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ABSTRACT

In this paper we show that interest rate rules lead to multiple equilibria when the central bank faces a limit to its ability to print money, or when private agents are limited in the amount of bonds that can be pledged to the central bank in exchange for money. Some of the equilibria are familiar and common to the environments where limits to money growth are not considered. However, new equilibria emerge, where money growth and inflation are higher. These equilibria involve a run on the central bank's interest target: households borrow as much as possible from the central bank, and the shadow interest rate in the private market is different from the policy target.

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1 Introduction

Until the last couple of years, most central banks around the world conducted monetary policy by setting targets for short-term interest rates, and letting the quantity of money adjust in response to demand. Maneuvering interest rates as a way to achieve low and stable inflation is now regarded as a success story. Yet this was not always the case. As mentioned by Sargent [11], the German Reichsbank also discounted treasury and commercial bills at fixed nominal interest rates in 1923; but, rather than contributing to stabilizing the value of the mark, the policy added fuel to the hyperinflation by causing the Reichsbank to greatly increase the money supply and transferring this money to the government and to those private entities lucky enough to borrow from the Reichsbank at the official discount rate. In our paper, we study the extent to which setting a short-term interest rate can be used as a way of implementing a unique equilibrium in a monetary economy.

We conduct our analysis in a simple environment that features flexible prices and a standard cash-in-advance constraint, where the intuition for our results is simple and transparent; however, our results would extend to models with frictions. In this setup, we consider the properties of an interest rate rule, whereby the central bank sets a price at which private agents are free to trade currency for one-period debt; this price need not be fixed, but rather may depend in arbitrary ways on all the information that the central bank has at the moment it makes its decision. We show that setting a policy rate in this way leads to multiple equilibria when the central bank faces a limit to its ability to print money, or when private agents are limited in the amount of bonds that can be pledged to the central bank in exchange for money.[1] Some of the equilibria are familiar and common to the environments where limits to money growth are not considered. However, new equilibria emerge, where money growth and inflation are higher. These equilibria involve a run on the central bank's interest target: households borrow as much as possible from

[1] The source of this multiplicity is very different from (and complementary to) that identified by Benhabib, Schmitt-Grohé and Uribe [5]. While they identify a class of interest-rate rules for which an undesired low-inflation equilibrium emerges, the speculative runs that we identify involve high inflation and more severe monetary distortions than the equilibria that do not feature a run.

the central bank, and the shadow interest rate in the private market is different from the policy target.

To the extent that monetary policy is primarily conducted by open market operations that exchange money for government bonds (or government-backed bonds), fiscal policy plays a prominent role in defining the characteristics of equilibria that feature runs. This happens because the amount of bonds held by the private sector determines the size of the run in the event of a run. This is a new channel by which excessive deficits threaten price stability, and is independent of the familiar unpleasant monetarist arithmetic of Sargent and Wallace [13] and the fiscal theory of the price level (Leeper [9], Sims [14], Woodford [15]). In fact, we deliberately rule out these alternative channels of monetary-fiscal interaction by postulating fiscal rules that ensure long-term budget balance independently of the path of inflation.

Our research implies that interest-rate targets are an incomplete description of the way modern central banks have succeeded in establishing low and stable inflation, and suggests a new role for the "twin-pillar" doctrine of paying attention to monetary aggregates (both broad and narrow) as well as interest rates in designing appropriate monetary policy rules.[2]

2 The basic cash-in-advance model

Consider a version of the cash-in-advance model. There are a continuum of households of unit mass and a government/monetary authority. Time is discrete with dates $t \in \{0, 1, 2, \ldots\}$. In each period, the timing is as follows: First, households pay lump sum nominal taxes T_t levied by the government and asset markets open. In these asset markets, households can buy (or sell) government bonds, acquire money, as well as trade zero-net supply securities with other households. At this same time, the government can print and destroy money, borrow and lend.

After the asset markets, a goods market opens. In the goods market, households produce the consumption good using their own labor for the use of other households (but, as usual, not their own household) and the government. Each household has one unit of time and a constant-returns-to-scale technology that converts units of time into units of the consumption good one

[2]For a discussion of the twin-pillar doctrine, see Lucas [10].

for one. Households use money to purchase units of the consumption good produced by other households. The government uses either money or bonds (it is immaterial which) to purchase $G_t = \overline{G} \in (0, 1)$ units of the consumption good.

Let M_t denote the amount of money in circulation at the end of the asset market in period t, after taxes are paid. Let B_{t-1} be the nominal amount of government bonds payable at date t. (If $B_{t-1} < 0$ then it represents a debt that households owe the government at date t.) The households start with initial nominal claims W_{-1} against the government.[3]

Consider a price sequence $\{P_t, R_t, \hat{R}_t\}_{t=0}^{\infty}$, where P_t is the nominal price of a unit of the consumption good at date t, R_t is the nominal risk-free rate between period t and $t+1$ at which the government trades with private agents, and $\hat{R}_t$ is the rate at which households trade with each other. A government policy $\{T_t, M_t, B_t\}_{t=0}^{\infty}$ is said to be feasible given $\{P_t, R_t, \hat{R}_t\}_{t=0}^{\infty}$ if for all $t > 0$

$$B_t = (1 + R_t)\Big[P_{t-1}\overline{G} - T_t - M_t + M_{t-1} + B_{t-1}\Big], \tag{1}$$

with the initial condition

$$B_0 = (1 + R_0)[W_{-1} - M_0 - T_0]. \tag{2}$$

In what follows, we use lower-case letters to indicate individual household choices and upper-case variables to indicate aggregates: as an example, m_t are individual money holdings, and M_t are aggregate money holdings. In equilibrium, lower and upper-case variables will coincide, since we consider a representative household.

Households are subject to a cash-in-advance constraint: their consumption must be purchased with money. A household's path is given by $\{c_t, y_t, \hat{b}_t, b_t, m_t\}_{t=0}^{\infty}$, where $\hat{b}_t$ are holdings of privately-issued bonds maturing in period $t+1$.[4] In addition, households are potentially constrained in their holdings of government securities to a set $\mathcal{B}_t$. We will first explore the case in which $\mathcal{B}_t$ is the entire real line, and we will then explore the implications of setting a limit to private indebtedness against the government.

[3]These claims represent money and maturing bonds, before paying period 0 taxes.

[4]In equilibrium, $\hat{b}_t \equiv 0$.

A household path is feasible if for all $t > 0$

$$\frac{\hat{b}_t}{1+\hat{R}_t} + \frac{b_t}{1+R_t} = P_{t-1}(y_{t-1} - c_{t-1}) - T_t - m_t + m_{t-1} + \hat{b}_{t-1} + b_{t-1}, \tag{3}$$

$$m_t \geq P_t c_t, \tag{4}$$

together with the initial condition

$$\frac{\hat{b}_0}{1+\hat{R}_0} + \frac{b_0}{1+R_0} = W_{-1} - m_0 - T_0 \tag{5}$$

and the no-Ponzi condition

$$\begin{aligned} &\hat{b}_t + b_t \geq \underline{A}_{t+1} := -P_t - m_t + T_{t+1} + \\ &\sum_{j=1}^{\infty} \left\{ \left(\prod_{v=1}^{j} \frac{1}{1+\hat{R}_{t+v}}\right) \left[T_{t+j+1} - P_{t+j} - \max_{\hat{b} \in \mathcal{B}_t} [\hat{b} \left(\frac{1}{1+\hat{R}_{t+j}} - \frac{1}{1+R_{t+j}} \right)] \right] \right\}. \end{aligned} \tag{6}$$

Equation (6) imposes that households cannot borrow more than the present value of working 1 unit of time while consuming nothing, holding no money in every period after t, and maximally exploiting any price discrepancy between government-issued and private securities. This present value is evaluated at the sequence of intertemporal prices $\{\hat{R}_s\}_{t=0}^{\infty}$.

When $\mathcal{B}_t = \mathbb{R}$, a no-arbitrage condition will ensure $\hat{R}_{t+j} = R_{t+j}$, making the corresponding term disappear from (6). When limits to household indebtedness against the government are present, we will study equilibria where government securities have a different price than equivalent privately-issued securities, in which case household can profit from the mispricing (at the expense of the government), and the corresponding profits are part of their budget resources.[5] Facing prices $\{P_t, R_t, \hat{R}_t\}_{t=0}^{\infty}$, tax policy $\{T_t\}_{t=0}^{\infty}$, and given initial nominal wealth, a household's problem is to choose $\{c_t, y_t, \hat{b}_t, b_t, m_t\}_{t=0}^{\infty}$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, y_t) \tag{7}$$

[5]Of course, in equilibrium the aggregate profits of the households from this activity are matched by lump-sum taxes that the government has to impose, so that in the aggregate this limited arbitrage opportunity is a zero-sum game.

subject to (3), (4), (5), (6), and $b_t \in \mathcal{B}_t$. We assume that u is continuously differentiable, that both consumption and leisure are normal goods, and that the following conditions hold:

$$\lim_{c \to 0} u_c(c, y) = \infty \quad \forall\, y > 0, \quad \lim_{y \to 1} u_y(c, y) = -\infty \quad \forall\, c > 0, \tag{8}$$

and

$$\forall\, y > 0 \; \exists\, \underline{u}_y(y) > 0 : |u_y(c, y)| > \underline{u}_y(y) \quad \forall\, c \geq 0. \tag{9}$$

Equation (8) is a standard Inada condition; it will ensure an interior solution to our problem. Equation (9) imposes that the marginal disutility of labor is bounded away from zero in equilibria in which production is also bounded away from zero.

3 An interest rate policy

In this section, we construct equilibria for an economy in which the government/monetary authority sets an interest rate rule, without imposing limits to household trades with the central bank. In particular, suppose the central bank offers to buy or sell any amount of promises to pay \$1 at date $t+1$ for $1/(1+R_t) < 1$ dollars at date t.[6] This interest rate R_t can be an arbitrary function of past history, and $\mathcal{B}_t = \mathbb{R}$.

We suppose that the government sets a "Ricardian" fiscal rule, i.e., a rule such that the set of equilibrium price levels is not restricted by the requirement of the present-value budget constraint of the government. We choose such a fiscal policy because we are interested in the set of equilibria that can arise when money is not directly backed by tax revenues, as it happens when the fiscal theory of the price level holds. We will specify a class of fiscal rules that satisfies sufficient conditions for this requirement below.

An equilibrium is then a sequence $\{P_t, \hat{R}_t, R_t, T_t, C_t, Y_t, \hat{B}_t, B_t, M_t\}_{t=0}^{\infty}$ such that $\{C_t, Y_t, \hat{B}_t, B_t, M_t\}_{t=0}^{\infty}$ solves the household's problem taking $\{P_t, \hat{R}_t, R_t, T_t\}_{t=0}^{\infty}$ as given, and such that markets clear for all $t \geq 0$:

$$C_t = Y_t - \overline{G} \tag{10}$$

[6] We assume that nominal interest rates remain strictly positive ($R_t > 0$). This greatly simplifies the analysis, since the cash-in-advance constraint will always be binding, but it does not play an essential role in our analysis.

and

$$\hat{B}_t = 0. \tag{11}$$

In order for the household problem to have a finite solution, it is necessary that the prices of government and private assets be the same:

$$\hat{R}_t = R_t. \tag{12}$$

When (12) fails, households can exploit the difference in price to make infinite profits. In addition to (6) and (12), necessary and sufficient conditions from the household optimization problem yield the following conditions for all $t \geq 0$:

$$-\frac{u_y(C_t, Y_t)}{u_c(C_t, Y_t)} = \frac{1}{1+\hat{R}_t}, \tag{13}$$

$$\frac{u_y(C_{t+1}, Y_{t+1})}{u_y(C_t, Y_t)} = \frac{1}{\beta(1+\hat{R}_{t+1})}\frac{P_{t+1}}{P_t}, \tag{14}$$

$$M_t/P_t = C_t, \tag{15}$$

and the transversality condition

$$\lim_{t\to\infty}\left(\prod_{j=0}^{t}\frac{1}{1+\hat{R}_j}\right)(\hat{B}_t + B_t - \underline{A}_{t+1}) = 0. \tag{16}$$

Substituting (10) and (12) into (13), we obtain

$$-\frac{u_y(C_t, C_t+\overline{G})}{u_c(C_t, C_t+\overline{G})} = \frac{1}{1+R_t}. \tag{17}$$

We now turn to constructing equilibria. The initial price level, P_0, is not determined. For each initial price P_0, one can use the interest rate rule R_t and equations (1), (2), (10), (14), (15), and (17) to sequentially solve for a *unique* candidate equilibrium allocation and price system.[7] That is, given R_0, the fiscal policy rule determines T_0, equation (17) solves for C_0 and equation (10) then implies Y_0 and equation (15) implies M_0. Finally, equation (2) determines B_0. With all

[7]The Inada condition and the assumptions of normal goods ensure that an interior solution can be found and that (17) is strictly monotone in C_t. In our analysis, we do not rule out explosive paths, for the reasons highlighted in Cochrane [6].

time-0 variables now determined, the monetary policy rule determines R_1, which by no arbitrage is equal to $\hat{R}_1$ when $\mathcal{B} = \mathbb{R}$. As in period 0, equation (17) solves then for C_1 and equation (10) for Y_1. Knowing C_1 and Y_1, equation (14) can be solved for P_1, and equation (15) for M_1. Equation (1) then yields B_1, and from there the process continues to period 2 and on.

To verify whether the candidate equilibrium allocation and price system we derived above is an equilibrium, we need only to check that the household transversality and no-Ponzi conditions (6) and (16) hold. To this end, we first restrict fiscal policy to a (broad) class which ensures the policy is Ricardian, and second, we make the following assumption:

Assumption 1 $\exists\ \overline{R} : R_t \leq \overline{R}$.

Assumption 1 imposes an upper bound on nominal interest rates. The appendix studies more general cases where Assumption 1 is not necessary; in those cases, it may not be possible to find equilibria with a perfectly anticipated run on the central bank's interest rate peg, such as the one we will study in section 4, but there will instead be equilibria where runs occur with positive probability.

The role of Assumption 1 is to ensure that the amount of seigniorage revenues that the government can raise remains bounded, which (together with the path of fiscal policy specified below) ensures that the household budget constraint is well specified.

As a specific example of Ricardian fiscal policy, we assume T_t satisfies

Assumption 2 *There exist finite* $\overline{B} > 0$ *and* $\overline{T}$ *such that*

- *if* $B_{t-1} \in [-\overline{B}P_{t-1}, \overline{B}P_{t-1}]$, T_t *is unrestricted except* $|T_t|P_{t-1} \leq \overline{T}$,
- *if* $B_{t-1} > \overline{B}P_{t-1}$, $T_t \in [\alpha B_{t-1}, B_{t-1}]$, *and*
- *if* $B_{t-1} < -\overline{B}P_{t-1}$, $T_t \in [-B_{t-1}, -\alpha B_{t-1}]$.

Essentially, we require that if real debt is neither too high nor too low, taxes may be any function of past information subject only to a uniform bound in real terms. But when real debt exceeds a threshold (in absolute value), taxes cover at least a fraction α of debt, putting the brakes to a debt spiral.

We relegate the proof that (6) and (16) hold (and thus the candidate equilibrium is an equilibrium) to the appendix.

In the construction we just completed, P_0 is indeterminate, but once a value of P_0 is specified, there exists a unique equilibrium allocation and price system. Moreover, whenever the nominal rate set by the central bank is low, so is inflation. In particular, if $R_t = \frac{1}{\beta} - 1$ for all $t \geq 0$, then inflation is exactly zero in all periods.

The appendix considers a more general case, in which uncertainty is present and sunspot equilibria may arise (particularly if assumption 1 is retained). But, even in that case, a low official interest rate translates into a limit on expected inflation. To see this, note that the intratemporal optimization condition (13) and the market clearing condition (10) still hold in a world with sunspots, so equation (17) still holds. Thus consumption and labor in each period are pinned down by the interest rate policy. If R_t is constant then consumption and labor are constant. If $R_t = \frac{1}{\beta} - 1$, the stochastic version of the consumption Euler equation becomes

$$E_t \frac{P_t}{P_{t+1}} = 1. \tag{18}$$

The expected real value of a dollar remains constant into the future. Furthermore, if we assume a bound ϵ on how fast the price level can drop (i.e., we impose $P_t/P_{t+1} < 1/\epsilon$ almost surely $\forall t$), then the law of large numbers will apply, and average inverse inflation over long horizons will be 0:

$$\lim_{T \to \infty} \frac{1}{T} \sum_{s=1}^{T} \frac{P_s}{P_{s+1}} = 1 \text{ almost surely.} \tag{19}$$

In the next section, we show that a very different type of equilibrium emerges when households are not allowed to borrow unlimited funds from the central bank. In these equilibria, a low interest rate set by the central bank may well lead to high inflation instead.

4 Limits to Central Bank Lending

Suppose now we impose the additional constraint on the households that $B_t \geq 0$, $t \geq 0$: households are not allowed to borrow from the government/central bank (or, equivalently, they are allowed to borrow from the central bank only by posting government bonds as collateral). That the borrowing limit is precisely zero is not central to our analysis, but simplifies exposition somewhat. In this section, we construct additional deterministic equilibria which do not exist when $\mathcal{B}_t = \mathbb{R}$.

With the no-borrowing limit we just imposed, the official rate R_t only becomes a lower bound for the private-sector rate $\hat{R}_t$. When households are at the borrowing limit with the central bank, private nominal interest rates may exceed the official rate. The no-arbitrage condition (12) becomes

$$\hat{R}_t \geq R_t, \quad B_t > 0 \Longrightarrow \hat{R}_t = R_t. \tag{20}$$

All other equilibrium conditions remain the same, except that the private rate $\hat{R}_t$ replaces the government rate R_t in equation (17):

$$-\frac{u_y(C_t, C_t + \overline{G})}{u_c(C_t, C_t + \overline{G})} = \frac{1}{1 + \hat{R}_t}. \tag{21}$$

The allocation of section 3 remains part of an equilibrium even when the central bank limits its lending, provided that households have nonnegative bond holdings in all periods. For a given sequence of prices, interest rates, consumption and work levels, household holdings of government debt in this equilibrium depend on the sequence of taxes. Government debt will be strictly positive in each period $t > 0$ if and only if the following condition is satisfied:

$$\frac{T_t}{P_{t-1}} < \overline{G} + \frac{B_{t-1}}{P_{t-1}} + \frac{M_{t-1}}{P_{t-1}} - \frac{\beta \hat{c}(R_t)(1 + R_t)\hat{u}_y(R_t)}{\hat{u}_y(R_{t-1})}, \tag{22}$$

where $\hat{c}(R)$ is the consumption implied by equation (21) when $\hat{R}_t = R$ and $\hat{u}_y(R) := u_y(\hat{c}(R), \overline{G} + \hat{c}(R))$. It is straightforward to see that there are fiscal rules that satisfy (22) and Assumption 2.[8] We assume that fiscal policy is run by one such rule.

In period 0, government debt will be nonnegative if

$$T_0 \leq W_{-1} - \hat{c}(R_0)P_0. \tag{23}$$

[8] As an example, choose $T_t = (1 - \alpha)(B_{t-1}/P_{t-1}) + \hat{T}_t$, with $\hat{T}_t < P_{t-1}\overline{G} - M_{t-1} - \frac{P_{t-1}\beta\hat{c}(0)(1+\overline{R})\hat{u}_y(\overline{R})}{\hat{u}_y(0)}$ and $\alpha \in (0, 1)$.

An interior equilibrium will only exist if

$$T_0 < W_{-1}, \tag{24}$$

which we will assume. While P_0 can take any positive value in section 3, now equation (23) imposes a ceiling.

4.1 Additional Equilibria: A Single Run

The simplest equilibrium that may arise when a limit to private indebtedness is introduced is a run on government debt where $B_s = 0$ for a single date $s > 0$. We now provide conditions under which such an equilibrium exists.

Assumption 3 *Define*

$$\overline{u}_y := \max_{R \in [0,\overline{R}]} \hat{c}(R)(1+R)|\hat{u}_y(R)|.$$

We assume that fiscal policy satisfies the following stronger version of (22):

$$\frac{T_t}{P_{t-1}} < \overline{G} + \frac{B_{t-1}}{P_{t-1}} + \frac{M_{t-1}}{P_{t-1}} - \frac{\beta \overline{u}_y}{\underline{u}_y(\overline{G})}. \tag{25}$$

Equation (22) guarantees that in each period there are positive bonds that can be converted into money and initiate a speculative run. The stronger condition (25) ensures that, *after* a period in which a run occurred and thus previous government debt was monetized, there are enough new bonds for the economy to return to a path where households hold positive amounts of government debt and equation (14) holds.

Proposition 1 *Let $\{P_t, \hat{R}_t, R_t, T_t, C_t, Y_t, \hat{B}_t, B_t, M_t\}_{t=0}^{s-1}$ be determined as in the equilibrium of section 3, with P_0 satisfying (23), and let fiscal policy satisfy Assumption 2. A necessary and sufficient condition for the existence of a different (deterministic) equilibrium in which $B_s = 0$ is that the following equation admits a solution for $\hat{R}_s > R_s$:*

$$\beta \hat{u}_y(\hat{R}_s)(1+\hat{R}_s)\hat{c}(\hat{R}_s)\left(\frac{P_{s-1}}{M_{s-1} + B_{s-1} + P_{s-1}\overline{G} - T_s}\right) = \hat{u}_y(R_{s-1}). \tag{26}$$

A sufficient condition (based on preferences alone) for (26) to have a solution with $\hat{R}_s > R_s$ is

$$\lim_{R \to \infty} |\hat{u}_y(R)|(1+R)\hat{c}(R) \to \infty. \tag{27}$$

Proof: The proof works by construction. Starting from an arbitrary price level P_0 that satisfies (23), the equilibrium allocation, price system, and government policy are solved as in section 3 up to period $s-1$. Specifically, we use the interest rate rule R_t and the fiscal policy rule with equations (10), (14), (15), and (17) to sequentially solve for the unique candidate equilibrium allocation and price system.

In period s, in order for $\hat{R}_s > R_s$ to be an equilibrium, the constraint $B_s \geq 0$ must be binding, which implies

$$\frac{M_{s-1}+B_{s-1}}{P_{s-1}} + \overline{G} = \frac{T_s}{P_{s-1}} + \hat{c}(\hat{R}_s)\frac{P_s}{P_{s-1}}. \tag{28}$$

Furthermore, equations (14) and (21) require

$$\beta(1+\hat{R}_s)\hat{u}_y(\hat{R}_s)\frac{P_{s-1}}{P_s} = \hat{u}_y(R_{s-1}). \tag{29}$$

Substituting (28) into (29), we obtain (26), which is a single equation to be solved for $\hat{R}_s$. If this equation does not admit a solution for $\hat{R}_s > R_s$, then it is impossible to satisfy all of the necessary conditions for an equilibrium with $B_s = 0$. If a solution exists, then we can retrieve consumption in period s as $C_s = \hat{c}(\hat{R}_s)$ (the unique solution that satisfies equation (21)), and hence (by market clearing) $Y_s = C_s + \overline{G}$. We can then solve equation (28) for the candidate equilibrium level of P_s. Equation (22) ensures that the solution for P_s is strictly positive.

From period $s+1$ onwards, the allocation and price system is once again uniquely determined (sequentially) by the interest rate rule R_t, the fiscal policy rule, and equations (10), (14), (15), and (17). Equation (25) ensures that the resulting sequence for government debt is strictly positive. Once again, the proof that (6) and (16) hold is relegated to the general proof in the appendix.

Finally, to verify the sufficient condition (27), set $\hat{R}_s = R_s$. Equations (14) and (22) imply

$$\beta|\hat{u}_y(R_s)|(1+R_s)\hat{c}(R_s)\left(\frac{P_{s-1}}{M_{s-1}+B_{s-1}+P_{s-1}\overline{G}-T_s}\right) < |\hat{u}_y(R_{s-1})|. \tag{30}$$

Since $|\hat{u}_y(R)|(1+R)\hat{c}(R)$ is a continuous function of R, when equation (27) holds, equation (30) ensures the existence of a solution of (26) with $\hat{R}_s > R_s$. QED.

To be concrete, consider the following numerical example. Let the monetary authority set $R_t = \frac{1}{\beta} - 1$, where $\beta = 1/1.01$, for all t and all histories (and thus we can set $\overline{R} = \frac{1}{\beta} - 1$ as well.)

Next, let $u(c_t, y_t) = \frac{c^{1-\sigma}}{1-\sigma} - y^{\psi}$, with $\sigma = 3$ and $\psi = 1.1$, and let $\overline{G} = .1$. Given these, equation (25) becomes

$$T_t < B_{t-1} + M_{t-1} - 1.12P_{t-1}. \tag{31}$$

Thus we assume $T_t = .5(B_{t-1} + M_{t-1}) - 1.12P_{t-1}$ which satisfies (31) whenever $B_{t-1} + M_{t-1} > 0$ (which holds throughout the example). Finally, assume $P_0 = 1$ and $W_{-1} = 2.57$.

Given these assumptions, one equilibrium of this economy is a steady state: In each period $t \geq 0$, $P_t = 1$, $C_t = M_t = .96$, $Y_t = 1.06$ and $B_t = 1.5$. And for the given P_0, when households have an unlimited ability to borrow from the government, this is the unique deterministic equilibrium.

Next suppose households face a restriction that $B_t \geq 0$ for all $t \geq 0$. Then, the following is a deterministic equilibrium for any date $s > 0$. In the first $s-1$ periods, all variables are equal to their values under the steady state equilibrium just defined. At date s, the run occurs. For the chosen parameters, if $B_s = 0$, then $P_s = 4.07$, $M_s = 2.46$, $C_s = .6$, $Y_s = .7$, and $\hat{R}_s = 3.28$. In all subsequent periods $t > s$, $\hat{R}_t = R_t$, $P_t = 4.24$, $M_t = 4.09$, and real variables C_t and Y_t return to their pre-run steady-state values. Government debt B_t then gradually approaches a new steady state from below, where B_t/P_t returns to its previous steady state value.

To see why this is an equilibrium, notice first that, when the run occurs, all government debt is converted into money; this largely increases the money supply. Furthermore, if a run occurs, then the private-sector interest rate $\hat{R}_t$ must be greater than the interest rate set by the central bank, which is constant at $1/\beta - 1$. The intratemporal optimality condition (21) implies that consumption decreases in period s when the run occurs. With consumption down and the money supply up, the price level must jump up so that the (binding) cash-in-advance constraint holds. Whether such a candidate allocation can be supported as an equilibrium depends on whether these changes can be made consistent with the household Euler equations for leisure and consumption, which are respectively (14) and

$$\frac{u_c(C_{t+1}, Y_{t+1})}{u_c(C_t, Y_t)} = \frac{1}{\beta(1 + \hat{R}_t)} \frac{P_{t+1}}{P_t}. \tag{32}$$

Specifically, in order to have a perfectly anticipated run in period s (and not before), it must be the case that households are willing to lend to the government in period $s-1$ (i.e., $\hat{R}_{s-1} = R_{s-1}$) even though the nominal interest rate by the central bank is constant and expected inflation between period $s-1$ and period s is high. Since households expect a consumption drop between periods $s-1$ and s, this can be the case, but only if either the drop in consumption (and, by market clearing, in the labor supply) is very steep or the intertemporal elasticity of substitution of consumption is sufficiently low. Equation (21) implies that the consumption drop is steeper, the less curvature there is in the marginal disutility of labor and in the marginal utility of consumption. So, less curvature in $u_y(c, c+\overline{G})$ unambiguously helps in satisfying equation (32). Less curvature in $u_c(c, c+\overline{G})$ has an ambiguous effect, since (for given $\hat{R}_s$) it creates a bigger drop in consumption, but it also implies a greater intertemporal elasticity of substitution. The second effect turns out to be the relevant one, so that a perfectly anticipated run can happen when the curvature is low and hence the function $\hat{c}$ is not very responsive to R. From these observations, we can thus understand the role of assumption A2. We can also understand why a run can happen under much weaker assumptions if it occurs with probability smaller than one, as described in the appendix: in this case, the potentially negative effect of a run on the households' willingness to save between periods $s-1$ and s is tempered by the lower probability of the occurrence. In the limit, as the probability of a run goes to 0, households are content to save at the rate $1/\beta - 1$ between periods $s-1$ and s when the no-run allocation remains at the steady state throughout.

Next, we consider the other intertemporal choice that households face in their decision to save between periods $s-1$ and s, i.e., their labor supply. Because of the cash-in-advance timing, this decision is related to the household labor supply in periods $s-2$ and $s-1$, as shown by equation (14). Since the allocation and inflation are at the no-run steady state values in these two periods, the relevant Euler equation for leisure is automatically satisfied. For this reason, the intertemporal elasticity of substitution of leisure does not play the same role as the one of consumption in determining whether a perfectly anticipated run can occur.

Having discussed the economic forces that lead households to save between periods $s-1$ and s, we next consider the elements that pertain to the private-market interest rate between periods s and $s+1$, in the period of the run. This time, it is simpler to start from the Euler equation for labor, equation (14). The relevant margin of choice for households is their labor supply in period $s-1$ (paid in period s) vs. period s. Here, it is straightforward to see why households optimally choose not to invest in government bonds in period s at the nominal rate $1/\beta - 1$. First, the nominal wage (which is equal to the price level) increases from period $s-1$ to period s, which yields an incentive to postpone labor when the nominal interest rate does not adjust correspondingly. Second, the equilibrium features actually a lower labor supply in period s than in period $s-1$, providing a further incentive not to save in period $s-1$ and to postpone work. Both of these channels imply that the interest rate offered by the government within the equilibrium allocation is too low for households to be willing to lend to the government, and that the private-market interest rate that justifies the labor decision is instead higher. Similarly, on the consumption side (where the relevant margin is once again shifted one period forward), households look forward to an increase in consumption between periods s and $s+1$, and hence they require a higher real interest rate to be willing to save than the one offered by the government. This is particularly true because further inflation occurs between periods s and $s+1$, as we establish next, in our discussion of how the run ends.

After the run ends, households resume lending to the government at the rate $R_{s+1} = 1/\beta - 1$ in period $s+1$. With a fixed nominal interest rate, inflation between period s and $s+1$ must adjust so that households find it optimal to increase their labor supply between the crisis period s and the return to normalcy in period $s+1$. By equation (14), this requires further inflation between periods s and $s+1$. The increase in both prices and production (and consumption) between periods s and $s+1$ implies that money supply must also grow. Since the crisis wiped out government debt, households cannot acquire this additional money by selling government debt. While part of the money can be acquired through the sales of output to the government in period s, a crisis will also require that fiscal policy generates new nominal liabilities through a tax cut at the beginning of period $s+1$, as implied by Assumption 3.

From that point onward, output and consumption return to their pre-run steady state, while government debt (in real terms) converges back to the steady state gradually.

4.2 Other Equilibria

By repeating the steps outlined in section 4.1, it is easy to construct equilibria in which runs occur repeatedly, and it is also possible to construct equilibria in which runs last for more than one period. The conditions under which such equilibria exist are similar to those for a single run (in particular, Assumptions 1, 2, and 3 are sufficient conditions). In more general cases, the appendix considers stochastic equilibria, where runs can emerge with probability less than 1. In these stochastic run equilibria, even when the official interest R_t is constant, the levels of consumption and labor are not constant because the effective interest rate in the household optimization conditions, $\hat{R}_t$, is not constant. Further, when $R_t = \frac{1}{\beta} - 1$, it is no longer the case that on average, $P_t/P_{t+1} = 1$. Setting a low nominal rate no longer guarantees low average real depreciation of the currency.

4.3 Alternative Strategies to Mitigate and/or Prevent Runs

The simplest way to prevent runs is of course not to adopt an interest rate rule in the first place. As an example, if preferences are $u(c, l) = \log c - \kappa(l)$, a fixed money supply will deliver a unique equilibrium. More in general, Atkeson et al. [3], following the methods in Bassetto [4], devise more sophisticated strategies to achieve unique implementation by reverting to money supply rules when the inflation rate deviates from its target.[9] But, as is well known (see e.g. Woodford [15]), money supply rules may also be subject to multiple equilibria. More importantly, our aim is not to assess the relative merits of interest-rate rules compared to money supply rules or other, more sophisticated strategies, but rather to point out a danger that arises specifically when a central bank commits to any given interest rate.

The presence of runs generates a new channel of interaction between monetary and fiscal policy. Note that, when we restrict discussion to Ricardian fiscal policies and equilibria without borrowing limits, fiscal policy is irrelevant in determining equilibrium consumption and labor levels. (In fact, this is the entire point of Ricardian equivalence.) When limits are present and

[9] Atkeson et al. also consider sophisticated strategies that only rely on interest rates, but those would be subject to the runs described in this paper.

the equilibrium features runs, this is no longer the case: intuitively, the consequences of a run will be more severe, the greater the pool of bonds that is available to be monetized. As an example, consider the run equilibrium of the section 4.1, but with a different tax policy. In particular, instead of $T_t = .5(B_{t-1} + M_{t-1}) - 1.12P_{t-1}$, let $T_t = .6(B_{t-1} + M_{t-1}) - 1.12P_{t-1}$. This leaves consumption and output unchanged in the no-run equilibrium, but decreases the steady state level of debt from 1.5 to 1.09. Now, at date s (when the run occurs), $B_s = 0$ (as before), but since B_{s-1} is now lower, there is less debt to convert into money, and thus the money rises less from period $s-1$ to period s. In this new example, P_s rises from 1 to 3.08 (instead of rising to 4.08), M_s rises from .96 to 2.04 (instead of rising to 2.46), C_s falls from .96 to .66 (instead of falling to $C_s = .6$), and $\hat{R}_s$ rises to 2.22 instead of rising to 3.28. Overall, that the increase in the money supply is smaller due to the smaller date $s-1$ debt causes smaller *real* effects (on consumption and output) from the run.

An alternative strategy to mitigate the consequences of a run is to assume that the central bank only sets the price of a subset of bonds. In the real world, this naturally happens due to the presence of long-term debt, whose price is not directly targeted by the central bank. Here, both to sharpen our point and to simplify the notation, we consider the case in which there are two types of bonds, "red" bonds and "blue" bonds, both with one-period maturity, whose only difference stems from their treatment by the central bank, as opposed to having the central bank target rates for some maturities and not others. Specifically, we assume that, when asset markets open, the central bank sets the interest rate on red bonds, being willing to purchase or sell them at a rate R_t (which may depend on past history, as before). In contrast, blue bonds are auctioned. From the fiscal perspective, red bonds and blue bonds are identical: both constitute a promise to deliver a dollar to the holder at the beginning of the subsequent period. We assume that taxes are set according to a fiscal policy rule that satisfies Assumptions 2 and 3, where B_t refers to the total amount of bonds (red and blue). In addition, we need to specify a rule that describes the supply of blue bonds at auction, as a function of past history. Letting B_t^B be the amount of blue bonds being auctioned in period t and maturing in period $t+1$, we assume that this rule satisfies the following assumption:[10]

Assumption 4

$$0 \leq B_t^B < P_{t-1}\overline{G} + B_{t-1} + M_{t-1} - \frac{\beta P_{t-1}\overline{u}_y}{\underline{u}_y(\overline{G})} - T_t. \tag{33}$$

It is straightforward to prove that Assumption 4 is sufficient for the existence of an interior equilibrium, in which private agents hold a strictly positive amount of red bonds. The allocation and price system in this equilibrium coincides with the one computed in Section 3. In this equilibrium, blue bonds, red bonds, and privately-issued bonds are perfect substitutes from the household perspective, and trade at the same interest rate. That the central bank targets a narrower segment of the bond market is thus immaterial for its ability to control inflation and real activity.

In the event of a run, the presence of blue bonds makes a difference. Households again perceive blue bonds, red bonds, and privately-issued bonds as perfect substitutes. But if a run occurs in period t, the interest rate R_t sanctioned by the central bank for red bonds is lower than the private-sector rate $\hat{R}_t$, and consequently households do not buy any red bonds. At the same time, if a positive amount of blue bonds is offered at auction, households will bid for them, at the interest rate $\hat{R}_t$. The evolution of money supply in period t will thus be governed by the following equation:[11]

$$M_t = P_{t-1}\overline{G} + M_{t-1} + B_{t-1} - \frac{B_t^B}{1 + \hat{R}_t}. \tag{34}$$

Ceteris paribus, the sale of blue bonds reduces the monetization of maturing government debt, alleviating the consequences of the run. We can illustrate this point using our numerical example once again. Let all the parameter values, the initial conditions, and the rules for T_t and R_t be those of Section 4, but assume that, in each period, blue bonds are supplied according to the following rule: $B_t^B = .4(B_{t-1} + M_{t-1})$, so that, in steady state, blue bonds represent roughly 2/3 of government debt. In this case, if a run occurs in period s, government debt B_s does not drop from 1.51 to 0, but to 0.99. Because of this, the increase in money supply is more contained: money supply rises from .96 to 2.18 (rather than 2.46). This in turn alleviates the effect on

[10] Assumption 3 ensures that the interval for B_t^B is nonempty after all histories.

[11] This equation is derived from (1), by assuming that in the event of a run red bonds are 0 and thus $B_t = B_t^B$.

consumption, that falls from .96 to .64 (rather than .6), on the nominal interest rate (rising to 2.56 rather than 3.28), and prices (rising on impact to 3.52 rather than 4.08).[12]

The blue bond-red bond model suggests that a central bank would be well advised to peg the interest rate in a narrow segment of the market, rather than across the entire spectrum of available bonds. When no run occurs, the two strategies implement the same set of equilibria. But, when the risk of runs is present, the consequences of a broad peg are more acute than those of a policy that sets the price in a narrower market. This conclusion provides a rationale for the widespread practice among central banks to set interest rate targets only for very short-term rates, rather than trying to impose an entire yield curve on the market. Even in recent times, when several central banks have tried to affect the yield curve by policies of "quantitative easing," it is noteworthy that they chose to do so by setting an *interest rate target* for the short end, and a *quantity target* for their purchases of longer-term securities.[13] (It is also noteworthy that the Fed's attempt to peg the entire yield curve in the 1940's ultimately led the Fed to be the sole purchaser of short-term Treasury debt.)

5 Discussion

In this paper, we have shown that considering *bounds* on open market operations may be crucial in determining the size of the set of monetary equilibria under interest rate rules. Policies which have unique equilibria in environments with no bounds may instead have many new equilibria when bounds are introduced. The particular bound we studied was on the size of privately held government debt – we assumed it must not be negative.

Suppose instead we had assumed that if a run is seen as occurring, the monetary authority stops it by not letting, say, government debt fall below 90% of its previous value. That is, the

[12] At first blush, the effect of blue bonds on the allocation and prices may seem surprisingly small, considering that they represent 2/3 of government debt in steady state. This happens because, according to the rule that we specified, the government auctions a *fixed nominal future repayment.* Given the very high nominal interest rates that prevail in a run, the real revenues raised by the auction in the event of a run are comparatively modest.

[13] In our simple model, of course, quantitative easing would have no effect on the equilibrium allocation and prices. But our results would apply equally well to richer environments where a preferred habitat is present.

central bank abandons the interest rate peg at that point. Then, of course, it is impossible for debt to go to zero in one period as in our examples. On the other hand, the same logic as our examples still holds, except that the lower bound on debt is no longer zero, but 90% of its previous value. What causes these additional equilibria is the existence of the bounds themselves, not their particular values.

Also, in the simple setup that we described, a run on an interest peg triggers immediate monetization of all of the government debt. This may be a good description of the experience of the Reichsbank during the German hyperinflation, but it is unlikely that a run would suddenly appear in this form in an economy that has previously experienced stable inflation and macroeconomic conditions. In practice, the unfolding of a run would be slowed by a number of frictions that may prevent households from immediately demanding cash for all of their government bond holdings; these frictions may take the form of limited participation in bond markets (see e.g. Grossman and Weiss [8], Alvarez and Atkeson [1], and Alvarez, Atkeson, and Edmond [2]), noisy information about other households' behavior, or the presence of long-term bonds whose price is not pegged by the central bank.

The questions we addressed are particularly important in the wake of quantitative easing. In our model, we do not distinguish between the monetary authority and the fiscal authority. In our run equilibrium, in essence, the monetary authority monetizes the debt. If that monetary authority proposed to limit such a run by not letting debt fall below 90% of its previous value, it could do this by simply abandoning the interest rate target and not buying government debt at some point. With quantitative easing, however, central banks *themselves* now owe large debts to private institutions in the form of excess bank reserves. We interpret excess reserves in our model to be part of B_t, not M_t, since in equilibrium they must pay the market rate of interest. While a central bank can refuse to turn government debt into cash by simply not purchasing it, it is unclear to us how a central bank can refuse to turn excess reserves into cash without explicitly or implicitly defaulting. Thus the dangers we outline in this paper may be more relevant now than ever.

A Analysis of the General Stochastic Case

A.1 The Environment with Sunspots

We modify the environment described in section 2 by introducing a sunspot variable s_t in each period. Without loss of generality, s_t is i.i.d. with a uniform distribution on $[0, 1]$. Its realization at time t is observed before any action takes place. All variables with a time-t subscript are allowed to be conditional on the history of sunspot realizations $\{s_j\}_{j=1}^t$.

We assume that the government only trades in one-period risk-free debt, but we allow the households to trade state-contingent assets, and we denote by a_{t+1} the amount of nominal claims that a household purchases in period t maturing in period $t+1$ (conditional on the sunspot realization s_{t+1}). Without uncertainty, $a_{t+1} \equiv \hat{b}_t$. Equation (3) is thus replaced by

$$E_t[a_{t+1}Q_{t+1}] + \frac{b_t}{1+R_t} = P_{t-1}(y_{t-1} - c_{t-1}) - T_t - m_t + m_{t-1} + a_t + b_{t-1}, \tag{35}$$

where Q_{t+1} is the stochastic discount factor of the economy. For the later analysis, it is convenient to define $\hat{R}_t := 1/E_tQ_{t+1} - 1$. This definition is consistent with the notation that we used in the main text for the deterministic case: $\hat{R}_t$ is the one-period nominal risk-free rate in the market for private credit.

In period 0, the household budget constraint becomes

$$E_0[a_1Q_1] + \frac{b_0}{1+R_0} = W_{-1} - m_0 - T_0. \tag{36}$$

The no-Ponzi condition (6) generalizes to

$$\begin{aligned} &a_{t+1} + b_t \geq \underline{A}_{t+1} := -P_t - m_t + T_{t+1} + \\ &E_{t+1}\sum_{j=1}^{\infty}\left\{\left(\prod_{v=1}^{j} Q_{t+v+1}\right)\left[T_{t+j+1} - P_{t+j} - \max_{\hat{b}\in\mathcal{B}_t}[\hat{b}\left(E_{t+j}Q_{t+j+1} - \frac{1}{1+R_{t+j}}\right)]\right]\right\}. \end{aligned} \tag{37}$$

With these changes, an equilibrium is defined as in section 3; the market-clearing condition (11) becomes

$$A_{t+1} = 0. \tag{38}$$

The conditions characterizing an equilibrium are given by (10), (15), (21), (38), the stochastic Euler equation

$$\frac{u_y(C_{t+1}, Y_{t+1})}{u_y(C_t, Y_t)} = \frac{Q_{t+1}(1+\hat{R}_t)}{\beta(1+\hat{R}_{t+1})}\frac{P_{t+1}}{P_t}, \tag{39}$$

the transversality condition, which in the stochastic case becomes[14]

$$\lim_{t\to\infty} E_0 \left[\left(\prod_{j=1}^{t+1} Q_j \right) (A_{t+1} + B_t - \underline{A}_{t+1}) \right] = 0, \tag{40}$$

and finally the no-arbitrage condition for interest rates. This last condition states $\hat{R}_t = R_t$ when $\mathcal{B} = \mathbb{R}$ and (20) when $B_t \geq 0$ is imposed.

In the main text, we adopted Assumption 1 to ensure that seigniorage revenues remain bounded and hence that the present-value budget constraint of the households is well defined. When Assumption 1 is violated, such as in the case of Taylor rules that have no upper bound on the interest rate, an alternative (sufficient) condition that we can adopt is given by

Assumption 5

$$\lim_{R\to\infty} \hat{c}(R)(1+R) = 0. \tag{41}$$

Notice that Assumption 5 is incompatible with the sufficient condition (27) in Proposition 1. When Assumption 5 is adopted, often perfectly anticipated runs will fail to exist (but probabilistic runs will continue to occur).

A.2 Verification of the Transversality and no-Ponzi conditions

Proposition 2 *Let a sequence* $\{P_t, Q_{t+1}, T_t, R_t, C_t, Y_t, A_{t+1}, B_t, M_t\}_{t=0}^{\infty}$ *satisfy equations (10), (11), (15), (21), (35), (36),and (39), and let fiscal policy satisfy Assumption 2. Assume also that either Assumption 1 or Assumption 5 holds. Then equations (37) and (40) hold.*

We prove this proposition in 3 steps. First, we prove that $\underline{A}_{t+1}$, as defined in (37), is well defined. Second, we prove that (40) holds, and finally that (37) holds.

[14]See Coşar and Green [7].

A.2.1 $\underline{A}_{t+1}$ is well defined.

We work backwards on the individual components of the sum defining $\underline{A}_{t+1}$ in equation (37). From (20) we obtain[15]

$$\max_{\hat{b}\in\mathcal{B}_t}[\hat{b}\left(E_{t+j}Q_{t+j+1}-\frac{1}{1+R_{t+j}}\right)]=0. \tag{42}$$

Next, use (39) to get

$$\begin{aligned}
&E_{t+1}\left\{\Big(\prod_{v=1}^{j}Q_{t+v+1}\Big)P_{t+j}\right\}\leq \hat{u}_y(0)E_{t+1}\left\{\Big(\prod_{v=1}^{j}Q_{t+v+1}\Big)\frac{P_{t+j}}{\hat{u}_y(\hat{R}_{t+j})}\right\}=\\
&\hat{u}_y(0)E_{t+1}\left\{\Big(\prod_{v=1}^{j-1}Q_{t+v+1}\Big)\frac{P_{t+j}}{\hat{u}_y(\hat{R}_{t+j})}E_{t+j}Q_{t+j+1}\right\}=\\
&\hat{u}_y(0)E_{t+1}\left\{\Big(\prod_{v=1}^{j-1}Q_{t+v+1}\Big)\frac{P_{t+j}}{\hat{u}_y(\hat{R}_{t+j})(1+\hat{R}_{t+j})}\right\}=\\
&\beta\hat{u}_y(0)E_{t+1}\left\{\Big(\prod_{v=1}^{j-2}Q_{t+v+1}\Big)\frac{P_{t+j-1}}{\hat{u}_y(\hat{R}_{t+j-1})(1+\hat{R}_{t+j-1})}\right\}=\\
&\beta^{j-1}\frac{\hat{u}_y(0)P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1+\hat{R}_{t+1})}
\end{aligned} \tag{43}$$

Equation (43) implies[16]

$$E_{t+1}\sum_{j=1}^{\infty}\left\{\Big(\prod_{v=1}^{j}Q_{t+v+1}\Big)P_{t+j}\right\}\leq\frac{\hat{u}_y(0)P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1+\hat{R}_{t+1})(1-\beta)}, \tag{44}$$

which proves that the second piece of the infinite sum defining $\underline{A}_{t+1}$ is well defined. From Assumption 2, we have $|T_{t+j+1}|\leq\overline{T}P_{t+j}+|B_{t+j}|$, and so

$$\left|E_{t+1}\sum_{j=1}^{\infty}\left\{\Big(\prod_{v=1}^{j}Q_{t+v+1}\Big)T_{t+j+1}\right\}\right|\leq\sum_{j=1}^{\infty}E_{t+1}\left\{\Big(\prod_{v=1}^{j}Q_{t+v+1}\Big)\Big[P_{t+j}\overline{T}+|B_{t+j}|\Big]\right\}. \tag{45}$$

We analyze equation (45) in pieces. Using (44), we have

$$\overline{T}\sum_{j=1}^{\infty}E_{t+1}\left\{\Big(\prod_{v=1}^{j}Q_{t+v+1}\Big)P_{t+j}\right\}\leq\frac{\overline{T}\hat{u}_y(0)P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1+\hat{R}_{t+1})(1-\beta)}. \tag{46}$$

[15]If the borrowing limit is not 0, the expression in (42) would not be 0, but it can be proven that $\underline{A}_{t+1}$ is nonetheless well defined.

[16]We can interchange the order of the sum and the expectations since all elements of the sum have the same sign.

To work on the sum of debt, notice first that equation (1) continues to hold even if we replace R_t by $\hat{R}_t$. This is because $B_t = 0$ in the periods and states of nature in which $\hat{R}_t > R_t$. If Assumption 1 is retained, define $\overline{S} := \max_{R\in[0,\overline{R}]}[\hat{c}(R)(1+R)]$; alternatively, if Assumption 5 is adopted instead, define $\overline{S} := \max_{R\in[0,\infty]}[\hat{c}(R)(1+R)]$. Finally, notice that Assumption 2 implies

$$|T_{t+j} - B_{t+j-1}| \leq P_{t+j-1}(\overline{T}+\overline{B}) + (1-\alpha)|B_{t+j-1}|. \tag{47}$$

We can then use (1), (15), (39), and (47) to get

$$\begin{aligned}
&E_{t+1}\left\{\left(\prod_{v=1}^{j} Q_{t+v+1}\right)|B_{t+j}|\right\} = E_{t+1}\Bigg\{\left(\prod_{v=1}^{j-1} Q_{t+v+1}\right)\Bigg|\Big[P_{t+j-1}\overline{G}- \\
&T_{t+j} + B_{t+j-1} + \hat{c}(\hat{R}_{t+j-1})P_{t+j-1} - \hat{c}(\hat{R}_{t+j})P_{t+j}\Big]\Bigg|\Bigg\} = \\
&E_{t+1}\Bigg\{\left(\prod_{v=1}^{j-1} Q_{t+v+1}\right)\Bigg|\Big[P_{t+j-1}\overline{G} - T_{t+j} + B_{t+j-1} + \hat{c}(\hat{R}_{t+j-1})P_{t+j-1}- \\
&\frac{\beta P_{t+j-1}\hat{c}(\hat{R}_{t+j})(1+\hat{R}_{t+j})\hat{u}_y(\hat{R}_{t+j})}{\hat{u}_y(\hat{R}_{t+j-1})}\Big]\Bigg|\Bigg\} \leq \\
&E_{t+1}\Bigg\{\left(\prod_{v=1}^{j-1} Q_{t+v+1}\right)\Bigg[\left(\overline{G}+\overline{T}+\overline{B}+\frac{\beta\hat{u}_y(0)\overline{S}}{\hat{u}_y(\hat{R}_{t+j-1})}+\hat{c}(0)\right)P_{t+j-1}+ \\
&(1-\alpha)\,|B_{t+j-1}|\Bigg]\Bigg\}.
\end{aligned} \tag{48}$$

Using (43) and (48), we obtain (for $j > 1$)

$$\begin{aligned}
&E_{t+1}\left\{\left(\prod_{v=1}^{j} Q_{t+v+1}\right)|B_{t+j}|\right\} \leq E_{t+1}\Bigg\{\sum_{s=2}^{j}(1-\alpha)^{j-s}\Bigg[\left(\prod_{v=1}^{s-1} Q_{t+v+1}\right)\cdot \\
&\Bigg[\left(\overline{G}+\overline{T}+\overline{B}+\frac{\beta\hat{u}_y(0)\overline{S}}{u_y(\hat{R}_{t+s-1})}+\hat{c}(0)\right)P_{t+s-1}\Bigg]\Bigg\} + (1-\alpha)^{j-1}\frac{|B_{t+1}|}{1+\hat{R}_{t+1}} \leq \\
&\frac{\hat{u}_y(0)P_{t+1}\left(\overline{G}+\overline{T}+\overline{B}+\beta\overline{S}+\hat{c}(0)\right)}{\hat{u}_y(\hat{R}_{t+1})(1+\hat{R}_{t+1})}\sum_{s=2}^{j}\left[\beta^{s-2}(1-\alpha)^{j-s}\right] + (1-\alpha)^{j-1}\frac{|B_{t+1}|}{1+\hat{R}_{t+1}} = \\
&\frac{\hat{u}_y(0)P_{t+1}\left[(1-\alpha)^{j-1}-\beta^{j-1}\right]\left(\overline{G}+\overline{T}+\overline{B}+\beta\overline{S}+\hat{c}(0)\right)}{\hat{u}_y(\hat{R}_{t+1})(1+\hat{R}_{t+1})(1-\alpha-\beta)} + (1-\alpha)^{j-1}\frac{|B_{t+1}|}{1+\hat{R}_{t+1}}.
\end{aligned} \tag{49}$$

Using (49) we get

$$\begin{aligned}&\sum_{j=1}^{\infty} E_{t+1}\left\{\left(\prod_{v=1}^{j} Q_{t+v+1}\right)|B_{t+j}|\right\} \leq \\ &\frac{\hat{u}_y(0)P_{t+1}\left(\overline{G}+\overline{T}+\overline{B}+\beta\overline{S}+\hat{c}(0)\right)}{\hat{u}_y(\hat{R}_{t+1})(1+\hat{R}_{t+1})\alpha(1-\beta)}+\frac{|B_{t+1}|}{\alpha(1+\hat{R}_{t+1})}\end{aligned} \tag{50}$$

Collecting all terms, equations (44), (46), and (50) imply

$$\begin{aligned}&|\underline{A}_{t+1}| \leq \frac{\hat{u}_y(0)P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1+\hat{R}_{t+1})(1-\beta)}\Bigg[1+\overline{T}+ \\ &\left(\frac{1}{\alpha}\right)\left(\overline{G}+\overline{T}+\overline{B}+\beta\overline{S}+\hat{c}(0)\right)\Bigg]+ \\ &\frac{|B_{t+1}|}{\alpha(1+\hat{R}_{t+1})}+P_t\left[1+\hat{c}(0)+\overline{T}\right]+|B_t|.\end{aligned} \tag{51}$$

A.2.2 Equation (40) holds.

Use (49) to obtain

$$\begin{aligned}&\lim_{t\to\infty} E_0\left[\left(\prod_{j=1}^{t+1} Q_j\right)|B_t|\right] \leq \\ &\frac{\hat{u}_y(0)P_0\left(\overline{G}+\overline{T}+\overline{B}+\beta\overline{S}+\hat{c}(0)\right)}{\hat{u}_y(\hat{R}_0)(1+\hat{R}_0)(1-\alpha-\beta)}\lim_{t\to\infty}\left[(1-\alpha)^t-\beta^t\right]+ \\ &\frac{|B_0|}{1+\hat{R}_0}\lim_{t\to\infty}(1-\alpha)^t=0.\end{aligned} \tag{52}$$

We then use (39), (51), and (52) to prove

$$\begin{aligned}&\lim_{t\to\infty} E_0\left[\left(\prod_{j=1}^{t+1} Q_j\right)|A_{t+1}|\right] \leq \frac{\hat{u}_y(0)}{1-\beta}\Bigg[1+\overline{T}+ \\ &\left(\frac{1}{\alpha}\right)\left(\overline{G}+\overline{T}+\overline{B}+\beta\overline{S}+\hat{c}(0)\right)\Bigg]\lim_{t\to\infty} E_0\left[\left(\prod_{j=1}^{t+2} Q_j\right)\frac{P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})}\right]+ \\ &\frac{1}{\alpha}\lim_{t\to\infty} E_0\left[\left(\prod_{j=1}^{t+2} Q_j\right)|B_{t+1}|\right]+\hat{u}_y(0)\left[1+\hat{c}(0)+\overline{T}\right]\lim_{t\to\infty} E_0\left[\left(\prod_{j=1}^{t+1} Q_j\right)\frac{P_t}{\hat{u}_y(\hat{R}_t)}\right]+ \\ &\lim_{t\to\infty} E_0\left[\left(\prod_{j=1}^{t+1} Q_j\right)|B_t|\right]=\frac{\hat{u}_y(0)P_0}{(1+\hat{R}_0)\hat{u}_y(\hat{R}_0)}\Bigg\{\frac{\beta}{1-\beta}\Bigg[1+\overline{T}+ \\ &\left(\frac{1}{\alpha}\right)\left(\overline{G}+\overline{T}+\overline{B}+\beta\overline{S}+\hat{c}(0)\right)\Bigg]+1+\hat{c}(0)+\overline{T}\Bigg\}\lim_{t\to\infty}\beta^t=0.\end{aligned} \tag{53}$$

Equations (11), (52), and (53) imply (16).

A.2.3 Equation (37) holds.

The same steps used to prove (52) can also be used to prove

$$\lim_{j\to\infty} E_t \left\{ \left(\prod_{v=1}^{j+1} Q_{t+v} \right) |B_{t+j}| \right\} = 0. \tag{54}$$

As previously noted, equation (1) continues to hold even if we replace R_t with $\hat{R}_t$, since the two values only differ when $B_t = 0$. We can then iterate (1) forward, taking expectations conditional on time-$t+1$ information, and use (54) to obtain

$$\begin{aligned} B_t = M_{t+1} - M_t - T_{t+1} - P_t\overline{G} + E_{t+1}\Bigg\{ \sum_{s=1}^{\infty} \Bigg[\Bigg(\prod_{v=1}^{s} Q_{t+v+1} \Bigg) \cdot \\ \big(M_{t+s+1} - M_{t+s} + T_{t+s+1} - P_{t+s}\overline{G} \big) \Bigg] \Bigg\} > \underline{A}_{t+1}, \end{aligned} \tag{55}$$

which completes the proof. Equation (55) relies on $\overline{G} < 1$ (government spending must be less than the maximum producible output) and on

$$E_{t+s}\left[M_{t+s}(1 - Q_{t+s+1}) \right] = \frac{\hat{R}_{t+s} M_{t+s}}{1 + \hat{R}_{t+s}} \geq 0.$$

This completes the proof of proposition 2.

B Other Equilibria of the Stochastic Economy

The perfectly anticipated run described in section 4.1 relies on strong assumptions about preferences. As an example, if we assume that preferences are given by $u(c_t, y_t) = \frac{c^{1-\sigma}}{1-\sigma} - y^{\psi}$, such an equilibrium will always fail to exist for $\sigma \leq 1$, since a solution to (26) cannot be found (with $\hat{R} > R$). Nonetheless, even for these preferences other equilibria that feature runs exist, provided that the occurrence of a run is sufficiently small. Moreover, these equilibria exist even when the central bank sets no upper bound to its interest rate (provided, of course, that preferences are such that the present value of seigniorage remains finite).

As is known since Sargent and Wallace [12], even without considering runs, setting monetary policy as an interest rate rule leaves open the possibility of sunspot equilibria. But equilibria with runs are qualitatively very different from these sunspot equilibria. In a standard environment where $\mathcal{B} = \mathbb{R}$ and no runs can occur, the nominal interest rate is closely related to expected (inverse) inflation, so that setting the nominal interest rate still allows the central bank a considerable degree of control, at least over long periods of time. This relationship between nominal interest rates and expected inflation is lost in equilibria that feature runs, and the dangers from relying purely on the nominal interest rate as a policy instrument are correspondingly more acute.

B.1 Sunspot Equilibria with no Runs

We can construct sunspot equilibria recursively as follows. For any arbitrary initial price P_0, the variables R_0, T_0, C_0, Y_0, M_0, and B_0 are determined as in section 3. The time-0 variables and the policy rules determine R_1 and T_1, also as in section 3, which then pin down C_1 and Y_1; this implies that C_1 and Y_1 are known as of period 0.[17] But now the deterministic Euler equation (14) is replaced by its stochastic counterpart, (39). In an equilibrium with no runs, we know that $\hat{R}_1 = R_1$. Substituting this into (39), rearranging and taking expected values we obtain

$$E_0 \frac{P_0}{P_1} = \frac{u_y(C_0, Y_0)}{\beta u_y(C_1, Y_1)(1 + R_1)}. \tag{56}$$

We can then pick P_1 as an arbitrary function of the sunspot s_1, subject to the single restriction (56) on its expected value. Given the realization of s_1 and thus P_1, equation (15) determines M_1, equation (1) yields B_1, and the process can be repeated for period 2.

Provided that either Assumption 1 or 5 hold, Proposition 2 ensures that the transversality and no-Ponzi conditions are satisfied for the sequences that we constructed: as discussed in Cochrane [6], in this model only fiscal policy can provide a boundary condition to rule out some of these arbitrary paths.[18]

[17] We assume that the monetary and fiscal authorities follow deterministic rules; this is immaterial to our results.

[18] Notice that uniqueness results based on the failure of both Assumptions 1 and 5 relate to fiscal policy: some sunspot paths can be ruled out because seigniorage revenues become infinite, making it impossible for fiscal policy

While sunspot equilibria imply that inflation is indeterminate, equilibria that feature no runs still display remarkable similarities across each other. As an example, suppose that monetary policy sets $R_t \equiv 1/\beta - 1$ unconditionally. It is straightforward to verify that equation (17) implies a constant allocation, and that (39) implies (18): the expected real value of a dollar remains constant. Equation (18) and the assumption of a uniform bound on P_t/P_{t+1} in turn imply (19).

B.2 A Probabilistic Run in Period $s > 0$.

We now construct an equilibrium where a run occurs in period s with probability $\phi \in (0, 1)$. As was the case in section 4.1, fiscal policy plays an important role in ensuring that households have enough nominal wealth to acquire their desired money balanced, and we assume that (25) holds. Starting from an arbitrary initial price level P_0, we construct recursively a deterministic allocation and price system up to period $s-1$ as we did in section 4.1. For period s, we consider an equilibrium with just two realizations of the allocation and price level: with probability ϕ, the price level is P_s^H and a run occurs ($\hat{R}_s^H > R_s$), and with probability $1-\phi$ the price level is P_s^L and the private nominal interest rate coincides with the public one: $\hat{R}_s^L = R_s$. In order for $\hat{R}_s^H > R_s$ to be an equilibrium, the constraint $B_s \geq 0$ must be binding, which implies

$$\frac{M_{s-1} + B_{s-1}}{P_{s-1}} + \overline{G} = \frac{T_s}{P_{s-1}} + \hat{c}(\hat{R}_s^H)\frac{P_s^H}{P_{s-1}}. \tag{57}$$

Given any arbitrary value $\hat{R}_s^H > R_s$, and given the predetermined time-$s-1$ variables and the fiscal policy rule for T_s, equation (57) can be solved for P_s^H/P_{s-1}, the level of inflation that will occur if a run on the interest rate peg materializes in period s. As was the case in section 4.1, since $\hat{c}$ is a decreasing function and taxes satisfy (22), inflation in the event of a run will necessarily be strictly greater than inflation in the equilibrium in which no run can take place.

To determine P_s^L/P_{s-1}, we rely on the household Euler equation (39). Rearranging terms and taking the expected value as of period $s-1$, we obtain

$$\beta\left[\phi\hat{u}_y(\hat{R}_s^H)(1+\hat{R}_s^H)\frac{P_{s-1}}{P_s^H} + (1-\phi)\hat{u}_y(R_s)(1+R_s)\frac{P_{s-1}}{P_s^L}\right] = \hat{u}_y(R_{s-1}). \tag{58}$$

to be Ricardian.

Generically, this equation can be solved for P_s^L/P_{s-1}. However, we need to ensure that the solution is nonnegative, and that it entails nonnegative bond holdings, i.e., that

$$M_{s-1} + B_{s-1} + P_{s-1}\overline{G} \geq \frac{T_s}{P_{s-1}} + \hat{c}(R_s)\frac{P_s^L}{P_{s-1}} \tag{59}$$

A sufficient condition for both is that ϕ be sufficiently small.[19]

If $\hat{u}_y$ does not decline too fast with R, then equation (58) will imply that P_s^L/P_{s-1} is lower than in the deterministic equilibrium with no runs. Because of this, the possibility of a run may cause the central bank to *under*shoot inflation while the run is not occurring, further undermining inflation stability.

From period s onwards, the characterization of the equilibrium proceeds again deterministically and recursively, separately for the branch that follows P_s^H and P_s^L; this follows the same steps as in section 4.1. The construction of the equilibrium is completed by Proposition 2 that ensures that the transversality and no-Ponzi conditions are satisfied for the sequences that we constructed.

The nature of the equilibrium that we constructed is quite different from those of section B.1. To see this more in detail, consider again the case in which the central bank sets $R_t \equiv 1/\beta - 1$ in every period. It is now no longer true that consumption is then fixed. If a run occurs, the relevant shadow cost of consumption in equation (21) is R_s^H, and consumption drops. This also implies that consumption is not predetermined, but it depends on the realization of the sunspot. Moreover, using (21) and (39), we obtain

$$u_c(C_{s-1}, C_{s-1} + \bar{G}) = \beta^2(1 + R_{s-1})E_{s-1}\left[(1 + \hat{R}_s)\frac{P_{s-1}}{P_{s+1}}u_c(C_{s+1}, C_{s+1} + \bar{G})\right].$$

With the constant interest rate above, and taking into account that the run occurs in period s only, consumption is the same in periods $s-1$ and $s+1$ and we thus find

$$1 = \beta E_{s-1}\left[(1 + \hat{R}_s)\frac{P_{s-1}}{P_{s+1}}\right]. \tag{60}$$

[19] Note that, as $\phi \to 0$, P_s^L/P_{s-1} converges to the inflation in the deterministic equilibrium with no runs, where (22) guarantees that (59) holds.

We know that $\beta(1+\hat{R}_s) \geq \beta(1+R_s) = 1$, and the inequality is strict with probability ϕ. This implies

$$1 > E_{s-1}\frac{P_{s-1}}{P_{s+1}}.$$

When runs can occur, setting the nominal interest rate is not sufficient to even control the expected real value of a dollar.

B.3 Recurrent Runs

We can generalize the example of subsection B.2 to construct equilibria in which runs can occur in any number of periods. As an example, there are equilibria in which runs occur with i.i.d. probability ϕ in each period. Once again, we construct the allocation and price system recursively, as we did in section B.2. In each period t, the history of runs up to period $t-1$ is taken as given, and (57) and (58) are used to solve for P_t^H/P_{t-1} and P_t^L/P_{t-1}.

To contrast these equilibria with the usual sunspot equilibria where no runs occur, consider again the interest rule $R_t \equiv 1/\beta - 1$, and assume that preferences are linear in leisure, i.e., $u(c,l) = v(c) - l$. In this case, equation (39) becomes

$$1 = \beta E_s\left[(1+\hat{R}_{s+1})\frac{P_s}{P_{s+1}}\right] \Longrightarrow 1 > E_s\frac{P_s}{P_{s+1}}.$$

We then get

$$\lim_{T\to\infty}\frac{1}{T}\sum_{s=1}^{T}\frac{P_s}{P_{s+1}} < 1 \text{ almost surely:}$$

if runs are a recurrent event, average inverse inflation is necessarily less than 1 over long horizons.

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