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ABSTRACT

We propose a method for solving and estimating linear rational expectations models that exhibit indeterminacy and we provide step-by-step guidelines for implementing this method in the Matlab-based packages Dynare and Gensys. Our method redefines a subset of expectational errors as new fundamentals. This redefinition allows us to treat indeterminate models as determinate and to apply standard solution algorithms. We prove that our method is equivalent to the solution method proposed by Lubik and Schorfheide, and using our methodology, we replicate the estimates from their work.

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1 Introduction

It is well known that linear rational expectations (LRE) models can have an indeterminate set of equilibria under realistic parameter choices. Lubik and Schorfheide (2003) provided an algorithm that computes the complete set of indeterminate equilibrium, but their approach has not yet been implemented in standard software packages and has not been widely applied in practice. In this paper, we propose an alternative methodology based on the idea that a model with an indeterminate set of equilibria is an incomplete model. We propose to close a model of this kind by treating a subset of the non-fundamental errors as newly defined fundamentals.

Our method builds on the approach of Sims (2001) who provided a widely used computer code, Gensys, implemented in Matlab, to solve for the reduced form of a general class of linear rational expectations (LRE) models. Sims's code classifies models into three groups; those with a unique rational expectations equilibrium, those with an indeterminate set of rational expectations equilibria, and those for which no bounded rational expectations equilibrium exists. By moving non-fundamental errors to the set of fundamental shocks, we select a unique equilibrium, thus allowing the modeler to apply standard solution algorithms. We provide step-by-step guidelines for implementing our method in the Matlab-based software programs Dynare (Adjemian et al., 2011) and Gensys (Sims, 2001).

Our paper is organized as follows. In Section 2, we provide a brief literature survey and in Section 3 we review solution methods for indeterminate models. In Section 4, we discuss the choice of which expectational errors to redefine as fundamental and we prove that all possible alternative selections have the same likelihood. Section 5 compares our method to the work of Lubik and Schorfheide (2003) and establishes an equivalence result between the two approaches. In Section 6, we apply our method to a simple New-Keynesian model, and in Section 7 we use our methodology to replicate the results of Lubik and Schorfheide (2004). Section 8 provides a brief conclusion.

2 Related Literature

Blanchard and Kahn (1980) showed that a LRE model can be written as a linear combination of backward-looking and forward-looking solutions. Since then, a number of alternative approaches for solving linear rational expectations models have emerged (King and Watson, 1998; Klein, 2000; Uhlig, 1999; Sims, 2001). These methods provide a solution if the equilibrium is unique, but there is considerable confusion about how to handle the indeterminate case. Some methods fail in the case of a non-unique solution, for example, Klein (2000), while others, e.g. Sims (2001), generate one solution with a warning message.

All of these solution algorithms are based on the idea that, when there is a unique determinate rational expectations equilibrium, the model's forecast errors are uniquely defined by the fundamental shocks. These errors must be chosen in a way that eliminates potentially explosive dynamics of the state variables of the model.

McCallum (1983) has argued that a model with an indeterminate set of equilibria is incompletely specified and he recommends a procedure, the minimal state variable solution, for selecting one of the many possible equilibria in the indeterminate case. Farmer (1999) has argued instead, that we should exploit the properties of indeterminate models to help understand data, but with the exceptions of an early piece by Farmer and Guo (1995) and a more recent literature (Belaygorod and Dueker, 2009; Bhattarai et al., 2012; Castelnuovo and Fanelli, 2013; Hirose, 2011; Zheng and Guo, 2013; Bilbiie and Straub, 2013) that follows the approach of Lubik and Schorfheide (2004), there has not been much empirical work that seeks to formally estimate indeterminate models. That is in contrast to a large body of theoretical work, surveyed in Benhabib and Farmer (1999), which demonstrates that the theoretical properties of models with indeterminacy present a serious challenge to conventional classical and new-Keynesian approaches.

The empirical importance of indeterminacy began with the work of Benhabib and Farmer (1994) who established that a standard one-sector growth model with increasing returns displays an indeterminate steady state and Farmer and Guo (1994) who exploited that property to generate business cycle models driven by self-fulfilling beliefs. More recent New-Keynesian models have been shown to exhibit indeterminacy if the monetary authority does not increase the nominal interest rate enough in response to higher inflation (see, for example, Clarida et al. (2000); Kerr and King (1996)). Our estimation method should be of interest to researchers in both literatures.

3 Solving LRE Models

Consider the following k-equation LRE model. We assume that $X_t \in R^k$ is a vector of deviations from means of some underlying economic variables. These may include predetermined state variables, for example, the stock of capital, non-predetermined control variables, for example, consumption; and expectations at date t of both types of variables.

We assume that z_t is an $l \times 1$ vector of exogenous, mean-zero shocks and η_t is a $p \times 1$ vector of endogenous shocks.[1] The matrices Γ_0 and Γ_1 are of dimension $k \times k$, possibly singular, Ψ and Π are respectively, $k \times l$ and $k \times p$ known matrices.

Using the above definitions, we will study the class of linear rational expectations models described by Equation (1),

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi \eta_t. \tag{1}$$

Sims (2001) shows that this way of representing a LRE is very general and most LRE models that are studied in practice by economists can be written

[1] Sims (2001) allows z_t to be autoregressive with non zero conditional expectation. We assume, instead, that z_t always has zero conditional mean. That assumption is unrestrictive since an autoregressive error can always be written in our form by defining a new state variable, $\tilde{z}_t$ and letting the innovation of the original variable, z_t, be the new fundamental shock.

in this form. We assume that

$$E_{t-1}\left(z_t\right) = 0, \quad \text{and} \qquad E_{t-1}\left(\eta_t\right) = 0. \tag{2}$$

We define the $l \times l$ matrix Ω,

$$E_{t-1}\left(z_t z_t^T\right) = \Omega_{zz}, \tag{3}$$

which represents the covariance matrix of the exogenous shocks. We refer to these shocks as predetermined errors, or equivalently, predetermined shocks. The second set of shocks, η_t, has dimension p. Unlike the z_t, these shocks are endogenous and are determined by the solution algorithm in a way that eliminates the influence of the unstable roots of the system. In many important examples, the $\eta_{i,t}$ have the interpretation of expectational errors and, in those examples,

$$\eta_{i,t} = X_{i,t} - E_{t-1}\left(X_{i,t}\right). \tag{4}$$

3.1 The QZ Decomposition

Sims (2001) shows how to write equation (1) in the form

$$\begin{aligned} \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{X}_{1,t-1} \\ \tilde{X}_{2,t-1} \end{bmatrix} \\ + \begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} z_t + \begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix} \eta_t, \end{aligned} \tag{5}$$

where the matrices S, T, $\tilde{\Psi}$ and $\tilde{\Pi}$ and the transformed variables $\tilde{X}_t$ are defined as follows. Let

$$\Gamma_0 = QSZ^T, \quad \text{and} \quad \Gamma_1 = QTZ^T, \tag{6}$$

be the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$ where Q and Z are $k \times k$ orthonormal matrices and S and T are upper triangular and possibly complex.

The QZ decomposition is not unique. The diagonal elements of S and T are called the *generalized eigenvalues* of $\{\Gamma_0, \Gamma_1\}$ and Sims's algorithm chooses one specific decomposition that orders the equations so that the absolute values of the ratios of the generalized eigenvalues are placed in increasing order that is,

$$|t_{jj}| \,/\, |s_{jj}| \geq |t_{ii}| \,/\, |s_{ii}| \text{ for } j > i. \tag{7}$$

Sims proceeds by partitioning S, T, Q and Z as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \tag{8}$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \tag{9}$$

where the first block contains all the equations for which $|t_{jj}| \,/\, |s_{jj}| < 1$ and the second block, all those for which $|t_{jj}| \,/\, |s_{jj}| \geq 1$. The transformed variables $\tilde{X}_t$ are defined as

$$\tilde{X}_t = Z^T X_t, \tag{10}$$

and the transformed parameters as

$$\tilde{\Psi} = Q^T \Psi, \quad \text{and} \quad \tilde{\Pi} = Q^T \Pi. \tag{11}$$

3.2 Using the QZ decomposition to solve the model

The model is said to be determinate if Equation (5) has a unique bounded solution. To establish existence of at least one bounded solution we must eliminate the influence of all of the unstable roots; by construction, these are contained in the second block,

$$\tilde{X}_{2,t} = S_{22}^{-1} T_{22} \tilde{X}_{2,t-1} + S_{22}^{-1} \left(\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t \right), \tag{12}$$

since the eigenvalues of $S_{22}^{-1} T_{22}$ are all greater than one in absolute value. Hence a bounded solution, if it exists, will set

$$\tilde{X}_{2,0} = 0, \tag{13}$$

and

$$\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t = 0. \tag{14}$$

Since the elements of $\tilde{X}_{2,t}$ are linear combinations of $X_{2,t}$, a necessary condition for the existence of a solution to equation (14) is that there are at least as many non-predetermined variables as unstable generalized eigenvalues. A sufficient condition is that the columns of $\tilde{\Pi}_2$ in the matrix,

$$\begin{bmatrix} \tilde{\Psi}_2 & \tilde{\Pi}_2 \end{bmatrix}, \tag{15}$$

are linearly independent so that there is at least one solution to Equation (14) for the endogenous shocks, η_t, as a function of the fundamental shocks, z_t. In the case that $\tilde{\Pi}_2$ is square and non-singular, we can write the solution for η_t as

$$\eta_t = -\tilde{\Pi}_2^{-1} \tilde{\Psi}_2 z_t. \tag{16}$$

More generally, Sims' code checks for existence using the singular value decomposition of (15).

To find a solution for $\tilde{X}_{1,t}$ we take equation (16) and plug it back into the first block of (5) to give the expression,

$$\tilde{X}_{1,t} = S_{11}^{-1} T_{11} \tilde{X}_{1,t-1} + S_{11}^{-1} \left(\tilde{\Psi}_1 - \tilde{\Pi}_1 \tilde{\Pi}_2^{-1} \tilde{\Psi}_2 \right) z_t. \tag{17}$$

Even if there is more than one solution to (14) it is possible that they all lead to the same solution for $\tilde{X}_{1,t}$. Sims provides a second use of the singular value decomposition to check that the solution is unique. Equations (13) and (17) determine the evolution of $\left\{\tilde{X}_t\right\}$ as functions of the fundamental shocks $\{z_t\}$ and, using the definition of $\left\{\tilde{X}_t\right\}$ from (10), we can recover the original sequence $\{X_t\}$.

3.3 The Indeterminate Case

There are many examples of sensible economic models where the number of expectational variables is larger than the number of unstable roots of the system. In that case, Gensys will find a solution but flag the fact that there are many others. We propose to deal with that situation by providing a statistical model for one or more of the endogenous errors.

The rationale for our procedure is based on the notion that agents situated in an environment with multiple rational expectations equilibria must still choose to act. And to act rationally, they must form *some* forecast of the future and, therefore, we can model the process of expectations formation by specifying how the forecast errors covary with the other fundamentals.

If a model has n unstable generalized eigenvalues and p non-fundamental errors then, under some regularity assumptions, there will be $m = p - n$ degrees of indeterminacy. In that situation we propose to redefine m non-fundamental errors as new fundamental shocks. This transformation allows us to treat indeterminate models as determinate and to apply standard solution and estimation methods.

Consider model (1) and suppose that there are m degrees of indeterminacy. We propose to partition the η_t into two pieces, $\eta_{f,t}$ and $\eta_{n,t}$ and to partition Π conformably so that,

$$\underset{k\times k}{\Gamma_0}\underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\underset{k\times 1}{X_{t-1}} + \underset{k\times l}{\Psi}\underset{l\times 1}{z_t} + \begin{bmatrix} \underset{k\times m}{\Pi_f} & \underset{k\times n}{\Pi_n} \end{bmatrix} \begin{bmatrix} \underset{m\times 1}{\eta_{f,t}} \\ \underset{n\times 1}{\eta_{n,t}} \end{bmatrix}. \tag{18}$$

Here, $\eta_{f,t}$ is an $m \times 1$ vector that contains the newly defined fundamental errors and $\eta_{n,t}$ contains the remaining n non-fundamental errors.

Next, we re-write the system by moving $\eta_{f,t}$ from the vector of expectational shocks to the vector of fundamental shocks:

$$\underset{k\times k}{\Gamma_0}\underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\underset{k\times 1}{X_{t-1}} + \begin{bmatrix} \underset{k\times l}{\Psi} & \underset{k\times m}{\Pi_f} \end{bmatrix} \underset{(l+m)\times 1}{\widetilde{z}_t} + \underset{k\times n}{\Pi_n}\underset{n\times 1}{\eta_{n,t}}, \tag{19}$$

where we treat

$$\underset{(l+m)\times 1}{\widetilde{z}_t} = \begin{bmatrix} \underset{l\times 1}{z_t} \\ \underset{m\times 1}{\eta_{f,t}} \end{bmatrix}, \tag{20}$$

as a new vector of fundamental shocks and $\eta_{n,t}$ as a new vector of non-fundamental shocks. To complete this specification, we define $\tilde{\Omega}$

$$\underset{(l+m)\times(l+m)}{\tilde{\Omega}} = E_{t-1}\left(\begin{bmatrix} \underset{l\times 1}{z_t} \\ \underset{m\times 1}{\eta_{f,t}} \end{bmatrix} \begin{bmatrix} \underset{l\times 1}{z_t} \\ \underset{m\times 1}{\eta_{f,t}} \end{bmatrix}^T \right) \equiv \begin{pmatrix} \underset{l\times l}{\Omega_{zz}} & \underset{l\times m}{\Omega_{zf}} \\ \underset{m\times l}{\Omega_{fz}} & \underset{m\times m}{\Omega_{ff}} \end{pmatrix}, \tag{21}$$

to be the new covariance matrix of fundamental shocks. This definition requires us to specify $m(m+1+2l)/2$ new variance parameters, these are the $m(m+1)/2$ elements of Ω_{ff}, and ml new covariance parameters, these are the elements of Ω_{zf}. By choosing these new parameters and applying Sims' solution algorithm, we select a unique bounded rational expectations equilibrium. The diagonal elements of $\tilde{\Omega}$ that correspond to η_f have the interpretation of a pure 'sunspot' component to the shock and the covariance of these terms with z_t represent the response of beliefs to the original set of fundamentals.

Our approach to indeterminacy is equivalent to defining a new model in which the indeterminacy is resolved by assuming that expectations are formed consistently using the same forecasting method in every period. For example, expectations may be determined by a learning mechanism as in

Evans and Honkapohja (2001) or using a belief function as in Farmer (2002). For our approach to be valid, we require that the belief function is time invariant and that shocks to that function can be described by a stationary probability distribution. Our newly transformed model can be written in the form of Equation (1), but the fundamental shocks in the transformed model include the original fundamental shocks z_t, as well as the vector of new fundamental shocks, $\eta_{f,t}$.

4 Choice of Expectational Errors

Our approach raises the practical question of which non-fundamentals should we choose to redefine as fundamental. Here we show that, given a relatively mild regularity condition, there is an equivalence between all possible ways of redefining the model.

Definition 1 (Regularity) *Let ε be an indeterminate equilibrium of model (1) and use the QZ decomposition to write the following equation connecting fundamental and non-fundamental errors.*

$$\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t = 0. \tag{22}$$

Let n be the number of generalized eigenvalues that are greater than or equal to 1 and let $p > n$ be the number of non-fundamental errors. Partition η_t into two mutually exclusive subsets, $\eta_{f,t}$ and $\eta_{n,t}$ such that $\eta_{f,t} \cup \eta_{n,t} = \eta_t$ and partition $\tilde{\Pi}_2$ conformably so that

$$\underset{n\times p}{\tilde{\Pi}_2}\ \underset{p\times 1}{\eta_t} = \begin{bmatrix} \underset{n\times m}{\tilde{\Pi}_{2f}} & \underset{n\times n}{\tilde{\Pi}_{2n}} \end{bmatrix} \begin{bmatrix} \underset{m\times 1}{\eta_{f,t}} \\ \underset{n\times 1}{\eta_{n,t}} \end{bmatrix}. \tag{23}$$

The indeterminate equilibrium, ε, is regular if, for all possible mutually exclusive partitions of η_t, $\tilde{\Pi}_{2n}$ has full rank.

Regularity rules out situations where there is a linear dependence in the non-fundamental errors and all of the indeterminate LRE models that we are aware of, that have been studied in the literature, satisfy this condition.

Theorem 1 *Let ε be an indeterminate equilibrium of model (1) and let* $\mathbf{P}$ *be an exhaustive set of mutually exclusive partitions of η_t into two non-intersecting subsets; where* $\left\{ \mathbf{p} \in \mathbf{P} \mid \mathbf{p} = \left(\underset{m\times 1}{\eta_{f,t}}, \underset{n\times 1}{\eta_{n,t}} \right)^T \right\}$. *Let* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *be elements of* $\mathbf{P}$ *and let* $\tilde{\Omega}_1$ *be the covariance matrix of the new set of fundamentals,* $\left[z_t, \eta_{f,t} \right]$ *associated with partition* $\mathbf{p}_1$. *If ε is regular then there is a covariance matrix* $\tilde{\Omega}_2$, *associated with partition* 2 *such that the covariance matrix*

$$\Omega = E\left(\begin{bmatrix} z_t \\ \eta_{f,t} \\ \eta_{n,t} \end{bmatrix} \begin{bmatrix} z_t \\ \eta_{f,t} \\ \eta_{n,t} \end{bmatrix}^T \right), \tag{24}$$

is the same for both partitions. $\mathbf{p}_1$ *and* $\mathbf{p}_2$, *parameterized by* $\tilde{\Omega}_1$ *and* $\tilde{\Omega}_2$, *are said to be equivalent partitions.*

Proof. See Appendix A. ■

Corollary 1 *The joint probability distribution over sequences* $\{X_t\}$ *is the same for all equivalent partitions.*

Proof. The proof follows immediately from the fact that the joint probability of sequences $\{X_t\}$, is determined by the joint distribution of the shocks. ■

The question of how to choose a partition $\mathbf{p}_i$ is irrelevant. All partitions have the same likelihood. However, the partition will matter, if the researcher imposes zero restrictions on the variance covariance matrix of fundamentals. A zero cross-correlation under one partition, $\mathbf{p}_i$ will imply a complicated non-linear cross-equation restriction on all other partitions $\mathbf{p}_j$. For this reason, we recommend that in practice, the VCV matrix of the shocks $\tilde{z}$ should be left unrestricted.

5 Lubik-Schorfheide and Farmer-Khramov-Nicolò Compared

The two papers by Lubik and Schorfheide, (Lubik and Schorfheide, 2003, 2004), are widely cited in the literature (Belaygorod and Dueker, 2009; Zheng and Guo, 2013; Lubik and Matthes, 2013) and their approach is the one most closely emulated by researchers who wish to estimate models that possess an indeterminate equilibrium. This section compares the Lubik-Schorfheide method to ours and proves an equivalence result.

We show in Theorem 2 that every LS equilibrium can be implemented as a Farmer-Khramov-Nicolò (FKN) equilibrium, and conversely, every Farmer-Khramov-Nicolò (FKN) equilibrium can be characterized using the Lubik-Schorfheide technique. Because our method can be implemented using standard algorithms, our method provides an easy way for applied researchers to simulate and estimate indeterminate models using widely available computer software. And Theorem 2 shows that the full set of indeterminate equilibria can be modeled using our approach.

5.1 The Singular Value Decomposition

Determinacy boils down to the following question: Does equation (14), which we repeat below as equation (25), have a unique solution for the $p \times 1$ vector of endogenous errors, η_t, as functions of the $\ell \times 1$ vector of fundamental errors, z_t?

$$\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{z_t} + \underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0. \tag{25}$$

To answer this question, LS apply the singular value decomposition to the matrix $\tilde{\Pi}_2$. The interesting case is when $p > n$, for which $\tilde{\Pi}_2$ has n singular values, equal to the positive square roots of the eigenvalues of $\tilde{\Pi}_2\tilde{\Pi}_2^T$. The singular values are collected into a diagonal matrix D_{11}. The matrices U and V in the decomposition are orthonormal and $m = p - n$ is the degree of indeterminacy.

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U} \left[\begin{array}{cc} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{array} \right] \underset{p\times p}{V^T}. \tag{26}$$

Replacing $\tilde{\Pi}_2$ in (25) with this expression and premultiplying by U^T leads to the equation

$$\underset{n\times n}{U^T} \underset{n\times \ell}{\tilde{\Psi}_2} \underset{\ell\times 1}{z_t} + \left[\begin{array}{cc} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{array} \right] \underset{p\times p}{V^T} \underset{p\times 1}{\eta_t} = 0. \tag{27}$$

Now partition V

$$V = \left[\begin{array}{cc} \underset{p\times n}{V_1} & \underset{p\times m}{V_2} \end{array} \right],$$

and premultiply (27) by D_{11}^{-1},

$$\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U^T} \underset{n\times \ell}{\tilde{\Psi}_2} \underset{\ell\times 1}{z_t} + \underset{n\times p}{V_1^T} \underset{p\times 1}{\eta_t} = 0. \tag{28}$$

Because $p > n$ this system has fewer equations than unknowns. Lubik and Schorfheide suggest that we supplement it with the following new $m = p - n$ equations,

$$\underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{m\times m}{M_\zeta} \underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2^T} \underset{p\times 1}{\eta_t}. \tag{29}$$

The $m \times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero and covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamentals, z_t.

$$E\left[\zeta_t\right] = 0, \quad E\left[\zeta_t z_t^T\right] = 0, \quad E\left[\zeta_t \zeta_t^T\right] = \Omega_{\zeta\zeta}. \tag{30}$$

Correlation of the forecast errors, η_t, with fundamentals, z_t, is captured by the matrix M_z. Because the parameters of $\Omega_{\zeta\zeta}$ cannot separately be identified from the parameters of M_ζ, LS choose the normalization

$$M_\zeta = I_m. \tag{31}$$

Appending equation (29) as additional rows to equation (28), premultiplying by V and rearranging terms leads to the following representation of the expectational errors as functions of the fundamentals, z_t and the sunspot shocks, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(-\underset{p\times n}{V_1} \underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2} + \underset{p\times m}{V_2} \underset{m\times \ell}{M_z} \right) \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}. \tag{32}$$

This is equation (25) in Lubik and Schorfheide (2003) using our notation for dimensions and where our M_z is what LS call $\tilde{M}$. More compactly

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1} \underset{n\times \ell}{N} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}, \tag{33}$$

where

$$\underset{n\times \ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2}.$$

is a function of the parameters of the model.

5.2 Equivalent characterizations of indeterminate equilibria

To define a unique sunspot equilibrium when the model is indeterminate, our method partitions η_t into two subsets; $\eta = \{\eta_f, \eta_n\}$. We refer to η_f as new fundamentals. A Farmer-Khramov-Nicolò (FKN) equilibrium is characterized by a parameter vector $\theta \in \Theta_{FKN}$ which has two parts. $\theta_1 \in \Theta_1$

$$\theta_1 \equiv vec\left(\Gamma_0, \Gamma_1, \Psi, \Omega_z\right)^T,$$

is a vector of parameters of the structural equations, including the variance covariance matrix of the original fundamentals. And $\theta_2 \in \Theta_2$

$$\theta_2 \equiv vec\left(\Omega_{zf}, \Omega_{ff}\right)^T,$$

is a vector of parameters that contains the variance covariance matrix of the new fundamentals and the covariances of these new fundamentals, η_f, with the original fundamentals, z.

A Farmer-Khramov-Nicolò representation of equilibrium is a vector $\theta_{FKN} \in \Theta_{FKN}$ where Θ_{FKN} is defined as,

$$\Theta_{FKN} \equiv \{\Theta_1, \Theta_2\}.$$

Theorem 1 establishes that there is an equivalence class of models, all with the same likelihood function, in which the $m \times 1$ vector η_f is selected as a new set of fundamentals and the VCV matrices Ω_{ff} and Ω_{zf} are additional parameters. To complete the model in this way we must add $m(m+1)/2$ new parameters to define the symmetric matrix Ω_{ff} and $m \times \ell$ new parameters to define the elements of Ω_{zf}.

In contrast a Lubik-Schorfheide equilibrium is characterized by a parameter vector

$$\Theta_{LS} \equiv \{\Theta_1, \Theta_3\},$$

where $\theta_3 \in \Theta_3$ is defined as

$$\theta_3 \equiv vec(\Omega_{\zeta\zeta}, M_z)^T. \tag{34}$$

These parameters characterize the additional equation,

$$\underset{m\times\ell}{M_z}\,\underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2^T}\,\underset{p\times 1}{\eta_t}, \tag{35}$$

where equation (35) adds the normalization (31) to equation (29).

The matrix $\Omega_{\zeta\zeta}$ has $m \times (m+1)/2$ new parameters; these are the variance covariances of the sunspot shocks and the matrix M_z has $m \times \ell$ new parameters, these capture the covariances of η with z. To establish the connection between the two characterizations of equilibrium, we establish the following lemma.

Lemma 1 *Let ε be a regular indeterminate equilibrium, characterized by $\theta_{FKN} = \{\theta_1, \theta_2\}$ and let $\mathbf{p} = \left\{\eta_{f,t}, \eta_{n,t}\right\}$ be an element of the set of partitions, $\mathbf{P}$. Let $\theta_{LS} = \{\theta_1, \theta_3\}$ be the parameters of a Lubik-Schorfheide representation of equilibrium. There is an $m \times m$ matrix G, and an $m \times \ell$ matrix H, where the elements of G and H, are functions of θ_1 and an $m \times \ell$ matrix S*

$$\underset{m\times\ell}{S} = \left(\underset{m\times\ell}{H} + \underset{m\times\ell}{M_z}\right),$$

such that the sunspots shocks in the LS representation of equilibrium are related to the fundamentals z_t and the newly defined FKN fundamentals, $\eta_{f,t}$ by the equation,

$$\underset{m\times 1}{\zeta_t} = \underset{m\times m}{G}\ \underset{m\times 1}{\eta_{f,t}} - \underset{m\times\ell}{S}\ \underset{\ell\times 1}{z_t}.$$

Proof. See Appendix B. ■

The following theorem, proved in Appendix C, uses this lemma to establish an equivalence between the LS and FKN methods.

Theorem 2 *Let θ_{LS} and θ_{FKN} be two alternative parameterizations of an indeterminate equilibrium in model (1). For every FKN equilibrium, parameterized by θ_{FKN}, there is a unique matrix M_z and a unique VCV matrix $\Omega_{\zeta\zeta}$ such that $\theta_3 = vec\left(\Omega_{\zeta\zeta}, M_z\right)^T$ and $\{\theta_1, \theta_3\} \in \Theta_{LS}$ defines an equivalent LS equilibrium. Conversely, for every LS equilibrium, parameterized by θ_{LS}, and every partition $\mathbf{p}_i \in \mathbf{P}$, there is a unique VCV matrix Ω_{ff} and a unique covariance matrix Ω_{zf} such that $\theta_2 = vec(\Omega_{ff}, \Omega_{zf})^T$ and $\{\theta_1, \theta_2\} \in \Theta_{FKN}$ defines an equivalent FKN equilibrium.*

Proof. See Appendix C. ■

Next, we turn to an example that shows how to use our results in practice.

6 Example: A Simple New-Keynesian Model

In this section, we apply our method to a simple form of the New-Keynesian model, discussed in Lubik and Schorfheide (2004), where we simplify the model by assuming that there is only one fundamental shock. This model has three equations:

$$E_t[x_{t+1}] + \sigma E_t[\pi_{t+1}] = x_t + \sigma R_t, \tag{36}$$

$$R_t = \psi \pi_t + \varepsilon_t, \tag{37}$$

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa x_t, \tag{38}$$

where x_t is output, π_t is inflation, R_t is the interest rate, and ε_t is a fundamental interest rate shock. The first equation is a consumption-Euler equation, the second is a monetary policy rule, and the third is a New-Keynesian Phillips curve. This model has two forward-looking variables and one fundamental shock.

Substituting R_t into Equation (36), this model can be reduced to the following system of two equations:

$$x_t + \sigma\psi\pi_t - E_t[x_{t+1}] - \sigma E_t[\pi_{t+1}] = -\sigma\varepsilon_t, \tag{39}$$

$$-\kappa x_t + \pi_t - \beta E_t[\pi_{t+1}] = 0. \tag{40}$$

Writing the system out in Sims' notation gives,

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi \eta_t, \tag{41}$$

where the parameter matrices Γ_0, Γ_1, Ψ and Π are given by the expressions,

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \sigma\psi & -1 & -\sigma \\ -\kappa & 1 & 0 & -\beta \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{42}$$

$$\Psi = \begin{bmatrix} 0 \\ 0 \\ -\sigma \\ 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{43}$$

and the vector of variables, X_t is,

$$X_t = [x_t, \pi_t, E_t[x_{t+1}], E_t[\pi_{t+1}]]^T . \tag{44}$$

The fundamental and non-fundamental shocks are

$$z_t = [\varepsilon_t], \quad \text{and,} \quad \eta_t = [\eta_{1,t}, \eta_{2,t}]^T, \tag{45}$$

where

$$\eta_{1,t} = x_t - E_{t-1}[x_t], \qquad \eta_{2,t} = \pi_t - E_{t-1}[\pi_t]. \tag{46}$$

6.1 The Determinate Case

The determinacy properties of this example are determined by the roots of the matrix $\Gamma_0^{-1}\Gamma_1$, and, in the determinate case, some tedious, but straightforward algebra, reveals that the vector of expectational errors is described by the following function of the fundamental shock:

$$\eta_t = -\frac{\sigma}{\kappa\sigma\psi + 1} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} z_t. \tag{47}$$

In this case, the first two equations of system (41), yield the following solution for x_t and π_t,

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = -\frac{\sigma}{\kappa\sigma\psi + 1} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} z_t. \tag{48}$$

Using the symbols σ_x^2, and σ_π^2 for the variances of x and π, and σ_z^2, for the variance of z, some further algebra gives,

$$\sigma_x^2 = \left(\frac{\sigma}{\kappa\sigma\psi + 1}\right)^2 \sigma_z^2, \quad \sigma_\pi^2 = \left(\frac{\sigma}{\kappa\sigma\psi + 1}\right)^2 \kappa^2 \sigma_z^2.$$

In this example, when there is a unique determinate equilibrium, the dynamics of real variables are completely determined by the dynamics of the fundamental shock.

6.2 The Indeterminate Case

Suppose instead that the equilibrium is indeterminate, a case which occurs if $0 < \psi < 1$ (Lubik and Schorfheide, 2004). When the model is indeterminate, we propose two new alternative models, described below by equations (49) and (50).

$$\begin{aligned} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \kappa\frac{\sigma}{\beta} + 1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ -a_2^{-1} a_0 \end{bmatrix} z_t + \begin{bmatrix} 1 \\ -\frac{\beta}{\beta\psi - 1} \frac{d+\beta+\kappa\sigma-1}{d+\beta+\kappa\sigma+1} \end{bmatrix} \eta_{1,t} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1}, \end{aligned} \tag{49}$$

$$\begin{aligned} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \kappa\frac{\sigma}{\beta} + 1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} \\ &+ \begin{bmatrix} -a_1^{-1} a_0 \\ 0 \end{bmatrix} z_t + \begin{bmatrix} -a_1^{-1} a_2 \\ 1 \end{bmatrix} \eta_{2,t} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1}, \end{aligned} \tag{50}$$

In this example, the analog of Equation (22) is (51), which links the fundamental and non-fundamental errors,

$$\eta_{2,t} = -a_2^{-1} \left[a_0 z_t + a_1 \eta_{1,t} \right], \tag{51}$$

and where a_0, a_1, a_2 and d are known functions of the underlying parameters.

This simple example illustrates that the choice of which expectational error to move to the set of fundamental shocks is irrelevant for identification purposes. By specifying $\eta_{1,t}$ as fundamental, Equation (51) determines the variance-covariance properties of $\eta_{2,t}$ with z_t and $\eta_{1,t}$. By picking $\eta_{2,t}$ as the fundamental, the same equation determines the variance-covariance properties of $\eta_{1,t}$. In both cases, the variance of the two expectational shocks and their covariance with the fundamental shock are linearly related.

Notice however, that the variance of the non-fundamental shock will, in general, depend on the coefficients a_0, a_1 and a_2 which are functions of all of the other parameters of the model. It follows that, when we place restrictions on one representation of the model, for example, by setting covariance terms to zero, those restrictions will have non-trivial implications for the behavior of the observables.

7 Applying Our Method in Practice: The Lubik-Schorfheide Example

This Section verifies that the approach proposed by FKN replicates the results of Lubik and Schorfheide (2004), who provide an algorithm to solve LRE models under indeterminacy. Section 7.1 implements our methodology for the New-Keynesian model described in Lubik and Schorfheide (2004) and Section 7.2 demonstrates that our approach delivers the same parameter estimates as theirs. The equivalence of these two sets of parameter estimates confirms the validity of Theorems 1 and 2 for a practical example.

7.1 Estimating the LS Model with the FKN Approach

The model of Lubik and Schorfheide (2004) consists of a dynamic IS curve

$$x_t = E_t(x_{t+1}) - \tau (R_t - E_t(\pi_{t+1})) + g_t, \tag{52}$$

a New Keynesian Phillips curve

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa (x_t - z_t), \tag{53}$$

and a Taylor rule,

$$R_t = \rho_R R_{t-1} + (1-\rho_R)[\psi_1 \pi_t + \psi_2 (x_t - z_t)] + \varepsilon_{R,t}. \tag{54}$$

The variable x_t represents log deviations of GDP from a trend path and π_t and R_t are log deviations from the steady state level of inflation and the nominal interest rate.

The shocks g_t and z_t follow univariate AR(1) processes

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \tag{55}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{56}$$

where the standard deviations of the fundamental shocks $\varepsilon_{g,t}$, $\varepsilon_{z,t}$ and $\varepsilon_{R,t}$ are defined as σ_g, σ_z and σ_R, respectively. We allow the correlation between shocks, ρ_{gz}, ρ_{gR} and ρ_{zR}, to be nonzero. The rational expectation forecast errors are defined as

$$\eta_{1,t} = x_t - E_{t-1}[x_t], \quad \eta_{2,t} = \pi_t - E_{t-1}[\pi_t]. \tag{57}$$

We define the vector of endogenous variables,

$$X_t = [x_t, \pi_t, R_t, E_t(x_{t+1}), E_t(\pi_{t+1}), g_t, z_t]^T$$

the vectors of fundamental shocks and non-fundamental errors,

$$\mathbf{z}_t = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}\right]^T, \qquad \boldsymbol{\eta}_t = \left[\eta_{1,t}, \eta_{2,t}\right]^T$$

and the vector of parameters

$$\theta = \left[\psi_1, \psi_2, \rho_R, \beta, \kappa, \tau, \rho_g, \rho_z, \sigma_g, \sigma_z, \sigma_R, \rho_{gz}, \rho_{gR}, \rho_{zR}\right]^T.$$

This leads to the following representation of the model,

$$\Gamma_0(\theta) X_t = \Gamma_1(\theta) X_{t-1} + \Psi(\theta)\mathbf{z}_t + \Pi(\theta)\boldsymbol{\eta}_t, \tag{58}$$

where Γ_0 and Γ_1 are represented by

$$\Gamma_0(\theta) = \begin{bmatrix} 1 & 0 & \tau & -1 & -\tau & -1 & 0 \\ \kappa & -1 & 0 & 0 & \beta & 0 & -\kappa \\ (1-\rho_R)\psi_2 & (1-\rho_R)\psi_1 & -1 & 0 & 0 & 0 & -(1-\rho_R)\psi_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and,

$$\Gamma_1(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_z \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

and the coefficients of the shock matrices Ψ and Π are given by,

$$\Psi(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Pi(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The last two rows of this system define the non-fundamental shocks and it is these rows that we modify when estimating the model with the FKN approach.

7.1.1 The Determinate Case

When the monetary policy is active, $|\psi| > 1$, the number of expectational variables, $\{E_t(x_{t+1}), E_t(\pi_{t+1})\}$, equals the number of unstable roots. The Blanchard-Kahn condition is satisfied and there is a unique sequence of non-fundamental shocks such that the state variables are bounded. In this case the model can be solved using Gensys which delivers the following system of equations

$$X_t = G_1(\theta)X_{t-1} + G_2(\theta)\mathbf{z}_t \tag{59}$$

where $G_1(\theta)$ represents the coefficients of the policy functions and $G_2(\theta)$ is the matrix which expresses the impact of fundamental errors on the variables of interest, X_t.

7.1.2 Indeterminate Models

A necessary condition for indeterminacy is that the monetary policy is passive, which occurs when

$$0 < |\psi_1| < 1. \tag{60}$$

A sufficient condition is that

$$0 < \psi_1 + \frac{(1-\beta)}{\kappa}\psi_2 < 1. \tag{61}$$

This condition is stronger than (60) but the two conditions are close, given our prior, which sets[2]

$$\frac{(1-\beta)}{\kappa}\psi_2 = 0.056.$$

When (61) holds, the number of expectational variables, $\{E_t(x_{t+1}), E_t(\pi_{t+1})\}$, exceeds the number of unstable roots and there is 1 degree of indeterminacy. Using our approach, one can specify two equivalent alternative models depending on choice of the partition $\mathbf{p}$.

Fundamental Output Expectations: Model 1 In our first specification, we choose $\eta_{1,t}$, the forecast error of output, as a new fundamental. We call this partition $\mathbf{p}_1$ and we write the new vector of fundamental shocks

$$\tilde{\mathbf{z}}_{1,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{1,t}\right]^T.$$

The model is defined as

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \hat{\Psi}_x(\theta)\tilde{\mathbf{z}}_{1,t} + \hat{\Pi}_x(\theta)\eta_{2,t}, \tag{62}$$

where

$$\hat{\Psi}_x(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \hat{\Pi}_x(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

[2] We thank one of the referees for pointing that the Taylor principle must be modified, when the central bank responds to the output gap as well as to inflation.

Notice that the matrices Γ_0 and Γ_1 are unchanged. We have simply redefined $\eta_{1,t}$ as a fundamental shock by moving one of the columns of Π to Ψ. Because the Blanchard-Kahn condition is satisfied under this redefinition, the model can be solved using Gensys to generate policy functions as well as the matrix which describes the impact of the re-defined vector of fundamental shocks on X_t.

Fundamental Inflation Expectations: Model 2 Following the same logic there is an alternative partition $\mathbf{p}_2$ where the new vector of fundamentals is defined as

$$\tilde{\mathbf{z}}_{2,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{2,t}\right]^T.$$

Here, the state equation is described by

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \hat{\Psi}_\pi(\theta)\tilde{\mathbf{z}}_{2,t} + \hat{\Pi}_\pi(\theta)\eta_{1,t}, \tag{63}$$

where now

$$\hat{\Psi}_\pi(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \hat{\Pi}_\pi(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using Gensys, we can find a unique series of non-fundamental shocks $\eta_{1,t}$ such that the state variables are bounded and the state variables X_t are then a function of X_{t-1} and the new vector of fundamental errors $\tilde{\mathbf{z}}_{2,t}$.

7.2 Estimation Results using the FKN Approach

Using the LS data we estimated models 1 and 2 in Dynare. The vector of observables, $\mathbf{y}_t = \{x_{obs,t}, \pi_{obs,t}, R_{obs,t}\}$, consists of

1. $x_{obs,t}$ the percentage deviations of (log) real GDP per capita from an HP-trend;

2. $\pi_{obs,t}$ the annualized percentage change in the Consumer Price Index for all Urban Consumers;

3. $R_{obs,t}$ the annualized percentage average Federal Funds Rate.

The measurement equation is given by,

$$\mathbf{y}_t = \begin{bmatrix} 0 \\ \pi^* \\ \pi^* + r^* \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix} X_t. \tag{64}$$

where π^* and r^* are annualized steady-state inflation and real interest rates expressed in percentages. The discount factor, β is a function of the annualized real interest rate in steady-state r^* (i.e. $\beta = (1 + r^*)^{-1/4}$).

Lubik and Schorfheide report estimates of their model for specifications in which the model is determinate and indeterminate, for sample periods before and after Volcker took over as Fed Chairman in 1979Q4 and for two different sets of priors. We replicated all of their results using our approach. Here we report just our results for the pre-Volcker period using two different permutations of the error vector and for one choice of priors.

Table 1 reports the prior distributions of the parameters used in our estimation. With the exception of the correlation terms between the fundamental shocks and the non-fundamental shock, we use the same prior distributions as Lubik and Schorfheide for the parameters defined under both methodologies.

Table 1: Prior Distribution for DSGE Model Parameters					
Name	Range	Density	Mean	Std. Dev.	90% interval
ψ_1	R^+	*Gamma*	1.1	0.50	[0.33,1.85]
ψ_2	R^+	*Gamma*	0.25	0.15	[0.06,0.43]
ρ_R	$[0,1)$	*Beta*	0.50	0.20	[0.18,0.83]
π^*	R^+	*Gamma*	4.00	2.00	[0.90,6.91]
r^*	R^+	*Gamma*	2.00	1.00	[0.49,3.47]
κ	R^+	*Gamma*	0.50	0.20	[0.18,0.81]
τ^{-1}	R^+	*Gamma*	2.00	0.50	[1.16,2.77]
ρ_g	$[0,1)$	*Beta*	0.70	0.10	[0.54,0.86]
ρ_z	$[0,1)$	*Beta*	0.70	0.10	[0.54,0.86]
σ_R	R^+	*Inverse Gamma*	0.31	0.16	[0.13,0.50]
σ_g	R^+	*Inverse Gamma*	0.38	0.20	[0.16,0.60]
σ_z	R^+	*Inverse Gamma*	1.00	0.52	[0.42,1.57]
σ_η	R^+	*Inverse Gamma*	0.25	0.13	[0.11,0.40]
ρ_{gz}	[-1,1]	*Normal*	0.00	0.40	[0.65,0.65]
ρ_{gR}	[-1,1]	*Normal*	0.00	0.40	[0.65,0.65]
ρ_{zR}	[-1,1]	*Normal*	0.00	0.40	[0.65,0.65]

Table 2 compares our estimates for this period with the results from Lubik and Schorfheide, under the assumption that the model displays an indeterminate equilibrium. The first column reports the LS results, columns two and three are our estimates for two alternative partitions $\mathbf{p}_1$ and $\mathbf{p}_2$. Partition $\mathbf{p}_1$ treats $\eta_{1,t}$ as fundamental and partition $\mathbf{p}_2$ treats $\eta_{2,t}$ as fundamental. We used a random walk Metropolis-Hastings algorithm to obtain 150,000 draws from the posterior mean and we report 90-percent confidence intervals of the estimated parameters.

Compare the mean parameter estimates across the three columns. Fifteen of these parameters are common to all three specifications; these are the parameters $\psi_1, \psi_2, \rho_R, \pi^*, r^*, \kappa, \tau^{-1}, \rho_g, \rho_z, \rho_R, \sigma_g, \sigma_z, \rho_{gz}, \rho_{gR}$ and ρ_{zR}. The remaining four parameters reported in columns 2 and 3, $\sigma_{\eta_i}, \sigma_{R\eta_i}, \sigma_{g\eta_i}$, and

$\rho_{z\eta_i}$ and the parameters $\sigma_\zeta, M_{R\zeta}, M_{g\zeta}$ and $M_{z\zeta}$ from column 1, represent variances and covariances that are not comparable across specifications.

Table 2: Pre-Volcker, Posterior Means and Confidence Intervals

L&S (prior 1)			FKN - Model 1			FKN - Model 2	
	Mean	90% interval		Mean	90% interval	Mean	90% interval
ψ_1	0.77	[0.64,0.91]	ψ_1	0.73	[0.53,0.98]	0.77	[0.60,0.97]
ψ_2	0.17	[0.04,0.30]	ψ_2	0.17	[0.01,0.33]	0.20	[0.02,0.38]
ρ_R	0.60	[0.42,0.78]	ρ_R	0.85	[0.75,0.96]	0.78	[0.65,0.92]
π^*	4.28	[2.21,6.21]	π^*	3.99	[1.60,6.31]	4.04	[1.56,6.33]
r^*	1.13	[0.63,1.62]	r^*	1.24	[0.58,1.88]	1.18	[0.59,1.76]
κ	0.77	[0.39,1.12]	κ	0.32	[0.10,0.54]	0.63	[0.27,0.97]
τ^{-1}	1.45	[0.85,2.05]	τ^{-1}	1.35	[0.71,1.93]	1.63	[0.91,2.28]
ρ_g	0.68	[0.54,0.81]	ρ_g	0.76	[0.66,0.86]	0.75	[0.66,0.84]
ρ_z	0.82	[0.72,0.92]	ρ_z	0.64	[0.55,0.73]	0.66	[0.58,0.73]
σ_R	0.23	[0.19,0.27]	σ_R	0.22	[0.19,0.25]	0.22	[0.19,0.24]
σ_g	0.27	[0.17,0.36]	σ_g	0.30	[0.21,0.40]	0.30	[0.21,0.38]
σ_z	1.13	[0.95,1.30]	σ_z	1.36	[1.01,1.71]	1.22	[1.01,1.42]
σ_ζ	0.20^+	[0.12,0.27]	σ_{η_i}	0.91^+	[0.80,1.01]	0.24^+	[0.17,0.31]
ρ_{gz}	0.14	[-0.40,0.71]	ρ_{gz}	-0.00	[-0.62,0.64]	0.00	[-0.61,0.64]
-	-	-	ρ_{gR}	0.40	[0.13,0.67]	0.39	[0.16,0.62]
-	-	-	ρ_{zR}	0.49	[0.34,0.66]	0.48	[0.33,0.64]
$M_{R\zeta}$	-0.68^+	[-1.58,0.23]	$\rho_{R\eta_i}$	-0.34^+	[-0.49,-0.19]	0.09^+	[-0.21,0.41]
$M_{g\zeta}$	1.74^+	[0.90,2.56]	$\rho_{g\eta_i}$	-0.72^+	[-0.93,-0.51]	-0.02^+	[-0.42,0.35]
$M_{z\zeta}$	-0.69^+	[-0.99,-0.39]	$\rho_{z\eta_i}$	-0.89^+	[-0.99,-0.79]	0.27^+	[-0.00,0.54]

$^+$Estimates are not comparable as they represent different parameters

With the exception of the parameter κ in specification 1, our results all lie within 90% confidence bounds when point estimates from one specification are compared to confidence bounds from another. This correspondence in parameter estimates across specifications is a consequence of Theorems 1 and 2 of our paper and it confirms that our method can be applied in practice to replicate the results of an important and influential paper that has become a benchmark for monetary models of indeterminacy.

8 Conclusion

We have shown how to solve and estimate indeterminate linear rational expectations models using standard software packages. Our method transforms indeterminate models by redefining a subset of the non-fundamental shocks and classifying them as new fundamentals. We illustrated our approach using the familiar New-Keynesian monetary model and we showed that, when monetary policy is passive, the new-Keynesian model can be closed in one of two equivalent ways.

Our procedure raises the question of which non-fundamental shocks to reclassify as fundamental. Our theoretical results demonstrate that the choice of parameterization is irrelevant since all parameterizations have the same likelihood function. We demonstrated that result in practice by estimating a model due to Lubik and Schorfheide in two different ways and recovering parameter estimates that are statistically indistinguishable from theirs. We caution that, in practice, it is important to leave the VCV matrix of errors unrestricted for our results to apply. Our work should be of interest to economists who are interested in estimating models that do not impose a determinacy prior.

A Appendix A

Proof of Theorem 1. Let A^1 and A^2 be two orthonormal row operators associated with partitions $\mathbf{p}_1$ and $\mathbf{p}_2$;

$$\begin{bmatrix} z_t \\ \eta^1_{f,t} \\ \eta^1_{n,t} \end{bmatrix} = A^1 \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}, \quad \begin{bmatrix} z_t \\ \eta^2_{f,t} \\ \eta^2_{n,t} \end{bmatrix} = A^2 \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}. \tag{A1}$$

We assume that the operators, A^i have the form

$$A^i = \begin{bmatrix} \underset{l\times l}{I} & 0 \\ 0 & \underset{p\times p}{\tilde{A}^i} \end{bmatrix}, \tag{A2}$$

where $\tilde{A}^i$ is a permutation of the columns of an I_p identity matrix. Premultiplying the vector $[z_t, \eta_t]^T$ by the operator A^i permutes the rows of η_t while leaving the rows of z_t unchanged. Define matrices Ω_{ff} and Ω_{zf} for $i \in \{1, 2\}$ to be the new terms in the fundamental covariance matrix,

$$E\left(\begin{bmatrix} z_t \\ \eta^i_{f,t} \end{bmatrix} \begin{bmatrix} z_t \\ \eta^i_{f,t} \end{bmatrix}^T \right) = \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}.$$

Next, use (22) and (23) to write the non-fundamentals as linear functions of the fundamentals,

$$\eta^i_{n,t} = \Theta^i_z z_t + \Theta^i_f \eta^i_{f,t}, \tag{A3}$$

where

$$\Theta^i_z \equiv -\tilde{\Pi}^{-1}_{2n} \tilde{\Psi}_2, \quad \text{and} \quad \Theta^i_f \equiv -\tilde{\Pi}^{-1}_{2n} \tilde{\Pi}_{2f}, \tag{A4}$$

and define the matrix D^i,

$$D^i = \begin{bmatrix} \underset{l\times l}{I} & \underset{l\times m}{0} \\ \underset{m\times l}{0} & \underset{m\times m}{I} \\ \underset{(p-m)\times l}{\Theta_z^i} & \underset{(p-m)\times m}{\Theta_f^i} \end{bmatrix}. \tag{A5}$$

Using this definition, the covariance matrix of all shocks, fundamental and non-fundamental, has the following representation,

$$E\left(\begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}\begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}^T\right) = D^i \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix} D^{iT}. \tag{A6}$$

We can also combine the last two row blocks of D^i and write D^i as follows

$$D^i = \begin{bmatrix} \underset{l\times l}{I} & \underset{l\times m}{0} \\ \underset{p\times l}{D_{21}^i} & \underset{p\times m}{D_{22}^i} \end{bmatrix}, \tag{A7}$$

where,

$$D_{21}^i = \begin{bmatrix} \underset{m\times l}{0} \\ \underset{(p-m)\times l}{\Theta_z^i} \end{bmatrix}, \quad D_{22}^i = \begin{bmatrix} \underset{m\times m}{I} \\ \underset{(p-m)\times m}{\Theta_f^i} \end{bmatrix}. \tag{A8}$$

Using (A1) and the fact that A^i is orthonormal, we can write the following expression for the complete set of shocks

$$\begin{bmatrix} z_t \\ \eta_t \end{bmatrix} = A^{iT} \begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}. \tag{A9}$$

Using equations (A6) and (A9), it follows that

$$E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T\right) = B^i W^i B^{iT}, \quad \text{for all } \mathbf{p}_i \in \mathbf{P}, \tag{A10}$$

where

$$W^i \equiv \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}, \tag{A11}$$

and

$$B^i \equiv A^{iT} D^i = \begin{bmatrix} I & 0 \\ 0 & \tilde{A}^i \end{bmatrix}\begin{bmatrix} I & 0 \\ D^i_{21} & D^i_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^i_{21} & B^i_{22} \end{bmatrix}. \tag{A12}$$

Using this expression, we can write out equation (A10) in full to give,

$$E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T\right) = \begin{bmatrix} I & 0 \\ B^i_{21} & B^i_{22} \end{bmatrix}\begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}\begin{bmatrix} I & B^{iT}_{21} \\ 0 & B^{iT}_{22} \end{bmatrix}. \tag{A13}$$

We seek to establish that for any partition $\mathbf{p}_i$, parameterized by matrices Ω_{ff}, and Ω_{zf} that there exist matrices Ω_{ff} and Ω_{zf} for all partitions $\mathbf{p}_j \in \mathbf{P}, \quad j \neq i,$ such that

$$\Omega = E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T\right) = B^i W^i B^{iT} = B^j W^j B^{jT}. \tag{A14}$$

To establish this proposition, we write out the elements of (A13) explicitly. Since W^i and B^i are symmetric we need consider only the upper-triangular elements which give three equations in the matrices of Ω_{zf} and Ω_{ff},

$$\begin{aligned}
\Omega_{11} &= \Omega_{zz}, \\
\Omega_{12} &= \Omega_{zz}^{i} B_{21}^{iT} + \Omega_{zf} B_{22}^{iT}, \\
\Omega_{22} &= B_{21}^{i} \Omega_{zz}^{i} B_{21}^{iT} + 2 B_{21}^{i} \Omega_{zf} B_{22}^{iT} + B_{22}^{i} \Omega_{ff} B_{22}^{iT}.
\end{aligned} \tag{A15}$$

The first of these equations defines the covariance of the fundamental shocks and it holds for all i, j. Now define

$$a = vec\left(\Omega_{zz}\right), \quad x^{i} = vec\left(\Omega_{zf}\right), \quad y^{i} = vec\left(\Omega_{ff}\right). \tag{A16}$$

Using the fact that

$$vec\left(ABC\right) = \left(C^{T} \otimes A\right)\ vec\left(B\right), \tag{A17}$$

we can pass the vec operator through equation (A15) and write the following system of linear equations in the unknowns x^{j} and y^{j},

$$S^{i} \begin{bmatrix} x^{i} \\ y^{i} \end{bmatrix} + T^{i} a = S^{j} \begin{bmatrix} x^{j} \\ y^{j} \end{bmatrix} + T^{j} a, \tag{A18}$$

$$S^{k} = \begin{bmatrix} \left(B_{22}^{k} \otimes I\right) & 0 \\ \left(B_{22}^{k} \otimes B_{21}^{j}\right) & \left(B_{22}^{k} \otimes B_{22}^{k}\right) \end{bmatrix}, \quad T^{k} = \begin{bmatrix} \left(B_{21}^{k} \otimes I\right) \\ \left(B_{21}^{k} \otimes B_{21}^{k}\right) \end{bmatrix}, \quad k \in \{i, j\}. \tag{A19}$$

It follows from the assumption that the equilibrium is regular that S^{j} has full rank for all j hence for any permutation $\mathbf{p}_{i}$, parameterized by $\{x^{i}, y^{i}\}$we can find an alternative permutation $\mathbf{p}_{j}$ with associated parameterization $\{x^{j}, y^{j}\}$,

$$\begin{bmatrix} x^{j} \\ y^{j} \end{bmatrix} = \left(S^{j}\right)^{-1} \left(S^{i} \begin{bmatrix} x^{i} \\ y^{i} \end{bmatrix} + \left[T^{i} - T^{j}\right] a \right), \tag{A20}$$

that gives the same covariance matrix $\tilde{\Omega}$ for the fundamental and non-fundamental shocks. ■

B Appendix B

Proof of Lemma 1. We seek to characterize the full set of solutions to the equation,

$$\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{z_t} + \underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0. \tag{B1}$$

Let U, V and D_{11} characterize the singular value decomposition of $\tilde{\Pi}_2$,

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U}\left[\begin{array}{cc} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{array}\right]\underset{p\times p}{V^T}, \tag{B2}$$

and let θ_{FKN} characterize a regular indeterminate equilibrium for some partition $\mathbf{p}_i$. From Appendix A, equation A3, we write the non-fundamentals $\eta_{n,t}$ as functions of the fundamentals, where we omit the superscript i to reduce notation, and where Θ_z and Θ_f are functions of θ_1,

$$\underset{n\times 1}{\eta_{n,t}} = \underset{n\times\ell}{\Theta_z}\underset{\ell\times 1}{z_t} + \underset{n\times m}{\Theta_f}\underset{m\times 1}{\eta_{f,t}}. \tag{B3}$$

Equation (B3) connects the non-fundamental shocks $\eta_{n,t}$ to the fundamental shocks $\left[z_t, \eta_{f,t}\right]$ in the FKN equilibrium. Equation (33) reproduced below as (B4), characterizes the additional equations that define an LS equilibrium,

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1}\underset{n\times\ell}{N}\underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2}\underset{m\times\ell}{M_z}\underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}, \tag{B4}$$

To establish the connection between the LS and FKN representations we partition V,

$$\underset{p\times p}{V} = \left[\begin{array}{cc} \underset{n\times n}{V_{11}} & \underset{n\times m}{V_{12}} \\ \underset{m\times n}{V_{21}} & \underset{m\times m}{V_{22}} \end{array}\right], \quad \underset{p\times n}{V_1} \equiv \left[\begin{array}{c} \underset{n\times n}{V_{11}} \\ \underset{m\times n}{V_{21}} \end{array}\right], \quad \underset{p\times m}{V_2} \equiv \left[\begin{array}{c} \underset{n\times m}{V_{21}} \\ \underset{m\times m}{V_{22}} \end{array}\right],$$

and split the equations of (B4) into two blocks,

$$\underset{n\times 1}{\eta_{n,t}} = \underset{n\times n}{V_{11}}\underset{n\times \ell}{N}\underset{\ell\times 1}{z_t} + \underset{n\times m}{V_{21}}\underset{m\times \ell}{M_z}\underset{\ell\times 1}{z_t} + \underset{n\times m}{V_{21}}\underset{m\times 1}{\zeta_t}, \tag{B5}$$

$$\underset{m\times 1}{\eta_{f,t}} = \underset{m\times n}{V_{21}}\underset{n\times \ell}{N}\underset{\ell\times 1}{z_t} + \underset{m\times m}{V_{22}}\underset{m\times \ell}{M_z}\underset{\ell\times 1}{z_t} + \underset{m\times m}{V_{22}}\underset{m\times 1}{\zeta_t}. \tag{B6}$$

Using (B3) to replacing $\eta_{n,t}$ in (B5),

$$\begin{bmatrix} \underset{n\times m}{\Theta_f} \\ I_m \end{bmatrix}\underset{m\times 1}{\eta_{f,t}} = \underset{p\times n}{V_1}\underset{n\times \ell}{N}\underset{\ell\times 1}{z_t} - \begin{bmatrix} \underset{n\times \ell}{\Theta_z} \\ \underset{m\times \ell}{\mathbf{0}} \end{bmatrix}\underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2}\underset{m\times \ell}{M_z}\underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}. \tag{B7}$$

Premultiplying (B7) by V_2^T and exploiting the fact that V is orthonormal, leads to the equation

$$\underset{m\times m}{G}\underset{m\times 1}{\eta_{f,t}} = \underset{m\times \ell}{H}\underset{\ell\times 1}{z_t} + \underset{m\times \ell}{M_z}\underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t}, \tag{B8}$$

where

$$\underset{m\times m}{G} = \underset{m\times n}{V_{21}^T}\underset{n\times m}{\Theta_f} + \underset{m\times m}{V_{22}^T}, \quad \text{and} \quad \underset{m\times \ell}{H} = \underset{m\times p}{V_2^T}\underset{p\times n}{V_1}\underset{n\times \ell}{N} - \underset{m\times n}{V_{21}^T}\underset{n\times \ell}{\Theta_z}. \tag{B9}$$

Rearranging (B8) and defining

$$\underset{m\times \ell}{S} = \underset{m\times \ell}{H} + \underset{m\times \ell}{M_z} \tag{B10}$$

gives

$$\underset{m\times 1}{\zeta_t} = \underset{m\times m}{G}\underset{m\times 1}{\eta_{f,t}} - \underset{m\times \ell}{S}\underset{\ell\times 1}{z_t}, \tag{B11}$$

which is the expression we seek. ■

C Appendix C

Proof of Theorem 2. Let $\theta_{FKN} = \{\theta_1, \theta_2\}$ characterize an FKN equilibrium. From (B8), which we repeat below,

$$\underset{m\times m}{G}\underset{m\times 1}{\eta_{f,t}} = \underset{m\times\ell}{H}\underset{\ell\times 1}{z_t} + \underset{m\times\ell}{M_z}\underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t}. \tag{C1}$$

Post-multiplying this equation by z_t^T and taking expectations gives

$$\underset{m\times m}{G}\underset{m\times\ell}{\Omega_{fz}} = \underset{m\times\ell}{H}\underset{\ell\times\ell}{\Omega_{zz}} + \underset{m\times\ell}{M_z}\underset{\ell\times\ell}{\Omega_{zz}}, \tag{C2}$$

which represents $m\times\ell$ linear equations in the $m\times\ell$ elements of $vec\,(M_z)$ as functions of the elements of H, G and Ω_{zz}, (these are functions of θ_1), and Ω_{fz} (these are elements of θ_2). Applying the vec operator to (C2), using the algebra of Kronecker products, and rearranging terms gives the following solution for the parameters $vec\,(M_z)$,

$$\underset{(m\times\ell)\times 1}{vec\,(M_z)} =$$

$$\underset{(m\times\ell)\times(\ell\times m)}{(\Omega_{zz}\otimes I_m)^{-1}}\left[\underset{(m\times\ell)\times(\ell\times m)}{\left(I_\ell\otimes G^T\right)}\underset{(m\times\ell)\times 1}{vec\,(\Omega_{fz})} - \underset{(m\times\ell)\times\ell^2}{\left(I_\ell\otimes H^T\right)}\underset{\ell^2\times 1}{vec\,(\Omega_{zz})}\right]. \tag{C3}$$

Using equation (C3) we can construct an expression for the elements of S as functions of θ_1 and θ_2. Post-multiplying equation (B11) by itself transposed, and taking expectations, we have

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \underset{m\times m}{G}\underset{m\times m}{\Omega_{ff}}\underset{m\times m}{G^T} - \underset{m\times m}{G}\underset{m\times\ell}{\Omega_{fz}}\underset{\ell\times m}{S^T} - \underset{m\times\ell}{S}\underset{\ell\times m}{\Omega_{zf}}\underset{m\times m}{G^T} + \underset{m\times\ell}{S}\underset{\ell\times\ell}{\Omega_{zz}}\underset{\ell\times m}{S^T}. \tag{C4}$$

The terms on the RHS of (C4) are all functions of the known elements of θ_1 and θ_2. Since the matrix $\Omega_{\zeta\zeta}$ is symmetric, this gives $m\times(m+1)/2$ equations that determine the parameters of $vec\,(\Omega_{\zeta\zeta})$. This establishes that every $\theta_{FKN}\in\Theta_{FKN}$ defines a unique parameter vector $\theta_{LS}\in\Theta_{LS}$. To prove the converse, solve equation (C3) for $vec(\Omega_{fz})$ as a function of θ_1 and the elements of M_z and apply the vec operator to (C4) to solve for $vec(\Omega_{ff})$ in terms of θ_1 and $vec(\Omega_{\zeta\zeta})$. ■

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