NBER WORKING PAPER SERIES

QUANTIFYING INTERNATIONAL PRODUCTION SHARING AT THE BILATERAL AND SECTOR LEVELS

Zhi Wang
Shang-Jin Wei
Kunfu Zhu

Working Paper 19677
http://www.nber.org/papers/w19677

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
November 2013

The views in the paper are those of the authors and may not reflect the views of the USITC and its Commissioners, the National Bureau of Economic Research, or any other organization that the authors are affiliated with. We thank Peter Dixon for constructive discussions and Ellen Lan Lin and Nikhil Patel for editorial assistance.

Quantifying International Production Sharing at the Bilateral and Sector Levels
Zhi Wang, Shang-Jin Wei, and Kunfu Zhu
NBER Working Paper No. 19677
November 2013, Revised March 2014
JEL No. F1,F15

ABSTRACT

This paper generalizes the gross exports accounting framework at the country level, recently proposed by Koopman, Wang, and Wei (2014), to one that decomposes gross trade flows (for both exports and imports) at the sector, bilateral, or bilateral sector level. We overcome major technical challenges for such generalization by allocating bilateral intermediate trade flows into their final destination of absorption. We also point out two major shortcomings associated with the VAX ratio concept often used in the literature and ways to overcome them. We present the dis-aggregated decomposition results for bilateral sector level gross trade flows among 40 trading nations in 35 sectors from 1995 to 2011 based on the WIOD database.

Zhi Wang
Research Division
Office of Economics
US International Trade Commission
500 E Street SW
Washington, DC 20436
zhi.wang@usitc.gov

Shang-Jin Wei
Graduate School of Business
Columbia University
Uris Hall 619
3022 Broadway
New York, NY 10027-6902
and NBER
shangjin.wei@columbia.edu

Kunfu Zhu
Academy of Mathematics and System Sciences
Chinese Academy of Sciences
Beijing, China
zhukunfu@gmail.com

1. Introduction

The paper aims to develop the first disaggregated accounting framework that decomposes gross trade, at the sector, bilateral, and bilateral sector levels, into the sum of various value added and double counted items. It generalizes all previous attempts in the literature on this topic and corrects some conceptual errors. Importantly, it goes beyond extracting value added exports from gross exports, and recovers additional useful information about the structure of international production sharing at a disaggregated level. Estimating value added exports can be accomplished by directly applying the original Leontief (1935) insight, which does not require decomposing international intermediate trade flows, and it has been successfully done in the literature. Recovering additional information on the structure of international production sharing from official statistics requires going beyond a simple application of the Leontief insight and finding a way to decompose international intermediate trade itself into value added and double counted terms at a disaggregated level, which has never been done (correctly) before in the literature. The additional structural information can be used to develop a measure of a sector's position in an international production chain that also varies by country, and a new and improved measure of revealed comparative advantage that takes into account both offshoring and domestic production sharing. Of course, the list of possible applications goes beyond these two examples. Finally, the paper produces several new panel trade databases covering 40 economies and 35 sectors over 1995-2011 by applying the disaggregated decomposition framework to the World Input-Output Database (WIOD). While the paper does not directly investigate the causes or consequences of patterns of international production sharing, the disaggregated accounting framework developed in the paper and the new databases that are derived from the framework should enrich the set of possible future research on these topics.

As more and more firms choose to offshore parts and services to suppliers in other countries and organize production on a global scale, production segmentation across national borders has become an important feature of contemporary world economy. An active and growing literature has been devoted to measuring different aspects of such cross-country production sharing phenomena including Feenstra (1998), Feenstra and Hanson (1998), Feenstra and Jensen (2009), Hummels, Ishii, and Yi (2001), Yi (2003), Daudin et al (2011), Johnson and Noguera (2012), Stehrer, Foster, and de Vries Lopez-Gonzalez (2012), Antras (2013), Antras and Chor (2013), Antras et al (2012), Baldwain and Lopez-Gonzalez (2013), Baldwain and Nicoud (2014), and Timmer, et. al (2013), among others. The key concepts proposed in these papers include vertical specialization (VS for short) or its variations such as VS1 and VS1*, and value added exports (VAX for short) or VAX to gross export ratio. The precise relationships among these concepts are established in a recent contribution by Koopman, Wang, and Wei (2014, subsequently referred to as KWW). A collection of papers in the volume edited by Mattoo, Wang, and Wei (2013) represents some of the latest thinking on the subject from both the scholarly community and international policy institutions such as the World Trade Organization, the OECD, the International Monetary Fund and the World Bank.

In a survey of work on quantifying global production sharing in recent years, Antras (2013, Chapter 1, page 6) calls the value added exports to gross export ratio, or the VAX ratio, as proposed by Johnson and Noguera (2012) – the "state of art" and "an appealing inverse measure of the importance of vertical specialization in the world production." However, the VAX ratio concept has two shortcomings. First, as we will point out, the VAX ratio, as currently defined in the literature, is not well behaved at either sector, bilateral, or bilateral sector level[1]. The key to understanding this point is a distinction between a forward-linkage based measure of value added exports, which includes indirect exports of a sector's value added via gross exports from other sectors of the same exporting country, and a backward-linkage based measure of value added exports, which is value added from all sectors of a given exporting country embodied in a given sector's gross exports. For example, a forward-linkage based measure of value added exports in the US electronics sector includes that sector's value added embodied in US gross exports from automobile and chemical sectors, but excludes the value added contributions from these sectors embodied in the gross exports of US electronics. In comparison, a backward-linkage based measure of US value added embodied in US electronics exports includes value added contributions from other US sectors such as services and automobiles to the production of US electronics gross exports, but excludes the value

[1] The VAX ratio at these levels is not upper-bounded by one. Indeed, it can take on the value of infinity when the gross exports are zero. An alternative measure that arises from our framework will be naturally bounded between zero and one at any level of disaggregation.

added contributions from the US electronics sector to the gross exports of other sectors such as US automobiles. Such a distinction is critical to properly define the VAX ratio at the sector, bilateral, or bilateral sector level (the VAX ratio cited in the literature is forward-linkages based, so it is not well behaved), but the distinction disappears at the country aggregate level (and hence the VAX ratio is only well behaved at this level). We advocate using the ratio of a country-sector's value added that is exported and stays abroad as a measure of international production sharing. Such a measure is always bounded between zero and 100% even at the bilateral, sector, or bilateral sector level. Since most cross-country production sharing occurs at the bilateral-sector or country-sector level, this may be a better "inverse measure of vertical specialization in the world production" at the sector or bilateral level.

Second, the VAX ratio, even after it is properly re-defined, still does not capture some of the important features of international production sharing. Let us consider a hypothetical example: both the US and Chinese electronics exports to the world can have an identical ratio of value added exports to gross exports (say, 50% for each) but for very different reasons. In the Chinese case, the VAX ratio is 50% because half of the Chinese gross exports reflect foreign value added (say value added from Japan, Korea, or even the United States). In contrast, for the US exports, half of the gross exports are US value added in intermediate goods that are used by other countries to produce goods that are exported back to the United States. So only half of the US value added that is initially exported is ultimately absorbed abroad; the US VAX ratio is 50% even if it does not use any foreign value added in the production of its electronics exports. In this example, China and the United States occupy very different positions on the global value chain but the two countries' VAX ratios would not reveal this important difference. To provide such additional information, the decomposition framework we propose will go beyond just simply extracting value added trade from gross trade statistics.

KWW (2014) have made the first effort in this direction by providing a unified framework to decompose a country's total gross exports into nine value-added and double counted components. Conceptually, the nine components can be grouped into four buckets. The first bucket gives a country's value added exports that are absorbed abroad, identical to "value added exports" as defined by Johnson and Noguera (2012). The second bucket gives the part of a country's domestic value added that is first exported but eventually returned home. While it is not a part of a country's exports of value added that stays abroad, it is a part of the exporting country's GDP. The third bucket is foreign value added that is used in the production of a country's exports and eventually absorbed by other countries. The forth bucket consists of what KWW call "pure double counted terms," arising from intermediate goods that cross border multiple times. Some of the terms in the fourth bucket double count value added originated in the home country, while others double count value added originated in foreign countries. Other measures of international production sharing in the existing literature such as VS, VS1, VS1*, and VAX ratio are shown to be some linear combinations of the terms in KWW's decomposition formula.

While the KWW method already has many useful applications, an important limitation of the approach is that the gross trade decomposition is only done at the aggregate level, not at the sector, bilateral, or bilateral sector level[2]. Major challenges exist to generalize the framework in that direction. In producing exports in any given sector, not only value added from other sectors in the same country will be used, but also value added produced by potentially all sectors in other countries also need to be accounted for. Such an accounting framework has never been developed in a consistent and comprehensive way before, which is the goal of this paper.

Generalizing the KWW approach to the bilateral/sector level is not a trivial exercise; it cannot be achieved by simply applying the KWW gross exports decomposition formula to bilateral/sector level data. Conceptually, domestic value added can be decomposed from both the producer (forward-linkage based) and the user's perspectives (backward linkage based). On one hand, domestic value added created in a home sector can be exported indirectly through other sectors' gross exports; on the other hand, domestic value added that is embedded in a sector's gross exports can include value added from other home sectors. These two concepts are obviously related but keeping track at the bilateral and sector details is challenging. Mathematically, additional adjustment terms have to be derived to properly account for other sectors/countries' value-added

[2] The calculation of domestic value added that is ultimately absorbed abroad can be done at the bilateral and sector level. Indeed, some examples are given in KWW (2014). However, the computations of the other three components that could sum to 100% bilateral/sector trade flows are not done in KWW.

contributions to a given country-sector's gross exports, in addition to properly measuring how that country-sector's value-added is used in its own intermediate and final goods exports, so that all its value added and double counted components can sum to 100% of gross exports at the bilateral-sector level. What makes the earlier work (KWW) at the country aggregate level relatively easier is that the difference between the decomposition from the producer and user's perspectives disappears after aggregating to the economy-wide level. A useful decomposition formula also has to have the property that all the decomposition terms from the bilateral/sector level gross trade flows must be internally consistent so that they can sum up to the decomposition equation given in KWW at the aggregate level.

This paper's main contribution is to provide a new and comprehensive methodological framework that decomposes bilateral sector level gross trade flows into various value-added and double counted terms. While it does not directly examine causes and consequences of changing structure of cross country production sharing, reliable measurements made possible by such an accounting methodology are necessary for investigating these research questions. The second contribution of the paper is to generate a new database on disaggregated bilateral, sector, and bilateral sector trade flows in both value added and various double counted terms.

The paper is organized as follows. Section 2 presents a derivation of our methodological framework, starting with two sectors and two and three country cases, and ending with the most general model of G countries and N sectors. Section 3 reports selected empirical decomposition results based on the World Input-Output Database (WIOD) and discusses how bilateral/sector level gross trade accounting results may help to measure international production sharing or a particular country/sector's position in global production network. Section 4 provides some concluding remarks.

2. Concepts and Methodology

2.1 The Leontief insight and its limitations

All the decomposition methods in the recent vertical specialization and trade in value added literatures are rooted in Leontief (1936). His work demonstrated that the amount and type of intermediate inputs needed in the production of one unit of output can be estimated based on the input-output (IO) structures across countries and industries. Using the linkages across industries and countries, gross output in all stages of production that is needed to produce one unit of final goods can be traced. When the gross output flows (endogenous in standard IO model) associated with a particular level of final demand (exogenous in standard IO model) are known, value added production and trade can be simply derived by multiplying these flows with the value added to gross output ratio in each country/industry.

To better understand how Leontief insight is applied, let us assume a two-country (home and foreign) world, in which each country produces goods in N differentiated tradable industries. Goods in each sector can be consumed directly or used as intermediate inputs, and each country exports both intermediate and final goods.

All gross output produced by Country *s* must be used as either an intermediate good or a final good at home or abroad, or

$$X^s = A^{ss}X^s + Y^{ss} + A^{sr}X^r + Y^{sr} \quad r, s = 1,2 \tag{1}$$

Where X^s is the N×1 gross output vector of Country s, Y^{sr} is the N×1 final demand vector that gives demand in Country r for final goods produced in s, and A^{sr} is the N×N IO coefficient matrix, giving intermediate use in r of goods produced in s. The two-country production and trade system can be written as an ICIO model in block matrix notation

$$\begin{bmatrix} X^s \\ X^r \end{bmatrix} = \begin{bmatrix} A^{ss} & A^{sr} \\ A^{rs} & A^{rr} \end{bmatrix} \begin{bmatrix} X^s \\ X^r \end{bmatrix} + \begin{bmatrix} Y^{ss} + Y^{sr} \\ Y^{rs} + Y^{rr} \end{bmatrix} \tag{2}$$

After rearranging terms, we have

$$\begin{bmatrix} X^s \\ X^r \end{bmatrix} = \begin{bmatrix} I - A^{ss} & -A^{sr} \\ -A^{rs} & I - A^{rr} \end{bmatrix}^{-1} \begin{bmatrix} Y^{ss} + Y^{sr} \\ Y^{rs} + Y^{rr} \end{bmatrix} = \begin{bmatrix} B^{ss} & B^{sr} \\ B^{rs} & B^{rr} \end{bmatrix} \begin{bmatrix} Y^s \\ Y^r \end{bmatrix} \tag{3}$$

where B^{sr} denotes the N×N block matrix, commonly known as a Leontief inverse, which is the total requirement matrix that gives the amount of gross output in producing Country s required for a one-unit increase in final demand in Country r. Y^s is an N×1 vector that gives global use of s' final goods, including domestic final goods sales Y^{ss} and final goods exports Y^{sr}. The relationship expressed in (3) is the Leontief insight. The intuition behind the expression is as follows: when $1 of export is produced, a first round of value added is generated. This is the direct domestic value added induced by the $1 export. To produce that export, intermediate inputs have to be used. The production of these intermediate inputs also generates value added. This is the second round or indirect domestic value added induced by the $1 export. Such a process to generate indirect value added continues and can be traced to additional rounds of production throughout the economy, as intermediate inputs are used to produce other intermediate inputs. The total domestic value added induced by the $1 export thus is equal to the sum of direct and all rounds of indirect domestic value added generated from the $1 of export production process. Expressing this process mathematically using the terms defined above, we have

$$\begin{aligned} &DVS = V + VA + VAA + VAAA + ... = V(I + A + A^2 + A^3 + ...) \\ &= V(I - A)^{-1} = VB \end{aligned} \tag{4}$$

It can be shown that the power series of matrix A is convergent and the inverse matrix $B = (I - A)^{-1}$ exists as long as A is in full rank (Miller and Jones, 2009).

Define V^s as a 1×N direct value-added coefficient vector. Each element of V^s gives the share of direct domestic value added in total output. This is equal to one minus the intermediate input share from all countries (including domestically produced intermediates):

$$V^s = u[I - A^{ss} - A^{rs}] \tag{5}$$

where u is a 1×N unity vector. When N=2, the corresponding inter-country input-output (ICIO) account can be described by Table 1 below.

Table 1: 2-Country and 2-Sector ICIO Table

	Country	Intermediate Use				Final Demand		Total gross output
		S		R		Y^s	Y^r	
Country	Sector	s1	s2	r1	r2			
S	s1	z_{11}^{ss}	z_{12}^{ss}	z_{11}^{sr}	z_{12}^{sr}	y_1^{ss}	y_1^{sr}	x_1^s
	s2	z_{21}^{ss}	z_{22}^{ss}	z_{21}^{sr}	z_{22}^{sr}	y_2^{ss}	y_2^{sr}	x_2^s
R	r1	z_{11}^{rs}	z_{12}^{rs}	z_{11}^{rr}	z_{12}^{rr}	y_1^{rs}	y_1^{rr}	x_1^r
	r2	z_{21}^{rs}	z_{22}^{rs}	z_{21}^{rr}	z_{22}^{rr}	y_2^{rs}	y_2^{rr}	x_2^r
Value-added		va_1^s	va_2^s	va_1^r	va_2^r			
Total input		x_1^s	x_2^s	x_1^r	x_2^r			

Where x_1^s is gross output of the 1st sector in Country s, va_1^s is direct value added of the 1st sector in Country s, y_1^{sr} is final goods produced by the 1st sector in Country s for consumption in Country r, and z_{11}^{sr} is intermediate goods produced in the 1st sector of Country s and used for the 1st sector production in Country r. Other variables can be interpreted similarly. Equations (2) and (3) can be re-written as follows:

$$\begin{bmatrix} x_1^s \\ x_2^s \\ x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} a_{11}^{ss} & a_{12}^{ss} & a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{ss} & a_{22}^{ss} & a_{21}^{sr} & a_{22}^{sr} \\ a_{11}^{rs} & a_{12}^{rs} & a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rs} & a_{22}^{rs} & a_{21}^{rr} & a_{22}^{rr} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \\ x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} y_1^{ss} + y_1^{sr} \\ y_2^{ss} + y_2^{sr} \\ y_1^{rs} + y_1^{rr} \\ y_2^{rs} + y_2^{rr} \end{bmatrix} \tag{2a}$$

$$\begin{aligned} \begin{bmatrix} x_1^s \\ x_2^s \\ x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} y_1^{ss} + y_1^{sr} \\ y_2^{ss} + y_2^{sr} \\ y_1^{rs} + y_1^{rr} \\ y_2^{rs} + y_2^{rr} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{ss} + y_1^{sr} \\ y_2^{ss} + y_2^{sr} \\ y_1^{rs} + y_1^{rr} \\ y_2^{rs} + y_2^{rr} \end{bmatrix} \end{aligned} \tag{3a}$$

where a_{11}^{sr} is the direct IO coefficient that gives units of the intermediate goods produced in the 1st sector of Country s that are used in the production of one unit of gross output in the 1st sector of Country r, b_{11}^{ss} is the total IO coefficient that gives the total amount of the gross output of 1st sector in Country s needed to produce an extra unit of the 1st sector's final good in Country s (which is for consumption in both Countries s and r). Other coefficients have similar economic interpretations.

The direct value added coefficient vector (equation 5) can be re-written as follows:

$$v_j^c \equiv va_j^c / x_j^c = 1 - \sum_i^2 a_{ij}^{sc} - \sum_i^2 a_{ij}^{rc} \quad (c = s, r \quad j = 1,2). \tag{5a}$$

Then we can define the total value added coefficient (*VB*) matrix, or the total value added multiplier as named in the input-output literature:

$$VB = \begin{bmatrix} v_1^s & v_2^s & v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} + v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} + v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \\ v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} + v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} + v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix}^T \tag{6}$$

where T denotes matrix transpose operation. Each element of the last term in VB equals unity.

Condensing the final demand vector in (3a) as:

$$\begin{bmatrix} y_1^{ss} + y_1^{sr} & y_2^{ss} + y_2^{sr} & y_1^{rs} + y_1^{rr} & y_2^{rs} + y_2^{rr} \end{bmatrix}^T = \begin{bmatrix} y_1^s & y_2^s & y_1^r & y_2^r \end{bmatrix}^T$$

The decomposition of the country/sector level value-added and final goods production as a direct application of the Leontief insight can be expressed as follows:

$$\begin{aligned} \hat{V}B\hat{Y} &= \begin{bmatrix} v_1^s & 0 & 0 & 0 \\ 0 & v_2^s & 0 & 0 \\ 0 & 0 & v_1^r & 0 \\ 0 & 0 & 0 & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^s & 0 & 0 & 0 \\ 0 & y_2^s & 0 & 0 \\ 0 & 0 & y_1^r & 0 \\ 0 & 0 & 0 & y_2^r \end{bmatrix} \\ &= \begin{bmatrix} v_1^s b_{11}^{ss} y_1^s & v_1^s b_{12}^{ss} y_2^s & v_1^s b_{11}^{sr} y_1^r & v_1^s b_{12}^{sr} y_2^r \\ v_2^s b_{21}^{ss} y_1^s & v_2^s b_{22}^{ss} y_2^s & v_2^s b_{21}^{sr} y_1^r & v_2^s b_{22}^{sr} y_2^r \\ v_1^r b_{11}^{rs} y_1^s & v_1^r b_{12}^{rs} y_2^s & v_1^r b_{11}^{rr} y_1^r & v_1^r b_{12}^{rr} y_2^r \\ v_2^r b_{21}^{rs} y_1^s & v_2^r b_{22}^{rs} y_2^s & v_2^r b_{21}^{rr} y_1^r & v_2^r b_{22}^{rr} y_2^r \end{bmatrix} \end{aligned} \tag{7}$$

This matrix gives the estimates of sector and country sources of value added in each country's final goods production. Each element in the matrix represents the value added from a source sector of a source country directly or indirectly used in the production of final goods (absorbed in both the domestic and foreign markets) in the source country. Looking at the matrix along the row yields the distribution of value added created from one country/sector used across all countries/sectors. For example, the first element of the first row, $v_1^s b_{11}^{ss}(y_1^{ss} + y_1^{sr})$ is value added created in Country s' 1st sector embodied in its final goods production for both the 1st sector's domestic sales and exports. The second element, $v_1^s b_{12}^{ss}(y_2^{ss} + y_2^{sr})$, is Country s' value added from the 1st sector embodied in its 2nd sector's final goods production. The third and fourth elements, $v_1^s b_{11}^{sr}(y_1^{rs} + y_1^{rr})$ and $v_1^s b_{12}^{sr}(y_2^{rs} + y_2^{rr})$, are Country s' value added from the 1st sector embodied in Country r's final goods production in its 1st and 2nd sector respectively. Therefore, summing up the first row of the matrix, we have Country s' total value added created by production factors employed in its 1st sector. In other words, it equals GDP by industry of the 1st sector in Country s. Expressing this mathematically,

$$
\begin{aligned}
& va_1^s \text{ or } GDP_1^s = v_1^s x_1^s = v_1^s (b_{11}^{ss} y_1^s + b_{12}^{ss} y_2^s + b_{11}^{sr} y_1^r + b_{12}^{sr} y_2^r) \\
& = \left[v_1^s b_{11}^{ss} y_1^{ss} + v_1^s b_{12}^{ss} y_2^{ss} + v_1^s b_{11}^{ss} y_1^{sr} + v_1^s b_{12}^{ss} y_2^{sr} \right] + \left[v_1^s b_{11}^{sr} y_1^{rs} + v_1^s b_{12}^{sr} y_2^{rs} + v_1^s b_{11}^{sr} y_1^{rr} + v_1^s b_{12}^{sr} y_2^{rr} \right]
\end{aligned} \tag{8}
$$

Looking at the $\hat{V}B\hat{Y}$ matrix along a column yields the contributions of value added from all countries/sectors to the final goods produced by a particular country/sector. For example, the second element in the first column, $v_2^s b_{21}^{sr} (y_1^{ss} + y_1^{sr})$,is Country s' value added created from the 2nd sector embodied in Country s' production of its 1st sector's final goods, and the third and fourth elements, $v_1^r b_{12}^{rs} (y_1^{ss} + y_1^{sr})$ and $v_2^r b_{21}^{rs} (y_1^{ss} + y_1^{sr})$ are Country r's (foreign) value added embodied in Country s' production of its 1st sector's final goods. Adding up all elements in the first column equals the total value of final goods production by Country s' 1st sector, i.e:

$$
(v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} + v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs}) y_1^s = y_1^s \tag{9}
$$

In summary, the sum of the $\hat{V}B\hat{Y}$ matrix across columns along a row accounts for how each country's domestic value-added originated in a particular sector is used by the sector itself and all its downstream countries/sectors. It traces forward linkages across all downstream countries/industries from a supply-side perspective. Since the sum of the $\hat{V}B\hat{Y}$ matrix across the rows along a column accounts for all upstream countries/sectors' value-added contributions to a specific country/sector's final goods output, it traces backward linkages across upstream countries/industries from a user's perspective. Based on the identity given by equation (6), all these sources should sum to 100% of the value of the final products for any given country/sector.

Therefore, the supply-side perspective (summing across columns along a row) decomposes how each country's GDP by industry is used, directly and indirectly to satisfy domestic and foreign final demand, while the user-side perspective (summing across rows along a column) decomposes a country/sector's final goods and services into its original country/sector sources. As an example, in the electronics sector, the supply-side perspective includes the value added created by production factors employed at the electronics sector and incorporated into the gross exports of electronics itself (direct domestic value-added exports), as well as in the exports of computers, consumer appliances, and automobiles (indirect domestic value-added exports). In other words, it decomposes GDP (domestic value-added) by industries according to where (i.e., which sector/country) it is used. Such a forward linkage perspective is consistent with the literature on factor content of trade. On the other hand, decomposition from a user's perspective includes all upstream sectors/countries' contributions to value added in a specific sector/country's exports. In the electronics sector, it includes value added in the electronics sector itself as well as value added in inputs from all other upstream sectors/countries (such as glass from country A, rubber from country B, transportation and design from the home country) used to produce electronics for exports by the home country (direct/indirect domestic value added in exports and foreign value-added in exports). Such a backward linkage based perspective aligns well with case studies of supply chains of specific sectors and products, as the iPod or iPhone examples frequently cited in the literature.

These two different ways to decompose value added and final goods production each have their own economic interpretations and thus different roles in economic analysis. While they are equivalent in the aggregate due to the identity of global value-added production equals global final demand[3], they are not equal at the sector, or bilateral sector level.

After understanding how value added (GDP) and final goods production at the sector level can be correctly decomposed based on the Leontief insight (equation (7) or the $\hat{V}B\hat{Y}$ matrix), we can better understand various decomposition methods proposed in the literature.

There are several attempts to estimate trade in value added and to decompose value added and final goods production based on the Leontief insight and ICIO database in recent years. Timmer et al (2013) decompose final goods production based on backward linkages. For example, their method provides estimates on how much contribution an unskilled worker employed in the Chinese steel industry makes to cars produced in Germany, or how much contribution a skilled US worker in the electronics industry made

[3] See the proofs in equation (17) in Koopman, Wang and Wei (2014) and Appendix F of this paper.

to a computer consumed by Chinese households. Johnson and Noguera (2012) estimate value-added content of trade based on forward linkages. However, they did not recognize that this forward linkage method would yield a VAX ratio concept that is not well defined at the sector or bilateral level, because domestic value added from other domestic sectors is not included in the forward linkage calculations, but are quantitatively important for a typical country/sector. In addition, the VAX ratio as defined by Johnson and Noguera (2012) at the sector level is not bounded by 100% for sectors with relatively small gross exports. Indeed, the VAX ratio as defined this way could be infinite for sectors that do not directly export. We will show that an alternative way of defining the VAX ratio based on backward linkages could yield a concept that is bounded between zero and one even at the sector, bilateral, and bilateral sector level.

In any case, if one is only interested in computing domestic value added embedded in a country/sector's gross exports that is ultimately absorbed abroad, applying the Leontief insight is sufficient. However, as pointed out in the introduction, for many economic applications, one needs to quantify other components of the gross exports at the sector, bilateral, and bilateral sector levels. In such circumstances, the original Leontief insight is not sufficient as it does not provide a way to decompose intermediate goods flows across countries into various value added terms according to their final absorption as it does to decompose sector-level value added and final goods production as illustrated by equation (7).

In Leontief's time from the1930s to the1960s, intermediate goods trade is relatively unimportant. Today, it is about two thirds of the world gross trade. So being able to decompose intermediate goods trade has become crucial in generating a complete value added accounting of gross trade flows. KWW has made a useful step to perform such decomposition at the country aggregate level. But as we pointed out earlier, it is not necessary to keep track of forward and backward linkages across countries and industries separately at that level, which makes the job easier. However, one has to confront such technical challenges in decomposing gross trade flows at the sector, bilateral, or bilateral sector level, which has to go beyond original Leontief insight as we will demonstrate in details below.

For ease of understanding, we continue our discussion with the two-country, two-sector ICIO model specified above to illustrate the Leontief insight. We first lay out the basic gross output and exports accounting identities at the sector level and then propose a way to fully decompose a country's gross exports into the sum of components that include both the country's domestic value added in exports and various double-counted components. We then use a three-country, two-sector model to discuss what additional components will be involved once third country effects are taken into account. Finally, we present the G-country N-sector model briefly and highlight how our decomposition formula in this most general case is different from that in the three-country two-sector model. We also provide numerical example following our analytical model to show intuitively how our accounting equation works.

2.2 Decompose intermediate and gross trade: the 2-country 2 Sector Case

The gross exports of Country s can be decomposed into two parts: final goods exports and intermediate goods exports based on the following accounting identity:

$$E^{sr} = \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} = \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \tag{10}$$

As we have already shown in previous section that the final goods exports can be easily decomposed into domestic and foreign value-added by directly applying the Leontief insight. However, the decomposition of intermediate goods exports is more complex. It cannot be achieved by simply multiplying the Leontief inverse with the gross intermediate exports (which leads to double counting) because the latter has to be solved from the ICIO models first for any given final demand level. To overcome this problem, all intermediate goods trade needs to be expressed as different countries' final demand according to where they are absorbed before they can be consistently decomposed. This is what we are going to do next.

Based on equation (3a), the gross output of Country r can be decomposed into the following four components according to where they are finally absorbed:

$$\begin{aligned} X^{r} &= \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} = \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} + y_1^{sr} \\ y_2^{ss} + y_2^{sr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} + y_1^{rs} \\ y_2^{rr} + y_2^{rs} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \end{aligned} \tag{11}$$

Insert equation (11) into the last term of equation (10), we can decompose Country s' gross intermediate goods exports according to where they are absorbed as:

$$A^{sr}X^{r} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix}$$
$$+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \quad (12)$$

The first term in equation (12) is the part of Country s' intermediate goods exports used by Country r to produce domestic final goods that are eventually consumed in Country r; the second term is the part of Country s' intermediate goods exports to Country r that are embedded in Country r's final goods exports back to Country s; the third term is the part of Country s' intermediate goods exports that are used by Country r to produce intermediate exports, shipped to and used by Country s to produce its domestic consumed final goods; the last term is the part of country s' intermediate goods exports used by Country r to produce intermediate goods exports that are shipped to Country s to produce final goods exports to Country r. These four terms collectively decompose completely Country s' intermediate exports according to where they are finally absorbed.

From equation (2), the gross output production and use balance conditions, we know

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \quad (13)$$

Re-arranging terms:

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \quad (14)$$

Define:

$L^{rr} = \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1}$ as the local Leontief inverse, then equation (14) can be re-written as

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \tag{15}$$

Insert equation (15) into the last term in equation (10), the intermediate goods exports by Country s can also be decomposed into the following two components according to where it is used, similar to a single country IO model:

$$A^{sr} X^r = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \tag{16}$$

Equations (12) and (16), each completely decomposing Country s' intermediate exports, are the key technical steps to fully decompose gross trade flows, since they convert gross output (and gross exports), usually endogenous variables in standard ICIO models, to exogenous variables in the gross trade accounting framework we develop. Together with the adding-up condition for the global value added multiplier defined in equation (6) and the local value added multipliers defined below, they are the major stepping stones for deriving our gross exports decomposition formula.

From equation (6), we obtain Country s' domestic and foreign value-added multipliers as follows:

$$V^s B^{ss} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} & v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \tag{17}$$

$$V^r B^{rs} = \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} = \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} & v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \tag{18}$$

Also from equation (6) we know that the sum of equations (17) and (18) equals a 1 by 2 vector of unity. In a single country IO model, domestic value-added multiplier can be calculated as

$$V^s (I - A^{ss})^{-1} = V^s L^{ss} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} & v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \tag{19}$$

Using equation (19), the identity of summing equations (17) and (18) equals to unity, and define "#" as element-wise matrix multiplication operation[4], the value of Country s' gross intermediate exports can be decomposed as

[4] For example, when a matrix is multiplied by $n \times 1$ column vector, each row of the matrix is multiplied by the corresponding row element of the vector.

$$
\begin{aligned}
A^{sr}X^{r} &= \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} = \begin{bmatrix} v_1^{s} l_{11}^{ss} + v_2^{s} l_{21}^{ss} \\ v_1^{s} l_{12}^{ss} + v_2^{s} l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} v_1^{s} b_{11}^{ss} + v_2^{s} b_{21}^{ss} \\ v_1^{s} b_{12}^{ss} + v_2^{s} b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^{s} l_{11}^{ss} + v_2^{s} l_{21}^{ss} \\ v_1^{s} l_{12}^{ss} + v_2^{s} l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \\ v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\}
\end{aligned} \tag{20}
$$

Finally, based on the Leontief insight, Country s' final goods exports can be decomposed into domestic and foreign value-added as follows:

$$
\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} = \begin{bmatrix} v_1^{s} b_{11}^{ss} + v_2^{s} b_{21}^{ss} \\ v_1^{s} b_{12}^{ss} + v_2^{s} b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \\ v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \tag{21}
$$

Inserting equations (12) and (16) into equation (20), and combining equations (20) and (21), we obtain Country s' gross exports decomposition equation as:

$$
\begin{aligned}
E^{sr} &= \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} = \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \\
&= \begin{bmatrix} v_1^{s} b_{11}^{ss} + v_2^{s} b_{21}^{ss} \\ v_1^{s} b_{12}^{ss} + v_2^{s} b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^{s} l_{11}^{ss} + v_2^{s} l_{21}^{ss} \\ v_1^{s} l_{12}^{ss} + v_2^{s} l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^{s} l_{11}^{ss} + v_2^{s} l_{21}^{ss} \\ v_1^{s} l_{12}^{ss} + v_2^{s} l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \right\} + \begin{bmatrix} v_1^{s} l_{11}^{ss} + v_2^{s} l_{21}^{ss} \\ v_1^{s} l_{12}^{ss} + v_2^{s} l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^{s} l_{11}^{ss} + v_2^{s} l_{21}^{ss} \\ v_1^{s} l_{12}^{ss} + v_2^{s} l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} v_1^{s} b_{11}^{ss} + v_2^{s} b_{21}^{ss} \\ v_1^{s} b_{12}^{ss} + v_2^{s} b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^{s} l_{11}^{ss} + v_2^{s} l_{21}^{ss} \\ v_1^{s} l_{12}^{ss} + v_2^{s} l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \\ v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \\ v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \\ v_1^{r} b_{11}^{rs} + v_2^{r} b_{21}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \right\}
\end{aligned} \tag{22}
$$
[5]

Similarly, we can derive the decomposition of Country r's gross exports in a similar way. To save space, we list the equation in appendix B.

Equation (22) indicates that the gross exports of a country can be completely

[5] Appendix A provides the relationship between global and local Leontief inverse matrix in the two-country, two-sector case in details.

decomposed into the sum of nine terms. It is a generalization of equation (13) in KWW for the case of a country's aggregate exports; with the domestic pure double counted term being further split into terms related to production of final and intermediate goods exports, respectively.

To better understand each term in this accounting equation, we provide the following economic interpretations: The first term is domestic value added embodied in the final exports of the 1st and 2nd sectors of Country s. Each of them has two parts: domestic value added created by the sector itself and domestic value added created by the other sector embodied in the sector's final exports. The second term is domestic value added embodied in Country s' 1st and 2nd sector's intermediate exports which are used by Country r to produce final goods, y_1^{rr} and y_2^{rr}, that are consumed in r. These two terms are domestic value added embodied in Country s' gross exports which are ultimately absorbed by Country r. They are value added exports of Country s.

The third term is domestic value added embodied in Country s' 1st and 2nd sector's intermediate exports used to produce Country r's final exports, i.e. Country s' imports of final goods from r. The fourth term is domestic value added embodied in Country s' 1st and 2nd sector's intermediate exports that are used by Country r to produce intermediate exports and return to Country s via its intermediate imports to produce its domestic final goods. These two terms are domestic value added embodied in the 1st and 2nd sector's gross exports, respectively, which return home and are finally consumed in Country s.

The first four terms are the domestic value added (GDP) embodied in the 1st and 2nd sectors' gross exports of Country s, which include value added created from all sectors in Country s.

The fifth term is the domestic value added of Country s' 1st and 2nd sector's intermediate exports which return home as its 1st and/or 2nd sector's intermediate imports and are used for production of Country s' both sectors' final exports that are finally consumed in Country r. They are parts of the value added in Country s' final exports and are already counted once by the first term of equation (22). For this reason they are a portion of domestic double counted terms caused by the back and forth intermediate goods trade in order to produce final goods exports in Country s.

The sixth term is domestic value added of Country s' 1st and 2nd sector's intermediate exports that return home as intermediate imports and are used for production of Country s' intermediate exports to Country r. It is also a domestic double counted portion caused by the back and forth intermediate goods trade but to produce intermediate goods exports in Country s.

The sum of the first to the sixth terms equals the domestic content of the 1st and 2nd sectors' gross exports, $\sum_i^2 v_i^s b_{i1}^{ss} e_1^{sr}$ and $\sum_i^2 v_i^s b_{i2}^{ss} e_2^{sr}$. A detailed proof is given in Appendix C.

The seventh term is foreign value added used in Country s' 1st and 2nd sector's final goods exports. Each of them also has two parts: foreign value added from the sector itself and from the other sector used to produce final exports from Country s. Adding up the first and the seventh terms accounts 100% of the value of the final exports in Country s by sector.

The eighth term is foreign value added used to produce the 1st and 2nd sector intermediate exports of Country s, which are then used by Country r to produce its domestic final goods. Summing the seventh and eighth terms yields the total foreign value added embodied in the 1st and 2nd sectors' gross exports of Country s, respectively.

The ninth term is foreign value added embodied in the 1st and 2nd sector's intermediate exports used by Country r to produce its final and intermediate exports, which is a pure foreign double counted term of Country s' exports. Adding up the eighth and ninth term yields the foreign content of the 1st and 2nd sector's intermediate exports.

Therefore, summing up the seventh to the ninth terms equals the foreign content of the 1st and 2nd sector's gross exports of Country s, $\sum_i^2 v_i^r b_{i1}^{rs} e_1^{sr}$ and $\sum_i^2 v_i^r b_{i2}^{rs} e_2^{sr}$, respectively.

It is easy to show that the aggregation of the two sectors in equation (22) results in equation (13) in KWW. A detailed proof is given in Appendix D. A numerical example to illustrate various concepts discussed in this section is provided in Appendix E.

2.3 Decompose intermediate and gross trade: 3-countries and 2-sectors

Examining a three-country case in detail is useful for two reasons: (i) it exhibits nearly all the richness of the fully general multi-country analysis, and (ii) analytical solutions remain tractable and continue to have intuitive explanations.

We use a superscript s, to represent the source country, r to represent the partner country, and t to represent the third country, and define the country set G = {s, r, t}. Based on the Leontief insight, from a three-country two-sector ICIO model we can decompose Country r's gross output into the following nine components according to where they are finally absorbed:

$$\begin{aligned} X^r &= B^{rs}Y^s + B^{rr}Y^r + B^{rt}Y^t \\ &= B^{rs}Y^{ss} + B^{rs}Y^{sr} + B^{rs}Y^{st} + B^{rr}Y^{rs} + B^{rr}Y^{rr} + B^{rr}Y^{rt} + B^{rt}Y^{ts} + B^{rt}Y^{tr} + B^{rt}Y^{tt} \end{aligned} \tag{23}$$

Where B^{rk} denotes a 2 by 2 block Leontief inverse matrix, which is total intermediate input requirement coefficients that specify the amount of gross output from Country r required for a one-unit increase in final demand in Country k. X^r and Y^k are vectors of Country r's gross output and Country k's final goods outputs respectively,

$$X^r = \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} x_1^{rs} + x_1^{rr} + x_1^{rt} \\ x_2^{rs} + x_2^{rr} + x_2^{rt} \end{bmatrix};\ B^{rk} = \begin{bmatrix} b_{11}^{sk} & b_{12}^{sk} \\ b_{21}^{sk} & b_{22}^{sk} \end{bmatrix}; Y^k = \begin{bmatrix} y_1^k \\ y_2^k \end{bmatrix} = \begin{bmatrix} y_1^{ks} + y_1^{kr} + y_1^{kt} \\ y_2^{ks} + y_2^{kr} + y_2^{kt} \end{bmatrix}; k \in G = \{s, r, t\}$$

Insert equation (23) into the last term of equation (10), we can decompose Country s' gross intermediate goods exports according to where and how they are absorbed as follows:

$$\begin{aligned} A^{sr}X^r = {} & A^{sr}B^{rr}Y^{rr} + A^{sr}B^{rt}Y^{tt} + A^{sr}B^{rr}Y^{rt} + A^{sr}B^{rt}Y^{tr} + A^{sr}B^{rr}Y^{rs} + A^{sr}B^{rt}Y^{ts} \\ & + A^{sr}B^{rs}Y^{ss} + A^{sr}B^{rs}(Y^{sr} + Y^{st}) \end{aligned} \tag{24}$$

Comparing equation (24) with equation (12), the intermediate goods exports decomposition equation in the 2-country, 2-sector case, the first, fifth, and seventh term in equation (24) are exactly the same as the first three terms in equation (12) and have similar economic interpretations, except that they are expressed in aggregate matrix notations[6]. They are part of Country s' intermediate goods exports used by partner Country r to produce final goods and are consumed in r; to produce final goods exports that are shipped back to Country s; and to produce intermediate exports that are shipped back to and used by Country s to produce domestically consumed final goods. The last term in the two equations also has similar economic interpretation. Both are part of Country s' intermediate goods exports used by importing Country r to produce

[6] The full derivation process in detailed matrix notation similar to section 2.2 (Equations 23 to31) is listed in Appendix H for interested readers.

intermediate goods exports that are shipped back to Country s to produce its final goods exports that are ultimately consumed abroad. However, in equation (12), the term only has final goods exports to Country r, while in equation (24), the term also includes final goods exports to the third country t.

Four additional terms appear in equation (24), namely, the second, third, fourth and sixth terms, all of which are related to the third Country t. The second term is Country s' intermediate exports used by the direct importer, Country r, to produce intermediate goods that are exported to the third Country t for production of finally goods consumed in t. The third term is Country s' intermediate exports used by the direct importer, Country r, to produce final exports which are ultimately absorbed by the third Country t. The fourth term is Country s' intermediate exports used by the direct importer, Country r, to produce intermediate exports to the third Country t for production of final exports that return to the direct importer (r). The sixth term is Country s' intermediate exports used by the direct importer (r) to produce intermediate exports to the third Country t for its production of final exports that return to Country s. These additional terms and their relative importance are measures of different patterns of production sharing that involve more than two countries and only can be observed in a setting with three or more countries.

The eight terms in equation (24) collectively decompose completely Country s' intermediate exports according to where and how they are finally absorbed.

In the three-country ICIO model, the gross output production and use balance, or the row balance condition becomes:

$$\begin{aligned} X^r &= A^{rs}X^s + A^{rr}X^r + A^{rt}X^t + Y^{rs} + Y^{rr} + Y^{rt} \\ &= A^{rr}X^r + Y^{rr} + E^{rs} + E^{rt} = A^{rr}X^r + Y^{rr} + E^{r*} \end{aligned} \tag{25}$$

Where $E^{r*} = E^{rs} + E^{rt}$ is total gross exports of Country r. Re-arrange equation (25)

$$X^r = (I - A^{rr})^{-1}Y^{rr} + (I - A^{rr})^{-1}E^{r*} \tag{26}$$

Therefore, the intermediate goods exports by Country s can also be decomposed into two components according to where it is used similar to a single-country IO model:

$$A^{sr}X^r = A^{sr}L^{rr}Y^{rr} + A^{sr}L^{rr}E^{r*} \tag{27}$$

Although expressed in aggregate matrix notation, equation (27) is almost the same as equation (17), except its last term on the RHS also includes Country s' exports to the third Country t in addition to its exports to Country r.

It is important to note that the value-added multipliers of Country s and r are exactly the same in the 3-country, 2-sector model as in the 2-country, 2-sector case as specified in equations (17) and (18). The value-added multiplier of Country t can be defined in a similar way:

$$V^t B^{ts} = \begin{bmatrix} v_1^t & v_2^t \end{bmatrix} \begin{bmatrix} b_{11}^{ts} & b_{12}^{ts} \\ b_{21}^{ts} & b_{22}^{ts} \end{bmatrix} = \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} & v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \tag{28}$$

and the sum of equations (17),(18) and (28) equals unity, i.e:

$$V^s B^{ss} + V^r B^{rs} + V^t B^{ts} = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{29}$$

Finally, based on the Leontief insight, Country s' final goods exports can be decomposed into Country s, r, and t 's value-added as follows:

$$Y^{sr} = (V^s B^{ss})^T \# Y^{sr} + (V^r B^{rs})^T \# Y^{sr} + (V^t B^{ts})^T \# Y^{sr} \tag{30}$$

Combine equations (24), (27), (29) and (30), we can obtain the gross exports decomposition equation in the 3-country, 2-sector model in a similar way as the 2-country 2-sector case. Detailed derivation is provided in Appendix G.

$$\begin{aligned} E^{sr} = Y^{sr} + A^{sr} X^r &= (V^s B^{ss})^T \# Y^{sr} + (V^s L^{ss})^T \# (A^{sr} B^{rr} Y^{rr}) + (V^s L^{ss})^T \# (A^{sr} B^{rt} Y^{tt}) \\ &+ (V^s L^{ss})^T \# (A^{sr} B^{rr} Y^{rt}) + (V^s L^{ss})^T \# (A^{sr} B^{rt} Y^{tr}) \\ &+ (V^s L^{ss})^T \# (A^{sr} B^{rr} Y^{rs}) + (V^s L^{ss})^T \# (A^{sr} B^{rt} Y^{ts}) + (V^s L^{ss})^T \# (A^{sr} B^{rs} Y^{ss}) \\ &+ (V^s L^{ss})^T \# [A^{sr} B^{rs} (Y^{sr} + Y^{st})] + [V^s (B^{ss} - L^{ss})]^T \# (A^{sr} X^r) \\ &+ (V^r B^{rs})^T \# Y^{sr} + (V^r B^{rs})^T \# (A^{sr} L^{rr} Y^{rr}) + (V^r B^{rs})^T \# (A^{sr} L^{rr} E^{r*}) \\ &+ (V^t B^{ts})^T \# Y^{sr} + (V^t B^{ts})^T \# (A^{sr} L^{rr} Y^{rr}) + (V^t B^{ts})^T \# (A^{sr} L^{rr} E^{r*}) \end{aligned} \tag{31}$$

Except for expressing in aggregate matrix notations, equation (31) is similar to equation (22) but with seven additional terms, all of which involve the third Country t.

Four of them are domestic value-added components. The third term is Country s' domestic value-added in its intermediate exports used by the direct importer (Country r) to produce intermediate exports to the third Country t for production of latter's domestic final goods; the fourth term is domestic value-added in Country s' intermediate exports used by the direct importer (r) for producing final goods exports to the third Country t; the fifth term is domestic value-added in Country s' intermediate exports used by the direct importer (r) to produce intermediate exports to the third Country t for its production of final goods exports that are shipped back to the direct importer (r); and the seventh term is domestic value-added in Country s' intermediate exports used by the direct importer (r) to produce intermediate exports to the third Country t for the latter's production of final goods exports that are shipped back to the source Country s.

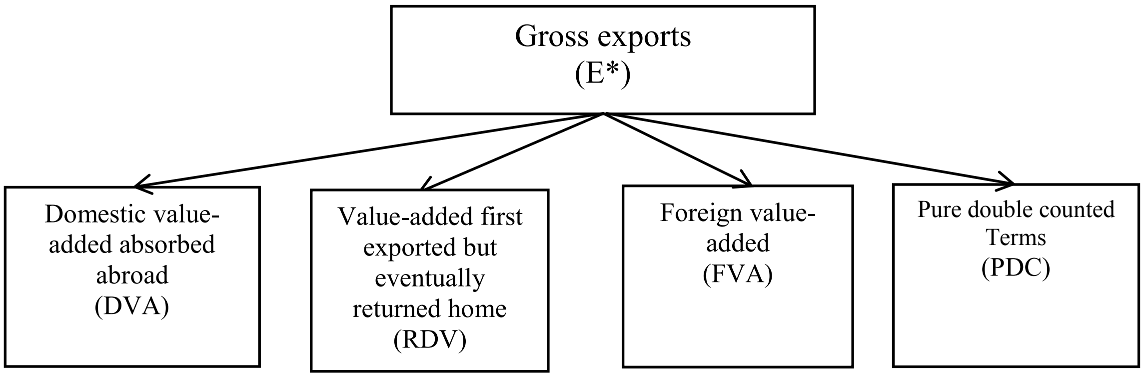
Two of the seven additional terms are foreign value added components. The fourteenth term is foreign value added from the third Country t used by Country s' 1st and 2nd sectors to produce final exports from Country s. Adding up the first (domestic value added from source Country s), eleventh (foreign value added from Country r) and fourteenth term (foreign value added from Country t) accounts for 100% of the value of the final exports in Country s by sector.

The fifteenth term is foreign value added from the third Country t used to produce the 1st and 2nd sectors' intermediate exports of Country s, which are then used by Country r to produce its domestic final goods. Summing the eleventh, twelfth, fourteenth and fifteenth terms yields the total foreign value added embodied in the 1st and 2nd sectors' gross exports of Country s respectively.

Similar to the ninth term in equation (22), the thirteenth and last terms in equation (31) are foreign value added (value-added from Country r and t) embodied in the 1st and 2nd sectors' intermediate exports used by Country r to produce its final and intermediate goods exports to the world (sum of exports to Country s and t), which are double counted terms in Country s' gross exports originated from foreign countries.

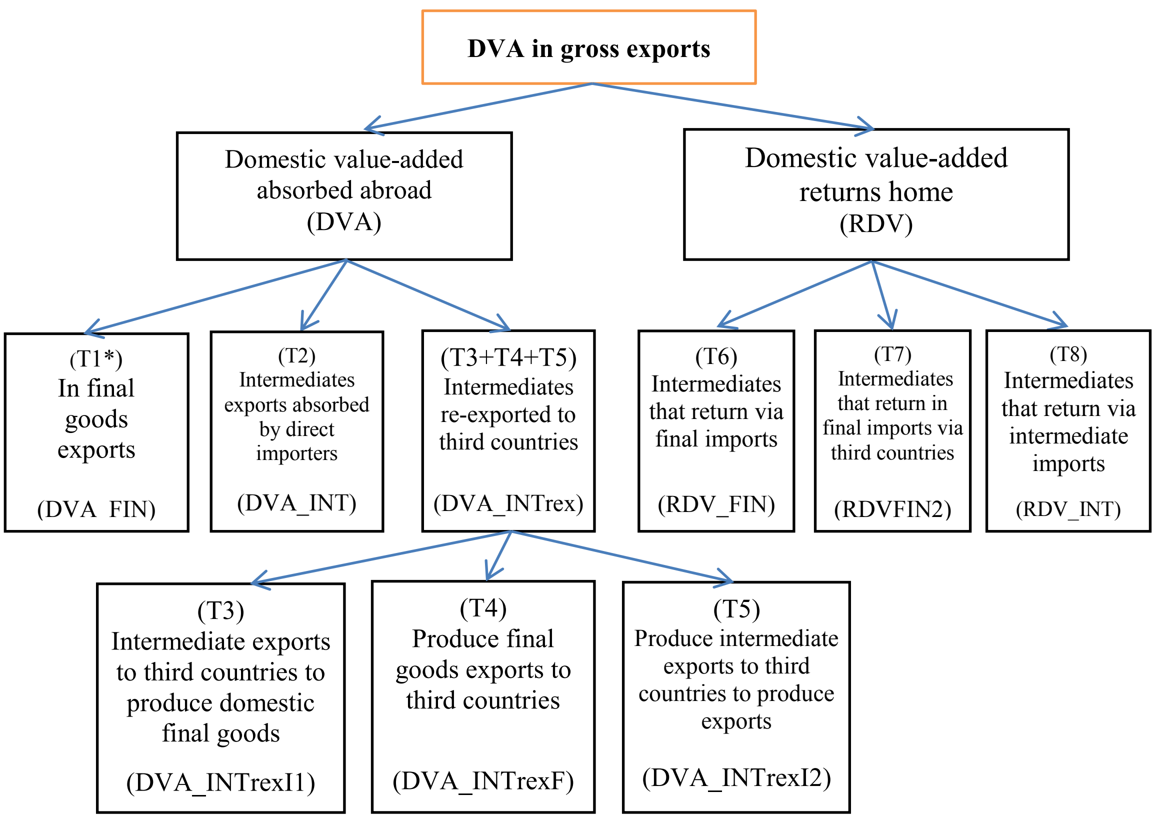
Coming to the rest of the six terms in equation (31), the first, second, sixth, eighth, ninth and tenth terms have similar economic interpretations as the first six terms in equation (22), so we do not repeat them here to save space. The sixteen terms completely decompose bilateral gross exports from Country s to Country r into different value-added and double counted components, and their sum equals 100% of bilateral trade flows at the sector level. The disaggregated accounting framework made by equation (31) is also diagrammed in Figure 1.

Figure 1a Gross Exports Accounting: Major Categories



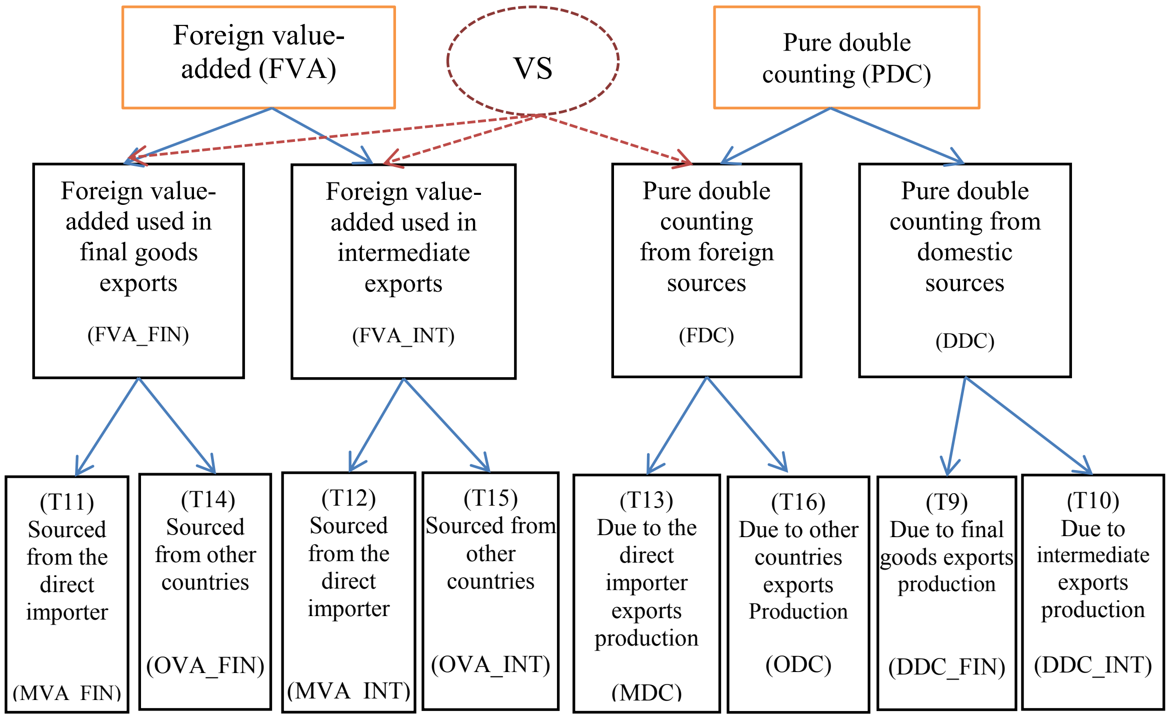
Note: E* can be at country/sector, country aggregate, bilateral /sector or bilateral aggregate; both DVA and RDV are based on backward linkages

Figure 1b Gross Exports Accounting: Domestic Value-Added



Note: *corresponds to terms in equation (31) in the main text.

Figure 1c Gross Exports Accounting: Foreign Value-Added



With our bilateral/sector gross exports decomposition equation (31) in hand, we can reflect on a proper definition of the value-added exports to gross exports ratio (the VAX ratio) and double counted measure at the bilateral/sector level.

Define domestic value added in bilateral exports in sector i from Country s to Country r that are ultimately absorbed by other countries as the sum of the first five terms in equation (31)

$$\begin{aligned} dva_i^{sr} &= (v_1^s b_{1i}^{ss} + v_2^s b_{2i}^{ss}) y_i^{sr} + (v_1^s l_{1i}^{ss} + v_2^s l_{2i}^{ss}) \sum_j^2 a_{ij}^{sr} \sum_k^2 b_{jk}^{rr} (y_k^{rr} + y_k^{rt}) \\ &+ (v_1^s l_{1i}^{ss} + v_2^s l_{2i}^{ss}) \sum_j^2 a_{ij}^{sr} \sum_k^2 b_{jk}^{rt} (y_k^{tr} + y_k^{tt}) \end{aligned} \tag{32}$$

Value-added exports from Country s to Country r based on backward linkages are

$$\begin{aligned} vt(e_i^{sr}) &= (v_1^s b_{1i}^{ss} + v_2^s b_{2i}^{ss}) y_i^{sr} + (v_1^s l_{1i}^{ss} + v_2^s l_{2i}^{ss}) \sum_j^2 a_{ij}^{sr} \sum_k^2 (b_{jk}^{rr} y_k^{rr} + b_{jk}^{rt} y_k^{tr}) \\ &+ (v_1^s l_{1i}^{ss} + v_2^s l_{2i}^{ss}) \sum_j^2 a_{ij}^{st} \sum_k^2 (b_{ij}^{tt} y_j^{tr} + b_{ij}^{tr} y_j^{rr}) \} \end{aligned} \tag{33}$$

Note that dva_i^{sr} includes value added absorbed by not only Country r, but also the third country t ($b_{ij}^{rr} y_j^{rt}$ and $b_{ij}^{rt} y_j^{tt}$ in equation (32)), while backward linkage based value added exports measure $vt(e_i^{sr})$ includes not only value added exports from Country s embodied in its own gross exports to Country r (the second term in equation (33)), but also value added exports by Country s embodied in its gross exports to the third Country t, that are finally absorbed by Country r (the last term in equation (33)).

Value added exports from Country s to Country r based on forward linkages are

$$vt_i^{sr} = v_i^s \sum_j^2 b_{ij}^{ss} y_j^{sr} + v_i^s \sum_j^2 b_i^{sr} y_j^{rr} + v_i^s \sum_j^2 b_{ij}^{st} y_j^{tr} \tag{34}$$

The following four propositions summarize our major analytical results[7]:

Proposition A: In a three-country world, dva_i^{sr}, $vt(e_i^{sr})$, and vt_i^{sr}, are not equal to each other in general except under special restrictions. In addition, the following aggregation relations (1)-(3) always hold:

(1) $\sum_{i=1}^{2} vt(e_i^{sr}) = \sum_{i=1}^{2} vt_i^{sr}$, (2) $\sum_{r \neq s}^{G} dva_i^{sr} = \sum_{r \neq s}^{G} vt(e_i^{sr})$, (3) $\sum_{r \neq s}^{G}\sum_{i=1}^{2} vt(e_i^{sr}) = \sum_{r \neq s}^{G}\sum_{i=1}^{2} vt_i^{sr} = \sum_{r \neq s}^{G}\sum_{i=1}^{2} dva_i^{sr}$; and (4) $\sum_{i=1}^{2} dva_i^{sr} \neq \sum_{i=1}^{2} vt_i^{sr}$, and $\sum_{r \neq s}^{G} dva_i^{sr} \neq \sum_{r \neq s}^{G} vt_i^{sr}$ hold in general except for special cases.

Proposition B: In a three-country world, dva_i^{sr} is always less than or equal to e_i^{sr}, the sector level gross bilateral exports. Therefore domestic value added absorbed abroad to gross exports ratio is upper-bounded at 1, i.e. $\frac{dva_i^{sr}}{e_i^{sr}} \leq 1$

Proposition C: $vt(e_i^{s*}) = \sum_{r \neq s}^{G} vt(e_i^{sr})$, the sector level value added exports measure based on backward linkages is always less than or equal to Country s' gross exports of sector i to the world: $e_i^{s*} = \sum_{r \neq s}^{G} e_i^{sr}$. Therefore, backward linkage based value-added exports to gross exports ratio is upper-bounded at 1, i.e $\frac{vt(e_i^{s*})}{e_i^{s*}} \leq 1$

Proposition D: vt_i^{s*} is always less than or equal to sector level value-added production. i.e. $vt_i^{s*} = \sum_{u \neq s}^{G} vt_i^{su} \leq v_i^s x_i^s$. Therefore, $\frac{vt_i^{s*}}{v_i^s x_i^s} \leq 1$, vt_i^{s*} to GDP by industry ratio is upper-bounded at 1.

[7] Due to space limitation, we leave the derivation of equations (32) to (34), mathematical proofs of the four propositions A-D, and detailed discussion of the relationship between the two type measures of value-added exports and domestic value-added in gross exports in Appendix H for interested readers.

The intuition behind these propositions is simple. As shown in Appendix F and H, the sector level direct value-added exports are the same for both value-added export measures, but the sector level indirect value-added exports can be very different. The indirect value-added exports in the forward linkage based measure are the sector's value added embodied in other sector's gross exports, which has no relation with the gross exports from that sector. Therefore, the value added exports to gross exports ratio defined by Johnson and Noguera (2012) at the sector level is not desirable since its denominator (sector gross exports) does not include the indirect value-added exports from other sectors. It is common in the data for some sectors to have no gross exports, but their products are used by other domestic industries as intermediate inputs and thus they can have indirect value added exports through other sectors. In such cases, the VAX ratio will become infinitive. Similarly, at the bilateral level, due to indirect value added trade via third countries, two countries can have a large volume of value added trade even when they have no gross trade. Therefore, it is not possible to define a well-behaved ratio of value-added exports to gross exports that is upper bounded at 1. However, because such indirect value-added exports are part of the total value-added created by the same sector, the forward linkage based value-added exports to GDP ratio can be properly defined at the sector level.

Analytical results based on propositions A-D show clearly that the VAX ratio defined by Johnson and Noguera (2012) cannot be used as a summary measure of value-added content and double counting of gross exports except at the country aggregate level. The correct measure at the bilateral/sector level is the share of domestic value-added that is absorbed abroad in gross exports. However, when the bilateral/sector decomposition is aggregated to the country/sector level, the VAX ratio computed based on backward linkage equals domestic value-added in exports that stay abroad (no need to separate different destinations), so the definition of value-added exports can be consistently maintained. In such a case, only the backward linkage based VAX ratio defined in this paper is upper bounded at 1, while the forward linkage based VAX ratio widely cited in the literature is not.

2.4 A numerical example

We now provide a numerical example to illustrate various concepts discussed so far. Suppose a simple 3-country, 2-sector ICIO table as summarized in Table 2 below:

Table 2 3-Country, 2-Sector Numerical ICIO Table

	Country	Intermediate Uses						Final Uses			Gross outputs
		s		R		T		Y^s	Y^r	Y^t	
Country	Sector	s1	s2	r1	r2	t1	t2				
s	s1	1	1	0	0	0	0	9/10	1/10	0	3
	s2	0	1	0	1	0	0	1	0	0	3
r	r1	0	0	1	1	0	0	0	1	0	3
	r2	0	0	1	1	0	0	0	1	1	4
t	t1	1	0	0	0	1	0	1	0	0	3
	t2	0	0	0	0	1	1	0	0	1	3
Value-added		1	1	1	1	1	2				
Total inputs		3	3	3	4	3	3				

The gross intermediate exports (EI) and final good exports (EF), the direct input coefficient matrix (A), Global Leontief inverse (B), Local Leontief inverse (L) and direct value-added coefficient vector (V) can be easily obtained from Table 2. By applying decomposition equation (31), we can fully decompose each of the three countries' gross bilateral exports into sixteen value added and double counted components as reported in table 3. All these detailed numerical computations are listed in Appendix I.

This example shows that one has to be careful about defining the VAX ratios at either bilateral, sector, or bilateral sector level. If one were to use the definition by Johnson and Noguera (2012) which is based on forward linkages, 6 out of the 12 VAX ratios at the bilateral sector level would be undefined (going to infinity) (shown in Column 18 of Table 3). If one were to define the VAX ratio based on backward linkages, it would be undefined for four cases (shown in column 20 of Table 3). At the aggregate bilateral, the VAX ratio based on either forward or backward linkages would be undefined in 2 out of the 6 cases (rows ST and TR in Table 3). We advocate computing the share of domestic value added that stays abroad (computed in column 16 of Table 3) as a summary statistics for inverse measure of vertical specialization in world production or international production sharing at any level of disaggregation. Such a measure is always bounded between zero and 100%. Note that at the country aggregate level, our measure coincides with the other two.

Table 3: Gross Exports Decomposition Results: 3-Country, 2-Sector Numerical Example

Trade Flows	T1	T2	T3	T4	T5-T6	T7	T8-T10	T11	T12-T13	T14	T15	T16	Gross exports	DVA	% of DVA	Value Added Exports_ Forward	VAX_F Ratio (J&N)	Value Added Exports _ Backward	VAX_B Ratio
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
sr1	1/20	0	0	0	0	0	0	0	0	1/20	0	0	1/10	1/20	50%	1/5	200%	1/5	200%
sr2	0	9/20	0	3/10	0	0	0	0	0	0	3/20	1/10	1	¾	75%	3/10	30%	3/10	30%
st1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/10	∞	0	0%
st2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/5	∞	3/10	∞
rt1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/5	∞	0	0%
rt2	3/5	0	0	0	0	0	0	1/10	0	3/10	0	0	1	3/5	60%	2/5	40%	3/5	60%
rs1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	0	0%	0	0%
rs2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	0	0%	0	0%
ts1	1	7/10	3/20	1/20	0	1/10	0	0	0	0	0	0	2	19/10	95%	17/20	43%	19/20	73%
ts2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	17/20	∞	1/4	∞
tr1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/10	∞	1/10	∞
tr2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/10	∞	1/10	∞
Sr	1/20	9/20	0	3/10	0	0	0	0	0	1/20	3/20	1/10	11/10	4/5	73%	½	45%	1/2	45%
St	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	3/10	∞	3/10	∞
Rt	3/5	0	0	0	0	0	0	1/10	0	3/10	0	0	1	3/5	60%	3/5	60%	3/5	60%
Rs	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	0	0%	0	0%
Ts	1	7/10	3/20	1/20	0	1/10	0	0	0	0	0	0	2	19/10	95%	17/10	85%	17/10	85%
Tr	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/5	∞	1/5	∞
S	1/20	9/20	0	3/10	0	0	0	0	0	1/20	3/20	1/10	11/10	**4/5**	**73%**	**4/5**	**73%**	**4/5**	**73%**
R	3/5	0	0	0	0	0	0	1/10	0	3/10	0	0	1	**3/5**	**60%**	**3/5**	**60%**	**3/5**	**60%**
T	1	7/10	3/20	1/20	0	1/10	0	0	0	0	0	0	2	**19/10**	**95%**	**19/10**	**95%**	**19/10**	**95%**

Note: 1. Row label "sr1" represents the 1st sector's trade flow from Country s to r; "rs2" represents the 2nd sector's trade flow from Country r to s … and so on.

2. T1-T16 is correspondent to the 16 terms in equation (31) and the 16 boxes in figure 1.

2.5 The general case of G countries and N sectors

Extending the 3-country, 2-sector model to a general model with G countries and N sectors is straightforward. Country s' intermediate exports to Country r can be split into eight terms similar to equation (24):

$$\begin{aligned} A^{sr}X^{r} &= A^{sr}B^{rr}Y^{rr} + A^{sr}\sum_{t\neq s,r}^{G}B^{rt}Y^{tt} + A^{sr}B^{rr}\sum_{t\neq s,r}^{G}Y^{rt} + A^{sr}\sum_{t\neq s,r}^{G}B^{rt}\sum_{u\neq s,t}^{G}Y^{tu} \\ &+ A^{sr}B^{rr}Y^{rs} + A^{sr}\sum_{t\neq s,r}^{G}B^{rt}Y^{ts} + A^{sr}B^{rs}Y^{ss} + A^{sr}\sum_{t\neq s}^{G}B^{rs}Y^{st} \end{aligned} \tag{35}$$

where A^{sr} is an N×N block input-output coefficient matrix, B^{rk} denotes the N×N block Leontief (global) inverse matrix, X^{r} is an N×1 vector that gives Country r' total gross output. Y^{sr} is also an N×1 vector gives final goods produced in s and consumed in r.

Although each of the 8 terms has the same economic interpretations as the corresponding terms in equation (24), all of the third-country terms in equation (35) include all other remaining counties.

Define total value-added multiplier for every country, similar to equations (17), (18), (19), and (28):

$$V^{s}B^{ss} = \begin{bmatrix} \sum_{i}^{N} v_i^s b_{i1}^{ss} \\ \sum_{i}^{N} v_i^s b_{i2}^{ss} \\ \vdots \\ \sum_{i}^{N} v_i^s b_{iN}^{ss} \end{bmatrix}^{T} \quad V^{r}B^{rs} = \begin{bmatrix} \sum_{i}^{N} v_i^r b_{i1}^{rs} \\ \sum_{i}^{N} v_i^r b_{i2}^{rs} \\ \vdots \\ \sum_{i}^{N} v_i^r b_{iN}^{rs} \end{bmatrix}^{T} \quad V^{t}B^{ts} = \begin{bmatrix} \sum_{i}^{N} v_i^t b_{i1}^{ts} \\ \sum_{i}^{N} v_i^t b_{i2}^{ts} \\ \vdots \\ \sum_{i}^{N} v_i^t b_{iN}^{ts} \end{bmatrix}^{T} \quad V^{s}L^{ss} = \begin{bmatrix} \sum_{i}^{N} v_i^s l_{i1}^{ss} \\ \sum_{i}^{N} v_i^s l_{i2}^{ss} \\ \vdots \\ \sum_{i}^{N} v_i^s l_{iN}^{ss} \end{bmatrix}^{T} \quad (36).$$

Using equations (27), (35) and the property of $\sum_{r}^{G} V^{r}B^{rs} = u$, and adding the decomposition of Country s' final goods exports to Country r based on the Leontief insight, we obtain the decomposition equation of gross bilateral exports from Country s to Country r for the general G-country N-sector setting. Step by step derivations are given in Appendix J.

$$
\begin{aligned}
E^{sr} &= (V^s B^{ss})^T \# Y^{sr} + (V^s L^{ss})^T \# (A^{sr} X^r) + (V^s B^{ss} - V^s L^{ss})^T \# (A^{sr} X^r) \\
&+ (V^r B^{rs})^T \# Y^{sr} + (V^r B^{rs})^T \# (A^{sr} X^r) + (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# Y^{sr} + (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# (A^{sr} X^r) \\
&= (V^s B^{ss})^T \# Y^{sr} + (V^s L^{ss})^T \# (A^{sr} B^{rr} Y^{rr}) + (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{tt}) \\
&+ (V^s L^{ss})^T \# (A^{sr} B^{rr} \sum_{t \neq s,r}^{G} Y^{rt}) + (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s,r}^{G} \sum_{u \neq s,t}^{G} B^{rt} Y^{tu}) \\
&+ (V^s L^{ss})^T \# (A^{sr} B^{rr} Y^{rs}) + (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{ts}) + (V^s L^{ss})^T \# (A^{sr} B^{rs} Y^{ss}) \\
&+ (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s}^{G} B^{rs} Y^{st}) + (V^s B^{ss} - V^s L^{ss})^T \# (A^{sr} X^r) \\
&+ (V^r B^{rs})^T \# Y^{sr} + (V^r B^{rs})^T \# (A^{sr} L^{rr} Y^{rr}) + (V^r B^{rs})^T \# (A^{sr} L^{rr} E^{r*}) \\
&+ (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# Y^{sr} + (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# (A^{sr} L^{rr} Y^{rr}) + (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# (A^{sr} L^{rr} E^{r*})
\end{aligned}
\tag{37}
$$

Summing up equation (37) across the G-1 trading partners and N sectors, we obtain a decomposition equation for total gross exports of Country s, which can be verified to be identical to equation (36) in Koopman, Wang, and Wei (2014). Detailed derivations are also given in Appendix J. This formally shows that our formula generalizes the one in KWW to any level of disaggregation.

3. Decomposition Results for 40 Economies during 1995-2011

In this section, we apply our disaggregated accounting framework to the World Input-output Database (WIOD). The WIOD, developed by a consortium of eleven European research institutions funded by the European Commission, provides a time series of inter-country input-output (ICIO) tables from 1995 to 2011, covering 40 economies including all major industrialized countries and major emerging trading nations. Timmer et al. (2012) provide a detailed description of this database.

Our disaggregated accounting framework produces a series of panel data sets, consisting of many GN (1435) by G (41) and GN by GN matrices each year, collectively taking up more than 20 gigabytes of storage space when the decomposition is computed at the most detailed level. To illustrate the estimation outcomes in a manageable manner, we provide a series of examples, which are selected and processed from subsets of the detailed results.

3.1. Decomposing gross exports into their four major components at the country-sector level

We first look at the decomposition for the gross exports of US transport equipment sector (WIOD sector 15). The sector includes automobiles and airplanes and is one of the most important export sectors for the United States. The decomposition for 1995-2011 is presented in Table 4a. Column 2 records the gross exports in millions of dollars (current price). In the next four columns, we report the four major components of gross exports: domestic value added that is ultimately absorbed abroad (DVA for short), foreign value added used in the production for the exports (FVA for short), returned value added or the portion of domestic value added that is initially exported but ultimately returned home by being embedded in the imports from other countries and consumed at home (RDV for short), and pure double counted terms (PDC for short) due to the back and forth intermediate goods trade, all expressed in percentage of gross exports. On average, DVA is under 70% of gross exports, while FVA that is embedded in US transport equipment exports is somewhere between 12-22% of the gross exports.

The remaining parts consist of returned domestic value added (RDV) and pure double counting (PDC). When we compare with the next example (Mexico's electric equipment exports), we will see that the RDV share for the US transport equipment exports is relatively high. This suggests that a fraction of the US exports are parts and components that are used as intermediate inputs in the production of transport equipment or refined components that are re-imported back to the United States.

Table 4a: Decomposition of US Transport Equipment Exports (WIOD sector 15)

Year (1)	Gross Exports (2)	DVA Share (% of (2)) (3)	FVA Share (% of (2)) (4)	RDV Share (% of (2)) (5)	PDC Share (% of (2)) (6)
1995	90,737	70.3	12.1	12.6	5.0
2000	124,345	67.2	12.0	14.5	6.4
2005	150,442	66.0	15.1	11.8	7.0
2007	194,374	67.8	16.6	8.9	6.7
2009	158,999	74.6	15.8	5.1	4.4
2010	179,540	68.1	20.7	5.2	6.0
2011	198,891	67.0	21.9	5.0	6.1

The four components exhibit different trends. Clearly, the FVA share has increased over time from 12.1% in 1995 to 21.9% in 2011. In comparison, the RDV share has declined from 12.6% to 5.0% during the same period.

We note that our definition of DVA at the sector level differs from that of value added exports defined by Johnson and Noguera (2012). The VAX measure describes the amount of domestic value added that originates in this sector (transport equipment) that is exported and absorbed abroad. It excludes domestic value added originated in other sectors (e.g., services) that is exported via the transport equipment sector. The ratio of VAX to gross exports for the US transport equipment sector is 33.3% in 1995 and 24.8% in 2011, which are less than half of the DVA shares in gross exports in the corresponding years.

We now look at the decomposition of Mexico electrical and optical equipment (WIOD sector 14) exports. The results are presented in Table 4b. An important feature of this example is the relatively high share of foreign value added in Mexico's exports in this sector. Indeed, FVA is often higher than DVA, driven in a significant part by imported components in the Maquiladora factories. Note that RDV is tiny in Table 4b (column (5)).

Table 4b: Decomposition of Mexico Electrical and Optical Equipment Exports (WIOD sector 14)

Year (1)	Gross exports (2)	DVA Share (% of (2)) (3)	FVA Share (% of (2)) (4)	RDV Share (% of (2)) (5)	PDC Share (% of (2)) (6)
1995	17,394	45.5	49.5	0.2	4.9
2000	46,483	44.6	49.1	0.3	6.0
2005	54,983	41.0	48.8	0.4	9.9
2007	69,083	41.9	47.8	0.3	9.9
2009	56,401	43.0	48.7	0.3	8.0
2010	67,893	40.1	50.6	0.3	9.1
2011	71,397	38.5	52.1	0.3	9.2

3.2 Further decomposing foreign value added (FVA) and imported content (VS) by source countries

Our decomposition formula allows us to trace the FVA in that sector to the original countries. The evolution of the top five foreign suppliers of value added in the US transport equipment sector is presented in Table 5a. In the 1990s and early 2000s, Japan and Canada are the top two suppliers of foreign value added. However, in more recent years, China has become the top supplier, followed by Canada and Japan.

Table 5a: Main Source Countries for Foreign Value Added in US Transport Equipment Exports (WIOD sector 15) (Unit: % of gross exports)

Year	1995	2000	2005	2010	2011
FVA (VS) share of gross exports	**11.4 (16.0)**	**11.0 (16.9)**	**13.5(20.9)**	**18.4(25.6)**	**19.3 (27.0)**
China	0.3 (0.4)	0.4 (0.7)	1.2 (1.9)	3.0 (4.1)	3.3 (4.5)
Canada	1.8 (2.5)	1.9 (2.9)	2.1 (3.2)	2.4 (3.2)	2.4 (3.3)
Japan	2.7 (3.7)	2.0 (2.8)	1.8 (2.6)	2.2 (2.8)	2.2 (2.7)
Mexico	0.6 (0.9)	0.8 (1.3)	1.2 (1.8)	2.0 (2.7)	2.1 (2.9)
Germany	0.9 (1.2)	0.9 (1.2)	1.2 (1.7)	1.4 (1.8)	1.5 (1.9)
Korea	0.5 (0.6)	0.3 (0.5)	0.5 (0.7)	0.8 (1.0)	0.8 (1.1)
United Kingdom	0.6 (0.9)	0.8 (1.1)	0.6 (0.9)	0.7 (0.9)	0.7 (0.9)

FVA share by top suppliers in Mexico's electrical and optical equipment exports is presented in Table 5b. The United States is the leading supplier of foreign value added to Mexico throughout the sample. However, in terms of relative changes, the most striking feature of Table 5b is the rapid rise of China, and a corresponding decline of the United States. The table suggests that, in a few years, China may very well overtake the United States as the leading foreign supplier of value added to Mexico's electrical and optical equipment industries.

Table 5b: Main Source Countries for Foreign Value Added in Mexico's Electronics Exports (WIOD sector 14)(Unit %)

Year	1995	2000	2005	2010	2011
FVA(VS) share of gross exports	**48.1(54.2)**	**47.6 (54.9)**	**45.9 (58.4)**	**46.5 (59.4)**	**47.5 (61.0)**
USA	31.3 (34.8)	30.0 (34.0)	16.9 (20.7)	15.2 (18.3)	15.1 (18.1)
China	0.6 (0.7)	1.1 (1.4)	5.3 (7.0)	10.9 (14.2)	11.9 (15.4)
Japan	4.1 (4.6)	3.3 (3.7)	4.8 (5.9)	3.4 (4.0)	3.1 (3.6)
Korea	1.1 (1.2)	1.6 (1.8)	2.4 (2.9)	2.7 (3.2)	2.9 (3.4)
Germany	1.9 (2.1)	1.7 (2.0)	2.4 (2.9)	2.0 (2.4)	2.1 (2.4)
Taiwan	0.9 (0.9)	0.9 (1.1)	1.7 (2.1)	1.6 (1.9)	1.5 (1.8)
Canada	1.0 (1.2)	1.2 (1.4)	1.0 (1.4)	1.1 (1.5)	1.0 (1.4)

Vertical specialization or VS, defined as foreign contents in a country's gross export is a summary statistics to measure international production sharing widely used in the literature (e.g., Hummels, Ishii, and Yi (2001), and Antras, (2013). We report its share as gross exports in parenthesis alongside with the share of FVA in Table 5a-b. The difference between FVA and VS share is the share of pure double counting due to the back and forth intermediate goods trade originated from foreign countries. Those intermediate trade transactions are not part of any country's GDP or final demand, similar to domestic inter-industry transactions that use one type intermediate input to produce another type intermediate inputs, just because all cross country trade transaction are recorded by each country's custom authority, so they show up in the official trade statistics, different from domestic intermediate input transactions that are deducted from total gross output when official GDP by industry statistics is account for. We will further discuss the structure of VS and its implications for cross country production sharing in the next sub-section.

3.3 Tracing structures of Vertical Specialization across countries and over time

As showed by our gross exports decomposition equation (31) and Figure 1c, there are different components within VS and each of these components has different economic meanings and represents different types of cross-country production sharing arrangement. For example, large share of foreign value-added in a country's final goods exports (FVA_FIN for short) may indicate that the country mainly engages in final assembling activities based on imported components and just participates in cross-country production sharing on the low end of a global value chain, while an increasing foreign value added share in a country's intermediate exports (FVA_INT for short) may imply the country is upgrading its industry to start producing intermediate goods for other countries, especially when more and more of these goods are exported to third countries for final goods production. The latter is a sign that the country is no longer at the bottom of the GVCs.

Pure double counting of foreign value-added in a country's exports (FDC for short) can only occur when there is back and forth trade of intermediate goods. An increasing weight of FDC share in VS indicates the deepening of cross-country production sharing. Intermediate goods have to cross national borders multiple times before they are used into final goods production. Therefore, knowing the relative importance of these components and their changing trend over time in a country's total VS can help us to gauge the depth and pattern of cross-country production sharing and discover the major driver of the general increase of VS in a country's gross exports during the last two decades.

As shown in Table 6, across all countries and all sectors, the total foreign content (VS) sourced from manufacturing and services sectors used in world manufacturing goods production has increased by 8.3 percentage points (from 22.5% in 1995 to 30.8% in 2011, column 3). Interestingly, the VS structure information reported in the last three columns indicates that this net increase is mainly driven by an increase of FDC. This suggests that the international production chain is getting longer; over time, a rising portion of trade reflects intermediate goods made and exported by one country, used in the production of the next-stage intermediate goods and exported by another country to be used by the next country to produce yet another intermediate good. This progressively more trade of intermediate goods that cross national borders multiple times is what gives the rising share of FDC.

Table 6: Average VS Structure of World Manufacturing Industries

Year	Gross exports	VS share in gross exports	% of VS		
			FVA_FIN	FVA_INT	FDC
1995	4,020,202	22.5	45.5	34.9	19.5
2000	4,916,605	26.5	45.7	32.2	22.2
2005	7,850,625	29.9	42.3	32.5	25.1
2007	10,472,405	31.6	40.7	32.4	26.9
2009	9,093,710	28.4	43.3	33.4	23.2
2010	10,878,662	30.3	41.7	33.6	24.7
2011	12,458,263	30.8	40.6	34.5	25.0

Note: VS is sourced from manufacturing and services sector only.

Because the share of foreign value-added in final goods exports in total VS has declined by about 5 percentage points during the same period (from 44.5% in 1995 to 40.6 in 2011), and because the share of foreign value-added in intermediate goods exports) in total VS stayed almost constant, the increase of VS share in world manufacturing exports is driven mainly by an increase in FDC share (from 19.5% in

1995 to 25% in 2011). If this trend continues, the FDC share may reach the level of the FVA share and become an important feature of cross-country production sharing. If we add the shares of FVA_INT and FDC, these two components involving intermediate goods trade have already accounted for about 60% of the total manufacture VS in 2011.

Of course, there is heterogeneity in the VS structure both across countries and across sectors, especially between industrialized and developing economies. Table 7 reports total VS and its structure in electrical and optical equipment exports for six Asian economies: Japan, Korea, Taiwan, China, India and Indonesia. The three industrialized Asian economies are reported in the right panel. Despite their difference in the level of total VS shares, their VS structure is very similar: lower and declining in FVA_FIN, relatively stable in FVA_INT and rapid expanding in FCD. Taiwan's VS structure is an informative example (presented in right bottom 5 rows in Table 7). Taiwan is an important supplier of parts and components, and crucially, as Taiwan often occupies several different positions on the global production chain (since it produces both inputs into chip making, memory chips themselves, and components that embed the chips), the collective shares of FDC and FVA_INT already exceed 80% of its total VS (or 40% of its gross exports) since 2005. In comparison, for other developing Asian countries such as China, India and Indonesia (presented in the left panel of Table 7), the share of FVA_FIN still accounts for about 50% their total VS until 2011. However, there are also interesting differences among the three emerging Asian giants: the VS structure change during the 17 years for China was mainly driven by the decline of FVA_FIN and increase of FDC, while FVA_INT stayed relatively stable. For Indonesia, it was driven by the rapid expanding of both FVA_INT and FDC, both of them increased more than 10 percentage points during this period, indicating that there was rapid upgrading of Indonesia's electrical and optical equipment industries during this period. While for India, the later-comer in Asian and global production network of electrical and optical equipment, its share of FVA-FIN rose (from 38.2% in 1995 to 52.8% in 2011) and FVA_INT share continued to decline (from 40.2% in 1995 to 25.3% in 2011), while FDC share stayed relatively stable in the last 17 years. This might result from a strategic shift from import substitution to export oriented development; it is also consistent with a move from the upper stream portion of the production chain to a more downstream position as China and Indonesia did decades ago. These empirical evidences indicate that the structure of VS in addition to its total sums offer additional information about each country's respective positions on the global value chain.

Table 7: VS Structure of Electrical and Optical Equipment Exports for Selected Asian Economies

Year	Gross Exports	VS share in Gross Exports	% of VS			Gross Exports	VS share in Gross Exports	% of VS		
			FAV_FIN	FVA_INT	FDC			FAV_FIN	FVA_INT	FDC
			China					Japan		
1995	34,032	22.1	56.9	27.5	15.6	124,265	6.7	44.6	34.8	20.6
2000	68,998	25.9	54.0	23.9	22.1	136,123	9.5	43.5	29.5	27.0
2005	296,936	37.6	52.3	24.4	23.3	143,324	11.8	35.5	31.4	33.1
2010	638,982	29.3	50.4	27.0	22.7	162,861	14.9	34.0	35.1	30.8
2011	721,417	28.9	50.2	27.7	22.1	166,935	16.0	33.1	37.5	29.4
			India					Korea		
1995	1,260	10.9	38.2	40.2	21.6	40,639	27.8	30.0	43.7	26.3
2000	1,927	17.8	41.7	32.2	26.1	60,434	35.1	40.3	30.9	28.7
2005	5,962	20.1	42.3	30.2	27.5	102,595	34.6	31.0	31.2	37.9
2010	23,994	19.0	54.1	24.0	21.9	147,823	36.9	24.8	39.3	36.0
2011	29,470	19.4	52.6	25.3	22.1	159,191	36.8	26.4	40.6	33.0
			Indonesia					Taiwan		
1995	2,831	28.7	70.2	19.1	10.7	41,818	43.8	40.2	39.1	20.7
2000	7,637	30.6	53.6	23.3	23.1	77,861	44.8	41.0	31.3	27.6
2005	8,387	29.7	43.6	26.8	29.6	100,957	49.0	22.2	32.8	45.0
2010	11,666	29.0	46.5	28.1	25.3	142,943	49.1	15.8	40.2	44.0
2011	12,558	30.7	48.1	29.1	22.8	147,646	48.2	17.4	41.7	40.9

Note: VS is sourced from manufacturing and services sector only.

3.4 Two concepts of exports of domestic value added at the country-sector level

As we discussed earlier, there is a backward-linkage based measure of domestic value added embedded in a country-sector's gross exports and a forward-linkage based measure of value added that is originated from a country-sector but is embedded in the gross exports from all sectors of that country. The distinction of the two concepts can be seen via an example of German business services exports.

The first measure of value added is from a recipient or importing country's perspective (user's perspective), and the domestic value added embedded in German business service exports includes German domestic value added from other German sectors used as inputs in the production of German business service exports. This notion of domestic value added exports is called a "backward linkage based" measure. Columns 2-5 of Table 8 provide a "backward-linkage based" decomposition, similar to the two examples in section 3.1. In particular, DVA is the domestic value added from all sectors in Germany that is embedded in its business service sector exports that are ultimately absorbed abroad. Unsurprisingly, all the other terms, RDV, FVA and PDC are relatively small. The DVA is about 93% of the gross exports for that sector.

A second measure, or a "forward linkage based" notion of value added exports takes into account all value added that is originated in the German business service industry but is either directly exported by the service sector or indirectly exported by other German sectors (producer's perspective). For example, if German automobile exports uses German business services, that constitutes indirect exports of value added from German business services, This particular indirect value added exports are a part of the forward-linkage based exports of value added from the German service sector (although they are also part of a backward-linkage exports of German value added that is embedded in German automobile gross exports).

Table 8: German Business Services Exports (WIOD sector 30)

Year	TEXP	Backward looking (Share)				Forward looking (Ratio)	
		DVA (% of (2))	FVA (% of (2))	RDV (% of (2))	PDC (% of (2))	VAX_F (% to (2))	RVA_F (% to (2))
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1995	14,725	92.9	2.7	3.2	1.3	377.3	7.4
2000	19,597	91.4	3.8	2.8	2.0	344.0	6.8
2005	43,240	92.5	3.8	2.0	1.7	293.2	5.2
2007	58,061	92.0	4.0	2.1	1.9	291.1	5.1
2009	59,629	92.5	3.4	2.3	1.8	278.7	4.8
2010	59,814	92.6	3.9	1.8	1.7	282.8	4.3
2011	62,854	92.4	4.0	1.8	1.8	291.6	4.7

If a sector does a lot of indirect exports of its sector-originated value added via other sectors, the forward linkage based measure of value added exports can in principle exceed that sector's direct gross exports because indirect exports of that sector's value added are not part of that sector's gross exports. This is indeed the case for the German business services sector. German exports in other sectors commonly embed value added that is originally from the German business service sector. As we see in column 7 of Table 8, as a result, the forward linkage based measure of value added exports from the German business services sector is often 280% to 377% of the sector's gross exports. (In contrast, the backward-looking measure of total domestic value added in a sector's gross exports is bounded between 0 and 100%.)

These two measures of value added exports at the sector level are useful for different purposes. If one wishes to understand the fraction of a country-sector's gross exports that reflects a country's domestic value added, one should look at the backward-linkage based value added for that sector, which by our decomposition formula is DVA = gross exports – FVA–RDV–PDC. If one wishes to understand the contribution of all value added from a given sector to the country's aggregate exports, one should look at the forward- linkage based measure of value added exports.

We also note briefly that our framework allows distinguishing forward-linkage based value-added exports measure (VAX_F) from GDP by industry in a sectors' gross exports, which also includes forward-linkage based measure of domestic value-added in a given sector but finally returns home (RDV_F) in addition to VAX_F. Such difference is particular important for countries located on the top of a global value chain. To save space, we report some selected industries examples from our decomposition results in Appendix K.

3.5 A new measure of Revealed Comparative Advantage which takes into account both domestic and international production sharing.

The discussion of the forward linkage based measure of value added in a sector's exports naturally leads to a revised notion of a country-sector's revealed comparative advantage. The traditional definition of a country-sector's revealed comparative advantage (traditional RCA, for short) is the share of that country-sector's gross exports in the country's total gross exports relative to that sector's gross exports from all countries as a share of the world total gross exports.

Formally, denoting e_i^{r*} to be the export of good i of Country r, and assuming that there are N commodities and G countries engaged in trade, then the traditional RCA is defined as:

$$TRCA_i^r = \frac{e_i^{r*}}{\sum_{i=1}^{n} e_i^{r*}} \Bigg/ \frac{\sum_{r}^{G} e_i^{r*}}{\sum_{i}^{n}\sum_{r}^{G} e_i^{r*}} \tag{38}$$

When the RCA exceeds one, the country is said to have a revealed comparative advantage in that sector; when the RCA is below one, the country is said to have a revealed comparative disadvantage in that sector.

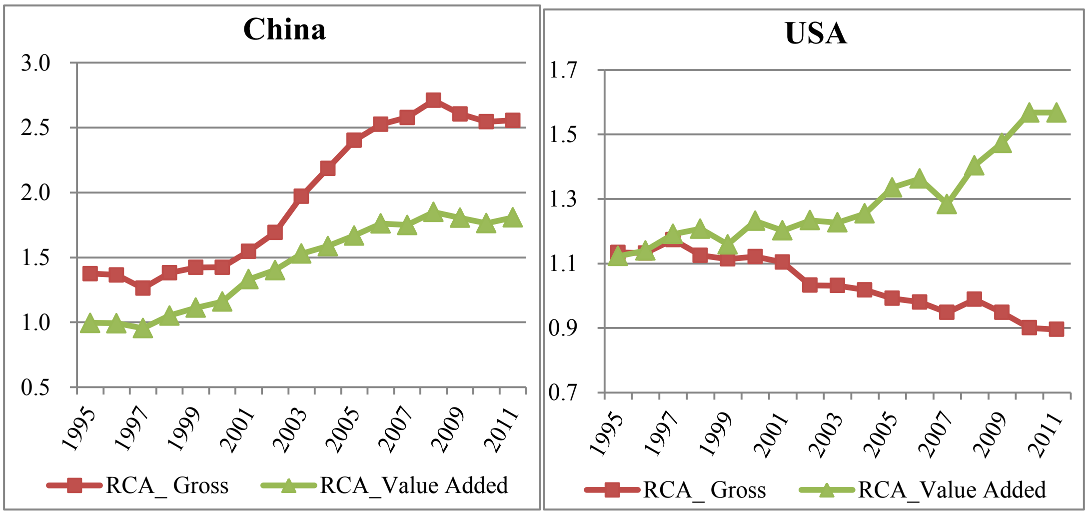
The traditional RCA ignores both domestic production sharing and international production sharing. To be more specific, first, it ignores the fact that a country-sector's value added may be exported indirectly via the country's exports in other sectors. Indirect exports of a sector's value added should be included in a conceptually correct measure of a country's sector's comparative advantage. Second, it also ignores the fact that a country-sector's gross exports partly reflect foreign contents (which show up in both FVA and a portion of PDC). A conceptually correct measure of comparative advantage needs to exclude foreign-originated value added and pure double counted terms in gross exports but include indirect exports of a sector's value added through other sectors of the exporting country.

Taking into account both domestic and international production sharing, we propose to define a new measure of a country sector's revealed comparative advantage (NRCA for short) as the share of a country-sector's forward linkage based measure of domestic value added in exports in the country's total domestic value added in exports relative to that sector's total forward linkage based domestic value added in exports from all countries as a share of global value added in exports. The new RCA measure, or NRCA, is:

$$NRCA_i^r = \frac{vax\_f_i^r + rva\_f_i^r}{\sum_{i=1}^{n}(vax\_f_i^r + rva\_f_i^r)} \Bigg/ \frac{\sum_{r}^{G}(vax\_f_i^r + rva\_f_i^r)}{\sum_{i}^{n}\sum_{r}^{G}(vax\_f_i^r + rva\_f_i^r)} \tag{39}$$

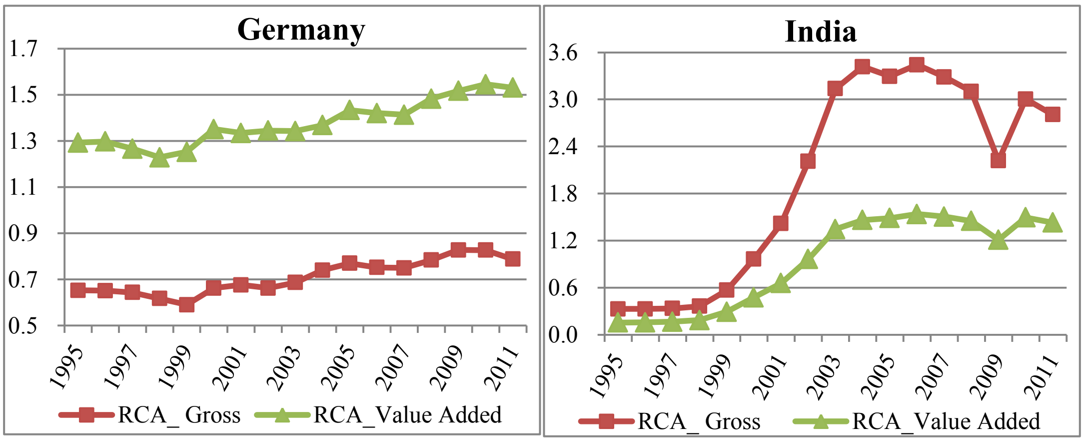
We report two pairs of examples to demonstrate the difference between the traditional and the new measures. First, we compute and plot both RCA for China and the United States in the sector of electric and optimal equipment respectively. The time series profiles of the RCA for China, computed by both methods are presented in the left graph of Figure 2. If one looks at the traditional measure of RCA, this is a strong comparative advantage sector for China, with the RCA exceeding 2.5 since 2007. In contrast, when the new measure of RCA is used, the RCA takes on a much lower value, about 1.8 in recent years.

Figure 2: RCA Indexes for Electrical and Optical Equipment Exports



The RCA for the US in this sector is plotted in the right graph of Figure 2. We see an even bigger divergence between the new and traditional measures. By the traditional measure, electrical and optical equipment has become a comparative disadvantage sector for the United States since 2003. However, by the new measure, not only this sector remains to be a comparative advantage sector for the United States, the strength of the RCA has in fact increased in recent years. The divergent trends in the new and traditional measures of the RCA illustrate the potential misleading nature of the traditional measure. While the traditional measure based on the gross trade data tells a seemingly sobering story of a decline in the US competitiveness in the manufacture of electrical and optical equipment, our new measure reveals the continued robustness of the US comparative advantage in the industry.

Figure 3: RCA Indexes for Business Services Exports



For the second pair of examples, we look at the RCA for India and Germany, respectively, in the business services sector. The traditional and new measures of the RCA for India are plotted in the left graph of Figure 3, whereas the two measures for Germany are plotted in the right graph. India's business services exports are legendary due to media reports about Infosys, Wipro, and call centers. Interestingly, the strength of the RCA for Indian business services is weaker under the new measure than under the traditional measure. In contrast, German business services exports attract less media attention than its manufacturing sector export successes. However, while the business services appear to be a comparative disadvantage sector for Germany based on gross exports (with traditional RCA < 1 throughout 1995-2011), it is a comparative advantage sector by our new measure that takes into account domestic and international production sharing. For India, the domestic business services sector contributes relatively little to the production and exports of other sectors. For Germany, the opposite is the case; the domestic business services sector is a significant contributor to the production and exports of automobiles, machineries, and other products. Once indirect exports of domestic business services are taken into account, India's business service exports become much less impressive relative to Germany and many other developed countries.

3.6 Decomposing bilateral-sector level exports

We consider the US – China bilateral trade in electrical and optical equipment.

Among all bilateral sector level trade flows in recent years, this is the single biggest product group measured by the volume of gross trade, with the sum of the two-way flows reaching 212 billion dollars in 2011. By the gross statistics, presented in column 1 of Table 9, the trade is highly imbalanced, with the Chinese exports ($176.9 billion in 2011) being five times that of the US exports to China ($35.1 billion in 2011). If we separate exports of final goods versus that of intermediate goods (reported in columns 2a and 2b of Table 9), we see that most of the Chinese exports consist of final goods, whereas most of the US exports consist of intermediate goods.

We provide a decomposition of the trade flows for selected years (1995, 2005, and 2011) in columns (3)-(7) of Table 9. More precisely,

Column (1) = (3)+(4)+(5)+(6)+(7), where column (3), DVA, represents the exporter's domestic value added that is ultimately absorbed by other countries, including both the direct importing country and other foreign countries; column (4), RDV, is the part of domestic value added initially exported but ultimately returned home and is absorbed at home; column (5), MVA, is the part of the FVA that comes from the direct importing country; column (6), OVA, is the part of the FVA that comes from third countries; and finally, column (7) is the pure double counted items.

Column (3) = (3a) + (3b) + (3c), that is, the DVA part is further decomposed into DVA in final goods, DVA in intermediate goods absorbed by the direct importer, and DVA in intermediate goods re-exported and ultimately absorbed in third countries.

The decomposition results show that the US and Chinese exports have very different value-added structures. First, the DVA as a share of the gross exports is much higher for the US exports (81% in 2011) than for the Chinese exports (about 70% in 2011)[8]. Second, correspondingly, the FVA share (MVA+OVA) is higher for the Chinese exports than for the US exports. This is especially true for the OVA share in China. In other words, the US exports rely overwhelmingly on its own value added (only 2.1% from China and 5.8% from other countries in 2011), whereas the Chinese

[8] Because WIOD data do not distinguish processing and normal trade, the domestic value added share for China is likely to be overestimated (Koopman, Wang and Wei, 2012).

exports use more foreign value added, especially value added from third countries (with 3.2% from the United States and 23.1% from Japan, Korea, and all other countries). Third, whereas the RDV share is trivial for China, it is non-negligible for the United States (7.0% in 2011). This again reflects the different positions the two countries occupy on the global production chain. As the United States produces and exports parts and components, it is on the upstream of the chain; part of its value added in its exports returns home as embedded in imports from other countries. In comparison, China is on the downstream of the chain; very few of its value added comes home as intermediate goods in other countries' exports. This also evidenced by China having a much higher proportion of FVA used in producing its final goods exports to the US, while the US has a higher share of FVA in producing its intermediate goods exports to China.

The decomposition of DVA into (3a), (3b) and (3c) also reveals differences between the two exporters. In particular, the DVA in the Chinese exports to the United States is dominated by DVA in the final goods, whereas the DVA in the US exports is dominated by DVA in intermediate goods that is absorbed by China and other countries.

As a consequence of these differences in the structure of value added composition, the China – US trade balance in this sector looks much smaller when computed in terms of domestic value added than in terms of gross exports. In column (8), we report forward-linkage based value added exports, or VAX_F. Because this concept captures value added originated in that sector in all downstream sectors of exports from the exporting country but excludes contributions of value added from other (upstream) domestic sectors to the electric and optical equipment sector, it is generally not the same as DVA at the bilateral sector level, and in our application, VAX_F is smaller than DVA (This is generally true for downstream sectors).

In column (9) of Table 9, we report backward-linkage based value added exports, VAX_B, reflecting all the exporting country's value added (from all upstream sectors) that is exported via this sector and absorbed by the direct importing country, including value-added embodied in the source country's gross exports to third countries, but finally absorbed by the partner country. Because the exporter's domestic value added that is exported to and absorbed by a particular partner country indirectly via third countries can be either larger or smaller than the exporter's domestic value-added embodied in its intermediate goods re-exported by the partner country and absorbed by third countries, VAX_B (e.g., 76.4% and 85.3% for Chinese and US exports in 2011 respectively) at the bilateral sector level is generally different from DVA (69.6% and 80.8% of Chinese and US gross exports in 2011 respectively).

We report the US-China bilateral balance of trade in electrical and optical equipment sector by gross and the two value-added export measures in Figure 4. It is important to understand that at the bilateral/sector level, DVA, different from both VAX_F and VAX_B (both of them deviate from gross trade flows), is only part of gross trade flows (so it is the only value-added measure that is consistent with bilateral gross trade flows), but not a measure of bilateral value-added trade flows, because it includes a portion of value-added that is absorbed by third countries (while both VAX_F and VAX_B are absorbed by the importing countries).

Figure 4: China and USA Trade Balance in Electrical and Optical Equipment.
Unit: millions USD

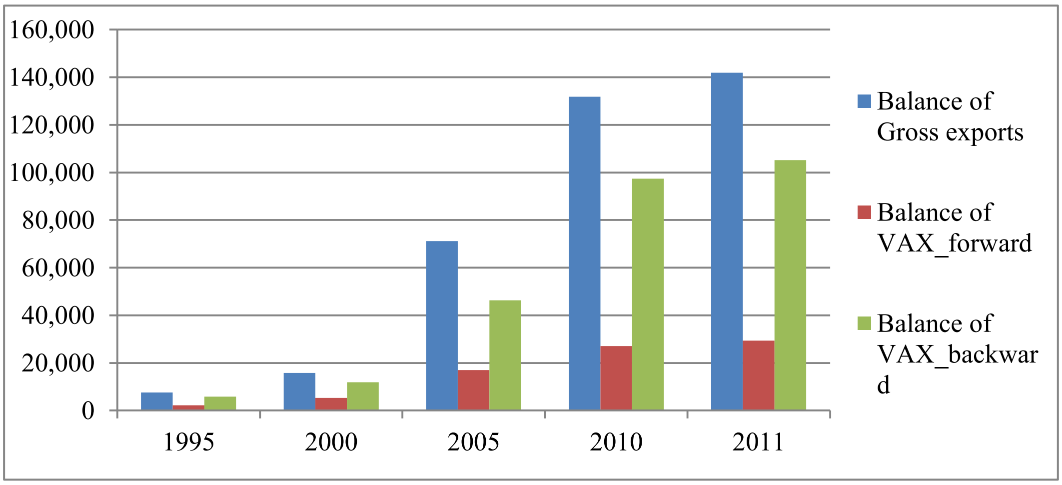


Table 9: US-China Trade in Electrical and Optical Equipment (WIOD C14)

Year		TEXP	TEXPF	TEXPI	DVA	DVA_FIN	DVA_INT	DVA_Intrex	RDV	MVA	OVA	PDC	VAX_F	VAX_B
		(1) =2a +2b =3+4+5 +6+7	(2a)	(2b)	(3) =3a+3b+3c	(3a)	(3b)	(3c)	(4)	(5)	(6)	(7)	(8)	(9)
China exports to the United States														
1995	Value	10,998	7,634	3,364	8,544	5,947	2,046	552	16	314	1,948	176	3,922	9,069
	Share	100	69.4	30.6	77.7	54.1	18.6	5.0	0.1	2.9	17.7	1.6	35.7	82.5
2005	Value	87,608	53,492	34,116	53,784	33,399	16,329	4,056	341	3,665	26,332	3,485	25,682	60,108
	Share	100	61.1	38.9	61.4	38.1	18.6	4.6	0.4	4.2	30.1	4.0	29.3	68.6
2011	Value	176,924	104,156	72,769	123,187	74,043	39,801	9,344	1,296	5,581	40,915	5,946	53,078	135,132
	Share	100	58.9	41.1	69.6	41.8	22.5	5.3	0.7	3.2	23.1	3.4	30.0	76.4
US exports to China														
1995	Value	3,400	1,284	2,116	2,691	1,097	1,215	379	182	13	383	130	1,746	3,288
	Share	100	37.8	62.2	79.2	32.3	35.7	11.2	5.4	0.4	11.3	3.8	51.4	96.7
2005	Value	16,402	3,845	12,556	11,926	3,264	5,072	3,591	1,777	231	1,251	1,216	8,748	13,779
	Share	100	23.4	76.6	72.7	19.9	30.9	21.9	10.8	1.4	7.6	7.4	53.3	84.0
2011	Value	35,059	10,584	24,475	28,314	9,377	12,195	6,742	2,470	718	2,044	1,513	23,754	29,896
	Share	100	30.2	69.8	80.8	26.7	34.8	19.2	7.0	2.1	5.8	4.3	67.8	85.3

Note: (3a) and (3b) equal T1 and T2, (3c) equals the sum of T3 to T5, (4) equals the sum of T6 to T8, (5) equals T11+T13, (6) equals T12+T14 and (7) equals the sum of T9, T10, T15 and T16 in equation (37) of this paper

TEXP notes value of total export goods, TEXPF and TEXPI note value of total final export goods and total intermediate export goods respectively, MVA is sum of MVA_FIN and MVA_INT, OVA is sum of OVA_FIN and OVA_INT.

4. Concluding Remarks

The major contribution of this paper is to provide a disaggregated accounting method for gross exports at either sector, bilateral, or bilateral sector level. It generalizes the framework recently proposed by KWW (2014) for a country's aggregate exports. Our new framework decomposes gross trade flows at any level of disaggregation into four major parts: (a) domestic value added that is absorbed abroad, (b) domestic value added that is initially exported but eventually returned home, (c) foreign value added, and (d) pure double counting terms. The framework in fact allows one to further decompose each of the four major parts above into finer components with economic interpretations. For example, we can decompose FVA in a country-sector's exports into different source countries; we can also trace exports of value added by channels, whether they are embedded in final goods exports, intermediate goods exports that are absorbed in the direct importing countries, or intermediate goods exports that are re-exported and absorbed outside the direct importing countries.

In order to do the decomposition at such disaggregated level, we have to solve a major technical challenge of how to decompose bilateral gross intermediate trade flows based on their final destination of absorption. This goes beyond the initial Leontief insight that has been applied in the existing literature on the decomposition of final demand and GDP by industry. We also point out two major shortcomings associated with the VAX ratio concept widely cited in the literature and ways to overcome them.

By applying our disaggregated decomposition framework to bilateral sector gross trade flows among 40 trading nations in 35 sectors from 1995 to 2011 in the WIOD database, we produce a sequence of large panel data sets that can be used by other researchers for a variety of topics. Because the full decomposition results takes up 20 gigabytes of space, we illustrate potential usefulness of the resulting data by a series of examples that utilize different subsets of the overall decomposition output. For example, we show how we may meaningfully trace the structural changes in the widely used measure of vertical specialization, initially proposed by Hummels, Ishii, and Yi (2001), over time. We distinguish a forward-linkage based measure of domestic value added exports from a sector (that indirectly exports its value added through other sectors' gross exports) from a backward-linkage based measure of value added exports (that includes the value added contributions from other domestic sectors). Based on the decomposition results, we can also correct some shortcomings of a popular measure of revealed comparative advantage and derive a new measure that takes into account both domestic and international production sharing.

In principle, when new countries or years are added to the WIOD database, or an alternative inter-country input-output table becomes available, our accounting framework can be applied as well. So the accounting framework developed in this paper is not inherently tied to the WIOD database and can be a stand-alone tool to help us extract useful information from official trade and national account statistics.

Reference

Antras, Pol, 2013, *Firms, Contracts, and Global Production*, CREI Lectures in Macroeconomics, manuscript on the author's website.

Antràs, Pol, and Davin Chor. 2013."Organizing the Global Value Chain." *Econometrica* 81(6): 2127–2204.

Antras, Pol, Davin Chor, Thibault Fally, and Russell Hillberry, 2012. "Measuring the Upstreamness of Production and Trade Flows," *American Economic Review* 102(3): 412-16, May.

Balassa, Bela. 1965. "Trade Liberalization and 'Revealed' Comparative Advantage." *Manchester School of Economic and Social Studies*, 33, 99–123.

Baldwin, Richard, and Javier Lopez-Gonzalez, 2013, "Supply-Chain Trade: A Portrait of Global Patterns and Several Testable Hypotheses," NBER Working Paper 18957.

Baldwin, Richard, and Robert-Nicoud, 2014, "Trade-in-goods and trade-in-task: An integrating framework." *Journal of International Economics*. 92:51-62.

Daudin, Guillaume, Christine Rifflart, and Danielle Schweisguth. 2011. "Who Produces for Whom in the World Economy?" *Canadian Journal of Economics 44(4):1409-1538, November*.

Feenstra, Robert, 1998, "Integration of Trade and Disintegration of Production in the Global Economy," *Journal of Economic Perspectives*, Fall, 31-50.

Feenstra, Robert, and Gordon Hanson, 1999, "The Impact of Outsourcing and

High-Technology Capital on Wages: Estimates for the U.S., 1972-1990", *Quarterly Journal of Economics*, 114(3): 907-940

Feenstra, Robert and J. Bradford Jensen. 2009. "Evaluating Estimates of Materials Offshoring from U.S. Manufacturing." NBER Working Paper 17916.

Hummels, David, Jun Ishii, and Kei-Mu Yi. 2001. "The Nature and Growth of Vertical Specialization in World Trade." *Journal of International Economics* 54:75–96.

Johnson, Robert, and Guillermo Noguera. 2012. "Accounting for Intermediates: Production Sharing and Trade in Value-added." *Journal of International Economics*: *86:224-236*

Koopman, Robert, Zhi Wang and Shang-Jin Wei. 2012. "Estimating domestic content in exports when processing trade is pervasive," *Journal of Development Economics,* (99):178-189.

Koopman, Robert, Zhi Wang and Shang-Jin Wei, 2014, "Tracing Value-added and Double Counting in Gross Exports," *American Economic Review,* 104(2):459-494. Also available as NBER Working Paper No. 18579.

Leontief W. (1936) "Quantitative Input and Output Relations in the Economic System of the United States" *Review of Economics and Statistics* Vol. 18 pp. 105-125.

Mattoo, Aaditya, Zhi Wang, and Shang-Jin Wei, 2013, *Trade in Value Added: Developing New Measures of Cross-border Trade*, e-book by the CEPR and the World Bank, Washington DC: World Bank.

Miller, R.E. and P.D. Blair (2009). Input–output Analysis: Foundations and Extensions, Cambridge. University Press, Cambridge.

Stehrer, Robert, Neil Foster, and Gaaitzen de Vries, 2012. "Value Added and Factors in Trade: A Comprehensive Approach," WIOD Working Paper No. 7.

Timmer, M. P., Los, B., Stehrer, R. and de Vries, G. J. (2013), Fragmentation, incomes and jobs: an analysis of European competitiveness. Economic Policy, 28: 613–661.

Timmer, M., A.A. Erumban, J. Francois, A. Genty, R. Gouma, B. Los, F. Neuwahl, O. Pindyuk, J. Poeschl, J.M. Rueda-Cantuche, R. Stehrer, G. Streicher, U. Temurshoev, A. Villanueva and G.J. de Vries (2012), The World Input-Output Database (WIOD): Contents, Sources and Methods, WIOD Background document available at www.wiod.org.

Yi, Kei-Mu, 2003, "Can Vertical Specialization Explain the Growth of World Trade?" *Journal of Political Economy*, 111(1): 52–102.

Appendix A: Relation between Global and Local Leontief Inverse Matrices

Define local Leontief inverse of Country s as:

$$L^{ss} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}$$

and note that it is actually the inverse of the block diagonal matrix in the global IO coefficient matrix A in equation (2) of the main text. Then the following proposition gives the mathematical relation between the block diagonals of global Leontief inverse matrix $B^{ss} = \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}$ and the local Leontief inverse L^{ss}:

Proposition: if both B^{ss} and L^{ss} exist then

$$B^{ss} \geq L^{ss} \text{ and } B^{ss} - L^{ss} = L^{ss} A^{sr} B^{rs} \geq 0. \tag{A1}$$

$B^{ss} = L^{ss}$ if and only if when $A^{sr} = 0$ or $B^{rs} = 0$.

This proposition plays an important role in the decomposition of gross exports at the sector level and to the understanding of the difference between trade in value-added estimates from an ICIO table and a national IO table. We give a step by step proof in the 2-country, 2-sector setting bellow and extend it to an N-sector and G-country setting in Appendix J.

Proof:

Based on the property of inverse matrix:

$$\begin{aligned} &\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \end{aligned} \tag{A2}$$

From (A2), we can obtain the relationship between the block diagonals in the global (inter-country) Leontief inverse matrix B and the local (country) Leontief inverse matrix L as the following equation:

$$\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \tag{A3}$$

Multiply L^{ss} for both sides, we have

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}-\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}=\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \tag{A4}$$

Because all elements in $\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}$, $\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}$ and $\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}$ are non-negative,

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}-\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}=\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\geq 0 .$$

B^{ss} is total output requirement coefficients of Country s by one unit increase of its production of final goods, L^{ss} is total output requirement coefficients of Country s by one unit increase of its production of final good using domestic intermediate goods, and $B^{ss}-L^{ss}$ is total output requirement coefficients of Country s by one unit increase of its production of final goods via its intermediate goods trade. $A^{sr}=0$ or $B^{rs}=0$ means there is only one country exports intermediate goods and only in such condition trade in value-added estimates from an ICIO table will be the same as that from a national IO table.

Appendix B: Derivation of decomposition equation of Country r's gross exports

The gross exports of Country r can be decomposed into two parts: final goods exports and intermediate goods exports:

$$E^{rs}=\begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}=\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix}+\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix} \tag{B1}$$

Based on equation (3a) in the main text, Country s' gross output can be decomposed as

$$\begin{aligned}\begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix}&=\begin{bmatrix} x_1^{ss}+x_1^{sr} \\ x_2^{ss}+x_2^{sr} \end{bmatrix}\\ &=\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}+\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}+\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}+\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix}\end{aligned} \tag{B2}$$

Inserting equation (B2) into the last term of equation (B1), we can decompose

Country r's gross intermediate goods exports according to where they are absorbed:

$$\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}$$
$$+ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \tag{B3}$$

From equation (2a) in the main text, Country s' gross output production and use balance condition, we know

$$\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} a_{11}^{ss} & a_{12}^{ss} \\ a_{21}^{ss} & a_{22}^{ss} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}^{ss} & a_{12}^{ss} \\ a_{21}^{ss} & a_{22}^{ss} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \tag{B4}$$

Re-arrange:

$$\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \tag{B5}$$

Define $L^{ss} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}$ as local Leontief inverse, then equation (B5) can be re-written as

$$\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \tag{B6}$$

Therefore, the intermediate goods exports by Country r can also be decomposed into two components according to where it is used similar to a single country IO model:

$$\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \tag{B7}$$

From equation (6) in the main text, we can obtain Country r's domestic and foreign value-added multiplier as follows:

$$V^r B^{rr} = \begin{bmatrix} v_1^r & v_2^r \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} & v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} \tag{B8}$$

$$V^s B^{sr} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} & v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \tag{B9}$$

In a single country IO model, Country r's domestic value-added multiplier can be calculate as

$$V^r(I-A^{rr})^{-1}=V^rL^{rr}=\begin{bmatrix}v_1^r & v_2^r\end{bmatrix}\begin{bmatrix}l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}=\begin{bmatrix}v_1^rl_{11}^{rr}+v_2^rl_{21}^{rr} & v_1^rl_{12}^{rr}+v_2^rl_{22}^{rr}\end{bmatrix} \tag{B10}$$

Using equations (B8)-(B10), and defining "#" as element-wise matrix multiplication operation, the value of Country r's gross intermediate exports can be decomposed as

$$\begin{aligned}
\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}x_1^s\\ x_2^s\end{bmatrix} &= \left\{\begin{bmatrix}v_1^rb_{11}^{rr}+v_2^rb_{21}^{rr}\\ v_1^rb_{12}^{rr}+v_2^rb_{22}^{rr}\end{bmatrix}+\begin{bmatrix}v_1^sb_{11}^{sr}+v_2^sb_{21}^{sr}\\ v_1^sb_{12}^{sr}+v_2^sb_{22}^{sr}\end{bmatrix}\right\}\#\left\{\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}x_1^s\\ x_2^s\end{bmatrix}\right\}\\
&= \begin{bmatrix}v_1^rb_{11}^{rr}+v_2^rb_{21}^{rr}\\ v_1^rb_{12}^{rr}+v_2^rb_{22}^{rr}\end{bmatrix}\#\left\{\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}x_1^s\\ x_2^s\end{bmatrix}\right\}+\begin{bmatrix}v_1^sb_{11}^{sr}+v_2^sb_{21}^{sr}\\ v_1^sb_{12}^{sr}+v_2^sb_{22}^{sr}\end{bmatrix}\#\left\{\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}x_1^s\\ x_2^s\end{bmatrix}\right\}\\
&= \begin{bmatrix}v_1^rl_{11}^{rr}+v_2^rl_{21}^{rr}\\ v_1^rl_{12}^{rr}+v_2^rl_{22}^{rr}\end{bmatrix}\#\left\{\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}x_1^s\\ x_2^s\end{bmatrix}\right\}\\
&+\left\{\begin{bmatrix}v_1^rb_{11}^{rr}+v_2^rb_{21}^{rr}\\ v_1^rb_{12}^{rr}+v_2^rb_{22}^{rr}\end{bmatrix}-\begin{bmatrix}v_1^rl_{11}^{rr}+v_2^rl_{21}^{rr}\\ v_1^rl_{12}^{rr}+v_2^rl_{22}^{rr}\end{bmatrix}\right\}\#\left\{\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}x_1^s\\ x_2^s\end{bmatrix}\right\}\\
&+\begin{bmatrix}v_1^sb_{11}^{sr}+v_2^sb_{21}^{sr}\\ v_1^sb_{12}^{sr}+v_2^sb_{22}^{sr}\end{bmatrix}\#\left\{\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}x_1^s\\ x_2^s\end{bmatrix}\right\}
\end{aligned} \tag{B11}$$

Finally, based on the Leontief insight, Country r's final goods exports can be decomposed into domestic and foreign value-added as follows:

$$\begin{bmatrix}y_1^{rs}\\ y_2^{rs}\end{bmatrix}=\begin{bmatrix}v_1^rb_{11}^{rr}+v_2^rb_{21}^{rr}\\ v_1^rb_{12}^{rr}+v_2^rb_{22}^{rr}\end{bmatrix}\#\begin{bmatrix}y_1^{rs}\\ y_2^{rs}\end{bmatrix}+\begin{bmatrix}v_1^sb_{11}^{sr}+v_2^sb_{21}^{sr}\\ v_1^sb_{12}^{sr}+v_2^sb_{22}^{sr}\end{bmatrix}\#\begin{bmatrix}y_1^{rs}\\ y_2^{rs}\end{bmatrix} \tag{B12}$$

Insert equations (B3) and (B7) into equation (B11), and combining equations (B11) and (B12), we obtain Country r's gross exports decomposition equation as:

$$
\begin{aligned}
\begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} &= \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} - \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \qquad \text{(B13)}
\end{aligned}
$$

Appendix C: Domestic content of Country s

Since the 2nd-6th terms of equation (22) sum to domestic content of the 1st and 2nd sector's intermediate exports of Country s.

$$
\begin{aligned}
&\begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \qquad \text{(C1)} \\
&= \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} + \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \\
&= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\}
\end{aligned}
$$

Adding the first term of equation (22) into equation (C1), we obtain the domestic contents of Country s' gross exports:

$$\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\}$$

$$= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} = \begin{bmatrix} \sum_i^2 v_i^s b_{i1}^{ss} e_1^{sr} \\ \sum_i^2 v_i^s b_{i2}^{ss} e_2^{sr} \end{bmatrix} \tag{C2}$$

Appendix D: Consistency between Equation (22) in this paper and Equation (13) in KWW (2014)

From equation (22), the decomposition of Country s' exports from its 1st sector can also be presented in scalar and summation notations:

$$\begin{aligned} e_1^{sr} &= (v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss}) y_1^{sr} + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &+ (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\ &+ (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + [(v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss}) - (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss})] \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \\ &+ (v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs}) y_1^{sr} + (v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs}) \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &+ (v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs}) \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \end{aligned}$$

Re-arrange

$$\begin{aligned} e_1^{sr} &= \sum_i^2 v_i^s b_{i1}^{ss} y_1^{sr} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rr} y_k^{rr} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rr} y_k^{rs} \\ &+ \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rs} y_k^{ss} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rs} y_k^{sr} + \sum_i^2 v_i^s (b_{i1}^{ss} - l_{i1}^{ss}) \sum_j^2 a_{1j}^{sr} x_j^r \\ &+ \sum_i^2 v_i^r b_{i1}^{rs} y_1^{sr} + \sum_i^2 v_i^r b_{i1}^{rs} \sum_j^2 \sum_k^2 a_{1j}^{sr} l_{jk}^{rr} y_k^{rr} + \sum_i^2 v_i^r b_{i1}^{rs} \sum_j^2 \sum_k^2 a_{1j}^{sr} l_{jk}^{rr} e_k^{rs} \end{aligned} \tag{D1}$$

In matrix notation

$$
\begin{aligned}
e_1^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}
\end{aligned}
\tag{D2}
$$

Similarly, the decomposition of Country s' 2nd sector exports to Country r can be presented as

$$
\begin{aligned}
e_2^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}
\end{aligned}
\tag{D3}
$$

Adding up equation (D2) and (D3), we can get the decomposition of Country S's total gross exports:

$$
\begin{aligned}
e_1^{sr}+e_2^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\right\}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}
\end{aligned}
\tag{D4}
$$

Based on the definition of Leontief Inverse matrix in equation (3a) in the main text, the following identity holds:

$$
\begin{aligned}
&\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}
\end{aligned}
\tag{D5}
$$

Express in block matrix, equation (D5) becomes

$$
\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} - \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = 0
$$

$$
\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix} - \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

With rearrange, we have

$$\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}$$
$$= \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \quad \text{(D6)}$$

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}$$
$$= \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \quad \text{(D7)}$$

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} = \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \quad \text{(D8)}$$

Combine equation (D7) and (D8):

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}$$
$$= \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \quad \text{(D9)}$$

Insert equation (D6) and (D9) into equation (D4):

$$e_1^{sr} + e_2^{sr} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}$$
$$+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}$$
$$+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}$$
$$+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \quad \text{(D10)}$$
$$+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}$$
$$+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}$$

Re-arrange:

$$e^{sr} = e_1^{sr} + e_2^{sr} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}$$
$$+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}$$
$$+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \tag{D11}$$
$$+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}$$
$$+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}$$

It is an extension of equation (13) in KWW from a 2-country, 1-sector case into a 2-country, 2-sector case.

Appendix E: Numerical Example: the 2-country, 2-sector case

Suppose a simple 2-country, 2-sector ICIO table as summarized in table below:

	Country	Intermediate Uses				Final Uses		Gross Output
		s		r		Y^s	Y^r	
Country	Sector	s1	s2	r1	r2			
s	s1	1	1	0	0	1	0	3
	s2	0	1	0	1	0	1	3
r	r1	1	0	1	0	0	1	3
	r2	0	0	1	1	1	0	3
Value-added		1	1	1	1			
Total input		3	3	3	3			

Gross intermediate and final good exports matrix is:

$$E = EI + EF = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

The direct input coefficient matrix A, Global Leontief inverse Matrix B and Local Leontief inverse matrix L and direct value-added coefficient vector V are

$$A=\begin{bmatrix}1/3 & 1/3 & 0 & 0\\ 0 & 1/3 & 0 & 1/3\\ 1/3 & 0 & 1/3 & 0\\ 0 & 0 & 1/3 & 1/3\end{bmatrix} \qquad B=\begin{bmatrix}8/5 & 4/5 & 1/5 & 2/5\\ 1/5 & 8/5 & 2/5 & 4/5\\ 4/5 & 2/5 & 8/5 & 1/5\\ 2/5 & 1/5 & 4/5 & 8/5\end{bmatrix}$$

$$L=\begin{bmatrix}3/2 & 3/4 & 0 & 0\\ 0 & 3/2 & 0 & 0\\ 0 & 0 & 3/2 & 0\\ 0 & 0 & 3/4 & 3/2\end{bmatrix} \qquad V=\begin{bmatrix}1/3 & 1/3 & 1/3 & 1/3\end{bmatrix}$$

The block matrixes are defined below：

$$E^{sr}=EI^{sr}+EF^{sr}=\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}, E^{rs}=EI^{rs}+EF^{rs}=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},$$

$$Y^{ss}=\begin{bmatrix}1\\0\end{bmatrix}, Y^{rr}=\begin{bmatrix}1\\0\end{bmatrix},\ Y^{sr}=EF^{sr}=\begin{bmatrix}0\\1\end{bmatrix}, Y^{rs}=EF^{rs}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$A^{ss}=\begin{bmatrix}1/3 & 1/3\\ 0 & 1/3\end{bmatrix},\ A^{sr}=\begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}, A^{rs}=\begin{bmatrix}1/3 & 0\\ 0 & 0\end{bmatrix},\ A^{rr}=\begin{bmatrix}1/3 & 0\\ 1/3 & 1/3\end{bmatrix},$$

$$B^{ss}=\begin{bmatrix}8/5 & 4/5\\ 1/5 & 8/5\end{bmatrix},\ B^{sr}=\begin{bmatrix}1/5 & 2/5\\ 2/5 & 4/5\end{bmatrix},\ B^{rs}=\begin{bmatrix}4/5 & 2/5\\ 2/5 & 1/5\end{bmatrix},\ B^{rr}=\begin{bmatrix}8/5 & 1/5\\ 4/5 & 8/5\end{bmatrix},$$

$$L^{ss}=\begin{bmatrix}3/2 & 3/4\\ 0 & 3/2\end{bmatrix},\ L^{rr}=\begin{bmatrix}3/2 & 0\\ 3/4 & 3/2\end{bmatrix},\ V^{s}=\begin{bmatrix}1/3 & 1/3\end{bmatrix}, V^{r}=\begin{bmatrix}1/3 & 1/3\end{bmatrix}$$

Based on equations (17)-(19) in the main text, the total value added coefficients can be computed as

$$\left(V^{s}B^{ss}\right)^{T}=\left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}8/5 & 4/5\\ 1/5 & 8/5\end{bmatrix}\right\}^{T}=\begin{bmatrix}3/5 & 4/5\end{bmatrix}^{T}=\begin{bmatrix}3/5\\ 4/5\end{bmatrix}$$

$$\left(V^{s}B^{sr}\right)^{T}=\left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}1/5 & 2/5\\ 2/5 & 4/5\end{bmatrix}\right\}^{T}=\begin{bmatrix}1/5 & 2/5\end{bmatrix}^{T}=\begin{bmatrix}1/5\\ 2/5\end{bmatrix}$$

$$\left(V^{r}B^{rs}\right)^{T}=\left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}4/5 & 2/5\\ 2/5 & 1/5\end{bmatrix}\right\}^{T}=\begin{bmatrix}2/5 & 1/5\end{bmatrix}^{T}=\begin{bmatrix}2/5\\ 1/5\end{bmatrix}$$

$$\left(V^{r}B^{rr}\right)^{T}=\left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}8/5 & 1/5\\ 4/5 & 8/5\end{bmatrix}\right\}^{T}=\begin{bmatrix}4/5 & 3/5\end{bmatrix}^{T}=\begin{bmatrix}4/5\\ 3/5\end{bmatrix}$$

$$\left(V^{s}L^{ss}\right)^{T}=\left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 3/4\\ 0 & 3/2\end{bmatrix}\right\}^{T}=\begin{bmatrix}1/2 & 3/4\end{bmatrix}^{T}=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}$$

$$\left(V^r L^{rr}\right)^T = \left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 0\\ 3/4 & 3/2\end{bmatrix}\right\}^T = \begin{bmatrix}3/4 & 1/2\end{bmatrix}^T = \begin{bmatrix}3/4\\ 1/2\end{bmatrix}$$

Based on equation (12) in the main text, Country s' intermediate exports to Country r can be split as

$$A^{sr}B^{rr}Y^{rr} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}8/5 & 1/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 4/15\end{bmatrix}$$

$$A^{sr}B^{rr}Y^{rs} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}8/5 & 1/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 8/15\end{bmatrix}$$

$$A^{sr}B^{rs}Y^{ss} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}4/5 & 2/5\\ 2/5 & 1/5\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 2/15\end{bmatrix}$$

$$A^{sr}B^{rs}Y^{sr} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}4/5 & 2/5\\ 2/5 & 1/5\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 1/15\end{bmatrix}$$

Adding up the four ABY above, we can get the Country s' intermediate exports to Country r, $EI^{sr} = \begin{bmatrix}0\\ 1\end{bmatrix}$.

Country s' intermediate exports to Country r can be also split as

$$A^{sr}L^{rr}Y^{rr} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 0\\ 3/4 & 3/2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 1/4\end{bmatrix}$$

$$A^{sr}L^{rr}E^{rs} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 0\\ 3/4 & 3/2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 3/4\end{bmatrix}$$

Similarly, Country r's intermediate exports to Country s can be split as

$$A^{rs}B^{ss}Y^{ss} = \begin{bmatrix}8/15\\ 0\end{bmatrix}, A^{rs}B^{ss}Y^{sr} = \begin{bmatrix}4/15\\ 0\end{bmatrix},$$

$$A^{rs}B^{sr}Y^{rr} = \begin{bmatrix}1/15\\ 0\end{bmatrix}, A^{rs}B^{sr}Y^{rs} = \begin{bmatrix}2/15\\ 0\end{bmatrix},$$

$$A^{rs}L^{ss}Y^{ss} = \begin{bmatrix}1/2\\ 0\end{bmatrix}, A^{rs}L^{ss}E^{sr} = \begin{bmatrix}1/2\\ 0\end{bmatrix}$$

Using decomposition equation (22), we can fully decompose Country s and r's gross exports into the nine value-added and double counted components as reported in following table

Gross exports decomposition results: 2-country, 2-sector numerical example

E	T1	T2	T3	T4	T5	T6	T7	T8	T9	VAX_F J&N	VAX_B WWZ	VAX_B Ratio	VAX_F Ratio
s1	0	0	0	0	0	0	0	0	0	1/3	0	0%	∞
s2	4/5	1/5	2/5	1/10	1/20	1/20	1/5	1/20	3/20	2/3	1	50%	33%
r1	0	2/5	1/5	1/20	1/10	1/20	0	1/10	1/10	1/3	2/5	40%	33%
r2	3/5	0	0	0	0	0	2/5	0	0	2/3	3/5	60%	67%
s	4/5	1/5	2/5	1/10	1/20	1/20	1/5	1/20	3/20	1	1	50%	50%
r	3/5	2/5	1/5	1/20	1/10	1/20	2/5	1/10	1/10	1	1	50%	50%

Detailed computation is listed below:

$$T_1^{sr} = \left(V^s B^{ss}\right)^T \# Y^{sr} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/5 \end{bmatrix}$$

$$T_2^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rr} Y^{rr}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 4/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$T_3^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rr} Y^{rs}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 8/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \end{bmatrix}$$

$$T_4^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rs} Y^{ss}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 2/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/10 \end{bmatrix}$$

$$T_5^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rs} Y^{sr}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/20 \end{bmatrix}$$

$$T_6^{sr} = \left(V^s B^{ss} - V^s L^{ss}\right)^T \# \left(EI^{sr}\right) = \left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \right\} \# \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/20 \end{bmatrix}$$

$$T_7^{sr} = \left(V^r B^{rs}\right)^T \# Y^{sr} = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$T_8^{sr} = \left(V^r B^{rs}\right)^T \# \left(A^{rs} L^{ss} Y^{ss}\right) = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/20 \end{bmatrix}$$

$$T_9^{sr} = \left(V^r B^{rs}\right)^T \# \left(A^{rs} L^{ss} E^{sr}\right) = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/20 \end{bmatrix}$$

Adding up the nine components above, we can get the Country s' sectoral exports to Country r, $E^{sr} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Similarly, Country r's sectoral exports to Country s can be fully decomposed as

$$T_1^{rs} = \begin{bmatrix} 0 \\ 3/5 \end{bmatrix},\ T_2^{rs} = \begin{bmatrix} 2/5 \\ 0 \end{bmatrix},\ T_3^{rs} = \begin{bmatrix} 1/5 \\ 0 \end{bmatrix},\ T_4^{rs} = \begin{bmatrix} 1/20 \\ 0 \end{bmatrix},\ T_5^{rs} = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix},$$

$$T_6^{rs} = \begin{bmatrix} 1/20 \\ 0 \end{bmatrix},\ T_7^{rs} = \begin{bmatrix} 0 \\ 2/5 \end{bmatrix},\ T_8^{rs} = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix},\ T_9^{rs} = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix}$$

Adding up the nine components above, we can get the Country r's sectoral exports to Country s, $E^{rs} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Country s' forward linkage and backward linkage based VAX to Country r can be estimated as:

$$VAX\_F^{sr} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \# \left\{ \begin{bmatrix} 8/5 & 4/5 \\ 1/5 & 8/5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$VAX\_B^{sr} = T_1^{sr} + T_2^{sr} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the VAX ratio for Country s' exports to Country r can be estimated as

$$VAX\_F^{sr} ratio = \begin{bmatrix} \infty \\ 1/3 \end{bmatrix} \qquad VAX\_B^{sr}\ \text{ratio} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

This example shows that the two measures of value-added exports only equal to each other at the aggregate level and are different at the sector level. It also shows that when a sector does not have gross exports (S1), but its output is used as intermediate inputs for the other domestic sector that exports (S2), we will have an infinitive VAX_F ratio. Only the new backward linkage based VAX_B ratio defined in this paper has the desired property and economic interpretations at the country-sector level as we demonstrate both analytically and numerically.

Appendix F: The relation between domestic value-added in a sector's gross exports and exports of value-added created from that sector: 2-country, 2-sector case

As discussed in the main text, there are two type measures of domestic value-added exports at the sector level. Technically, the two type measures are computed by aggregating the $\hat{V}B\hat{Y}$ matrix along different directions. The first measure is based on the forward linkages in the IO literature by summing up the off-diagonal elements across the columns along the rows in the $\hat{V}B\hat{Y}$ matrix. It measures how a country's GDP by industry is used to produce exports that are absorbed by the destination countries. It is consistent with the factor content method in the international trade literature and is the same as what is defined in Johnson and Norgera (2012)

The second measure is based on backward linkages in the IO literature by summing up the off-diagonal elements across the rows along the columns in the $\hat{V}B\hat{Y}$ matrix. It decomposes a particular sector's final products according to its value-added sources. It measures each source country's value-added embodied in particular sector's gross export flows absorbed by each destination country, regardless of value-added creating sectors in the source country. This measure is consistent with the GVC case studies in the literature. At the country aggregate bilateral level, these two value-added trade flow measures are exactly the same, but at the sector level they are quite different.

Without loss of generality, let us use vt_1^{sr} to denote value-added exports by the 1st sector of Country s (producer's perspective, forward linkage based) and $vt(e_1^{sr})$ as Country s' domestic value-added in the 1st sector's gross exports that is absorbed in Country r (user perspective, backward linkage based). Then vt_1^{sr} and $vt\left(e_1^{sr}\right)$ can be fully decomposed as follows:

$$\begin{aligned} vt_1^{sr} &= \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &= v_1^s b_{11}^{ss} y_1^{sr} + v_1^s b_{12}^{ss} y_2^{sr} + v_1^s b_{11}^{sr} y_1^{rr} + v_1^s b_{12}^{sr} y_2^{rr} \end{aligned} \qquad \text{(F1)}$$

which is an extension of equation (9) of KWW in the two sector case.

$$vt\left(e_1^{sr}\right)=\begin{bmatrix} v_1^s b_{11}^{ss}+v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss}+v_2^s b_{22}^{ss}\end{bmatrix}^T \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix}+\begin{bmatrix} v_1^s l_{11}^{ss}+v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss}+v_2^s l_{22}^{ss}\end{bmatrix}^T \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr}\end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr}\end{bmatrix}\right\}$$
$$= v_1^s b_{11}^{ss} y_1^{sr} + v_2^s b_{21}^{ss} y_1^{sr} + v_1^s l_{11}^{ss}\sum_i^2\sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr} + v_2^s l_{21}^{ss}\sum_i^2\sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr} \quad \text{(F2)}$$

which is part of the first two terms in equation (22) in the main text.

Based on equation (D6), we have:

$$B^{sr}=\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}=\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr}\end{bmatrix}=L^{ss}A^{sr}B^{rr} \quad \text{(F3)}$$

Inserting equation (F3) into equation (F1) and re-arrange, we have

$$vt_1^{sr} = V_1^s B^{ss} Y^{sr} + V_1^s L^{ss} A^{sr} B^{rr} Y^{rr}$$
$$= v_1^s b_{11}^{ss} y_1^{sr} + v_1^s b_{12}^{ss} y_2^{sr} + v_1^s l_{11}^{ss}\sum_i^2\sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr} + v_1^s l_{12}^{ss}\sum_i^2\sum_j^2 a_{2i}^{sr} b_{ij}^{rr} y_j^{rr} \quad \text{(F4)}$$

Comparing equations (F2) and (F6), the first and third terms of the two measures of value-added exports are the same. They are value-added created from the 1st sector of Country s embodied in Country s' 1st sector's gross exports, or the direct value-added exports of the 1st sector. However, the second and the last term of the two measures are different. Therefore, the difference between vt_1^{sr} and $vt\left(e_1^{sr}\right)$ equals

$$vt_1^{sr} - vt\left(e_1^{sr}\right)=\left[v_1^s b_{12}^{ss} y_2^{sr} + v_1^s l_{12}^{ss}\sum_j^2\sum_k^2 a_{2j}^{sr} b_{jk}^{rr} y_k^{rr}\right]-\left[v_2^s b_{21}^{ss} y_1^{sr} + v_2^s l_{21}^{ss}\sum_j^2\sum_k^2 a_{1j}^{sr} b_{jk}^{rr} y_k^{rr}\right] \text{(F5)}$$

The two terms in the first square brackets of equation (F5) are the second and last terms from equation (F4), representing value added created by the 1st sector of Country s but embodied in gross exports of the 2nd sector in Country s and are finally consumed in Country r (indirect value-added exports of the 2nd sector embodied in the 1st sector's gross exports) and hence has no relation with the 1st sector's gross exports. The two terms in the second square brackets of equation (F5) are the second and last term from equation (F4), representing the 2nd sector's value added that is embodied in the gross exports of the 1st sector produced by Country s and is finally consumed in Country r (indirect value-added exports of the 2nd sector embodied in the 1st sector's gross exports). Unless these indirect value-added exports terms equal to each other, Country s' value-added exports from its 1st sector cannot equal to its domestic value-added embodied in the 1st sector's gross exports absorbed in Country r.

Similarly, the difference between vt_2^{sr} and $vt(e_2^{sr})$ equals

$$vt_2^{sr} - vt(e_2^{sr}) = \left[v_2^s b_{21}^{ss} y_1^{sr} + v_2^s l_{21}^{ss} \sum_i^2 \sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr} \right] - \left[v_1^s b_{12}^{ss} y_2^{sr} + v_1^s l_{12}^{ss} \sum_i^2 \sum_j^2 a_{2i}^{sr} b_{ij}^{rr} y_j^{rr} \right] \quad \text{(F6)}$$

It is easy to show that the sum of equations (F5) and (F6) equals 0. This means that when aggregating the two sectors together, the difference between Country s' value added exports and Country s' domestic value-added in gross exports absorbed in Country r at the sector level cancels out. Therefore, at country aggregate, the two value-added exports measures should equal each other.

Extending the equation (F5) and (F6) to the n-sector case, the value-added exports to Country r produced by sector k of Country s, vt_k^{sr} and the Country s' domestic value-added in sector k's gross exports absorbed in Country r, $vt(e_k^{sr})$ can be expressed as

$$vt_k^{sr} = \sum_i^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_i^n v_k^s b_{ki}^{sr} y_i^{rr} = \sum_i^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_i^n \sum_j^n \sum_u^n v_k^s l_{ki}^{ss} a_{ij}^{sr} b_{ju}^{rr} y_u^{rr} \quad \text{(F7)}$$

$$vt(e_k^{sr}) = \sum_i^n v_i^s b_{ik}^{ss} y_k^{sr} + \sum_i^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} \quad \text{(F8)}$$

It is easy to show:

$$vt(e_k^{sr}) - vt_k^{sr} = \left[\sum_{i \neq k}^n v_i^s b_{ik}^{ss} y_k^{sr} + \sum_{i \neq k}^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} \right] - \left[\sum_{i \neq k}^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_{i \neq k}^n \sum_j^n \sum_u^n v_k^s l_{ki}^{ss} a_{ij}^{sr} b_{ju}^{rr} y_u^{rr} \right] \text{(F9)}$$

The two terms in the first square brackets of equation (F9) are other sectors' value added embodied in sector k's gross exports produced by Country s and finally consumed in Country r in final and intermediate goods respectively. They increase the domestic value-added in sector k's gross exports. Similarly, the two terms in the second square brackets of equation (F9) are the value added created by sector k but embodied in other sectors' intermediate goods respectively. Thus they reduce the value-added created in sector k that can be embodied in sector k's gross exports. Therefore, the two measures of value-added exports at the sector level are not equal in general. Understanding this fact is important for us to define the value-added to gross exports ratio at the country/sector level properly. We will discuss this later in more details when the third country effect can be explicitly accounted in a three country model.

Following KWW (2014), we define a country's value-added exports differently from "domestic value added in a country's gross exports". The latter is the sum of a country's value-added exports and its domestic value-added that was first exported but eventually returns and is consumed at home. The second concept only considers where the value added is originated regardless where it is ultimately absorbed. In comparison, a country's "value added exports" refers to a subset of "domestic value added in a country's gross exports" that is ultimately absorbed abroad. Such a conceptual difference naturally extends to the sector level measures and can be computed from two different directions as the sector level measure of value-added exports.

Based on the Leontief insight, and using equation (F3) and (A4) country s' GDP by sector can be decomposed as

$$\begin{aligned} GDP^s &= \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \right\} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \right\} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \right\} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \end{aligned} \tag{F10}$$

Country r's GDP can be expressed in a similar way:

$$\begin{aligned} GDP^r &= \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \right\} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \end{aligned} \tag{F11}$$

Subtracting global GDP from global gross exports yields residuals as follows:

$$
\begin{aligned}
&E^{sr}+E^{rs}-GDP^{s}-GDP^{r}\\
&=\begin{bmatrix} v_1^s l_{11}^{ss}+v_2^s l_{21}^{ss}\\ v_1^s l_{12}^{ss}+v_2^s l_{22}^{ss}\end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr}\\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs}\\ b_{21}^{rs} & b_{22}^{rs}\end{bmatrix}\begin{bmatrix} y_1^{sr}\\ y_2^{sr}\end{bmatrix}\right\}\\
&+\left\{\begin{bmatrix} v_1^s b_{11}^{ss}+v_2^s b_{21}^{ss}\\ v_1^s b_{12}^{ss}+v_2^s b_{22}^{ss}\end{bmatrix}-\begin{bmatrix} v_1^s l_{11}^{ss}+v_2^s l_{21}^{ss}\\ v_1^s l_{12}^{ss}+v_2^s l_{22}^{ss}\end{bmatrix}\right\}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr}\\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} x_1^{r}\\ x_2^{r}\end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs}\\ v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs}\end{bmatrix}\#\begin{bmatrix} y_1^{sr}\\ y_2^{sr}\end{bmatrix}+\begin{bmatrix} v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs}\\ v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs}\end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr}\\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\begin{bmatrix} y_1^{rr}\\ y_2^{rr}\end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs}\\ v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs}\end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr}\\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\begin{bmatrix} e_1^{rs}\\ e_2^{rs}\end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^r l_{11}^{rr}+v_2^r l_{21}^{rr}\\ v_1^r l_{12}^{rr}+v_2^r l_{22}^{rr}\end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr}\\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix} y_1^{rs}\\ y_2^{rs}\end{bmatrix}\right\}\\
&+\left\{\begin{bmatrix} v_1^r b_{11}^{rr}+v_2^r b_{21}^{rr}\\ v_1^r b_{12}^{rr}+v_2^r b_{22}^{rr}\end{bmatrix}-\begin{bmatrix} v_1^r l_{11}^{rr}+v_2^r l_{21}^{rr}\\ v_1^r l_{12}^{rr}+v_2^r l_{22}^{rr}\end{bmatrix}\right\}\#\left\{\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix} x_1^{s}\\ x_2^{s}\end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^s b_{11}^{sr}+v_2^s b_{21}^{sr}\\ v_1^s b_{12}^{sr}+v_2^s b_{22}^{sr}\end{bmatrix}\#\begin{bmatrix} y_1^{rs}\\ y_2^{rs}\end{bmatrix}+\begin{bmatrix} v_1^s b_{11}^{sr}+v_2^s b_{21}^{sr}\\ v_1^s b_{12}^{sr}+v_2^s b_{22}^{sr}\end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix} y_1^{ss}\\ y_2^{ss}\end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^s b_{11}^{sr}+v_2^s b_{21}^{sr}\\ v_1^s b_{12}^{sr}+v_2^s b_{22}^{sr}\end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix} e_1^{sr}\\ e_2^{sr}\end{bmatrix}\right\}\\
&-\begin{bmatrix} v_1^s\\ v_2^s\end{bmatrix}\#\left\{\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix} y_1^{ss}\\ y_2^{ss}\end{bmatrix}\right\}-\begin{bmatrix} v_1^r\\ v_2^r\end{bmatrix}\#\left\{\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\begin{bmatrix} y_1^{rr}\\ y_2^{rr}\end{bmatrix}\right\}
\end{aligned}
\tag{F12}
$$

Multiply u=[1,1], the unit vector with each terms at the left hand side of (F12) and conceal similar terms in the right hand side, we obtain:

$$
\begin{aligned}
&uE^{sr}+uE^{rs}-uGDP^{s}-uGDP^{r}\\
&=\begin{bmatrix} v_1^s & v_2^s\end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr}\\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs}\\ b_{21}^{rs} & b_{22}^{rs}\end{bmatrix}\begin{bmatrix} y_1^{sr}\\ y_2^{sr}\end{bmatrix}+\begin{bmatrix} v_1^s & v_2^s\end{bmatrix}\left\{\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss}\\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}-\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\right\}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr}\\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} x_1^{r}\\ x_2^{r}\end{bmatrix}\\
&+\begin{bmatrix} v_1^r & v_2^r\end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs}\\ b_{21}^{rs} & b_{22}^{rs}\end{bmatrix}\left\{\begin{bmatrix} y_1^{sr}\\ y_2^{sr}\end{bmatrix}+\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr}\\ a_{21}^{sr} & a_{22}^{sr}\end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\begin{bmatrix} y_1^{rr}\\ y_2^{rr}\end{bmatrix}\right\}+\begin{bmatrix} v_1^s & v_2^s\end{bmatrix}\left\{\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss}\\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}-\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\right\}\begin{bmatrix} e_1^{sr}\\ e_2^{sr}\end{bmatrix}\\
&+\begin{bmatrix} v_1^r & v_2^r\end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr}\\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix} y_1^{rs}\\ y_2^{rs}\end{bmatrix}+\begin{bmatrix} v_1^r & v_2^r\end{bmatrix}\left\{\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr}\\ b_{21}^{rr} & b_{22}^{rr}\end{bmatrix}-\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\right\}\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix} x_1^{s}\\ x_2^{s}\end{bmatrix}\\
&+\begin{bmatrix} v_1^s & v_2^s\end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr}\\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\left\{\begin{bmatrix} y_1^{rs}\\ y_2^{rs}\end{bmatrix}+\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs}\\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}^{-1}\begin{bmatrix} y_1^{ss}\\ y_2^{ss}\end{bmatrix}\right\}+\begin{bmatrix} v_1^r & v_2^r\end{bmatrix}\left\{\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr}\\ b_{21}^{rr} & b_{22}^{rr}\end{bmatrix}-\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\right\}\begin{bmatrix} e_1^{rs}\\ e_2^{rs}\end{bmatrix}\\
&-\begin{bmatrix} v_1^s & v_2^s\end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss}\\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix} y_1^{ss}\\ y_2^{ss}\end{bmatrix}-\begin{bmatrix} v_1^r & v_2^r\end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr}\\ l_{21}^{rr} & l_{22}^{rr}\end{bmatrix}\begin{bmatrix} y_1^{rr}\\ y_2^{rr}\end{bmatrix}
\end{aligned}
\tag{F13}
$$

Using the property of inverse matrix similar to equation (D9), we have

$$
\begin{aligned}
&\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} - \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} = \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \\
&= \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}
\end{aligned} \tag{F14}
$$

Inserting equations (D9) and (F14) into (F13), and re-arrange

$$
\begin{aligned}
&uE^{sr} + uE^{rs} - uGDP^{s} - uGDP^{r} \\
&= 2\begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ 2\begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} - \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \right\} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}^{-1} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} - \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}
\end{aligned} \tag{F15}
$$

Equation (F15) shows clearly that besides the value added produced and consumed at home (the last two terms), which is not a part of either country's gross exports, the seventh and eighth term in equation (22) of the main text (the second term in (F15)), and the seventh and eighth terms in the gross exports decomposition equations of Country r (the forth term in (F15)) given in equation (B13), are double counted only once as foreign value-added in the other country's gross exports. Because the third and fourth terms in (22) reflect part of the countries' GDP, they are not double counted from the global GDP point of view. In comparison, the first and third term in equation (F15) (they are the same as the fifth, sixth and ninth term in equations (22)) are counted twice relative to the global GDP since they are not a part of either country's GDP. This explains the reason why we would like to label the fifth, sixth and ninth term in equations (22) as the "pure" double counted terms to differentiate them from those double counted domestic and foreign value-added in gross export statistics (the third, fourth, seventh and eighth term in the gross exports decomposition equations). The pure double counted terms are greater than zero only when there is two-way intermediate goods trade as pointed by KWW (2014).

Just as the sector level measures of value-added exports can be defined from either the supply-side or user's perspective (i.e., based on forward or backward linkage), the sector level domestic value-added in gross exports can also be defined in these two different directions.

The user's perspective measure for the sector level domestic value-added in sector gross exports for Country s can be defined by directly taking the first four terms from equation (22) as follows:

$$\begin{aligned} dv(E^{sr}) = &\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\ &+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \right\} \\ &+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \end{aligned} \tag{F16}$$

Obviously, it is the domestic value-added portion of Country s' gross exports.

To define measure of sector level domestic value-added in country s' gross exports from the producer's perspective, we use the first four terms of equation (F10) and insert equation (D9) into the forth term:

$$\begin{aligned} dv^{sr} = &\begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \end{aligned} \tag{F17}$$

In fact, they are equivalent to summing up the $\hat{V}B\hat{Y}$ matrix across columns along the rows and then subtracting the part of domestic value added that is directly consumed at home, as the last item is not a part of the either country's exports as shown in equation (F15). Equation (F17) is a generalization of equation (22) in KWW to a two-country and two-sector setting. Detailed derivation is given bellow.

Based on equation (F17), the aggregation of domestic value added in the two sectors' exports can be presented as

$$
\begin{aligned}
udv^{sr} &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix}\#\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{sr} \\ y_2^{sr}\end{bmatrix}+\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix}\#\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rr} \\ y_2^{rr}\end{bmatrix} \\
&+\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix}\#\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rs} \\ y_2^{rs}\end{bmatrix}+\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix}\#\left\{\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}-\begin{bmatrix}l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\right\}\begin{bmatrix}y_1^{ss} \\ y_2^{ss}\end{bmatrix} \\
&=\begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{sr} \\ y_2^{sr}\end{bmatrix}+\begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rr} \\ y_2^{rr}\end{bmatrix} \\
&+\begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rs} \\ y_2^{rs}\end{bmatrix}+\begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\left\{\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}-\begin{bmatrix}l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\right\}\begin{bmatrix}y_1^{ss} \\ y_2^{ss}\end{bmatrix}
\end{aligned}
\tag{F18}
$$

Insert equation (D9) into (F18)

$$
\begin{aligned}
udv^{sr} &= \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{sr} \\ y_2^{sr}\end{bmatrix}+\begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rr} \\ y_2^{rr}\end{bmatrix} \\
&+\begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rs} \\ y_2^{rs}\end{bmatrix}+\begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{ss} \\ y_2^{ss}\end{bmatrix}
\end{aligned}
\tag{F19}
$$

It is easy to see that equation (F19) is the extension of equation (22) in KWW into a 2-country, 2-sector case.

Appendix G: Derivation of Equation (31) in the main text using detailed matrix notation

From a three-country two-sector ICIO model we can obtain Country r's gross output decomposition in terms of all country's final demand as follows

$$
\begin{bmatrix}x_1^r \\ x_2^r\end{bmatrix}=\begin{bmatrix}x_1^{rs}+x_1^{rr}+x_1^{rt} \\ x_2^{rs}+x_2^{rr}+x_2^{rt}\end{bmatrix}=\begin{bmatrix}\sum_u^G\sum_g^G\sum_j^2 b_{1j}^{rg}y_j^{gu} \\ \sum_u^G\sum_g^G\sum_j^2 b_{2j}^{rg}y_j^{gu}\end{bmatrix} \qquad u,g\in G=\{s,r,t\} \tag{G1}
$$

Therefore, the gross output of Country r can be decomposed into the following nine components according to where they are finally absorbed:

$$
\begin{aligned}
\begin{bmatrix}x_1^r \\ x_2^r\end{bmatrix} &= \begin{bmatrix}b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs}\end{bmatrix}\begin{bmatrix}y_1^{ss} \\ y_2^{ss}\end{bmatrix}+\begin{bmatrix}b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs}\end{bmatrix}\begin{bmatrix}y_1^{sr} \\ y_2^{sr}\end{bmatrix}+\begin{bmatrix}b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs}\end{bmatrix}\begin{bmatrix}y_1^{st} \\ y_2^{st}\end{bmatrix} \\
&+\begin{bmatrix}b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr}\end{bmatrix}\begin{bmatrix}y_1^{rs} \\ y_2^{rs}\end{bmatrix}+\begin{bmatrix}b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr}\end{bmatrix}\begin{bmatrix}y_1^{rr} \\ y_2^{rr}\end{bmatrix}+\begin{bmatrix}b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr}\end{bmatrix}\begin{bmatrix}y_1^{rt} \\ y_2^{rt}\end{bmatrix} \\
&+\begin{bmatrix}b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt}\end{bmatrix}\begin{bmatrix}y_1^{ts} \\ y_2^{ts}\end{bmatrix}+\begin{bmatrix}b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt}\end{bmatrix}\begin{bmatrix}y_1^{tr} \\ y_2^{tr}\end{bmatrix}+\begin{bmatrix}b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt}\end{bmatrix}\begin{bmatrix}y_1^{tt} \\ y_2^{tt}\end{bmatrix}
\end{aligned}
\tag{G2}
$$

Insert equation (G2) into the last term of equation (10) in the main text, we can decompose country s' gross intermediate goods exports according to where and how they are absorbed as follows:

$$
\begin{aligned}
\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} &= \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \\
&+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \\
&+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{ts} \\ y_2^{ts} \end{bmatrix} \\
&+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\left\{\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} y_1^{st} \\ y_2^{st} \end{bmatrix}\right\}
\end{aligned}
\tag{G3}
$$

which is equation (24) in the main text expressed in detailed matrix notation.

In the three-country ICIO model, the gross output production and use balance, or the row balance condition becomes:

$$
\begin{aligned}
\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} &= \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix} + \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix}\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} + \begin{bmatrix} a_{11}^{rt} & a_{12}^{rt} \\ a_{21}^{rt} & a_{22}^{rt} \end{bmatrix}\begin{bmatrix} x_1^{t} \\ x_2^{t} \end{bmatrix} + \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix}\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} + \begin{bmatrix} e_1^{rt} \\ e_2^{rt} \end{bmatrix} = \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix}\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix}
\end{aligned}
\tag{G4}
$$

Re-arrange:

$$
\begin{aligned}
\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} &= \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1}\begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix} \\
&= \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix}
\end{aligned}
\tag{G5}
$$

$$
\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix}
\tag{G6}
$$

which is equation (27) in the main text expressed in detailed matrix notation.

Using equations (17), (18), (28) in the main text and $\sum_{u=s,r,t}^{G} V^{u}B^{us} = u$, the value of country s' gross intermediate exports in the 3-country, 2-sector model can be decomposed in a similar way as the 2-country 2-sector case as follows:

$$
\begin{aligned}
\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\} + \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\}
\end{aligned}
\tag{G7}
$$

Finally, based on the Leontief insight, Country s' final goods exports can be decomposed into country s, r, and t 's value-added as follows:

$$
\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}
\tag{G8}
$$

Insert equations (G3), (G6) into (G7) and combine with (G8), we obtain the gross exports decomposition equation expressed in detailed matrix notation similar to equation (31) in the main text:

$$
\begin{aligned}
\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} &= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{ts} \\ y_2^{ts} \end{bmatrix} \right\} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\left(\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} y_1^{st} \\ y_2^{st} \end{bmatrix} \right) \right\} \\
&+ \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix} \right\}
\end{aligned}
\tag{G9}
$$

Appendix H: Forward/backward linkage based value-added exports and domestic value-added in exports stay abroad: 3-country, 2-sector case

Without loss of generality, define vt_1^{sr} as value-added exports by the first sector of Country s (producer's perspective, forward linkage based) to Country r, then

$$\begin{aligned} vt_1^{sr} &= V_1^s B^{ss} Y^{sr} + V_1^s B^{sr} Y^{rr} + V_1^s B^{st} Y^{tr} \\ &= \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{st} & b_{12}^{st} \\ b_{21}^{st} & b_{22}^{st} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \\ &= v_1^s \sum_j^2 b_{1j}^{ss} y_j^{sr} + v_1^s \sum_j^2 b_{1j}^{sr} y_j^{rr} + v_1^s \sum_j^2 b_{1j}^{st} y_j^{tr} \end{aligned} \tag{H1}$$

which shows how equation (34) in the main text is derived, and it is an extension of equation (F1) into 3-country 2-sector setting. The three terms in equation (H1) represent three different ways that value-added created from the 1st sector of the source Country s is absorbed by the destination Country r: The first term is sector 1's value-added embodied in Country s' final goods exports (of both sectors) consumed by Country r, the second term is sector 1's value-added embodied in Country s' intermediate goods exports (of both sectors) used by Country r to produce its domestic final goods and consumed there. The last term is sector 1's value-added embodied in Country s' intermediate goods exports (of both sectors) to third Country t and used by t to produce final goods exports to Country r. This last third country term is the only difference between equations (F1) and (H1).

Denote $vt(e_1^{sr})$ as Country s' domestic value-added in the 1st sector's gross exports that is absorbed in Country r (user perspective, backward linkage based).

$$
\begin{aligned}
& vt(e_1^{sr}) = V^s B^{ss} Y_1^{sr} + V^s L^{ss} A_1^{sr} B^{rr} Y^{rr} + V^s L^{ss} A_1^{sr} B^{rt} Y^{tr} \\
& + V^s L^{ss} A_1^{st} B^{tr} Y^{rr} + V^s L^{ss} A_1^{st} B^{tt} Y^{tr} \\
& = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
& + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
& = (v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss}) y_1^{sr} + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss})(a_{11}^{sr} b_{11}^{rr} y_1^{rr} + a_{11}^{sr} b_{12}^{rr} y_2^{rr} + a_{12}^{sr} b_{21}^{rr} y_1^{rr} + a_{12}^{sr} b_{22}^{rr} y_2^{rr}) \\
& + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss})(a_{11}^{sr} b_{11}^{rt} y_1^{tr} + a_{11}^{sr} b_{12}^{rt} y_2^{tr} + a_{12}^{sr} b_{21}^{rt} y_1^{tr} + a_{12}^{sr} b_{22}^{rt} y_2^{tr}) \\
& + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss})(a_{11}^{st} b_{11}^{tt} y_1^{tr} + a_{11}^{st} b_{12}^{tt} y_2^{tr} + a_{12}^{st} b_{21}^{tt} y_1^{tr} + a_{12}^{st} b_{22}^{tt} y_2^{tr}) \\
& + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss})(a_{11}^{st} b_{11}^{tr} y_1^{rr} + a_{11}^{st} b_{12}^{tr} y_2^{rr} + a_{12}^{st} b_{21}^{tr} y_1^{rr} + a_{12}^{st} b_{22}^{tr} y_2^{rr}) \\
& = (v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss}) y_1^{sr} + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \sum_j^2 a_{1j}^{sr} \sum_i^2 b_{ij}^{rr} (y_j^{rr} + y_j^{rt}) \\
& + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \sum_j^2 a_{1j}^{sr} \sum_i^2 b_{ij}^{rt} (y_j^{tr} + y_j^{tt})
\end{aligned} \tag{H2}
$$

which shows how equation (33) in the main text is derived. It is an extension of equation (F2) into 3-country 2-sector setting. The three terms in equation (H2) represent three different ways that value-added (of both sectors) created from the source Country s is absorbed by the destination Country r: The first term is Country s' exports of its final goods from sector 1 that consumed in Country r. The second term is Country s' exports of its intermediate goods from sector 1 to Country r and used by Country r to produce domestic final goods or intermediate goods exports to third Country t to produce final goods shipped back to r and consumed there; the third term is Country s' exports of its intermediate goods from sector 1 to third Country t, used by Country t to produce final goods export to Country r or to produce intermediate exports to Country r and used by r to produce its domestically consumed final goods. The difference between equations (H2) and (F2) is also due to the third country effect, but in three different ways: Country s' domestic value-added could be embodied into its intermediate goods exports to Country r and used by r to produce another type intermediate goods re-export to third Country t to produce final goods shipped back to Country r; Country s' domestic value-added could also be embodied into its intermediate goods exports to third Country t first, then used by Country t either produce final goods export to Country r or produce intermediate goods to Country r as inputs for its production of domestically consumed final goods.

Denote dva_1^{sr}, the additional value-added trade measure that is different from the two bilateral value-added export measures and only show up in a model with three or more countries, as the sum of first five terms in equation (31) in the main text:

$$
\begin{aligned}
dva_1^{sr} &= V^s B^{ss} Y_1^{sr} + V^s L^{ss} A_1^{sr} B^{rr} Y^{rr} + V^s L^{ss} A_1^{sr} B^{rt} Y^{tt} + V^s L^{ss} A_1^{sr} B^{rr} Y^{rt} + V^s L^{ss} A_1^{sr} B^{rt} Y^{tr} \\
&= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} + \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&= (v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss}) y_1^{sr} + (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \sum_j^2 a_{1j}^{sr} \sum_i^2 b_{ij}^{rr} (y_j^{rr} + y_j^{rt}) \\
&+ (v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss}) \sum_j^2 a_{1j}^{sr} \sum_i^2 b_{ij}^{rt} (y_j^{tr} + y_j^{tt})
\end{aligned}
\tag{H3}
$$

which shows how equation (32) in the main text is derived. It is the only value-added trade measure consistent to bilateral gross trade flows (the two value-added export measures divert from bilateral gross trade flows due to either indirect exports through other domestic sectors or indirect exports through third countries). The difference between equations (H2) and (H3) is obvious, since (H2) not only includes value-added exports from Country s embodied in its own gross exports to Country r (second term), but also include value-added exports by Country s embodied in its gross exports to third Country t, but finally absorbed by Country r(last term), while (H3) only concern value-added embodied in Country s' gross exports to country r, regardless these value-added is finally absorbed by Country r or not.

With these three value-added trade measures precisely defined in mathematics, we are ready to proof proposition A in the main text.

The difference between $vt(e_1^{sr})$ and dva_1^{sr} can be derived as:

$$
\begin{aligned}
vt(e_1^{sr}) - dva_1^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \right\}
\end{aligned}
\tag{H4}
$$

The positive term in equation (H4) is value-added from the 1st sector of Country s embodied in its gross exports to Country t and finally consumed by Country r, while the negative term in (H4) is value-added from the 1st sector of Country s embodied in its gross exports to Country r and finally consumed by Country t, if and only if these two indirect value-added trade terms equals each other, $vt(e_1^{sr}) = dva_1^{sr}$. Obviously, this is not true in general.

Similarly, the difference between $vt(e_1^{st})$ and dva_1^{st} equals:

$$vt(e_1^{st}) - dva_1^{st} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \right\}$$
$$- \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \quad \text{(H5)}$$

The positive term in equation (H5) is value-added from the 1st sector of Country s embodied in its gross exports to Country r and finally consumed by Country t, while the negative term in (H5) is value-added from the 1st sector of Country s embodied in its gross exports to Country t and finally consumed by Country r, if and only if these two terms equals each other, $vt(e_1^{st}) = dva_1^{st}$. Obviously, this is also not true in general.

Compare equations (H4) and (H5), the positive term in (H4) is exactly the negative term in (H5) and vs visa. Therefore, adding the two equations together, i.e. aggregate $vt(e_1^{su})$ and dva_1^{su} over trading partner r and t, we have

$$vt(e_1^{sr}) + vt(e_1^{st}) = dva_1^{sr} + dva_1^{st} \qquad \text{i.e. } \sum_{r \neq s}^{G} dva_i^{sr} = \sum_{r \neq s}^{G} vt(e_i^{sr}) \qquad \text{（H6）}$$

The difference between vt_1^{sr} and $vt(e_1^{sr})$ is not so obvious if we compare equations (H1) and (H2) directly, so we first transform equation (H1) by using following properties of Leontief Inverse matrix:

From $\begin{bmatrix} I - A^{ss} & -A^{sr} & -A^{st} \\ -A^{rs} & I - A^{rr} & -A^{rt} \\ -A^{ts} & -A^{tr} & I - A^{tt} \end{bmatrix} \begin{bmatrix} B^{ss} & B^{sr} & B^{st} \\ B^{rs} & B^{rr} & B^{rt} \\ B^{ts} & B^{tr} & B^{tt} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$

We have:

$(I - A^{ss})B^{sr} - A^{sr}B^{rr} - A^{st}B^{tr} = 0$; $(I - A^{ss})B^{st} - A^{sr}B^{rt} - A^{st}B^{tt} = 0$

Rearrange:

$$B^{sr} = L^{ss}A^{sr}B^{rr} + L^{ss}A^{st}B^{tr}\,;\;\; B^{st} = L^{ss}A^{sr}B^{rt} + L^{ss}A^{st}B^{tt} \tag{H7}$$

Inserting equation (H7) into equation (H1) and re-arrange, we obtain:

$$\begin{aligned}
vt_1^{sr} &= V_1^s B^{ss}Y^{sr} + V_1^s L^{ss}A^{sr}B^{rr}Y^{rr} + V_1^s L^{ss}A^{st}B^{tr}Y^{rr} + V_1^s L^{ss}A^{sr}B^{rt}Y^{tr} + V_1^s L^{ss}A^{st}B^{tt}Y^{tr} \\
&= \begin{bmatrix} v_1^s & 0 \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & 0 \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix}\right\}
\end{aligned} \tag{H8}$$

Comparing equations (H2) and (H8), the two measures have the same BY block matrix and same block local inverse. However, Equation (H8) only include value-added from the 1st sector of Country s, including 1st sector's value-added embodied in its 2nd sector's final and intermediate exports, while equation (H2) only measure Country s' value-added embodied in its 1st sector's final and intermediate goods exports, regardless the value-added come from which sectors. The direct value-added exports of the 1st sector are the same in both measures, but the indirect value-added measured by the two equations are very different. The difference between vt_1^{sr} and $vt\left(e_1^{sr}\right)$ can be derived as

$$\begin{aligned}
vt_1^{sr} - vt\left(e_1^{sr}\right) &= \begin{bmatrix} v_1^s & 0 \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} - \begin{bmatrix} 0 & v_2^s \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix}\right\} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix}\right\} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix}\right\}
\end{aligned} \tag{H9}$$

The three positive terms in equation (H9) represent value added created by the 1st sector of Country s but embodied in final (the first term) and intermediate (the third and fifth terms) gross exports of the 2nd sector in Country s and are finally consumed in Country r (indirect value-added exports of the 1st sector that is embodied in the 2nd sector's gross exports) and hence has no relation with the 1st sector's gross exports.

The three negative terms in equation (H9) are the 2nd sector's value added that is embodied in the final (the second term) and intermediate (the fourth and sixth terms) gross exports of the 1st sector produced by Country s and is finally consumed in Country r (indirect value-added exports of the 2nd sector embodied in the 1st sector's gross exports). Unless these indirect value-added exports terms equal to each other, Country s' value-added exports from its 1st sector cannot be equal to its domestic value-added embodied in its 1st sector's gross exports absorbed in Country r. Therefore, vt_1^{sr} and $vt\left(e_1^{sr}\right)$ do not equal in general.

Similarly, the difference between vt_2^{sr} and $vt\left(e_2^{sr}\right)$ equals

$$
\begin{aligned}
vt_2^{sr} - vt\left(e_2^{sr}\right) = & \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} - \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} \\
& + \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
& - \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
& + \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
& - \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\}
\end{aligned} \tag{H10}
$$

Compare equations（H9）and（H10), the positive terms in (H9) are exactly the same as the negative terms in (H10). This indicates that the difference between vt_1^{sr} and $vt\left(e_1^{sr}\right)$is the difference between value-added produced by the 1st sector embodied in the 2nd sector's gross exports and value-added produced by the 2nd sector embodied in the 1st sector's gross exports, which is exactly the same as the difference between $vt\left(e_2^{sr}\right)$ and vt_2^{sr}. Therefore, when we aggregate over sectors, the difference between vt_i^{sr} and $vt(e_i^{sr})$ will cancels out, the two bilateral value-added exports measures equal each other at country aggregate level:

$$vt_1^{sr} + vt_2^{sr} = vt(e_1^{sr}) + vt(e_2^{sr}) \qquad \text{i.e. } \sum_{i=1}^{n} vt(e_i^{sr}) = \sum_{i=1}^{2} vt_i^{sr} \tag{H11}$$

Based on equation (H6), we are able to infer

$$vt(e_1^{sr}) + vt(e_1^{st}) = dva_1^{sr} + dva_1^{st} \quad \text{and} \quad vt(e_2^{sr}) + vt(e_2^{st}) = dva_2^{sr} + dva_2^{st}$$

Therefore, $vt(e_1^{sr}) + vt(e_1^{st}) + vt(e_2^{sr}) + vt(e_2^{st}) = dva_1^{sr} + dva_1^{st} + dva_2^{sr} + dva_2^{st}$

$$\text{i.e. } \sum_{i=1}^{2}\sum_{r\neq s}^{G} vt(e_i^{sr}) = \sum_{i=1}^{2}\sum_{r\neq s}^{G} dva_i^{sr} \tag{H12}$$

Based on equation (H11), we are able to infer

$$vt_1^{sr} + vt_2^{sr} = vt(e_1^{sr}) + vt(e_2^{sr}) \quad \text{and} \quad vt_1^{st} + vt_2^{st} = vt(e_1^{st}) + vt(e_2^{st})$$

Therefore, $vt_1^{sr} + vt_2^{sr} + vt_1^{st} + vt_2^{st} = vt(e_1^{sr}) + vt(e_2^{sr}) + vt(e_1^{st}) + vt(e_2^{st})$

$$\text{i.e. } \sum_{r\neq s}^{G}\sum_{i=1}^{2} vt_i^{sr} = \sum_{r\neq s}^{G}\sum_{i=1}^{2} vt(e_i^{sr}) \tag{H13}$$

Combine (H12) and (H13), we obtain:

$$\sum_{r\neq s}^{G}\sum_{i=1}^{2} vt(e_i^{sr}) = \sum_{r\neq s}^{G}\sum_{i=1}^{2} vt_i^{sr} = \sum_{r\neq s}^{G}\sum_{i=1}^{2} dva_i^{sr}. \tag{H14}$$

The difference between dva_1^{sr} and vt_1^{sr} equals

$$\begin{aligned}
dva_1^{sr} - vt_1^{sr} &= \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} - \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\}
\end{aligned} \tag{H15}$$

Where the first two positive terms in equation (H15) represent value added created by the 2nd sector of Country s embodied in the 1st sector exports from Country s to Country r and consumed in r. The first two negative terms represent value added created by the 1st sector of Country s embodied in the 2nd sector exports from Country s to Country r and consumed in Country r. The final positive term represent value added created by Country s (both sectors) embodied in the 1st sector exports from Country s to Country r and consumed in third Country t. The final negative term represent value added created by the 1st sector of Country s embodied in exports (both sectors) from Country s to third Country t and consumed in t. These positive and negative terms should not equal each other except very special cases, so dva_1^{sr} and vt_1^{sr} do not equal each other in general.

Similarly, the difference between dva_2^{sr} and vt_2^{sr} equals

$$
\begin{aligned}
dva_2^{sr} - vt_2^{sr} &= \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} - \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\}
\end{aligned}
\tag{H16}
$$

Where the first two positive terms in equation (H16) are the first two negative terms in equation (H15) exactly, and the first two negative terms in equation (H16) are the first two positive terms in equation (H15). Therefore, when we aggregate over sectors, the difference between dva_i^{sr} and vt_i^{sr} will partly cancels out.

$$
\begin{aligned}
&(dva_1^{sr} + dva_2^{sr}) - (vt_1^{sr} + vt_2^{sr}) = \\
&\begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\}
\end{aligned}
\tag{H17}
$$

Where the first term (positive) represents value added created by Country s embodied in Country s' intermediate exports to Country r and absorbed in third Country t. The second term (negative) represents value added created by Country s embodied in Country s' intermediate exports to the third Country t and finally consumed in r. These two indirect value-added exports via other countries (the positive and negative terms) should not equal each other except very special cases, so $(dva_1^{sr}+dva_2^{sr})$ and $(vt_1^{sr}+vt_2^{sr})$ do not equal each other in general. i.e. $\sum_{i=1}^{2} vt_i^{sr} \neq \sum_{i=1}^{2} dva_i^{sr}$ due to indirect value-added trade via third countries.

Similarly, the difference between $(dva_1^{st}+dva_2^{st})$ and $(vt_1^{st}+vt_2^{st})$ equals

$$
\begin{aligned}
&(dva_1^{st}+dva_2^{st})-(vt_1^{st}+vt_2^{st})= \\
&\begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}+\begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix}\right\} \\
&-\begin{bmatrix} v_1^s & v_2^s \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\left\{\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix}+\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix}\right\}
\end{aligned} \tag{H18}
$$

Compare equations（H17）and（H18）, the difference between $(dva_1^{sr}+dva_2^{sr})$ and $(vt_1^{sr}+vt_2^{sr})$ is exactly the same as the difference between $(vt_1^{st}+vt_2^{st})$ and $(dva_1^{st}+dva_2^{st})$, therefore, when one aggregate the two measures over both sector and trading partners, these difference in indirect value-added trade via third countries cancel each other. i.e. $dva_1^{sr}+dva_2^{sr}+dva_1^{st}+dva_2^{st}=vt_1^{sr}+vt_2^{sr}+vt_1^{st}+vt_2^{st}$, we get last two terms of equation (H14) again.

Similar to equation (H15), we can obtain the difference between dva_1^{st} and vt_1^{st} as

$$
\begin{aligned}
dva_1^{st} - vt_1^{st} &= \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{st} \\ 0 \end{bmatrix} - \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{st} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\}
\end{aligned} \tag{H19}
$$

Comparing equation (H15) and (H19), we can see that only the last terms can be partly cancelled out when aggregate dva_1^{su} and vt_1^{su} over trading partners, other terms will be fully reserved.

$$
\begin{aligned}
&(dva_1^{sr} + dva_1^{st}) - (vt_1^{sr} + vt_1^{st}) \\
&= \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} + y_1^{st} \\ 0 \end{bmatrix} - \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} + y_2^{st} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&- \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\}
\end{aligned} \tag{H20}
$$

In the right side of equation (H20), all positive terms represent value added created by the 2nd sector of Country s embodied in the 1st sector exports of Country s and consumed abroad, all negative terms represent value added created by the 1st sector of Country s embodied in the 2nd sector exports of Country s and consumed at abroad. Therefore, $\sum_{r \neq s}^{G} dva_i^{sr} \neq \sum_{r \neq s}^{G} vt_i^{sr}$ except very special cases.

Similarly, aggregate dva_2^{su} and vt_2^{su} over trading partners,

$$
\begin{aligned}
&(dva_2^{sr} + dva_2^{st}) - (vt_2^{sr} + vt_2^{st}) \\
&= \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} + y_2^{st} \end{bmatrix} - \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} + y_1^{st} \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{st} & a_{22}^{st} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\} \qquad \text{(H21)} \\
&- \begin{bmatrix} 0 & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{st} & a_{12}^{st} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{tr} & b_{12}^{tr} \\ b_{21}^{tr} & b_{22}^{tr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{tt} & b_{12}^{tt} \\ b_{21}^{tt} & b_{22}^{tt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \right\}
\end{aligned}
$$

In the right side of equation (H21), all positive terms represent value added created by the 1st sector of Country s embodied in the 2nd sector exports of Country s and consumed abroad, all negative terms represent value added created by the 2nd sector of Country s embodied in the 1st sector exports of Country s and consumed abroad. Therefore, adding up equation (H20) and (H21), the difference will be fully canceled out.

By proof of proposition A, the relationship among the three value-added trade measures is made clear analytically. Then, the proof of proposition C-D is straightforward.

Because equation (H3) is the sum of first five terms in equation (31) of the main text by definition, i. e. dva_i^{sr} is always part of e_i^{sr} and $dva_i^{sr} \le e_i^{sr}$, therefore, $\frac{dva_i^{sr}}{e_i^{sr}} \le 1$ always holds. i.e. Proposition B is valid.

From on Proposition A, we have $\sum_{r \ne s}^{G} vt(e_i^{sr}) = \sum_{r \ne s}^{G} dva_i^{sr}$; and from Proposition B we have $dva_i^{sr} \le e_i^{sr}$, therefore,

$vt(e_i^{s*}) = \sum_{r \ne s}^{G} vt(e_i^{sr}) = \sum_{r \ne s}^{G} dva_i^{sr} \le \sum_{r \ne s}^{G} e_i^{sr} = e_i^{s*}$, i.e. $\frac{vt(e_i^{s*})}{e_i^{s*}} \le 1$. Therefore,

Proposition C is true.

From equation (H1), the definition of value-added exports based on forward linkage, we have

$$\begin{aligned} vt_i^{s*} &= \sum_{u \ne s}^{G} vt_i^{su} = vt_i^{sr} + vt_i^{st} \\ &= V_i^s B^{ss} Y^{sr} + V_i^s B^{sr} Y^{rr} + V_i^s B^{st} Y^{tr} + V_i^s B^{ss} Y^{st} + V_i^s B^{sr} Y^{rt} + V_i^s B^{st} Y^{tt} \\ &= V_i^s B^{ss}(Y^{sr} + Y^{st}) + V_i^s B^{sr}(Y^{rr} + Y^{rt}) + V_i^s B^{st}(Y^{tr} + Y^{tt}) \end{aligned} \tag{H16}$$

where V_i^s is a vector with its ith element equals v_i^s and all other elements equal to 0.

Based on the definition of value-added (GDP) by industry, we have

$$\begin{aligned} GDP_i^s &= v_i^s x_i^s = V_i^s X^s \\ &= V_i^s B^{ss}(Y^{ss} + Y^{sr} + Y^{st}) + V_i^s B^{sr}(Y^{rs} + Y^{rr} + Y^{rt}) + V_i^s B^{st}(Y^{ts} + Y^{tr} + Y^{tt}) \end{aligned} \tag{H17}$$

Subtract equation (H16) from (H17)

$$\begin{aligned} & GDP_i^s - \sum_{u \ne s}^{G} vt_i^{su} \\ &= V_i^s B^{ss}(Y^{ss} + Y^{sr} + Y^{st}) + V_i^s B^{sr}(Y^{rs} + Y^{rr} + Y^{rt}) + V_i^s B^{st}(Y^{ts} + Y^{tr} + Y^{tt}) \\ &- V_i^s B^{ss}(Y^{sr} + Y^{st}) - V_i^s B^{sr}(Y^{rr} + Y^{rt}) - V_i^s B^{st}(Y^{tr} + Y^{tt}) \\ &= V_i^s B^{ss} Y^{ss} + V_i^s B^{sr} Y^{rs} + V_i^s B^{st} Y^{ts} \ge 0 \end{aligned} \tag{H18}$$

Therefore, $vt_i^{s*} = \sum_{u \neq s}^{G} vt_i^{su} \leq v_i^s x_i^s$ and $\frac{\sum_{u \neq s}^{G} vt_i^{su}}{v_i^s x_i^s} \leq 1$ i.e. Proposition D is proved.

Appendix I: Numerical Example: the 3-country, 2-sector case

The 3-country, 2-sector ICIO table

	Country	Intermediate Uses						Final Uses			Gross outputs
		s		r		t		Y^s	Y^r	Y^t	
Country	**Sector**	**s1**	**s2**	**r1**	**r2**	**t1**	**t2**				
S	**s1**	1	1	0	0	0	0	9/10	1/10	0	3
	s2	0	1	0	1	0	0	1	0	0	3
R	**r1**	0	0	1	1	0	0	0	1	0	3
	r2	0	0	1	1	0	0	0	1	1	4
T	**t1**	1	0	0	0	1	0	1	0	0	3
	t2	0	0	0	0	1	1	0	0	1	3
Value-added		1	1	1	1	1	2				
Total inputs		3	3	3	4	3	3				

Gross intermediate and final good exports matrix:

$$E = EI + EF = \begin{bmatrix} 0 & 1/10 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/10 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The direct input coefficient matrix A, Global Leontief inverse Matrix B, Local Leontief inverse matrix L, and direct value-added coefficient vector V can be easily computed as

$$A = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 1/4 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix} \qquad B = \begin{bmatrix} 3/2 & 3/4 & 3/20 & 3/10 & 0 & 0 \\ 0 & 3/2 & 3/10 & 3/5 & 0 & 0 \\ 0 & 0 & 9/5 & 3/5 & 0 & 0 \\ 0 & 0 & 4/5 & 8/5 & 0 & 0 \\ 3/4 & 3/8 & 3/40 & 3/20 & 3/2 & 0 \\ 3/8 & 3/16 & 3/80 & 3/40 & 3/4 & 3/2 \end{bmatrix}$$

$$L = \begin{bmatrix} 3/2 & 3/4 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9/5 & 3/5 & 0 & 0 \\ 0 & 0 & 4/5 & 8/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 3/2 \end{bmatrix} \qquad V = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1/4 & 1/3 & 2/3 \end{bmatrix}$$

The block direct input-output coefficients matrixes:

Name	A^{ss}		A^{sr}		A^{st}	
Block	1/3	1/3	0	0	0	0
matrix	0	1/3	0	1/4	0	0
Name	A^{rs}		A^{rr}		A^{rt}	
Block	0	0	1/3	1/4	0	0
matrix	0	0	1/3	1/4	0	0
Name	A^{ts}		A^{tr}		A^{tt}	
Block	1/3	0	0	0	1/3	0
matrix	0	0	0	0	1/3	1/3

The block global Leontief inverse matrixes:

Name	B^{ss}		B^{sr}		B^{st}	
Block	3/2	3/4	3/20	3/10	0	0
matrix	0	3/2	3/10	3/5	0	0
Name	B^{rs}		B^{rr}		B^{rt}	
Block	0	0	9/5	3/5	0	0
matrix	0	0	4/5	8/5	0	0
Name	B^{ts}		B^{tr}		B^{tt}	
Block	3/4	3/8	3/40	3/20	3/2	0
matrix	3/8	3/16	3/80	3/40	3/4	3/2

The block Local Leontief inverse matrixes

Name	L^{ss}		L^{rr}		L^{tt}	
Block	3/2	3/4	9/5	3/5	3/2	0
matrix	0	3/2	4/5	8/5	3/4	3/2

The block Value Added Coefficients Vectors

Name	V^{s}		V^{r}		V^{t}	
Vectors	1/3	1/3	1/3	1/4	1/3	2/3

Based on equations (17)-(19) and (28) in the main text, the total value added coefficients can be computed as

Name	V^sB^{ss}		V^rB^{rr}		V^tB^{tt}	
Vectors	1/2	3/4	4/5	3/5	1	1
Name	V^sL^{ss}		V^rL^{rr}		V^tL^{tt}	
Vectors	1/2	3/4	4/5	3/5	1	1
Name	V^rB^{rs}		V^tB^{tr}		V^sB^{st}	
Vectors	0	0	1/20	1/10	0	0
Name	V^tB^{ts}		V^sB^{sr}		V^rB^{rt}	
Vectors	1/2	1/4	3/20	3/10	0	0

Based on equation (24), Country s' intermediate exports to Country r can be split into following 8 parts:

$$A^{sr}B^{rr}Y^{rr}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}9/5 & 3/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 3/5\end{bmatrix}$$

$$A^{sr}B^{rt}Y^{tt}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$A^{sr}B^{rr}Y^{rt}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}9/5 & 3/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 2/5\end{bmatrix}$$

$$A^{sr}B^{rt}Y^{tr}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$A^{sr}B^{rr}Y^{rs}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}9/5 & 3/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$A^{sr}B^{rt}Y^{ts}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$A^{sr}B^{rs}Y^{ss}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}9/10\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$A^{sr}B^{rs}(Y^{sr}+Y^{st})=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\left\{\begin{bmatrix}1/10\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right\}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Adding up the eight ABY terms above, we obtain Country s' intermediate exports to Country r $EI^{sr}=\begin{bmatrix}0\\ 1\end{bmatrix}$.

Based on equation (27) in the main text, Country s' intermediate exports to

Country r can also be split as

$$A^{sr}L^{rr}Y^{rr}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}9/5 & 3/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 3/5\end{bmatrix}$$

$$A^{sr}L^{rr}E^{r*}=\begin{bmatrix}0 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}9/5 & 3/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 2/5\end{bmatrix}$$

Applying decomposition equation (31), we can fully decompose each of the three countries' gross bilateral exports into the 16 value-added and double counted components as reported in table 3 of the main text. Detailed computation is listed below:

$$T_1^{sr}=\left(V^sB^{ss}\right)^T\#Y^{sr}=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}1/10\\ 0\end{bmatrix}=\begin{bmatrix}1/20\\ 0\end{bmatrix}$$

$$T_2^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rr}Y^{rr}\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 3/5\end{bmatrix}=\begin{bmatrix}0\\ 9/20\end{bmatrix}$$

$$T_3^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rt}Y^{tt}\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$T_4^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rr}Y^{rt}\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 2/5\end{bmatrix}=\begin{bmatrix}0\\ 3/10\end{bmatrix}$$

$$T_5^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rt}Y^{tr}\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$T_6^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rr}Y^{rs}\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$T_7^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rt}Y^{ts}\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$T_8^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rs}Y^{ss}\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$T_9^{sr}=\left(V^sL^{ss}\right)^T\#\left(A^{sr}B^{rs}(Y^{sr}+Y^{st})\right)=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\#\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$T_{10}^{sr}=\left(V^sB^{ss}-V^sL^{ss}\right)^T\#\left(EI^{sr}\right)=\left\{\begin{bmatrix}1/2\\ 3/4\end{bmatrix}-\begin{bmatrix}1/2\\ 3/4\end{bmatrix}\right\}\#\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$T_{11}^{sr} = \left(V^r B^{rs}\right)^T \# Y^{sr} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \# \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{12}^{sr} = \left(V^r B^{rs}\right)^T \# \left(A^{rs} L^{ss} Y^{ss}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \# \begin{bmatrix} 0 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{13}^{sr} = \left(V^r B^{rs}\right)^T \# \left(A^{rs} L^{ss} E^{s*}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \# \begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{14}^{sr} = \left(V^t B^{ts}\right)^T \# Y^{sr} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \# \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/20 \\ 0 \end{bmatrix}$$

$$T_{15}^{sr} = \left(V^t B^{ts}\right)^T \# \left(A^{rs} L^{ss} Y^{ss}\right) = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/20 \end{bmatrix}$$

$$T_{16}^{sr} = \left(V^t B^{ts}\right)^T \# \left(A^{rs} L^{ss} E^{s*}\right) = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/10 \end{bmatrix}$$

Adding up the 16 components above, we can get the Country s' sectorial exports to Country r $E^{sr} = \begin{bmatrix} 1/10 \\ 1 \end{bmatrix}$.

In the same way, other bilateral trade flows can be fully decomposed as reported in Table 3 in the main text.

Appendix J: The General Case of G Countries and N Sectors

This appendix specifies the general case with any arbitrary number of countries and sectors. The ICIO model, the gross output decomposition matrix based on the Leontief insight, and the total value-added multiplier or value added share by source matrix can be specified as follows:

$$\begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^G \end{bmatrix} = \begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1G} \\ A^{21} & A^{22} & \cdots & A^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ A^{G1} & A^{G2} & \cdots & A^{GG} \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^G \end{bmatrix} + \begin{bmatrix} Y^{11} & Y^{12} & \cdots & Y^{1G} \\ Y^{21} & Y^{22} & \cdots & Y^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{G1} & Y^{G2} & \cdots & Y^{GG} \end{bmatrix} \tag{J1}$$

$$\begin{bmatrix} X^{11} & X^{12} & \cdots & X^{1G} \\ X^{21} & X^{22} & \cdots & X^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ X^{G1} & X^{G2} & \cdots & X^{GG} \end{bmatrix} = \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix} \begin{bmatrix} Y^{11} & Y^{12} & \cdots & Y^{1G} \\ Y^{21} & Y^{22} & \cdots & Y^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{G1} & Y^{G2} & \cdots & Y^{GG} \end{bmatrix}$$

(J2)

$$VB = \begin{bmatrix} V^1 & 0 & \cdots & 0 \\ 0 & V^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V^G \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix}$$

$$= \begin{bmatrix} V^1B^{11} & V^1B^{12} & \cdots & V^1B^{1G} \\ V^2B^{21} & V^2B^{22} & \cdots & V^2B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ V^GB^{G1} & V^GB^{G2} & \cdots & V^GB^{GG} \end{bmatrix}^T \tag{J3}$$

The sum of value added share from all countries in Country s' production equals to unity.

$$\sum_t^G V^t B^{ts} = \mu \tag{J4}$$

With G countries and N sectors, A, and B are GN×GN matrices. A^{sr} is an N×N block input-output coefficient matrix, and B^{sr} denotes the N×N block Leontief (global) inverse matrix, which is the total requirement matrix that describes the amount of gross output in producing Country s required for a one-unit increase in the final demand in destination Country r. V^s is a 1 by N vector of direct value-added coefficients of Country s. X^{sr} is an N×1 gross output vector that gives gross output produced in s and absorbed in r. $X^s = \sum_r^G X^{sr}$ is also an N×1 vector that gives Country s' total gross output. Y^{sr} is an N×1 vector gives final goods produced in s and consumed in r. $Y^s = \sum_r^G Y^{sr}$ is also an N×1 vector that gives the global use of s' final goods. The final demand matrix Y in equation (J1), the gross output decomposition matrix X in equation (J2) and the total value-added multiplier matrix VB are all GN×G matrices.

Country s' gross exports to Country r include intermediate and final goods exports:

$$E^{sr} = Y^{sr} + A^{sr} X^r \tag{J5}$$

where E^{sr} is an N by1 vector of Country s' gross exports to country r.

Based on equation (J2), Country r's gross output can be decomposed as

$$
\begin{aligned}
X^r &= \sum_t^G X^{rt} = \sum_t^G \sum_u^G B^{rt} Y^{tu} \\
&= B^{rr}Y^{rr} + B^{rr}\sum_{t\neq s,r}^G Y^{rt} + B^{rr}Y^{rs} + \sum_{t\neq s,r}^G B^{rt}Y^{tt} + \sum_{t\neq s,r}^G \sum_{u\neq s,t}^G B^{rt}Y^{tu} + \sum_{t\neq s,r}^G B^{rt}Y^{ts} \\
&+ B^{rs}Y^{ss} + \sum_{t\neq s}^G B^{rs}Y^{st}
\end{aligned}
\tag{J6}
$$

Insert equation (J6) in to Country s' intermediate exports to country r, the last term in equation (J5) can be expressed as:

$$
\begin{aligned}
A^{sr}X^r &= A^{sr}B^{rr}Y^{rr} + A^{sr}\sum_{t\neq s,r}^G B^{rt}Y^{tt} + A^{sr}B^{rr}\sum_{t\neq s,r}^G Y^{rt} + A^{sr}\sum_{t\neq s,r}^G B^{rt}\sum_{u\neq s,t}^G Y^{tu} \\
&+ A^{sr}B^{rr}Y^{rs} + A^{sr}\sum_{t\neq s,r}^G B^{rt}Y^{ts} + A^{sr}B^{rs}Y^{ss} + A^{sr}\sum_{t\neq s}^G B^{rs}Y^{st}
\end{aligned}
\tag{J7}
$$

In the right side of equation (J7), Country s' intermediate exports are split into eight terms, similar to equation (24) in the three country model. The 1st term ($A^{sr}B^{rr}Y^{rr}$), 5th term ($A^{sr}B^{rr}Y^{rs}$), and 7th term($A^{sr}B^{rs}Y^{ss}$) are Country s' intermediate exports which are direct absorbed by the importing country to produce its domestic consumed final goods; used by the direct importing country to produce its final goods exports and shipped back to the source country; and used by the direct importing country to produce intermediate goods exports and shipped back to the source country for production of source country's final goods for domestic consumption, respectively. They are exactly the same as the three terms in equation (24). The 2nd term ($A^{sr}\sum_{t\neq s,r}^G B^{rt}Y^{tt}$), 3rd term ($A^{sr}B^{rr}\sum_{t\neq s,r}^G Y^{rt}$), 4th term ($A^{sr}\sum_{t\neq s,r}^G \sum_{u\neq s,t}^G B^{rt}Y^{tu}$), and the 6th term ($A^{sr}\sum_{t\neq s,r}^G B^{rt}Y^{ts}$) are Country s' intermediate exports which are used by the direct importing country to produce intermediate inputs re-exports to the third Countries t in production of its domestic consumed final goods; used by direct importing country to produce its final exports to third Countries (t) (but do not return back to the source country); used by the direct importing country to produce intermediate exports to third countries t for production of final exports shipped to other countries including those returning back to the direct importer (Country r); used by the direct importing country to produce intermediate exports to third countries for production of final exports that return back to the source country respectively. Although with a very similar economic interpretation as those terms in equation (24), all of these four third-country effect terms in equation (J7) include all other G-2 counties, not only a single third Country t as that in equation (24). This means all other counties besides the two partner countries that are the final destinations of the source Country s' intermediate exports, are aggregated together as one group in equation (J7)[1]. The final term, $A^{sr}\sum_{t\neq s}^{G} B^{rs}Y^{st}$, is Country s' intermediate exports used by the direct importing country to produce intermediate goods exports that are shipped back to source country for production of its own final goods exports(from all sectors), similar to the last term in equation (24) for the three-country model.

Based on equation (27) in the main text, we can decompose Country s' intermediate exports to Country r into two parts by using the gross output supply and use balance condition:

$$A^{sr}X^{r} = A^{sr}L^{rr}Y^{rr} + A^{sr}L^{rr}E^{r*} \tag{J8}$$

Where L^{rr} is the N by N local Leontief inverse matrix E^{r*} is a N by 1 vector of total gross exports by Country r, $E^{r*} = \sum_{t\neq r}^{G} E^{rt}$.

From equation (J3), we can obtain the total value-added multiplier for every country

[1] The 1st , the 5th and the 7th terms have only one *ABY* term; the 2nd, the 3rd and the 6th have G-2 *ABY* terms; the 4th term has (G-2)* (G-2) *ABY* terms; the 8th term has G-1 *ABY* terms. Summing up these eight terms, the total number of *ABY* terms are:

$$1+(G-2)+(G-2)+(G-2)*(G-2)+1+(G-2)+1+(G-1)=G*G$$

$$V^sB^{ss}=\begin{bmatrix}\sum_{i}^{N}v_i^s b_{i1}^{ss}\\ \sum_{i}^{N}v_i^s b_{i2}^{ss}\\ \vdots \\ \sum_{i}^{N}v_i^s b_{iN}^{ss}\end{bmatrix}^T \quad V^rB^{rs}=\begin{bmatrix}\sum_{i}^{N}v_i^r b_{i1}^{rs}\\ \sum_{i}^{N}v_i^r b_{i2}^{rs}\\ \vdots \\ \sum_{i}^{N}v_i^r b_{iN}^{rs}\end{bmatrix}^T \quad V^tB^{ts}=\begin{bmatrix}\sum_{i}^{N}v_i^t b_{i1}^{ts}\\ \sum_{i}^{N}v_i^t b_{i2}^{ts}\\ \vdots \\ \sum_{i}^{N}v_i^t b_{iN}^{ts}\end{bmatrix}^T \quad V^sL^{ss}=\begin{bmatrix}\sum_{i}^{N}v_i^s l_{i1}^{ss}\\ \sum_{i}^{N}v_i^s l_{i2}^{ss}\\ \vdots \\ \sum_{i}^{N}v_i^s l_{iN}^{ss}\end{bmatrix}^T \quad \text{(J9)}$$

Using equation (J9), the value of Country s' gross intermediate exports to r can be decomposed as

$$\begin{aligned}A^{sr}X^r &= (V^sL^{ss})^T\#(A^{sr}X^r)+(V^sB^{ss}-V^sL^{ss})^T\#(A^{sr}X^r)\\ &+(V^rB^{rs})^T\#(A^{sr}X^r)+(\sum_{t\neq s,r}^{G}V^tB^{ts})^T\#(A^{sr}X^r)\end{aligned} \quad \text{(J10)}$$

Inserting equations (J7) and (J8) into (J10), we can obtain the decomposition equation of Country s' gross intermediate exports to Country r as

$$\begin{aligned}A^{sr}X^r &= (V^sL^{ss})^T\#(A^{sr}B^{rr}Y^{rr})+(V^sL^{ss})^T\#(A^{sr}\sum_{t\neq s,r}^{G}B^{rt}Y^{tt})\\ &+(V^sL^{ss})^T\#(A^{sr}B^{rr}\sum_{t\neq s,r}^{G}Y^{rt})+(V^sL^{ss})^T\#(A^{sr}\sum_{t\neq s,r}^{G}\sum_{u\neq s,t}^{G}B^{rt}Y^{tu})\\ &+(V^sL^{ss})^T\#(A^{sr}B^{rr}Y^{rs})+(V^sL^{ss})^T\#(A^{sr}\sum_{t\neq s,r}^{G}B^{rt}Y^{ts})+(V^sL^{ss})^T\#(A^{sr}B^{rs}Y^{ss})\\ &+(V^sL^{ss})^T\#(A^{sr}\sum_{t\neq s}^{G}B^{rs}Y^{st})+(V^sB^{ss}-V^sL^{ss})^T\#(A^{sr}X^r)\\ &+(V^rB^{rs})^T\#(A^{sr}L^{rr}Y^{rr})+(V^rB^{rs})^T\#(A^{sr}L^{rr}E^{r*})\\ &+(\sum_{t\neq s,r}^{G}V^tB^{ts})^T\#(A^{sr}L^{rr}Y^{rr})+(\sum_{t\neq s,r}^{G}V^tB^{ts})^T\#(A^{sr}L^{rr}E^{r*})\end{aligned} \quad \text{(J11)}$$

Finally, Country s' final goods exports to r can be decomposed into domestic and foreign value-added as follows:

$$Y^{sr}=(V^sB^{ss})^T\#Y^{sr}+(V^rB^{rs})^T\#Y^{sr}+(\sum_{t\neq s,r}^{G}V^tB^{ts})^T\#Y^{sr} \quad \text{(J12)}$$

Adding up equations (J11) and (J12), we obtain the decomposition equation of gross bilateral exports from Country s to Country r in the most general G-country N-sector case as equation (37) in the main text.

$$E^{sr} = (V^s B^{ss})^T \# Y^{sr} + (V^s L^{ss})^T \# (A^{sr} B^{rr} Y^{rr}) + (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{tt})$$
$$+ (V^s L^{ss})^T \# (A^{sr} B^{rr} \sum_{t \neq s,r}^{G} Y^{rt}) + (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s,r}^{G} \sum_{u \neq s,t}^{G} B^{rt} Y^{tu})$$
$$+ (V^s L^{ss})^T \# (A^{sr} B^{rr} Y^{rs}) + (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{ts}) + (V^s L^{ss})^T \# (A^{sr} B^{rs} Y^{ss}) \tag{J13}$$
$$+ (V^s L^{ss})^T \# (A^{sr} \sum_{t \neq s}^{G} B^{rs} Y^{st}) + (V^s B^{ss} - V^s L^{ss})^T \# (A^{sr} X^r)$$
$$+ (V^r B^{rs})^T \# Y^{sr} + (V^r B^{rs})^T \# (A^{sr} L^{rr} Y^{rr}) + (V^r B^{rs})^T \# (A^{sr} L^{rr} E^{r*})$$
$$+ (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# Y^{sr} + (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# (A^{sr} L^{rr} Y^{rr}) + (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# (A^{sr} L^{rr} E^{r*})$$

The economic interpretations for the 16 terms in equations (J13) are similar to equation (31) as listed in table J1. The only difference is that all the third-country related terms become a sum of G-2 countries except for the two trading partner countries, instead of just one third Country t, as in equation (31).

Summing up all the G-1 trading partners, we obtain the decomposition equation of Country s' gross exports to the world:

$$E^{s*} = \sum_{r \neq s}^{G} E^{sr} = (V^s B^{ss})^T \# \sum_{r \neq s}^{G} Y^{sr} + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rr} Y^{rr}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{tt})$$
$$+ (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rr} \sum_{t \neq s,r}^{G} Y^{rt}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} \sum_{u \neq s,t}^{G} B^{rt} Y^{tu})$$
$$+ (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rr} Y^{rs}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{ts}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss})$$
$$+ (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rs} Y^{st}) + (V^s B^{ss} - V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} X^r)$$
$$+ (\sum_{r \neq s}^{G} V^r B^{rs})^T \# Y^{sr} + (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# Y^{sr} + (\sum_{r \neq s}^{G} V^r B^{rs})^T \# (A^{sr} L^{rr} Y^{rr})$$
$$+ (\sum_{t \neq s,r}^{G} V^t B^{ts})^T \# (A^{sr} L^{rr} Y^{rr}) + \sum_{r \neq s}^{G} (V^r B^{rs})^T \# (A^{sr} L^{rr} E^{r*}) + \sum_{t \neq s,r}^{G} (V^t B^{ts})^T \# (A^{sr} L^{rr} E^{r*})$$
$$= (V^s B^{ss})^T \# \sum_{r \neq s}^{G} Y^{sr} + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rr} Y^{rr}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{tt})$$
$$+ (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rr} \sum_{t \neq s,r}^{G} Y^{rt}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} \sum_{u \neq s,t}^{G} B^{rt} Y^{tu})$$
$$+ (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rr} Y^{rs}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{ts}) + (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss}) \tag{J14}$$
$$+ (V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rs} Y^{st}) + (V^s B^{ss} - V^s L^{ss})^T \# (\sum_{r \neq s}^{G} A^{sr} X^r)$$
$$+ (\sum_{r \neq s}^{G} \sum_{t \neq s}^{G} V^t B^{ts})^T \# Y^{sr} + (\sum_{r \neq s}^{G} \sum_{t \neq s}^{G} V^t B^{ts})^T \# (A^{sr} L^{rr} Y^{rr}) + (\sum_{r \neq s}^{G} \sum_{t \neq s}^{G} V^t B^{ts})^T \# (A^{sr} L^{rr} E^{r*})$$

As a sum of domestic value-added in gross exports to all other G-1 countries, the first 10 terms that decompose Country s' domestic value-added in exports have the same economic interpretations as the first 10 terms in equation (J13). However, the 6 terms that decompose foreign content in bilateral gross exports are summed to three terms with no distinction between direct importing country and all other countries.

Summing up equation (J14) by sectors, we can obtain a decomposition equation for total gross exports of Country s, which is exactly the same as equation (36) in KWW. Detailed math proof is given below.

Table J1 Definition of the 16 Terms in Equation (J13)

Label	Description
T1	DVA exports in final goods exports
T2	DVA in intermediate exports to the direct importer and is absorbed there
T3	DVA in intermediate exports used by the direct importer to produce intermediate exports for production of third countries' domestic used final goods
T4	DVA in Intermediate exports used by the direct importer producing final exports to third countries
T5	DVA in Intermediate exports used by the direct importer producing intermediate exports to third countries
T6	Returned DVA in final goods imports -from the direct importer
T7	Returned DVA in final goods imports -via third countries
T8	Returned DVA in intermediate imports
T9	Double counted DVA used to produce final goods exports
T10	Double counted DVA used to produce intermediate exports
T11	Direct importer's VA in source country's final goods exports
T12	Direct importer's VA in source country's intermediate goods exports
T13	Direct importer's VA double counted in exports production
T14	Third countries' VA in final goods exports
T15	Third countries' countries' VA in intermediate goods exports
T16	Third countries' VA double counted in exports production

Note: These 16 terms are the same as the 16 terms in Equations (31) and (37) as well as table 3 in the main text

Summing up equation (J14) over all sectors, we obtain following equation:

$$\mu E^{s*} = V^s B^{ss} \sum_{r \neq s}^{G} Y^{sr} + \left(V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rr} Y^{rr} + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{tt} \right)$$
$$+ \left(V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rr} \sum_{t \neq s,r}^{G} Y^{rt} + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} B^{rt} \sum_{u \neq s,t}^{G} Y^{tu} \right)$$
$$+ \left(V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rr} Y^{rs} + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s,r}^{G} B^{rt} Y^{ts} \right) + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss} \quad \text{(J15)}$$
$$+ V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rs} Y^{st} + (V^s B^{ss} - V^s L^{ss}) \sum_{r \neq s}^{G} A^{sr} X^r$$
$$+ \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} Y^{sr} + \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} A^{sr} L^{rr} Y^{rr} + \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} A^{sr} L^{rr} E^{r*}$$

Summing up each bracket in equation (J15), and re-arrange

$$\mu E^{s*} = V^s B^{ss} \sum_{r \neq s}^{G} Y^{sr} + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} Y^{tt} + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} \sum_{u \neq s,t}^{G} Y^{tu}$$
$$+ V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} Y^{ts} + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss}$$
$$+ V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} \sum_{t \neq s}^{G} Y^{st} + V^s (B^{ss} - L^{ss}) \sum_{r \neq s}^{G} A^{sr} X^r \quad \text{(J16)}$$
$$+ \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} Y^{sr} + \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} A^{sr} L^{rr} Y^{rr} + \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} A^{sr} L^{rr} E^{r*}$$

Because $\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} = \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr}$

Therefore,

$$V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} Y^{tt} = V^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} Y^{rr}$$
$$V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} \sum_{u \neq s,t}^{G} Y^{tu} = V^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} \sum_{u \neq s,r}^{G} Y^{ru}$$
$$V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} Y^{ts} = V^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} Y^{rs}$$
$$V^s (B^{ss} - L^{ss}) \sum_{r \neq s}^{G} A^{sr} X^r = V^s (B^{ss} - L^{ss}) \sum_{t \neq s}^{G} A^{st} X^t$$

So, equation (J16) can be re-arranged as

$$\mu E^{s*} = V^s B^{ss} \sum_{r \neq s}^{G} Y^{sr} + V^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} Y^{rr} + V^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} \sum_{u \neq s,r}^{G} Y^{ru}$$
$$+ V^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} Y^{rs} + V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss}$$
$$+ V^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} \sum_{t \neq s}^{G} Y^{st} + V^s (B^{ss} - L^{ss}) \sum_{t \neq s}^{G} A^{st} X^t$$
$$+ \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} Y^{sr} + \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} A^{sr} L^{rr} Y^{rr} + \sum_{t \neq s}^{G} V^t B^{ts} \sum_{r \neq s}^{G} A^{sr} L^{rr} E^{r*} \tag{J17}$$

Based on the definition of global Leontief Inverse matrix, following identity holds:

$$\begin{bmatrix} I - A^{11} & -A^{12} & \cdots & -A^{1G} \\ -A^{21} & I - A^{22} & \cdots & -A^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ -A^{G1} & -A^{G2} & \cdots & I - A^{GG} \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$
$$= \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix} \begin{bmatrix} I - A^{11} & -A^{12} & \cdots & -A^{1G} \\ -A^{21} & I - A^{22} & \cdots & -A^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ -A^{G1} & -A^{G2} & \cdots & I - A^{GG} \end{bmatrix} \tag{J18}$$

From (J18) we can obtain following two equations:

$$(I - A^{ss}) B^{sr} - \sum_{t \neq s}^{G} A^{st} B^{tr} = 0 \tag{J19}$$

$$(I - A^{ss}) B^{ss} - \sum_{r \neq s}^{G} A^{sr} B^{rs} = I = B^{ss} (I - A^{ss}) - \sum_{r \neq s}^{G} B^{sr} A^{rs} \tag{J20}$$

Re-arrange equation (J19) and (J20)

$$B^{sr} = (I - A^{ss})^{-1} \sum_{r \neq s}^{G} A^{st} B^{tr} = L^{ss} \sum_{r \neq s}^{G} A^{st} B^{tr} \tag{J21}$$

$$L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} = B^{ss} - L^{ss} = \sum_{r \neq s}^{G} B^{sr} A^{rs} L^{ss} \tag{J22}$$

Because L^{ss}, B^{sr} and $A^{rs} \geq 0$, Therefore $B^{ss} - L^{ss} \geq 0$.

Inserting equation (J21) and (J22) into equation (J17)

$$uE^{s*} = V^s B^{ss}\sum_{r\neq s}^{G} Y^{sr} + V^s \sum_{r\neq s}^{G} B^{sr}Y^{rr} + V^s \sum_{r\neq s}^{G} B^{sr} \sum_{u\neq s,r}^{G} Y^{ru} + V^s \sum_{r\neq s}^{G} B^{sr}Y^{rs}$$

$$+ V^s \sum_{r\neq s}^{G} B^{sr} A^{rs} L^{ss} Y^{ss} + \left(V^s \sum_{r\neq s}^{G} B^{sr} A^{rs} L^{ss} \sum_{t\neq s}^{G} Y^{st} + V^s \sum_{r\neq s}^{G} B^{sr} A^{rs} L^{ss} \sum_{t\neq s}^{G} A^{st} X^t \right) \quad \text{(J23)}$$

$$+ \sum_{t\neq s}^{G} V^t B^{ts} \sum_{r\neq s}^{G} Y^{sr} + \sum_{t\neq s}^{G} V^t B^{ts} \sum_{r\neq s}^{G} A^{sr} L^{rr} Y^{rr} + \sum_{t\neq s}^{G} V^t B^{ts} \sum_{r\neq s}^{G} A^{sr} L^{rr} E^{r*}$$

Re-arrange

$$uE^{s*} = V^s B^{ss}\sum_{r\neq s}^{G} Y^{sr} + V^s \sum_{r\neq s}^{G} B^{sr}Y^{rr} + V^s \sum_{r\neq s}^{G} B^{sr} \sum_{u\neq s,r}^{G} Y^{ru} + V^s \sum_{r\neq s}^{G} B^{sr}Y^{rs}$$

$$+ V^s \sum_{r\neq s}^{G} B^{sr} A^{rs} L^{ss} Y^{ss} + V^s \sum_{r\neq s}^{G} B^{sr} A^{rs} L^{ss} E^{s*} \quad \text{(J24)}$$

$$+ \sum_{t\neq s}^{G} V^t B^{ts} \sum_{r\neq s}^{G} Y^{sr} + \sum_{t\neq s}^{G} V^t B^{ts} \sum_{r\neq s}^{G} A^{sr} L^{rr} Y^{rr} + \sum_{t\neq s}^{G} V^t B^{ts} \sum_{r\neq s}^{G} A^{sr} L^{rr} E^{r*}$$

It is the same as equation (36) in KWW.

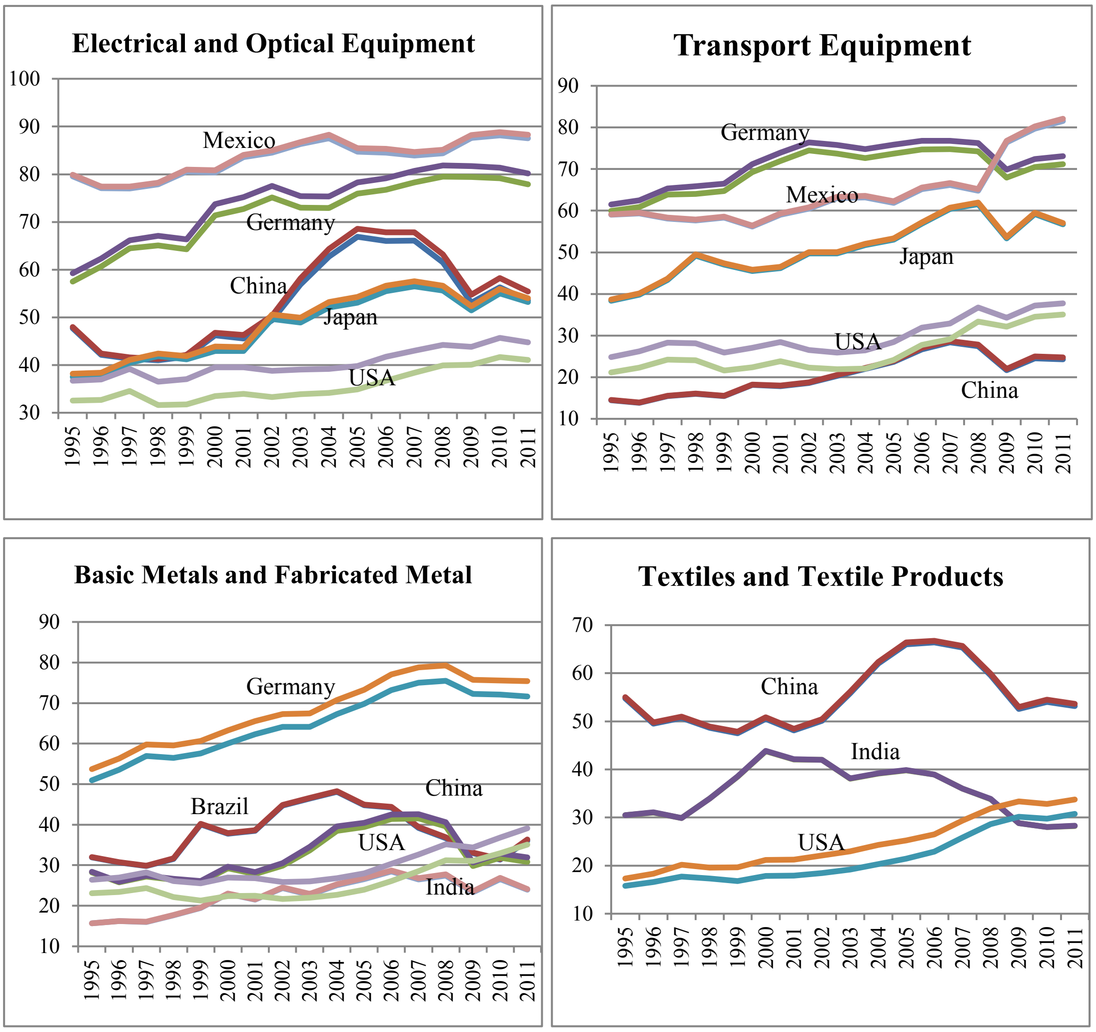
Appendix K: The difference between Value-added exports and GDP by Industry in Gross Exports at the Country-sector Level

As pointed out in KWW, domestic value-added in a country's exports and value-added exports are, in general, not equal to each other. They are related but different concepts. The former only looks where the value added is originated regardless where it is ultimately absorbed. While a country's "value added exports" refers to a subset of "domestic value added in a country's exports" that is ultimately absorbed abroad.

Figure K1 plots the time trend of "value-added exports"(VAX_F) and "domestic value-added" in exports to GDP ratios (both of them are forward linkage based) for four selected industries based on estimates from WIOD. These graphs show clearly domestic value-added in exports to GDP ratios are constantly higher than sector value-added exports to GDP ratios, especially for advanced economies. For instance, the difference between these two ratios is around 4%, 5% and 4% of sector total value-added for the United States, and 3.5%, 2.5% and 2% for Germany in basic mental, electric and optical equipment, and transportation equipment industries, respectively, during the 17 years of our sample period. Even in the textile and textile industries, there is also a 2-3% difference consistently between these two ratios for the

U.S. and Germany during the same period. While the difference between these two ratios for most developing countries is generally tiny.

Figure K1 The Difference between Value-added exports to GDP and Domestic Value-added in exports to GDP ratio



Appendix L Patterns of Production Sharing by Country and Sectors

As our decomposition formula allows us to not only capture the vertical specialization (VS) share of a country's gross exports similar to HIY (2001) VS measure but also each source countries' VS share, we can use the information to characterize the type of production sharing arrangements by country and sector.

In Table L2, we report the average values of VS shares across all countries in 1995 and 2011, for each of the 35 sectors, in Columns 2 and 3, respectively. We sort the sectors in descending order of the value of the average VS share in 2011. The sectors with the highest VS shares in 2011 are electric (and optical) equipment, transport equipment, basic medals, machinery, and rubber and plastics. The sectors with the lowest VS shares are private household (services), education, real estate, public administration, and retail trade. These numbers and the sector order are hardly surprising.

For a given sector, we summarize the distribution of countries in each type of production sharing in Columns 4-6. If the VS share is less than 5%, we label that country-sector as following a national production arrangement. If the VS share exceeds 5%, we label the country-sector as following a cross-country production sharing. We further divide the latter into two categories: if intra-regional sourcing accounts for 60% or more of the VS, we label it as using regional production sharing; otherwise, we label it as using global production sharing. As an example, for the electric (and optical) equipment sector, there is no country in the WIOD database follows a national production arrangement, all 40 countries have significant cross-country production sharing. Within the latter group, 16 countries follow a regional sharing arrangement, and 24 countries follow a global sharing arrangement. By this set of definitions, we find that it is common to see a global production sharing arrangement in the electric equipment, transport equipment, machinery, rubber and plastics, air transport, water transport, textile and leather and footwear industries.

We can also get more details about any particular sector. As an illustration, in Table L3, we zoom in on the transport equipment sector. We list the major developed and emerging market economies in the first column, ordered by the volume of gross exports in that sector in 2011 (recorded in the second column). The largest exporters of transport equipment are Germany, United States, Japan, France, Korea, and China. For each country, we report the top markets for their transport equipment exports in Column 3. For example, for Germany, the largest markets are France, the United States and China. For the United States, the largest markets are Canada, Mexico, and China. In Column 4, we report vertical specialization as a share of the gross exports. All countries on this list have a relatively high VS share, often in excess of 30%. This confirms that transport equipment sector is highly integrated both regionally and globally; production in most countries relies on parts and components made in some foreign countries. In Columns 5 and 6, we report the share of VS coming from within the same region of the country and from outside the region, respectively. Generally speaking, European countries source heavily from other European countries, though they also import value added from outside the region. Most countries outside Europe tend to source globally, with value added from countries outside the region accounting for more than half of the overall VS.

If one wishes to test theories about determinants of offshoring and outsourcing, such information can be very useful.

Table L1 WIOD Country and Region

Label	Country	Region	Label	Country	Region
AUS	Australia	Asia-Pacific	**IRL**	Ireland	Europe
AUT	Austria	Europe	**ITA**	Italy	Europe
BEL	Belgium	Europe	**JPN**	Japan	Asia-Pacific
BGR	Bulgaria	Europe	**KOR**	South Korea	Asia-Pacific
BRA	Brazil	American	**LTU**	Lithuania	Europe
CAN	Canada	American	**LUX**	Luxembourg	Europe
CHN	China	Asia-Pacific	**LVA**	Latvia	Europe
CYP	Cyprus	Europe	**MEX**	Mexico	American
CZE	Czech Republic	Europe	**MLT**	Malta	Europe
DEU	Germany	Europe	**NLD**	Netherlands	Europe
DNK	Denmark	Europe	**POL**	Poland	Europe
ESP	Spain	Europe	**PRT**	Portugal	Europe
EST	Estonia	Europe	**ROM**	Romania	Europe
FIN	Finland	Europe	**RUS**	Russia	Europe
FRA	France	Europe	**SVK**	Slovak Republic	Europe
GBR	United Kingdom	Europe	**SVN**	Slovenia	Europe
GRC	Greece	Europe	**SWE**	Sweden	Europe
HUN	Hungary	Europe	**TUR**	Turkey	Europe
IDN	Indonesia	Asia-Pacific	**TWN**	Taiwan	Asia-Pacific
IND	India	Asia-Pacific	**USA**	United States	American

Table L2: Patterns of International Production Sharing by Sector

	VS share in gross exports		Numbers of countries in each type of production sharing arrangements in 2011		
Sector	1995	2011	National Production	Regional Sharing	Global Sharing
c14: Electrical Equipment	28.6	33.5	0	16	24
c15: Transport Equipment	26.8	33.4	0	23	17
c12: Basic Metals	24.7	28.7	0	28	12
c13: Machinery	26.0	28.6	0	23	17
c10: Rubber and Plastics	26.4	28.5	0	23	17
c09: Chemical Products	24.2	26.9	0	25	15
c04: Textiles Products	25.0	25.6	0	23	17
c25: Air Transport	21.0	25.5	0	19	21
c16: Recycling	20.6	24.5	0	27	13
c24: Water Transport	22.3	23.6	0	22	18
c08: Refined Petroleum	23.3	22.9	5	22	13
c07: Paper and Printing	21.1	22.4	0	30	10
c05: Leather and Footwear	21.1	20.9	2	20	18
c06: Wood Products	17.5	19.7	0	33	7
c11: Other Non-Metal	17.3	18.9	0	30	10
c18: Construction	16.5	17.9	2	25	13
c03: Food	13.6	15.9	0	28	12
c17: Electricity, Gas and Water	13.7	15.4	2	25	13
c23: Inland Transport	11.7	15.3	1	26	13
c26: Other Transport	13.2	15.1	3	23	14
c01: Agriculture	10.6	13.5	4	29	7
c19: Sale of Vehicles and Fuel	11.4	13.3	7	23	10
c27:Post and Telecommunications	9.3	12.7	3	21	16
c02: Mining	11.9	12.5	7	28	5
c34: Other Services	10.8	12.1	4	24	12
c20: Wholesale Trade	10.4	11.7	6	22	12
c30: Business services	11.2	11.7	5	20	15
c33: Health and Social Work	9.9	11.6	6	29	5
c22: Hotels and Restaurants	9.6	10.6	4	28	8
c28: Financial Intermediation	8.0	9.9	14	17	9
c21: Retail Trade	8.5	9.5	6	26	8
c31: Public Admin	8.2	8.9	5	25	10
c29: Real Estate	4.2	6.0	19	18	3
c32: Education	4.8	4.9	23	16	1
c35: Private Households	0.3	0.3	39	1	0

Note: VS is sourced from manufacturing and services sector only. National sharing defined as VS <10%; Regional sharing defined as VS > 5%, regional VS > 60% of total VS; Global sharing defined as VS >5%, regional VS<60% of total VS.

Table L3: Production Sharing Patterns in the Transport Equipment Sector (WIOD 15) for Selected Countries in 2011

Country (1)	Gross exports (2)	Top 3 Destinations and its Share (3)	VS (4)	Region % of (4) (5)	Extra -regional % of (4) (6)	Top 3 Suppliers of FVA (7)
Germany	312,488	FRA(13.1),USA(9.0) CHN(8.9)	31.08	62.80	37.20	FRA(8.9),CHN(8.2), ITA(7.8)
USA	198,891	CAN(24.8),MEX(11.5) CHN(7.4)	23.73	25.70	74.30	CHN(16.9),CAN(12.6), JPN(11.3),MEX(11.1)
Japan	178,412	USA(23.6),CHN(11.1) RUS(8.4)	11.00	44.72	55.28	CHN(25.0),USA(13.3) KOR(9.0)
France	127,659	DEU(20.4)ESP(8.7) GBR(5.9)	36.78	65.79	34.21	DEU(27.3),USA(9.6) ITA(7.0)
Korea	121,150	USA(12.0),RUS(9.4) DEU(7.9)	24.63	50.01	49.99	CHN(22.2), JPN(19.5) USA(12.4)
China	96,956	USA(12.4), DEU(8.4) RUS(5.5)	16.68	36.52	63.48	JPN(17.9), USA(13.1) DEU(11.8)
UK	84,809	DEU(16.6), USA(9.4) FRA(7.2)	33.83	56.15	43.85	DEU(18.8),USA(15.1) CHN(7.3)
Canada	75,047	USA(79.7), MEX(2.8) DEU(1.9)	31.51	57.91	42.09	USA(50.5),CHN(8.7) JPN(6.6)
Italy	50,463	DEU(17.5), FRA(9.6)) GBR(7.5)	26.38	62.03	37.97	DEU(19.6),CHN(8.7) FRA(7.9)
Poland	34,410	DEU(25.8), ITA(12.2) GBR(7.8)	42.60	70.85	29.15	DEU(28.4),ITA(8.6) CHN(6.8)
Czech	28,520	DEU(31.5), RUS(8.6) FRA(7.3)	49.18	72.52	27.48	DEU(29.6),POL(6.5) CHN(6.4)
Brazil	24,792	USA(8.2), MEX(5.3) CHN(4.6)	17.01	22.91	77.09	USA(18.4), CHN(11.2) DEU(10.1)
India	21,383	GBR(10.8), USA(6.6) DEU(3.8)	15.52	32.73	67.27	CHN(18.9),USA(12.1) DEU(6.7)
Russia	2,551	POL(3.2), DEU(2.5) FRA(0.9)	30.76	50.07	49.93	DEU(18.1), JPN(16.0) CHN(9.3)

Note：Regional division is defined in“WIOD Country and Region”table in the appendix. VS is sourced from manufacturing and services sector only.

Table L4 WIOD Sectors

Code	NACE	Industry	Description
C01	**AtB**	Agriculture	Agriculture, Hunting, Forestry and Fishing
C02	**C**	Mining	Mining and Quarrying
C03	**15t16**	Food	Food, Beverages and Tobacco
C04	**17t18**	TextilesProducts	Textiles and Textile Products
C05	**19**	Leather and Footwear	Leather, Leather and Footwear
C06	**20**	WoodProducts	Wood and Products of Wood and Cork
C07	**21t22**	Paper and Printing	Pulp, Paper, Paper, Printing and Publishing
C08	**23**	Refined Petroleum	Coke, Refined Petroleum and Nuclear Fuel
C09	**24**	Chemical Products	Chemicals and Chemical Products
C10	**25**	Rubber and Plastics	Rubber and Plastics
C11	**26**	Other Non-Metal	Other Non-Metallic Mineral
C12	**27t28**	Basic Metals	Basic Metals and Fabricated Metal
C13	**29**	Machinery	Machinery, Nec
C14	**30t33**	Electrical Equipment	Electrical and Optical Equipment
C15	**34t35**	Transport Equipment	Transport Equipment
C16	**36t37**	Recycling	Manufacturing, Nec; Recycling
C17	**E**	Electricity, Gas and Water	Electricity, Gas and Water Supply
C18	**F**	Construction	Construction
C19	**50**	Sale of Vehicles andFuel	Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel
C20	**51**	Wholesale Trade	Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles
C21	**52**	Retail Trade	Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods
C22	**H**	Hotels and Restaurants	Hotels and Restaurants
C23	**60**	Inland Transport	Inland Transport
C24	**61**	Water Transport	Water Transport
C25	**62**	Air Transport	Air Transport
C26	**63**	Other Transport	Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies
C27	**64**	Post and Telecommunications	Post and Telecommunications
C28	**J**	Financial Intermediation	Financial Intermediation
C29	**70**	Real Estate	Real Estate Activities
C30	**71t74**	Business Activities	Renting of M&Eq and Other Business Activities
C31	**L**	Public Admin	Public Admin and Defense; Compulsory Social Security
C32	**M**	Education	Education
C33	**N**	Health and Social Work	Health and Social Work
C34	**O**	OtherServices	Other Community, Social and Personal Services
C35	**P**	Private Households	Private Households with Employed Persons

Appendix M Some Additional graphics

Figure M1a: Structure of US Transport EquipmentExports Decomposition

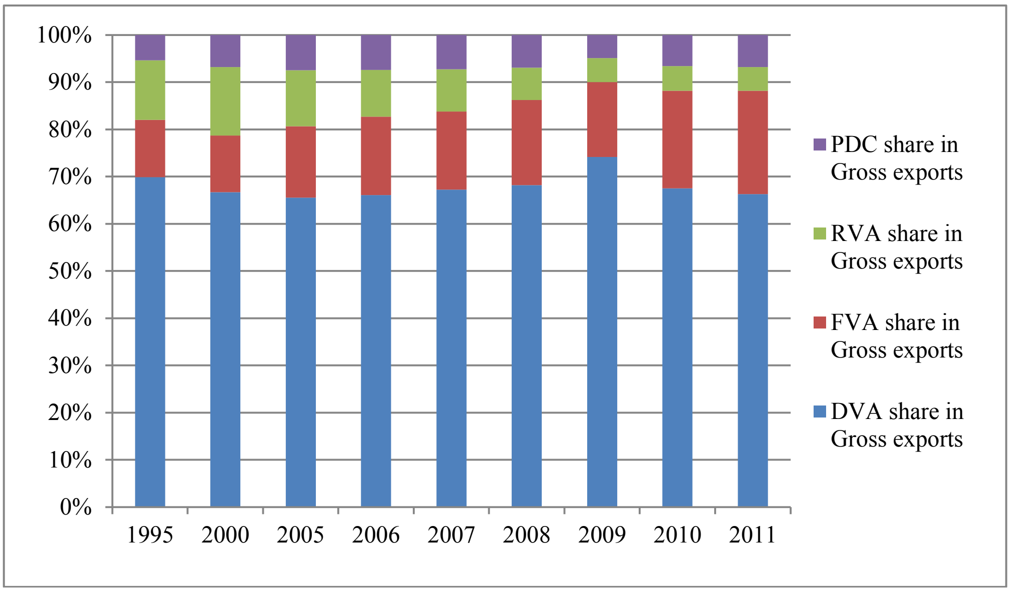
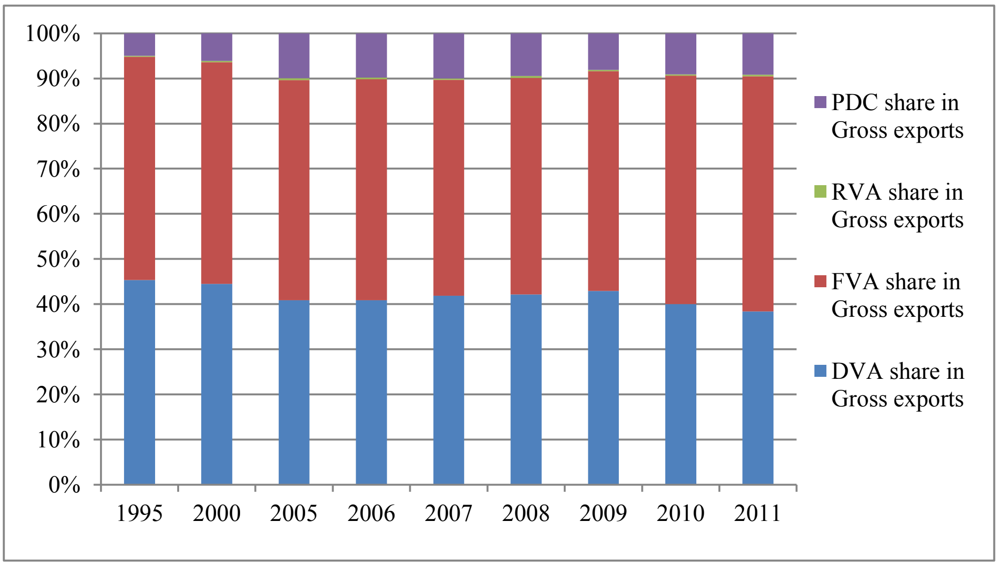


Figure M1b: Structure of Mexico Electrical and Optical Equipment Exports Decomposition



Note: Corresponding to Tables4a and 4b in the main text.

Figure M2a: VS Share by source in US transport equipment exports(Unit %)

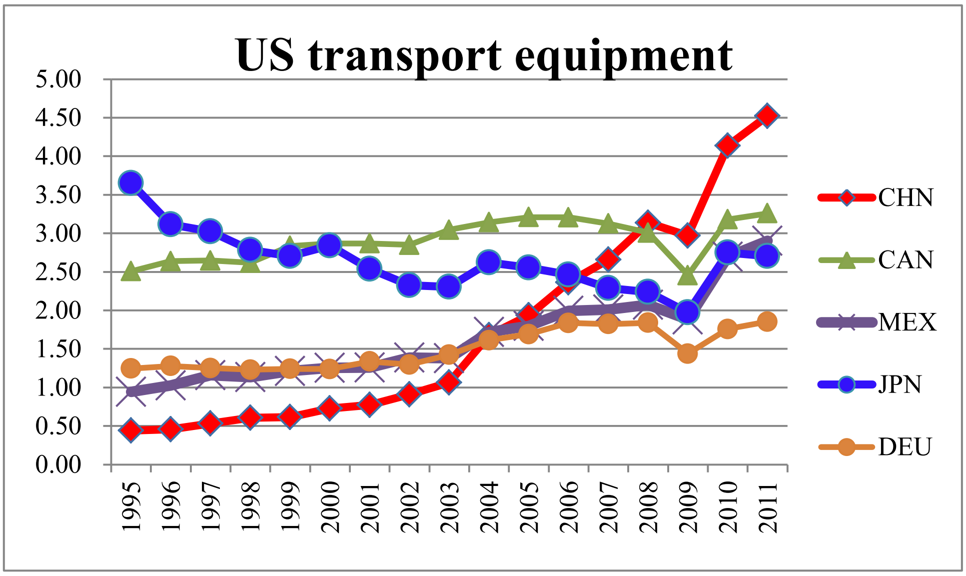
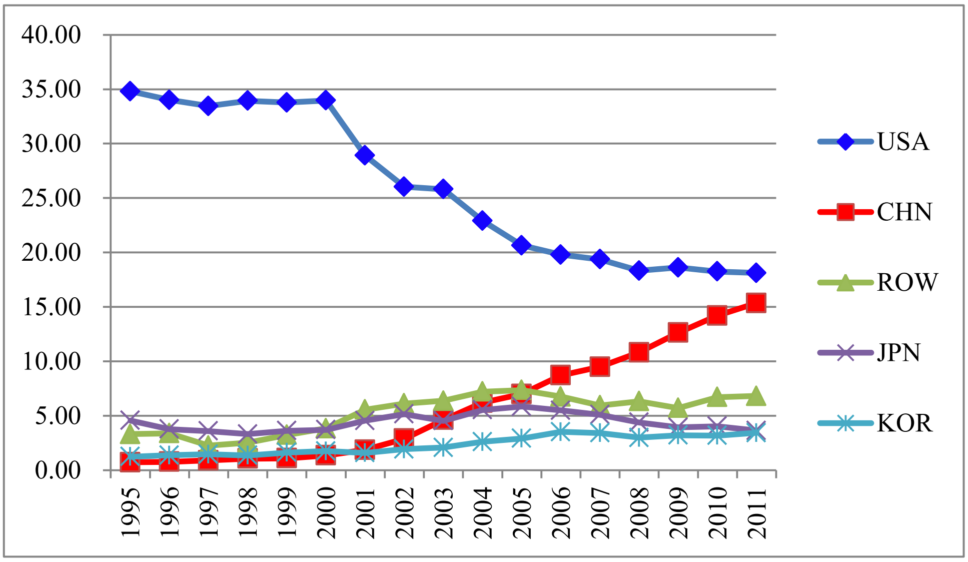


Figure M2b: VS share by source in Mexico's electronics exports (Unit %)



Note: Corresponding to Tables5a and 5b in the main text.

Figure M2c: VS Share by source in DEU transport equipment exports (Unit %)

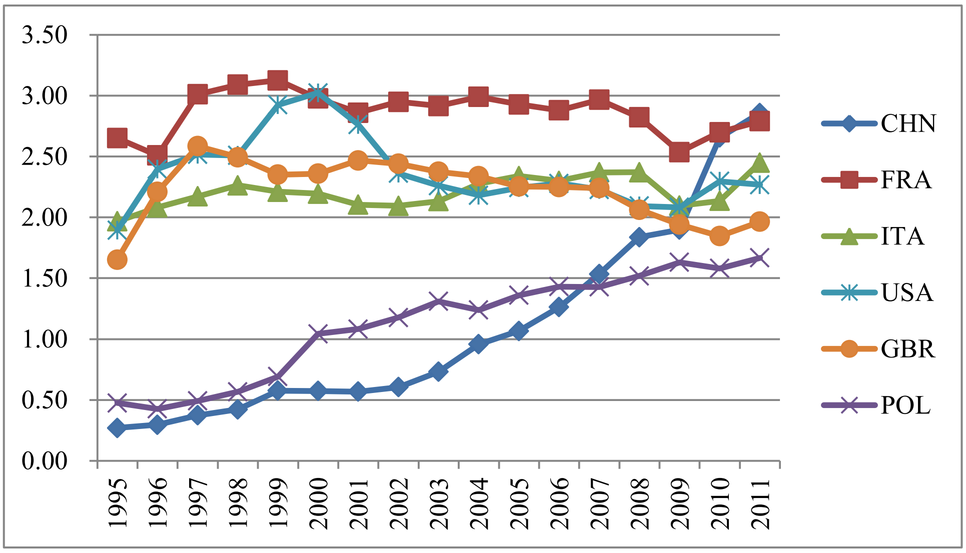
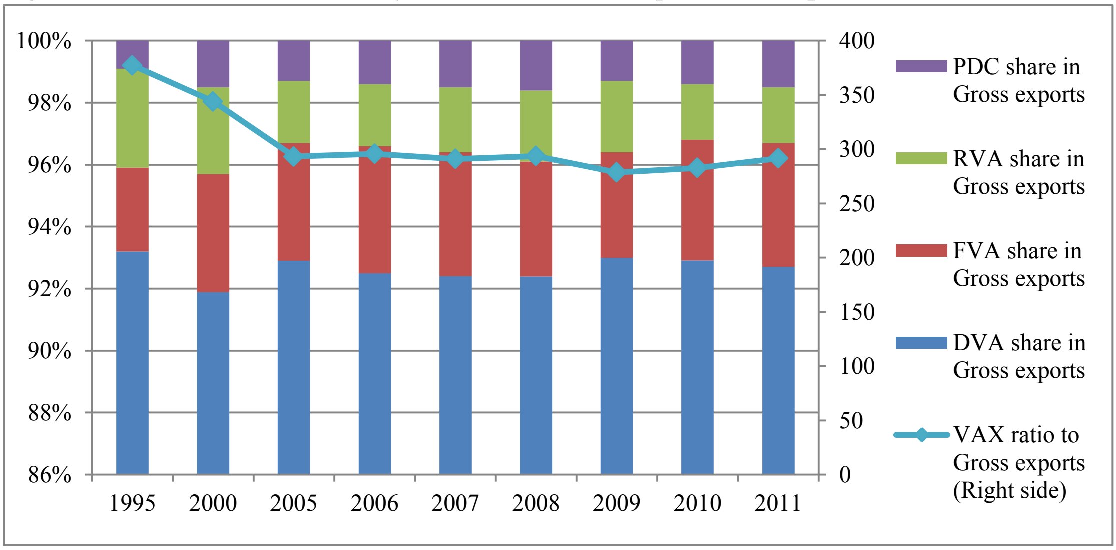


Figure M3: Structure of Germany Business ServicesExports Decomposition and VAX ratio



Note: Corresponding to Tables8 in the main text.

Figure M4 plot the three different types of trade balance measures for China-Japan bilateral trade in rubber and plastics in gross exports, VAX_F, and VAX_B, respectively, similar to figure 2 in the main text on China-US bilateral trade in electrical and optical equipment: As we can see, due to the vast differences in the structure of value added in exports by the two countries, the trade balance looks different, often with a sign switch, as we move from one measure to the other.

Figure M4: China and Japan Bilateral trade balance in Rubber and Plastics
Unit: millions USD

Appendix N: Notations and Important Decomposition Relations

1. At the country aggregate level

(1) $E^s = DVA^s + FVA^s + RDV^s + PDC^s$

(2) $DVA^s = VAX\_F^s = VAX\_B^s$

2. At the country-sector level

(3) $E_j^s = DVA_j^s + FVA_j^s + RDV_j^s + PDC_j^s$

(4) $DVA_j^s = VAX\_B_j^s \neq VAX\_F_j^s$

(5) $GDPinE_j^s = VAX\_F_j^s + RDV\_F_j^s \neq DVA_j^s + RDV_j^s$

3. At the bilateral aggregate level

(6) $E^{sr} = DVA^{sr} + FVA^{sr} + RDV^{sr} + PDC^{sr}$

(7) $DVA^{sr} \neq VAX\_B^{sr} = VAX\_F^{sr}$

4. At the bilateral-sector level

(8) $E_j^{sr} = DVA_j^{sr} + FVA_j^{sr} + RDV_j^{sr} + PDC_j^{sr}$

(9) $DVA_j^{sr} \neq VAX\_B_j^{sr} \neq VAX\_F_j^{sr}$

Where E^s is Country s' gross exports. (time subscript is omitted for simplicity.);

DVA^s is domestic value-added that is exported by Country s and ultimately absorbed abroad; FVA^s is foreign value-added in Country s' exports; RDV^s is returned domestic value-added in Country s' exports, or domestic value added that is initially exported by Country s but eventually returned and is consumed at home; PDC^s is pure double counted component due to double counting of the previous terms in Country s' exports (or back-and-forth intermediate goods trade).

VAX_F^s is forward-linkages based value added exports, equaling the sum of $VAX\_F_j^s$ across all sectors; $RDV\_F_j^s$ is forward-linkages based domestic value-added that is first exported but finally returns and is consumed at home; VAX_B^s is backward-linkages based value added in exports of Country s, equaling sum of $VAX\_B_j^s$ across all sectors.

E_j^s is total exports of sector j from Country s; DVA_j^s, FVA_j^s, RDV_j^s, and PDC_j^s are the four major components of sector j's gross exports, backward-linkage based;

$GDPinE_j^s$ is GDP by industry in exports. This concept of value-added created by production factors (labor, capital) employed in sector j of Country s and embed in the sector's gross exports, is only concerned with where the value-added is created, but not where it is absorbed;

$VAX\_F_j^s$ is forward-linkages based value added exports of sector j from Country s, which is sector j's value added embedded in all sectors gross exports from Country s (including indirect exports of sector j's value added through gross exports of Country s' other sectors); $VAX\_B_j^s$ is backward-linkage based value added exports of sector j of Country s, which is value added from all sectors in Country s that is embedded in its sector j's gross exports.

5. **Finer Decompositions**:

(10) $DVA_j^{sr} = DVA\_Fin_j^{sr} + DVA\_Int_j^{sr} + DVA\_Intrex_j^{sr}$

(11) $VAX\_F_j^{sr} = VAX\_F\_Fin_j^{sr} + VAX\_F\_Int_j^{sr} + VAX\_F\_Intrex_j^{sr}$

$$VAX\_F_j^{sr} = \sum_{r \neq s}^{G} V_j^s B^{ss} Y^{sr} + \sum_{r \neq s}^{G} V_j^s B^{sr} Y^{rr} + \sum_{r \neq s}^{G} V_j^s B^{sr} \sum_{t \neq s,r}^{G} Y^{rt}$$

Where $V_j^s = \begin{bmatrix} 0 & \cdots & v_j^s & \cdots & 0 \end{bmatrix}$

(12) $VAX\_B_j^{sr} = VAX\_B\_Fin_j^{sr} + VAX\_B\_Int_j^{sr} + VAX\_B\_Intrex_j^{sr}$

$$VAX\_B_j^{sr} = \sum_{r \neq s}^{G} V^s B^{ss} Y_j^{sr} + \sum_{r \neq s}^{G} V^s B^{sr} Y_j^{rr} + \sum_{r \neq s}^{G} V_j^s B^{sr} \sum_{t \neq s,r}^{G} Y_j^{rt}$$

(13) $PDC_j^{sr} = DDC_j^{sr} + FDC_j^{sr} = DDC\_Fin_j^{sr} + DDC\_Int_j^{sr} + MDC_j^{sr} + ODC_j^{sr}$

(14)
$$\begin{aligned} & FVA_j^{sr} = FVA\_Fin_j^{sr} + FVA\_Int_j^{sr} \\ & = MVA\_Fin_j^{sr} + MVA\_Int_j^{sr} + OVA\_Fin_j^{sr} + OVA\_Int_j^{sr} \end{aligned}$$

Where $DVA\_Fin_j^{sr}$ is domestic value-added in final goods exports consumed by direct importers; $DVA\_Int_j^{sr}$ is domestic value-added in intermediate goods exports absorbed by direct importers; $DVA\_Intrex_j^{sr}$ is domestic value-added in intermediate goods re-exported to third countries.

Similar to the three sub-components for $VAX\_F_j^{sr}$ and $VAX\_B_j^{sr}$, we have the following sub-components: DDC_j^{sr} is domestic value-added pure double counting in production of exports; FDC_j^{sr} is foreign value-added pure double counting in production of exports; MVA_j^{sr} is foreign value-added sourced from the direct importer; OVA_j^{sr} is foreign value-added sourced from third countries; MDC_j^{sr} is the direct importer's VA double counted in exports production; ODC_j^{sr} is third countries' VA double counted in exports production.

<u>At the country aggregate level</u>

$$DVA\_Fin^s = VAX\_B\_Fin^s = VAX\_F\_Fin^s,$$

$$DVA\_Int^s = VAX\_B\_Int^s = VAX\_F\_Int^s$$

$$DVA\_Intrex^{s} = VAX\_B\_Intrex^{s} = VAX\_F\_Intrex^{s}$$

$$RDV\_Fin^{s} = RDV\_B\_Fin^{s} = RDV\_F\_Fin^{s}$$

$$RDV\_Int^{s} = RDV\_B\_Int^{s} = RDV\_F\_Int^{s}$$

At the country-sector level

$$DVA\_Fin_{j}^{s} = VAX\_B\_Fin_{j}^{s} \neq VAX\_F\_Fin_{j}^{s},$$

$$DVA\_Int_{j}^{s} = VAX\_B\_Int_{j}^{s} \neq VAX\_F\_Int_{j}^{s}$$

$$DVA\_Intrex_{j}^{s} = VAX\_B\_Intrex_{j}^{s} \neq VAX\_F\_Intrex_{j}^{s}$$

$$RDV\_Fin_{j}^{s} = RDV\_B\_Fin_{j}^{s} \neq RDV\_F\_Fin_{j}^{s}$$

$$RDV\_Int_{j}^{s} = RDV\_B\_Int_{j}^{s} \neq RDV\_F\_Int_{j}^{s}$$

At the bilateral-sector level

$$DVA\_Fin_{j}^{sr} \neq VAX\_B\_Fin_{j}^{sr} \neq VAX\_F\_Fin_{j}^{sr}$$

$$DVA\_Int_{j}^{sr} \neq VAX\_B\_Int_{j}^{sr} \neq VAX\_F\_Int_{j}^{sr}$$

$$DVA\_Intrex_{j}^{sr} \neq VAX\_B\_Intrex_{j}^{sr} \neq VAX\_F\_Intrex_{j}^{sr}$$

Appendix O Decompose bilateral intermediate trade flows based on where it is finally absorbed – implementation into Computer code

Correctly decompose bilateral intermediate trade flows at sector level into major groups according to their destination of final absorption are the key technical step to fully decompose gross bilateral trade flows, which transfer gross output (gross exports is part of it), usually as endogenous variable in standard IO models, to exogenous variables in our gross trade accounting framework. As discussed in the main text, we decompose bilateral intermediate trade flows into following 8 groups, as specified in equation (35) of the main text.

Group 1: $A_{sr}B_{rr}Y_{rr}$, $\begin{bmatrix} 0 & A_{12}B_{22}Y_{22} & A_{13}B_{33}Y_{33} & A_{14}B_{44}Y_{44} \\ A_{21}B_{11}Y_{11} & 0 & A_{23}B_{33}Y_{33} & A_{24}B_{44}Y_{44} \\ A_{31}B_{11}Y_{11} & A_{32}B_{22}Y_{22} & 0 & A_{34}B_{44}Y_{44} \\ A_{41}B_{11}Y_{11} & A_{42}B_{22}Y_{22} & A_{43}B_{33}Y_{33} & 0 \end{bmatrix}$, 1 term

Group 2: $A_{sr}B_{rt}Y_{tt}$, $t \neq r,s$ $\begin{bmatrix} 0 & A_{12}(B_{23}Y_{33}+B_{24}Y_{44}) & A_{13}(B_{32}Y_{22}+B_{34}Y_{44}) & A_{14}(B_{42}Y_{22}+B_{43}Y_{33}) \\ A_{21}(B_{13}Y_{33}+B_{14}Y_{44}) & 0 & A_{23}(B_{31}Y_{11}+B_{34}Y_{44}) & A_{24}(B_{41}Y_{11}+B_{43}Y_{33}) \\ A_{31}(B_{12}Y_{22}+B_{14}Y_{44}) & A_{32}(B_{21}Y_{11}+B_{24}Y_{44}) & 0 & A_{34}(B_{42}Y_{22}+B_{41}Y_{11}) \\ A_{41}(B_{13}Y_{33}+B_{12}Y_{22}) & A_{42}(B_{23}Y_{33}+B_{21}Y_{11}) & A_{43}(B_{32}Y_{22}+B_{31}Y_{11}) & 0 \end{bmatrix}$, 2 terms

Group 3: $A_{sr}B_{rr}Y_{rt}$, $t \neq r,s$, $\begin{bmatrix} 0 & A_{12}B_{22}(Y_{23}+Y_{24}) & A_{13}B_{33}(Y_{32}+Y_{34}) & A_{14}B_{44}(Y_{42}+Y_{43}) \\ A_{21}B_{11}(Y_{13}+Y_{14}) & 0 & A_{23}B_{33}(Y_{31}+Y_{34}) & A_{24}B_{44}(Y_{41}+Y_{43}) \\ A_{31}B_{11}(Y_{12}+Y_{14}) & A_{32}B_{22}(Y_{21}+Y_{24}) & 0 & A_{34}B_{44}(Y_{42}+Y_{41}) \\ A_{41}B_{11}(Y_{13}+Y_{12}) & A_{42}B_{22}(Y_{23}+Y_{21}) & A_{43}B_{33}(Y_{32}+Y_{31}) & 0 \end{bmatrix}$, 2 terms

Group 4: $A_{sr}B_{rt}Y_{tu}$, $t \neq r,s$, $u \neq t,s$

$$\begin{bmatrix} 0 & A_{12}[B_{23}(Y_{32}+Y_{34})+B_{24}(Y_{42}+Y_{43})] & A_{13}[B_{32}(Y_{23}+Y_{24})+B_{34}(Y_{42}+Y_{43})] & A_{14}[B_{42}(Y_{23}+Y_{24})+B_{43}(Y_{32}+Y_{34})] \\ A_{21}[B_{13}(Y_{31}+Y_{34})+B_{14}(Y_{41}+Y_{43})] & 0 & A_{23}[B_{31}(Y_{13}+Y_{14})+B_{34}(Y_{43}+Y_{41})] & A_{24}[B_{41}(Y_{13}+Y_{14})+B_{43}(Y_{31}+Y_{34})] \\ A_{31}[B_{12}(Y_{21}+Y_{24})+B_{14}(Y_{41}+Y_{42})] & A_{32}[B_{21}(Y_{12}+Y_{14})+B_{24}(Y_{42}+Y_{41})] & 0 & A_{34}[B_{42}(Y_{21}+Y_{24})+B_{43}(Y_{32}+Y_{34})] \\ A_{41}[B_{13}(Y_{31}+Y_{32})+B_{12}(Y_{21}+Y_{23})] & A_{42}[B_{23}(Y_{32}+Y_{31})+B_{21}(Y_{12}+Y_{13})] & A_{43}[B_{32}(Y_{23}+Y_{21})+B_{31}(Y_{13}+Y_{12})] & 0 \end{bmatrix}, \text{ 4 terms}$$

Group 5: $A_{sr}B_{rr}Y_{rs}$,

$$\begin{bmatrix} 0 & A_{12}B_{22}Y_{21} & A_{13}B_{33}Y_{31} & A_{14}B_{44}Y_{41} \\ A_{21}B_{11}Y_{12} & 0 & A_{23}B_{33}Y_{32} & A_{24}B_{44}Y_{42} \\ A_{31}B_{11}Y_{13} & A_{32}B_{22}Y_{23} & 0 & A_{34}B_{44}Y_{43} \\ A_{41}B_{11}Y_{14} & A_{42}B_{22}Y_{24} & A_{43}B_{33}Y_{34} & 0 \end{bmatrix}, \text{ 1 term}$$

Group 6: $A_{sr}B_{rt}Y_{ts}$, $t \neq r,s$

$$\begin{bmatrix} 0 & A_{12}(B_{23}Y_{31}+B_{24}Y_{41}) & A_{13}(B_{32}Y_{21}+B_{34}Y_{41}) & A_{14}(B_{42}Y_{21}+B_{43}Y_{31}) \\ A_{21}(B_{13}Y_{32}+B_{14}Y_{42}) & 0 & A_{23}(B_{31}Y_{12}+B_{34}Y_{42}) & A_{24}(B_{41}Y_{12}+B_{43}Y_{32}) \\ A_{31}(B_{12}Y_{23}+B_{14}Y_{43}) & A_{32}(B_{21}Y_{13}+B_{24}Y_{43}) & 0 & A_{34}(B_{42}Y_{23}+B_{41}Y_{13}) \\ A_{41}(B_{12}Y_{24}+B_{13}Y_{34}) & A_{42}(B_{23}Y_{34}+B_{21}Y_{14}) & A_{43}(B_{32}Y_{24}+B_{31}Y_{14}) & 0 \end{bmatrix}, \text{ 2 terms}$$

Group 7: $A_{sr}B_{rs}Y_{ss}$,

$$\begin{bmatrix} 0 & A_{12}B_{21}Y_{11} & A_{13}B_{31}Y_{11} & A_{14}B_{41}Y_{11} \\ A_{21}B_{12}Y_{22} & 0 & A_{23}B_{32}Y_{22} & A_{24}B_{42}Y_{22} \\ A_{31}B_{13}Y_{33} & A_{32}B_{23}Y_{33} & 0 & A_{34}B_{43}Y_{33} \\ A_{41}B_{14}Y_{44} & A_{42}B_{24}Y_{44} & A_{43}B_{34}Y_{44} & 0 \end{bmatrix}, \text{ 1 terms}$$

Group 8: $A_{sr}B_{rs}Y_{st}$, $t \neq s$

$$\begin{bmatrix} 0 & A_{12}B_{21}(Y_{12}+Y_{13}+Y_{14}) & A_{13}B_{31}(Y_{12}+Y_{13}+Y_{14}) & A_{14}B_{41}(Y_{12}+Y_{13}+Y_{14}) \\ A_{21}B_{12}(Y_{23}+Y_{21}+Y_{24}) & 0 & A_{23}B_{32}(Y_{23}+Y_{21}+Y_{24}) & A_{24}B_{42}(Y_{23}+Y_{21}+Y_{24}) \\ A_{31}B_{13}(Y_{32}+Y_{31}+Y_{34}) & A_{32}B_{23}(Y_{32}+Y_{31}+Y_{34}) & 0 & A_{34}B_{43}(Y_{32}+Y_{31}+Y_{34}) \\ A_{41}B_{14}(Y_{43}+Y_{41}+Y_{42}) & A_{42}B_{24}(Y_{43}+Y_{41}+Y_{42}) & A_{43}B_{34}(Y_{43}+Y_{41}+Y_{42}) & 0 \end{bmatrix}, \text{ 3 terms}$$

The total ABY terms can be computed as $G \times G = 1 + (G-2) + (G-2) + (G-2)*(G-2) + 1 + (G-2) + 1 + (G-1)$; in 4 country case, there are total 16 ABY terms.

Gross intermediate exports can be expressed as *AX* matrix and the sum of the 8 ABY groups exactly equal *AX*:

$$AX = \begin{bmatrix} 0 & A_{12}X_2 & A_{13}X_3 & A_{14}X_4 \\ A_{21}X_1 & 0 & A_{23}X_3 & A_{24}X_4 \\ A_{31}X_1 & A_{32}X_2 & 0 & A_{34}X_4 \\ A_{41}X_1 & A_{42}X_2 & A_{43}X_3 & 0 \end{bmatrix}, \quad A_{sr}X_r = A_{sr}\sum_t^G B_{rt}\sum_k^G Y_{tk} = A_{sr}\sum_t^G B_{rt}Y_{t*} \text{ and } \qquad Y_{t*} = \sum_k^G Y_{tk}$$

Gross intermediate exports AX can first be decomposed according to where they are used to produce final good as three major groups:

(1) Used in the direct importing Country r; (2) re-exported by Country r and used in third countries; (3) return to the exporting Country s and used there.

$$A_{sr}X_r = A_{sr}\sum_t^G B_{rt}Y_{t*} = A_{sr}B_{rr}Y_{r*} + A_{sr}\sum_{t\neq s,r}^G B_{rt}Y_{t*} + A_{sr}B_{rs}Y_{s*}$$

Gross output can also be decomposed in similar way:

$$X_r = \sum_t^G B_{rt}Y_{t*} = B_{rr}Y_{r*} + \sum_{t\neq s,r}^G B_{rt}Y_{t*} + B_{rs}Y_{s*}$$

Each of the three groups can be further decomposed according to where these final goods are consumed:

(1) **Intermediate exports from Country s used to produce final goods in Country r can be further decomposed into three groups**:

$$A_{sr}B_{rr}Y_{r*} = A_{sr}B_{rr}\sum_k^G Y_{rk} = A_{sr}B_{rr}Y_{rr} + A_{sr}B_{rr}\sum_{t\neq s,r}^G Y_{rt} + A_{sr}B_{rr}Y_{rs}$$

Corresponding gross output can also be decomposed in similar way:

$$B_{rr}Y_{r*} = B_{rr}\sum_k^G Y_{rk} = B_{rr}Y_{rr} + B_{rr}\sum_{t\neq s,r}^G Y_{rt} + B_{rr}Y_{rs}$$

$A_{sr}B_{rr}Y_{rr}$ is Group 1， $A_{sr}B_{rr}\sum_{t\neq s,r}^{G}Y_{rt}$ is Group 3， $A_{sr}B_{rr}Y_{rs}$ is Group 5

The detailed computer implementation is given below:

Group 1: $A_{sr}B_{rr}Y_{rr}$ $s \neq r$ Country s' intermediate goods exports used by partner Country r to produce its domestically consumed final goods

Step 1:

$$\begin{bmatrix} 0 & B_{22} & B_{33} & B_{44} \\ B_{11} & 0 & B_{33} & B_{44} \\ B_{11} & B_{22} & 0 & B_{44} \\ B_{11} & B_{22} & B_{33} & 0 \end{bmatrix}\begin{bmatrix} Y_{11} & 0 & 0 & 0 \\ 0 & Y_{22} & 0 & 0 \\ 0 & 0 & Y_{33} & 0 \\ 0 & 0 & 0 & Y_{44} \end{bmatrix} = \begin{bmatrix} 0 & B_{22}Y_{22} & B_{33}Y_{33} & B_{44}Y_{44} \\ B_{11}Y_{11} & 0 & B_{33}Y_{33} & B_{44}Y_{44} \\ B_{11}Y_{11} & B_{22}Y_{22} & 0 & B_{44}Y_{44} \\ B_{11}Y_{11} & B_{22}Y_{22} & B_{33}Y_{33} & 0 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{22}Y_{22} & B_{33}Y_{33} & B_{44}Y_{44} \\ B_{11}Y_{11} & 0 & B_{33}Y_{33} & B_{44}Y_{44} \\ B_{11}Y_{11} & B_{22}Y_{22} & 0 & B_{44}Y_{44} \\ B_{11}Y_{11} & B_{22}Y_{22} & B_{33}Y_{33} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{12}B_{22}Y_{22} & A_{13}B_{33}Y_{33} & A_{14}B_{44}Y_{44} \\ A_{21}B_{11}Y_{11} & 0 & A_{23}B_{33}Y_{33} & A_{24}B_{44}Y_{44} \\ A_{31}B_{11}Y_{11} & A_{32}B_{22}Y_{22} & 0 & A_{34}B_{44}Y_{44} \\ A_{41}B_{11}Y_{11} & A_{42}B_{22}Y_{22} & A_{43}B_{33}Y_{33} & 0 \end{bmatrix} \backslash$$

Or in one step

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix}\begin{bmatrix} Y_{11} & 0 & 0 & 0 \\ 0 & Y_{22} & 0 & 0 \\ 0 & 0 & Y_{33} & 0 \\ 0 & 0 & 0 & Y_{44} \end{bmatrix}\begin{bmatrix} B_{11} & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 \\ 0 & 0 & B_{33} & 0 \\ 0 & 0 & 0 & B_{44} \end{bmatrix} = \begin{bmatrix} 0 & A_{12}B_{22}Y_{22} & A_{13}B_{33}Y_{33} & A_{14}B_{44}Y_{44} \\ A_{21}B_{11}Y_{11} & 0 & A_{23}B_{33}Y_{33} & A_{24}B_{44}Y_{44} \\ A_{31}B_{11}Y_{11} & A_{32}B_{22}Y_{22} & 0 & A_{34}B_{44}Y_{44} \\ A_{41}B_{11}Y_{11} & A_{42}B_{22}Y_{22} & A_{43}B_{33}Y_{33} & 0 \end{bmatrix}$$

Group 3: $A_{sr}B_{rr}Y_{rt}$, $t \neq r,s$, Country s' intermediate exports used by partner Country r, to produce intermediate good that is exported to third Countries (t) for production of final goods consumed in t.

Step 1:

$$\begin{bmatrix} 0 & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & 0 & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & 0 & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & Y_{12}+Y_{13}+Y_{14} & Y_{12}+Y_{13}+Y_{14} & Y_{12}+Y_{13}+Y_{14} \\ Y_{21}+Y_{23}+Y_{24} & 0 & Y_{21}+Y_{23}+Y_{24} & Y_{21}+Y_{23}+Y_{24} \\ Y_{31}+Y_{32}+Y_{34} & Y_{31}+Y_{32}+Y_{34} & 0 & Y_{31}+Y_{32}+Y_{34} \\ Y_{41}+Y_{42}+Y_{43} & Y_{41}+Y_{42}+Y_{43} & Y_{41}+Y_{42}+Y_{43} & 0 \end{bmatrix} - \begin{bmatrix} 0 & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & 0 & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & 0 & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & Y_{13}+Y_{14} & Y_{12}+Y_{14} & Y_{12}+Y_{13} \\ Y_{23}+Y_{24} & 0 & Y_{21}+Y_{24} & Y_{21}+Y_{23} \\ Y_{32}+Y_{34} & Y_{31}+Y_{34} & 0 & Y_{31}+Y_{32} \\ Y_{42}+Y_{43} & Y_{41}+Y_{43} & Y_{41}+Y_{42} & 0 \end{bmatrix}$$

Step2:

$$\begin{bmatrix} B_{11} & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 \\ 0 & 0 & B_{33} & 0 \\ 0 & 0 & 0 & B_{44} \end{bmatrix} \begin{bmatrix} 0 & Y_{13}+Y_{14} & Y_{12}+Y_{14} & Y_{12}+Y_{13} \\ Y_{23}+Y_{24} & 0 & Y_{21}+Y_{24} & Y_{21}+Y_{23} \\ Y_{32}+Y_{34} & Y_{31}+Y_{34} & 0 & Y_{31}+Y_{32} \\ Y_{42}+Y_{43} & Y_{41}+Y_{43} & Y_{41}+Y_{42} & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_{11}(Y_{13}+Y_{14}) & B_{11}(Y_{12}+Y_{14}) & B_{11}(Y_{13}+Y_{12}) \\ B_{22}(Y_{23}+Y_{24}) & 0 & B_{22}(Y_{21}+Y_{24}) & B_{22}(Y_{23}+Y_{21}) \\ B_{33}(Y_{32}+Y_{34}) & B_{33}(Y_{31}+Y_{34}) & 0 & B_{33}(Y_{32}+Y_{31}) \\ B_{44}(Y_{42}+Y_{43}) & B_{44}(Y_{41}+Y_{43}) & B_{44}(Y_{42}+Y_{41}) & 0 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 0 & B_{11}(Y_{13}+Y_{14}) & B_{11}(Y_{12}+Y_{14}) & B_{11}(Y_{13}+Y_{12}) \\ B_{22}(Y_{23}+Y_{24}) & 0 & B_{22}(Y_{21}+Y_{24}) & B_{22}(Y_{23}+Y_{21}) \\ B_{33}(Y_{32}+Y_{34}) & B_{33}(Y_{31}+Y_{34}) & 0 & B_{33}(Y_{32}+Y_{31}) \\ B_{44}(Y_{42}+Y_{43}) & B_{44}(Y_{41}+Y_{43}) & B_{44}(Y_{42}+Y_{41}) & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & B_{22}(Y_{23}+Y_{24}) & B_{33}(Y_{32}+Y_{34}) & B_{44}(Y_{42}+Y_{43}) \\ B_{11}(Y_{13}+Y_{14}) & 0 & B_{33}(Y_{31}+Y_{34}) & B_{44}(Y_{41}+Y_{43}) \\ B_{11}(Y_{12}+Y_{14}) & B_{22}(Y_{21}+Y_{24}) & 0 & B_{44}(Y_{42}+Y_{41}) \\ B_{11}(Y_{13}+Y_{12}) & B_{22}(Y_{23}+Y_{21}) & B_{33}(Y_{32}+Y_{31}) & 0 \end{bmatrix}$$

Step 4:

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{22}(Y_{23}+Y_{24}) & B_{33}(Y_{32}+Y_{34}) & B_{44}(Y_{42}+Y_{43}) \\ B_{11}(Y_{13}+Y_{14}) & 0 & B_{33}(Y_{31}+Y_{34}) & B_{44}(Y_{41}+Y_{43}) \\ B_{11}(Y_{12}+Y_{14}) & B_{22}(Y_{21}+Y_{24}) & 0 & B_{44}(Y_{42}+Y_{41}) \\ B_{11}(Y_{13}+Y_{12}) & B_{22}(Y_{23}+Y_{21}) & B_{33}(Y_{32}+Y_{31}) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A_{12}B_{22}(Y_{23}+Y_{24}) & A_{13}B_{33}(Y_{32}+Y_{34}) & A_{14}B_{44}(Y_{42}+Y_{43}) \\ A_{21}B_{11}(Y_{13}+Y_{14}) & 0 & A_{23}B_{33}(Y_{31}+Y_{34}) & A_{24}B_{44}(Y_{41}+Y_{43}) \\ A_{31}B_{11}(Y_{12}+Y_{14}) & A_{32}B_{22}(Y_{21}+Y_{24}) & 0 & A_{34}B_{44}(Y_{42}+Y_{41}) \\ A_{41}B_{11}(Y_{13}+Y_{12}) & A_{42}B_{22}(Y_{23}+Y_{21}) & A_{43}B_{33}(Y_{32}+Y_{31}) & 0 \end{bmatrix}$$

Group 5: $A_{sr}B_{rr}Y_{rs}$, $s \neq r$ Country s' intermediate goods exports used by partner Country r to produce final goods exports that are shipped back to the source Country s.

Step1:

$$\begin{bmatrix} B_{11} & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 \\ 0 & 0 & B_{33} & 0 \\ 0 & 0 & 0 & B_{44} \end{bmatrix} \begin{bmatrix} 0 & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & 0 & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & 0 & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & B_{22}Y_{21} & B_{33}Y_{31} & B_{44}Y_{41} \\ B_{11}Y_{12} & 0 & B_{33}Y_{32} & B_{44}Y_{42} \\ B_{11}Y_{13} & B_{22}Y_{23} & 0 & B_{44}Y_{43} \\ B_{11}Y_{14} & B_{22}Y_{24} & B_{33}Y_{34} & 0 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} 0 & B_{11}Y_{12} & B_{11}Y_{13} & B_{11}Y_{14} \\ B_{22}Y_{21} & 0 & B_{22}Y_{23} & B_{22}Y_{24} \\ B_{33}Y_{31} & B_{33}Y_{32} & 0 & B_{33}Y_{34} \\ B_{44}Y_{41} & B_{44}Y_{42} & B_{44}Y_{43} & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & B_{22}Y_{21} & B_{33}Y_{31} & B_{44}Y_{41} \\ B_{11}Y_{12} & 0 & B_{33}Y_{32} & B_{44}Y_{42} \\ B_{11}Y_{13} & B_{22}Y_{23} & 0 & B_{44}Y_{43} \\ B_{11}Y_{14} & B_{22}Y_{24} & B_{33}Y_{34} & 0 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{22}Y_{21} & B_{33}Y_{31} & B_{44}Y_{41} \\ B_{11}Y_{12} & 0 & B_{33}Y_{32} & B_{44}Y_{42} \\ B_{11}Y_{13} & B_{22}Y_{23} & 0 & B_{44}Y_{43} \\ B_{11}Y_{14} & B_{22}Y_{24} & B_{33}Y_{34} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{12}B_{22}Y_{21} & A_{13}B_{33}Y_{31} & A_{14}B_{44}Y_{41} \\ A_{21}B_{11}Y_{12} & 0 & A_{23}B_{33}Y_{32} & A_{24}B_{44}Y_{42} \\ A_{31}B_{11}Y_{13} & A_{32}B_{22}Y_{23} & 0 & A_{34}B_{44}Y_{43} \\ A_{41}B_{11}Y_{14} & A_{42}B_{22}Y_{24} & A_{43}B_{33}Y_{34} & 0 \end{bmatrix}$$

(2) Intermediate exports of Country s used to produce final goods in third countries (t) also can be further decomposed into three groups

$$A_{sr}\sum_{t\neq r,s}^{G} B_{rt}Y_{t*} = A_{sr}\sum_{t\neq r,s}^{G} B_{rt}\sum_{k}^{G} Y_{tk} = A_{sr}\sum_{t\neq r,s}^{G} B_{rt}Y_{tt} + A_{sr}\sum_{t\neq r,s}^{G} B_{rt}\sum_{u\neq s,t}^{G} Y_{tu} + A_{sr}\sum_{t\neq r,s}^{G} B_{rt}Y_{ts}$$

Corresponding gross output can also be decomposed in similar way:

$$\sum_{t\neq r,s}^{G} B_{rt}Y_{t*} = \sum_{t\neq r,s}^{G} B_{rt}\sum_{k}^{G} Y_{tk} = \sum_{t\neq r,s}^{G} B_{rt}Y_{tt} + \sum_{t\neq r,s}^{G} B_{rt}\sum_{u\neq s,t}^{G} Y_{tu} + \sum_{t\neq r,s}^{G} B_{rt}Y_{ts}$$

Group 2: $A_{sr}B_{rt}Y_{tt}$, $t \neq r,s$, Country s' intermediate exports used by partner Country r, to produce intermediate good that is exported to third Countries (t) for production of final goods consumed in t, including all the G-2 countries that are not the two direct trading partners.

Step 1:

$$\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{bmatrix} \begin{bmatrix} 0 & Y_{11} & Y_{11} & Y_{11} \\ Y_{22} & 0 & Y_{22} & Y_{22} \\ Y_{33} & Y_{33} & 0 & Y_{33} \\ Y_{44} & Y_{44} & Y_{44} & 0 \end{bmatrix} = \begin{bmatrix} B_{12}Y_{22}+B_{13}Y_{33}+B_{14}Y_{44} & B_{13}Y_{33}+B_{14}Y_{44} & B_{12}Y_{22}+B_{14}Y_{44} & B_{12}Y_{22}+B_{13}Y_{33} \\ B_{23}Y_{33}+B_{24}Y_{44} & B_{21}Y_{11}+B_{23}Y_{33}+B_{24}Y_{44} & B_{21}Y_{11}+B_{24}Y_{44} & B_{21}Y_{11}+B_{23}Y_{33} \\ B_{32}Y_{22}+B_{34}Y_{44} & B_{31}Y_{11}+B_{34}Y_{44} & B_{31}Y_{11}+B_{32}Y_{22}+B_{34}Y_{44} & B_{31}Y_{11}+B_{32}Y_{22} \\ B_{42}Y_{22}+B_{43}Y_{33} & B_{41}Y_{11}+B_{43}Y_{33} & B_{41}Y_{11}+B_{42}Y_{22} & B_{41}Y_{11}+B_{42}Y_{22}+B_{43}Y_{33} \end{bmatrix}$$

Step2:

Make all diagonal elements to 0 and transpose

$$\begin{bmatrix} 0 & B_{23}Y_{33}+B_{24}Y_{44} & B_{32}Y_{22}+B_{34}Y_{44} & B_{42}Y_{22}+B_{43}Y_{33} \\ B_{13}Y_{33}+B_{14}Y_{44} & 0 & B_{31}Y_{11}+B_{34}Y_{44} & B_{41}Y_{11}+B_{43}Y_{33} \\ B_{12}Y_{22}+B_{14}Y_{44} & B_{21}Y_{11}+B_{24}Y_{44} & 0 & B_{41}Y_{11}+B_{42}Y_{22} \\ B_{12}Y_{22}+B_{13}Y_{33} & B_{21}Y_{11}+B_{23}Y_{33} & B_{31}Y_{11}+B_{32}Y_{22} & 0 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{23}Y_{33}+B_{24}Y_{44} & B_{32}Y_{22}+B_{34}Y_{44} & B_{42}Y_{22}+B_{43}Y_{33} \\ B_{13}Y_{33}+B_{14}Y_{44} & 0 & B_{31}Y_{11}+B_{34}Y_{44} & B_{41}Y_{11}+B_{43}Y_{33} \\ B_{12}Y_{22}+B_{14}Y_{44} & B_{21}Y_{11}+B_{24}Y_{44} & 0 & B_{41}Y_{11}+B_{42}Y_{22} \\ B_{12}Y_{22}+B_{13}Y_{33} & B_{21}Y_{11}+B_{23}Y_{33} & B_{31}Y_{11}+B_{32}Y_{22} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A_{12}(B_{23}Y_{33}+B_{24}Y_{44}) & A_{13}(B_{32}Y_{22}+B_{34}Y_{44}) & A_{14}(B_{42}Y_{22}+B_{43}Y_{33}) \\ A_{21}(B_{13}Y_{33}+B_{14}Y_{44}) & 0 & A_{23}(B_{31}Y_{11}+B_{34}Y_{44}) & A_{24}(B_{41}Y_{11}+B_{43}Y_{33}) \\ A_{31}(B_{12}Y_{22}+B_{14}Y_{44}) & A_{32}(B_{21}Y_{11}+B_{24}Y_{44}) & 0 & A_{34}(B_{41}Y_{11}+B_{42}Y_{22}) \\ A_{41}(B_{12}Y_{22}+B_{13}Y_{33}) & A_{42}(B_{21}Y_{11}+B_{23}Y_{33}) & A_{43}(B_{31}Y_{11}+B_{32}Y_{22}) & 0 \end{bmatrix}$$

Group 4: $A_{sr}B_{rt}Y_{tu}$, $t \neq r,s$, $u \neq t,s$ Country s' intermediate exports used by partner Country r, to produce intermediate exports to the third Countries(t) for production of final exports to countries other than Country t and the source Country s.

Step 1:

$$\begin{bmatrix} 0 & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & 0 & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & 0 & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & Y_{12}+Y_{13}+Y_{14} & Y_{12}+Y_{13}+Y_{14} & Y_{12}+Y_{13}+Y_{14} \\ Y_{21}+Y_{23}+Y_{24} & 0 & Y_{21}+Y_{23}+Y_{24} & Y_{21}+Y_{23}+Y_{24} \\ Y_{31}+Y_{32}+Y_{34} & Y_{31}+Y_{32}+Y_{34} & 0 & Y_{31}+Y_{32}+Y_{34} \\ Y_{41}+Y_{42}+Y_{43} & Y_{41}+Y_{42}+Y_{43} & Y_{41}+Y_{42}+Y_{43} & 0 \end{bmatrix} - \begin{bmatrix} 0 & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & 0 & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & 0 & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & Y_{13}+Y_{14} & Y_{12}+Y_{14} & Y_{12}+Y_{13} \\ Y_{23}+Y_{24} & 0 & Y_{21}+Y_{24} & Y_{21}+Y_{23} \\ Y_{32}+Y_{34} & Y_{31}+Y_{34} & 0 & Y_{31}+Y_{32} \\ Y_{42}+Y_{43} & Y_{41}+Y_{43} & Y_{41}+Y_{42} & 0 \end{bmatrix}$$

Step2:

$$\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{bmatrix} \begin{bmatrix} 0 & Y_{13}+Y_{14} & Y_{12}+Y_{14} & Y_{12}+Y_{13} \\ Y_{23}+Y_{24} & 0 & Y_{21}+Y_{24} & Y_{21}+Y_{23} \\ Y_{32}+Y_{34} & Y_{31}+Y_{34} & 0 & Y_{31}+Y_{32} \\ Y_{42}+Y_{43} & Y_{41}+Y_{43} & Y_{41}+Y_{42} & 0 \end{bmatrix}$$ and make the diagonal elements of the resulted matrix to 0, then transpose

Step 3:

$$\begin{bmatrix} 0 & B_{13}(Y_{31}+Y_{34})+B_{14}(Y_{41}+Y_{43}) & B_{12}(Y_{21}+Y_{24})+B_{14}(Y_{41}+Y_{42}) & B_{12}(Y_{21}+Y_{23})+B_{13}(Y_{31}+Y_{32}) \\ B_{23}(Y_{32}+Y_{34})+B_{24}(Y_{42}+Y_{43}) & 0 & B_{21}(Y_{12}+Y_{14})+B_{24}(Y_{41}+Y_{42}) & B_{21}(Y_{12}+Y_{13})+B_{23}(Y_{31}+Y_{32}) \\ B_{32}(Y_{23}+Y_{24})+B_{34}(Y_{42}+Y_{43}) & B_{31}(Y_{13}+Y_{14})+B_{34}(Y_{41}+Y_{43}) & 0 & B_{31}(Y_{12}+Y_{13})+B_{32}(Y_{21}+Y_{23}) \\ B_{42}(Y_{23}+Y_{24})+B_{43}(Y_{32}+Y_{34}) & B_{41}(Y_{13}+Y_{14})+B_{43}(Y_{31}+Y_{34}) & B_{41}(Y_{12}+Y_{14})+B_{42}(Y_{21}+Y_{24}) & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & B_{23}(Y_{32}+Y_{34})+B_{24}(Y_{42}+Y_{43}) & B_{32}(Y_{23}+Y_{24})+B_{34}(Y_{42}+Y_{43}) & B_{42}(Y_{23}+Y_{24})+B_{43}(Y_{32}+Y_{34}) \\ B_{13}(Y_{31}+Y_{34})+B_{14}(Y_{41}+Y_{43}) & 0 & B_{31}(Y_{13}+Y_{14})+B_{34}(Y_{41}+Y_{43}) & B_{41}(Y_{13}+Y_{14})+B_{43}(Y_{31}+Y_{34}) \\ B_{12}(Y_{21}+Y_{24})+B_{14}(Y_{41}+Y_{42}) & B_{21}(Y_{12}+Y_{14})+B_{24}(Y_{41}+Y_{42}) & 0 & B_{41}(Y_{12}+Y_{14})+B_{42}(Y_{21}+Y_{24}) \\ B_{12}(Y_{21}+Y_{23})+B_{13}(Y_{31}+Y_{32}) & B_{21}(Y_{12}+Y_{13})+B_{23}(Y_{31}+Y_{32}) & B_{31}(Y_{12}+Y_{13})+B_{32}(Y_{21}+Y_{23}) & 0 \end{bmatrix}$$

Step 4:

$$
\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{23}(Y_{32}+Y_{34})+B_{24}(Y_{42}+Y_{43}) & B_{32}(Y_{23}+Y_{24})+B_{34}(Y_{42}+Y_{43}) & B_{42}(Y_{23}+Y_{24})+B_{43}(Y_{32}+Y_{34}) \\ B_{13}(Y_{31}+Y_{34})+B_{14}(Y_{41}+Y_{43}) & 0 & B_{31}(Y_{13}+Y_{14})+B_{34}(Y_{41}+Y_{43}) & B_{41}(Y_{13}+Y_{14})+B_{43}(Y_{31}+Y_{34}) \\ B_{12}(Y_{21}+Y_{24})+B_{14}(Y_{41}+Y_{42}) & B_{21}(Y_{12}+Y_{14})+B_{24}(Y_{41}+Y_{42}) & 0 & B_{41}(Y_{12}+Y_{14})+B_{42}(Y_{21}+Y_{24}) \\ B_{12}(Y_{21}+Y_{23})+B_{13}(Y_{31}+Y_{32}) & B_{21}(Y_{12}+Y_{13})+B_{23}(Y_{31}+Y_{32}) & B_{31}(Y_{12}+Y_{13})+B_{32}(Y_{21}+Y_{23}) & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & A_{12}[B_{23}(Y_{32}+Y_{34})+B_{24}(Y_{42}+Y_{43})] & A_{13}[B_{32}(Y_{23}+Y_{24})+B_{34}(Y_{42}+Y_{43})] & A_{14}[B_{42}(Y_{23}+Y_{24})+B_{43}(Y_{32}+Y_{34})] \\ A_{21}[B_{13}(Y_{31}+Y_{34})+B_{14}(Y_{41}+Y_{43})] & 0 & A_{23}[B_{31}(Y_{13}+Y_{14})+B_{34}(Y_{41}+Y_{43})] & A_{24}[B_{41}(Y_{13}+Y_{14})+B_{43}(Y_{31}+Y_{34})] \\ A_{31}[B_{12}(Y_{21}+Y_{24})+B_{14}(Y_{41}+Y_{42})] & A_{32}[B_{21}(Y_{12}+Y_{14})+B_{24}(Y_{41}+Y_{42})] & 0 & A_{34}[B_{41}(Y_{12}+Y_{14})+B_{42}(Y_{21}+Y_{24})] \\ A_{41}[B_{12}(Y_{21}+Y_{23})+B_{13}(Y_{31}+Y_{32})] & A_{42}[B_{21}(Y_{12}+Y_{13})+B_{23}(Y_{31}+Y_{32})] & A_{43}[B_{31}(Y_{12}+Y_{13})+B_{32}(Y_{21}+Y_{23})] & 0 \end{bmatrix}
$$

Group 6: $A_{sr}B_{rt}Y_{ts}$, $t \neq r,s$, Country s' intermediate exports used by partner Country r to produce intermediate exports to the third Countries t for its production of final exports that return back to the source Country s.

Step 1

$$
\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{bmatrix} \begin{bmatrix} 0 & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & 0 & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & 0 & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & 0 \end{bmatrix}
$$

and make the diagonal elements of the resulted matrix to 0, then transpose

Step 2:

$$
\begin{bmatrix} 0 & B_{13}Y_{32}+B_{14}Y_{42} & B_{12}Y_{23}+B_{14}Y_{43} & B_{12}Y_{24}+B_{13}Y_{34} \\ B_{23}Y_{31}+B_{24}Y_{41} & 0 & B_{21}Y_{13}+B_{24}Y_{43} & B_{21}Y_{14}+B_{23}Y_{34} \\ B_{32}Y_{21}+B_{34}Y_{41} & B_{31}Y_{12}+B_{34}Y_{42} & 0 & B_{31}Y_{14}+B_{32}Y_{24} \\ B_{42}Y_{21}+B_{43}Y_{31} & B_{41}Y_{12}+B_{43}Y_{32} & B_{41}Y_{13}+B_{42}Y_{23} & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & B_{23}Y_{31}+B_{24}Y_{41} & B_{32}Y_{21}+B_{34}Y_{41} & B_{42}Y_{21}+B_{43}Y_{31} \\ B_{13}Y_{32}+B_{14}Y_{42} & 0 & B_{31}Y_{12}+B_{34}Y_{42} & B_{41}Y_{12}+B_{43}Y_{32} \\ B_{12}Y_{23}+B_{14}Y_{43} & B_{21}Y_{13}+B_{24}Y_{43} & 0 & B_{41}Y_{13}+B_{42}Y_{23} \\ B_{12}Y_{24}+B_{13}Y_{34} & B_{21}Y_{14}+B_{23}Y_{34} & B_{31}Y_{14}+B_{32}Y_{24} & 0 \end{bmatrix}
$$

Step 3:

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{23}Y_{31} + B_{24}Y_{41} & B_{32}Y_{21} + B_{34}Y_{41} & B_{42}Y_{21} + B_{43}Y_{31} \\ B_{13}Y_{32} + B_{14}Y_{42} & 0 & B_{31}Y_{12} + B_{34}Y_{42} & B_{41}Y_{12} + B_{43}Y_{32} \\ B_{12}Y_{23} + B_{14}Y_{43} & B_{21}Y_{13} + B_{24}Y_{43} & 0 & B_{41}Y_{13} + B_{42}Y_{23} \\ B_{12}Y_{24} + B_{13}Y_{34} & B_{21}Y_{14} + B_{23}Y_{34} & B_{31}Y_{14} + B_{32}Y_{24} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A_{12}(B_{23}Y_{31} + B_{24}Y_{41}) & A_{13}(B_{32}Y_{21} + B_{34}Y_{41}) & A_{14}(B_{42}Y_{21} + B_{43}Y_{31}) \\ A_{21}(B_{13}Y_{32} + B_{14}Y_{42}) & 0 & A_{23}(B_{31}Y_{12} + B_{34}Y_{42}) & A_{24}(B_{41}Y_{12} + B_{43}Y_{32}) \\ A_{31}(B_{12}Y_{23} + B_{14}Y_{43}) & A_{32}(B_{21}Y_{13} + B_{24}Y_{43}) & 0 & A_{34}(B_{42}Y_{23} + B_{41}Y_{13}) \\ A_{41}(B_{12}Y_{24} + B_{13}Y_{34}) & A_{42}(B_{23}Y_{34} + B_{21}Y_{14}) & A_{43}(B_{32}Y_{24} + B_{31}Y_{14}) & 0 \end{bmatrix}$$

(3) Intermediate exports of Country s that return home and used to produce final goods in Country s can be further decomposed into two groups

$$A_{sr}B_{rs}Y_{s*} = A_{sr}B_{rs}\sum_{k}^{G}Y_{sk} = A_{sr}B_{rs}Y_{ss} + A_{sr}B_{rs}\sum_{t\neq s}^{G}Y_{st}$$

Corresponding gross output can be decomposed in a similar way

$$B_{rs}Y_{s*} = B_{rs}\sum_{k}^{G}Y_{sk} = B_{rs}Y_{ss} + B_{rs}\sum_{t\neq s}^{G}Y_{st}$$

Group 7: $A_{sr}B_{rs}Y_{ss}$, They are part of Country s' intermediate goods exports used by partner Country r to produce intermediate exports that shipped back to Country s and used by Country s to produce final goods consumed at home.

Step 1:

$$\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{bmatrix} \begin{bmatrix} Y_{11} & 0 & 0 & 0 \\ 0 & Y_{22} & 0 & 0 \\ 0 & 0 & Y_{33} & 0 \\ 0 & 0 & 0 & Y_{44} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & B_{21}Y_{11} & B_{31}Y_{11} & B_{41}Y_{11} \\ B_{12}Y_{22} & 0 & B_{32}Y_{22} & B_{42}Y_{22} \\ B_{13}Y_{33} & B_{23}Y_{33} & 0 & B_{43}Y_{33} \\ B_{14}Y_{44} & B_{24}Y_{44} & B_{34}Y_{44} & 0 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} 0 & B_{12}Y_{22} & B_{13}Y_{33} & B_{14}Y_{44} \\ B_{21}Y_{11} & 0 & B_{23}Y_{33} & B_{24}Y_{44} \\ B_{31}Y_{11} & B_{32}Y_{22} & 0 & B_{34}Y_{44} \\ B_{41}Y_{11} & B_{42}Y_{22} & B_{43}Y_{33} & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & B_{21}Y_{11} & B_{31}Y_{11} & B_{41}Y_{11} \\ B_{12}Y_{22} & 0 & B_{32}Y_{22} & B_{42}Y_{22} \\ B_{13}Y_{33} & B_{23}Y_{33} & 0 & B_{43}Y_{33} \\ B_{14}Y_{44} & B_{24}Y_{44} & B_{34}Y_{44} & 0 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{21}Y_{11} & B_{31}Y_{11} & B_{41}Y_{11} \\ B_{12}Y_{22} & 0 & B_{32}Y_{22} & B_{42}Y_{22} \\ B_{13}Y_{33} & B_{23}Y_{33} & 0 & B_{43}Y_{33} \\ B_{14}Y_{44} & B_{24}Y_{44} & B_{34}Y_{44} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{12}B_{21}Y_{11} & A_{13}B_{31}Y_{11} & A_{14}B_{41}Y_{11} \\ A_{21}B_{12}Y_{22} & 0 & A_{23}B_{32}Y_{22} & A_{24}B_{42}Y_{22} \\ A_{31}B_{13}Y_{33} & A_{32}B_{23}Y_{33} & 0 & A_{34}B_{43}Y_{33} \\ A_{41}B_{14}Y_{44} & A_{42}B_{24}Y_{44} & A_{43}B_{34}Y_{44} & 0 \end{bmatrix}$$

Group 8: $A_{sr}B_{rs}Y_{st}$, $t \neq s$ Country s' intermediate goods exports used by importing Country r to produce intermediate goods exports that are shipped back to Country s to produce its final goods exports that are consumed abroad.

Step 1:

$$\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{bmatrix} \begin{bmatrix} Y_{12}+Y_{13}+Y_{14} & 0 & 0 & 0 \\ 0 & Y_{21}+Y_{23}+Y_{24} & 0 & 0 \\ 0 & 0 & Y_{31}+Y_{32}+Y_{34} & 0 \\ 0 & 0 & 0 & Y_{41}+Y_{42}+Y_{43} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & B_{12}(Y_{23}+Y_{21}+Y_{24}) & B_{13}(Y_{32}+Y_{31}+Y_{34}) & B_{14}(Y_{43}+Y_{41}+Y_{42}) \\ B_{21}(Y_{12}+Y_{13}+Y_{14}) & 0 & B_{23}(Y_{32}+Y_{31}+Y_{34}) & B_{24}(Y_{43}+Y_{41}+Y_{42}) \\ B_{31}(Y_{12}+Y_{13}+Y_{14}) & B_{32}(Y_{23}+Y_{21}+Y_{24}) & 0 & B_{34}(Y_{43}+Y_{41}+Y_{42}) \\ B_{41}(Y_{12}+Y_{13}+Y_{14}) & B_{42}(Y_{23}+Y_{21}+Y_{24}) & B_{43}(Y_{32}+Y_{31}+Y_{34}) & 0 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 0 & B_{12}(Y_{23}+Y_{21}+Y_{24}) & B_{13}(Y_{32}+Y_{31}+Y_{34}) & B_{14}(Y_{43}+Y_{41}+Y_{42}) \\ B_{21}(Y_{12}+Y_{13}+Y_{14}) & 0 & B_{23}(Y_{32}+Y_{31}+Y_{34}) & B_{24}(Y_{43}+Y_{41}+Y_{42}) \\ B_{31}(Y_{12}+Y_{13}+Y_{14}) & B_{32}(Y_{23}+Y_{21}+Y_{24}) & 0 & B_{34}(Y_{43}+Y_{41}+Y_{42}) \\ B_{41}(Y_{12}+Y_{13}+Y_{14}) & B_{42}(Y_{23}+Y_{21}+Y_{24}) & B_{43}(Y_{32}+Y_{31}+Y_{34}) & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & B_{21}(Y_{12}+Y_{13}+Y_{14}) & B_{31}(Y_{12}+Y_{13}+Y_{14}) & B_{41}(Y_{12}+Y_{13}+Y_{14}) \\ B_{12}(Y_{23}+Y_{21}+Y_{24}) & 0 & B_{32}(Y_{23}+Y_{21}+Y_{24}) & B_{42}(Y_{23}+Y_{21}+Y_{24}) \\ B_{13}(Y_{32}+Y_{31}+Y_{34}) & B_{23}(Y_{32}+Y_{31}+Y_{34}) & 0 & B_{43}(Y_{32}+Y_{31}+Y_{34}) \\ B_{14}(Y_{43}+Y_{41}+Y_{42}) & B_{24}(Y_{43}+Y_{41}+Y_{42}) & B_{34}(Y_{43}+Y_{41}+Y_{42}) & 0 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix} \# \begin{bmatrix} 0 & B_{21}(Y_{12}+Y_{13}+Y_{14}) & B_{31}(Y_{12}+Y_{13}+Y_{14}) & B_{41}(Y_{12}+Y_{13}+Y_{14}) \\ B_{12}(Y_{23}+Y_{21}+Y_{24}) & 0 & B_{32}(Y_{23}+Y_{21}+Y_{24}) & B_{42}(Y_{23}+Y_{21}+Y_{24}) \\ B_{13}(Y_{32}+Y_{31}+Y_{34}) & B_{23}(Y_{32}+Y_{31}+Y_{34}) & 0 & B_{43}(Y_{32}+Y_{31}+Y_{34}) \\ B_{14}(Y_{43}+Y_{41}+Y_{42}) & B_{24}(Y_{43}+Y_{41}+Y_{42}) & B_{34}(Y_{43}+Y_{41}+Y_{42}) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A_{12}B_{21}(Y_{12}+Y_{13}+Y_{14}) & A_{13}B_{31}(Y_{12}+Y_{13}+Y_{14}) & A_{14}B_{41}(Y_{12}+Y_{13}+Y_{14}) \\ A_{21}B_{12}(Y_{23}+Y_{21}+Y_{24}) & 0 & A_{23}B_{32}(Y_{23}+Y_{21}+Y_{24}) & A_{24}B_{42}(Y_{23}+Y_{21}+Y_{24}) \\ A_{31}B_{13}(Y_{32}+Y_{31}+Y_{34}) & A_{32}B_{23}(Y_{32}+Y_{31}+Y_{34}) & 0 & A_{34}B_{43}(Y_{32}+Y_{31}+Y_{34}) \\ A_{41}B_{14}(Y_{43}+Y_{41}+Y_{42}) & A_{42}B_{24}(Y_{43}+Y_{41}+Y_{42}) & A_{43}B_{34}(Y_{43}+Y_{41}+Y_{42}) & 0 \end{bmatrix}$$