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INVESTMENT, TOBIN'S Q, AND INTEREST RATES

Chong Wang
Neng Wang
Jinqiang Yang

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Chong Wang, Neng Wang, and Jinqiang Yang
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ABSTRACT

The interest rate is a key determinant of firm investment. We integrate a widely used term structure model of interest rates, CIR (Cox, Ingersoll, and Ross (1985)), with the q theory of investment (Hayashi (1982) and Abel and Eberly (1994)). We show that stochastic interest rates have significant effects on investment and firm value because capital is medium/long lived. Capital adjustment costs have a first-order effect on investment and firm value. We use duration to measure the interest rate sensitivity of firm value, decompose a firm into assets in place and growth opportunities, and value each component. By extending the model to allow for endogenous capital liquidation, we find that the liquidation option provides a valuable protection against the increase of interest rates. We further generalize the model to incorporate asymmetric adjustment costs, a price wedge between purchasing and selling capital, fixed investment costs, and irreversibility. We find that inaction is often optimal for an empirically relevant range of interest rates for firms facing fixed costs or price wedges. Finally, marginal q is equal to average q in our stochastic interest rate settings, including one with serially correlated productivity shocks.

Chong Wang
Naval Postgraduate School
Graduate School of Business and Public Policy
1 University Circle
Monterey, CA 93943
cwang@nps.edu

Neng Wang
Columbia Business School
3022 Broadway, Uris Hall 812
New York, NY 10027
and NBER
nw2128@columbia.edu

Jinqiang Yang
Shanghai University of Finance
   and Economics
Guoding Rd. 777
Shanghai, 200433
China
huda518@gmail.com

1 Introduction

If a firm can frictionlessly adjust its capital stock, its investment in each period is essentially a static choice of "target" capital stock, which optimally equates the marginal product of capital with the user cost of capital (Jorgenson (1963)). However, changing capital stock often incurs various adjustment costs. Installing new equipment or upgrading capital may require time and resources, and lead to disruptions in production lines. Workers need to go through a costly learning process to operate newly installed capital. The complete/partial irreversibility of business projects is another type of adjustment cost. Lacking secondary markets for capital may generate a price wedge between purchasing and selling capital. Additionally, informational asymmetries and agency conflicts distort investment, which may be captured by adjustment costs as an approximation. These frictions, modeled by various capital adjustment costs, prevent the firm from instantaneously adjusting its capital stock to the target level, and make the firm's investment decision intrinsically dynamic.

The intertemporal optimizing framework with capital adjustment costs has become known as the q theory of investment.[1] Almost all existing work in the q literature assumes that interest rates are constant over time. However, interest rates are persistent, volatile, and carry risk premia. Additionally, physical capital is medium- and/or long-term lived, making the value of capital sensitive to movements of interest rates. Moreover, adjustment costs make capital illiquid and hence capital carries an additional illiquidity premium, which depends on the interest rate level, persistence, and risk premium.

Theoretically, it is appealing that stochastic interest rates have an important effect on corporate investment. We incorporate a widely-used term structure model of interest rates, Cox, Ingersoll, and Ross (1985), henceforth CIR, into a widely used q model of investment, a stochastic version of the seminal Hayashi (1982). Our model incorporates an interest rate risk premium, rules out arbitrage, and is quantitatively suitable to value a firm. We show that Tobin's q is stochastic and depends on the interest rate and the risk-adjusted expectation of its future evolution. Additionally, our work is also inspired by empirical work on the relation between the cost of capital and investment.

Abel and Blanchard (1986) document that variations in the expected present value of

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[1] Abel and Eberly (1994) develop a unified neoclassical q theory of investment. Lucas and Prescott (1971), Mussa (1977), Lucas (1981), Hayashi (1982), and Abel (1983) are important early contributors. See Caballero (1999) for a survey on investment.

marginal profits, i.e. marginal q, are more due to variation in the cost of capital than to variations in marginal profit. Gilchrist and Zakrajsek (2007) report that one percentage point increase in the user cost of capital implies a reduction in investment of 50 to 75 basis points and in the long run, a one percent reduction in the stock of capital. Guiso, Kashyap, Panetta, and Terlizzese (2002) find that investment is very sensitive to interest rate changes, using a unique Italian dataset with 30,000 firms over 10 years. Cross sectionally, Dew-Becker (2011) provides evidence that high term spread is associated with low average duration for investment. Despite theoretical appeal and some empirical evidence, the profession lacks consensus on the effect of the cost of capital on investment.[2] We point out that to fundamentally address the effects of interest rates on investment, incorporating a term structure model with the neoclassical q theory of investment is necessary.

Our baseline model includes minimal but essential elements. The firm faces convex capital adjustment costs and operates a constant return to scale production technology with independently and identically distributed (iid) productivity shocks. For simplicity, we assume that the adjustment cost function is homogeneous in investment and capital as in Lucas and Prescott (1971) and Hayashi (1982). The interest rate process is governed by the CIR term structure model. Even with stochastic interest rates, our framework generates the result that the marginal q is equal to Tobin's average q,[3] thus extending the condition for this equality result given by Hayashi (1982) in a deterministic setting to a stochastic interest rate environment. Our parsimonious framework yields tractable solutions for investment and firm value. We derive an ordinary differential equation (ODE) for Tobin's q. As we expect, investment and firm value are decreasing and convex in interest rates.

Existing q models generate rich investment behavior from interactions between persistent productivity shocks and adjustment costs, but under constant interest rates. These models work through the cash flow channel. Unlike them, we focus on the effects of stochastic interest rates on firm value and investment, by intentionally choosing iid productivity shocks to rule out the effects of time-varying investment opportunities. Time-series variation of investment and q in our model thus is driven by interest rates. Empirically, both productivity shocks and interest rates are likely to have significant effects. Our work thus complements the existing

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[2]See Chirinko (1993) and Caballero (1999) for surveys.

[3]Tobin's average q is the ratio between the market value of capital to its replacement cost, which was originally proposed by Brainard and Tobin (1968) and Tobin (1969) to measure a firm's incentive to invest.

literature by demonstrating the importance of the interest rate channel.

Calibrating our model to the US data, we find that interest rates and adjustment costs interact with each other and have quantitatively significant effects on investment and firm value. As in fixed-income analysis, we use duration to measure the interest rate sensitivity of firm value. We decompose a firm into assets in place and growth opportunities (GO). While the value of assets in place decreases with interest rates for the standard discount rate effect, the value of GO may either decrease or increase with interest rates due to two opposing effects. In addition to the standard discount rate effect, there is also a cash flow effect for GO: increasing interest rates discourages investment, lowers adjustment costs, and thus increases the firm's expected cash flows and the value of GO, *ceteris paribus*. As adjustment costs increase, capital becomes more illiquid and the relative weight of assets in place in firm value increases. In the limit, with infinity adjustment costs and thus completely illiquid capital, the firm is simply its assets in place with no GO.

For simplicity, we have chosen the widely-used convex adjustment costs for the baseline model. However, investment frictions may not be well captured by symmetric convex adjustment costs. For example, increasing capital stock is often less costly than decreasing capital stock, thus suggesting an asymmetric adjustment cost. Additionally, the firm may pay fixed costs when investing,[4] may face a price wedge between purchase and sale prices of capital, and investment may be completely or partially irreversible. Optimal investment may thus be lumpy and inaction may sometimes be optimal. Abel and Eberly (1994) develop a unified q theory of investment with a rich specification of adjustment costs.[5] We further generalize our baseline model with stochastic interest rates by incorporating a much richer specification of adjustment costs as in Abel and Eberly (1994).

If a firm can liquidate its capital at a scrap value, it will optimally choose the liquidation strategy which provides a valuable protection against the increase of interest rates. For a firm facing either fixed costs or a price wedge between purchase and sale of capital, the firm's

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[4]There are two forms of fixed costs, *stock* and *flow* fixed costs. In this paper, we focus on flow fixed costs as in Abel and Eberly (1994). Quoting the analogy for the two forms of fixed costs in Caballero and Leahy (1996), "stock fixed costs are the costs of turning on a tap independent of how much water flows through it or how long the water flows, whereas flow fixed costs are the costs of running the tap per unit of time water flows and is independent of how much water flows." Caballero and Leahy (1996) show that investment is no longer monotonically increasing with marginal q in the presence of stock fixed costs of adjustment.

[5]Stokey (2009) provides a modern textbook treatment of the economics of optimal inaction in a continuous-time framework.

optimal investment policy is generally characterized by three regions: positive investment, inaction, and divestment, with endogenously determined interest rate cutoff levels. Positive investment is optimal when interest rates are sufficiently low. At high interest rates, the firm optimally divests. For intermediary interest rates between these two cutoff levels, inaction is optimal. We further extend our model to a regime-switching setting with persistent productivity shocks. Despite stochastic interest rates and a wide array of adjustment costs, our model has the property that the marginal q is equal to Tobin's average q in all settings.

Cochrane (1991, 1996), Jermann (1998), and Zhang (2005) study the implications of the firm's intertemporal production decisions on asset pricing. No arbitrage implies that all traded claims (including firm value) earns risk-free rate after proper risk adjustments. While the production-based asset pricing literature often study equity returns in a q-theoretic framework, we focus on the effects of term structure of interest rates (level, persistence, and risk premium) on corporate investment and Tobin's q.

2   Model

We generalize the neoclassic q theory of investment to incorporate the effects of stochastic interest rates on investment and firm value.

Physical production and investment technology.   A firm uses its capital to produce output.[6] Let K and I denote respectively its capital stock and gross investment. Capital accumulation is given by

$$dK_t = (I_t - \delta K_t)\, dt, \quad t \geq 0, \tag{1}$$

where $\delta \geq 0$ is the rate of depreciation for capital stock.

The firm's operating revenue over time period $(t, t+dt)$ is proportional to its time-t capital stock K_t, and is given by $K_t dX_t$, where dX_t is the firm's productivity shock over the same time period $(t, t + dt)$. After incorporating the systematic risk for the firm's productivity

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[6]The firm may use both capital and labor as factors of production. As a simple example, we may embed a static labor demand problem within our dynamic optimization. We will have an effective revenue function with optimal labor demand. The remaining dynamic optimality will be the same as the one in q theory. See Abel and Eberly (2011) for an example of such a treatment.

shock, we may write the productivity shock dX_t under the risk-neutral measure[7] as follows,

$$dX_t = \pi dt + \epsilon dZ_t, \quad t \geq 0, \tag{2}$$

where Z is a standard Brownian motion. The productivity shock dX_t specified in (2) is independently and identically distributed (iid). The constant parameters π and $\epsilon > 0$ give the corresponding (risk-adjusted) productivity mean and volatility per unit of time.

The firm's operating profit dY_t over the same period $(t, t + dt)$ is given by

$$dY_t = K_t dX_t - C(I_t, K_t)dt, \quad t \geq 0, \tag{3}$$

where $C(I, K)$ is the total cost of the investment including both the purchase cost of the investment good and the additional adjustment costs of changing capital stock. The firm may sometimes find it optimal to divest and sell its capital, $I < 0$. Importantly, capital adjustment costs make installed capital more valuable than new investment goods. The ratio between the market value of capital and its replacement cost, often referred to as Tobin's q, provides a measure of rents accrued to installed capital. The capital adjustment cost plays a critical role in the neoclassical q theory of investment.

Stochastic interest rates. While much work in the q theory context assumes constant interest rates, empirically, there is much time-series variation in interest rates. Additionally, the interest rate movement is persistent and has systematic risk. Moreover, the investment payoffs are often long term in nature and hence cash flows from investment payoffs are sensitive to the expected change and volatility of interest rates. In sum, interest rate dynamics and risk premium have significant impact on investment and firm value.

Researchers often analyze effects of interest rates via comparative statics with respect to interest rates (using the solution from a dynamic model with a constant interest rate). While potentially offering insights, the comparative static analysis is unsatisfactory because it ignores the dynamics and the risk premium of interest rates. By explicitly incorporating a term structure of interest rates, we analyze the persistence and volatility effects of interest rates on investment and firm value in a fully specified dynamic stochastic framework.

We choose the widely-used CIR model, which specifies the following dynamics for r:

$$dr_t = \mu(r_t)dt + \sigma(r_t)dB_t, \quad t \geq 0, \tag{4}$$

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[7]The risk-neutral measure incorporates the impact of the interest rate risk on investment and firm value.

where B is a standard Brownian motion under the risk-neutral measure, and the risk-neutral drift $\mu(r)$ and volatility $\sigma(r)$ are respectively given by

$$\mu(r) = \kappa(\xi - r), \tag{5}$$

$$\sigma(r) = \nu\sqrt{r}. \tag{6}$$

Both the (risk-adjusted) conditional mean and the conditional variance of the interest rate change are linear in r. The parameter κ measures mean reversion of interest rates. The implied first-order autoregressive coefficient in the corresponding discrete-time model is $e^{-\kappa}$. The higher κ, the more mean-reverting. The parameter ξ is the long-run mean of interest rates. The CIR model captures the mean-reversion and conditional heteroskedasticity of interest rates, and belongs to the widely-used affine models of interest rates.[8]

For simplicity, we assume that interest rate risk and the productivity shock are uncorrelated, i.e. the correlation coefficient between the Brownian motion B driving the interest rate process (4) and the Brownian motion Z driving the productivity process (2) is zero.

Firm's objective. While our model features stochastic interest rates and real frictions such as capital adjustment costs, financial markets are frictionless and the Modigliani-Miller theorem holds. The firm chooses investment I to maximize its market value defined below:

$$\mathbb{E}\left[\int_0^\infty e^{-\int_0^t r_s ds} dY_t\right], \tag{7}$$

where the interest rate process r under the risk-neutral measure is given by (4) and the risk-adjusted cash-flow process dY is given by (3). The expectation in (7) is for the risk-neutral measure, which incorporates the interest rate risk premium. The infinite-horizon setting keeps the model stationary and allows us to focus on the effect of stochastic interest rates.

3 Solution

With stochastic interest rates, the firm's investment decision naturally depends on the current value and future evolution of interest rates. Hence, both investment and the value of

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[8]Vasicek (1977) is the other well known one-factor model. However, this process is less desirable because it implies conditionally homoskedastic (normally distributed) shocks and allow interest rates to be unbounded from below. Vasicek and CIR models belong to the "affine" class of models. See Duffie and Kan (1996) for multi-factor affine term-structure models and Dai and Singleton (2000) for estimation of three-factor affine models. Piazzesi (2010) provides a survey on affine term structure models.

capital are time-varying even when firms face iid productivity shocks.

Investment and Tobin's q in the interior interest rate region $0 < r < \infty$. Let $V(K, r)$ denote firm value. Using the standard principle of optimality, we have the following Hamilton-Jacobi-Bellman (HJB) equation,

$$rV(K,r) = \max_{I} \ (\pi K - C(I,K)) + (I - \delta K) V_K(K,r) + \mu(r)V_r(K,r) + \frac{\sigma^2(r)}{2}V_{rr}(K,r). \quad (8)$$

The first term on the right side of (8) gives the firm's risk-adjusted expected cash flows. The second term gives the effect of adjusting capital on firm value. The last two terms give the drift and volatility effects of interest rate changes on $V(K, r)$. The firm optimally chooses investment I by setting its expected rate of return to the risk-free rate after risk adjustments.

Let $q(K, r)$ denote the marginal value of capital, which is also known as the firm's marginal q, $q(K, r) = V_K(K, r)$. The first-order condition (FOC) for investment I is

$$V_K(K, r) = C_I(I, K), \quad (9)$$

which equates $q(K, r)$ with the marginal cost of investing $C_I(I, K)$. With convex adjustment costs, the second-order condition (SOC) is satisfied, and hence the FOC characterizes investment optimality. Let I^* denote the optimal investment implied by (9). The firm's marginal q, $q(K, r)$, solves the following differential equation,

$$(r+\delta)q(K,r) = (\pi - C_K(I^*,K)) + (I^* - \delta K) q_K(K,r) + \mu(r)q_r(K,r) + \frac{\sigma^2(r)}{2}q_{rr}(K,r). \quad (10)$$

Homogeneity of the adjustment cost function $C(I, K)$. For analytical simplicity, we further assume that the firm's total investment cost is homogeneous of degree one in I and K. We may write $C(I, K)$ as follows,

$$C(I, K) = c(i)K, \quad (11)$$

where $i = I/K$ is the investment-capital ratio, and $c(i)$ is an increasing and convex function.[9] The convexity of $c(\cdot)$ implies that the marginal cost of investing $C_I(I, K) = c'(i)$

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[9]Lucas (1981), Hayashi (1982), and Abel and Blanchard (1983) specify the adjustment cost to be convex and homogenous in I and K. While in this paper, we have specified the adjustment cost on the "cost" side, we can also effectively specify the effect of adjustment costs on the "revenue" side by choosing a concave installation function in the "drift" of the capital accumulation equation (1) and obtain effectively similar results. See Lucas and Prescott (1971), Baxter and Crucini (1993), and Jermann (1998) for examples which specify the adjustment cost via a concave installation function for capital from one period to the next.

is increasing in i, and hence encourages the firm to smooth investment over time, *ceteris paribus*. The production specification (1)-(11) features the widely used "AK" technology and the homogeneous adjustment cost function in macroeconomics.

Investment and Tobin's q at the boundaries: $r = 0$ and $r \to \infty$. First, consider the situation at $r = 0$. Equation (8) implies the following boundary condition,

$$\max_I \pi K - C(I, K) + (I - \delta K) V_K(K, 0) + \kappa \xi V_r(K, 0) = 0. \tag{12}$$

As $r \to \infty$, the time value of money vanishes and firm value approaches zero, i.e.

$$\lim_{r \to \infty} V(K, r) = 0. \tag{13}$$

We next use the homogeneity property to simplify our analysis.

The homogeneity property of firm value $V(K, r)$. There are two state variables: capital K and interest rate r. Despite the stochastic interest rates, our model features the homogeneity property. We may write firm value as follows:

$$V(K, r) = K \cdot q(r), \tag{14}$$

where $q(r)$ is both average and marginal q. The homogeneity property implies that $V(K, r)$ is proportional to K. We now characterize $q(r)$, firm value per unit of capital.

For expositional simplicity, we specify $c(i)$ as the following quadratic function,

$$c(i) = i + \frac{\theta}{2} i^2, \tag{15}$$

where the price of the investment good is normalized to unity and the quadratic term gives the capital adjustment costs with θ as the adjustment cost parameter. The next theorem summarizes the main results on optimal investment and $q(r)$.

Theorem 1 *Tobin's q, $q(r)$, solves the following ordinary differential equation (ODE),*

$$(r + \delta) q(r) = \pi + \frac{(q(r) - 1)^2}{2\theta} + \mu(r) q'(r) + \frac{\sigma^2(r)}{2} q''(r), \tag{16}$$

subject to the following boundary conditions,

$$\pi - \delta q(0) + \frac{(q(0) - 1)^2}{2\theta} + \kappa \xi q'(0) = 0, \tag{17}$$

$$\lim_{r \to \infty} q(r) = 0. \tag{18}$$

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The optimal investment $i(r)$ is linearly related to $q(r)$ as follows,

$$i(r) = \frac{q(r) - 1}{\theta}. \tag{19}$$

Equation (17) describes Tobin's q at $r = 0$, and (18) states that $q = 0$ as $r \to \infty$. The ODE (16) and the boundary conditions (17)-(18) jointly characterize $q(r)$. Equation (19) gives the optimal $i(r)$ as an increasing function of $q(r)$. Before analyzing the impact of stochastic interest rates on q, we summarize the results with constant interest rates.

4   A benchmark: constant interest rates

We now provide closed-form solutions for investment and Tobin's q when $r_t = r$ for all t. This special case is effectively Hayashi (1982) with iid productivity shocks. To ensure that investment opportunities are not too attractive so that firm value is finite, we assume

$$(r + \delta)^2 - 2 \left(\pi - (r + \delta) \right) / \theta > 0. \tag{20}$$

The following proposition summarizes the main results.

Proposition 1 *With constant interest rate, $r_t = r$ for all t, and under the convergence condition (20), firm value equals $V = qK$, where Tobin's q is given by*

$$q = 1 + \theta i, \tag{21}$$

and the optimal investment-capital ratio $i = I/K$ is constant and equals

$$i = r + \delta - \sqrt{(r + \delta)^2 - \frac{2}{\theta} \left(\pi - (r + \delta) \right)}. \tag{22}$$

First, due to the homogeneity property, marginal q equals average q as in Hayashi (1982). Second, if and only if the expected productivity π is higher than $(r+\delta)$, the gross investment is positive, the installed capital earns rents, and hence Tobin's q is greater than unity. Third, the idiosyncratic volatility ϵ has no effects on investment and q. This is the certainty equivalence result for a linear-quadratic regulator applied to our setting.[10] The firm grows at a constant rate regardless of past realized productivity shocks.

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[10]See Ljungqvist and Sargent (2004) for a macroeconomics textbook treatment of the certainty equivalence result for linear-quadratic regulators in discrete-time settings.

5 The general case: Stochastic interest rates

First, we specify the risk premia. We then calibrate the model and provide a quantitative analysis of the effects of stochastic interest rates on investment and firm value. Finally, we value the firm by decomposing it into assets in place and growth opportunities.

5.1 Risk premia

As in CIR, we assume that the interest rate risk premium is given by $\lambda\sqrt{r}$, where λ is a constant that measures the sensitivity of risk premium with respect to r. By the no-arbitrage principle, we have the following dynamics for the interest rate under the physical measure,[11]

$$dr_t = \mu^P(r_t)dt + \sigma(r_t)dB_t^P,\tag{23}$$

where B^P is a standard Brownian motion, and the drift $\mu^P(r)$ is given by

$$\mu^P(r) = \kappa(\xi - r) + \nu\lambda r = \kappa^P(\xi^P - r),\tag{24}$$

and

$$\kappa^P = \kappa - \lambda\nu,\tag{25}$$
$$\xi^P = \frac{\kappa\xi}{\kappa - \lambda\nu}.\tag{26}$$

The parameter κ^P given in (25) measures the speed of mean reversion under the physical measure. The higher κ^P, the more mean-reverting. We require $\kappa^P > 0$ to ensure stationarity. The parameter ξ^P given in (26) measures the long-run mean of the interest rate under the physical measure. Note that the volatility function under the physical measure is $\sigma(r) = \nu\sqrt{r}$, the same as the one under the risk-neutral measure given by (6). Note that under both the physical and the risk-neutral measures, the interest rate follows a square-root process.

We now specify the risk premium associated with the productivity shock. Let ρ denote the correlation coefficient between the firm's productivity shock and the aggregate productivity shock. Write the firm's productivity shock dX_t under the physical measure as follows,

$$dX_t = \pi^P dt + \epsilon dZ_t^P,\tag{27}$$

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[11]Using the Girsanov theorem, we relate the Brownian motion under the physical measure, B^P, to the Brownian motion under the risk-neutral measure, B, by $dB_t = dB_t^P + \lambda\sqrt{r_t}dt$. See Duffie (2002).

where Z_t^P is a standard Brownian motion driving X under the physical measure. The drift for X under the physical measure, π^P, is linked to the risk-neutral drift π as follows,

$$\pi^P = \pi + \rho\eta\epsilon, \tag{28}$$

where η captures the aggregate risk premium per unit of volatility.[12]

5.2 Parameter choices

We now choose the parameter values. Downing, Jaffee and Wallace (2009) estimate the parameter values for the CIR interest rate process, using the methodology of Pearson and Sun (1994) and daily data on constant maturity 3-month and 10-year Treasury rates for the period 1968-2006.[13] Whenever applicable, all parameter values are annualized. Their estimates are: the persistence parameter $\kappa^P = 0.1313$, the long-run mean $\xi^P = 0.0574$, the volatility parameter is $\nu = 0.0604$, and the risk premium parameter $\lambda = -1.2555$. Negative interest rate premium ($\lambda < 0$) implies that the interest rate is more persistent ($\kappa < \kappa^P$) and is higher on average ($\xi > \xi^P$) after risk adjustments. Under the risk-neutral measure, we have the persistence parameter $\kappa = 0.0555$, the long-run mean $\xi = 0.1359$, and the volatility parameter $\nu = 0.0604$. No arbitrage/equilibrium implies that the volatility parameter remains unchanged.

We choose the capital depreciation rate $\delta = 0.09$. The mean and volatility of the risk-adjusted productivity shock are $\pi = 0.18$ and $\epsilon = 0.09$, respectively, which are in line with the estimates of Eberly, Rebelo, and Vincent (2009) for large US firms. We consider three levels of the adjustment cost parameter, $\theta = 2, 5, 20$, which span the range of empirical estimates in the literature.[14]

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[12]As for the interest rate analysis, we apply the Girsanov theorem to link the Brownian motions for the productivity shocks under the risk-neutral and physical measures via $dZ_t = dZ_t^P + \rho\eta dt$.

[13]Stanton (1995) uses a similar strategy in testing a prepayment model for mortgage-backed securities.

[14]The estimates of the adjustment cost parameter vary significantly in the literature. Procedures based on neoclassic (homogeneity-based) q theory of investment (e.g. Hayashi (1982)) and aggregate data on Tobin's q and investment typically give a high estimate for the adjustment cost parameter θ. Gilchrist and Himmelberg (1995) estimate the parameter to be around 3 using unconstrained subsamples of firms with bond rating. Hall (2004) specifies quadratic adjustment costs for both labour and capital, and finds a low average (across industries) value of $\theta = 1$ for capital. Whited (1992) estimates the adjustment cost parameter to be 1.5 in a q model with financial constraints. Cooper and Haltiwanger (2006) estimate a value of the adjustment cost parameter lower than 1 in a model with fixed costs and decreasing returns to scale. Eberly, Rebelo, and Vincent (2009) estimate a value θ around 7 for large US firms in a homogeneous stochastic framework of Hayashi (1982) with regime-switching productivity shocks.
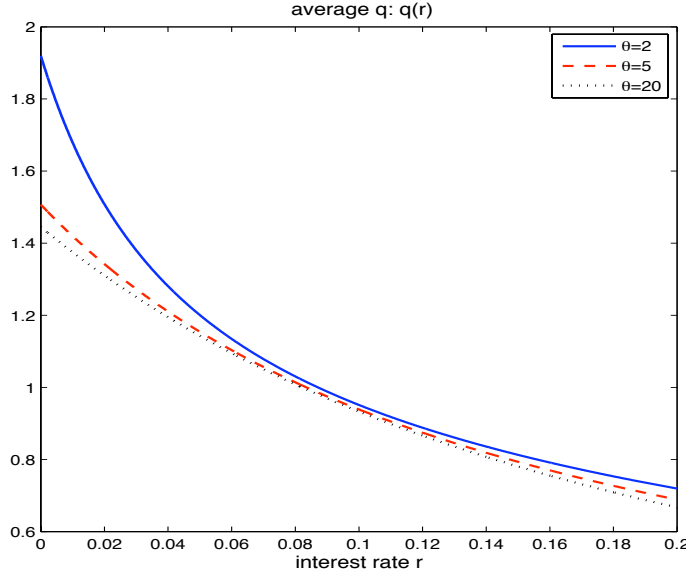
Figure 1: **Tobin's average q**

With constant r, both the optimal investment-capital ratio i and Tobin's q are constant, as in Hayashi (1982). Fix $r = \xi^P = 0.0574$. For $\theta = 5$, we have $i = 0.054$ and $q = 1.271$. With a high adjustment cost ($\theta = 20$), $i = 0.011$, and Tobin's q is lowered to $q = 1.221$. For $\theta = 2$ and with constant r, firm value is no longer finite because of the low adjustment costs. Firm value, however, becomes finite when interest rates are stochastic.

5.3 Investment, Tobin's q, and duration

Figure 1 plots Tobin's $q(r)$ for $\theta = 2, 5, 20$. First, the lower the adjustment cost parameter θ, the more productive capital and hence the higher Tobin's $q(r)$. Second, $q(r)$ is decreasing and convex in r. As we expect, firm value is quite sensitive to interest rate movements. For example, with $\theta = 2$, Tobin's q at $r = 0$ is $q(0) = 1.921$, which is significantly higher than $q(\xi^P) = 1.151$ at its long-run mean, $\xi^P = 0.0574$. The firm loses 40% of its value (from 1.921 to 1.151) when the interest rate increases from 0 to 0.0574.

Panels A and B of Figure 2 plots the optimal investment-capital ratio $i(r)$ and investment sensitivity with respect to interest rate r, respectively, also for $\theta = 2, 5, 20$. First, high interest rates discourage investment, and thus $i(r)$ decreases in r. Second, investment changes more with respect to changes in r with a lower adjustment cost θ. For example, with $\theta = 2$,
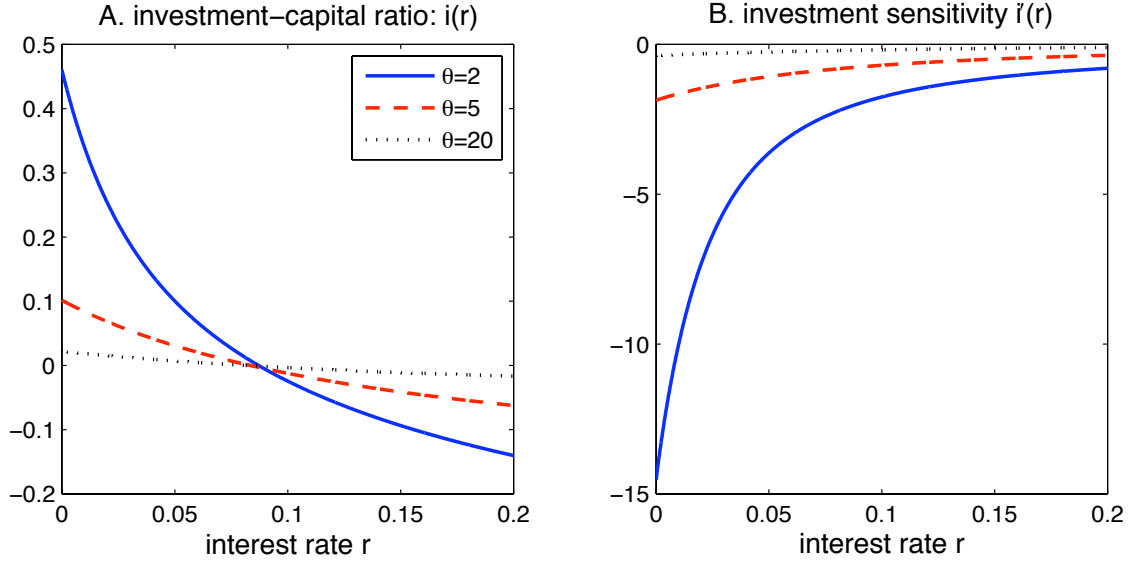
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Figure 2: **The investment-capital ratio $i(r)$ and its interest sensitivity $i'(r)$**

investment decreases significantly from 0.461 to 0.075 when r increases from 0 to 0.0574. With $\theta = 20$, while investment still decreases in r, the rate of change is much smaller (flatter). As a result, at high interest rates, a firm with a higher θ invests more, while the opposite holds at low interest rates, i.e. a firm with a higher θ invests less. Panel B plots $i'(r)$. The sensitivity $i'(r)$ critically depends on the level of r and capital illiquidity measured by θ. For example, at the long-run mean $r = \xi^p = 0.0574$, $i'(r) = -3.17$ when $\theta = 2$, but $i'(r) = -0.24$ when $\theta = 20$. That is, interest sensitivity may sometimes be difficult to detect empirically, simply because of the adjustment cost. With a sufficiently high adjustment cost, the firm adjusts much less in response to changes in interest rates.

To measure the sensitivity of firm value with respect to changes of interest rates, we define duration for firm value as follows (motivated by duration in fixed-income analysis),

$$D(r) = -\frac{1}{V(K,r)}\frac{dV(K,r)}{dr} = -\frac{q'(r)}{q(r)}, \qquad (29)$$

where the last equality follows from the homogeneity property, $V(K,r) = q(r)K$. Figure 3 plots duration for firm value, $D(r)$, as a function of r for $\theta = 2, 5, 20$. At low interest rates, duration is higher for a firm with a lower adjustment cost because higher investment in the future makes firm value more sensitive to interest rates. However, the opposite holds true at high interest rates. That is, duration is higher for a firm with a higher adjustment cost when
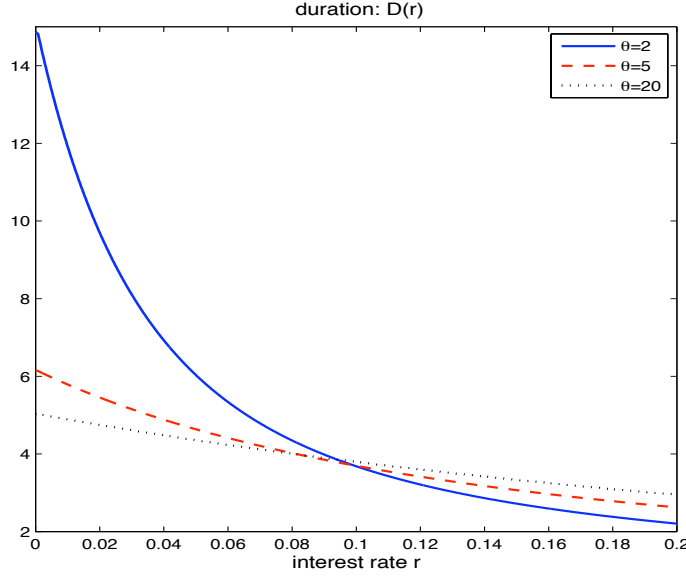
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Figure 3: **Duration for firm value,** $D(r)$

interest rates are high, because such a firm divests less and hence installed capital is longer-lived, *ceteris paribus.* The quantitative effects of r on duration are significant, particularly with a low adjustment cost θ. We next provide a decomposition of firm value.

5.4   Decomposition: Assets in place and growth opportunities

Using this decomposition, we separate the impact of interest rates on assets in place and growth opportunities (GO), and quantify their separate contributions to firm value.

Assets in place.   Let $A(K, r)$ denote the value of assets in place, which is the present discounted value of future cash flows generated by existing capital stock without any further investment/divestment in the future, i.e. by permanently setting $I = 0$. Using the homogeneity property, we have $A(K, r) = a(r) \cdot K$. No gross investment ($I = 0$) implies that $a(r)$ solves the following linear ODE,

$$(r + \delta)\, a(r) = \pi + \mu(r)a'(r) + \frac{\sigma^2(r)}{2} a''(r)\,. \tag{30}$$

As $r \to \infty$, assets are worthless, i.e. $a(r) \to 0$. At $r = 0$, we have $\pi - \delta a(0) + \kappa \xi a'(0) = 0\,.$

Intuitively, the value of assets in place (per unit of capital) for an infinitely-lived firm can be viewed as a perpetual bond with a discount rate given by $(r + \delta)$, the sum of interest rate
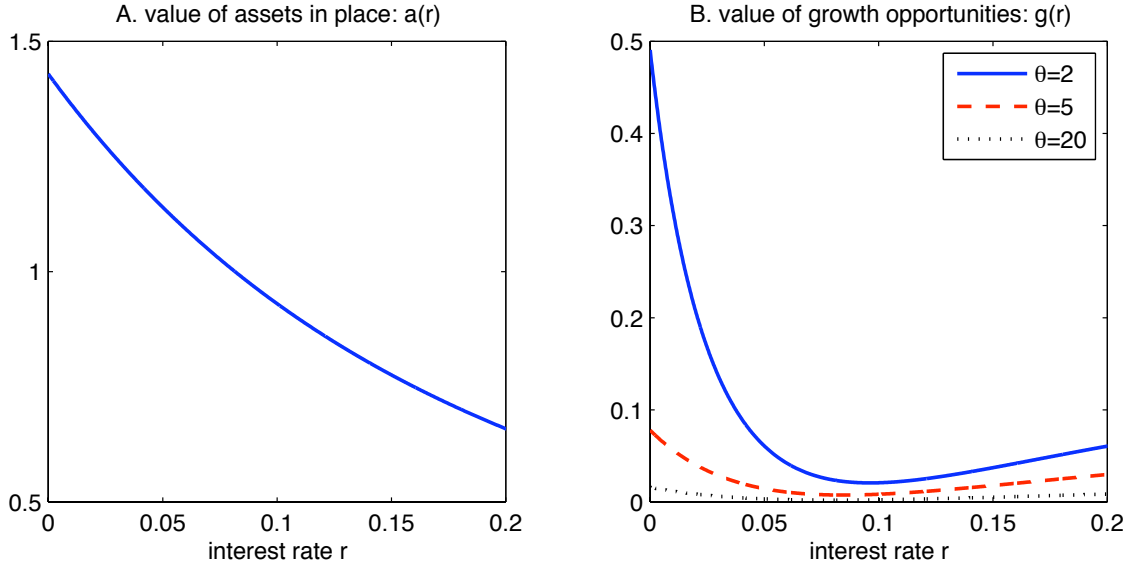
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Figure 4: **The values of assets in place, $a(r)$, and of growth opportunities, $g(r)$**

r and capital depreciation rate δ. Using the perpetual bond interpretation, the "effective" coupon for this asset in place is the firm's constant expected productivity π after the risk adjustment (i.e. under the risk-neutral probability). The value of assets in place $a(r)$ is equal to Tobin's $q(r)$ if and only if no investment is the firm's optimal decision making, i.e. when the adjustment cost is infinity, $\theta = \infty$.

Panel A of Figure 4 plots the scaled value of assets in place, $a(r)$. By definition, $a(r)$ is independent of GO and the adjustment cost parameter θ. By the perpetual bond interpretation, we know that $a(r)$ is decreasing and convex in r. Quantitatively, $a(r)$ accounts for a significant fraction of firm value. For example, at its long-run mean $\xi^P = 0.0574$, $a(\xi^P) = 1.105$, which accounts for about 96% of total firm value, i.e. $a(\xi^P)/q(\xi^P) = 0.96$.

The value of assets in place generally is not equal to the "book" value or replacement costs of capital, contrary to the conventional wisdom. The value of assets in place is $A(K,r) = a(r)K$, while the book value of capital is K. In general, $a(r) \neq 1$. However, the value of assets in place does not account for growth opportunities (GO), to which we now turn.

Growth opportunities. The value of GO, $G(K,r)$ given by $G(K,r) = V(K,r) - A(K,r)$, accounts for the value of optimally adjusting investment in response to changes in interest

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rates. Let $g(r)$ denote the scaled value of GO, $g(r) = G(K, r)/K$. We have

$$g(r) = q(r) - a(r). \tag{31}$$

Panel B of Figure 4 plots the value of GO, $g(r)$, for $\theta = 2, 5, 20$. First, we note that the impact of r on $g(r)$ is strong, especially with a low adjustment cost. For example, with $\theta = 2$, $g(r)$ drops by 91% from 0.490 to 0.046 when r increases from 0 to its long-run mean $\xi^P = 0.0574$. Second, $g(r)$ decreases in r for low r but increases in r for high r, unlike the value of assets in place $a(r)$, which always decreases in r. Intuitively, interest rates have two opposing effects on $g(r)$. The standard discount rate effect suggests that the higher the r, the lower $g(r)$. However, the expected future cash flows of GO critically depend on r. The higher the interest rate r, the lower investment and hence the higher the expected cash flows, which may be referred to as the cash flow effect of r. For sufficiently high r, the cash flow effect overturns the discount rate effect, causing $g(r)$ to increase in r. This cash flow effect does not exist for $a(r)$; its expected cash flow is π, a constant.

6 User cost of capital

Jorgenson (1963) introduces the concept of user cost of capital in his seminal neoclassical model of investment with no adjustment costs. Abel (1990) provides an in-depth discussion on the user cost of capital and further extends this concept to the adjustment-cost-based q theory of investment. Abel and Eberly (1996) calculate the user cost of capital in their stochastic framework with partial irreversibility. Building on Jorgenson (1963), Hall and Jorgenson (1967), Abel (1990), and Abel and Eberly (1996), we extend the definition of the user cost of capital to our setting with stochastic interest rates.

Let u denote the user cost of capital. Incorporating the risk premia into Abel (1990)'s analysis, we define the user cost of capital u via the following present value (PV) formula,

$$q_t = \mathbb{E}_t \left[\int_t^\infty e^{-\int_t^s (r_v + \delta) dv} u_s ds \right]. \tag{32}$$

Equation (32) states that time-t marginal q equals the *risk-adjusted* PV of the stream of marginal cash flow attributable to a unit of capital installed at time t. By risk adjustment, we mean that the expectation operator $\mathbb{E}_t [\cdot]$ in (32) is under the risk-neutral measure, which incorporates the effects of risk premia for interest rate and productivity shocks.

The definition of the user cost of capital u given in (32) implies[15]

$$u(r) = (r + \delta) \, q(r) - \mathcal{D}q(r), \tag{33}$$

where $\mathcal{D}q(r)$, the risk-adjusted expected change of $q(r)$ due to r, is given by

$$\mathcal{D}q(r) = \mu(r)q'(r) + \frac{\sigma^2(r)}{2}q''(r). \tag{34}$$

Importantly, $\mathcal{D}q(r)$ uses the risk-neutral drift $\mu(r)$, not the physical drift $\mu^P(r)$, so that we account for risk premia when calculating the user cost of capital. The intuition for (33) is as follows. Consider someone who owns the capital and rents it out. For each unit of time, the owner collects u from the user of the capital, faces the risk-adjusted expected change of value $\mathcal{D}(r)$ given in (34), less $\delta q(r)$, the loss due to depreciation. In equilibrium and after risk adjustments, the owner earns the risk-free rate of return r on the value of capital, $q(r)$. That is, $rq(r) = u + \mathcal{D}q(r) - \delta q(r)$ and (33) holds at all times.

Comparing with (16), the ODE for Tobin's q, we obtain the following simple formula for the user cost of capital $u(r)$:

$$u(r) = \pi + \frac{(q(r) - 1)^2}{2\theta}. \tag{35}$$

The first term π in (35) is the expected risk-neutral productivity, which differs from the expected productivity π^P by the risk premium $\rho\eta\epsilon$. The second term reflects the additional value of installed capital due to future profitable investment opportunities.

Equivalently, we may also derive the user cost of capital by extending the insight and analysis of Abel (1990) in the deterministic setting to our stochastic framework with risk premia. Installing a unit of capital yields an incremental risk-adjusted marginal product of capital π and also lowers the marginal cost of adjustment by $-C_K(I, K) > 0$. Therefore, the user cost of capital equals the sum of the risk-adjusted marginal product of capital, π, and the reduction of the adjustment cost by amount $-C_K(I, K)$, in that

$$u(r) = \pi - C_K(I, K) = \pi + \frac{1}{2}\theta i(r)^2 = \pi + \frac{(q(r) - 1)^2}{2\theta}. \tag{36}$$

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[15]Technically, the stochastic process $\{M_t : t \geq 0\}$, where M_t is defined by

$$M_t \equiv \int_0^t e^{-\int_0^t (r_v + \delta)dv} u_s ds + e^{-\int_0^t (r_v + \delta)dv} q_t = \mathbb{E}_t \left[\int_0^\infty e^{-\int_0^s (r_v + \delta)dv} u_s ds \right]$$

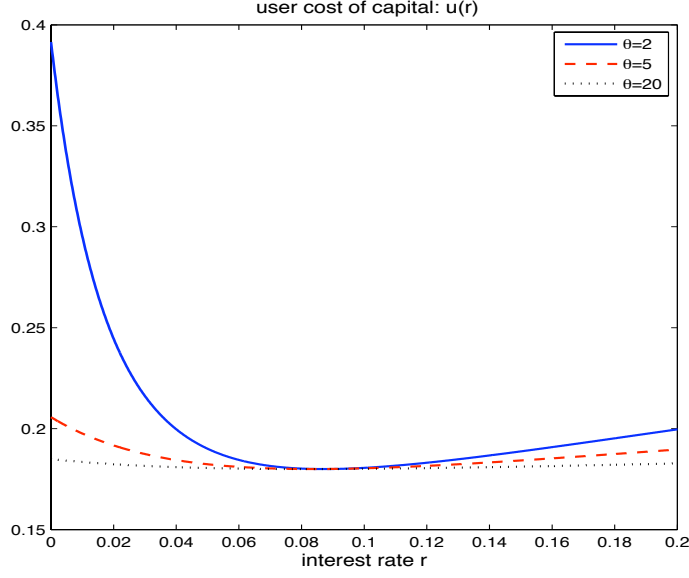is a martingale. Therefore, its drift is zero and hence the differential equation (33) holds.

Figure 5: **The user cost of capital** $u(r)$

As we expect, we obtain the same formula for $u(r)$ in two different ways. Importantly, $u(r)$ depends on future use of installed capital and the term structure of interest rates, which are reflected in the second non-linear term involving the forward-looking variable, $q(r)$.

Figure 5 plots the user cost of capital $u(r)$ for $\theta = 2, 5, 20$. Note that $u(r) \geq \pi = 18\%$ for all r. When r is low, $u(r)$ is high because the reduction in the marginal adjustment cost, the second term in $u(r)$, is large. For example, $u(0) = 0.392$ for $\theta = 2$. Note that $u(r)$ is non-monotonic in r. When $q'(r) < 0$, the firm divests, i.e. $i(r) < 0$. The higher the adjustment cost θ, the less sensitive the user cost of capital $u(r)$ is with respect to r because the reduction of the marginal adjustment cost, $-C_K(I, K)$, is smaller. As a special case, with infinite adjustment costs, $\theta \to \infty$, $u(r) = \pi = 18\%$.

7 Stationary distributions for Tobin's q and investment

We now turn to their stationary distributions of Tobin's q and the firm's investment-capital ratio. First, we recall that the stationary distributions of the interest rate for the CIR model under both the physical and risk-neutral measures are Gamma but with different parameter
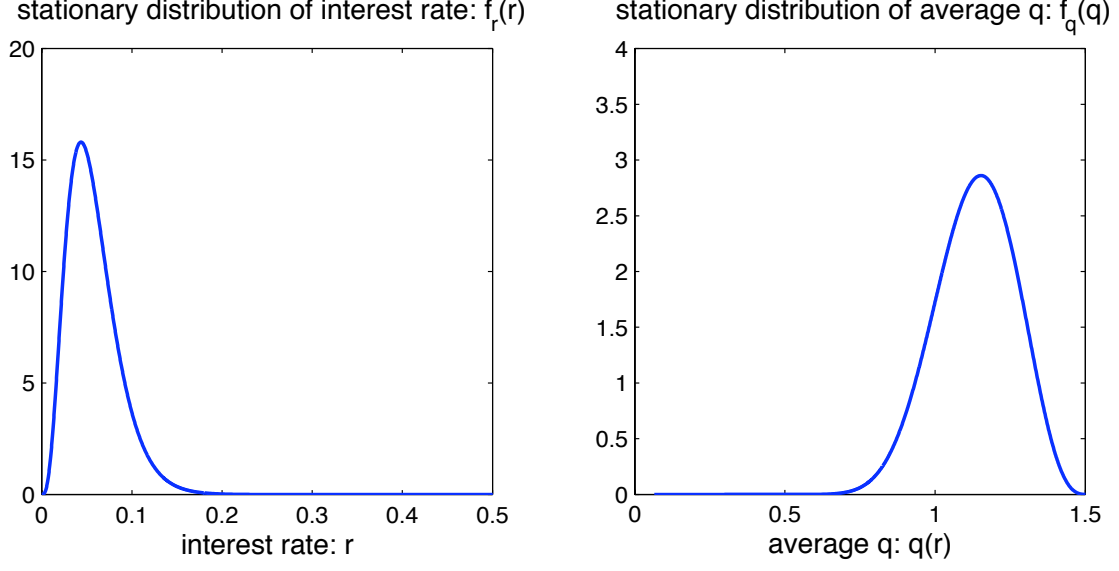
Figure 6: **The stationary distribution for interest rate $f_r(r)$ and average q $f_q(q)$**

values. The probability density function (pdf) under the physical measure, $f_r(r)$, is given by

$$f_r(r) = \frac{1}{\Gamma\left(2\kappa^P\xi^P/\nu^2\right)} \left(2\kappa^P/\nu^2\right)^{2\kappa^P\xi^P/\nu^2} r^{2\kappa^P\xi^P/\nu^2 - 1} e^{-2\kappa^P r/\nu^2} , \tag{37}$$

where $\Gamma(\cdot)$ is the Gamma function.

Applying the standard probability density transformation technique, we have the following probability density function for Tobin's q under the physical measure,

$$f_q(q) = \frac{f_r(r)}{|q'(r)|} . \tag{38}$$

Intuitively, the probability density function $f_q(q)$ depends on the probability density function $f_r(r)$ for the interest rate and inversely on the sensitivity of Tobin's q with respect to r. We plot the stationary distribution for $f_q(q)$ in Panel B of Figure 6.

Table 1 reports stationary moments of Tobin's q under stochastic interest rates for varying levels of the adjustment cost parameter θ. For comparison purposes, we also provide corresponding values of Tobin's q when the interest rate r equals its long-run mean $\xi^P = 5.74\%$. The quantitative effects of stochastic interest rates on Tobin's q and investment are significant. For $\theta = 5$, which is within various empirical estimates, the average value of Tobin's q is 1.133. Ignoring the stochastic interest rate and the risk premium, Tobin's q equals 1.27,

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Table 1: MOMENTS OF THE STATIONARY DISTRIBUTION OF TOBIN'S q AND INVESTMENT-CAPITAL RATIO

| θ | $r = \xi^P = 5.74\%$ | | Stochastic r | |
	$q(r)$	$i(r)$	mean($q(r)$)	mean($i(r)$)
2	-	-	1.192	0.096
3	-	-	1.150	0.050
4	1.295	0.074	1.139	0.035
5	1.271	0.054	1.133	0.027
8	1.247	0.031	1.126	0.016
10	1.241	0.024	1.124	0.012
20	1.230	0.012	1.120	0.006
∞	1.221	0	1.116	0

which is 12% higher. For $\theta = 2$, another commonly used value for the adjustment cost parameter, firm value does not converge under constant interest rates, while the average value of Tobin's q is 1.192 with stochastic interest rates and risk premium. Mean reversion of the interest rates lower Tobin's q from its deterministic benchmark. Also, the quantitative effect of interest rates on Tobin's q is large. Similarly, using $f_i(i) = \frac{f_r(r)}{|i'(r)|} = \theta f_q(q)$, we can also generate the stationary distribution of investment-capital ratio and compare with the benchmark value under constant interest rates. For space considerations, we leave out the discussions on the stationary distribution of i.

8 The value of the liquidation option

Capital often has an alternative use if deployed elsewhere. Empirically, there are significant reallocation activities between firms as well as between sectors.[16] We now extend the baseline model by endowing the firm an option to liquidate its capital stock at any time; doing so allows the firm to recover l per unit of capital where $l > 0$ is a constant. We focus on a single firm's decision and hence ignore the general equilibrium implications. We show that the optionality significantly influences firm investment and the value of capital.[17] The following

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[16]See Eisfeldt and Rampini (2006) and Eberly and Wang (2011) for equilibrium capital reallocation.

[17]McDonald and Siegel (1986) and Dixit and Pindyck (1994) develop the real options approach of investment. Abel, Dixit, Eberly, and Pindyck (1996) integrate the option pricing approach into the q theory of investment.

theorem summarizes the main results.

Theorem 2 *Tobin's q, $q(r)$, solves the ODE (16) subject to (17) and the following value-matching and smooth-pasting boundary conditions*

$$q(r^*) = l\,, \tag{39}$$

$$q'(r^*) = 0\,. \tag{40}$$

The optimal investment strategy $i(r)$ is given by (19).

The value-matching condition given in (39) states that $q(r)$ is equal to its opportunity cost l at liquidation. Because liquidation is optimal, we have the smooth-pasting condition given in (40). Intuitively, at the endogenously chosen interest rate threshold level r^* for liquidation, the marginal effect of changes in r on Tobin's q is zero. In summary, we obtain Tobin's q by solving the ODE (16) subject to the condition (17), and the two free boundary conditions (39) and (40), which characterize the optimal liquidation boundary r^*.

Liquidation gives the firm an exit option to collect the opportunity cost of its capital. This is an American-style option on interest rates. The firm effectively has a long position in assets in place, a long position in growth opportunities, and also a long position in the liquidation option. The liquidation option provides a protection for the value of capital against the interest rate increase by putting a lower bound l for Tobin's q.

For the quantitative exercise, we set the liquidation parameter value $l = 0.9$, i.e. the firm recovers 90 cents on a dollar of the book value of capital upon liquidation (Hennessy and Whited (2005)). We choose the adjustment cost parameter, $\theta = 2$. Panels A and B of Figure 7 plot $q(r)$ and $i(r)$, respectively. In our example, the firm liquidates all its capital stock if the interest rate is higher than $r^* = 0.1432$. Liquidating capital stock rather than operating the firm as a going concern is optimal for sufficiently high interest rates, i.e. $q(r) = l = 0.9$ for $r \geq 0.1432$. Compared with the baseline case (with no liquidation option), the liquidation option increases Tobin's q and investment $i(r)$ for all levels of r. The quantitative effects are much stronger for interest rates closer to the liquidation boundary $r^* = 0.1432$ due to the fact that the liquidation option is much closer to being in the money.
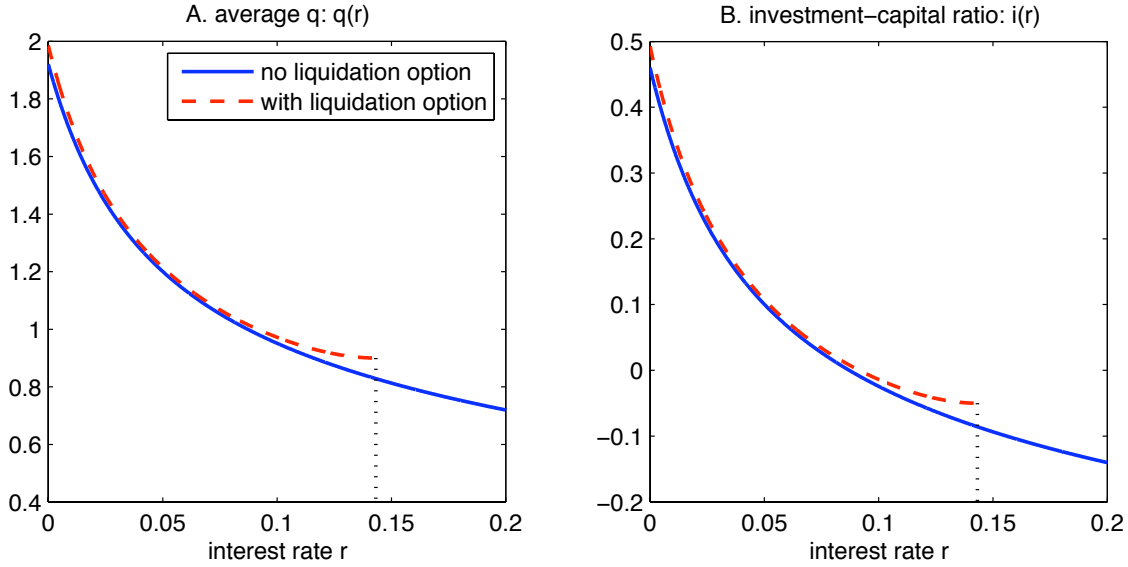
Figure 7: **Tobin's q and the investment-capital ratio $i(r)$ with and without the liquidation option**

9 Asymmetry, price wedge, and fixed costs

9.1 Model setup

We extend the convex adjustment cost $C(I, K)$ in our baseline model along three important dimensions. Empirically, downward adjustments of capital stock are often more costly than upward adjustments. We capture this feature by assuming that the firm incurs *asymmetric* convex adjustment costs in investment ($I > 0$) and divestment ($I < 0$) regions. Hall (2001) uses the asymmetric adjustment cost in his study of aggregate market valuation of capital and investment. Zhang (2005) uses this asymmetric adjustment cost in studying investment-based cross-sectional asset pricing.

Second, as in Abel and Eberly (1994, 1996), we assume a wedge between the purchase and sale prices of capital, for example due to capital specificity and illiquidity premium. There is much empirical work documenting the size of the wedge between the purchase and sale prices. Arrow (1968) stated that "there will be many situations in which the sale of capital goods cannot be accomplished at the same price as their purchase." The wedge naturally depends on the business cycles and market conditions.[18] Let p_+ and p_- denote the respective

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[18]The estimates range from 0.6 to 1, depending on data sources, estimation methods, and model specifi-

purchase and sale prices of capital. An economically sensible assumption is $p_+ \geq p_- \geq 0$ with an implied wedge $p_+ - p_-$.

Third, investment often incurs fixed costs. Fixed costs may capture investment indivisibilities, increasing returns to the installation of new capital, and organizational restructuring during periods of intensive investment. Additionally, fixed costs significantly improve the empirical fit of the model with the micro data. Inaction becomes optimal in certain regions. To ensure that the firm does not grow out of fixed costs, we assume that the fixed cost is proportional to its capital stock. See Hall (2004), Cooper and Haltiwanger (2006), and Riddick and Whited (2009) for the same size-dependent fixed cost assumption.

With the homogeneity property, we may write $c(i) = C(I, K)/K$. Following Abel and Eberly (1994), we write the region-dependent function $c(i)$ as follows,

$$c(i) = \begin{cases} 0, & \text{if } i = 0, \\ \phi_+ + p_+ i + \frac{\theta_+}{2} i^2, & \text{if } i > 0, \\ \phi_- + p_- i + \frac{\theta_-}{2} i^2, & \text{if } i < 0, \end{cases} \tag{41}$$

where ϕ_+ an ϕ_- parameterize the fixed costs of investing and divesting, p_+ and p_- are the respective price of purchasing and selling capital, and θ_+ and θ_- are the asymmetric convex adjustment cost parameters. For $i > 0$, $c(i)$ is increasing and convex in i. For $i < 0$, $c(i)$ is also convex. Panels A and B of Figure 8 plots $c(i)$ given in (41), and the marginal cost of investing $c'(i)$, respectively. Note that $c(i)$ is not continuous at $i = 0$ and hence $c'(i)$ is not defined at the origin ($i = 0$).

9.2 Model solution

In general, the model solution has three distinct regions: (positive) investment, inaction, and divestment regions. We use $q_+(r)$, $q_0(r)$ and $q_-(r)$ to denote Tobin's q in these three regions, respectively. The following theorem summarizes the main results.

Theorem 3 *Tobin's q in investment, inaction, and divestment regions, $q_+(r)$, $q_0(r)$, and*

cations. See Pulvino (1998), Hennessy and Whited (2005), Cooper and Haltiwanger (2006), and Warusawitharana (2008), for example.
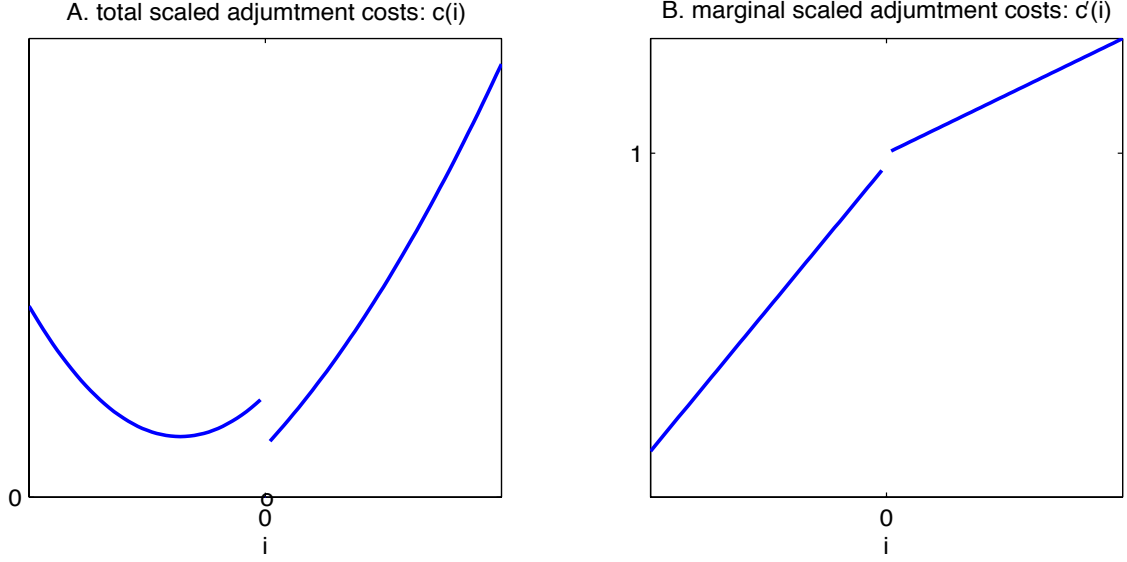
Figure 8: **The cost of investing $c(i)$ and marginal cost of investing $c'(i)$**

$q_-(r)$, *respectively, solve the following three linked ODEs,*

$$(r + \delta) q_+(r) = \pi - \phi_+ + \frac{(q_+(r) - p_+)^2}{2\theta_+} + \mu(r)q'_+(r) + \frac{\sigma^2(r)}{2}q''_+(r), \quad \text{if } r < \underline{r}, \quad (42)$$

$$(r + \delta)q_0(r) = \pi + \mu(r)q'_0(r) + \frac{\sigma^2(r)}{2}q''_0(r), \qquad \qquad \text{if } \underline{r} \leq r \leq \overline{r}, \quad (43)$$

$$(r + \delta) q_-(r) = \pi - \phi_- + \frac{(q_-(r) - p_-)^2}{2\theta_-} + \mu(r)q'_-(r) + \frac{\sigma^2(r)}{2}q''_-(r), \quad \text{if } r > \overline{r}. \quad (44)$$

The endogenously determined cutoff interest rate levels for these three regions, $\underline{r}$ and $\overline{r}$, satisfy the following boundary conditions,

$$\pi - \phi_+ - \delta q_+(0) + \frac{(q_+(0) - p_+)^2}{2\theta_+} + \kappa \xi q'_+(0) = 0, \qquad \qquad (45)$$

$$q_+(\underline{r}) = q_0(\underline{r}), \qquad q_0(\overline{r}) = q_-(\overline{r}), \qquad (46)$$

$$q'_+(\underline{r}) = q'_0(\underline{r}), \qquad q'_0(\overline{r}) = q'_-(\overline{r}), \qquad (47)$$

$$q''_+(\underline{r}) = q''_0(\underline{r}), \qquad q''_0(\overline{r}) = q''_-(\overline{r}), \qquad (48)$$

$$\lim_{r \to \infty} q_-(r) = 0. \qquad (49)$$

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The optimal investment-capital ratios, denoted as $i_+(r)$, $i_0(r)$, and $i_-(r)$, are given by

$$i_+(r) = \frac{q_+(r) - p_+}{\theta_+}, \qquad if \quad r < \underline{r}, \tag{50}$$

$$i_0(r) = 0, \qquad if \quad \underline{r} \leq r \leq \overline{r}, \tag{51}$$

$$i_-(r) = -\frac{p_- - q_-(r)}{\theta_-}, \qquad if \quad r > \overline{r}. \tag{52}$$

When r is sufficiently low ($r \leq \underline{r}$), the firm optimally chooses to invest, $I > 0$. Investment is proportional to $q_+(r) - p_+$, the wedge between Tobin's q and purchase price of capital, p_+. Tobin's q in this region, $q_+(r)$, solves the ODE (42). Condition (45) gives the firm behavior at $r = 0$. The right boundary $\underline{r}$ is endogenous. Tobin's q at $\underline{r}$, $q_+(\underline{r})$, satisfies the first set of conditions in (46)-(48), i.e. $q(r)$ is twice continuously differentiable at $\underline{r}$.

Similarly, when r is sufficiently high ($r \geq \overline{r}$), the firm divests, $I < 0$. Divestment is proportional to $p_- - q_-(r)$, the wedge between the sale price of capital of capital, p_-, and Tobin's q. Tobin's q in the divestment region, $q_-(r)$, solves the ODE (44). Condition (49) states that the firm is worthless as $r \to \infty$, the right boundary condition. The left boundary for the divestment region $\overline{r}$ is endogenous. Tobin's q at $\overline{r}$, $q_-(\overline{r})$, satisfies the second set of the conditions in (46)-(48), i.e. $q(r)$ is twice continuously differentiable at $\overline{r}$.

For r in the intermediate range ($\underline{r} \leq r \leq \overline{r}$), the firm optimally chooses to be inactive, $i(r) = 0$, and hence incurs no adjustment costs. Tobin's q in this region thus behaves likes assets in place and solves the linear ODE (43). The optimal thresholds $\underline{r}$ and $\overline{r}$ are endogenously determined by conditions (46)-(48), as we discussed previously.

Theorem 3 focuses on the case where all three regions exist, i.e. $0 < \underline{r} < \overline{r}$. In the appendix, we discuss the settings under which the model solution only has one or two regions.

9.3   Three special cases

We next study the impact of each friction on investment and Tobin's q. For the baseline case, we set $\theta_+ = \theta_- = 2$ (symmetric convex costs), $p_+ = p_- = 1$ (no price wedge) and $\phi_+ = \phi_- = 0$ (no fixed costs). For each special case, we only change the key parameter of interest and keep all other parameters the same as in the baseline case just described.

Asymmetric convex adjustment costs.   Much empirical evidence suggests that divestment is generally more costly than investment, i.e. $\theta_- > \theta_+$. We set the adjustment cost parameter $\theta_+ = 2$ for investment ($I > 0$) and $\theta_- = 2, 5, 20$ for divestment ($I < 0$).
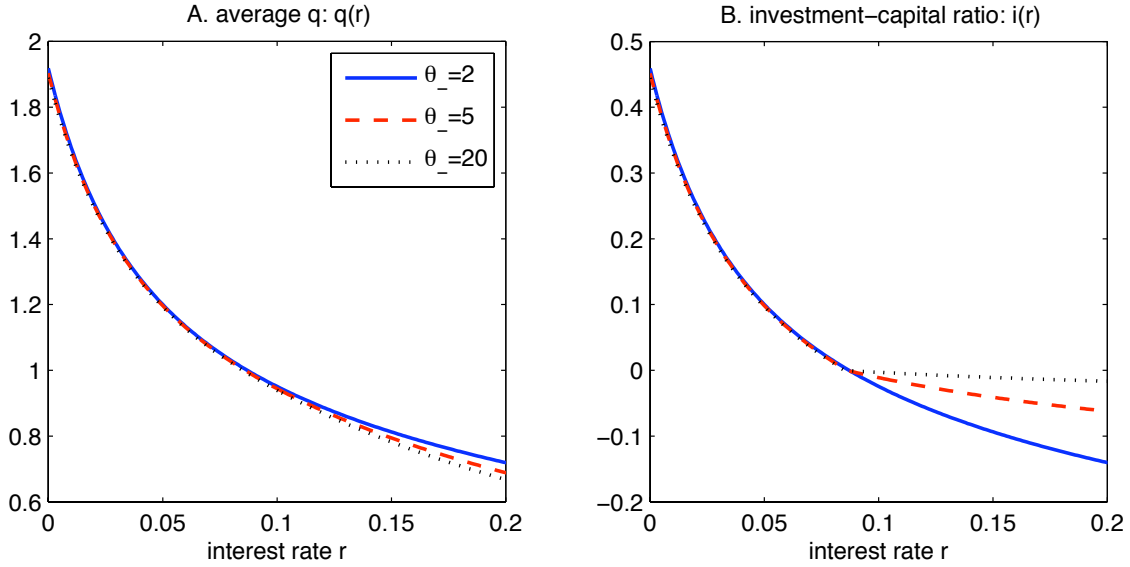
Figure 9: **Tobin's average q and the investment-capital ratio $i(r)$ with asymmetric convex adjustment costs**

Figure 9 shows that the divestment adjustment cost parameter θ_- has strong impact on Tobin's q and $i(r)$ in the divestment region (high r), but almost no impact on $q(r)$ and $i(r)$ in the positive investment region. When r is sufficiently high, the firm divests, and changing θ_- has first-order effects on divestment. The higher the value of θ_-, the more costly divestment and the less divestment activity. With $\theta_- = 20$, the firm is close to facing an irreversible investment option, and hence the optimal divestment level is close to zero. When r is sufficiently low, it is optimal to invest. The divestment option is far out of the money and thus changing θ_- has negligible effects on valuation and investment.

The wedge between purchase and sale prices of capital. We now turn to the effects of price wedge. We normalize the purchase price at $p_+ = 1$ and consider two sale prices, $p_- = 0.8, 0.9$, with implied wedge being 0.2 and 0.1, respectively. We also plot the baseline case with no price wedge as a reference.

Figure 10 plots Tobin's q and the investment-capital ratio $i(r)$ for a firm facing a price wedge. The price wedge leads to three distinct investment regions: investment ($I > 0$), inaction (zero), and divestment ($I < 0$). With low interest rates, the firm invests for growth and the asset sales option is sufficiently out of the money. Hence, price wedge has negligible
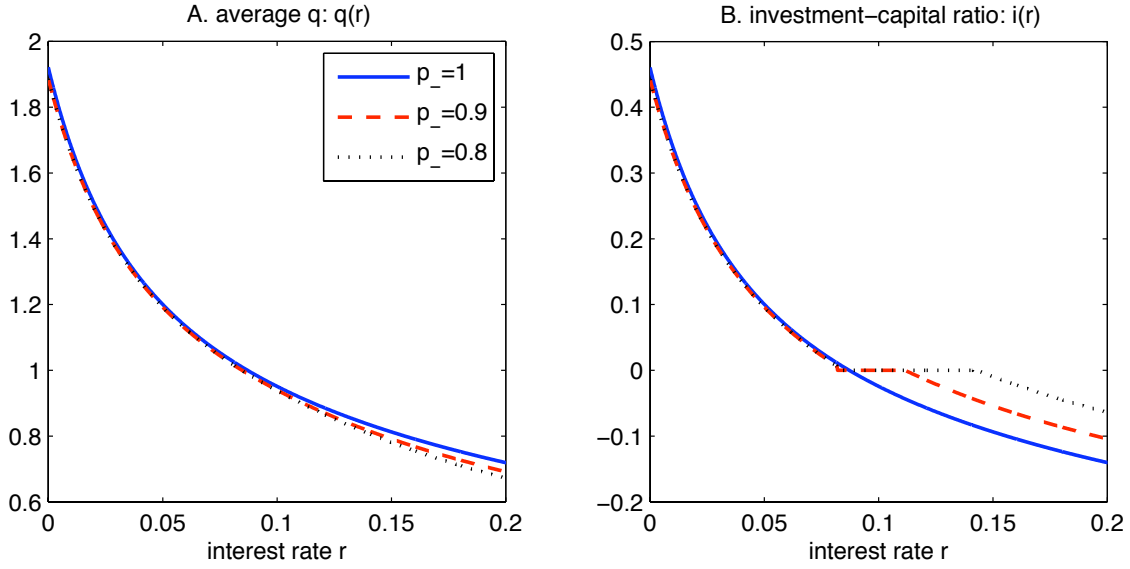
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Figure 10: **Tobin's q and the investment-capital ratio $i(r)$ with price wedges**

effects on Tobin's q and investment. However, with high interest rates, the asset sales option becomes in the money and divestment is optimal. The price wedge thus has significant effects on divestment and value. With wedge being $p_+ - p_- = 0.2$, the firm invests when $r \leq 0.082$ and divests when $r \geq 0.141$. For intermediate values of r ($0.082 \leq r \leq 0.141$), inaction is optimal. In this range, the marginal cost of investment/divestment justifies neither purchasing nor selling capital due to the price wedge. Note that inaction is generated here by the price wedge, not fixed costs. Finally, we note that investment/divestment activities and inaction significantly depend on the price wedge. For example, the inaction region narrows from (0.082, 0.141) to (0.082, 0.109) when the price wedge decreases from 20% to 10%.

Fixed costs and optimal inaction. We now study two settings with fixed costs: (a) fixed costs for divestment only ($\phi_+ = 0$, $\phi_- = 0.01$), and (b) symmetric fixed costs for both investment and divestment ($\phi_+ = \phi_- = 0.01$). We also plot the case with no fixed costs ($\phi_+ = \phi_- = 0$) as a reference.

Figure 11 plots Tobin's average q and the investment-capital ratio $i(r)$ under fixed costs. With fixed costs for divestment, $\phi_- > 0$, we have three regions for $i(r)$. For sufficiently low interest rates ($r \leq 0.082$), optimal investment is positive and is almost unaffected by ϕ_-. For sufficiently high r ($r \geq 0.142$), divestment is optimal. The firm divests more aggressively with
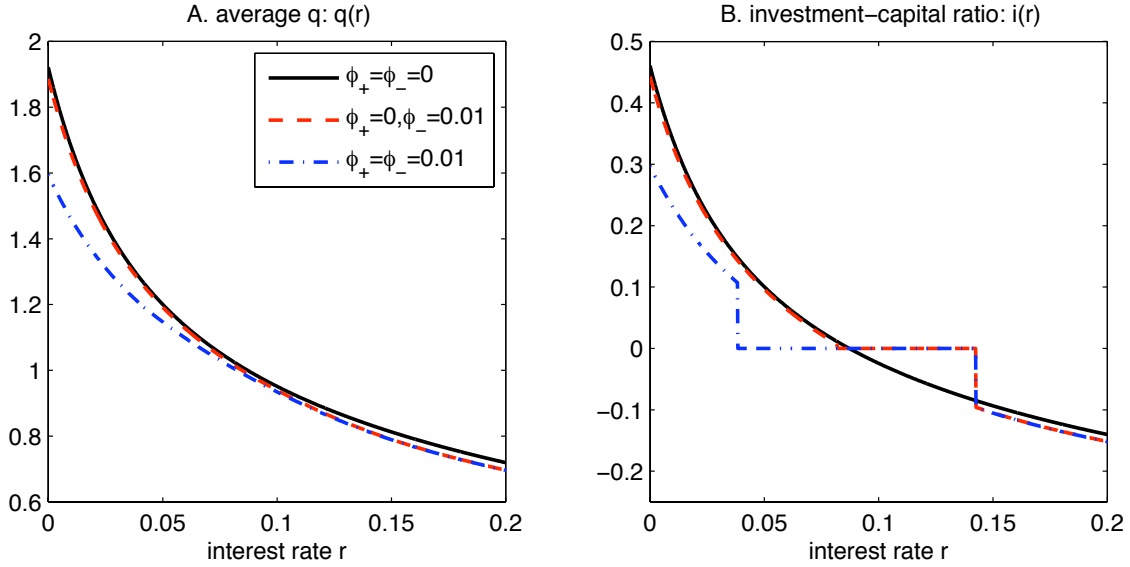
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Figure 11: **Tobin's q and investment-capital ratio $i(r)$ with fixed adjustment costs**

fixed costs of divestment than without. Intuitively, the firm's more aggressive divestment strategy economizes fixed costs of divestment. Additionally, fixed costs generates an inaction region, $0.082 \leq r \leq 0.142$. The impact of fixed costs of divestment is more significant on Tobin's q in medium to high r regions than in the low r region.

Now we incrementally introduce fixed costs for investment by changing ϕ_+ from 0 to 0.01, while holding $\phi_- = 0.01$. We have three distinct regions for $i(r)$. For high r, $r \geq 0.142$, the firm divests. Tobin's q and $i(r)$ in this region remain almost unchanged by ϕ_+. For low r, $r \leq 0.038$, the firm invests less with $\phi_+ = 0.01$ than with $\phi_+ = 0$.

Introducing the fixed costs ϕ_+ discourages investment, lowers Tobin's q, shifts the inaction region to the left, and widens the inaction region. The lower the interest rate, the stronger the effects of ϕ_+ on Tobin's q, investment, and the inaction region.

9.4   Irreversibility

Investment is often irreversible, or at least costly to reverse after capital is installed. There is much work motivated by the irreversibility of capital investment. Arrow (1968) is a pioneering study in a deterministic environment. Our model generates irreversible investment as a special case. We have three ways to deliver irreversibility within our general framework.
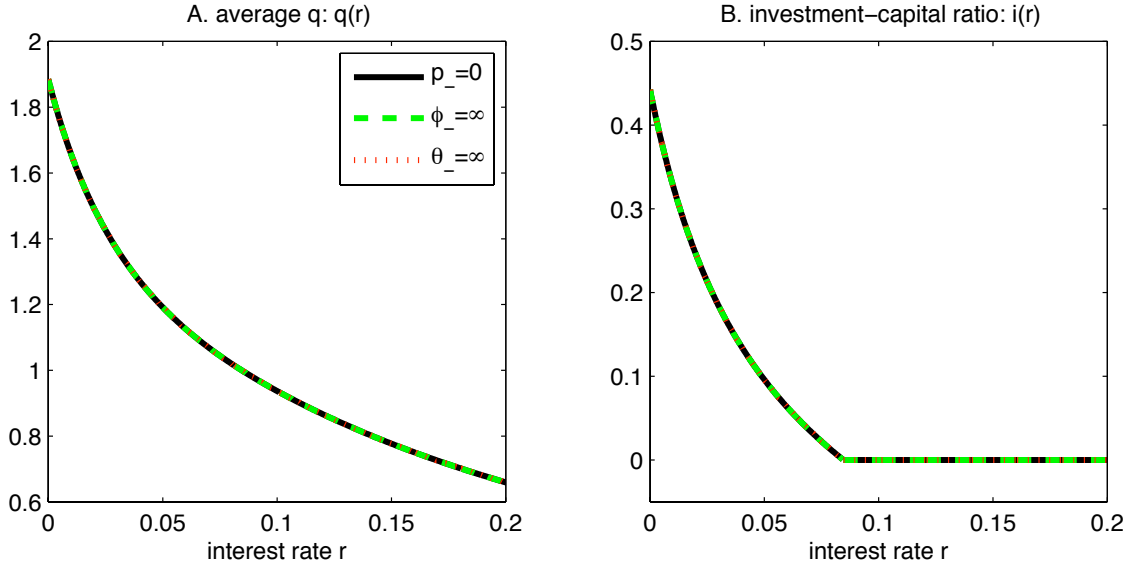
Figure 12: **Tobin's q and investment-capital ratio $i(r)$ with irreversible investment**

Intuitively, they all work to make divestment very costly. We may set the re-sale price of installed capital to zero ($p_- = 0$), making capital completely worthless if liquidated. Alternatively, we may choose the adjustment cost for either convex or lumpy divestment to infinity, (i.e. $\theta_- = \infty$, $\phi_- = \infty$). The three cases all deliver identical solutions for both the divestment and the positive investment regions. Figure 12 plots Tobin's q and the optimal investment-capital ratio $i(r)$ under irreversibility. As in our baseline model, investment varies significantly with the level of the interest rate. Ignoring interest rate dynamics induces significant error for Tobin's q and investment.

10   Serially correlated productivity shocks

We now extend our baseline convex model to allow for serially correlated productivity shocks. Let s_t denote the state (regime) at time t. The expected productivity in state s at any time t, $\pi(s_t)$, can only take on one of the two possible values, i.e. $\pi(s_t) \in \{\pi_L, \pi_H\}$ where $\pi_L > 0$ and $\pi_H > \pi_L$ are constant. Let s denote the current state and $s-$ refer to the other state. Over the time period $(t, t + \Delta t)$, under the risk-neutral measure, the firm's expected productivity changes from π_s to π_{s-} with probability $\zeta_s \Delta t$, and stays unchanged at π_s with the remaining probability $1 - \zeta_s \Delta t$. The change of the regime may be recurrent. That is, the transition
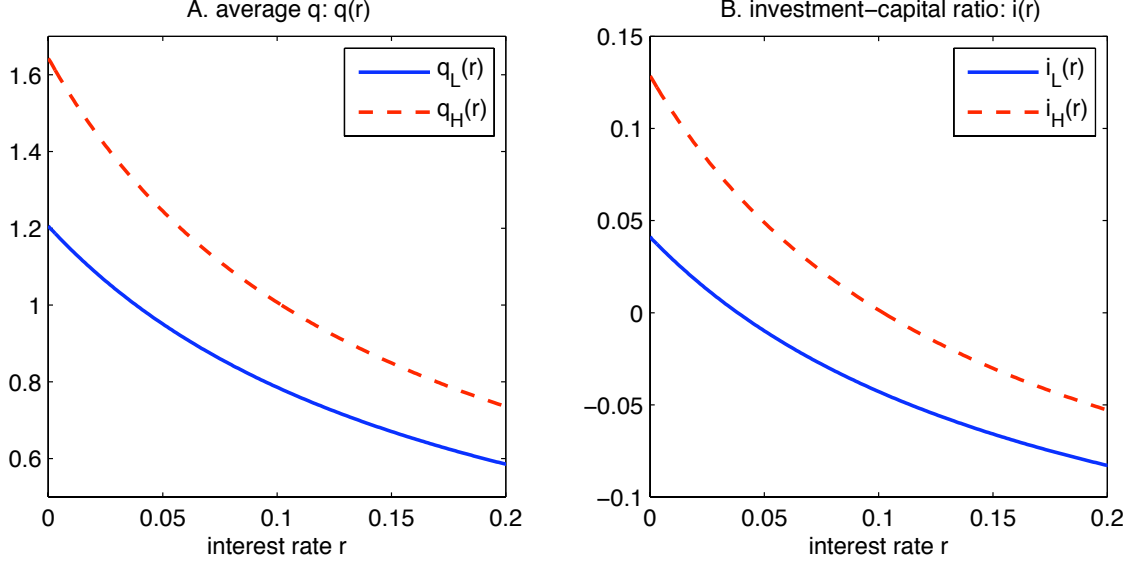
Figure 13: **Tobin's q and investment-capital ratio $i(r)$ with serially correlated productivity shocks**

intensities from either state, ζ_1 and ζ_2, are strictly positive. The incremental productivity shock dX after risk adjustments (under the risk neutral measure) is given by

$$dX_t = \pi(s_{t-})dt + \epsilon(s_{t-})dZ_t, \quad t \geq 0. \tag{53}$$

The firm's operating profit dY_t over the same period $(t, t + dt)$ is also given by (3) as in the baseline model. The homogeneity property continues to hold. The following theorem summarizes the main results.

Theorem 4 *Tobin's q in two regimes, $q_H(r)$ and $q_L(r)$, solves the following linked ODEs:*

$$(r + \delta)\, q_s(r) = \pi_s + \frac{(q_s(r) - 1)^2}{2\theta} + \mu(r)q_s'(r) + \frac{\sigma^2(r)}{2}q_s''(r) + \zeta_s(q_{s-}(r) - q_s(r)), \ \ s = H, L, \tag{54}$$

subject to the following boundary conditions,

$$\pi_s - \delta q_s(0) + \frac{(q_s(0) - 1)^2}{2\theta} + \kappa\xi q_s'(0) + \zeta_s(q_{s-}(0) - q_s(0)) = 0, \tag{55}$$

$$\lim_{r \to \infty} q_s(r) = 0. \tag{56}$$

The optimal investment-capital ratios in two regimes $i_H(r)$ and $i_L(r)$ are given by

$$i_s(r) = \frac{q_s(r) - 1}{\theta}, \quad s = H, L. \tag{57}$$

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Figure 13 plots Tobin's average q and the investment-capital ratio $i(r)$ for both the high- and the low-productivity regimes. We choose the expected (risk-neutral) productivity, $\pi_H = 0.2$ and $\pi_L = 0.14$, set the (risk-neutral) transition intensities at $\zeta_L = \zeta_H = 0.03$. The expected productivity has first-order effects on firm value and investment; both $q_H(r)$ and $i_H(r)$ are significantly larger than $q_L(r)$ and $i_L(r)$, respectively. Additionally, both $q_H(r)$ and $q_L(r)$ are decreasing and convex as in the baseline model. Our model with serially correlated productivity shocks can be extended to allow for richer adjustment cost frictions such as the price wedge and fixed costs as we have done in the previous section, and multiple-state Markov chain processes for productivity shocks.

11 Conclusion

A fundamental determinant of corporate investment and firm value is interest rates, which change stochastically over time and have time-varying risk premia. Existing q models focus on capital illiquidity induced by adjustment costs but with constant interest rates. We recognize the importance of stochastic interest rates and incorporate a widely-used CIR term structure model into the neoclassic q theory of investment (Hayashi (1982) and Abel and Eberly (1994, 1996)). We capture the impact of interest rate mean reversion, volatility, and risk premia on investment and the value of capital. We provide analytical solutions for Tobin's q as a function of the interest rate by deriving and solving an ODE. As in fixed-income analysis, we use duration to measure the interest rate sensitivity of firm value, and find that the duration decreases and varies significantly with interest rates.

We decompose a firm into its assets in place and growth opportunities (GO). While the value of assets in place decreases with the interest rate, the value of GO may either increase or decrease with the interest rate. When the firm has an option to endogenously liquidate its capital at a scrap value, it will optimally exercise this exit option (an "American" style put option on interest rates) to protect itself against the increase of interest rates.

Motivated by empirical evidence on lumpy and partially irreversible investment, we generalize our model with convex adjustment costs to incorporate asymmetric adjustment costs, a price wedge between purchasing and selling capital, fixed costs, and irreversibility. We find that the optimal inaction region critically depends on the interest rate and is quantitatively important. We further extend our model to incorporate persistent productivity shocks. We

show that marginal q is equal to average q even with stochastic interest rates and persistent productivity shocks, extending Hayashi (1982)'s result that average q is equal to marginal q obtained under homogeneity conditions, constant interest rates, and deterministic investment opportunities.

For simplicity, we have chosen a one-factor term structure model for interest rates. Much empirical evidence shows that term structure is much richer (see Piazzesi (2010) for a survey). While our one-factor term structure model captures dynamics and risk premium of the interest rate, any one factor model by definition implies a one-to-one relation between the short rate and the long rate for any horizon. There is a noted debate in the empirical literature on whether it is the long rate or the short rate that determines investment and the value of capital.[19] A multi-factor term structure model of interest rates is naturally suited to conceptually and quantitatively address such interest rate maturity related issues.

We may extend our homogeneous framework to incorporate decreasing returns to scale and a more general non-homogenous adjustment cost specification, either of which will generate a wedge between marginal q and average q. Finally, for a financially constrained firm (where the Modigliani-Miller theorem does not hold), the feedback between corporate decision making and firm-level interest rates is important because financing and investment decisions become intertwined, and moreover, the interest rate risk and the credit risk may interact with each other.[20] We leave these economically motivated but technically involved extensions for future research.

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[19]See Hall (1977) and the discussion of such an issue in Abel (1990).

[20]See Cooley and Quadrini (2001), Gomes (2001), and Whited (1992), among others, for quantitative assessments of financial frictions on corporate investment. See Gourio and Michaux (2011) on the effects of stochastic volatility on corporate investment under imperfect capital markets.

Appendices

A   Sketch of technical details

For Theorem 1.   Using the homogeneity property of $V(K, r)$, we conjecture that $V(K, r) = Kq(r)$ as in (14), which implies $V_K(K, r) = q(r)$, $V_r(K, r) = Kq'(r)$, and $V_{rr}(K, r) = Kq''(r)$. Substituting these into the PDE (8) for $V(K, r)$ and simplifying, we obtain

$$rq(r) = \max_i \ (\pi - c(i)) + (i - \delta) q(r) + \mu(r)q'(r) + \frac{\sigma^2(r)}{2}q''(r). \tag{A.1}$$

Using the FOC (9) for investment I and simplifying, we obtain (19) for the optimal $i(r)$. Substituting the optimal $i(r)$ given by (19) into the ODE (A.1), we have the ODE (16) for $q(r)$. Evaluating the ODE (16) at $r = 0$ gives the boundary condition (17) at $r = 0$. Finally, $V(K, r)$ approaches zero as $r \to \infty$, which implies $\lim_{r \to \infty} q(r) = 0$ given in (18).

For Proposition 1.   With constant interest rates, we may simplify (A.1) as follows,

$$rq(r) = \max_i \ (\pi - c(i)) + (i - \delta) q(r). \tag{A.2}$$

Substituting the optimal i into (A.2) and using economic intuition (higher productivity leads to higher investment and value), we explicitly solve $i = I/K$, which is given by (22).

The value of assets in place, $a(r)$.   For $A(K, r)$, we have the following HJB equation:

$$rA(K, r) = \pi K - \delta K A_K(K, r) + \mu(r)A_r(K, r) + \frac{\sigma^2(r)}{2}A_{rr}(K, r). \tag{A.3}$$

Using $A(K, r) = K \cdot a(r)$ and substituting it into (A.3), we obtain the ODE(30) for $a(r)$. The value of assets in place $A(K, r)$ vanishes as $r \to \infty$, i.e. $\lim_{r \to \infty} A(K, r) = 0$, which implies $\lim_{r \to \infty} a(r) = 0$. Equation (30) implies that the natural boundary condition at $r = 0$ should be $\pi - \delta a(0) + \kappa \xi a'(0) = 0$.

For Theorem 2.   With a liquidation option, the firm optimally exercises its option so that $V(K, r)$ satisfies the value matching condition $V(K, r^*) = lK$, and the smooth pasting condition $V_r(K, r^*) = 0$. With $V(K, r) = q(r)K$, we obtain $q(r^*) = l$ and $q'(r^*) = 0$, given by (39) and (40), respectively.

For Theorem 3. With homogeneity property, we conjecture that there are three regions (positive, zero, and negative investment regions), separated by two endogenous cutoff interest-rate levels $\underline{r}$ and $\overline{r}$. Firm value in the three regions can be written as follows,

$$V(K, r) = \begin{cases} K \cdot q_- (r), & if \quad r > \overline{r}, \\ K \cdot q_0 (r), & if \quad \underline{r} \le r \le \overline{r}, \\ K \cdot q_+ (r), & if \quad r < \underline{r}, \end{cases} \tag{A.4}$$

Importantly, at $\underline{r}$ and $\overline{r}$, $V(K, r)$ satisfies value-matching, smooth-pasting, and super contact conditions, which imply (46), (47), and (48), respectively. Note that (45) is the natural boundary condition at $r = 0$ and (49) reflects that firm value vanishes as $r \to \infty$. Other details are essentially the same as those in Theorem 1.

When the fixed cost for investment ϕ_+ is sufficiently large, there is no investment region, i.e. $\underline{r} = 0$. Additionally, the condition at $r = 0$, (45), is replaced by the following condition,

$$\pi - \delta q_0(0) + \kappa \xi q_0'(0) = 0. \tag{A.5}$$

In sum, for the case with inaction and divestment regions, the solution is given by the linked ODEs (43)-(44) subject to (A.5), the free-boundary conditions for the endogenous threshold $\overline{r}$ given as the second set of conditions in (46)-(48), and the limit condition (49).

Similarly, if the cost of divestment ϕ_- is sufficiently high, the firm has no divestment region, i.e. $\overline{r} = \infty$. The model solution is given by the linked ODEs (42)-(43) subject to (45), the free-boundary conditions for $\underline{r}$ given as the first set of conditions, and $\lim_{r\to\infty} q_0(r) = 0$.

For Theorem 4. Firm value in the low and high productivity regimes, $V(K, r, \pi_L)$ and $V(K, r, \pi_H)$, jointly solve the following coupled HJB equations:

$$\begin{aligned} rV(K, r, \pi_L) &= \max_I \ (\pi_L K - C(I, K)) + (I - \delta K) V_K(K, r, \pi_L) + \mu(r)V_r(K, r, \pi_L) \\ &\quad + \frac{\sigma^2(r)}{2} V_{rr}(K, r, \pi_L) + \zeta_L(V(K, r, \pi_H) - V(K, r, \pi_L)). \end{aligned} \tag{A.6}$$

$$\begin{aligned} rV(K, r, \pi_H) &= \max_I \ (\pi_H K - C(I, K)) + (I - \delta K) V_K(K, r, \pi_H) + \mu(r)V_r(K, r, \pi_H) \\ &\quad + \frac{\sigma^2(r)}{2} V_{rr}(K, r, \pi_H) + \zeta_H(V(K, r, \pi_L) - V(K, r, \pi_H)). \end{aligned} \tag{A.7}$$

Using the homogeneity property, we conjecture $V(K, r, \pi_s) = K \cdot q_s(r)$, for $s = H, L$. The remaining details are essentially the same as those for Theorem 1.

References

Abel, A. B., 1983, "Optimal investment under uncertainty," *American Economic Review*, 73, 228-233.

Abel, A. B., and O. J. Blanchard, 1983, "An intertemporal model of saving and investment," *Econometrica*, 51, 675-692.

Abel, A. B., and O. J. Blanchard, 1986, "The present value of profits and cyclical movements in investment," *Econometrica*, 54, 249-273.

Abel, A. B., A. K. Dixit, J. C. Eberly and R. S. Pindyck, 1996, "Options, the value of capital, and investment," *Quarterly Journal of Economics*, 111, 753-777.

Abel, A. B., and J. C. Eberly, 1994, "A unified model of investment under uncertainty," *American Economic Review*, 84, 1369-1384.

Abel, A. B., and J. C. Eberly, 1996, "Optimal investment with costly reversibility," *Review of Economic Studies*, 63, 581-593.

Abel, A. B., and J. C. Eberly, 2011, "How Q and cash flow affect investment without frictions: An analytic explanation," *Review of Economic Studies*, forthcoming.

Arrow, K. J., 1968, "Optimal capital policy with irreversible investment," in J. N. Wolfe, ed., *Value, capital and growth. Papers in honour of Sir John Hicks.* Edinburgh: Edinburge University Press, 1-19.

Baxter, M., and M. J. Crucini, 1993, "Explaining saving-investment correlations," *American Economic Review*, 83, 416-436.

Blanchard, O., and S. Fisher, 1989, Lectures on macroeconomics, Princeton University Press, Princeton, NJ.

Brainard, W. C., and J. Tobin, 1968, "Pitfalls in Financial Model-Building," *American Economic Review*, 58, 99-122.

Caballero, R. J., 1999, "Aggregate investment,"in J. B. Taylor, and M. Woodford (ed.), Handbook of Macroeconomics, vol. 1 of *Handbook of Macroeconomics*, Ch. 12, pp. 813-862, Elsevier.

Caballero, R. J., and J. V. Leahy, 1996, "Fixed costs: The demise of marginal q,"Working papers, National Bureau of Economic Research.

Chirinko, R., 1993, "Business fixed investment spending: Modeling strategies, empirical results, and policy implications," *Journal of Economic Literature*, 31, 1875-1911.

Cochrane, J. H., 1991, "Production-based asset pricing and the link between stock returns and economic fluctuations," *Journal of Finance*, 46, 209-237.

Cochrane, J. H., 1996, "A cross-sectional test of an investment-based asset pricing model," *Journal of Political Economy*, 104, 572-621.

Cooley, T. F., and V. Quadrini, 2001, "Financial markets and firm dynamics," *American Economic Review*, 91, 1286-1310.

Cooper, R. S. and J. C. Haltiwanger, 2006, "On the nature of capital adjustment costs," *Review of Economic Studies*, 73, 611-633.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross, 1985, "A theory of term structure of interest rates," *Econometrica*, 53, 385-408.

Dai, Q., and K. Singleton, 2000, "Specification analysis of affine term structure models," *Journal of Finance,* 55, 1943-1978.

Dew-Becker, I., 2011, "Investment and the cost of capital in the cross-section: The term spread predicts the duration of investment,"Working paper, Harvard University.

Dixit, A. K., and R. S. Pindyck, 1994, Investment under uncertainty, Princeton University Press, Princeton, NJ.

Downing, C., D. Jaffee, and N. Wallace, 2009, "Is the market for mortgage-backed securities a market for lemons?" *Review of Financial Studies,* 22, 2457-2494.

Duffie, J. D., 2002, Dynamic asset pricing theory, Princeton University Press, Princeton, NJ.

Duffie, J. D., and R. Kan, 1996, "A yield-factor model of interest rates," *Mathematical Finance,* 6, 379-406.

Eberly, J. C., S. Rebelo, and N. Vincent, 2009, "Investment and value: A neoclassical benchmark," working paper, Northwestern University.

Eblery, J. C., and N. Wang, 2011, "Reallocating and pricing illiquid capital: Two productive trees," Working paper, Columbia University and Northwestern University.

Eisfeldt, A., L, and A. A. Rampini, 2006, "Capital reallocation and liquidity," *Journal of Monetary Economics*, 53, 369-399.

Gilchrist, S., and C. P. Himmelberg, 1995, "Evidence on the role of cash flow for investment," *Journal of Monetary Economics*, 36, 541-572.

Gilchrist, S., and E. Zakrajsek, 2007, "Investment and the cost of capital: new evidence from the corporate bond market," Working papers, National Bureau of Economic Research.

Gomes, J. F., 2001, "Financing investment," *American Economic Review*, 91, 1263-1285.

Gourio F., and M. Michaux, 2011, "Stochastic volatility, bond yields, and the Q theory of investment," Working paper, Boston University and USC.

Guiso, L., A. K. Kashyap, F. Panetta, and D. Terlizzese, 2002, "How interest sensitive is investment? Very (when the data are well measured)," Working paper, Chicago Booth.

Hall, R. E, 1977, "Investment, interest rates, and the effects of stabilization policies," *Brookings Papers on Economic Activity*, 1, 61-103.

Hall, R. E., 2001, "The stock market and capital accumulation," *American Economic Review*, 91, 1185-1202.

Hall, R. E., 2004, "Measuring factor adjustment costs," *Quarterly Journal of Economics*, 119, 899-927.

Hall, R. E., and D. W. Jorgenson, 1967, "Tax policy and investment behavior ," *American Economic Review*, 57, 391-414.

Hayashi, F., 1982, "Tobin's marginal Q and average Q: A neoclassical interpretation," *Econometrica*, 50, 215-224.

Hennessy, C. A., and T. M. Whited, 2005, "Debt dynamics," *Journal of Finance*, 60, 1129-1165.

Jermann, U. J., 1998, "Asset pricing in production economies," *Journal of Monetary Economics*, 41, 257-275.

Jorgenson, D. W., 1963, "Capital theory and investment behavior," *American Economic Review*, 53(2), 247-259.

Lucas, R. E., Jr., 1981, "Optimal investment with rational expectation," in R. E. Lucas, Jr., and T. J. Sargent, *Rational expectations and economic behavior,* Vol I., Minneapolis: University of Minnesota Press.

Lucas, R. E., Jr., and E. C. Prescott, 1971, "Investment under uncertainty," *Econometrica*, 39, 659-681.

Ljungqvist, L., and T. J. Sargent, 2004, Recursive macroeconomic theory, MIT Press, Cambridge, MA.

McDonald, R., and Siegel, D., 1986, "The value of waiting to invest," *Quarterly Journal of Economics*, 101, 707-728.

Mussa, M., 1977, "External and internal adjustment costs and the theory of aggregate and firm investment," *Economica*, 44, 163-178.

Pearson, N. D., and T. S. Sun, 1994, "Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll, and Ross model," *Journal of Finance,* 54, 1279-1304.

Piazzesi, M., 2010, "Affine term structure models," in Y. Ait-Sahalia and L. P. Hansen (eds.), Handbook of Financial Econometrics, Elsevier.

Pulvino, T. C., 1998, "Do asset fire sales exist? An empirical investigation of commercial aircraft transactions," *Journal of Finance,* 53, 939-978.

Riddick, L. A., and T. M. Whited, 2009, "The corporate propensity to save," *Journal of Finance,* 64, 1729-1766.

Stanton, R. 1995, "Rational prepayment and the value of mortgage-backed securities," *Review of Financial Studies*, 8, 677-708.

Stokey, N. L., 2009, The economics of inaction: Stochastic control models with fixed costs, Princeton University Press, Princeton, NJ.

Tobin, J., 1969, "A general equilibrium approach to monetary theory," *Journal of Money, Credit and Banking*, 1, 15-29.

Vasicek, O., 1977, "An equilibrium characterization of the term structure," *Journal of Financial Economics*, 5 (2), 177-188.

Warusawitharana, M., 2008, "Corporate asset purchases and sales: Theory and evidence," *Journal of Financial Economics*, 87, 471-497.

Whited, T. M., 1992, "Debt, liquidity constraints, and corporate investment: Evidence from panel data," *Journal of Finance*, 47, 1425-1460.

Zhang, L., 2005, "The value premium," *Journal of Finance*, 60, 67-104.