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THE NEW-KEYNESIAN LIQUIDITY TRAP

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ABSTRACT

In standard solutions, the new-Keynesian model produces a deep recession with deflation in a liquidity trap. Useless government spending, technical regress, and capital destruction have large positive multipliers. The recession prediction, and deflation and policy paradoxes are larger when prices are less sticky. I show that these puzzling predictions are artifacts of equilibrium selection. For the same interest-rate policy, different choices of multiple equilibria overturn all these results. A "local-to-frictionless" equilibrium, for the same interest rate policy, predicts mild inflation, no output reduction and negative multipliers during the liquidity trap, and its predictions approach the frictionless model smoothly.

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1 Introduction

Standard solutions of new-Keynesian models predict deep recessions and deflation in a liquidity trap, when the "natural" rate of interest is negative and the nominal rate is stuck at zero. The models assume that the level of future consumption and output will revert to trend, and today's consumption is that trend less the sum of all future growth rates, driven by the sum of all future real interest rates. Too-high real rates then mean too-strong expected consumption and output growth, which produce a level of consumption and output deeply below trend.

New-Keynesian models produce striking policy predictions at the zero bound. "Open-mouth operations," "forward guidance," and commitments to keep interest rates low for long periods, with no current action, lower expected future real interest rates, lower growth rates, and thus stimulate the current level of consumption. Fully-expected future inflation is a good thing, as it lowers real rates, lowers consumption growth and thus boosts the level of consumption. Deliberate destruction of output, capital, and productivity raise output, because they raise such inflation, and the models assume a reversion to an unchanged trend despite such destruction. Government spending, even if financed by current taxation, and even if completely wasted, can have huge output multipliers.

Even more puzzling, new-Keynesian models predict that deflation and depression during the trap gets worse, and these topsy-turvy policy prescriptions become stronger, as prices become *more* flexible, and they give no well-defined limit to the case of completely flexible prices. Thus, although price stickiness is the central friction keeping the economy from achieving its optimal output, policies that reduce price stickiness would make matters worse.

In short, all of economics seems to change sign at the zero bound.

New-Keynesian models have multiple equilibria. I show that these puzzling liquidity-trap predictions result from particular equilibrium choices. Choosing a different equilibrium, despite exactly the same interest-rate policies, the New-Keynesian model does not produce deep and deflationary depressions; the model solutions approach the frictionless limit in a smooth way; and the paradoxical (to some) or intoxicating (to others) policy predictions vanish.

Following Werning (2012), I suppose that the natural rate is negative up until period T, and the interest rate is zero. After period T, people expect the interest rate to follow the natural rate. I compute the range of equilibria allowed by this specification in a very simple continuous-time new-Keynesian model.

Figure 5 shows the standard equilibrium choice, which Werning presents. This case features a large recession, large deflation, and strong output growth. The thin lines show that the predicted recession is *worse* as price stickiness gets *better*, approaching a cliff. I show below that multipliers for useless government spending, promises of future inflation, and promises to delay the rise in interest rates strongly raise output in this equilibrium.

Figure 4 shows a "local to frictionless" equilibrium. Here, there is small positive inflation during the trap, matching the negative natural rate. Output is slightly above potential throughout. The thin lines show equilibria for steadily lower price frictions. These equilibria smoothly approach the frictionless case. I show below that the multiplier is small and negative in this equilibrium, and policies have the normal sign.

The equilibria shown in these figures have *exactly the same interest rate path.* They are each completely valid equilibria of the same model. They differ only by equilibrium choice.

At a minimum, this analysis shows that the choice of equilibrium matters enormously to the new-Keynesian model's predictions about what happens in a liquidity trap. One simply cannot say that the new-Keynesian model predicts deep depressions and paradoxical policies in a liquidity trap. Those predictions rely on one, arbitrary, equilibrium choice, and have no greater foundation than the statement that the new-Keynesian model predicts mild inflation, no reduction in output, and normal policy responses in a liquidity trap.

2 Literature

This paper owes an obvious debt to Werning (2012) whose structure I adopt. It is not a critique of Werning. Many of my points are acknowledged in his footnotes, and I do not claim any mistakes in his analysis. Kiley (2013) and Wieland (2013) nicely summarize the puzzling predictions of new-Keynesian zero-bound analyses. Kiley contrasts sticky-price and sticky-information models behavior with the policy paradoxes, and finds that sticky-information models also resolve some paradoxes.

Eggertsson and Woodford (2003) study monetary policy in a liquidity trap, emphasizing the dangers of the trap and the idea that the Fed commit to keep interest rates low after the trap ends, which is the centerpiece of Werning (2012).

Christiano, Eichenbaum and Rebelo (2011), Eggertsson (2011) Woodford (2011) Carlstrom, Fuerst, and Paustian (2012) all find large fiscal multipliers, and multipliers that increase with the duration of fiscal expansion, in new-Keynesian zero-bound models.

Eggertsson (2010) and Wieland (2013) analyzes the "paradox of toil" that negative productivity can be expansionary. Wieland shows that several cases of endowment destruction did not seem to have the predicted stimulative effects.

Braun, Körber, and Waki (2012) find that the paradoxial properties of new-Keynesian models are artifacts of linearization. Christiano and Eichenbaum (2012) argue that the criticism is not "E-learnable" and does not apply for small enough deviations. I use the linearized system entirely, so this controversy is not relevant to my points.

All the multiple equilibria in this paper are locally bounded as $t \to \infty$. Cochrane (2011) discusses that additional source of multiplicity. That is not the issue here – I play by standard rules of the new-Keynesian game and simply rule out equilibria that explode as $t \to \infty$.

3 Model

I follow Werning (2012) and study the continuous-time specification of the standard new-Keynesian model. Werning's brilliantly simple framework clarifies the issues relative to possibly more realistic but more complex environments. The model is

$$\frac{dx_t}{dt} = \sigma^{-1}(i_t - r_t - \pi_t) \tag{1}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa x_t. \tag{2}$$

Here, x_t is the output gap, i_t is the nominal rate of interest, r_t is the "natural" real rate of interest, and π_t is inflation. All variables are deviations from trend or steady state. I specialize to $\sigma = 1$ in what follows as variation in this parameter plays no role in the analysis. Equation (1) is the "IS" curve, which ought to be renamed the "intertemporal substitution" curve. It is derived from the first-order condition for allocation of consumption over time, and consumption equals output with no capital. Equation (2) is the new-Keynesian forward-looking Phillips curve. Solving it forwards, it expresses inflation in terms of expected future output gaps,

$$\pi_t = \kappa \int_{s=0}^{\infty} e^{-\rho s} x_{t+s} ds$$

Like Werning, I suppose that the economy suffers from a temporarily negative natural rate $r_t = -r < 0$, which lasts until time T before returning to a positive value. After time T, the Fed follows the policy $i_t = r_t > 0$, setting the interest rate equal to the natural rate. Before time T, however, the zero bound $i_t \geq 0$ is binding. Thus, interest-rate policy follows

$$\begin{aligned} t &< T : i_t = 0 \\ t &\geq T : i_t = r_t \geq 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} t &< T : i_t - r_t = r \\ t &\geq T : i_t - r_t = 0 \end{aligned} \tag{3}$$

I treat the case that interest rates follow a Taylor rule for $T > t$ below.

Next, I find the equilibrium paths of the output gap and inflation consistent with this forcing process, and I study how those equilibria behave as we vary the parameter κ, and in particular the limits as $\kappa \to \infty$ (price flexibility) and $\kappa \to 0$ (constant inflation).

3.1 The flexible-price case

To understand the flexible-price limit, it is best to write the Phillips curve (2) as

$$x_t = \frac{1}{\kappa}\left(\rho \pi_t - \frac{d\pi_t}{dt}\right).$$

As κ rises, prices become flexible, as the output gap x_t for any inflation path π_t becomes smaller and smaller. In the flexible-price limit, inflation can do whatever it wants, and the output gap is zero. Thus, in the flexible-price limit point is $\kappa = \infty$, where we must have $x_t = 0$.

Turning to the IS curve (1), if $x_t = 0$ then $dx_t/dt = 0$ and we must have

$$i_t - r_t = \pi_t.$$

This is just the linearized Fisher relationship, which is an arbitrage relationship in this perfect-foresight continuous-time economy. When prices are flexible, the Fed loses its power to affect real interest rates and the output gap. Since r_t is given, by changing nominal rates i_t, the Fed changes (expected) inflation, period.

Thus, the flexible-price solution to our liquidity-trap scenario (3) is

$$\begin{aligned} t &< T : \pi_t = r,\ x_t = 0 \\ t &\geq T : \pi_t = 0,\ x_t = 0. \end{aligned}$$

During the "liquidity trap" inflation is exactly equal to the negative "natural rate," so that the zero nominal rate of interest produces a real rate equal to the negative natural real rate.

3.2 The general case with price stickiness

Now, I solve the standard case $\kappa < \infty$. The model (1)-(2) is

$$\frac{d}{dt}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} ir_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -\kappa & \rho \end{bmatrix}\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \tag{4}$$

where $ir_t = i_t - r_t$, with $ir_t = r > 0$ for $t < T$ and $ir_t = 0$ for $t \geq T$.

The model is a standard first-order matrix differential equation. I solve it by finding the eigenvalue decomposition of the square matrix, and pasting together the solutions in each of $t < T$ and $t \geq T$. I present the algebra in the Appendix. The answer is

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir_t - \frac{1}{\lambda^p - \lambda^m}\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix}\begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} ir_t + \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_T e^{\lambda^m(t-T)}. \tag{5}$$

where, recall, $ir_t = ir$ for $t < T$ and $ir_t = 0$ for $t > T$, and

$$\begin{aligned} \lambda^p &= \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\kappa}\right) > 0 \\ \lambda^m &= \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\kappa}\right) < 0. \end{aligned} \tag{6}$$

The first term of (5) is the steady state. The second term describes mixed exponential dynamics. The sum of first and second terms vanish as $t \to T$. (You need the identity $\rho = \lambda^m + \lambda^p$ to see that fact.) Because $ir_t = 0,\ t > T$, the first and second terms also vanish for $t > T$ leaving only the third term.

In this perfect-foresight model, the parameter π_T in the third term of (5), inflation at the end of the liquidity trap, indexes all the possible multiple equilibria that are locally bounded as $t \to \infty$. The third term shows that such multiple equilibria add exponentially decaying dynamics.

There are in general two free constants. I impose the standard new-Keynesian assumption that equilibria must not explode as $t \to \infty$ to eliminate one of them, setting to zero the forward-explosive eigenvalue in the $t > T$ region. One could equivalently index the alternative equilibria by expectations about the output gap at T, x_T, but I follow tradition in specifying them as a function of inflation expectations. The resulting discussion follows the tradition of implying causality from inflation to output expectations that is not present in the equations.

As pricing frictions decrease and κ increases, both eigenvalues λ^p and λ^m increase in absolute value. Dynamics happen faster and faster as prices become more flexible.

3.3 The standard equilibrium

Werning (2012), like the rest of the literature, chooses the equilibrium $\pi_T = 0$. In this equilibrium, the economy is in the steady state as soon as the liquidity trap is lifted, and

$$t \geq T: \ x_t = \pi_t = 0 \tag{7}$$

Before T, during the liquidity trap episode, (5) then becomes

$$t \leq T: \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir - \frac{1}{\lambda^p - \lambda^m}\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix}\begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} ir \tag{8}$$

The $t < T$ solution loads on both eigenvalues, including $\lambda^m < 0$. These solutions explode *backward* in time, giving very strong deflation and output gaps early in a long liquidity-trap episode, and strong liquidity-trap dynamics.

As Werning shows, and I verify below, these solutions also explode as prices become *less* sticky, $\kappa \to \infty$. Both eigenvalues rise without limit, so the backwards explosions of (8) happen faster and faster. Thus, we are in the strange position that for prices very near frictionless, the model predicts huge deflation and output gaps, while for the frictionless limit point, it instead predicts steady inflation and no output gap.

3.4 The local-to-frictionless equilibrium

Another interesting choice of equilibrium, via π_T, is the one in which the economy approaches the steady state as t goes *backwards* in time. This is a natural counterpart to the criterion we used in the $t > T$ region, that the economy should converge to the steady state as t goes *forward* in time.

This equilibrium produces the frictionless solution in the limit as frictions disappear. The frictionless solution says that we are at the steady state $\pi_t = r$, $x_t = 0$ for the entire period $t \le T$. The backwards-stable equilibrium will approach this steady state. We could use that fact as a criterion to select this equilibrium: price-stickiness can only have finitely lived effects.

To impose this "two-sided boundedness" or "near-to-frictionless" criterion, we pick π_T so that the loading of the $t \le T$ solution on the backwards-explosive eigenvalue λ^m is zero. Writing (5) for $t < T$ in the form

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir_t + \left(\begin{bmatrix} \lambda^p & 0 \\ 1 & 0 \end{bmatrix} \pi_T - \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} ir \right) \begin{bmatrix} e^{\lambda^m (t-T)} \\ e^{\lambda^p (t-T)} \end{bmatrix},$$

we see that the choice

$$\pi_T = \frac{\lambda^p}{\lambda^p - \lambda^m} ir$$

puts no loading on the backwards-explosive root λ^m for $t < T$. Imposing that equilibrium choice, the solution (5) becomes

$$t < T : \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} ir + \frac{\lambda^m}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^m \\ 1 \end{bmatrix} e^{\lambda^p (t-T)} ir \tag{9}$$

$$t > T : \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{\lambda^p}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m (t-T)} ir \tag{10}$$

3.5 Pictures

A picture is worth a thousand equations.

Figure 1 presents the frictionless solution $\kappa = \infty$. I specify a $-r = -5\%$ natural rate that lasts until $T = 5$. As the figure shows, the frictionless solution gives steady 5% inflation during the zero-bound period, and 0% inflation thereafter. There is no output gap.

Figure 2 presents the local-to-frictionless solution (9)-(10). I use $\rho = 0.05$ and $\kappa = 1$. The solution is well-behaved in both directions. The economy starts with 5% inflation and a small

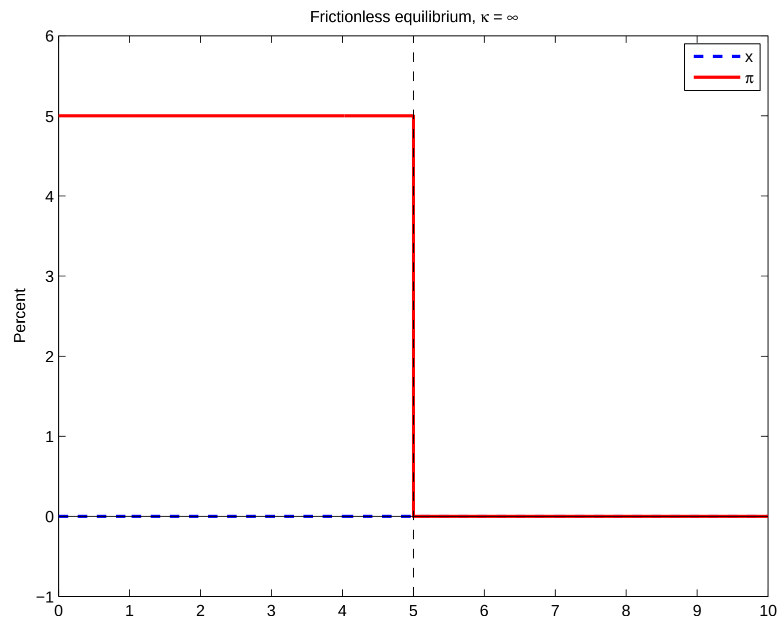

Figure 1: Solution to the liquidity-trap scenario in the frictionless case $\kappa = \infty$.

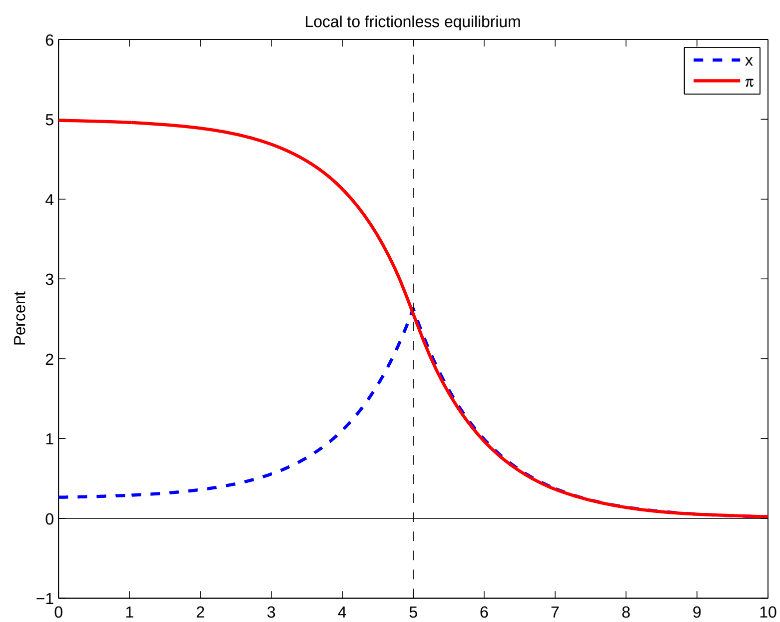

Figure 2: The local-to-frictionless equilibrum of the new-Keynesian model in a liquidity trap. $\rho = 0.05$, $\kappa = 1$, $\sigma = 1$. The 5% negative natural rate lasts until $t = T = 5$.

permanent output gap. As the end of the liquidity trap, inflation slowly comes down. There is a small increase in output towards the end of the liquidity trap. This is a standard and much-criticized feature of the usual new-Keynesian Phillips curve (2), that output rises when inflation is declining, i.e. when inflation today is higher than expected future inflation.

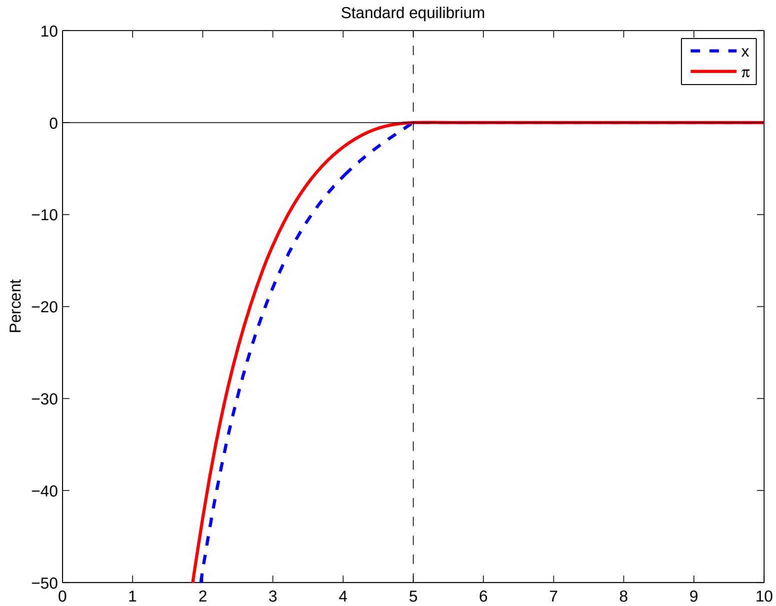

Figure 3: The standard choice of equilibrium in the new-Keynesian liquidity trap scenario.

Figure 3 presents the standard equilibrium choice in this liquidity trap scenario, (7) and (8). This equilibrium is driven by the specification that the output gap and inflation π_T will both be zero at $t = T$. This choice of equilibrium for $t \geq T$ has a drastic effect on the equilibrium for $t \leq T$. Now we see a dramatic deflation and catastrophic output gaps during the liquidity-trap period. We also see strong dynamics – deflation steadily improves, and output *growth* is strong. This model does not produce a "slump," a long period of a steady output gap.

Figure 2 and Figure 3 are both equilibria of the *same* model with the *same* parameters, and the *same interest rate policy*. The only difference between the figures is the choice of equilibrium. Nothing is theoretically "wrong" with either equilibrium, at least playing by the standard new-Keynesian rules.

The two equilibrium choices have drastically different implications for how the economy behaves in a liquidity trap. Figure 2 suggests that a liquidity trap is associated with a mild inflation and a small increase in output relative to "potential" as the trap ends. Figure 3 suggests that a liquidity trap is an economic calamity, with large output gap, strong expected output growth, and rampant deflation.

The comparison of the two equilibria emphasizes Werning's (2012) central message. The real problem of the liquidity trap is not so much the lower bound on contemporaneous interest rates, but expectations of what will happen once the trap passes. The only difference between a smooth glidepath and a disastrous recession is people's expectations of inflation in the immediate aftermath of the liquidity trap.

Most of all, one cannot argue that the choice of equilibrium (or, equivalently, but less transparently, the choice of expected Taylor-rule intercept) is an innocuous technical detail in new-Keynesian models!

Figure 4 shows how the local-to-frictionless solutions behave as we increase the parameter κ, lowering the pricing friction, in the sequence of thin dashed lines. The figure only shows how

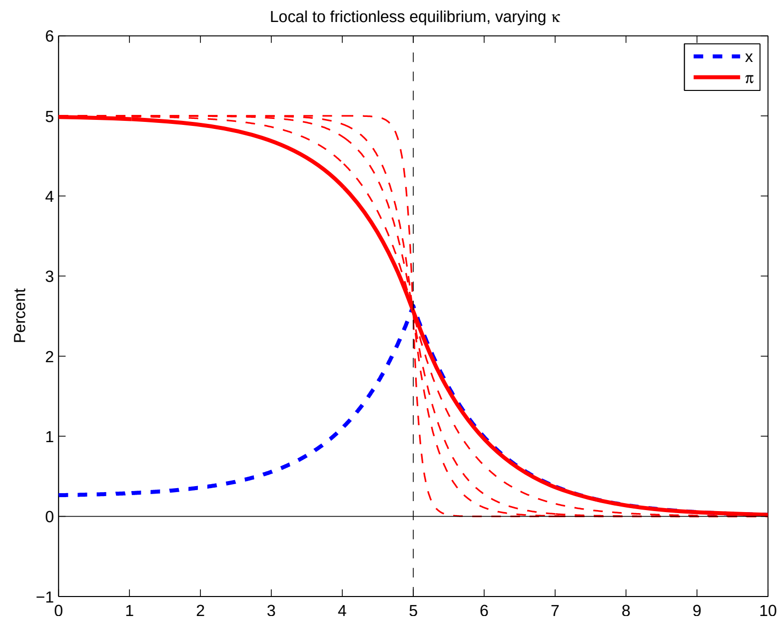

Figure 4: Local-to-frictionless equilibrium choice, varying price-stickiness. The price-stickiness paramter κ takes values 1,2,5,10,100. For clarity, the figure only shows inflation is shown for $\kappa \neq 1$.

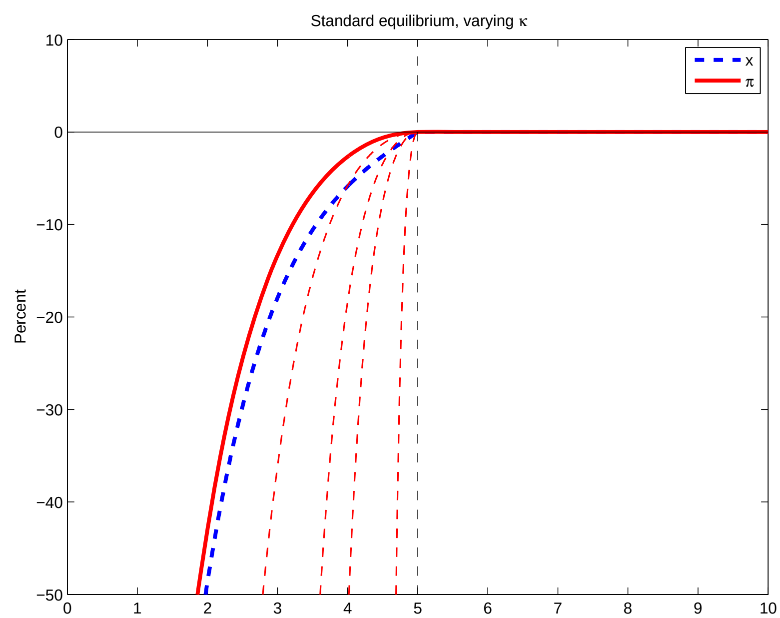

Figure 5: Standard equilibrium choice for different values of price stickiness κ. κ takes values 1,2,5,10,100. The figure only shows inflation is shown for $\kappa \neq 1$.

inflation varies as κ rises above 1 for clarity. Output gaps also smoothly approach the frictionless solution $x_t = 0$. As κ increases both eigenvalues increase, so the steady state is arrived at more quickly in both time directions.

Figure 5 shows how inflation behaves in the standard equilibrium choice as we as we increase κ and thus reduce the pricing friction. The solutions become steeper and steeper with greater and greater deflation, and larger and larger output gaps (not shown) as the price friction *decreases*. The eigenvalues increase with κ, but in this case that means the backward explosion happens faster and faster. The limit as $\kappa \to \infty$ is a cliff for both output and inflation.

The prefect-foresight new-Keynesian Phillips curve (2) does not allow inflation jumps. This local-to-frictionless solution smooths naturally with inflation following an S shape. But some of that smoothing must occur in the $t > T$ period. Insistence that inflation is zero *immediately* at $t = T$ is the crucial specification that drives the economic dislocation and puzzling limiting behavior of the standard solution.

Figure 6 and figure 7 show inflation and output gaps in a range of equilibria, indexed by different choices for π_T. The solid lines reprise the standard equilibrium choice, generated by $\pi_T = 0$, and the local-to-frictionless equilibrium choice. The dots at $T = 5$ represent many other possible values of π_T, each of which selects the equilibrium in my parameterization. The figures give an idea of the full range of bounded equilibria that can emerge in this model, all for fixed interest rate policy.

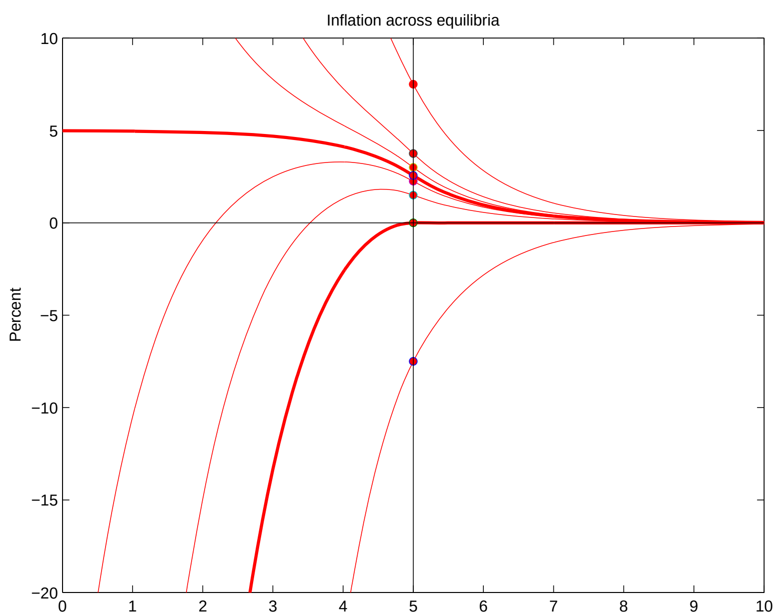

Figure 6: Inflation in all equilibria. Equilibria are indexed by the expected value of inflation π_T at $T = 5$, shown by the circles. The thicker lines show the standard deflation equilibrium and the local-to-frictionless equilibrum. Thinner lines show a range of equilibria indexed by different choices for π_T.

Figure 8 presents the local-to-frictionless equilibrium and the standard equilibrium in state space. The blue dashed line is the standard equilibrium choice, as in Werning (2012) Figure 1. Inflation and output gap approach from the bottom left, the region of deflation and depression. Dots indicate years. At $t = T$ the standard solution attains the central red dot and stays there. The solid lines display the two-way bounded or local-to-frictionless equilibrium choice. The red part is $t \geq T$. Once we eliminate the explosive solution or $t > T$, there is a whole range of non-explosive solutions that converge to the origin along the red ray. Equilibrium choice comes

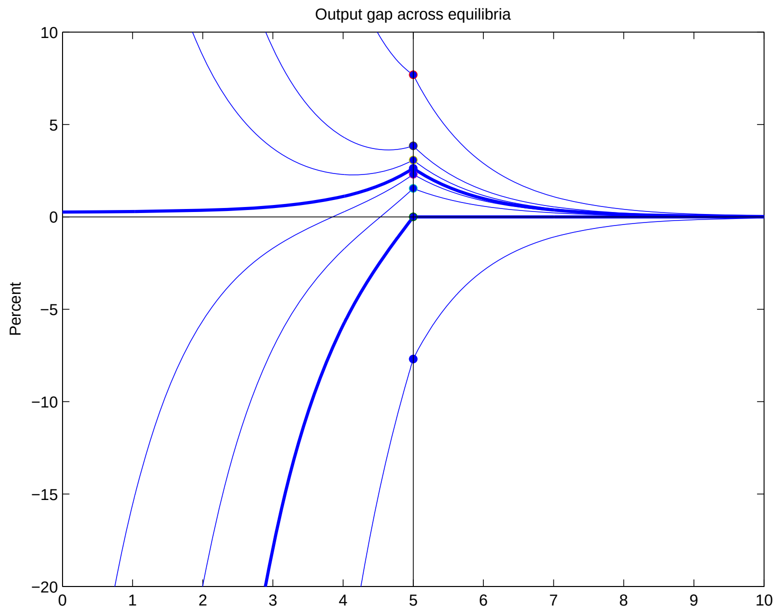

Figure 7: Output gaps in all equilibria. Alterniatve equilibria are indexed by the value of the output gap at $T = 5$, shown by the circles. The thicker lines show the standard deflation equilibrium and the local-to-frictionless equilibrum. Thinner lines show a range of equilibria..

down to where we specify that the $t < T$ solution will join this path. The standard choice picks the origin itself. The local-to-frictionless or two-way-bounded choice merges at a point to the northeast of the origin at $t = T$, at just the right place so that the blue line is non explosive. Going forward in time, this solution starts at the right end of the blue line and works left, hitting the output gap peak at $t = T$ and then converging back to the steady state at the origin.

4 Magical multipliers and paradoxical policies

To analyze the fiscal multipliers and paradoxical policies, I add a shifter variable to the Phillips curve

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa\left(x_t + g_t\right). \tag{11}$$

The variable g_t can represent totally useless (since it does not affect utility) government spending – hiring people to dig ditches and fill them up or construct defenses against imaginary alien invasions, to use two classic examples. It also can represent destruction of capital or technological regress – throwing away ATM machines to employ bank tellers, idling bulldozers to employ people with shovels, or even spoons, breaking glass or welcoming hurricanes, to use classic examples. These policy steps matter in this model because they increase inflation π_t for a given output gap, and thus reduce the real interest rate and consumption growth. In the new-Keynesian model where we always revert to an unchanged potential, again, reducing growth is god and increases the current output level.

Again, I imagine an economy in a liquidity trap, with $ir_t = ir$ for $t < T$ and $ir_t = 0$ thereafter. Now I assume that the government sets $g_t = g$ for $t < T$ and $g_t = 0$ thereafter, and

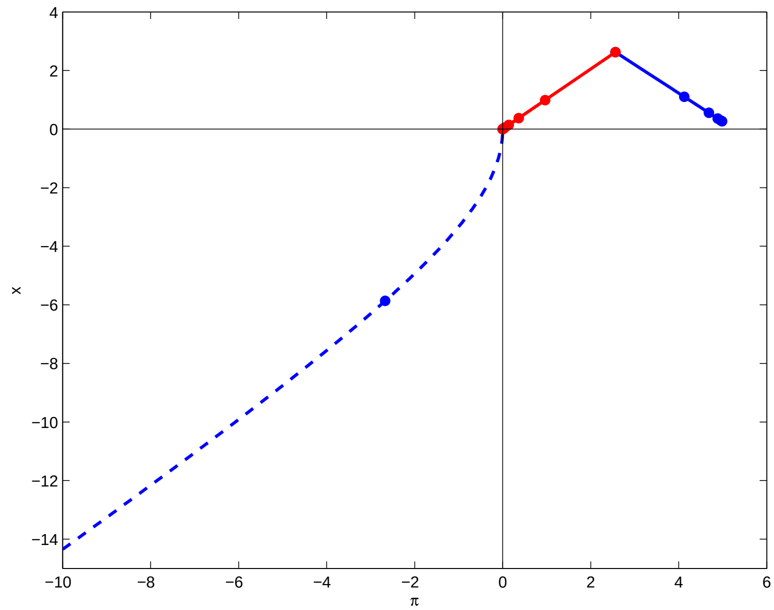

Figure 8: The two-way bounded and standard solutions compared in state space. The dashed blue line gives the standard solution for $t < T$. For $t > T$, the standard solution stops at the central dot. The solid lines give the local to frictionless or two-way bounded equilibrium. Dots indicate each years. The blue line is for $t < T$ and the red line is for $t > T$. The solution starts on the right and proceeds to the left

I examine how increasing g affects equilibrium output and employment.

In the steady state, (11) shows that a unit increase in g means a unit decrease in x_t, and the same inflation as before. The IS curve (1) is unaffected. However, the dynamics can paint a different picture.

The general solution is now

$$\begin{aligned}\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} &= \begin{bmatrix}\rho i r_t - \kappa g_t \\ i r_t\end{bmatrix} - \frac{1}{\lambda^p - \lambda^m}\left(\begin{bmatrix}\lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m\end{bmatrix} i r_t - \begin{bmatrix}\lambda^p & -\lambda^m \\ 1 & -1\end{bmatrix}\kappa g_t\right)\begin{bmatrix}e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)}\end{bmatrix} \\ &\quad + \begin{bmatrix}\lambda^p \\ 1\end{bmatrix}\pi_T e^{\lambda^m(t-T)}.\end{aligned}$$

(The algebra is in the Appendix.) The standard equilibrium choice with $\pi_T = 0$ is

$$\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} = \begin{bmatrix}\rho i r_t - \kappa g_t \\ i r_t\end{bmatrix} - \frac{1}{\lambda^p - \lambda^m}\left(\begin{bmatrix}\lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m\end{bmatrix} i r_t - \begin{bmatrix}\lambda^p & -\lambda^m \\ 1 & -1\end{bmatrix}\kappa g_t\right)\begin{bmatrix}e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)}\end{bmatrix}. \tag{12}$$

The local-to-frictionless or two-way-bounded solution, which rules out backwards explosions in $t < T$, setting to zero the loading on λ^m, picks

$$\pi_T = \frac{\lambda^p i r - \kappa g}{\lambda^p - \lambda^m} \tag{13}$$

and is therefore[1], for $t < T$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} + \frac{1}{\lambda^p - \lambda^m}\left(\begin{bmatrix} \lambda^{m2} \\ \lambda^m \end{bmatrix} ir - \begin{bmatrix} \lambda^m \\ 1 \end{bmatrix} \kappa g \right) e^{\lambda^p (t-T)} \tag{14}$$

and for $t > T$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{\lambda^p ir - \kappa g}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m (t-T)}.$$

Now we can evaluate the multiplier. In the standard solution (12),

$$\begin{aligned} t &< T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \frac{1}{\lambda^p - \lambda^m}\left(\begin{bmatrix} \lambda^p & -\lambda^m \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} e^{\lambda^m (t-T)} \\ e^{\lambda^p (t-T)} \end{bmatrix} \\ t &> T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

while in the local-to-frictionless solution, (14)

$$\begin{aligned} t &< T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^m \\ 1 \end{bmatrix} e^{\lambda^p (t-T)} \\ t &> T : \frac{\partial}{\partial g} \begin{bmatrix} x_t \\ \frac{1}{\kappa}\pi_t \end{bmatrix} = \frac{-1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m (t-T)} \end{aligned}$$

Recall that the root $\lambda^m < 0$ leads to $e^{\lambda^m (t-T)}$ that explodes as we go backward in time. Hence, the standard-solution multiplier depends on the length of time the liquidity trap and g policy are expected to last, and can be arbitrarily large. Since the eigenvalue λ^m increases in absolute value as price stickiness declines, $\kappa \to \infty$, the multiplier grows larger as frictions disappear and the multiplier approaches infinity in the frictionless limit! These are standard magical results.

By contrast, the root $\lambda^p > 0$ is the only one that appears in (14), and $e^{\lambda^p (t-T)}$ converges to zero as we go backwards in time. Hence, the local-to frictionless multiplier is at its largest in absolute value just before the liquidity trap ends, and is lower the earlier it is applied. Furthermore, the local-to-frictionless multiplier is negative throughout. And the multiplier approaches -1 smoothly and sensibly as we approach the frictionless limit.

Figure 9 presents these multipliers. The thick red dashed line presents the multipliers in the standard equilibrium choice. The exercise is an increase in g that lasts from the indicated time t, until $t = T$ when the liquidity trap ends. The graph shows very large multipliers, and how multipliers increase drastically as the time of the liquidity trap and g policy increases, moving to the left.

The thin dashed red lines present multipliers from the standard equilibrium choice, for steadily less price stickiness. They show how multipliers paradoxically *increase* as price stickiness is *reduced*. In the limit that price stickiness goes to zero, the multiplier goes to infinity.

[1]

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} - \frac{1}{\lambda^p - \lambda^m}\left(\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} ir - \begin{bmatrix} \lambda^p & -\lambda^m \\ 1 & -1 \end{bmatrix} \kappa g \right) \begin{bmatrix} e^{\lambda^m (t-T)} \\ e^{\lambda^p (t-T)} \end{bmatrix} + \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \frac{\lambda^p ir - \kappa g}{\lambda^p - \lambda^m} e^{\lambda^m (t-T)} \\ \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} - \frac{1}{\lambda^p - \lambda^m}\left(\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} ir - \begin{bmatrix} \lambda^p & -\lambda^m \\ 1 & -1 \end{bmatrix} \kappa g - (\lambda^p ir - \kappa g) \begin{bmatrix} \lambda^p & 0 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} e^{\lambda^m (t-T)} \\ e^{\lambda^p (t-T)} \end{bmatrix} \end{aligned}$$

and simplify

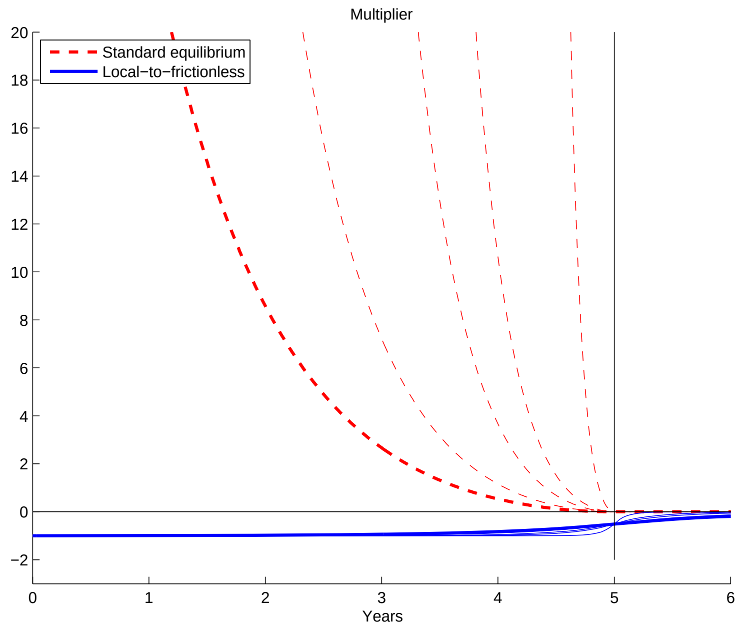

Figure 9: Multpliers. I modify the Phillips curve to $d\pi_t/dt = \rho\pi_t - \kappa(x_t + g_t)$. The graph plots the multiplier dx_t/dg for an increase in g that lasts from the indicated time until the end of the liquidity trap at $T = 5$. The dashed lines present the standard solution, the solid lines present the local-to-frictionless solution. The thin lines show what happens as price stickiness is steadily reduced. Parameters are $\kappa = 1, \rho = 0.05, r = -5\%$ and the thin lines present $\kappa = 2, 5, 10, 100$.

The thick solid blue line is the multiplier in the local-to-frictionless solution. The multiplier is negative throughout and it asymptotes smoothly to the frictionless limit -1, and also as price stickiness is reduced.

(I present private output = private consumption multipliers $\partial x_t/\partial g_t$. If g represents government spending, conventional multipliers would add g itself and all be larger by one. I present private-outut multipliers since g can represent other Phillips curve shocks, and because when g is completely wasted spending – the relevant case – the private-output multiplier better captures the economic issue. Additionally, if the equations represent log-linearized deviations, $y = x + g$ doesn't hold.)

In sum, magical multipliers and paradoxical policies are direct results of *equilibrium choice*, holding interest rates constant. The local-to-frictionless equilibrium produces a perfectly normal sign and magnitude of multipliers.

4.1 Forward guidance and commitments

Many authors have advocated "forward guidance" and "commitment," "open-mouth" policies, or promises to temporarily raise future inflation targets, as polices to ameliorate a liquidity trap. If the Fed could only commit itself to keep rates low for some time after the negative natural rate passes, we would get a little bit of inflation at that time, and the discounted effect of lower future real rates would stimulate output today. Werning's (2012) optimal policy is of this sort.

To address the power of this sort of policy, I now assume that people can be induced to expect that the interest rate will remain at zero $i_t = 0$ for a time interval τ after T, when the natural rate rises to a new constant and positive value r_τ. In the previous simulations, people expected the Fed to raise rates to $i_t = r_\tau$ immediately at $t = T$, so $\tau = 0$.

I solve the model as before. The same general solution (5) holds, and I pick non-explosive solutions at $t > T + \tau$, eliminating one eigenvalue in that region. To work backwards, I now paste twice, once at $t = T + \tau$ when interest rates rise from $i_t = 0$ to $i_t = r_\tau$ and again at $t = T$ when the natural rate switches from $-r$ to $+r_\tau$. The standard solution chooses the equilibrium with $\pi_{T+\tau} = 0$. The local-to-frictionless solution chooses the equilibrium that is bounded as time goes backwards, though this allows some inflation to continue past $\pi_{T+\tau}$.

Figure 10 presents the predictions of the standard equilibrium-selection choice for a variety of time intervals for the continuation of zero nominal rates τ. In the top left, I present the previous solution with $\tau = 0$, which reminds us of the deep recession and deflation baseline. The top right panel supposes that people expect the interest-rate rise to be delayed. This delay allows a little inflation and output gap to emerge between $t = T$ and $t = T + \tau$. Multiplying small changes in terminal conditions by backward-explosive eigenvalues has large effects. The vertical difference between the top left and top right equilibria at $t = 2$, for example, is quite large, and larger still at $t = 1$. So, the further in the future the same promise is, the more its effect today. Moving to the bottom left and bottom right, allowing one year and then two years of delayed interest rate rises has a dramatic effect. Now the deflation and depression during the liquidity trap is turned into an inflation with a boom! Not shown, as pricing frictions decrease, and the backward-explosive eigenvalue increases, all of these effects become *larger.*

The completely frictionless solution (not shown) requires $x_t = 0$ at all times. Thus, we will see inflation $\pi_t = r$ for $t < T$, a period of *de*flation $\pi_t = -r_\tau$ during the period of zero nominal rate and positive natural rate $r_\tau > 0$, and then no inflation when $i_t = r_\tau$ for $t > T + \tau$.

Figure 11 presents the local-to-frictionless equilibrium choice in the same situation. As we might have expected, The "local-to-frictionless" equilibrium shows very little current ($t < T$) effect of the open-mouth policy. As the length of the negative real interest rate period between T and $T + \tau$ and stretches out, the equilibrium begins to approach the frictionless prediction of deflation during this period. Since the model is stable backward, promises about the further-off future have smaller and smaller effects, the opposite of the standard solution.

The main point remains: equilibrium choice is vitally important to analyzing predictions of this model, and all the usual dramatic properties disappear in the local-to-frictionless solution which is bounded backward.

In equations, the standard solutions shown in Figure 10 are, for $t < T$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r - \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \left\{ \begin{bmatrix} e^{\lambda^m (t-T)} \\ e^{\lambda^p (t-T)} \end{bmatrix} r + \begin{bmatrix} e^{\lambda^m (t-T)} \left[1 - e^{-\lambda^m \tau} \right] \\ e^{\lambda^p (t-T)} \left[1 - e^{-\lambda^p \tau} \right] \end{bmatrix} r_\tau \right\}.$$

We recognize the solution before, plus the r_τ term. The r_τ term thus capture the effects of the interest-rate-rise postponement policy. The $e^{\lambda^m (t-T)}$ terms explode backward so are the most important. Thus, the $\left[e^{\lambda^m (t-T)} \right] r$ multiplied by λ^p and λ^{p2} are the terms that drive the negative explosion going back in time in the standard solution. Now, $e^{-\lambda^m \tau} > 1$, so the top right term is negative, offsetting the $\left[e^{\lambda^m (t-T)} \right] r$ term. For larger enough τ, the total effect multiplying the $e^{-\lambda^m (t-T)}$ root becomes positive, which gives the inflationary solutions in a deflationary trap show in Figure 10 for $\tau = 1$ and $\tau = 2$.

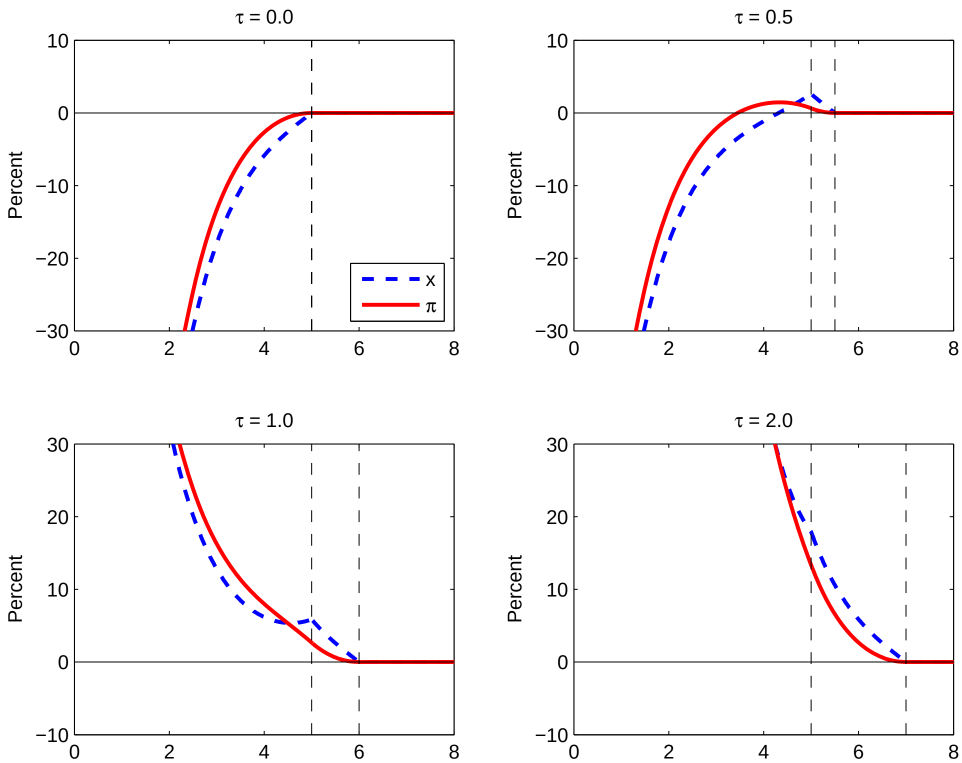

Figure 10: Equilibrium values of output gap x and inflation π in the standard equilibrium choice of the new-Keynesian model. At $t = T = 5$, marked by the left dashed line, the natural rate changes from -5% to +5%. At $t = T + \tau$ people expect the nominal interest rate to rise from $i = 0$ to $i = 5\%$, for τ as indicated. The standard selecton criterion is $\pi_{T+\tau} = 0$.

Between T and $T + \tau$, the standard solution is

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m (t-(T+\tau))} - 1 \\ e^{\lambda^p (t-(T+\tau))} - 1 \end{bmatrix} r_\tau$$

Both terms are positive, so inflation and output gap are positive during this period. Beyond $t > T + \tau$, both inflation and output gap are zero.

The local-to-frictionless solution is, for $t < T$

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{m2} \\ \lambda^m \end{bmatrix} \left[r + \left(1 - e^{-\lambda^p \tau}\right) r_\tau \right] e^{\lambda^p (t-T)}$$

We see the backwards-stable root $e^{\lambda^p (t-T)}$ is the only one left, and the system approaches the steady state going back in time. For $t > T + \tau$ this solution leaves some slowly decaying inflation and output gap, now controlled by the forward-stable root $e^{\lambda m (t-T)}$,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} \\ \lambda^p \end{bmatrix} \left[r + r_\tau \left(1 - e^{-\lambda^m \tau}\right) \right] e^{\lambda^m (t-T)}.$$

The behavior between T and $T + \tau$ is not very interesting (well, I couldn't make it pretty), but

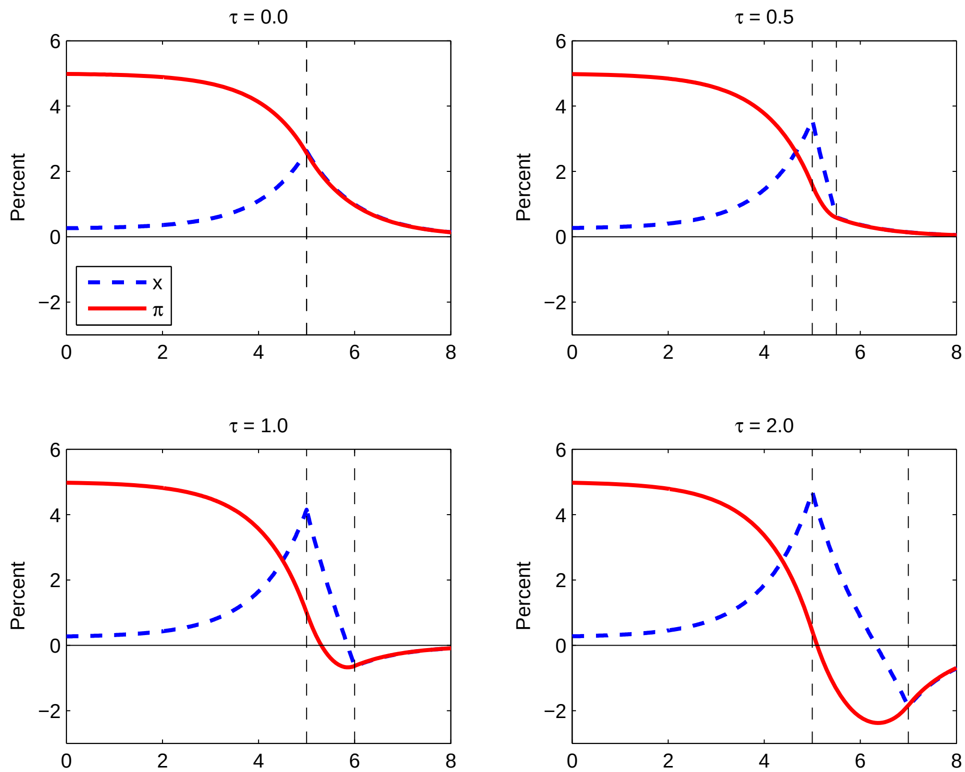

Figure 11: Equilibrium values of output gap x and inflation π in the local-to-frictionless equilibrium choice of the new-Keynesian model. At $t = 5$, marked by the left dashed line, the natural rate changes from -5% to +5%. At $t = T + \tau$ the Fed raises the nominal interest rate from $i = 0$ to $i = 5\%$, for τ as indicated.

for completeness it is

$$\begin{aligned}\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m (t-(T+\tau))} - 1 \\ e^{\lambda^p (t-(T+\tau))} - 1 \end{bmatrix} r_\tau \\ &+ \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} \\ \lambda^p \end{bmatrix} \left[r + r_\tau \left(1 - e^{-\lambda^m \tau} \right) \right] e^{\lambda^m (t-T)}. \end{aligned}$$

5 A Taylor rule

How do we choose which is the right equilibrium? In the standard new-Keynesian approach, the Fed chooses the equilibrium – or, more precisely, people's expectations of Fed equilibrium-selection policies choose the equilibrium. The Fed chooses the interest rate path $\{i_t^*\}$. It then conducts an equilibrium-selection policy to choose which of the many possible equilibria $\{\pi_t\}$ (and $\{x_t\}$) consistent with that $\{i_t^*\}$ will emerge as the equilibrium $\{\pi_t^*\}$.

Regarding equilibrium selection this way, then, the disastrous predictions for the economic consequences of a liquidity trap, and the prediction that reverse-sign policies can counteract those consequences, rest centrally on expectations about the Fed's "equilibrium selection" policies, at least as much as they do on interest rate policies – where it sets $\{i_t^*\}$ – which are the conventional focus.

But just how does the Fed execute "equilibrium selection policy?" In the new-Keynesian tradition, the Fed selects the equilibrium π_t^* by following for $t > T$ a Taylor-rule inspired policy of the form

$$i_t = i_t^* + \phi_\pi(\pi_t - \pi_t^*). \tag{15}$$

This policy de-stabilizes the economy. With $\|\phi_\pi\| > 1$ in this model, all the equilibria other than $\{\pi_t^*\}$ explode as $t \to \infty$. The new-Keynesian tradition adopts as an equilibrium-selection principle that the economy will not choose non-locally bounded equilibria, and thus predicts that π_t^* is the observed equilibrium. As Woodford (2004, p.128) explains, "The equilibrium ..[π^*].. is nonetheless *locally* unique, which may be enough to allow expectations to coordinate upon that equilibrium rather than on one of the others."

(The form (15), due to King (2000), may seem unusual. The interest rate rule is often parameterized as a simple Taylor rule

$$i_t = \bar{\imath}_t + \phi_\pi \pi_t, \tag{16}$$

where $\bar{\imath}_t$ is a potentially time-varying intercept or policy shock, in which the Fed responds directly to other events or shocks in the economy. Woodford (2004) advocates such an intercept as "Wicksellian policy" and shows its desirable properties.

This formulation is equivalent to (15). In equilibrium, (16) is

$$i_t^* = \bar{\imath}_t + \phi_\pi \pi_t^*.$$

Then we can rewrite (16) in the form (15)

$$i_t = (i_t^* - \phi_\pi \pi_t^*) + \phi_\pi \pi_t = i_t^* + \phi_\pi (\pi_t - \pi_t^*).$$

Thus, saying that expectations about equilibrium selection in (15) drive model predictions and policy implications is the same as saying that expectations about the intercept in the standard Taylor rule (16) drive model predictions and policy implications.

The Taylor rule policy often includes output responses $\phi_y(y_t - y_t^*)$, lagged responses ρi_{t-1} and other complexities. That generality is inessential to the current point.

There are many actions the Fed could take to enforce its inflation target π_t^* ; Atkeson, Chari,Varadarajan and Kehoe (2010), but Taylor-inspired rules dominate applied analysis.

I call (15) "inspired by" because Taylor introduced the rule in fundamentally different old-Keynesian models, where it serves an opposite purpose, stabilizing rather than destabilizing model dynamics, providing stability rather than determinacy.)

To examine this idea, let us ask: suppose that after the liquidity trap period ends at T, the natural rate rises to $r > 0$, the Fed wishes to produce $i_t^* = r$ *and* choose the equilibrium $\pi_t^* = 0$, and it does so with a Taylor rule of this form for $t > T$, i.e. the Fed follows

$$i_t = r + \phi_\pi \pi_t. \tag{17}$$

This is also an important calculation if you're dubious of all this equilibrium-selection business. Many analyses of liquidity traps pose the problem this way: People's expectations of Fed behavior after the liquidity trap ends are that it will revert to a Taylor rule of the form (17). If you objected to my previous calculations that described post-trap policy by separate i_t^* and π_t^* choices, this is the calculation you want.

There are two flies in the ointment. First, we can't use rules of the form (15) or (17) in continuous-time models, as explained in the Appendix. Therefore, I follow Sims (2004) and Fernández-Villaverde, Posch, and Rubio-Ramírez (2012) by writing the continuous-time Taylor rule in partial-adjustment form, which is also how it is typically estimated,

$$\frac{d\left(i_t - i_t^*\right)}{dt} = \theta\left[\phi\left(\pi_t - \pi_t^*\right) - \left(i_t - i_t^*\right)\right] \tag{18}$$

i.e.,

$$i_t - i_t^* = \phi\theta \int_{s=0}^{\infty} e^{-\theta s}(\pi_{t-s} - \pi_{t-s}^*)ds.$$

The parameter ϕ is the usual Taylor parameter, measuring the eventual response of interest rates to inflation. The parameter θ controls speed of adjustment. I use $\theta = 1$ or a half-life of one year.

Second, in many equilibria a rule of the form (18) soon passes the lower bound $i_t = 0$. To make a more reasonable specification of what the Fed will do, I therefore modify (18) to respect the zero bound. Also specializing to $\pi_t^* = 0$ and $i_t^* = r > 0$, and allowing the interest rate to rise above the zero bound, we then have

$$\frac{di_t}{dt} = \left\{ \begin{array}{cc} \theta\left[\phi\pi_t - (i_t - r)\right] & \text{if } i_t > 0 \\ \max\left\{\theta\left[\phi\pi_t - (i_t - r)\right], 0\right\} & \text{if } i_t = 0 \end{array} \right\}. \tag{19}$$

This modification just cuts off solutions which otherwise spiral to $i_t \to -\infty$. Since my point is that expectation of such policy is not credible, I don't want to stack the deck by supposing impossible interest rates. Most analyses of liquidity traps with Taylor rules likewise specify that the Taylor principle ceases to hold at the lower bound.

Since $i_t = 0$ for $t < T$, this modification makes no difference to the solutions previously presented in the $t < T$ region. The difference is, they are now pasted to different expectations about what will happen for $t > T$. The Appendix details the calculations.

Figure 12 presents inflation. Compare it to figure 6 – the figures are the same for $t < T$, but now we see the effect of the Taylor-rule equilibrium selection efforts for $t > T$. Figure 13 graphs the output gap x_t, and compare it to Figure 7. Figure 14 presents the interest rate. I assume a $r = 5\%$ natural rate of interest in the $t > T$ period. The corresponding value in the previous exercise was simply $i_t = 5\%$ for $t > T$.

The comparison between the sets of figures verify that the Taylor rule is doing what it's supposed to do: $\phi > 1$ induces explosive dynamics in all variables except for the unique locally bounded equilibrium $x_t^* = 0, \pi_t^* = 0$, $t > T$, where before there were multiple locally-bounded equilibria for $t > T$. If people's expectations are coordinated on locally-bounded equilibria, this Taylor rule will coordinate expectations on the $x_t^* = 0$, $\pi_t^* = 0$ equilibrium.

The inflation and interest paths are interesting – they overshoot. For example, to rule out the local-to-frictionless equilibrium, the Fed starts raising the interest rate i_t to combat the unwanted inflation at $t = T$. This interest rate rise makes output grow more quickly, and the higher output level pushes down inflation more quickly through the Phillips curve $d\pi/dt = \rho\pi_t - \kappa x_t$ than it otherwise would do.

But when inflation hits zero it doesn't stop. This equilibrium path overshoots and send inflation off to negative territory, while shooting output off to positive infinity. This Fed ends up sending the economy into a deflationary boom. The inflation and output paths looks much the same without the $i \geq 0$ restriction.

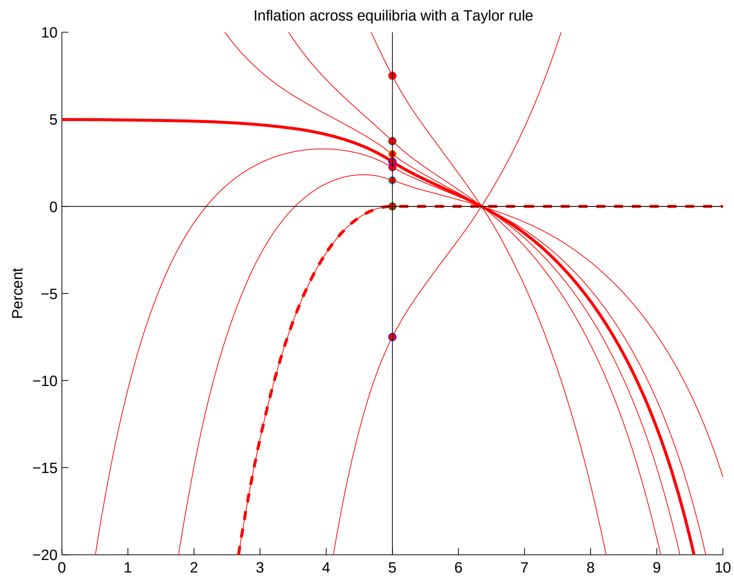

Figure 12: Path of inflation in all equilibria with a Taylor rule. At $t = T = 5$ people expect the Fed to follow a Taylor rule to enforce the $\pi^* = x^* = 0$ equilibrium. The Taylor parameter is $\phi = 1.1$ and the half-life of adjustment is $\theta = 1$. The thick dashed line is the conventional equilibrium. The upper solid line is the local-to-frictionless equilibrium.

Now, let us ponder the lessons of these figures. All of the equilibria to the left of $t < T$ remain unchanged. Our question is, how do new-Keynesian modelers decide that, given the interest rate path, the economy will follow the disastrous standard equilibrium rather than, say, the benign local-to-frictionless equilibrium?

The answer is given by the figures. If the economy were to follow the local-to-frictionless equilibrium, people believe that after $t > T$, the Fed would not tolerate the "glide path" to zero inflation shown in Figure 2, and would instead start aggressively raising interest rates above the natural rate. But the problem is not that people expect the Fed to push the glide path more quickly to zero. If that were the expectation, then this equilibrium would remain bounded, and would remain a valid equilibrium by new-Keynesian rules. The problem is that people expect the Fed to over-react so drastically that the economy takes off explosively to a spiraling deflation.

We have to ask ourselves, is this at all a reasonable specification of how people think the Fed would react to inflation above its target? Would the Fed, seeing the end of the liquidity trap approaching and inflation declining, really not allow a glide path back to its target? Even if people expect the Fed to over-react, do people really think the Fed would over-react so strongly as to send the economy to a "non-locally-bounded" *deflationary boom* equilibrium? It would be an interesting exercise for survey researchers to find out how many people's expectations of what the Fed would do if its inflation target were violated correspond to these figures.

In the new-Keynesian model, people are thought to believe that Fed policy deliberately destabilizes the economy. The policy $\phi_\pi > 0$ takes an eigenvalue less than one, which leaves the

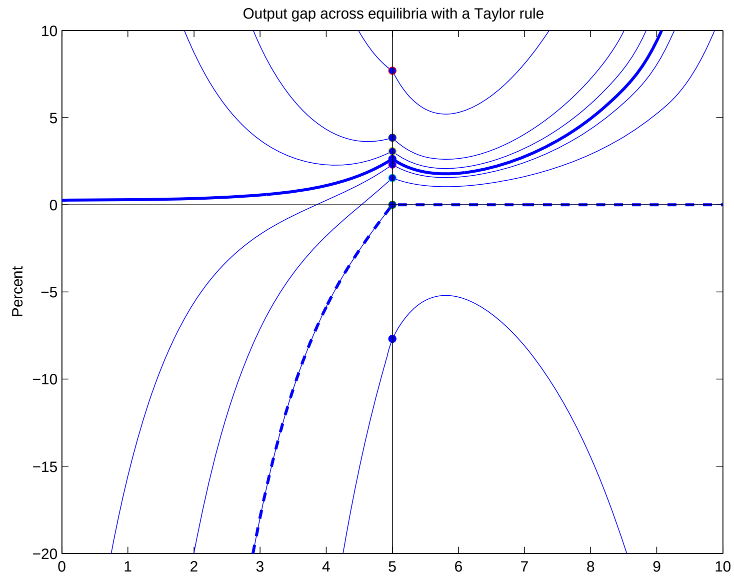

Figure 13: Path of output gap in all equilibria with a Taylor rule for $t > T$. The top solid line gives the local-to-frictionless equilibrium for $t < T$, and the thick dashed line gives the standard equilibrium.

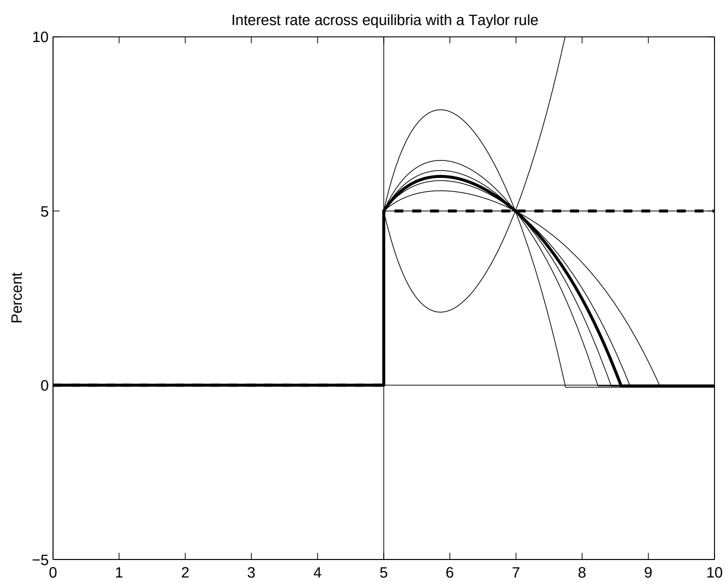

Figure 14: Path of interest rates in all equilibria with a Taylor rule. The Taylor parameter is $\phi = 1.1$ and the half-life of adjustment is $\theta = 1$. The solid lines give the local-to-frictionless equilibrium and the thick dashed line gives the standard equilibrium. I assume the natural rate is 5% for $t > T$.

stable dynamics of Figure 6 and turns it in to an eigenvalue greater than one, generating through policy the explosive dynamics of Figure 12. If the Fed were just to leave interest rates alone $i_t = r$ then the local-to-frictionless equilibrium would be perfectly acceptable and the glide path would emerge. So, the key question is, why would people's expectations of the future be built on the idea that the Fed operates interest rate policy in such a way as to deliberately destabilize the economy? Especially when every statement every Fed official has ever made emphasizes stabilizing, rather than de-stabilizing dynamics?

The point does not violate Woodford's (2004, p. 128) admonishment on how to read new-Keynesian dynamics

> ... the equation [explosive inflation dynamics] indicates how the equilibrium inflation rate in period t is determined by *expectations* regarding inflation in the following period... The equilibria that involve initial inflation rates near (but not equal to) Π^* can only occur as a result of expectations of *future* inflation rates (at least in some states) that are even *further* from the target inflation rate. Thus the economy can only move to one of these alternative paths if expectations about the future change significantly, something that one may suppose would not easily occur.

I am explicitly reading from expectations of the future to equilibrium choice today. In new-Keynesian thinking, the benign local-to-frictionless equilibrium is ruled out, because it requires an expectation that $\pi_T > 0$ at the end of the trap as shown in Figure 2. In turn, $\pi_T > 0$ at the end of the trap is ruled out because the only way for $\pi_T > 0$ to be an equilibrium is if people expect the crazy dynamics of Figure 12 to be their future. The analysis is just as Woodford recommends. But are these really people's expectations?

5.1 Which equilibrium will the Fed choose?

In sum, the new-Keynesian prediction that we will see the disastrous standard equilibrium emerge posits that people expect the Fed to follow a deliberate equilibrium-selection policy to select $\pi_t^* = 0$ $t > T$ instead of the equally possible local-to-frictionless glide path, and the nature of that policy is to induce explosive dynamics into an otherwise stable economy. My point so far is to seriously question whether this set of "equilibrium-selection" expectations make any sense at all.

But this analysis does not answer the central question. If one accepts that this or some other equilibrium-selection policy is a reasonable description of Fed behavior and people's expectations of Fed behavior, why would people expect the Fed to focus its equilibrium-selection policies on the disastrous standard path shown in Figure 3, rather than the benign local-to-frictionless glide path of Figure 2? We could plug the local-to-frictionless inflation path π_t^* $t > T$ in (15) or write the algebraically equivalent (17) as

$$i_t = r - (\phi_\pi \pi_t^*) + \phi_\pi \pi_t. \tag{20}$$

In this interpretation, the first term reflects an expectation of a time-varying intercept or policy error, or more simply an expectation that the Fed will allow a glide path back to its eventual $\pi_t^* = 0$ target while it ramps back in to a Taylor principle.

Figures 15 and 16 make this point graphically. In these figures, I imagine that the Fed wishes to select the local-to-frictionless equilibrium, and centers its equilibrium-selection policy there.

Now, those equilibria are the only bounded ones and all the others, including the standard equilibrium, explode. (The interest rate graph is so visually similar to figure 14 that it is not worth repeating.) If you think the Fed selects equilibria by Taylor-inspired rules, why not this one?

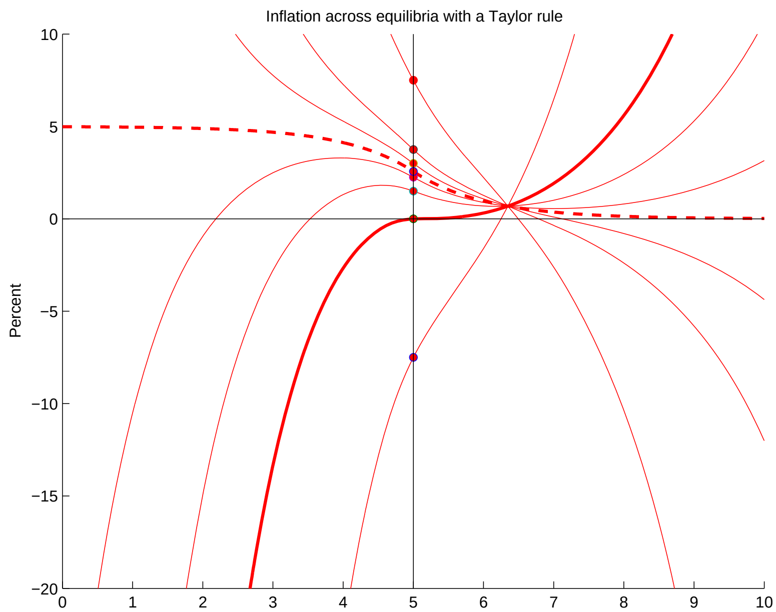

Figure 15: Path of inflation in all equilibria with a Taylor rule, when the Fed selects the local-to-frictionless equilibrium. The thick dashed line is the local-to-frictionless equilibrium. The solid line is the standard equilibrium.

Werning's (2012) analysis, the expectation that the Fed will insist on $\pi_t^* = 0 \;\; t > T$ follows a logical and reasonable principle: People believe that the Fed cannot precommit, so it will do ex-post what looks best going forward no matter what last year's "forward guidance" was. Werning finds the optimal *discretionary* policy. His Fed maximizes a standard loss function

$$L = \frac{1}{2}\int_0^\infty e^{-\rho t}\left(x_t^2 + \lambda \pi_t^2\right) dt, \tag{21}$$

choosing i_t and π_t at each date in a completely forward-looking manner, subject to the model dynamics (1)-(2). Since there are multiple equilibria, Werning's monetary policy consists of *both* the specification of i_t and of π_t, with an unstated equilibrium-selection device for the latter. (This is clear in Werning's footnote 7, p. 9.) At time T, after the liquidity trap is over, such a Fed can achieve $x_t = 0, \pi_t = 0 \; \forall \; t \geq T$ immediately, and it will do so.

However, there is a curious paradox (or perhaps a form of subgame imperfection) in this specification. All equilibria except the selected one are disastrous from the Fed's objective. Thus, to select an equilibrium, the Fed must completely precommit to follow an equilibrium-selection strategy which, ex-post, is disastrous for its objectives. For example, if the local-to-frictionless equilibrium emerged nonetheless, all the Fed has to is to give up the Taylor-rule threat, revert to $i_t = r$ and the economy will settle down to $\pi_t = 0$, $x_t = 0$ soon enough. But no, it has precommitted to a Taylor rule, and now it must follow that rule to explosive deflation. Why would people think the Fed is incapable of precommitment to its interest rate and inflation

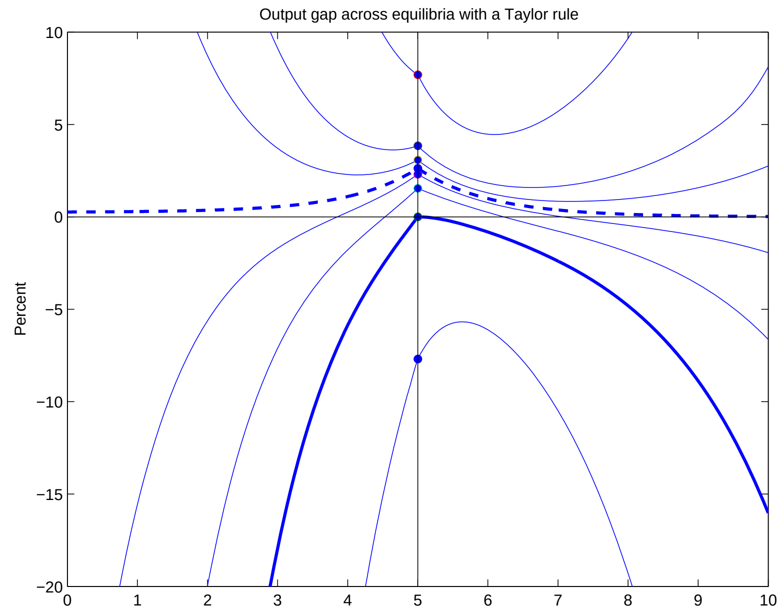

Figure 16: Path of output gap in all equilibria with a Taylor rule for $t > T$, when the Fed selects the local-to-frictionless equilibrium. The thick dashed line gives the local-to-frictionless equilibrium, and the thick solid line gives the standard equilibrium.

targets, but the victim of such monstrous precommitment in its equilibrium-selection policy? (Werning only analyzes discretion and commitment in the choice of interest rate target and inflation target. While recognizing that an equilibrium-selection policy is necessary, he does not specify a specific equilibrium-selection policy, and thus does not analyze whether it requires commitment.)

Equilibrium-selection discretion is also particularly unfortunate in light of Werning's objective. Given the interest rate policy, the local-to-frictionless solution maximizes the Fed's objective in the limit of a long liquidity trap

$$L = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-\rho t}\left(x_t^2 + \lambda \pi_t^2\right) dt.$$

In fact, the-local-to-frictionless solution is the only one that gives a finite value to this objective, as all the others spiral off to infinity as $t \to -\infty$.

5.2 Inflation targets

One might view the equations as validating calls that the Fed adopt and commit to an explicit inflation target *separately* from its interest rate policy announcements. After all, the local-to-frictionless solution differs only from the disastrous standard solution in that the Fed changes its inflation target, changing π^* in $i_t = i_t^* + \phi(\pi_t - \pi_t^*)$ while leaving i^* alone. Equation (20) looks especially like a Taylor rule with a temporarily higher inflation target.

This is true, but the whole enterprise hinges on the question, what does the Fed do if the world doesn't follow its announcements of what inflation it would like to see? What is the stick corresponding to the "speak loudly" advice?

In this case the answer is the Taylor-inspired rule, in its off-equilibrium threat form. People are assumed to expect that if the wrong equilibrium emerges, the Fed will drive the economy off a cliff. I don't think that people who advocate a higher inflation target as a means to stimulus, such as Blanchard, Dell'Ariccia, and Mauro (2010), have this in mind. I think they have in mind a rather old-Keynesian view that the central bank will keep *equilibrium* interest rates low for a while as inflation picks up, mixing new and old Keynesian intuition in a way that cannot be captured in these models, or perhaps as captured by my exercises that postpone the interest rate rise. They do not mean a different equilibrium-selection policy for the *same* interest rate path, and they do not mean "and if the desired inflation does not occur, the Fed should precommit to drive the economy off a cliff."

6 Equilibrium choice

This discussion should leave us hungry for better principles on which to select equilibria. In a liquidity trap, either the benign local-to-frictionless or the disastrous standard equilibrium will emerge. Which is right? Is there no better way to think about this question than to imagine the Fed introduces explosive dynamics to select equilibria?

In models with multiple equilibria – bounded or not – a wide range of essentially philosophical principles have been advocated to select equilibria. Examples include the "minimum state variable" criterion, "E-stability" and "learnability" (Evans and Hohkapohja (2001), McCallum (2003), (2009), and in this context Christiano and Eichenbaum (2012)). The new-Keynesian procedure is in essence a philosophical criterion as well, resting primarily on the principle that expectations should "coordinate" on "locally bounded" equilibria, as explained in the above Woodford (2004) quote and King (2000, p. 58-59). Efforts to turn locally bounded into an economic criterion – to establish global uniqueness by economic rather than philosophical principles – have, as far as I was able to ascertain in an extensive survey (Cochrane 2011), simply failed, and that includes the standard citation to Obsfteld and Rogoff (1983).

One can make a similar philosophical case that we should choose the local-to-frictionless equilibrium. It has a sensible frictionless limit, and it does not produce magical multipliers and policy paradoxes. It is locally bounded in *both* directions. It is a minimal perturbation of the underlying real equilibrium to account for price stickiness – it minimizes the sum of squared deviations from the frictionless equilibrium. Any of these could be turned in to pleasant axioms for its selection.

However, I won't try to shoehorn this analysis into one of these categories and make an economic or philosophical case for a particular equilibrium, nor introduce a new philosophical criterion. Doing so would require me to examine a wide range of models, and stochastic ones in particular.

In Cochrane (2011) I advocated that fiscal considerations could help to select equilibria of this kind of model. In the model so far, fiscal policy is passive, meaning that the Treasury is assumed to raise whatever revenue monetary policy requires by lump sum taxation. The huge deflation in the new-Keynesian equilibrium choice would put that assumption to the test. Deflation would have been an unexpected bonanza to the holders of long-term U.S. government debt, requiring huge new taxes to transfer wealth to bondholders, at a time when the fiscal solvency of the US was already in question, and large unexpected income transfers from taxpayers to typical Treasury bondholders not exactly high on the political agenda. The local-to-frictionless equilibrium, by contrast, with a small inflation, has little fiscal impact. However, using fiscal considerations to specify exactly which of the slightly inflationary solutions would emerge would require a lengthy digression into monetary-fiscal coordination, government debt maturity structure, and the actual structure of the underlying model rather than the simple loglinearized form here. For now, it can be said that the need for fiscal support of any monetary policy at least pushes us away from equilibria that involve deep deflations.

7 Concluding comments

I build on a standard new-Keynesian analysis of the zero bond: A negative natural rate will last until time T, and the nominal rate will be stuck at zero. After that, policy returns to normal, meaning that people expect interest rates to rise again and thereafter follow the natural rate. I calculate the new-Keynesian model equilibria in this circumstance. I find there are many locally-bounded (nonexplosive as $t \to \infty$) equilibria, which we can index by the value of expected inflation at the end of the trap π_T. These equilibria all share the *same* interest rate path.

The new-Keynesian literature analyzes one equilibrium: it features a huge recession, deflation, and strong expected output growth, which is why the level of output is so low. It predicts large multipliers to wasted government spending, output destruction, and announcements about far-off future policy. These predictions grow larger the longer the period of the liquidity trap, and as the degree of price stickiness is *reduced*. An ε price stickiness produces paleolithic output levels and nearly infinite multipliers. Though price stickiness is the central friction causing the economy to deviate from efficient output levels, reducing price frictions would lower output further, and increasing frictions would raise output.

Another interesting equilibrium is the "local-to-frictionless" equilibrium. It predicts mild *in*flation during the liquidity trap, no reduction in output relative to potential, small negative multipliers, and little effects of promises of far-off policies. Its predictions smoothly approach the frictionless limit as pricing frictions are reduced.

The equilibrium we will observe during the trap depends on people's expectations of what will happen following the trap. If people think we must have exactly zero inflation (deviation from long run trend) as soon as the trap ends, then we will experience the new-Keynesian recession and its paradoxical policy implications. If people expect that we can retain a mild inflation – about half the negative "natural rate" in the trap – and then a smoothly declining inflation "glide path" as graphed in Figure 2, then we will experience a benign period during the liquidity trap and magical policies will not work.

At a minimum, this analysis shows that equilibrium selection, rather than just interest rate policy, is vitally important for understanding these models' predictions for a liquidity trap and the effectiveness of stimulative policies. In usual interpretations of new-Keynesian model results, authors feel that interest rate policy is central, and equilibrium-selection policy by the Fed, or equilibrium-selection criteria, are details relegated to technical footnotes (as in Werning 2012), game-theoretic foundations, or philosophical debates, which can all safely be ignored in applied research. These results deny that interpretation.

The standard new-Keynesian approach specifies that expectations about Fed behavior select equilibria. In that analysis, the Fed has two important policy tools – interest rate policy, which specifies the equilibrium interest rate path $\{i_t^*\}$ (which may covary systematically with equilibrium output x_t^* and inflation π_t^*), and an equilibrium-selection policy which selects one particular equilibrium path $\{\pi_t^*\}$ from the set consistent with $\{i_t^*\}$. Equilibrium selection is accomplished by a Taylor-inspired rule, $i_t = i_t^* + \phi\left(\pi_t - \pi_t^*\right),\ \phi > 1$. People believe that the Fed selects equilibria by this method, and the Fed will explode the economy should undesired equilibria emerge.

You may say, "this is nuts, the Fed doesn't have an equilibrium-selection policy, and does not threaten to explode the economy for equilibria it doesn't like." You may object that none of this has any relation to the Fed's clear statements of how it would stabilize the economy if inflation came out higher than the Fed's target, not the opposite. You may object that nobody has written any op-eds or policy essays castigating the Fed for its equilibrium-selection policies. You may look at my graphs of the expectations for alternative equilibria required to support a given equilibrium choice, and feel they are nutty assumptions to make about what people expect. If so, you conclude that the Fed really doesn't have an "equilibrium selection" policy, and this class of models needs a different equilibrium-selection mechanism in order to provide a definite prediction for the data and reliable policy prescriptions.

And that discussion still leaves unanswered why, even if people did expect the Fed to follow an equilibrium-selection policy, people would expect the Fed choose an equilibrium with such disastrous consequences, when the same sort of threats could support benign equilibria. "Discretion" won't do it, as equilibrium-selection requires prodigious precommitment.

I have throughout added what may have seemed unnecessary words to emphasize that equilibria are selected not by what the Fed will do, could do, or promises to do, but by what people *expect* to happen. The logic of these models is strong – the expectation of future inflation (or output) paths is what selects equilibria today. In evaluating which equilibrium emerges today it is not just desirable but necessary to think about what people actually expect to happen when the liquidity trap ends.

Leaving out such selection mechanisms – if the Fed really just picks the equilibrium interest rate path (which, again, may covary with equilibrium inflation and output) – then there really are multiple equilibria and choosing one vs. another is simply an arbitrary choice. Since there *is* an equilibrium with no depression and deflation, and no magical policy predictions, one cannot say that the new-Keynesian model makes a definite prediction of depression and policy impact.

I have not advocated a specific alternative equilibrium selection criterion. Obviously, the local-to-frictionless equilibrium has some points to commend it: It is bounded in both directions, it produces normal policy predictions, it has a smooth limit as price stickiness is reduced, and it does not presume an enormous fiscal support for deflation. But this is not yet economic proof that it is the "right" equilibrium choice.

We might consider which equilibrium choice is more consistent with the data. The US economy 2009-2013 features steady but slow growth, a level of output stuck about 6-7% below the previous trendline and the CBO's assessment of "potential," a stagnant employment-population ratio, and steady positive 2-2.5% inflation.

The local-to-frictionless equilibrium as shown in Figure 2 can produce this stagnant outcome, but only if one thinks that current output is about equal to potential, i.e. that the problem is "supply" rather than "demand," and that the CBO and other calculations of "potential" or non-inflationary output and employment are optimistic, as they were in the 1970s, and do not reflect new structural impediments to output.

The standard equilibrium choice as shown in Figure 3 cannot produce stagnation. It counterfactually predicts deflation (Coibion and Gorodnichenko (2013)), and it counterfactually predicts strong growth. One would have imagine a steady stream of unexpected negative shocks – that each year, the expected duration of the negative natural rate increases unexpectedly by one more year – to rescue the model. But five tails in a row is pretty unlikely.

The problem in generating stagnation is central to the new-Keynesian model. The "IS" curve and the assumption that we return to trend means that we can only have a low *level* of output and consumption if we expect strong *growth*. The Phillips curve says that to have a large output gap, we must have inflation today much below expected inflation tomorrow and thus growing inflation (or declining deflation). Thus if we are to return to a low-inflation steady state, we must experience sharp deflation today. If one wants a model with stagnation resulting from perpetual lack of "demand," this model isn't it. Static old-Keynesian models produce slumps, but dynamic intertemporal new-Keynesian models do not. Of course, the very stylized model used here is too simplistic for a serious comparison to data. But the trouble is so essential to a forward-looking IS and Phillips curve, it does not seem likely to be solved by epicycles around those main ingredients.

We might consider which equilibrium choice is correct by examining data and policies from longer periods in history. Magical multipliers and paradoxical policies have been found in similar models any time interest rates do not respond to changes in equilibrium output and inflation. Thus, they should occur in periods such as the great depression, at the zero bound, the late 1940s and early 1950s, when interest rates were explicitly fixed, and in the 1970s, when new-Keynesian thought claims that interest rates did not respond enough to output and inflation. At a casual level, attempts to deliberately inflate, output destruction, technical regress, useless government spending, and promises all seem to have been tried in those periods without the large effects claimed for them now. That too, however, is a suggestion, not a claim, and one that requires much more careful documentation.

I close with a few kinds words for the new-Keynesian model. This paper is really an argument to *save* the core of the new-Keynesian model – proper, forward-looking intertemporal behavior in its IS and price-setting equations – rather than to attack it. Inaccurate predictions for data (deflation, depression, strong growth), crazy-sounding, or at least sign-changing, policy predictions, a paradoxical limit as price stickiness declines, and explosive off-equilibrium expectations, are *not* essential results of the model's core ingredients. A model with the core ingredients can give a very conventional view of the world, if one only picks the local-to-frictionless equilibrium. That model will build neatly on a stochastic growth model, represented here in part by the forward-looking "IS" equation and changes in "potential." Its price stickiness will modify dynamics in small but sensible ways and allow a description of the effects of monetary policy. This was the initial vision for new-Keynesian models, and it remains true.

Really, the fault is not in the core of the new-Keynesian model. The fault is in its application, which failed to take seriously the fundamental problem of nominal indeterminacy, i.e. multiple nominal equilibria. Interest rate targets, even those that vary with output and inflation, or money supply control with interest-elastic demand, simply do not determine the price level or inflation. (Cochrane (2011).) In a model with price stickiness, nominal indeterminacy spills over in to real indeterminacy.

In that context, this paper shows there is an equilibrium choice that leads to sensible results. Alas, those sensible results are non-intoxicating. In that equilibrium, our present (2013) economic troubles cannot be chalked up to one big simple story, a "negative natural rate" (whatever that means) facing a lower bound on short term nominal rates; and our economic troubles cannot be solved by promises, or a sign reversal of all the dismal parts of our dismal science.

Technical regress, wasted government spending, and deliberate capital destruction do not work. Growth is good, not bad. That outcome is bad news for those who found magical policies an intoxicating possibility, but good news for a realistic and sober macroeconomics.

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9 Appendix

9.1 Derivation of the model solution

The model is

$$\begin{aligned}\frac{dx_t}{dt} &= ir_t - \pi_t \\ \frac{d\pi_t}{dt} &= \rho\pi_t - \kappa\left(x_t + g_t\right).\end{aligned}$$

where $ir_t = ir$ for $t < T$ and $ir = 0$ for $t > T$, and g_t is a constant g for $t < T$ and $g_t = 0$ for $t > T$. It is convenient to scale x_t by κ, yielding

$$\frac{d}{dt}\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} = \begin{bmatrix}\kappa ir_t \\ -\kappa g_t\end{bmatrix} + \begin{bmatrix}0 & -\kappa \\ -1 & \rho\end{bmatrix}\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix}. \tag{22}$$

The steady state of (22) is

$$\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} = \begin{bmatrix}\rho ir - \kappa g \\ ir\end{bmatrix}$$

Thus, we find solutions to the homogenous part – without the first term in (22) – and then add back this steady state.

Eigenvalue decomposing the transition matrix, we have

$$\begin{bmatrix}0 & -\kappa \\ -1 & \rho\end{bmatrix} = \begin{bmatrix}\lambda^p & \lambda^m \\ 1 & 1\end{bmatrix}\begin{bmatrix}\lambda^m & 0 \\ 0 & \lambda^p\end{bmatrix}\begin{bmatrix}\lambda^p & \lambda^m \\ 1 & 1\end{bmatrix}^{-1}$$

where

$$\begin{aligned}\lambda^p &= \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\kappa}\right) \geq 0 \\ \lambda^m &= \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\kappa}\right) \leq 0.\end{aligned}$$

For simplifying algebra it is convenient to note

$$\begin{aligned}\lambda^p - \lambda^m &= \sqrt{\rho^2 + 4\kappa} \\ \lambda^m + \lambda^p &= \rho \\ \lambda^m\lambda^p &= -\kappa\end{aligned} \tag{23}$$

Define

$$\begin{bmatrix}z_t \\ w_t\end{bmatrix} = \begin{bmatrix}\lambda^p & \lambda^m \\ 1 & 1\end{bmatrix}^{-1}\left(\begin{bmatrix}\kappa x_t \\ \pi_t\end{bmatrix} - \begin{bmatrix}ir - \kappa g \\ ir\end{bmatrix}\right)$$

and thus

$$\begin{bmatrix}x_t \\ \pi_t\end{bmatrix} = \begin{bmatrix}\rho ir - \kappa g \\ ir\end{bmatrix} + \begin{bmatrix}\lambda^p & \lambda^m \\ 1 & 1\end{bmatrix}\begin{bmatrix}z_t \\ w_t\end{bmatrix}.$$

Then (22) reduces to

$$\frac{d}{dt}\begin{bmatrix}z_t \\ w_t\end{bmatrix} = \begin{bmatrix}\lambda^m & 0 \\ 0 & \lambda^p\end{bmatrix}\begin{bmatrix}z_t \\ w_t\end{bmatrix}$$

with solutions

$$\begin{bmatrix} z_t \\ w_t \end{bmatrix} = \begin{bmatrix} e^{\lambda^m (t-T)} & 0 \\ 0 & e^{\lambda^p (t-T)} \end{bmatrix} \begin{bmatrix} z_T \\ w_T \end{bmatrix}$$

where and z_T and w_T are constants indexing multiple equilibria.

Transforming back to x and π, the general solution to (22) when g and ir are constant is therefore

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} + \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m (t-T)} & 0 \\ 0 & e^{\lambda^p (t-T)} \end{bmatrix} \begin{bmatrix} z_T \\ w_T \end{bmatrix} \tag{24}$$

For $t \geq T$, I follow the standard procedure by requiring that x_t and π_t do not explode as $t \to \infty$. This requirement means that the constant w_T multiplying the explosive eigenvalue λ^p must be zero. We also have $ir = 0$ and $g = 0$ for $t \geq T$, and so we have

$$t \geq T: \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} z_T e^{\lambda^m (t-T)}.$$

The single parameter z_T controls both x_t and π_t equilibria, and $z_T = \pi_T$ so I use that notation.

$$t \geq T: \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_T e^{\lambda^m (t-T)}.$$

For $t \leq T$, we pick a different set of constants z_T^* and w_T^*. We pick these constants so that the solution $(\kappa x_t, \pi_t)$ pastes in to the $t \geq T$ solution at $t = T$. This consideration does not rule out solutions that load on the positive eigenvalue λ^p. The pasting condition is

$$\begin{bmatrix} \kappa x_T \\ \pi_T \end{bmatrix} = \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} + \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} = \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_T \tag{25}$$

We can solve (25) for $\begin{bmatrix} z_T^* & w_T^* \end{bmatrix}'$ and substitute into (24).

$$\begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} = \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} z_T - \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} \right)$$

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} \\ &+ \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m (t-T)} & 0 \\ 0 & e^{\lambda^p (t-T)} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_T - \begin{bmatrix} \rho ir - \kappa g \\ ir \end{bmatrix} \right) \end{aligned}$$

Using (23) we can write each of the rightmost terms as

$$\begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m (t-T)} & 0 \\ 0 & e^{\lambda^p (t-T)} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_T = \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m (t-T)} \pi_T,$$

$$\begin{aligned} &\begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m (t-T)} & 0 \\ 0 & e^{\lambda^p (t-T)} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^m + \lambda^p \\ 1 \end{bmatrix} ir \\ &= \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m (t-T)} \\ e^{\lambda^p (t-T)} \end{bmatrix} ir \end{aligned}$$

and

$$\begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} & 0 \\ 0 & e^{\lambda^p(t-T)} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \kappa g = \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^p & -\lambda^m \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} \kappa g$$

Putting it all back together,

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho i r_t - \kappa g_t \\ i r_t \end{bmatrix} \\ &- \frac{1}{\lambda^p - \lambda^m} \left(\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} i r_t - \begin{bmatrix} \lambda^p & -\lambda^m \\ 1 & -1 \end{bmatrix} \kappa g_t \right) \begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} \\ &+ \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_T e^{\lambda^m(t-T)} \end{aligned} \tag{26}$$

9.2 Solution with a postponed interest rate rise

Here I consider a policy in which the Fed keeps the interest rate at zero from period T to period $T+\tau$, though the natural rate has risen from $-r$ to $+r$. The model is

$$\begin{aligned} \frac{dx_t}{dt} &= i r_t - \pi_t \\ \frac{d\pi_t}{dt} &= \rho \pi_t - \kappa x_t. \end{aligned}$$

where $ir_t = r$ for $t < T$; $ir = -r_\tau$ for $T < t < t + \tau$ and and $ir = 0$ for $t > T$

The general solution to (22) when ir is constant is given in (24). We just paste twice.

For $t \geq T + \tau$, we again set to zero the explosive root,

$$t \geq T + \tau : \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m[t-(T+\tau)]} \pi_{T+\tau}.$$

For $T < t < T + \tau$, since $ir = -r_\tau$, we use the pasting condition at $t = T + \tau$

$$\begin{bmatrix} \kappa x_{T+\tau} \\ \pi_{T+\tau} \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} (-r_\tau) + \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m \tau} & 0 \\ 0 & e^{\lambda^p \tau} \end{bmatrix} \begin{bmatrix} z_T^{**} \\ w_T^{**} \end{bmatrix} = \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_{T+\tau}$$

where r_τ is the positive natural rate that holds between time T and time $T + \tau$. We solve this equation for $\begin{bmatrix} z_{T+\tau}^{**} & w_{T+\tau}^{**} \end{bmatrix}'$ and substitute into (24).

$$\begin{bmatrix} z_{T+\tau}^{**} \\ w_{T+\tau}^{**} \end{bmatrix} = \begin{bmatrix} e^{-\lambda^m \tau} & 0 \\ 0 & e^{-\lambda^p \tau} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_{T+\tau} + \begin{bmatrix} \rho \\ 1 \end{bmatrix} r_\tau \right)$$

so between T and $T + \tau$,

$$\begin{aligned} \begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_\tau + \\ & \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} & 0 \\ 0 & e^{\lambda^p(t-(T+\tau))} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_{T+\tau} + \begin{bmatrix} \rho \\ 1 \end{bmatrix} r_\tau \right) \end{aligned}$$

Using (23) we can write each of the rightmost terms as

$$\begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} & 0 \\ 0 & e^{\lambda^p(t-(T+\tau))} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_{T+\tau} = \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} \pi_{T+\tau} e^{\lambda^m(t-(T+\tau))},$$

$$\begin{aligned} & \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} & 0 \\ 0 & e^{\lambda^p(t-(T+\tau))} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^m + \lambda^p \\ 1 \end{bmatrix} r_\tau \\ = & \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} \\ e^{\lambda^p(t-(T+\tau))} \end{bmatrix} r_\tau \end{aligned}$$

Putting it all back together, for $T < t < T + \tau$

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_\tau + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} \\ e^{\lambda^p(t-(T+\tau))} \end{bmatrix} r_\tau + \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m(t-(T+\tau))} \pi_{T+\tau}. \tag{27}$$

Now, we paste again at $t = T$, with the general solution for $t < T$ matching this solution for $T < t < T + \tau$ at $t = T$,

$$\begin{aligned} \begin{bmatrix} \kappa x_T \\ \pi_T \end{bmatrix} &= \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} \\ &= -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_\tau + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{-\lambda^m\tau} \\ e^{-\lambda^p\tau} \end{bmatrix} r_\tau + \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{-\lambda^m\tau} \pi_{T+\tau} \end{aligned}$$

Solving for z_T^* and w_T^*,

$$\begin{aligned} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix} = & -\begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho \\ 1 \end{bmatrix} (r_\tau + r) \\ & + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{-\lambda^m\tau} \\ e^{-\lambda^p\tau} \end{bmatrix} r_\tau \\ & + \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{-\lambda^m\tau} \pi_{T+\tau} \end{aligned}$$

We substitute in (24) once again, repeated here and specialized,

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} & 0 \\ 0 & e^{\lambda^p(t-T)} \end{bmatrix} \begin{bmatrix} z_T^* \\ w_T^* \end{bmatrix}.$$

The terms are, first using $\lambda^p + \lambda^m = \rho$

$$\begin{aligned} & \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} & 0 \\ 0 & e^{\lambda^p(t-T)} \end{bmatrix} \left(-\begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho \\ 1 \end{bmatrix} (r_\tau + r) \right) \\ = & \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} -\lambda^{p2} & \lambda^{m2} \\ -\lambda^p & \lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} (r_\tau + r); \end{aligned}$$

next,

$$\begin{aligned} & \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} & 0 \\ 0 & e^{\lambda^p(t-T)} \end{bmatrix} \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{-\lambda^m\tau} \\ e^{-\lambda^p\tau} \end{bmatrix} r_\tau \\ = & \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} \\ e^{\lambda^p(t-(T+\tau))} \end{bmatrix} r_\tau \end{aligned}$$

and finally

$$
\begin{aligned}
& \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} & 0 \\ 0 & e^{\lambda^p(t-T)} \end{bmatrix} \begin{bmatrix} \lambda^p & \lambda^m \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{-\lambda^m \tau} \pi_{T+\tau} \\
= & \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m(t-(T+\tau))} \pi_{T+\tau}.
\end{aligned}
$$

Putting it all together, for $t < T$,

$$
\begin{aligned}
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} -\lambda^{p2} & \lambda^{m2} \\ -\lambda^p & \lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} (r_\tau + r) \\
& + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} \\ e^{\lambda^p(t-(T+\tau))} \end{bmatrix} r_\tau + \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m(t-(T+\tau))} \pi_{T+\tau}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \\
& + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} -\lambda^{p2} & \lambda^{m2} \\ -\lambda^p & \lambda^m \end{bmatrix} \left\{ \begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} r + \begin{bmatrix} e^{\lambda^m(t-T)} \left[1 - e^{-\lambda^m \tau}\right] \\ e^{\lambda^p(t-T)} \left[1 - e^{-\lambda^p \tau}\right] \end{bmatrix} r_\tau \right\} + \\
& + \begin{bmatrix} \lambda^p \\ 1 \end{bmatrix} e^{\lambda^m(t-(T+\tau))} \pi_{T+\tau}
\end{aligned} \tag{28}
$$

Standard solution

The standard solution just sets $\pi_{T+\tau} = 0$ as usual. For $\tau < T$,

$$
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r - \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \left\{ \begin{bmatrix} e^{\lambda^m(t-T)} \\ e^{\lambda^p(t-T)} \end{bmatrix} r + \begin{bmatrix} e^{\lambda^m(t-T)} \left[1 - e^{-\lambda^m \tau}\right] \\ e^{\lambda^p(t-T)} \left[1 - e^{-\lambda^p \tau}\right] \end{bmatrix} r_\tau \right\}
$$

while for $T < t < T + \tau$,

$$
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = - \begin{bmatrix} \rho \\ 1 \end{bmatrix} r_\tau + \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} \\ e^{\lambda^p(t-(T+\tau))} \end{bmatrix} r_\tau
$$

and with $\rho = \lambda^p + \lambda^m$,

$$
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix} \begin{bmatrix} e^{\lambda^m(t-(T+\tau))} - 1 \\ e^{\lambda^p(t-(T+\tau))} - 1 \end{bmatrix} r_\tau
$$

and of course $\pi_t = 0, x_t = 0$ for $t > T + \tau$.

Local-to-frictionless solution

To make the loading on $e^{\lambda^m(t-T)}$ equal to zero in the $t < T$ period, we need to choose $\pi_{T+\tau}$ by

$$
e^{-\lambda^m \tau} \pi_{T+\tau} = \frac{\lambda^p}{\lambda^p - \lambda^m} \left[r + r_\tau \left(1 - e^{-\lambda^m \tau} \right) \right].
$$

So, for $t > T + \tau$ we have

$$
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \frac{1}{\lambda^p - \lambda^m} \begin{bmatrix} \lambda^{p2} \\ \lambda^p \end{bmatrix} \left[r + r_\tau \left(1 - e^{-\lambda^m \tau} \right) \right] e^{\lambda^m(t-T)}
$$

Substituting in (27) we have for $T < t < T + \tau$,

$$\begin{aligned}
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= -\begin{bmatrix} \rho \\ 1 \end{bmatrix} r_\tau + \\
&+\frac{1}{\lambda^p - \lambda^m}\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix}\begin{bmatrix} e^{\lambda^m(t-(T+\tau))} \\ e^{\lambda^p(t-(T+\tau))} \end{bmatrix} r_\tau \\
&+\begin{bmatrix} \lambda^p \\ 1 \end{bmatrix}\frac{\lambda^p}{\lambda^p - \lambda^m}\left[r + r_\tau\left(1 - e^{-\lambda^m \tau}\right)\right] e^{\lambda^m(t-T)}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} &= \frac{1}{\lambda^p - \lambda^m}\begin{bmatrix} \lambda^{p2} & -\lambda^{m2} \\ \lambda^p & -\lambda^m \end{bmatrix}\begin{bmatrix} e^{\lambda^m(t-(T+\tau))} - 1 \\ e^{\lambda^p(t-(T+\tau))} - 1 \end{bmatrix} r_\tau + \\
&+\frac{1}{\lambda^p - \lambda^m}\begin{bmatrix} \lambda^{p2} \\ \lambda^p \end{bmatrix}\left[r + r_\tau\left(1 - e^{-\lambda^m \tau}\right)\right] e^{\lambda^m(t-T)}
\end{aligned}$$

and substituting in (28) for $t < T$ gives

$$\begin{bmatrix} \kappa x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \rho \\ 1 \end{bmatrix} r + \frac{1}{\lambda^p - \lambda^m}\begin{bmatrix} \lambda^{m2} \\ \lambda^m \end{bmatrix}\left\{r + \left(1 - e^{-\lambda^p \tau}\right) r_\tau\right\} e^{\lambda^p(t-T)}$$

9.3 Solution with a Taylor rule

I want to add a Taylor rule of the form $i_t = i_t^* + \phi_\pi(\pi_t - \pi_t^*)$ by which people expect the Fed to select equilibria in the $t > T$ period. Unfortunately, this simple form of Taylor rule which works in discrete time cannot be directly transported to continuous time. The problem is easiest to see in a frictionless example. In a discrete-time frictionless perfect foresight model, the IS curve and Taylor rule are

$$\begin{aligned}
i_t &= \pi_{t+1} \\
i_t &= \phi_\pi \pi_t
\end{aligned}$$

leading to equilibrium dynamics

$$\pi_{t+1} = \phi_\pi \pi_t.$$

The Taylor coefficient is the system eigenvalue, and a coefficient greater than one gives a unique locally bounded solution, $\pi_t = 0$ in this case. If we take this example directly to continuous time, however, we have

$$\begin{aligned}
i_t &= \pi_t \\
i_t &= \phi_\pi \pi_t
\end{aligned}$$

and now the corresponding eigenvalue is infinite! In discrete time there is a one period lag between the inflation that the interest rate controls –$E_t\pi_{t+1}$– and the inflation that the Fed responds to – π_t. To keep reasonable dynamics in the continuous time limit, we have to reintroduce that lag somehow.

I follow Sims (2004) and Fernández-Villaverde, Posch, and Rubio-Ramírez (2012) in writing the continuous-time Taylor rule in partial-adjustment form – which is also how it is typically estimated –

$$\frac{d\left(i_t - i_t^*\right)}{dt} = \theta\left[\phi\left(\pi_t - \pi_t^*\right) - \left(i_t - i_t^*\right)\right]$$

i.e.,

$$i_t - i_t^* = \phi\theta \int_{s=0}^{\infty} e^{-\theta s}(\pi_{t-s} - \pi_{t-s}^*)ds.$$

With this specification, a permanent unit increase in $\pi_t - \pi_t^*$ thus leads to a ϕ increase in $i_t - i_t^*$, so ϕ still has a natural interpretation. By specifying that the speed of adjustment parameters $\theta = 1$, we specify that interest rates adapt with a one-year half life, and generate approximately the same dynamics as we would in applying the standard new-Keynesian model at an annual horizon.

I also impose the zero bound on these off-equilibrium solutions. This turns out to have very little effect on output and inflation dynamics – undesired equilibria spiral off to infinity just as well when the interest rate hits the zero bound as they do without it. But since the point is that these expectations are not reasonable, it makes sense to give them the best shot by imposing the reasonable zero bound in the future as well as the past. Since I have $r_t = r$ a constant throughout $t > T$, my Taylor rule becomes

$$\frac{di_t}{dt} = \begin{cases} \theta\phi\left(\pi_t - \pi_t^*\right) - \theta\left(i_t - r\right) & i > 0 \\ \max\left[\theta\phi\left(\pi_t - \pi_t^*\right) - \theta\left(i_t - r\right), 0\right] & i = 0 \end{cases}$$

or,

$$\frac{dir_t}{dt} = \begin{cases} \theta\phi\left(\pi_t - \pi_t^*\right) - \theta ir_t & i > 0 \\ \max\left[\theta\phi\left(\pi_t - \pi_t^*\right) - \theta ir_t, 0\right] & i = 0 \end{cases} \tag{29}$$

where $ir_t = i_t - r_t$.

The model is now, for $t > T$,

$$\begin{aligned} \frac{dx_t}{dt} &= ir_t - \pi_t \\ \frac{d\pi_t}{dt} &= \rho\pi_t - \kappa x_t. \end{aligned}$$

along with (29).

Putting it all together, when $i_t > 0$ or $i_t = 0$ and $\theta\phi\left(\pi_t - \pi_t^*\right) - \theta ir_t > 0$ we have

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & \theta\phi & -\theta \end{bmatrix}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \phi\theta \end{bmatrix}\pi_t^*.$$

π_t^* and x_t^* are also solutions to this model with $ir_t = ir_t^* = 0$,

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \end{bmatrix} = \begin{bmatrix} 0 & -\kappa \\ -1 & \rho \end{bmatrix}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \end{bmatrix},$$

and thus

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & \theta\phi & -\theta \end{bmatrix}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \phi\theta \end{bmatrix}\pi_t^*.$$

Subtracting, it is easiest to describe deviations from the desired equilibrium,

$$\frac{d}{dt}\begin{bmatrix} \kappa\left(x_t - x_t^*\right) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & \theta\phi & -\theta \end{bmatrix}\begin{bmatrix} \kappa\left(x_t - x_t^*\right) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} \tag{30}$$

When $i_t = 0$ and $\theta\phi(\pi_t - \pi_t^*) - \theta i r_t < 0$ we have instead

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \kappa x_t \\ \pi_t \\ ir_t \end{bmatrix}.$$

The desired π_t^* and x_t^* are also solutions to this model with $ir_t = ir_t^* = 0$,

$$\frac{d}{dt}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \kappa x_t^* \\ \pi_t^* \\ 0 \end{bmatrix},$$

so we have in this case

$$\frac{d}{dt}\begin{bmatrix} \kappa(x_t - x_t^*) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & \kappa \\ -1 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \kappa(x_t - x_t^*) \\ \pi_t - \pi_t^* \\ ir_t \end{bmatrix} \tag{31}$$

We already know the unique forward-bounded solutions of these systems. The point is to look at the off equilibrium behavior. Thus, I simulate *forward*, from the known π_T^*, x_T^*, to see what would happen to alternative multiple equilibria.

One could simply discretize these derivatives. However, for greater accuracy, I simulate forward the solutions. Each of (30)-(31) is of the form

$$\frac{d}{dt}X_t = AX_t = Q\Lambda Q^{-1}X_t$$

with solution

$$X_t = Qe^{\Lambda t}Q^{-1}X_0$$

Thus, I simulate forward starting at time T,

$$X_{t+\Delta} = Qe^{\Lambda\Delta}Q^{-1}X_t$$

at each date t I check which case holds, and simulate forward to date $t+\Delta$ using the appropriate transition matrix. I find eigenvalues and eigenvectors numerically.