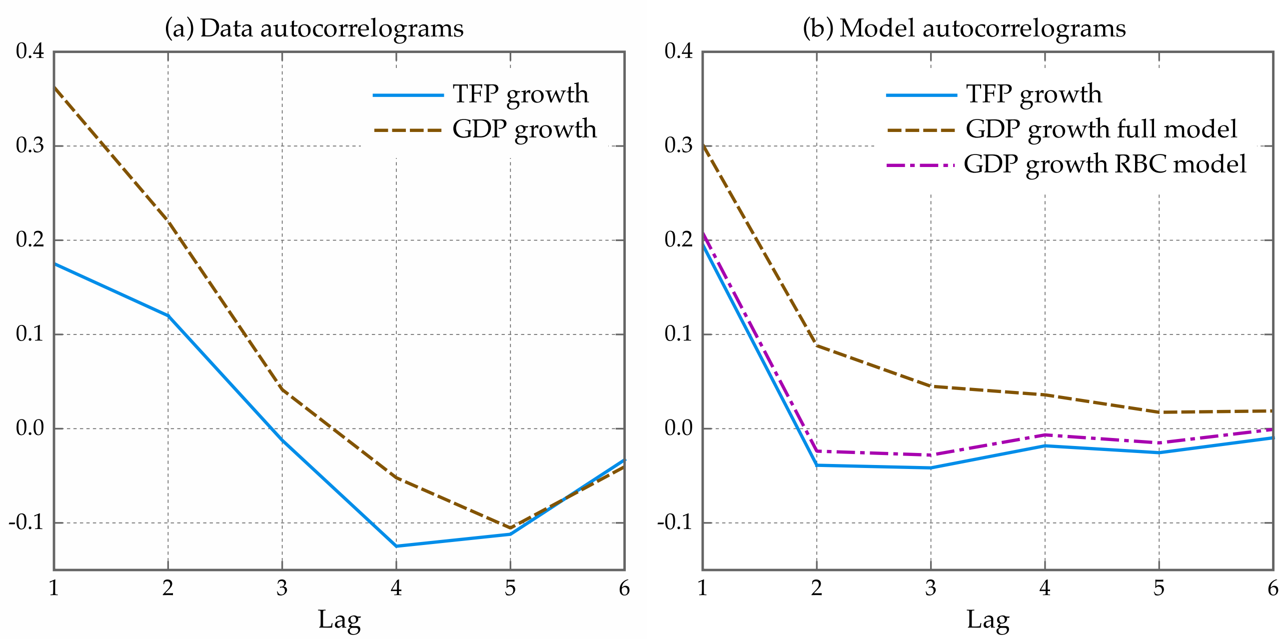
B.1 Persistence

Figure 10 shows the autocorrelogram of output growth and TFP growth in the data, in the full model and in the RBC model.[26]

B.2 Non-linearity

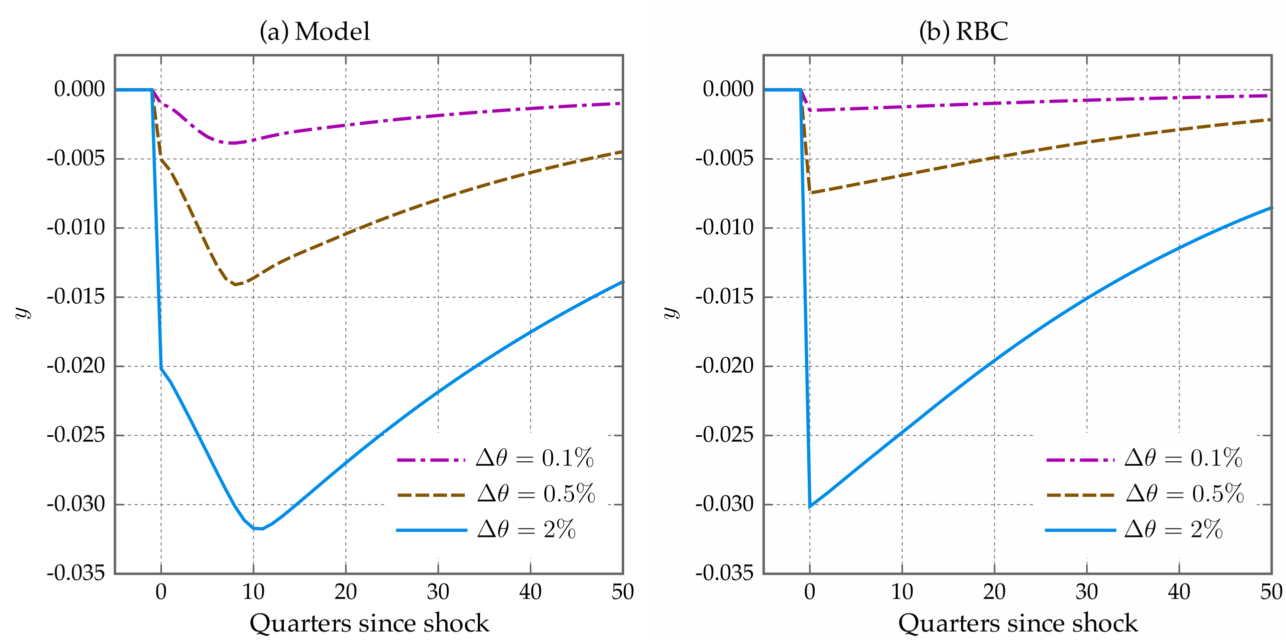
This section contains impulse response functions from the model and the RBC framework together with impulse responses from VARs estimated on the US data. Figure 11 shows the impulse responses of output to shocks to productivity θ in the benchmark model and an RBC model. Figure 12 shows the path of GDP centered at the peak of each recession. Blue curves correspond to high-TFP fall recessions, while brown lines correspond to low-TFP fall recessions. Figure 13 shows the impulse responses from the two VARs described in section 5.3.

[26]Panel (b) is similar when we use the version of the full model with risk-aversion and variable labor.



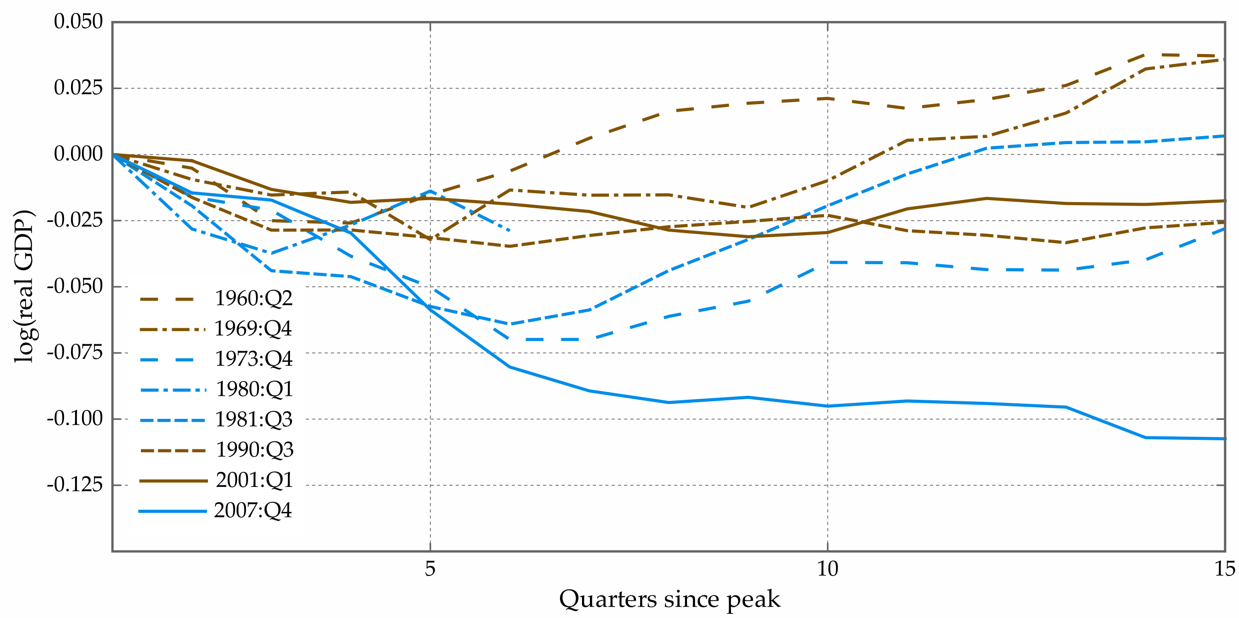
Notes: (a) GDP is the detrended (linear) log of the seasonally adjusted Bureau of Economic Analysis real GDP from 1960:Q1 to 2014:Q2. TFP is from Fernald (2014), linearly detrended and seasonally adjusted by removing quarterly dummies; (b) TFP is drawn from the calibrated process, which is the same in both models. The full model is the benchmark model simulated with θ shocks only to allow comparison with the RBC model. The RBC model is calibrated as in Appendix B.

Figure 10: Autocorrelogram of output and TFP



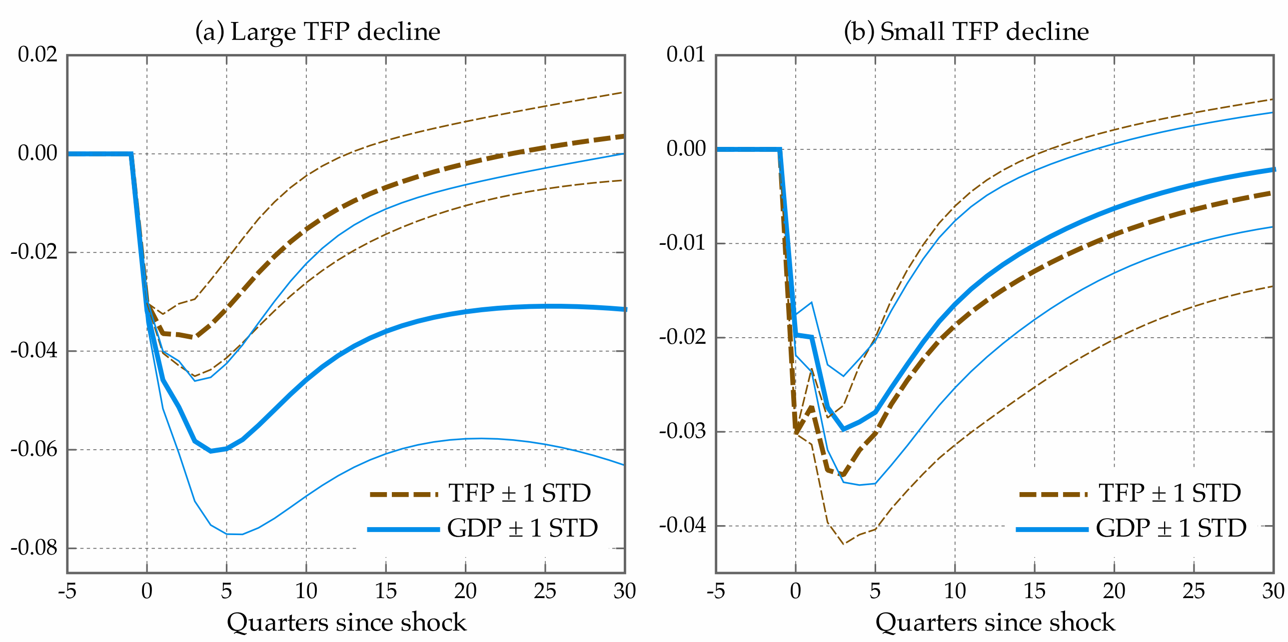
Notes: (a) Impulse responses of output to shocks in θ in the benchmark model (b) Impulse responses of output to shocks in θ in an RBC model calibrated as in Appendix B

Figure 11: Impulse responses in the benchmark model and the RBC model



Notes: The blue curves represent NBER recessions with TFP declines above median (in absolute value) over the period 1960-2014, while the brown curves present recessions with TFP declines below median. The curves display the detrended (linear) log of real GDP for each recession, centered at the preceding peak.

Figure 12: NBER recessions over 1960-2014 classified according to TFP falls



Notes: Impulse responses to a minus one-standard deviation TFP shock (3.02% on impact) in a bivariate VAR of TFP and output with TFP ordered first. The data is in logs and detrended using a linear trend over 1960-2014. It includes the 8 NBER recessions over the period 1960-2014. Each recession is included from its peak plus five additional years. Panel (a) reports the response of the VAR estimated for the 4 recessions with large TFP declines: 1973Q4-1975Q1, 1980Q1-1980Q3, 1981Q3-1982Q4 and 2007Q4-2009Q2. Panel (b) reports the response of the VAR estimated for the 4 recessions with small TFP declines: 1960Q2-1961Q1, 1969Q4-1970Q4, 1990Q3-1991Q1 and 2001Q1-2001Q4. The error bands are computed by bootstrap and correspond to +/- 1 standard deviation around the point estimates.

Figure 13: Impulse response from the estimated VAR

C Sensitivity Analysis

This appendix shows how economies with different parameters respond to the main shock of Section 5. In all cases, the parameters are as in table 1 except if stated otherwise. The figures of this section are discussed in the body of the text.

C.1 Precision of signals

Figure 14 and 15 show the response of output to a 5% negative shock to μ when γ_x and γ_y vary.

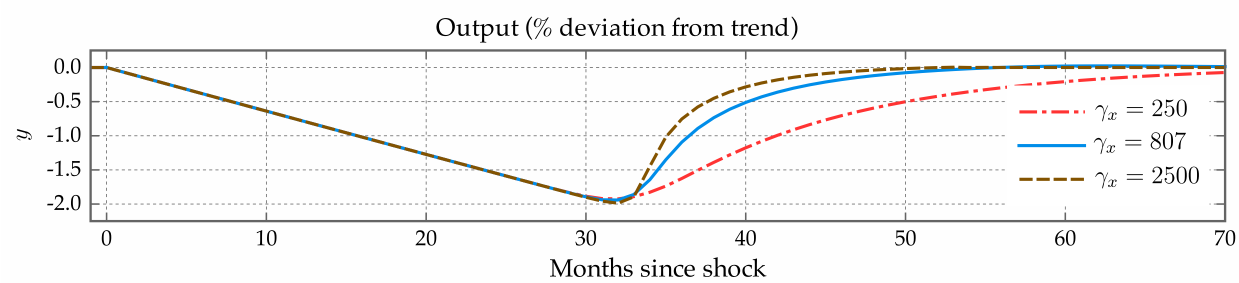


Figure 14: Impact of a -5% shock to μ with various precisions of the individual signal γ_x

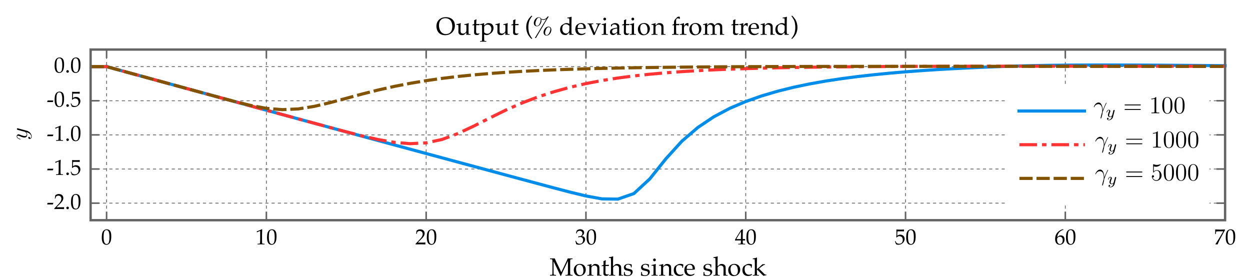


Figure 15: Impact of a -5% shock to μ with various precisions of the public signal γ_y

C.2 Dispersion of fixed costs

Figure 16 shows the response of output to a 5% negative shock to μ when σ_f varies.

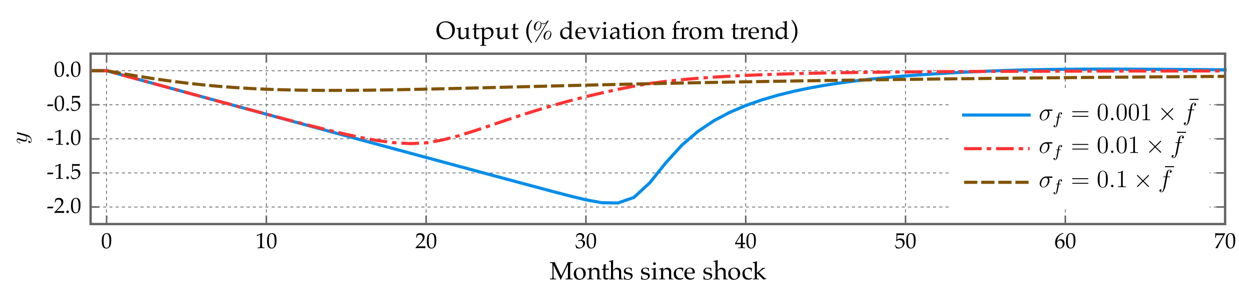


Figure 16: Impact of a -5% shock to μ with various dispersions of fixed costs σ_f

C.3 Destruction of investment opportunities

We extend the benchmark model to allow for the destruction of investment opportunities. We assume that a firm with an opportunity loses it with probability $\underline{q}$ every period. Equation 16, the value of a firm with an opportunity that decides to wait, therefore becomes

$$\begin{aligned} V_1^W(\mu,\gamma,K,Q,k) &= \max_{i_1^c\in[\underline{i},\bar{i}]} \mathbb{E}\left\{-pc\left(i_1^c\right)k+\beta\left(\left(1-\underline{q}\right)\int V_1\left(\mu',\gamma',K',Q',k',f'\right)dF\left(f'\right)\right)\right. \\ &+ \left.\underline{q}V_0\left(\mu,\gamma,K,Q,k\right)\Big|\mu,\gamma\right\} \end{aligned}$$

Similarly, we need to adjust the laws of motion of Q to

$$Q' = \left(1-\delta+i_0^c\right)\bar{q}\left(K-Q\right)+\left(1-\delta+i_1^c\right)\left(1-n\right)Q\left(1-\underline{q}\right)+\left(1-\delta+i_1\right)\bar{q}nQ.$$

To evaluate the impact of the destruction of opportunities on our quantitative result, we solve the competitive equilibrium for various values of $\underline{q}$ and plot the response of output to a μ shock in each equilibrium. As we can see in Figure 17, $\underline{q}$ needs to reach very high levels to substantially affect the impulse response.

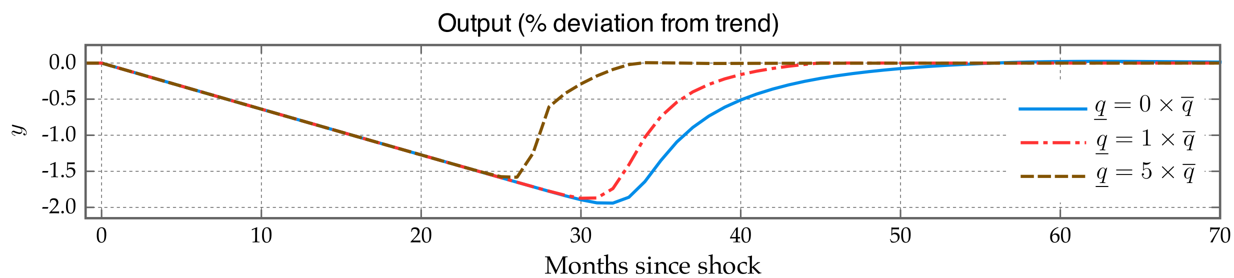


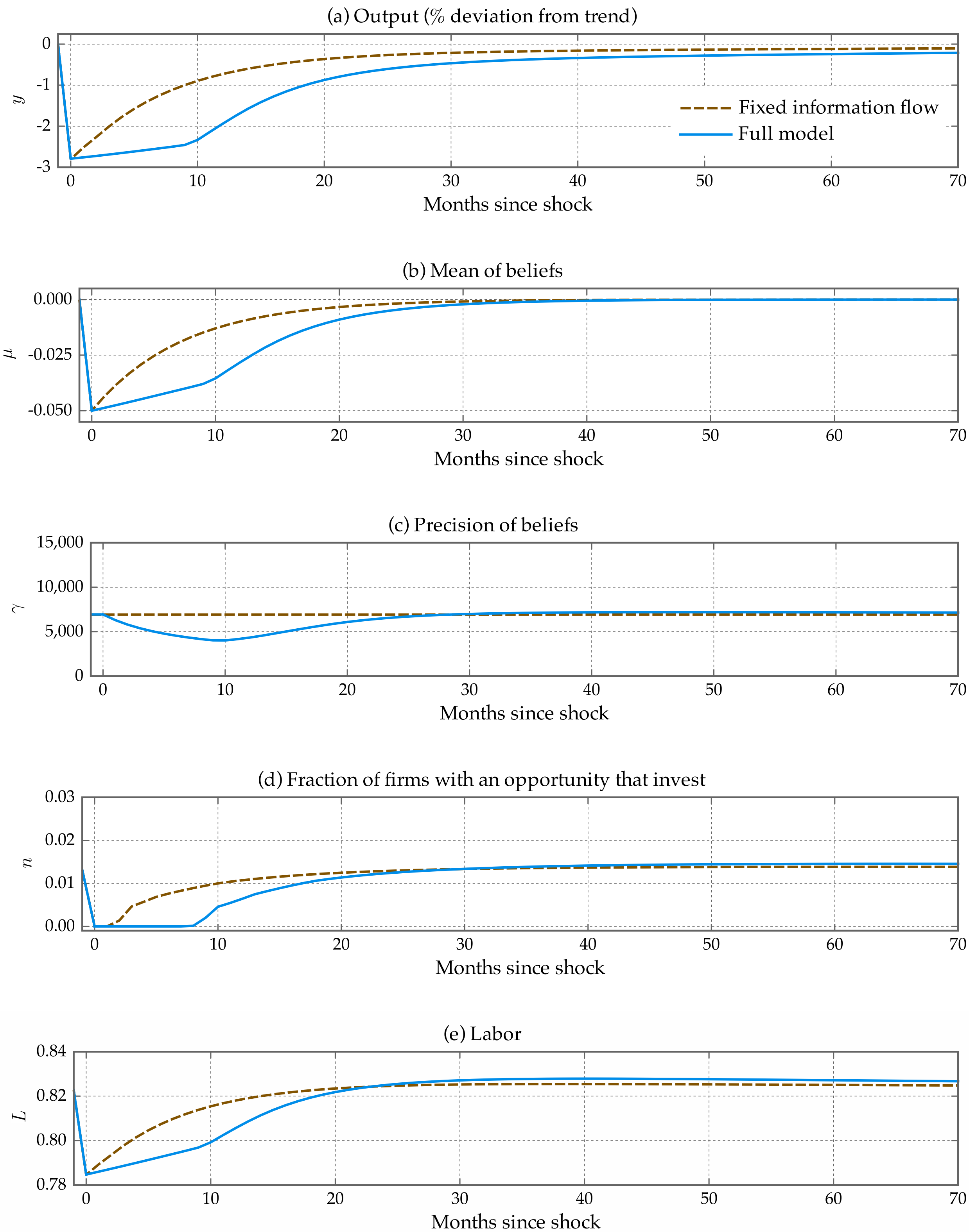
Figure 17: Impact of a -5% shock to μ with various probabilities of opportunity destruction $\underline{q}$

C.4 Risk aversion and variable labor

Figure 18 shows the response of the economy to a 5% negative shock to μ when the household is endowed with log preferences and a preference for leisure. The preferences of the household are

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left[\log\left(C_t\right)-\frac{L_t^{1+\nu}}{1+\nu}\right]$$

and we set $\nu = 0.25$ for a Frisch elasticity of 4. As we can see in Figure 18, the uncertainty traps mechanism still generates a longer downturn.



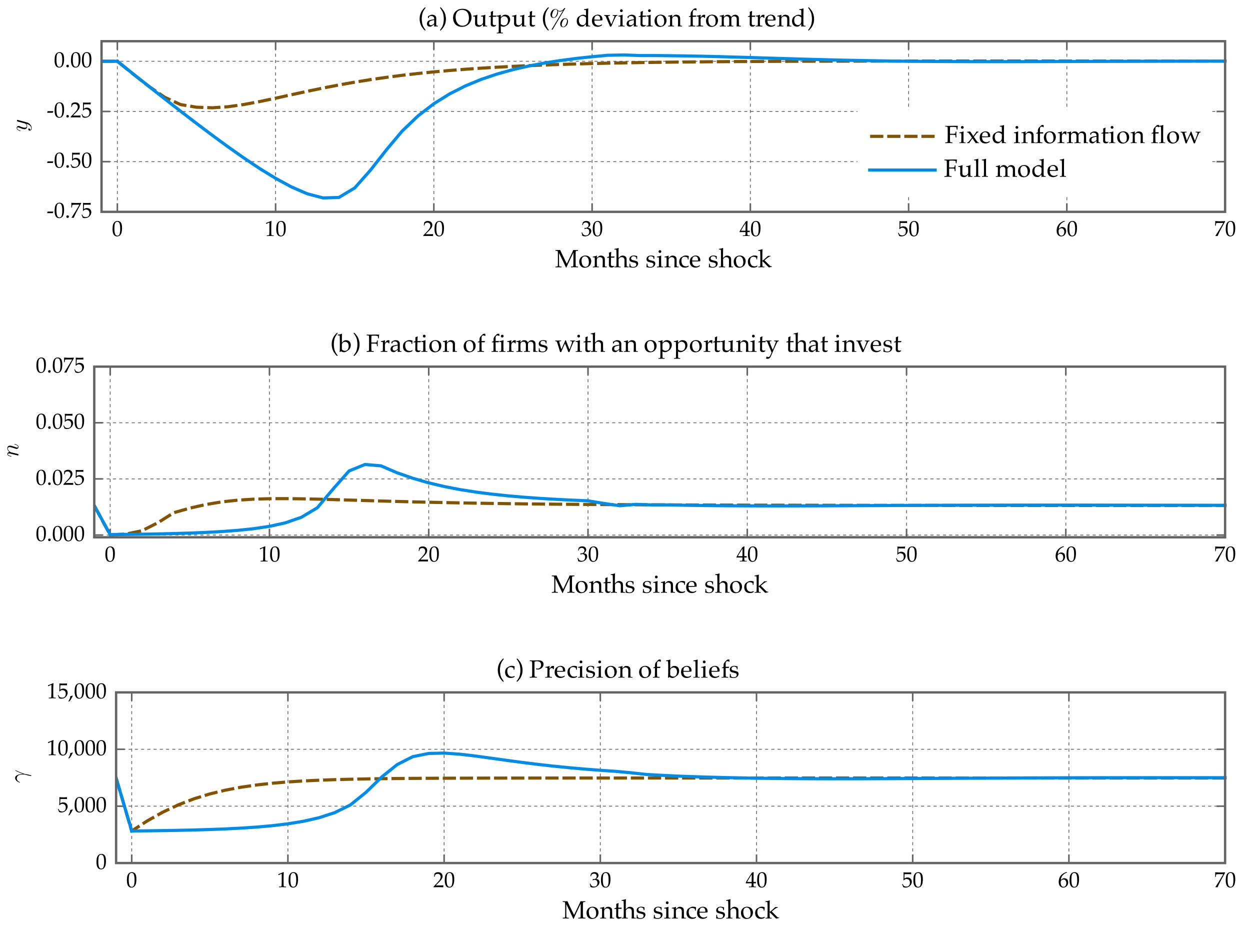
Notes: Impact of a -5% shock to μ on an economy with risk aversion and variable labor. The solid curves show the evolution of the economy according to the full model while the dashed curves show the evolution of a control economy in which the flow of public information is fixed at its steady-state level.

Figure 18: Impact of a shock in an economy with risk-aversion and variable labor

D Further Applications

D.1 Uncertainty shocks

Figure 19 displays the impact of an uncertainty shocks on the economy. We model these shocks as exogenous zero-probability events that raise uncertainty and that agents do not anticipate.

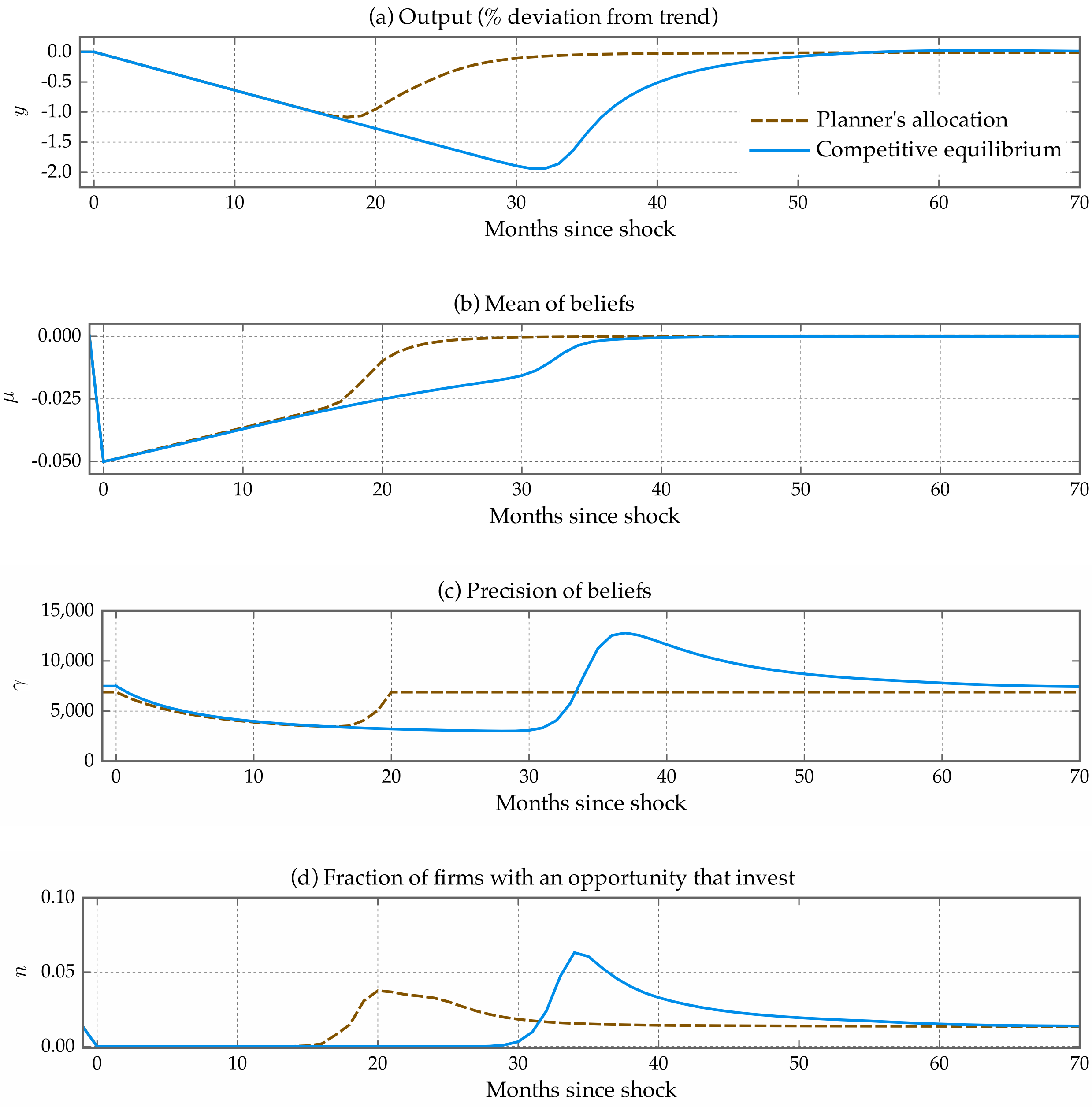


Notes: Evolution of the economy after an exogenous 60% increase in the standard deviation of the prior. The solid curve shows the evolution of the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.

Figure 19: Impact of an exogenous uncertainty shock

D.2 Social planner

Figure 20 shows the response of the economy to a 5% negative shock to μ in the competitive equilibrium and in the planner's allocation.



Notes: Impact of a -5% shock to μ. The solid curves shows the evolution of the competitive equilibrium while the dashed curves show the evolution of the planner's allocation.

Figure 20: Impact of a shock in the competitive equilibrium and the planner's allocation

E Data Appendix

This section details the different data sources that we use for the measurement of uncertainty and for the calibration.

- Our series for Total Factor Productivity in the US is the quarterly Business Sector TFP, adjusted for labor quality from Fernald (2014) over the period 1947:Q2-2013:Q2, seasonally adjusted using quarterly dummies.
- We use the quarterly Real GDP series from the Bureau of Economic Analysis over the period 1960:Q1-2014:Q2 in constant prices, seasonally adjusted.
- We use the quarterly series from Compustat over the period 1975:Q1-2011:Q4. We define capital as the variable "Property, Plant and Equipment - Total (Gross) - Quarterly" (PPE). We drop all firms outside of manufacturing and only keep firms with observations throughout the entire sample. Investment is defined as the change in PPE between two quarters.
- We use the mean probability forecasts from the Survey of Professional Forecasters (PRGDP) which provides the mean responses for the probability that annual-average over annual-average percent change in real GDP falls in particular bins. The time series are at quarterly frequency and cover the period 1992:Q1-2015:Q1. We also use the real GNP series over the period 1981:Q3-1991:Q4 in Table 3.
- We use the uncertainty series H1, H3 and H12 constructed by Jurado et al. (2015) over the period 1960:Q3-2014:Q4.
- We use the VXO time series from the Chicago Board Options Exchange over the period 1986:Q1-2014:Q2 averaged over monthly periods.
- We use the Michigan Survey of Consumers over the period 1960:Q1-2015:Q1 and construct series with the number of respondents answering "Uncertain Future" as the main reason why it is a bad time to purchase big household goods.

F Limit economy when $\overline{N} \to \infty$ (ONLINE APPENDIX)

In the baseline model introduced in part 2, firms face the same ex-ante probability of investing. Yet, the aggregate number of investing firms N is random because of the finiteness in the number of firms. This sample risk is of no relevance in respect to the uncertainty trap mechanism that we propose. We thus take the limit $\overline{N} \to \infty$ in part 4, so that the fraction of investing firms becomes a deterministic function of the state variables (μ, γ). This section exposes in detail how this limit is to be taken and how the limit economy is defined.

The only potential difficulty that we face as $\overline{N} \to \infty$ is that the social learning channel could become fully revealing. To be more precise, if we kept the precision of each individual signal γ_x constant, a law of large number would apply and $X_N \equiv \frac{1}{N}\sum_{j=1}^{N} x_j \to \theta$. The fundamental θ would be revealed for sure and no uncertainty would remain in the economy.

Instead, we assume that $\gamma_x\left(\overline{N}\right)$ evolves with $\overline{N}$ in the following manner,

$$\gamma_x\left(\overline{N}\right) = \gamma_x/\overline{N}. \tag{22}$$

This assumption captures the idea that the information gathered by each agent is proportional to its size. Possible microfoundations may include: i) the amount of information is proportional to the market size of the firm, its number of clients, etc; ii) agents use similar sources of information and information is correlated, implying that the precision of information brought by each agent decreases with $\overline{N}$.

Specification (22) displays the great advantage that the Bayesian updating rules for beliefs do not depend on $\overline{N}$. In particular, the learning dynamics follow the same rule as in the finite $\overline{N}$ case, thus ensuring that the intuition behind uncertainty traps remains the same. We define $n(\mu,\gamma) = N(\mu,\gamma)/\overline{N}$ the fraction of investing firm. The law of motion for beliefs satisfy

$$\begin{aligned}
\mu' &= \rho_\theta \frac{\gamma\mu + \gamma_y Y + N\gamma_x\left(\overline{N}\right)X}{\gamma + \gamma_y + N\gamma_x\left(\overline{N}\right)} = \rho_\theta \frac{\gamma\mu + \gamma_y Y + n\gamma_x X}{\gamma + \gamma_y + n\gamma_x}, &(23)\\
\gamma' &= \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + N\gamma_x\left(\overline{N}\right)} + \frac{1-\rho_\theta^2}{\gamma_\theta}\right)^{-1} = \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + n\gamma_x} + \frac{1-\rho_\theta^2}{\gamma_\theta}\right)^{-1}, &(24)
\end{aligned}$$

where $X = \frac{1}{N}\sum_{j=1}^{N} x_n \sim \theta + \mathcal{N}\left(0, (n\gamma_x)^{-1}\right)$. The number of investing firms is now deterministic as

$$n = \frac{N}{\overline{N}} = \frac{1}{\overline{N}} \sum_{j=1}^{\overline{N}} \mathbb{1}\left(f_j \le f_c(\mu,\gamma)\right) \underset{\text{a.s}}{\longrightarrow} p(\mu,\gamma) = F\left(f_c(\mu,\gamma)\right),$$

by a law of large number since the investment costs $\{f_j\}_{j\ge1}$ are i.i.d and distributed according to F.

We may now define an equilibrium for the limit economy:

Definition 4. A recursive equilibrium of the limit economy consists of a policy function $f_c(\mu,\gamma)$, value functions $V(\mu,\gamma,f)$, $V^W(\mu,\gamma)$, $V^I(\mu,\gamma)$, laws of motions for aggregate beliefs $\{\mu',\gamma'\}$, and a fraction of investing firms $n(\mu,\gamma)$, such that

1. The value function $V(\mu,\gamma,f)$ solves (6), with $V^W(\mu,\gamma)$ and $V^I(\mu,\gamma)$ defined according to (7) and (8), with the corresponding cutoff rule $f_c(\mu,\gamma)$;
2. The aggregate beliefs (μ,γ) evolve according to (23) and (24);
3. The number $n(\mu,\gamma)$ of firms that invest is given by $n(\mu,\gamma)=F(f_c(\mu,\gamma))$.

Since the limit economy is in many aspects simpler than the economy considered in part 2, all the lemmas and propositions derived in appendix G extend to this new environment. Given the similarity in the arguments, the proofs are omitted.

G Proofs (ONLINE APPENDIX)

G.1 Assumptions and Definitions

It is useful for the propositions and definitions below to define the following mapping which sums over the distribution of investment costs:

Definition 5. Let S be the mapping such that

$$[S(G)](\mu,\gamma)=\int G(\mu,\gamma,f)\,dF(f),$$

where $G:\mathbb{R}^3\longrightarrow\mathbb{R}$ and $S(G):\mathbb{R}^2\longrightarrow\mathbb{R}$.

Definition 6. Define the following bounds and set:

1. Let $\overline{\gamma}$ be the unique strictly positive solution of

$$\overline{\gamma}=\left(\frac{\rho_\theta^2}{\overline{\gamma}+\gamma_y+\overline{N}\gamma_x}+\left(1-\rho_\theta^2\right)\sigma_\theta^2\right)^{-1}=\Gamma\left(\overline{N},\overline{\gamma}\right),\tag{25}$$

and $\underline{\gamma}$ the unique strictly positive solution of

$$\underline{\gamma}=\left(\frac{\rho_\theta^2}{\underline{\gamma}+\gamma_y}+\left(1-\rho_\theta^2\right)\sigma_\theta^2\right)^{-1}=\Gamma\left(0,\underline{\gamma}\right),\tag{26}$$

2. Let $\mathcal{S}=[\underline{\mu},\overline{\mu}]\times\left[\underline{\gamma},\overline{\gamma}\right]$, where $\underline{\mu}$ and $\overline{\mu}$ are some arbitrary but large bounds on μ.

We define the set $\mathcal{P}$ in which the probability $p(\mu,\gamma)=F(f_c(\mu,\gamma))$ that a firm invests will lie:

Definition 7. Let $\mathcal{P}$ be the set of twice-differentiable functions $p:(\mu,\gamma)\in\mathcal{S}\longrightarrow\mathbb{R}$ such that p has bounded first and second derivatives: $\forall(\mu,\gamma)\in\mathcal{S}, |p_\mu(\mu,\gamma)|\leq\overline{p}_\mu$, $|p_\gamma(\mu,\gamma)|\leq\overline{p}_\gamma$, and $|p_{xy}(\mu,\gamma)|\leq\overline{p}_{xy}$ for $(x,y)\in\{\mu,\gamma\}^2$.

We also define the set $\mathcal{G}$ in which the firm's surplus of waiting compared to investing will lie:

Definition 8. Let $\mathcal{G}$ be the set of continuous functions G of $(\mu, \gamma, f) \in \mathcal{S} \times \left[\underline{f}, \overline{f}\right] \longrightarrow \mathbb{R}$ such that

1. G is bounded by $\overline{G}$,

2. G is weakly decreasing and convex in μ,

3. G is weakly decreasing in γ,

4. G is Lipschitz continuous of constant 1 in f, and

5. G is such that $[S(G)](\mu, \gamma)$ is twice-differentiable with bounded first and second derivatives: $\forall (\mu, \gamma), \left|\frac{\partial}{\partial x}[S(G)](\mu, \gamma)\right| \leq \overline{G}_x$ and $\left|\frac{\partial}{\partial xy}[S(G)](\mu, \gamma)\right| \leq \overline{G}_{xy}$ for $(x, y) \in \{\mu, \gamma\}^2$.

We define the mapping $\mathcal{T}$ that corresponds to the waiting decision of a firm in partial equilibrium, taking a probability of investment for other firms $p \in \mathcal{P}$ as given:

Definition 9. For a given probability of investment $p \in \mathcal{P}$, define the mapping $\mathcal{T}^p : G \in \mathcal{G} \longrightarrow \mathcal{G}$

$$[\mathcal{T}^p G](\mu, \gamma, f) = \max\{C^p(G(\mu, \gamma, f)), 0\},$$

where $C^p(G)$ is the value in the continuation region, defined by:

$$\begin{aligned}
[C^p(G)](\mu, \gamma, f) = \; & \frac{1}{a} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{a(1-\rho_\theta)\mu - \frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right) \\
& - \frac{1}{a}(1-\beta) + f - \beta\omega^f + \beta\mathbb{E}_p\{[S(G)](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma))\}.
\end{aligned}$$

In the recursive equilibrium from 1, the probability that each firm invests satisfies $p(\mu, \gamma) = F(f_c(\mu, \gamma))$. Therefore, we define the following mapping:

Definition 10. Let $\mathcal{M}$ be the mapping from $p : \mathcal{P} \longrightarrow \mathcal{P}$ such that, for all $\mu, \gamma \in \mathcal{S}$,

$$(\mathcal{M}p)(\mu, \gamma) = F(f_c^p(\mu, \gamma))$$

where $f_c^p(\mu, \gamma)$ is defined by

$$\begin{aligned}
f_c^p(\mu, \gamma) \quad = \quad & -\frac{1}{a} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{a(1-\rho_\theta)\mu - \frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right) \\
& + \frac{1}{a}(1-\beta) + \beta\omega^f - \beta\mathbb{E}_p\left\{[S(G^p)]\left(\mu', \gamma'\right)\right\},
\end{aligned}$$

where G^p is the unique fixed point of the mapping

$$G^p = \mathcal{T}^p G^p.$$

Assumption 1. *The parameters and bounds are chosen so that*

$$1 \geqslant \beta e^{a(1-\rho_\theta)\overline{\mu} - \frac{a^2}{2}\frac{1-\rho_\theta^2}{\overline{\gamma}} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}.$$

This assumption is a necessary condition to guarantee that the real option channel is active. The first two terms in the exponential require that the mean reversion in μ and γ is small enough not to dominate the wait-and-see effects. A persistence in the fundamental ρ_θ sufficiently close to 1 ensures that these terms are small. The last term requires that the variance in the fundamental σ_θ^2 is not too large to prevent risk aversion to eliminate the option value of waiting.

Assumption 2. *F is a continuous, twice differentiable cumulative distribution function with bounded first and second derivatives. F has bounded support $\left[\underline{f}, \overline{f}\right]$, mean ω^f and standard deviation σ^f.*

These regularity conditions on the cumulative distribution of investment costs guarantee that the equilibrium number of investing firms $N(\mu,\gamma) \sim Bin\left(\overline{N}, F(f_c(\mu,\gamma))\right)$ is well-behaved.

G.2 Two Useful Lemmas

Lemma 1. *For a given N, mean beliefs μ follow an AR(1) process with time-varying volatility s,*

$$\mu^{'} = \rho_\theta \mu + s(N,\gamma)\varepsilon,$$

where $s(N,\gamma) = \rho_\theta \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{\frac{1}{2}}$ and $\varepsilon \sim \mathcal{N}(0,1)$.

Proof. We use (4), (3) and (2) to compute the mean and the variance of the next period mean beliefs μ' given current-period information, (μ,γ), and a given realization of N:

$$\begin{aligned} \mathbb{E}\left[\mu'|\mu,\gamma,N\right] &= \rho_\theta \mu, \\ V\left[\mu'|\mu,\gamma,N\right] &= \rho_\theta^2 \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N\gamma_x}\right). \end{aligned}$$

Being the sum of normally distributed variables, μ' is also normally distributed and can therefore be expressed by $\mu^{'} = \rho_\theta\mu + \rho_\theta \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{\frac{1}{2}} \varepsilon$ with $\varepsilon \sim \mathcal{N}(0,1)$. $\square$

Lemma 2. *The precision of next-period beliefs, γ', increases with N and γ. For a given N, there exists a unique positive fixed point in the law of motion for the precision of beliefs $\gamma' = \Gamma(N,\gamma)$.*

Proof. The fact that γ' increases with N and γ follows by inspection of (5). Given N, uniqueness of a positive fixed point follows from noting that the all fixed points γ must satisfy:

$$0 = \gamma^2 + \gamma(\gamma_y + N\gamma_x) - \frac{1}{\sigma_\theta^2}(\gamma_y + N\gamma_x).$$

Because the quadratic function of γ on the right-hand side is negative at $\gamma = 0$, it necessarily has a unique positive root. $\square$

G.3 Propositions

We start in Proposition 1 by demonstrating that the individual firm problem is well defined for a given $p(\mu, \gamma)$ and characterize its properties. Then we show that there is a unique $p(\mu, \gamma)$ in Proposition 2.

Proposition 1. *For sufficiently low volatility and high persistence of the fundamental process such that Assumption 1 is satisfied , given a random number of investing firms $N \sim Bin\left(\overline{N}, p(\mu, \gamma)\right)$ for some $p \in \mathcal{P}$, and for γ_x sufficiently low, there exists a unique solution to the firm's problem and the resulting cutoff $f_c(\mu, \gamma)$ is strictly increasing in μ and γ.*

Proof. First, we demonstrate that the difference between the value of waiting and investing is uniquely determined. Then, we characterize properties of that difference that guarantee the existence of the cutoff and its properties.

To proceed it is useful to define some notation. Note that the distributions of μ' and γ' defined in (4) and (5) depend on the random variable N with binomial distribution given by (11). In particular, the probability that N firms invest when the total number of firms in the economy is $\overline{N}$ and the individual probability of investing is p is

$$\pi_N^{\overline{N}}(p) = \binom{\overline{N}}{N} p^N (1-p)^{\overline{N}-N}. \tag{27}$$

Since $\pi_N^{\overline{N}}(p)$ is a polynomial in p of degree $\overline{N}$, it is bounded on $[0, 1]$. Denote $\overline{\pi}$ its upper bound, as well as $\overline{\pi}_p$ ($\overline{\pi}_{pp}$) the upper bound of its first (second) derivative.

For a given value function V, we define the surplus of waiting $G(\mu, \gamma, f) \equiv V(\mu, \gamma, f) - \left[V^I(\mu, \gamma) - f\right]$. In particular, using the definition of V from (6), G must satisfy the recursive relation

$$G(\mu, \gamma, f) = \max\left\{\beta\mathbb{E}\left[G\left(\mu', \gamma', f'\right) + V^I\left(\mu', \gamma'\right) - f'\right] - \left(V^I(\mu, \gamma) - f\right), 0\right\}.$$

Substituting the stopping value $V^I(\mu, \gamma) = \frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\right)$, using (1) and (5) and some manipulations give

$$G(\mu, \gamma, f) = \max\left\{C^p(G(\mu, \gamma, f)), 0\right\}, \tag{28}$$

where C^p is the value in the continuation region, defined by:

$$[C^p(G)](\mu,\gamma,f) = \frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}\left(1-\rho_\theta^2\right)\sigma_\theta^2}\right]$$
$$-\frac{1}{a}(1-\beta)+f-\beta\omega^f+\beta\mathbb{E}_{p,f}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]. \quad (29)$$

In other words, G is a fixed point of the mapping $\mathcal{T}^p$ from Definition 9.[27] The expectation in the last term is with respect to the shock ε to average beliefs, the number of investing firms N and the fixed cost f'. Notice for future reference that the term $\mathbb{E}_{p,f}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]$ is equal to $\mathbb{E}_p\left\{[S(G)]\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma)\right)\right\}$. It depends on the individual probability of investing, p:

$$\mathbb{E}_p\left\{[S(G)](\mu+s\varepsilon,\Gamma)\right\} = \sum_{N=1}^{\overline{N}}\pi_N^{\overline{N}}(p)\,g_N(\mu,\gamma), \quad (30)$$

where

$$g_N(\mu,\gamma) \equiv \int [S(G)]\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma)\right)d\Phi(\varepsilon) \quad (31)$$

where $\Phi(\varepsilon)$ is the CDF of a standard normal, and where $\Gamma=\Gamma(N,\gamma)$.

Note that $\mathcal{T}^p$ trivially satisfies the Blackwell conditions for a contraction so that, if it is a well defined mapping from $\mathcal{G}$ to $\mathcal{G}$, it admits a unique fixed point. To prove uniqueness of $\mathcal{T}^p$ it remains to show that it is indeed a well defined mapping from $\mathcal{G}$ to $\mathcal{G}$, i.e. that if G is an element of the set $\mathcal{G}$ defined in (8) then so is $\mathcal{T}^pG$. We do so next:

1. $\mathcal{T}^pG$ **is bounded and continuous**: continuity follows easily from the definition of the mapping $\mathcal{T}^p$ as it is the maximum of two continuous functions. Boundedness follows from the fact that we can bound $C^p(G)$ as follows:

$$\left|[C^p(G)](\mu,\gamma,f)\right| \le \frac{1}{a}e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{\underline{\gamma}}+\frac{1}{\gamma_x}\right)}\left(1+\beta e^{a(1-\rho_\theta)\overline{\mu}-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\overline{\gamma}}+\frac{a^2}{2}\left(1-\rho_\theta^2\right)\sigma_\theta^2}\right)$$
$$+\frac{1-\beta}{a}+\overline{f}+\beta\omega^f+\beta\overline{G}.$$

Thus, $\mathcal{T}^pG$ is bounded as long as $\overline{G}$ is chosen large enough that

$$\overline{G} \ge (1-\beta)^{-1}\left(\frac{1}{a}e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{\underline{\gamma}}+\frac{1}{\gamma_x}\right)}\left(1+\beta e^{a(1-\rho_\theta)\overline{\mu}-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\overline{\gamma}}+\frac{a^2}{2}\left(1-\rho_\theta^2\right)\sigma_\theta^2}\right)+\frac{1-\beta}{a}+\overline{f}+\beta\omega^f\right).$$

[27]Note that G is defined for μ over the interval $\left[\underline{\mu},\overline{\mu}\right]$ but the expectation for μ' is computed using a normal distribution with unbounded support. Therefore, for the expectation term in the second line of (29), we extend the definition of G to values of μ not in $\left[\underline{\mu},\overline{\mu}\right]$ by assuming that the bounds are absorbing, i.e., that $\forall\mu>\overline{\mu}$, $G(\mu,\gamma,f)=G(\overline{\mu},\gamma,f)$ and $\forall\mu<\underline{\mu}$, $G(\mu,\gamma,f)=G\left(\underline{\mu},\gamma,f\right)$. This assumption guarantees the validity of our proofs. In numerical simulations, the bounds can be chosen to be sufficiently large so as to have no impact on the results.

2. $\mathcal{T}^p G$ **is decreasing with** μ: within the continuation region,

$$\begin{aligned}
\frac{\partial}{\partial \mu}\left[C^p(G)\right](\mu,\gamma,f) &= -ae^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right] \\
&+e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left[-a(1-\rho_\theta)\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right] \\
&+\beta\mathbb{E}_p\left[\frac{\partial}{\partial\mu}S(G)\right]+\beta\frac{\partial p}{\partial \mu}\frac{\partial}{\partial p}\mathbb{E}_p\left\{[S(G)](\mu+s\varepsilon,\Gamma)\right\}. \qquad (32)
\end{aligned}$$

We must prove that this expression is negative. The first term is negative and bounded away from 0 since

$$\begin{aligned}
& -ae^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right] \\
\leqslant\ & -ae^{-a\overline{\mu}+\frac{a^2}{2}\left(\frac{1}{\overline{\gamma}}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\overline{\mu}-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\overline{\gamma}}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right]<0
\end{aligned}$$

The second term is always negative. The third term, $\frac{\partial}{\partial\mu}S(G)$, is also negative because $G \in \mathcal{G}$. To conclude, we show that the last term is $O(\gamma_x)$ and therefore negligible compared to the first term, so that $\frac{\partial}{\partial\mu}\left[C^p(G)\right]<0$ when γ_x is small. For that, note first that from Definition 7 the term $\frac{\partial p}{\partial \mu}$ is bounded above by some constant $\overline{p}_\mu$. Therefore, it remains to show that $\frac{\partial}{\partial p}\mathbb{E}_p\left\{[S(G)](\mu+s\varepsilon,\Gamma)\right\}=O(\gamma_x)$. For that, let $\Pi_N^{\overline{N}}(p)=\sum_{n=1}^{N}\pi_n^{\overline{N}}(p)$, and sum by parts in (30) to write:

$$\mathbb{E}_p\left\{[S(G)](\mu+s\varepsilon,\Gamma)\right\}=g_{\overline{N}}(\mu,\gamma)-\sum_{N=1}^{\overline{N}-1}\Pi_N^{\overline{N}}(p)\cdot(g_{N+1}-g_N)(\mu,\gamma), \qquad (33)$$

which implies:

$$\frac{\partial}{\partial p}\mathbb{E}_p\left\{[S(G)](\mu+s\varepsilon,\Gamma)\right\}=-\sum_{N=1}^{\overline{N}-1}\left[\frac{\partial}{\partial p}\Pi_N^{\overline{N}}(p)\right]\cdot(g_{N+1}-g_N)(\mu,\gamma). \qquad (34)$$

Note in addition that

$$\begin{aligned}
|(g_{N+1}-g_N)(\mu,\gamma)| = \int & [S(G)(\mu+s(N+1,\gamma)\varepsilon,\Gamma(N+1,\gamma)) \\
& -S(G)(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma))]\,d\Phi(\varepsilon) \\
& \leq \overline{G}_\mu\,|(s(N+1,\gamma)-s(N,\gamma))|+\overline{G}_\gamma\,|\Gamma(N+1,\gamma)-\Gamma(N,\gamma)|,
\end{aligned}$$

where the last line follows from the fact that G has bounded derivatives. From the expressions for s and Γ obtained in lemmas (1) and (2) and using the concavity of s, we note that the terms in absolute value on the second inequality are $O(\gamma_x)$,

$$\begin{aligned}
&|s(N+1,\gamma)-s(N,\gamma)| \\
&\leq s_N(N,\gamma)=\frac{\rho_\theta}{2}\left(\frac{1}{\gamma}-\frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{-\frac{1}{2}}\frac{\gamma_x}{(\gamma+\gamma_y+N\gamma_x)^2} \\
&\leq \frac{\rho_\theta}{2}\left(\frac{\gamma_y}{\overline{\gamma}(\overline{\gamma}+\gamma_y)}\right)^{-\frac{1}{2}}\frac{\gamma_x}{(\underline{\gamma}+\gamma_y)^2}\equiv B_s\gamma_x=O(\gamma_x),
\end{aligned}$$

$$\begin{aligned}
& |\Gamma(N+1,\gamma)-\Gamma(N,\gamma)| \\
= & \frac{\rho_\theta^2\gamma_x/\sigma_\theta^4}{[\rho_\theta^2/\sigma_\theta^2+(1-\rho_\theta^2)(\gamma+\gamma_y+N\gamma_x)][\rho_\theta^2/\sigma_\theta^2+(1-\rho_\theta^2)(\gamma+\gamma_y+(N+1)\gamma_x)]} \\
\leqslant & \rho_\theta^2\frac{\overline{\gamma}^2}{(\underline{\gamma}+\gamma_y)^2}\gamma_x\equiv B_\Gamma\gamma_x=O(\gamma_x),
\end{aligned}$$

implies that

$$|(g_{N+1}-g_N)(\mu,\gamma)|=O(\gamma_x) \tag{35}$$

and therefore $\left|\frac{\partial}{\partial p}\mathbb{E}_p\{[S(G)](\mu+s\varepsilon,\Gamma)\}\right|=O(\gamma_x)$, where we have used the fact that

$$\left|\frac{\partial}{\partial p}\Pi_N^{\overline{N}}(p)\right|\leq\overline{N}\overline{\pi}_p.$$

3. $\mathcal{T}^pG$ **is decreasing in** γ: This follows from the same argument as the one developed above to show that $\mathcal{T}^pG$ is decreasing in μ. Following these arguments, we have that

$$\begin{aligned}
& \frac{\partial}{\partial\gamma}[C^p(G)](\mu,\gamma,f) \\
\leqslant & -\frac{a}{2}\frac{1}{\gamma^2}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right] \\
& -\beta\frac{1}{a}\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma^2}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2} \\
& +\underbrace{\beta\mathbb{E}_p\left[\frac{\partial}{\partial\gamma}S(G)\right]}_{\leq 0}+\beta\underbrace{\frac{\partial}{\partial p}\mathbb{E}_p\{[S(G)](\mu+s\varepsilon,\Gamma)\}}_{O(\gamma_x)}\underbrace{\frac{\partial p}{\partial\gamma}}_{\leq\overline{p}_\gamma},
\end{aligned}$$

where the first term is strictly negative, bounded away from 0,

$$\begin{aligned}
& -\frac{a}{2}\frac{1}{\gamma^2}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right] \\
\leqslant & -\frac{a}{2\overline{\gamma}^2}e^{-a\overline{\mu}+\frac{a^2}{2}\frac{1}{\overline{\gamma}}}\left[1-\beta e^{a(1-\rho_\theta)\overline{\mu}-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\overline{\gamma}}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right]
\end{aligned}$$

so that, for γ_x small enough, the derivative is strictly negative and bounded away from 0.

4. $\mathcal{T}^p G$ **is Lipschitz in** f **of constant** 1**:** Choosing $f_1 < f_2$ then from (28) this is trivially satisfied because

$$
\begin{aligned}
\left|\mathcal{T}^p G\left(\mu,\gamma,f_2\right) - \mathcal{T}^p G\left(\mu,\gamma,f_1\right)\right| &\leq \left|C^p\left(G\left(\mu,\gamma,f_2\right)\right) - C^p\left(G\left(\mu,\gamma,f_1\right)\right)\right| \\
&= \left|f_2 - f_1\right|.
\end{aligned}
$$

5. $\mathcal{T}^p G$ **is convex in** μ: From (32), the second derivative of the continuation value with respect to μ is:

$$
\begin{aligned}
\frac{\partial^2}{\partial\mu^2}C^p\left(G\right) = &\underbrace{a^2 e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}\left(1-\rho_\theta^2\right)\sigma_\theta^2}\right]}_{\geq a^2 e^{-a\overline{\mu}+\frac{a^2}{2}\frac{1}{\overline{\gamma}}}\left(1-\beta e^{a(1-\rho_\theta)\overline{\mu}-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\overline{\gamma}}+\frac{a^2}{2}\left(1-\rho_\theta^2\right)\sigma_\theta^2}\right)>0} \\
&+\underbrace{\beta a^2\left(1-\rho_\theta^2\right)e^{-a\rho_\theta\mu+\frac{a^2}{2}\left(\frac{\rho_\theta^2}{\gamma}+\left(1-\rho_\theta^2\right)\sigma_\theta^2+\frac{1}{\gamma_x}\right)}}_{\geqslant 0} \\
&+\underbrace{\beta\mathbb{E}_p\left[\frac{\partial^2}{\partial\mu^2}S\left(G\right)\right]}_{\geq 0}+\beta\underbrace{\frac{\partial p}{\partial\mu}}_{\leq\overline{p}_\mu}\underbrace{\frac{\partial^2}{\partial p\partial\mu}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}}_{O(\gamma_x)} \\
&+\beta\underbrace{\frac{\partial^2 p}{\partial\mu^2}}_{\leq\overline{p}_{\mu\mu}}\underbrace{\frac{\partial^2}{\partial p^2}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}}_{O(\gamma_x)},
\end{aligned}
$$

where $\frac{\partial^2}{\partial p^2}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\} = O\left(\gamma_x\right)$ follows from (34) and the fact that $\Pi_N''\left(p\right)$ is a polynomial of degree $\overline{N}-2$ in p and is therefore bounded on $[0,1]$. To see that

$$
\frac{\partial^2}{\partial p\partial\mu}\mathbb{E}_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] = O\left(\gamma_x\right),
$$

note from (33) that

$$
\frac{\partial}{\partial\mu}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\} = \frac{\partial g_{\overline{N}}}{\partial\mu} - \sum_{N=1}^{\overline{N}-1}\Pi_N^{\overline{N}}\left(p\right)\cdot\left(\frac{\partial g_{N+1}}{\partial\mu}-\frac{\partial g_N}{\partial\mu}\right)\left(\mu,\gamma\right),
$$

which, taking derivative with respect to p and using (31), gives

$$
\frac{\partial^2}{\partial p\partial\mu}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\} = -\sum_{N=1}^{\overline{N}-1}\frac{\partial\Pi_N^{\overline{N}}\left(p\right)}{\partial p}\cdot\left(\frac{\partial g_{N+1}}{\partial\mu}-\frac{\partial g_N}{\partial\mu}\right)\left(\mu,\gamma\right)
$$

Following the same arguments as before, we obtain that

$$
\begin{aligned}
\left|\frac{\partial g_{N+1}}{\partial\mu}-\frac{\partial g_N}{\partial\mu}\right| &\leq \overline{G}_{\mu\mu}\left|\left(s\left(N+1,\gamma\right)-s\left(N,\gamma\right)\right)\right| \\
&\quad+\overline{G}_{\mu\gamma}\left|\Gamma\left(N+1,\gamma\right)-\Gamma\left(N,\gamma\right)\right| = O\left(\gamma_x\right),
\end{aligned}
$$

which completes the proof.

6. $[S(\mathcal{T}^p G)](\mu,\gamma)$ **is twice-differentiable with bounded first and second derivatives:** Notice, first, that since $G \in \mathcal{G}$, then $C^p(G)$ is twice-differentiable in (μ,γ) and linear in f:

$$\begin{aligned} C^p(G) &= \frac{1}{a}\left[e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right)\right] \\ &\quad -\frac{1}{a}(1-\beta)+f-\beta\omega^f+\beta\mathbb{E}_p\left\{[S(G)](\mu+s\varepsilon,\Gamma(N,\gamma))\right\}. \end{aligned}$$

We show below that investment takes the form of a cutoff rule $f_c^p(\mu,\gamma)$ that satisfies $f_c^p(\mu,\gamma) = f - [C^p(G)](\mu,\gamma,f)$. Hence, $f_c^p(\mu,\gamma)$ is twice-differentiable in (μ,γ) and independent of f. Therefore,

$$\begin{aligned} [S(\mathcal{T}^p G)](\mu,\gamma) &\equiv \int [\mathcal{T}^p G](\mu,\gamma,f)\,dF(f) \\ &= \int \max\{[C^p(G)](\mu,\gamma,f),0\}\,dF(f) \\ &= \int_{f_c^p(\mu,\gamma)}^{\infty} [C^p(G)](\mu,\gamma,f)\,dF(f). \end{aligned}$$

Since $C^p(G)$ and f_c^p are twice-differentiable in (μ,γ), so is $S(\mathcal{T}^p G)$. To finish the proof, it only remains to show that the first and second derivatives of $S(\mathcal{T}^p G)$ are bounded. For $(x,y) = \{\mu,\gamma\}$,

$$\frac{\partial}{\partial x} S(\mathcal{T}^p G) = \int_{-\infty}^{f_c^p(\mu,\gamma)} \frac{\partial}{\partial x}[C^p(G)](\mu,\gamma,f)\,dF(f).$$

According to previous results and after some manipulations,

$$\begin{aligned} \left|\frac{\partial}{\partial\mu}[C^p(G)](\mu,\gamma,f)\right| &\le a e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{\underline{\gamma}}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\underline{\mu}-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\underline{\gamma}}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right] \\ &\quad + a(1-\rho_\theta)e^{-a\rho_\theta\underline{\mu}+\frac{a^2}{2}\left(\frac{\rho_\theta^2}{\underline{\gamma}}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2+\frac{1}{\gamma_x}\right)} \\ &\quad + \beta\overline{G}_\mu + \beta\overline{p}_\mu\overline{\pi}_p\overline{N}^2\gamma_x\left(B_s\overline{G}_\mu + B_\Gamma\overline{G}_\gamma\right), \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial}{\partial\gamma}[C^p(G)](\mu,\gamma,f)\right| &\le \frac{a}{2}\frac{1}{\underline{\gamma}^2} e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{\underline{\gamma}}+\frac{1}{\gamma_x}\right)}\left[1-\beta e^{a(1-\rho_\theta)\underline{\mu}-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\underline{\gamma}}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right] \\ &\quad + \beta\frac{a}{2}\frac{1-\rho_\theta^2}{\underline{\gamma}^2}e^{-a\rho_\theta\underline{\mu}+\frac{a^2}{2}\left(\frac{\rho_\theta^2}{\underline{\gamma}}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2+\frac{1}{\gamma_x}\right)} \\ &\quad + \beta\overline{G}_\gamma + \beta\overline{p}_\gamma\overline{\pi}_p\overline{N}^2\gamma_x\left(B_s\overline{G}_\mu + B_\Gamma\overline{G}_\gamma\right). \end{aligned}$$

Boundedness of the derivatives is guaranteed if

$$X(\gamma_x) + \beta A(\gamma_x) \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \leq \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \Leftrightarrow X(\gamma_x) \leq (I - \beta A(\gamma_x)) \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix}$$

where

$$A(\gamma_x) = \begin{bmatrix} 1 + \overline{p}_\mu \overline{\pi}_p \overline{N}^2 \gamma_x B_s & \overline{p}_\gamma \overline{\pi}_p \overline{N}^2 \gamma_x B_\Gamma \\ \overline{p}_\mu \overline{\pi}_p \overline{N}^2 \gamma_x B_\Gamma & 1 + \overline{p}_\gamma \overline{\pi}_p \overline{N}^2 \gamma_x B_\Gamma \end{bmatrix}$$

and

$$X(\gamma_x) = \begin{bmatrix} a e^{-a\underline{\mu} + \frac{a^2}{2}\left(\frac{1}{\underline{\gamma}} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{a(1-\rho_\theta)\underline{\mu} - \frac{a^2}{2}\frac{1-\rho_\theta^2}{\underline{\gamma}} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right) + a(1-\rho_\theta) e^{-a\rho_\theta \underline{\mu} + \frac{a^2}{2}\left(\frac{\rho_\theta^2}{\underline{\gamma}} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2 + \frac{1}{\gamma_x}\right)} \\ \frac{a}{2}\frac{1}{\underline{\gamma}^2} e^{-a\underline{\mu} + \frac{a^2}{2}\left(\frac{1}{\underline{\gamma}} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{a(1-\rho_\theta)\underline{\mu} - \frac{a^2}{2}\frac{1-\rho_\theta^2}{\underline{\gamma}} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right) + \beta \frac{a}{2} \frac{1-\rho_\theta^2}{\underline{\gamma}^2} e^{-a\rho_\theta \underline{\mu} + \frac{a^2}{2}\left(\frac{\rho_\theta^2}{\underline{\gamma}} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2 + \frac{1}{\gamma_x}\right)} \end{bmatrix}.$$

Note that the matrix $I - \beta A(\gamma_x)$ satisfies $I - \beta A(\gamma_x) \underset{\gamma_x \to 0}{\longrightarrow} (1-\beta) I$. Thus, with γ_x small enough, the following is satisfied

$$[I - \beta A(\gamma_x)] \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \geq \frac{1}{2}(1-\beta) \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix}.$$

We can then choose positive bounds $\{\overline{G}_\mu, \overline{G}_\gamma\}$ such that

$$\begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \geq 2(1-\beta)^{-1} X(\gamma_x) \geq 0,$$

which guarantees the boundedness of the first derivatives. Regarding the second derivatives, manipulations of a similar nature as above yield bounds $\{\overline{G}_{\mu\mu}, \overline{G}_{\mu\gamma}, \overline{G}_{\gamma\gamma}\}$ for the second derivatives of $S(G)$ as long as γ_x is small enough.

It remains to show existence and monotonicity of $f_c(\mu, \gamma)$. A firm invests if and only if

$$\begin{aligned} [C^p(G)](\mu,\gamma,f) &= \frac{1}{a} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{a(1-\rho_\theta)\mu - \frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right) \\ &\quad - \frac{1}{a}(1-\beta) + f - \beta\omega^f + \beta \mathbb{E}_p \{[S(G)](\mu + s\varepsilon, \Gamma(N,\gamma))\} \leq 0, \end{aligned}$$

i.e., when its fixed cost satisfies

$$\begin{aligned} f \quad & \leq -\frac{1}{a} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{a(1-\rho_\theta)\mu - \frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma} + \frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right) \\ & + \frac{1}{a}(1-\beta) + \beta\omega^f + \beta \mathbb{E}_p \{[S(G)](\mu + s\varepsilon, \Gamma(N,\gamma))\} \\ & \equiv f_c^p(\mu,\gamma) \end{aligned} \tag{36}$$

Notice, furthermore, that $f_c^p(\mu,\gamma) = f - C^p(G(\mu,\gamma,f))$. Thus, the threshold inherits a number of properties from the continuation value. In particular, $f_c(\mu,\gamma)$ is strictly increasing in μ and γ, and strictly concave in μ for γ_x small enough.

$\square$

Proposition 2. *Under assumptions 1 and 2 and for γ_x small enough, a recursive equilibrium exists and is unique. The equilibrium $p(\mu,\gamma)$ is increasing in the mean of beliefs μ and the precision γ.*

Proof. Proving uniqueness of the recursive equilibrium is equivalent to showing that there is a unique fixed point $p^*(\mu,\gamma)\in\mathcal{P}$ such that $\mathcal{M}p^*=p^*$ for the mapping $\mathcal{M}$ in Definition 10.

We establish first that $\mathcal{M}$ is a well-defined mapping from $\mathcal{P}$ to $\mathcal{P}$. This follows from the definition of $f_c^p(\mu,\gamma)$, which inherits the properties of C^p. In particular, it is twice-differentiable with bounded first and second derivatives. Under assumption 2, $\mathcal{M}p=F\left(f_c^p\right)$ preserves these properties and it is possible to find bounds on the first and second derivatives of p that are preserved by the mapping using a similar argument as the one developed in proposition (1).

Next, we show that $\mathcal{M}$ defines a contraction from $\mathcal{P}$ to $\mathcal{P}$.[28] For $p_1,p_2\in\mathcal{P}$, by the mean value theorem, the mapping $\mathcal{M}$ satisfies

$$\begin{aligned}\left|\left(\mathcal{M}p_2-\mathcal{M}p_1\right)(\mu,\gamma)\right| &= \left|F\left(f_c^{p_2}(\mu,\gamma)\right)-F\left(f_c^{p_1}(\mu,\gamma)\right)\right|\\ &= \left|F'\left(\tilde{f}\right)\left(f_c^{p_2}(\mu,\gamma)-f_c^{p_1}(\mu,\gamma)\right)\right|\end{aligned}$$

for some $\tilde{f}\in\left[f_c^{p_1}(\mu,\gamma),f_c^{p_2}(\mu,\gamma)\right]$. Therefore, if

$$\left|f_c^{p_2}(\mu,\gamma)-f_c^{p_1}(\mu,\gamma)\right|\leq A\gamma_x\parallel p_2-p_1\parallel \tag{37}$$

for some constant A, we reach

$$\left|\left(\mathcal{M}p_2-\mathcal{M}p_1\right)(\mu,\gamma)\right|\leq A\gamma_x\parallel F'\left(\tilde{f}\right)\parallel\cdot\parallel p_2-p_1\parallel, \tag{38}$$

implying that the mapping $\mathcal{M}$ is continuous as long as F' is bounded, which is guaranteed by assumption 2. We can then choose γ_x such that $A\gamma_x\parallel F'\left(\tilde{f}\right)\parallel<1$ and use (38) to guarantee that $\mathcal{M}$ is indeed a contraction. By the contraction mapping theorem, this implies that the equilibrium exists and is unique for γ_x sufficiently small.

Therefore, to prove existence and uniqueness it remains to establish that (37) holds for some constant A. From the definition of $f_c^p(\mu,\gamma)$, the left-hand side of (37) can be expressed as:

$$\begin{aligned}&\left|f_c^{p_2}(\mu,\gamma)-f_c^{p_1}(\mu,\gamma)\right|\\ &=\beta\left|\mathbb{E}_{p_2}\left\{\left[S\left(G^{p_2}\right)\right](\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma))\right\}-\mathbb{E}_{p_1}\left\{\left[S\left(G^{p_1}\right)\right](\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma))\right\}\right|\\ &\leq\beta\left|\mathbb{E}_{p_2}\left\{\left[S\left(G^{p_2}\right)\right](\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma))-\left[S\left(G^{p_1}\right)\right](\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma))\right\}\right| && (39)\\ &+\beta\left|\mathbb{E}_{p_2}\left\{\left[S\left(G^{p_1}\right)\right](\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma))\right\}-\mathbb{E}_{p_1}\left\{\left[S\left(G^{p_1}\right)\right](\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma))\right\}\right| && (40)\end{aligned}$$

To prove that (37) holds we will control each term in this expression. We start with the term in

[28]We cannot prove that $\mathcal{M}$ satisfies monotonicity and therefore we cannot apply the Blackwell conditions. Instead, we directly show that $\mathcal{M}$ satisfies the definition of a contraction.

(40). For any $G \in \mathcal{G}$, we can use (36) to write

$$\begin{aligned}
\left| \begin{array}{c} \mathbb{E}_{p_2}\left\{\left[S\left(G\right)\right]\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right)\right)\right\} \\ -\mathbb{E}_{p_1}\left\{\left[S\left(G\right)\right]\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right)\right)\right\} \end{array} \right| &= \left| \sum_{N=1}^{\overline{N}-1} \left[\Pi_N^{\overline{N}}\left(p_2\right)-\Pi_N^{\overline{N}}\left(p_1\right)\right] \cdot \left(g_{N+1}-g_N\right)\left(\mu,\gamma\right) \right| \\
&\leq \gamma_x \left[B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma\right] \sum_{N=1}^{\overline{N}-1} \left| \Pi_N^{\overline{N}}\left(p_2\right)-\Pi_N^{\overline{N}}\left(p_1\right) \right| \\
&\leq B\gamma_x \parallel p_2 - p_1 \parallel \qquad (41)
\end{aligned}$$

where B is some constant. The second line follows from the results established in Proposition 1, and the third line follows from noting that $\Pi_N^{\overline{N}}(p)$ is a polynomial in p of degree $\overline{N}$ and therefore continuous on the compact set $[0,1]$. In particular, we can control the term by $\parallel \Pi_N^{\overline{N}}\left(p_2\right) - \Pi_N^{\overline{N}}\left(p_1\right) \parallel \leq \parallel \frac{\partial}{\partial p}\Pi_N^{\overline{N}} \parallel \parallel p_2 - p_1 \parallel$.

To conclude, we move to the term in (39). For that, we need to evaluate the norm of $\parallel G^{p_2} - G^{p_1} \parallel$. We first consider the term $\left[\mathcal{T}^{p_2}\left(G\right) - \mathcal{T}^{p_1}\left(G\right)\right]\left(\mu,\gamma,f\right)$, starting from some common function $G \in \mathcal{G}$. Assuming w.l.o.g. that

$$\left[\mathcal{T}^{p_2}\left(G\right)\right]\left(\mu,\gamma,f\right) \geq \left[\mathcal{T}^{p_1}\left(G\right)\right]\left(\mu,\gamma,f\right),$$

from the definition of $\mathcal{T}^p$ it follows that only the next scenarios are possible:

1. $\left[\mathcal{T}^{p_2}\left(G\right)\right]\left(\mu,\gamma,f\right) = \left[\mathcal{T}^{p_1}\left(G\right)\right]\left(\mu,\gamma,f\right) = 0$;

2. $\left[\mathcal{T}^{p_2}\left(G\right)\right]\left(\mu,\gamma,f\right) = \left[C^{p_2}\left(G\right)\right]\left(\mu,\gamma,f\right)$, in which case (41) implies

$$\begin{aligned}
& \left|\left[\mathcal{T}^{p_2}\left(G\right) - \mathcal{T}^{p_1}\left(G\right)\right]\left(\mu,\gamma,f\right)\right| \\
\leq \quad & \beta\left\{\mathbb{E}_{p_2}\left\{\left[S\left(G\right)\right]\left(\mu',\gamma'\right)\right\} - \mathbb{E}_{p_1}\left\{\left[S\left(G\right)\right]\left(\mu',\gamma'\right)\right\}\right\} \\
\leq \quad & \beta B\gamma_x \parallel p_2 - p_1 \parallel .
\end{aligned}$$

Following similar arguments, we can recursively show that, for $k > 1$,

$$\parallel \left(\mathcal{T}^{p_2}\right)^k G - \left(\mathcal{T}^{p_1}\right)^k G \parallel \leq \beta \frac{1-\beta^k}{1-\beta} B\gamma_x \parallel p_2 - p_1 \parallel$$

Since, from Proposition 1, the operator $\mathcal{T}^p$ is a contraction, we have in the limit that:

$$\parallel G^{p_2} - G^{p_1} \parallel \leq \frac{\beta}{1-\beta} B\gamma_x \parallel p_2 - p_1 \parallel, \qquad (42)$$

implying

$$\begin{aligned}
& \left|\mathbb{E}_{p_2}\left\{\left[S\left(G^{p_2}\right)\right]\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right)\right) - \left[S\left(G^{p_1}\right)\right]\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right)\right)\right\}\right| \\
& \leq \frac{\beta}{1-\beta} B\gamma_x \parallel p_2 - p_1 \parallel \qquad (43)
\end{aligned}$$

Combining (41) with (39) and (40) implies

$$\left| f_c^{p_2} (\mu, \gamma) - f_c^{p_1} (\mu, \gamma) \right| \leq \frac{1}{1-\beta} B \gamma_x \parallel p_2 - p_1 \parallel$$

so that (37) indeed holds, implying uniqueness.

Showing that the expected number of investing firms is increasing in the mean of beliefs μ and in precision γ is equivalent to showing the threshold $f_c^{p^*}(\mu, \gamma)$ is strictly increasing in μ and γ. This immediately follows from the properties of the continuation value C^p demonstrated in the proof of the previous proposition. □

Proposition 3. *Under the conditions of Proposition 2 and for σ^f small enough, there exists a non-empty interval $[\mu_l, \mu_h]$ such that, for all $\mu \in (\mu_l, \mu_h)$, the economy features an uncertainty trap with at least two regimes $\gamma_l(\mu) < \gamma_h(\mu)$. Regime γ_l is characterized by high uncertainty and low investment while regime γ_h is characterized by low uncertainty and high investment.*

Proof. In the limit case where the number of firms is large enough that the approximation $n = N/\overline{N} = F(f_c)$ is valid, we can define the function

$$\begin{aligned} \varphi_\mu^n(\gamma) &= \Gamma\left(n(\mu,\gamma), \gamma\right) - \gamma \\ &= \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + n(\mu,\gamma)\gamma_x} + \left(1-\rho_\theta^2\right)\sigma_\theta^2 \right)^{-1} - \gamma \end{aligned}$$

where Γ is the law of motion for γ defined in (5). By continuity of $n = F(f_c(\mu,\gamma))$, the function $\varphi_\mu^n(\gamma)$ is continuous in γ. From the definition of $\left\{\overline{\gamma}, \underline{\gamma}\right\}$ in (25) and (26), we have that $\varphi_\mu^n(\underline{\gamma}) \geq 0 \geq \varphi_\mu^n(\overline{\gamma})$.

Consider a distribution of fixed investment costs F^1 with mean ω^f and standard deviation 1 that satisfies Assumption 2. Let

$$F^{\sigma^f}(f) = F^1\left[\left(\sigma^f\right)^{-1} \left(f - \omega^f\right) + \omega^f \right]$$

be a mean-preserving, rescaled version of that distribution with standard deviation σ^f. We are going to show that when σ_f is low, there exists a range $[\mu_l, \mu_h]$ such that for any $\mu^* \in (\mu_l, \mu_h)$, we can always find two points $\gamma_1 < \gamma_2$ with $\gamma_1, \gamma_2 \in \left(\underline{\gamma}, \overline{\gamma}\right)$ such that $\varphi_{\mu^*}^n(\gamma_1) < 0$ and $\varphi_{\mu^*}^n(\gamma_2) > 0$. This will imply, by the Intermediate Value Theorem, that there exist two values $\gamma_l^* < \gamma_h^*$ with $\underline{\gamma} \leq \gamma_l^* < \gamma_1$ and $\gamma_2 < \gamma_h^* \leq \overline{\gamma}$ such that $\varphi_\mu^n(\gamma_l^*) = \varphi_\mu^n(\gamma_h^*) = 0$, i.e. two distinct stationary points in the dynamics of precision γ.

An important step in this proof is established in lemma 3 from Appendix H, where we prove that as σ^f goes to 0 the cutoff $f_c^{\sigma^f}$ corresponding to the variance σ^f of the fixed-cost distribution converges uniformly towards some limit f_c^0 and that the number of investing firms converges pointwise to the limit $n^0(\mu,\gamma) = \mathbb{I}\left(\omega^f \leq f_c^0(\mu,\gamma)\right)$.

We must first find a range of values for μ in which we are guaranteed to have multiple stationary points for γ. We are going to use the fact that $f_c^{\sigma^f}$ is strictly increasing in μ and γ at a bounded rate. In what follows, we denote G^{σ^f} the general equilibrium surplus of investing for a given dispersion of costs σ^f, i.e., $\mathcal{T}^{p(\sigma^f)}G^{\sigma^f} = G^{\sigma^f}$ where $\mathcal{M}^{\sigma^f}\left[p\left(\sigma^f\right)\right] = p\left(\sigma^f\right)$. Recall the definition:

$$\begin{aligned} f_c^{\sigma^f}(\mu,\gamma) &= -\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{a(1-\rho_\theta)\mu-\frac{a^2}{2}\frac{1-\rho_\theta^2}{\gamma}+\frac{a^2}{2}(1-\rho_\theta^2)\sigma_\theta^2}\right) \\ &\quad +(1-\beta)\frac{1}{a}+\beta\omega^f-\beta\mathbb{E}\left\{\left[S^{\sigma^f}\left(G^{\sigma^f}\right)\right](\mu',\gamma')\right\}. \end{aligned}$$

Since G^{σ^f} has bounded derivatives, we can find upper and lower bounds for the derivatives of $f_c^{\sigma^f}$ in μ and γ that are strictly positive, as we did in proposition 1 for γ_x low enough. Denote these bounds $\overline{f}_\mu$, $\underline{f}_\mu$ and $\overline{f}_\gamma$, $\underline{f}_\gamma$. The derivatives are:

$$\begin{aligned} 0<\underline{f}_\mu &\le \frac{\partial}{\partial\mu}f_c^{\sigma^f}(\mu,\gamma)\le\overline{f}_\mu \\ 0<\underline{f}_\gamma &\le \frac{\partial}{\partial\gamma}f_c^{\sigma^f}(\mu,\gamma)\le\overline{f}_\gamma \end{aligned}$$

Since f_c^0 is the uniform limit of continuous functions, it is continuous. The limit f_c^0 may not be differentiable, but it is bi-Lipschitz continuous with Lipschitz constants $\left(\underline{f}_\mu,\overline{f}_\mu\right)$ and $\left(\underline{f}_\gamma,\overline{f}_\gamma\right)$. We know therefore that for the bounds $\left[\underline{\mu},\overline{\mu}\right]$ chosen wide enough, we can find a $\mu\in\left[\underline{\mu},\overline{\mu}\right]$ low enough such that $f_c^0(\mu,\overline{\gamma})<\omega^f$ (remember that ω^f is the mean of the fixed cost distribution), and that for some $\mu\in\left[\underline{\mu},\overline{\mu}\right]$ high enough, $f_c^0(\mu,\overline{\gamma})>\omega^f$. By the Intermediate Value theorem, we know that there exists a point μ_l at which $f_c^0(\mu_l,\overline{\gamma})=\omega^f$. Since f_c^0 is strictly increasing in γ, we have that $f_c^0\left(\mu_l,\underline{\gamma}\right)<\omega^f$. Using the fact that f_c^0 is bi-Lipschitz continuous, we have the following inequality:

$$f_c^0\left(\mu,\underline{\gamma}\right) \le f_c^0\left(\mu_l,\underline{\gamma}\right)+\overline{f}_\mu\cdot(\mu-\mu_l).$$

Define $\mu_h=\mu_l+\frac{\omega^f-f_c\left(\mu_l,\underline{\gamma}\right)}{\overline{f}_\mu}>\mu_l$. Then, for any $\mu\in(\mu_l,\mu_h)$:

$$f_c^0\left(\mu,\underline{\gamma}\right)\le f_c^0\left(\mu_l,\underline{\gamma}\right)+\overline{f}_\mu\cdot(\mu-\mu_l)<\omega^f<f_c^0(\mu,\overline{\gamma}).$$

We will now show that the interval (μ_l,μ_h) is a range of values for μ in which we are guaranteed to have two steady-states. Pick any $\mu^*\in(\mu_l,\mu_h)$. Then $f_c^0\left(\mu^*,\underline{\gamma}\right)<\omega^f$ (meaning that $n^0\left(\mu^*,\underline{\gamma}\right)=0$) and $f_c^0(\mu^*,\overline{\gamma})>\omega^f$ (meaning $n^0(\mu^*,\overline{\gamma})=1$). By continuity of f_c^0, we can pick (γ_1,γ_2) with $\underline{\gamma}<\gamma_1<\gamma_2<\overline{\gamma}$, such that $f_c^0(\mu^*,\gamma_1)<\omega^f$ and $f_c^0(\mu^*,\gamma_2)>\omega^f$. Therefore, $n^0(\mu^*,\gamma_1)=0$ and

$n^0(\mu^*, \gamma_2) = 1$. We have:

$$\begin{aligned}
\varphi_{\mu^*}^{n^0}(\gamma_1) &= \left(\frac{\rho_\theta^2}{\gamma_1 + \gamma_y + n^0(\mu^*, \gamma_1)\gamma_x} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)^{-1} - \gamma_1 \\
&= \left(\frac{\rho_\theta^2}{\gamma_1 + \gamma_y} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)^{-1} - \gamma_1 \\
&< \left(\frac{\rho_\theta^2}{\underline{\gamma} + \gamma_y} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)^{-1} - \underline{\gamma} = 0
\end{aligned}$$

$$\begin{aligned}
\varphi_{\mu^*}^{n^0}(\gamma_2) &= \left(\frac{\rho_\theta^2}{\gamma_2 + \gamma_y + n^0(\mu^*, \gamma_2)\gamma_x} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)^{-1} - \gamma_2 \\
&= \left(\frac{\rho_\theta^2}{\gamma_2 + \gamma_y + \gamma_x} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)^{-1} - \gamma_2 \\
&> \left(\frac{\rho_\theta^2}{\overline{\gamma} + \gamma_y + \gamma_x} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)^{-1} - \overline{\gamma} = 0.
\end{aligned}$$

Since $n^{\sigma^f}(\mu, \gamma) \underset{\sigma^f \to 0}{\longrightarrow} n^0(\mu, \gamma)$, for σ^f small enough, we will have: $\varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_1) < 0$ and $\varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_2) > 0$, which implies that there exists at least two locally stable steady-states γ_l^* and γ_h^* $\left(\varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_l^*) = \varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_h^*) = 0\right)$ with $\underline{\gamma} \leq \gamma_l^* < \gamma_1$ and $\gamma_2 < \gamma_h^* \leq \overline{\gamma}$ (one can pick at least 2 locally stable steady-states because $\varphi_{\mu^*}^{n^{\sigma^f}}$ must cross the x-axis from above at least twice). □

Proposition 4. *The recursive competitive equilibrium is constrained inefficient and the efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$ and a uniform tax. In turn, when γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

Proof. 1. In the limit case where the number of firms is large enough that the approximation $n = N/\overline{N} = F(f_c)$ is valid, we can write the constrained planner's decision as a choice over the optimal cutoff $f_c^{eff} \in \mathbb{R} \cup \{-\infty, \infty\}$ under which firms should invest:

$$\begin{aligned}
W(\mu, \gamma) &= \max_{f_c^{eff}} \int_{-\infty}^{f_c^{eff}} \left(\mathbb{E}\left[u(\theta + \varepsilon^x) \mid \mu, \gamma\right] - \tilde{f}\right) dF\left(\tilde{f}\right) \\
&\qquad + \beta \mathbb{E}\left[W\left(\mu', \gamma'\right)\right] \\
\text{s.t} \quad & \mu' = \rho_\theta \frac{\gamma\mu + \gamma_y Y + n\gamma_x X}{\gamma + \gamma_y + n\gamma_x} \\
& \gamma' = \left(\frac{\rho_\theta^2}{\gamma + \gamma_y + n\gamma_x} + \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)^{-1} \\
& n = F\left(f_c^{eff}\right)
\end{aligned}$$

with $\theta' = \rho_\theta\theta + \varepsilon^\theta, \varepsilon^\theta \sim \mathcal{N}\left(0, \left(1 - \rho_\theta^2\right)\sigma_\theta^2\right)$, $Y = \theta + \varepsilon^y, \varepsilon^y \sim \mathcal{N}\left(0, \gamma_y^{-1}\right)$ and $X = \theta + \varepsilon^X, \varepsilon^X \sim$

$\mathcal{N}\left(0,(n\gamma_x)^{-1}\right)$. The first order condition with respect to the cutoff is

$$F'\left(f_c^{eff}\right)\left(\mathbb{E}\left[u\left(\theta+\varepsilon^x\right)\mid\mu,\gamma\right]-f_c^{eff}+\beta\frac{d}{dn}\mathbb{E}\left[W\left(\rho_\theta\mu+s\left(n,\gamma\right)\varepsilon,\Gamma\left(n,\gamma\right)\right)\right]\right)=0,$$

where ε is a unit normal, so that we can derive an expression for the efficient cutoff:

$$f_c^{eff}\left(\mu,\gamma\right)=\mathbb{E}\left[u\left(\theta+\varepsilon^x\right)\mid\mu,\gamma\right]+\beta\frac{d}{dn}\mathbb{E}\left[W\left(\rho_\theta\mu+s\left(n,\gamma\right)\varepsilon,\Gamma\left(n,\gamma\right)\right)\right].$$

We show that this optimal cutoff is implementable using beliefs-dependent investment subsidies $\tau\left(\mu,\gamma\right)$ and a uniform tax $T\left(\mu,\gamma\right)$ levied on all firms at the beginning of the period. Let us write the problem of firms facing these policy instruments:

$$V^\tau\left(\mu,\gamma,f\right)=\max\left\{\mathbb{E}\left[u\left(\theta+\varepsilon^x\right)\mid\mu,\gamma\right]-f+\tau\left(\mu,\gamma\right),\beta\mathbb{E}\left[V^\tau\left(\mu',\gamma',f'\right)\right]\right\}-T\left(\mu,\gamma\right)$$

which yields the individual cutoff rule f_c:

$$f_c^\tau\left(\mu,\gamma\right)=\mathbb{E}\left[u\left(\theta+\varepsilon^x\right)\mid\mu,\gamma\right]+\tau\left(\mu,\gamma\right)-\beta\mathbb{E}\left[V^\tau\left(\mu',\gamma',f'\right)\right].$$

Requiring that the government's budget constraint balances implies

$$\tau\left(\mu,\gamma\right)F\left(f_c^\tau\left(\mu,\gamma\right)\right)=T\left(\mu,\gamma\right).$$

To implement the efficient allocation, we must identify the two cutoffs

$$\begin{aligned}
& f_c^\tau\left(\mu,\gamma\right)=f_c^{eff}\left(\mu,\gamma\right)\\
\Leftrightarrow\quad & \tau\left(\mu,\gamma\right)=\underbrace{\beta\frac{d}{dn}\mathbb{E}\left[W\left(\rho_\theta\mu+s\left(n,\gamma\right)\varepsilon,\Gamma\left(n,\gamma\right)\right)\right]}_{\text{information externality}}+\underbrace{\beta\mathbb{E}\left[V^\tau\left(\mu',\gamma',f'\right)\right].}_{\text{option value of waiting}}
\end{aligned}\tag{44}$$

Expression (44) is a functional equation in τ because V^τ depends implicitly on τ. To show that this functional equation has a solution, we define the following mapping T on the set of continuous and bounded functions to itself such that

$$\begin{aligned}
\left[T\left(V\right)\right]\left(\mu,\gamma,f\right) &= \max\left\{\mathbb{E}\left[u\left(\theta+\varepsilon^x\right)\mid\mu,\gamma\right]-f+\tau\left(\mu,\gamma\right),\beta\mathbb{E}\left[V\left(\mu',\gamma',f'\right)\right]\right\}\\
&\qquad -T\left(\mu,\gamma\right)\\
\text{s.t.}\quad & \tau\left(\mu,\gamma\right)=\beta\frac{d}{dn}\mathbb{E}\left[W\left(\mu',\gamma'\right)\right]+\beta\mathbb{E}\left[V\left(\mu',\gamma',f'\right)\right]\\
& T\left(\mu,\gamma\right)=n^{eff}\left(\mu,\gamma\right)\tau\left(\mu,\gamma\right)\\
& \mu'=\rho_\theta\mu+s\left(n^{eff}\left(\mu,\gamma\right),\gamma\right)\varepsilon\\
& \gamma'=\Gamma\left(n^{eff}\left(\mu,\gamma\right),\gamma\right).
\end{aligned}$$

By standard arguments, this mapping defines a contraction. The maximization yields the following decision: invest if and only if

$$\begin{aligned} f &\leq \mathbb{E}\left[u\left(\theta+\varepsilon^{x}\right) \mid \mu, \gamma\right]+\tau\left(\mu, \gamma\right)-\beta \mathbb{E}\left[V\left(\mu^{\prime}, \gamma^{\prime}, f^{\prime}\right)\right] \\ &\leq \mathbb{E}\left[u\left(\theta+\varepsilon^{x}\right) \mid \mu, \gamma\right]+\beta \frac{d}{d n} \mathbb{E}\left[W\left(\mu^{\prime}, \gamma^{\prime}\right)\right], \end{aligned}$$

which coincides with the efficient cutoff f_c^{eff}. Thus, denoting V^* the only fixed point of this mapping, the investment subsidy $\tau(\mu,\gamma) = \beta \frac{d}{dn}\mathbb{E}_{n^{eff}}\left[W(\mu',\gamma')\right] + \beta\mathbb{E}_{n^{eff}}\left[S(V^*)(\mu',\gamma')\right]$ is a solution to the functional equation (44). It implements the efficient cutoff rule and balances the government budget by construction. An explicit expression for the optimal subsidy can be derived by noticing that V^* satisfies

$$\begin{aligned} \left[S\left(V^{*}\right)\right](\mu, \gamma) &= F\left(f_{c}^{eff}(\mu, \gamma)\right)\left(\mathbb{E}\left[u\left(\theta+\varepsilon^{x}\right) \mid \mu, \gamma\right]-\mathbb{E}\left[f \mid f \leq f_{c}^{eff}(\mu, \gamma)\right]\right) \\ &\quad +\left(1-F\left(f_{c}^{eff}(\mu, \gamma)\right)\right) \beta \mathbb{E}\left[S\left(V^{*}\right)\left(\mu^{\prime}, \gamma^{\prime}\right)\right], \end{aligned}$$

which can be computed from primitives once f_c^{eff} is known.

We have shown that the efficient allocation can be implemented by transfers to investing firms. To complete the proof, we show that these transfers are positive and non-zero in non-trivial cases. More precisely, rewrite the mapping satisfied by these transfers:

$$\tau(\mu, \gamma)=\underbrace{\beta \frac{d}{d n} \mathbb{E}\left[W\left(\mu^{\prime}, \gamma^{\prime}\right)\right]}_{\equiv A(\mu, \gamma)}+\underbrace{\beta \mathbb{E}\left\{\left[S\left(V^{*}\right)\right]\left(\mu^{\prime}, \gamma^{\prime}\right)\right\}}_{\equiv B(\mu, \gamma)}.$$

As long as the efficient allocation is not trivial, i.e. that there exists some (μ,γ,f) at which firms invest (which is guaranteed since f has an unbounded support), term $B(\mu,\gamma)$ is strictly positive for some (μ,γ).

We now prove that A is non-negative. The effect of an increase of n in $\mathbb{E}\left[W(\mu',\gamma')\right]$ is proportional to that of an exogenous arrival of information. The following discussion thus focuses on the impact on welfare of an exogenous arrival of information. It is useful for our purpose to rewrite the planner's problem in a sequential way. A strategy for the planner is a collection of cutoff functions $\{f_0, f_1, \ldots, f_t, \ldots\}$ such that for each date t, f_t maps the set of all past histories of signals up to time t, $\{Y_s, X_s\}_{s=0}^{t}$, to the real line. Pick some date t_0. We are going to show that the exogenous arrival of a signal S of precision γ_S at date t_0 allows the planner to do at least as well as without it, because it can ignore it. Denote $\mathcal{F}_t$ the information set $\{Y_s, X_s\}_{s=0}^{t}$ of the planner at each date without the exogenous signal, and $\mathcal{F}_t^S$ the information set $\left\{Y_s, X_s^S\right\}_{s=0}^{t}$ of the planner when the arrival of the exogenous signal is known and anticipated. Let $\{f_{c,t}\}$ be any strategy considered by the planner without the exogenous signal. Construct the following strategy for the case with exogenous arrival of information:

$$\forall t < t_0, \qquad f^S_{c,t}\left(\{Y_s, X_s\}^t_{s=0}\right) = f_{c,t}\left(\{Y_s, X_s\}^t_{s=0}\right),$$
$$\forall t \geq t_0, \qquad f^S_{c,t}\left(\{Y_s, X_s\}^t_{s=0}, S\right) = f_{c,t}\left(\{Y_s, X_s\}^t_{s=0}\right),$$

so that the two strategies and the information sets $\mathcal{F}_t$ and $\mathcal{F}^S_t$ coincide up to time $t_0 - 1$. After date t_0, strategy f^S_c deliberately ignores the new information. Therefore, by the law of iterated expectations, the two strategies have the same ex-ante payoffs. Welfare can only be increased with the arrival of new information, hence term $A(\mu, \gamma)$ is non-negative.

We conclude that the symmetric, efficient allocation can be implemented with positive transfers. In non-trivial cases, these transfers are strictly positive, which implies that the decentralized economy without transfers is inefficient.

2. The proof that the efficient allocation is subject to uncertainty traps follows closely that of the decentralized case. Thus, we only state the major steps of the proof:

- The optimal cutoff for the planner is defined by:

$$f^{eff}_c(\mu, \gamma) = \mathbb{E}\left[u(\theta + \varepsilon^x) \mid \mu, \gamma\right] + \beta \frac{d}{dN} \mathbb{E}\left[W\left(\rho_\theta \mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)\right)\right].$$

  The first step of the proof is to show that $\frac{d}{dN}\mathbb{E}\left[W\left(\rho_\theta \mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)\right)\right]$ is a $O(\gamma_x)$, so that for γ_x low enough f^{eff}_c is strictly increasing in μ and γ with derivatives that can be bounded away from 0;

- In a second step, show that when $\sigma^f \to 0$, then f^{eff,σ^f}_c converges uniformly to some limit $f^{eff,0}_c$ that is bi-Lipschitz continuous, strictly increasing in μ and γ with derivatives bounded away from 0. Thus, we have the pointwise limit:

$$\forall(\mu, \gamma), \qquad n^{eff,\sigma^f}(\mu, \gamma) \to n^{eff,0}(\mu, \gamma) = \mathbb{I}\left(\omega^f \leq f^{eff,0}_c(\mu, \gamma)\right);$$

- Conclude identically to proposition 3 that for σ^f sufficiently small there are at least two locally stable steady-states in the dynamics of γ.

□

H Additional Lemmas (ONLINE APPENDIX)

This online appendix contains the proofs of the two technical lemmas 3 and 4.

First, we prove the technical lemma that establishes the continuity of the cutoff $f^{\sigma^f}_c$ in σ^f.

Lemma 3. *As $\sigma^f \to 0$, the equilibrium cutoff value $f^{\sigma^f}_c$ converges uniformly towards some limit f^0_c:*

$$\sup_{(\mu,\gamma)\in\mathcal{S}} \left| f^{\sigma^f}_c(\mu, \gamma) - f^0_c(\mu, \gamma) \right| \underset{\sigma^f \to 0}{\longrightarrow} 0$$

and the fraction of investing firms converges pointwise to the following limit:

$$\forall(\mu,\gamma), \quad n^{\sigma^f}(\mu,\gamma) = F^{\sigma^f}\left(f_c^{\sigma^f}(\mu,\gamma)\right) \underset{\sigma^f\to 0}{\longrightarrow} n^0(\mu,\gamma) \equiv \mathbb{1}\left(\omega^f \leq f_c^0(\mu,\gamma)\right).$$

Proof. This proof is similar to the argument developed in proposition 2. Since $n = p = F(f_c(\mu,\gamma))$, we use n and p interchangeably from now on and abuse notation in saying that $\mathcal{M}$ is a mapping for $n : \mathcal{N} \longrightarrow \mathcal{N}$. Pick two different variances for the fixed cost σ_1^f and σ_2^f. The notation $\mathcal{T}^{n,\sigma_i^f}$ denotes the mapping $\mathcal{T}$ for the value function G when n is the aggregate number of investing firms perceived by agents and the fixed costs are distributed according to $F^{\sigma_i^f}$.

Outline of the proof: Starting with the same initial aggregate law n, we compare the objects f_c^{n,σ_1^f} and f_c^{n,σ_2^f} after the first iteration of the mappings $\mathcal{M}^{\sigma_1^f}$ and $\mathcal{M}^{\sigma_2^f}$. In a second step, we establish a recursive relationship to compare the same objects after an arbitrary number of iterations. We then conclude that the limits of both contractions $n^{\sigma_i^f} = \lim_{k\to\infty}\left(\mathcal{M}^{\sigma_i^f}\right)^k n$ produce equilibrium cutoffs that are close in the following sense:

$$\| f_c^{n^{\sigma_2^f},\sigma_2^f} - f_c^{n^{\sigma_1^f},\sigma_1^f} \| \leq \overline{A}\left|\sigma_2^f - \sigma_1^f\right|$$

for some strictly positive constant $\overline{A}$, which suffices to establish the result.

Step 1. Start with some functions G and N, identical for both mappings. Denote $G_k^{n,\sigma_i^f} \equiv \left(\mathcal{T}^{n,\sigma_i^f}\right)^k G$. Let me prove by recursion that:

$$\left|\left(G_k^{n,\sigma_2^f} - G_k^{n,\sigma_1^f}\right)(\mu,\gamma,f)\right| \leq \beta\frac{1-\beta^k}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|.$$

This is trivially true for $k = 0$. Assume it is true for until $k \geq 0$, then:

$$\begin{aligned}
& \left|\left(G_{k+1}^{n,\sigma_2^f} - G_{k+1}^{n,\sigma_1^f}\right)(\mu,\gamma,f)\right| \\
\leq\ & \beta\left|E\left\{\left[S^{\sigma_2^f}\left(G_k^{n,\sigma_2^f}\right)\right](\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma))\right\} - E\left\{\left[S^{\sigma_1^f}\left(G_n^{N,\sigma_1^f}\right)\right](\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma))\right\}\right| \\
\leq\ & \beta\left|E\left\{\left[S^{\sigma_2^f}\left(G_k^{n,\sigma_2^f}\right)\right](\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma))\right\} - E\left\{\left[S^{\sigma_1^f}\left(G_n^{N,\sigma_2^f}\right)\right](\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma))\right\}\right| \\
& +\beta\left|E\left\{\left[S^{\sigma_1^f}\left(G_k^{n,\sigma_2^f}\right)\right](\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma))\right\} - E\left\{\left[S^{\sigma_1^f}\left(G_k^{n,\sigma_1^f}\right)\right](\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma))\right\}\right| \\
\leq\ & \beta\left|\int\left(G_k^{n,\sigma_2^f}\left(\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma), \omega^f + \sigma_2^f\upsilon\right)\right.\right. \\
& \left.\left. -G_k^{n,\sigma_2^f}\left(\rho_\theta\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma), \omega^f + \sigma_1^f\upsilon\right)\right) d\Phi(\varepsilon)\, dF^1\left(\upsilon + \omega^f\right)\right| + \beta\times\beta\frac{1-\beta^k}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| \\
\leq\ & \beta\int\left|\sigma_2^f - \sigma_1^f\right||\upsilon|\, dF^1\left(\omega^f + \upsilon\right) + \beta^2\frac{1-\beta^k}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| \quad \text{(Lipschitz of constant 1 in } f\text{)} \\
\leq\ & \beta\left|\sigma_2^f - \sigma_1^f\right| + \beta^2\frac{1-\beta^k}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| = \beta\frac{1-\beta^{k+1}}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|
\end{aligned}$$

which proves the recursion. Taking the limit $G^{n,\sigma_i^f}=\lim_{k\to\infty}\left(\mathcal{T}^{n,\sigma_i^f}\right)^k G$:

$$\parallel G^{n,\sigma_2^f}-G^{n,\sigma_1^f}\parallel\leq\frac{\beta}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right|. \tag{45}$$

Turning to the equilibrium cutoff rule and using the same argument:

$$\begin{aligned}
&\left|f_c^{n,\sigma_2^f}(\mu,\gamma)-f_c^{n,\sigma_1^f}(\mu,\gamma)\right|\\
=\;&\beta\left|E\left\{\left[S^{\sigma_2^f}\left(G^{n,\sigma_2^f}\right)\right](\rho_\theta\mu+s(n,\gamma)\varepsilon,\Gamma(n,\gamma))\right\}-E\left\{\left[S^{\sigma_1^f}\left(G^{n,\sigma_1^f}\right)\right](\rho_\theta\mu+s(n,\gamma)\varepsilon,\Gamma(n,\gamma))\right\}\right|\\
\leq\;&\frac{\beta}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right|
\end{aligned} \tag{46}$$

Let us now consider the number of investing firms n. Denote $n_k^{\sigma_i^f}\equiv\left(\mathcal{M}^{\sigma_i^f}\right)^k n$, starting from the same arbitrary initial n.

$$\begin{aligned}
&\left|\left(n_1^{\sigma_2^f}-n_1^{\sigma_1^f}\right)(\mu,\gamma)\right|\leq\left|F^{\sigma_2^f}\left(f_c^{n,\sigma_2^f}(\mu,\gamma)\right)-F^{\sigma_1^f}\left(f_c^{n,\sigma_1^f}(\mu,\gamma)\right)\right|\\
&\leq\left|F^{\sigma_2^f}\left(f_c^{n,\sigma_2^f}(\mu,\gamma)\right)-F^{\sigma_2^f}\left(f_c^{n,\sigma_1^f}(\mu,\gamma)\right)+F^{\sigma_2^f}\left(f_c^{n,\sigma_1^f}(\mu,\gamma)\right)-F^{\sigma_1^f}\left(f_c^{n,\sigma_1^f}(\mu,\gamma)\right)\right|
\end{aligned}$$

where we see that $n^{\sigma_2^f}$ may not always be close to $n^{\sigma_1^f}$ under the sup norm. The problem is that the above expression could be close to 1 for a few of points if σ_i^f is low and $f_c^{n,\sigma_2^f}\neq f_c^{n,\sigma_1^f}$. However, we now show that this is not a problem as they will be close on *average*. The only thing we need for the final result is pointwise convergence for n^{σ^f} as $\sigma^f\to 0$.

Step 2. We will now establish a recursive relationship to compare the two objects $f_c^{n_k^{\sigma_1^f},\sigma_1^f}$ and $f_c^{n_k^{\sigma_2^f},\sigma_2^f}$. Assume that after k iterations of the mapping $\mathcal{M}$, we have two different functions $n_k^{\sigma_2^f}$ and $n_k^{\sigma_1^f}$ and that

$$\forall(\mu,\gamma),\quad\left|f_c^{n_k^{\sigma_2^f},\sigma_2^f}(\mu,\gamma)-f_c^{n_k^{\sigma_1^f},\sigma_1^f}(\mu,\gamma)\right|\leq A_k\left|\sigma_2^f-\sigma_1^f\right|.$$

Let us study the following term:

$$\begin{aligned}
&\left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f}-G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\\
\leq\;&\left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f}-G^{n_{k+1}^{\sigma_2^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|+\left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_1^f}-G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\\
\leq\;&\frac{\beta}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right|+\left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_1^f}-G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|
\end{aligned} \tag{47}$$

where we have controlled the first term by the same argument as in (45). We need to study the second term:

$$
\begin{aligned}
& \left| \left(G^{n^{\sigma_2^f}_{k+1}, \sigma_1^f} - G^{n^{\sigma_1^f}_{k+1}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\
= & \left| \left(\lim_{m \to \infty} \left(\mathcal{T}^{n^{\sigma_2^f}_{k+1}, \sigma_1^f} \right)^m G - \lim_{m \to \infty} \left(\mathcal{T}^{n^{\sigma_1^f}_{k+1}, \sigma_1^f} \right)^m G \right) (\mu, \gamma, f) \right| \\
= & \left| \left(\lim_{m \to \infty} G_m^{n^{\sigma_2^f}_{k+1}, \sigma_1^f} - \lim_{m \to \infty} G_m^{n^{\sigma_1^f}_{k+1}, \sigma_1^f} \right) (\mu, \gamma, f) \right| .
\end{aligned}
$$

Starting with the first iteration:

$$
\begin{aligned}
& \left| \left(G_1^{n^{\sigma_2^f}_{k+1}, \sigma_1^f} - G_1^{n^{\sigma_1^f}_{k+1}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\
\leq \ & \beta \left| \int \left[G \left(\rho_\theta \mu + s \left(n^{\sigma_2^f}_{k+1}, \gamma \right) \varepsilon, \Gamma \left(n^{\sigma_2^f}_{k+1}, \gamma \right), f' \right) \right. \right. \\
& \left. \left. - G \left(\rho_\theta \mu + s \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \varepsilon, \Gamma \left(n^{\sigma_1^f}_{k+1}, \gamma \right), f' \right) \right] d\Phi(\varepsilon) \, dF^{\sigma_1^f} (f') \right| \\
\leq \ & \beta \left| \int \left[G \left(\rho_\theta \mu + s \left(n^{\sigma_2^f}_{k+1}, \gamma \right) \varepsilon, \Gamma \left(n^{\sigma_2^f}_{k+1}, \gamma \right), f' \right) \right. \right. \\
& \left. - G \left(\rho_\theta \mu + s \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \varepsilon, \Gamma \left(n^{\sigma_2^f}_{k+1}, \gamma \right), f' \right) \right] d\Phi(\varepsilon) \, dF^{\sigma_1^f} (f') \\
& + \int \left[G \left(\rho_\theta \mu + s \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \varepsilon, \Gamma \left(n^{\sigma_2^f}_{k+1}, \gamma \right), f' \right) \right. \\
& \left. \left. - G \left(\rho_\theta \mu + s \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \varepsilon, \Gamma \left(n^{\sigma_1^f}_{k+1}, \gamma \right), f' \right) \right] d\Phi(\varepsilon) \, dF^{\sigma_1^f} (f') \right| \\
\leq \ & \beta \left[\int \overline{G}_\mu |\varepsilon| \left| s \left(n^{\sigma_2^f}_{k+1}, \gamma \right) - s \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \right| d\Phi(\varepsilon) \, dF^{\sigma_1^f} (f') \right. \\
& \left. + \int \overline{G}_\gamma \left| \Gamma \left(n^{\sigma_2^f}_{k+1}, \gamma \right) - \Gamma \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \right| d\Phi(\varepsilon) \, dF^{\sigma_1^f} (f') \right] \\
\leq \ & \beta \left[\overline{G}_\mu \left| s \left(n^{\sigma_2^f}_{k+1}, \gamma \right) - s \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \right| + \overline{G}_\gamma \left| \Gamma \left(n^{\sigma_2^f}_{k+1}, \gamma \right) - \Gamma \left(n^{\sigma_1^f}_{k+1}, \gamma \right) \right| \right] \\
\leq \ & \beta \gamma_x \left(\overline{G}_\mu B_s + \overline{G}_\gamma B_\Gamma \right) \left| \left(n^{\sigma_2^f}_{k+1} - n^{\sigma_1^f}_{k+1} \right) (\mu, \gamma) \right| \\
\leq \ & \beta C \gamma_x \left| \left(n^{\sigma_2^f}_{k+1} - n^{\sigma_1^f}_{k+1} \right) (\mu, \gamma) \right|
\end{aligned}
$$

where $C = B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma$ is a constant similar to the one we used in proposition 2. We now establish recursively that for $m \geq 2$:

$$
\begin{aligned}
\left| \left(G_m^{n^{\sigma_2^f}_{k+1}, \sigma_1^f} - G_m^{n^{\sigma_1^f}_{k+1}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \leq \ & \beta C \gamma_x \left| \left(n^{\sigma_2^f}_{k+1} - n^{\sigma_1^f}_{k+1} \right) (\mu, \gamma) \right| \\
& + \beta^2 \frac{1 - \beta^{m-1}}{1 - \beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|
\end{aligned}
$$

where constants C and D are those coming from lemma 4 below. Assuming the relationship is true until $m \geq 2$, we have*:*

$$
\begin{aligned}
&\left|\left(G_{m+1}^{n_{k+1}^{\sigma_2^f},\sigma_1^f}-G_{m+1}^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\leq\beta\left|E\left[G_m^{n_{k+1}^{\sigma_2^f},\sigma_1^f}-G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right](\mu',\gamma',f')\right|\\
&\leq\beta\left|\int\left[G_m^{n_{k+1}^{\sigma_2^f},\sigma_1^f}\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right),f'\right)\right.\right.\\
&\left.\left.-G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right|\\
&\leq\beta\left|\int\left[G_m^{n_{k+1}^{\sigma_2^f},\sigma_1^f}\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right),f'\right)\right.\right.\\
&\left.\left.-G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right|\\
&+\beta\left|\int\left[G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right),f'\right)\right.\right.\\
&\left.\left.-G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right|\\
&\leq\beta\int\left|\left(G_m^{n_{k+1}^{\sigma_2^f},\sigma_1^f}-G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right),f'\right)d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right|\\
&+\beta\int\left|\frac{\partial s}{\partial n}\varepsilon\frac{\partial G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}}{\partial\mu}+\frac{\partial\Gamma}{\partial n}\frac{\partial G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f}}{\partial\gamma}\right|\left|\left(n_{k+1}^{\sigma_2^f}-n_{k+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\\
&\leq\beta\left(\beta C\gamma_x\int\left|\left(n_{k+1}^{\sigma_2^f}-n_{k+1}^{\sigma_1^f}\right)\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right)\right)\right|d\Phi(\varepsilon)\right.\\
&\left.+\beta^2\frac{1-\beta^{m-1}}{1-\beta}C\gamma_x\left(A_kD+E\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|\right)\\
&+\beta C\gamma_x\left|\left(n_{k+1}^{\sigma_2^f}-n_{k+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|
\end{aligned}
$$

Using lemma 4, we can control the term:

$$
\int\left|\left(n_{k+1}^{\sigma_2^f}-n_{k+1}^{\sigma_1^f}\right)\left(\rho_\theta\mu+s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right)\right)\right|d\Phi(\varepsilon)\leq\left(A_kD+E\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|.
$$

Therefore:

$$
\begin{aligned}
&\left|\left(G_{m+1}^{n_{k+1}^{\sigma_2^f},\sigma_1^f}-G_{m+1}^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\\
\leq\quad&\beta C\gamma_x\left|\left(n_{k+1}^{\sigma_2^f}-n_{k+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|+\beta^2\frac{1-\beta^m}{1-\beta}C\gamma_x\left(A_kD+E\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|
\end{aligned}
$$

which establishes the recursion. Taking the limit as $m\to\infty$:

$$
\begin{aligned}
&\left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_1^f}-G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\\
\leq\quad&\beta C\gamma_x\left|\left(n_{k+1}^{\sigma_2^f}-n_{k+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|+\frac{\beta^2}{1-\beta}C\gamma_x\left(A_kD+E\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|.
\end{aligned}
\tag{48}
$$

We see that G may not converge pointwise. However, the expectation of G will, which is what we need for our final result. Going back to equation (47):

$$\begin{aligned}
& \left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f} - G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \\
\leq & \left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f} - G^{n_{k+1}^{\sigma_2^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| + \left|\left(G^{n_{k+1}^{\sigma_2^f},\sigma_1^f} - G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \\
\leq & \frac{\beta}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| + \beta C\gamma_x \left|\left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f}\right)(\mu,\gamma)\right| + \frac{\beta^2}{1-\beta} C\gamma_x \left(A_k D + E\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f - \sigma_1^f\right|.
\end{aligned}$$

where we have used equations (45) and (48). Let us turn to the cutoff value:

$$\begin{aligned}
& \left|f_c^{n_{k+1}^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{n_{k+1}^{\sigma_1^f},\sigma_1^f}(\mu,\gamma)\right| \\
= & \beta\left[\mathbb{E}\left\{\left[S^{\sigma_2^f}\left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f}\right)\right]\left(\rho_\theta\mu + s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right)\right)\right\}\right. \\
& \left. -\mathbb{E}\left\{\left[S^{\sigma_1^f}\left(G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)\right]\left(\rho_\theta\mu + s\left(n_{k+1}^{\sigma_1^f},\gamma\right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_1^f},\gamma\right)\right)\right\}\right] \\
\leq & \beta\left|\mathbb{E}\left\{\left[S^{\sigma_2^f}\left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f}\right)\right]\left(\rho_\theta\mu + s\left(n_{k+1}^{\sigma_2^f},\gamma\right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_2^f},\gamma\right)\right)\right\}\right. \\
& \left. -\mathbb{E}\left\{\left[S^{\sigma_2^f}\left(G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)\right]\left(\rho_\theta\mu + s\left(n_{k+1}^{\sigma_1^f},\gamma\right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_1^f},\gamma\right)\right)\right\}\right| \\
& +\beta\left|\mathbb{E}\left\{\left[S^{\sigma_2^f}\left(G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)\right]\left(\rho_\theta\mu + s\left(n_{k+1}^{\sigma_1^f},\gamma\right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_1^f},\gamma\right)\right)\right\}\right. \\
& \left. -\mathbb{E}\left\{\left[S^{\sigma_1^f}\left(G^{n_{k+1}^{\sigma_1^f},\sigma_1^f}\right)\right]\left(\rho_\theta\mu + s\left(n_{k+1}^{\sigma_1^f},\gamma\right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_1^f},\gamma\right)\right)\right\}\right| \\
\leq & \beta\left(\frac{\beta}{1-\beta} + \beta C\gamma_x\left(A_k D + E\sigma_1^f\sigma_2^f\right) + \frac{\beta^2}{1-\beta}C\gamma_x\left(A_k D + E\sigma_1^f\sigma_2^f\right)\right)\left|\sigma_2^f - \sigma_1^f\right| \\
& +\beta\left|\sigma_2^f - \sigma_1^f\right| \\
\leq & \beta\left(\frac{1}{1-\beta} + \frac{\beta}{1-\beta}C\gamma_x\left(A_k D + E\sigma_1^f\sigma_2^f\right)\right)\left|\sigma_2^f - \sigma_1^f\right| \\
\leq & \underbrace{\left[\frac{\beta}{1-\beta}\left(1 + \beta\gamma_x CE\sigma_1^f\sigma_2^f\right) + \frac{\beta^2}{1-\beta}\gamma_x CDA_k\right]}_{\equiv A_{k+1}}\left|\sigma_2^f - \sigma_1^f\right|
\end{aligned}$$

This expression defines a recursive relationship:

$$A_{k+1} = \frac{\beta}{1-\beta}\left(1 + \beta\gamma_x CE\sigma_1^f\sigma_2^f\right) + \frac{\beta^2}{1-\beta}\gamma_x CDA_k$$

which converges to a unique limit $\overline{A}$ as long as $\frac{\beta^2}{1-\beta}\gamma_x CD < 1$ which is true if γ_x is chosen sufficiently small. Taking the limit as $k \to \infty$, we have:

$$\begin{aligned}
\left|f_c^{n^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{n^{\sigma_1^f},\sigma_1^f}(\mu,\gamma)\right| &= \left|f_c^{\lim_{k\to\infty} n_k^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{\lim_{k\to\infty} n_k^{\sigma_1^f},\sigma_1^f}(\mu,\gamma)\right| \\
&\leq \overline{A}\left|\sigma_2^f - \sigma_1^f\right|.
\end{aligned}$$

This tells us that as $\sigma^f \to 0$, the equilibrium cutoff converges uniformly to some limit:

$$\forall\left(\mu,\gamma\right),\quad f_c^{n^{\sigma^f},\sigma^f}\left(\mu,\gamma\right)\to f_c^0\left(\mu,\gamma\right).$$

Turning to the equilibrium entry schedule, n converges pointwise towards the limit:

$$\forall\left(\mu,\gamma\right),\quad n^{\sigma^f}\left(\mu,\gamma\right)=F^{\sigma^f}\left(f_c^{n^{\sigma^f},\sigma^f}\left(\mu,\gamma\right)\right)\underset{\sigma^f\to 0}{\longrightarrow} n^0\left(\mu,\gamma\right)=\mathbb{1}\left(\omega^f\leq f_c^0\left(\mu,\gamma\right)\right).$$

$\square$

Lemma 4. *Suppose two functions f_1 and f_2 are such that $\sup\left|f_2\left(\mu,\gamma\right)-f_1\left(\mu,\gamma\right)\right|\leq A\left|\sigma_2^f-\sigma_1^f\right|$ for some strictly positive constant A. Assume also that both f_i's are continuously differentiable and that $\frac{\partial f_i}{\partial\mu}>\underline{f}_\mu$. Then, for $n^i=F^{\sigma_i^f}\left(f_i\right)$, there exists two strictly positive constants D and E such that for $i=1,2$:*

$$\int\left|\left(n^2-n^1\right)\left(\rho_\theta\mu+s\left(n^i,\gamma\right)\varepsilon,\Gamma\left(n^i,\gamma\right)\right)\right|d\Phi\left(\varepsilon\right)\leq\left(AD+E\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|.$$

Proof. Abusing notation slightly with the convention $s\equiv s\left(n_1^{\sigma_i^f},\gamma\right)$ and $\gamma'\equiv\Gamma\left(n_1^{\sigma_i^f},\gamma\right)$:

$$\begin{aligned}
&\int\left|\left(n^2-n^1\right)\left(\rho_\theta\mu+s\left(n^i,\gamma\right)\varepsilon,\Gamma\left(n^i,\gamma\right)\right)\right|d\Phi\left(\varepsilon\right)\\
=&\int\left|F^{\sigma_2^f}\left(f_2\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)-F^{\sigma_1^f}\left(f_1\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)\right|d\Phi\left(\varepsilon\right)\\
\leq&\underbrace{\int\left|F^{\sigma_2^f}\left(f_2\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)-F^{\sigma_2^f}\left(f_1\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)\right|d\Phi\left(\varepsilon\right)}_{\equiv A_1}\\
&+\underbrace{\int\left|F^{\sigma_2^f}\left(f_1\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)-F^{\sigma_1^f}\left(f_1\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)\right|d\Phi\left(\varepsilon\right)}_{\equiv A_2}
\end{aligned}$$

Let us take care of the first term:

$$\begin{aligned}
A_1&=\int\left|F^{\sigma_2^f}\left(f_2\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)-F^{\sigma_2^f}\left(f_1\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)\right)\right|d\Phi\left(\varepsilon\right)\\
&\leq\int\left[F^{\sigma_2^f}\left(f_2\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)+A\left|\sigma_2^f-\sigma_1^f\right|\right)-F^{\sigma_2^f}\left(f_2\left(\rho_\theta\mu+s\varepsilon,\gamma'\right)-A\left|\sigma_2^f-\sigma_1^f\right|\right)\right]d\Phi\left(\varepsilon\right)
\end{aligned}$$

using equation (46). f_2 is a continuously differentiable, strictly increasing function of μ, so we can proceed to the change of variable $x = f_2\left(\mu + s\varepsilon, \gamma'\right)$:

$$\begin{aligned}
A_1 &\leq \int \left[F^{\sigma_2^f}\left(x + A\left|\sigma_2^f - \sigma_1^f\right|\right) - F^{\sigma_2^f}\left(x - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right] \underbrace{\frac{\Phi'\left(\left(f_2\right)^{-1}(x)\right)dx}{s\cdot\left(f_2\right)'\left(\left(f_2\right)^{-1}(x)\right)}}_{\equiv d\varphi(x)} \\
&\leq \int_{x=-\infty}^{\infty}\int_{f=x-A\left|\sigma_2^f-\sigma_1^f\right|}^{x+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f} d\varphi(x) \leq \int_{f=-\infty}^{\infty}\int_{x=f-A\left|\sigma_2^f-\sigma_1^f\right|}^{f+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f} d\varphi(x) \\
&\leq \int_{f=-\infty}^{\infty}\left[\varphi\left(f + A\left|\sigma_2^f - \sigma_1^f\right|\right) - \varphi\left(f - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right] dF^{\sigma_2^f} \\
&\leq \int_{f=-\infty}^{\infty}\left[\varphi'\left(\tilde{f}\right)2A\left|\sigma_2^f - \sigma_1^f\right|\right] dF^{\sigma_2^f} \quad \text{(mean value theorem)} \\
&\leq 2A\left|\sigma_2^f - \sigma_1^f\right|\int_{f=-\infty}^{\infty}\left|\varphi'\left(\tilde{f}\right)\right| dF^{\sigma_2^f} \\
&\leq A\cdot 2\frac{\sup\left|\Phi'\right|}{\inf\left|s\cdot\left(f_2\right)'\right|}\left|\sigma_2^f - \sigma_1^f\right| \equiv AC\left|\sigma_2^f - \sigma_1^f\right|
\end{aligned}$$

where we have used the fact the PDF of a unit normal is bounded, $s \equiv s\left(n_1^{\sigma_i^f}, \gamma\right)$ is uniformly bounded from below and away from 0, and the derivative of f_2 is strictly positive, uniformly bounded away from 0 for γ_x small enough. Notice that the upper bound we derived is uniform: it does not depend on μ, γ, γ_x, etc. Let us control the second term A_2:

$$\begin{aligned}
A_2 &= \int\left|F^{\sigma_2^f}\left(f_1\left(\rho_\theta\mu + s\varepsilon, \gamma'\right)\right) - F^{\sigma_1^f}\left(f_1\left(\rho_\theta\mu + s\varepsilon, \gamma'\right)\right)\right| d\Phi(\varepsilon) \\
&\leq \int\left|F^{\sigma_2^f}(x) - F^{\sigma_1^f}(x)\right| d\varphi(x) \quad \text{(change of variable } x = f_1\left(\rho_\theta\mu + s\varepsilon, \gamma'\right)\text{)} \\
&\leq \int\left|\Phi\left(\frac{x-\omega^f}{\sigma_2^f}\right) - \Phi\left(\frac{x-\omega^f}{\sigma_1^f}\right)\right| d\varphi(x) \quad \text{(change of variable } x = \sigma_1^f\sigma_2^f\tilde{x} + \omega^f\text{)} \\
&\leq \int\left|\Phi\left(\sigma_1^f\tilde{x}\right) - \Phi\left(\sigma_2^f\tilde{x}\right)\right|\sigma_1^f\sigma_2^f d\varphi\left(\sigma_1^f\sigma_2^f\tilde{x} + \omega^f\right) \\
&\leq \left[\int\left|\Phi'(\hat{x})\,\tilde{x}\right| d\varphi\left(\sigma_1^f\sigma_2^f\tilde{x} + \omega^f\right)\right]\sigma_1^f\sigma_2^f\left|\sigma_2^f - \sigma_1^f\right| \equiv D\sigma_1^f\sigma_2^f\left|\sigma_2^f - \sigma_1^f\right|,
\end{aligned}$$

which concludes the proof of the lemma. □