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THE INEFFICIENT MARKETS HYPOTHESIS: WHY FINANCIAL MARKETS DO NOT WORK WELL IN THE REAL WORLD

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ABSTRACT

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Abstract. Existing literature continues to be unable to offer a convincing explanation for the volatility of the stochastic discount factor in real world data. Our work provides such an explanation. We do not rely on frictions, market incompleteness or transactions costs of any kind. Instead, we modify a simple stochastic representative agent model by allowing for birth and death and by allowing for heterogeneity in agents' discount factors. We show that these two minor and realistic changes to the timeless Arrow-Debreu paradigm are sufficient to invalidate the implication that competitive financial markets efficiently allocate risk. Our work demonstrates that financial markets, by their very nature, cannot be Pareto efficient, except by chance. Although individuals in our model are rational; markets are not.

I. Introduction

> Discount rates vary a lot more than we thought. Most of the puzzles and anomalies that we face amount to discount-rate variation we do not understand. Our theoretical controversies are about how discount rates are formed. Cochrane (2011, Page 1091).

Since the work of Paul Samuelson and Eugene Fama, writing in the 1960's, (Samuelson, 1963; Fama, 1963, 1965a,b), the efficient markets hypothesis (EMH) has been the starting point for any discussion of the role of financial markets in the allocation of risk. In his 1970 review article, Fama (1970) defines an efficient financial market as one that "reflects all available information". If markets are efficient in this sense,

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uninformed traders cannot hope to profit from clever trading strategies. To reflect that idea we say there is "no free lunch".

Although the efficient markets hypothesis is primarily about the inability to make money in financial markets, there is a second implication of the EMH that follows from the first welfare theorem of general equilibrium theory; this is the idea that complete, competitive financial markets lead to Pareto efficient allocations. Richard Thaler, (2009), writing in a review of Justin Fox's (2009) book, *The Myth of the Rational Market*, refers to this second dimension of the EMH as "the price is right".

We argue here that unregulated financial markets do not lead to Pareto efficient outcomes, except by chance, and that the failure of complete financial markets to deliver socially efficient allocations has nothing to do with financial constraints, transactions costs or barriers to trade. We show that the first welfare theorem fails in any model of financial markets that reflects realistic population demographics. Although individuals in our model are rational; markets are not.

In their seminal paper, Cass and Shell (1983) differentiate between uncertainty generated by shocks to preferences, technology or endowments – intrinsic uncertainty – and shocks that do not affect any of the economic fundamentals – extrinsic uncertainty. When consumption allocations differ in the face of extrinsic uncertainty, Cass and Shell say that *sunspots matter*. Our paper demonstrates that the existence of equilibria with extrinsic uncertainty has important practical implications for real world economies. We show that sunspots really do matter: And they matter in a big way in any model that is calibrated to fit realistic probabilities of birth and death.

The paper is structured as follows. Sections II and III explain how our findings are connected with the literature on the excess volatility of stock market prices. Section IV provides an informal description of our model along with a description of our main results. Section V provides a series of definitions, lemmas and propositions that formalize our results. Section VI discusses the implications of our work for the equity premium puzzle. In Section VII, we provide some computer simulations of the invariant distribution implied by our model for a particular calibration. Finally, Section VIII presents a short conclusion and a summary of our main ideas.

II. Related literature

Writing in the early 1980s, Leroy and Porter (1981) and Shiller (1981) showed that the stock market is too volatile to be explained by the asset pricing equations associated with complete, frictionless financial markets. The failure of the frictionless

Arrow-Debreu model to explain the volatility of asset prices in real world data is referred to in the literature as 'excess volatility'.

To explain excess volatility in financial markets, some authors introduce financial frictions that prevent rational agents from exploiting Pareto improving trades. Examples include, Bernanke and Gertler (1989, 2001); Bernanke, Gertler, and Gilchrist (1996) and Carlstom and Fuerst (1997) who have developed models where net worth interacts with agency problems to create a *financial accelerator*.

An alternative way to introduce excess volatility to asset markets is to drop aspects of the rational agents assumption. Examples of this approach include Barsky and DeLong (1993), who introduce noise traders, Bullard, Evans, and Honkapohja (2010) who study models of learning where agents do not have rational expectations and Lansing (2010), who describes bubbles that are 'near-rational' by dropping the transversality condition in an infinite horizon framework.

It is also possible to explain excess volatility by moving away from a standard representation of preferences as the maximization of a time separable Von-Neuman Morgenstern expected utility function. Examples include the addition of habit persistence in preferences as in Campbell and Cochrane (1999), the generalization to non time-separable preferences as in Epstein and Zin (1989, 1991) and the models of behavioral finance surveyed by Barberis and Thaler (2003).

In a separate approach, a large body of literature follows Kiyotaki and Moore (1997) who developed a model where liquidity matters as a result of credit constraints. A list of papers, by no means comprehensive, that uses related ideas to explain financial volatility and its effects on economic activity would include the work of Abreu and Brunnermeier (2003); Brunnermeir (2012); Brunnermeir and Sannikov (2012); Farmer (2013); Fostel and Geanakoplos (2008); Geanakoplos (2010); Miao and Wang (2012); Gu and Wright (2010) and Rochetau and Wright (2010).

There is a further literature which includes papers by Caballero and Krishnamurthy (2006); Fahri and Tirole (2011) and Martin and Ventura (2011, 2012), that explains financial volatility and its effects using the overlapping generations model. Our work differs from this literature. Although we use a version of the overlapping generations framework, our results do not rely on frictions of any kind.

Models of financial frictions have received considerable attention in the wake of the 2008 recession. But models in this class have not yet been able to provide a convincing explanation for the size and persistence of the rate of return shocks that are required to explain large financial crises. The importance of shocks of this kind is highlighted by the work of Christiano, Motto, and Rostagno (2012), who estimate a

dynamic stochastic general equilibrium model with a financial sector. They find that a shock they refer to as a "risk shock" is the most important driver of business cycles. In effect, the risk shock changes the rate at which agents discount the future.

New Keynesian explanations of financial crises also rely on a discount rate shock and, to explain the data following major financial crises, this shock must be large and persistent (Eggertsson and Woodford, 2002; Eggertsson, 2011). Eggertsson (2011), for example, requires a 5.47% annualized shock to the time preference factor to account for the large output and inflation declines that occurred following the stock market crash of 1929.

The literature that we have reviewed in this section continues to be unable to offer a convincing explanation for volatility of the stochastic discount factor of the magnitude that is required to explain real world data. Our work provides such an explanation. Our explanation is simple and general and the logic of our argument applies to any model of financial markets with realistic population demographics.

We do not rely on frictions, market incompleteness or transactions costs of any kind. Instead, we modify a simple stochastic representative agent model by allowing for birth and death and by allowing for heterogeneity in agents' discount factors. We show that these two minor, and realistic, changes to the timeless Arrow-Debreu paradigm are sufficient to invalidate the implication that competitive financial markets efficiently allocate risk. Our work demonstrates that financial markets, by their very nature, cannot be Pareto efficient, except by chance. Financial markets do not work well in the real world.

III. Why equilibria are inefficient

Inefficiency occurs in overlapping generations models for two reasons. The first, is dynamic inefficiency that occurs because there is a double infinity of agents and commodities. The second is sunspot inefficiency that occurs because agents are unable to insure against events that occur before they are born.

The fact that the equilibria of overlapping generations models may be dynamically inefficient has been known since Samuelson's (1958) seminal paper. The cause of that inefficiency was identified by Shell (1971) who showed that, even if all agents could trade contingent commodities at the beginning of time, the non-stochastic OLG model would still contain equilibria that are dynamically inefficient. The first welfare theorem fails in that environment because the wealth of all individuals is finite in an inefficient equilibrium even when social wealth is unbounded. We do not rely on

dynamic inefficiency in this paper and, in the absence of uncertainty, our model has a unique dynamically efficient equilibrium.

The second source of inefficiency in overlapping generations models arises from the absence of insurance opportunities. In their (1983) paper, Cass and Shell showed that equilibria may be inefficient if some agents are unable to participate in markets that open before they are born and Azariadis (1981) provided a dynamic example of a model where sunspots influence economic activity.[1] The example that Cass and Shell provided in the body of their paper relied on the existence of multiple equilibria in the underlying, non-stochastic economy. As a result, the majority of the work on sunspots that followed Azariadis and Cass and Shell has sought to construct examples of models where there are multiple equilibria in the underlying economy as in the work of Farmer and Woodford (1984, 1997), Benhabib and Farmer (1994); Farmer and Guo (1994) and Wen (1998).

We depart from this literature. Unlike previous papers that have constructed calibrated examples of sunspot models, our work does not rely on randomizing over the multiple equilibria of an underlying non-stochastic model. Instead, as in Farmer (2012b), and the example constructed in the appendix to Cass and Shell (1983), equilibrium in the non-stochastic version of our model is unique.

Angeletos and La'O (2011) and Benhabib, Wang, and Wen (2012) also construct sunspot models where there is a unique underlying equilibrium. Unlike their work, however, our model does not rely on informational frictions, nor do we assume that there are credit constraints, borrowing constraints or liquidity constraints. Our only departure from a frictionless, timeless, Arrow Debreu model is the assumption that agents cannot participate in financial markets that open before they are born.

When agents have realistic death probabilities and discount factors ranging from 2% to 10%, we find that the human wealth of new-born agents can differ by a factor of 25% depending on whether they are born into a boom or into a recession. These numbers are similar in magnitude to the long-term costs of job loss reported by Davis and Von Wachter (2012) in their study of the effects of severe recessions. Although we do not provide an explicit model of unemployment in this paper, related work by Farmer (2012a,c, 2013), Farmer and Plotnikov (2012) and Plotnikov (2012) does

[1]Cass and Shell (1983) distinguished between ex ante and ex post optimality. Ex post optimality distinguishes between the same person, call him Mr. $A(S)$ who is born into state of the world S and Mr. $A(S')$ who is the same person born into the state of the world S'. Using an ex post Pareto criterion, sunspot equilibria are Pareto optimal because people born into different states of the world are different people. In this paper, we adopt an ex-ante definition of Pareto efficiency.

provide a mechanism that translates asset price shocks into persistent unemployment. In conjunction with these related papers, our work provides an explanation for the large welfare costs of business cycles that Davis and Von-Wachter find in the data.

Our work has important implications for the appropriate role of central bank intervention in financial markets. Farmer (2012b) argues that the inefficiency of competitive financial markets provides a justification for central bank intervention to stabilize asset prices. That argument applies, a fortiori, to the environment we develop here.

IV. An informal description of the environment

This section provides an informal description of our model. We study a pure trade economy with a stochastic aggregate endowment, ω_t, that we refer to interchangeably as *income* or *GDP*. Our economy is populated by patient type 1 agents and impatient type 2 agents. Time is discrete and, as in Blanchard (1985), both types survive into period $t+1$ with age invariant probability π. Each type maximizes the expected present discounted value of a logarithmic utility function. These assumptions allow us to find simple expressions for the aggregate consumption of type $i \in \{1,2\}$ as a function of type i's wealth.

We model a stationary population by assuming that in every period a fraction $(1-\pi)$ of each type dies and is replaced by a fraction $(1-\pi)$ of newborns of the same type. Agents are selfish and do not leave bequests to their descendents. Type 1 agents own a fraction μ of the aggregate endowment and type 2 agents own a fraction $1-\mu$.

We assume that there is a perfect annuities market, mediated by a set of competitive, zero-profit, financial intermediaries. Agents borrow and lend to financial intermediaries at a gross interest rate that exceeds the market rate. If an agent dies with positive financial wealth, the agent's assets are returned to the financial intermediary and, on the other side of this market, agents who borrow are required to take out life insurance contracts that settle their debts when they die.

IV.1. **Our main results.**

In an earlier paper, (Farmer, Nourry, and Venditti, 2011), we derived an explicit expression for the price of an Arrow security (Arrow, 1964) that can be applied to stochastic versions of Blanchard's (1985) perpetual youth model. Here, we apply our earlier result to characterize equilibria as a pair of stochastic difference equations in two state variables that we call $z_{1,t}$ and z_t.

The variable $z_{1,t}$ is the present discounted value of the endowment of all living type 1 agents, divided by aggregate GDP. The variable z_t is the present discounted value of the endowments of all living agents, (both type 1 and type 2) divided by aggregate

GDP. We call these variables, the type 1 human wealth ratio and the aggregate human wealth ratio.

Let S_t be a vector of random variables realized at date t that may be influenced by either intrinsic or extrinsic uncertainty and let $S^t \equiv \{S_0, S_1 ... S_t\}$ be the history of realizations of S from date 0 to date t. Subscripts denote date t realizations of S and superscripts denote histories.

We define the *pricing kernel*, $Q_t^{t+1}\left(S^{t+1}\right)$ to be the price paid at date t in history S^t, in units of consumption, for delivery of one unit of the consumption commodity at date $t+1$ in state S_{t+1}. We define a second variable

$$\tilde{Q}_t^{t+1}\left(S^{t+1}\right) = Q_t^{t+1}\left(S^{t+1}\right)\frac{\gamma\left(S^{t+1}\right)}{\psi\left(S^{t+1}\right)}, \tag{1}$$

where

$$\gamma\left(S^{t+1}\right) \equiv \frac{\omega_{t+1}\left(S^{t+1}\right)}{\omega_t\left(S^t\right)}, \tag{2}$$

is endowment growth in history S^{t+1} and

$$\psi\left(S^{t+1}\right), \tag{3}$$

is the probability that state S_{t+1} occurs conditional on history S^t. We refer to $\tilde{Q}_t^{t+1}\left(S^{t+1}\right)$ as the *normalized pricing kernel*.

Using the results of Farmer, Nourry, and Venditti (2011) we derive an expression for the normalized pricing kernel as a function of the aggregate human wealth ratio at date t and the type 1 human wealth ratio at date $t+1$,

$$\tilde{Q}_t^{t+1} = \tilde{Q}\left(z_t, z_{1,t+1}\right). \tag{4}$$

By applying this expression to the definitions of the type 1 human wealth ratio and to the aggregate human wealth ratio, we are able to characterize equilibria as solutions to the following pair of stochastic difference equations,

$$z_{1,t} = \mu + E_t\left\{\pi\tilde{Q}\left(z_t, z_{1,t+1}\right) z_{1,t+1}\right\}, \tag{5}$$

$$z_t = 1 + E_t\left\{\pi\tilde{Q}\left(z_t, z_{1,t+1}\right) z_{t+1}\right\}. \tag{6}$$

Notice that, although the endowment fluctuates, Equations (5) and (6) do not explicitly involve terms in the random aggregate endowment. Although human wealth is a random variable, there is an equilibrium in which the human wealth ratio is not. This equilibrium is represented by a non-stochastic solution to Equations (5) and (6).

Not all sequences that solve equations (5) and (6) are consistent with market clearing because very high or very low values of human wealth would require negative

consumption of one of the two types. If a sequence is consistent with an interior equilibrium at all points in time we say that the solution is admissible. We prove that the non-stochastic system represented by the equations

$$z_{1,t} = \mu + \pi \tilde{Q}\left(z_t, z_{1,t+1}\right) z_{1,t+1}, \tag{7}$$

$$z_t = 1 + \pi \tilde{Q}\left(z_t, z_{1,t+1}\right) z_{t+1}, \tag{8}$$

has a unique admissible steady state which is a saddle. We show further that the model has a single initial condition represented by the financial assets of type 1 agents at date 0. It follows, that the model has a unique fundamental equilibrium, represented by the stable branch of the saddle.

We derive an explicit closed-form solution for the equation that characterizes this equilibrium. This solution is a first order difference equation in z_t, found by replacing $z_{1,t}$ in Equations (7) and (8) with the equality,

$$z_{1,t} = \mu z_t, \tag{9}$$

at all dates. This substitution leads to a function, $g\left(\cdot\right)$ for the stable branch of the saddle which is found by solving the equation

$$z_t = 1 + \pi \tilde{Q}\left(z_t, \mu z_{t+1}\right) z_{t+1}, \tag{10}$$

for z_{t+1} as a function of z_t. Given this function, the sequence $\{z_t\}$, defined as the unique solution to the difference equation

$$z_{t+1} = g\left(z_t\right), \quad z_0 = \bar{z}_0, \tag{11}$$

is an equilibrium of our model economy. The initial condition is determined by asset and goods market clearing in the first period and it is natural to impose an initial condition where agents of type 1 and type 2 are each born with zero financial obligations. We refer to the sequence $\{z_t\}$, constructed in this way, as the *fundamental equilibrium* of our model economy.

IV.2. **Properties of the fundamental equilibrium.** The fundamental equilibrium has the following properties. Given the initial value z_0, human wealth converges to a unique steady state value, z^*, and once this steady state has been reached, the normalized pricing kernel remains constant at a fixed value $\tilde{Q}^*$.

Recall that the pricing kernel is defined by the expression,

$$Q_t^{t+1} = \tilde{Q}^* \frac{\gamma}{\psi}, \tag{12}$$

where $\tilde{Q}^*$ is the value of the normalized pricing kernel at the steady state. This equation implies that, in the fundamental equilibrium, the price of an Arrow security will fluctuate in proportion to shocks to the stochastic endowment process. This mirrors the pricing equation associated with a representative agent economy where the agent has logarithmic preferences and where $\tilde{Q}^*$ plays the role of the representative agent's discount factor.

In the fundamental equilibrium, all uncertainty is intrinsic. Newborn agents trade a complete set of Arrow securities with financial intermediaries and, depending on type, these agents may start life as net borrowers (these are the type 2 agents) or net lenders, (these are the type 1 agents). As time progresses, the measure of agents born at date t shrinks exponentially. Long-lived type 1 agents eventually consume more than their endowments as they accumulate financial assets. Long-lived type 2 agents eventually consume less than their endowments as they devote an ever larger fraction of their incomes to debt repayment.

IV.3. **Equilibria where sunspots matter.** In addition to the unique fundamental equilibrium, our model has many sunspot equilibria, represented by stochastic processes for z_t that satisfy the following analog of Equation (10).

$$z_t = 1 + E_t \left\{ \pi \tilde{Q} \left(z_t, \mu z_{t+1} \right) z_{t+1} \right\}. \tag{13}$$

For example, let $\varepsilon_{s,t+1}$ be a bounded random variable, with mean 1, and consider the equation

$$\left(z_t - 1 \right) \varepsilon_{s,t+1} = \pi \tilde{Q} \left(z_t, \mu z_{t+1} \right) z_{t+1}. \tag{14}$$

Let $\{z_t\}$ be a sequence of random variables generated by the expression,

$$z_{t+1} = g \left(z_t, \varepsilon_{s,t+1} \right), \tag{15}$$

where the function $g(z, \varepsilon)$ is obtained by solving Equation (14) for z_{t+1} as a function of z_t and $\varepsilon_{s,t+1}$. By taking expectations of Equation (14), using the assumption that the conditional mean of $\varepsilon_{s,t+1}$ is equal to one, it follows that this sequence satisfies Equation (13). Since this equation completely characterizes equilibrium sequences, it follows that our economy admits sunspot equilibria.

Business cycles in our model are generated, not only by intrinsic shocks to GDP growth, but also by sunspot shocks. For plausible values of the parameters of the model, we show that the aggregate human wealth ratio can differ by 25% at different points of the business cycle. If we think of a low value of the human wealth ratio as a recession, a person of either type who is born into a recession, will find that the net

present value of their life-time earnings is 25% lower than if they had been born into a boom.

V. A formal description of the environment

In this section we provide a formal description of the model. Uncertainty each period is indexed by a finite set of states $\mathbf{S} = \{S_1, \ldots, S_n\}$. Define the set of t-period histories $\mathbf{S}^t$ recursively as follows:

$$\begin{aligned} \mathbf{S}^1 &= \mathbf{S} \\ \mathbf{S}^t &= \mathbf{S}^{t-1} \times \mathbf{S}, \quad t = 2, \ldots \end{aligned} \tag{16}$$

We will use S_t to denote a generic element of $\mathbf{S}$ realized at date t, S^t to denote an element of $\mathbf{S}^t$ realized at t and $|\mathbf{S}^t|$ to denote the number of elements in $\mathbf{S}^t$. Let the probability that S_{t+1} occurs at date $t+1$, conditional on history S^t, be given by $\psi(S^{t+1})$ and assume that this probability is independent of time.

We define β_i to be the discount factor of type i and we assume

$$0 < \beta_2 < \beta_1 < 1. \tag{17}$$

Throughout the paper, we use the following transformed parameters,

$$B_i \equiv (1 - \beta_i \pi), \tag{18}$$

and from Equation (17) it follows that,

$$B_2 > B_1. \tag{19}$$

A household of type i, born at date j, solves the problem,

$$\max E_j \left\{ \sum_{t=j}^{\infty} (\pi \beta_i)^{t-j} \log c_{i,t}^j \left(S^t\right) \right\}, \tag{20}$$

such that

$$\sum_{S_{t+1} \in \mathbf{S}}^{|\mathbf{S}|} \pi Q_t^{t+1} \left(S^{t+1}\right) a_{i,t+1}^j \left(S^{t+1}\right) \leq a_{i,t}^j \left(S^t\right) + \omega_{i,t} \left(S^t\right) - c_{i,t}^j \left(S^t\right), \quad t = j, \ldots \tag{21}$$

$$a_{i,j}^j \left(S^j\right) = 0. \tag{22}$$

The solution to this problem satisfies the Euler equation

$$Q_t^{t+1} \left(S^{t+1}\right) = \frac{\psi(S^{t+1}) \beta_i c_{i,t}^j \left(S^t\right)}{c_{i,t+1}^j \left(S^{t+1}\right)}, \tag{23}$$

for each history S^t and each of its $|\mathbf{S}|$ successors S_{t+1}, where $c^j_{i,t}(S^t)$ is the consumption at date t in history S^t, of a member of type i, born at date j, and $a^j_{i,t}(S^t)$ is the agent's financial wealth.

Let $h_{i,t}(S^t)$ be type $i's$ human wealth, defined as

$$h_{i,t}\left(S^t\right) = \omega_{i,t}\left(S^t\right) + \pi \sum_{S_{t+1}} Q_t^{t+1}\left(S^{t+1}\right) h_{i,t+1}\left(S^{t+1}\right), \quad t = 0, ... \tag{24}$$

Since each member of type i has the same endowments and the same probability of dying, the human wealth of all members of type i will be the same across generations. We assume that

$$\lim_{T\to\infty} \pi^{T-1} Q_{T-1}^T\left(S^T\right) \omega_{i,T}\left(S^T\right) = 0, \text{ for all } S^T \in \mathbf{S}^T, \tag{25}$$

which implies that human wealth is well defined and can be represented as the net present value of future endowments summed over all possible future histories,

$$h_{i,t}\left(S^t\right) = \sum_{\tau=t}^{\infty} \sum_{S^\tau \in \mathbf{S}^\tau}^{|\mathbf{S}^\tau|} \pi^{\tau-t} Q_t^\tau\left(S^\tau\right) \omega_{i,\tau}\left(S^\tau\right). \tag{26}$$

Using these results and the properties of logarithmic preferences, we have that,

$$c^j_{i,t}\left(S^t\right) = B_i\left[a^j_{i,t}\left(S^t\right) + h^j_{i,t}\left(S^t\right)\right]. \tag{27}$$

Next, we apply the methods developed in Farmer, Nourry, and Venditti (2011) to find the following expression for the pricing kernel,

Proposition 1. The pricing kernel can be expressed as

$$Q_t^{t+1}\left(S^{t+1}\right) = \frac{\psi\left(S^{t+1}\right)(1-B_i)\, c_{i,t}\left(S^t\right)}{c_{i,t+1}\left(S^{t+1}\right) - B_i\,(1-\pi)\, h_{i,t+1}\left(S^{t+1}\right)}, \tag{28}$$

where $c_{i,t}(S^t)$ is the aggregate consumption of all agents of type i alive at date t in history S^t and $h_{i,t+1}(S^{t+1})$ is the human wealth of agents of type i at date $t+1$ in history S^{t+1}.

Proof. See Appendix A. □

V.1. **Competitive equilibria.** In this section, we find simple expressions for the equations that define an equilibrium. We begin by normalizing the variables of our model by the aggregate endowment, $\omega_t(S^t)$. Since this is an endowment economy, this variable is our measure of GDP, equal to income; hence we refer to this procedure as normalizing by income.

Let $\mathcal{A}_t$ be the index set of all agents alive at date t. Using this definition, we aggregate the consumption function, Equation (27) over all agents of type i alive at date t, and divide by income to generate the following expression,

$$\lambda_{i,t}\left(S^t\right) = B_i\left[\alpha_{i,t}\left(S^t\right) + z_{i,t}\left(S^t\right)\right]. \tag{29}$$

The terms

$$\lambda_{i,t}\left(S^t\right) = \frac{\sum_{j\in\mathcal{A}_t} c^j_{i,t}\left(S^t\right)}{\omega_t\left(S^t\right)}, \quad \alpha_{i,t}\left(S_t\right) = \frac{\sum_{j\in\mathcal{A}_t} a^j_{i,t}\left(S^t\right)}{\omega_t\left(S^t\right)}, \tag{30}$$
$$\text{and} \quad z_{i,t}\left(S^t\right) = \frac{\sum_{j\in\mathcal{A}_t} h^j_{i,t}\left(S^t\right)}{\omega_t\left(S^t\right)},$$

represent consumption, financial wealth, and human wealth of all members of type i, expressed as fractions of GDP. We refer to these variables as the *consumption share,* the *asset ratio* and the *human wealth ratio* for type i.

Since there are two types of agents, we define

$$\lambda_t\left(S^t\right) \equiv \lambda_{1,t}\left(S^t\right), \tag{31}$$

and we refer to $\lambda_t\left(S^t\right)$ as simply, *the* consumption share. From the goods market clearing equation, the consumption shares of the two types must sum to unity, which implies that the consumption share of type 2 agents is given by the expression,

$$\lambda_{2,t}(S^t) = 1 - \lambda_t(S^t). \tag{32}$$

Similarly, we refer to

$$\alpha_t\left(S^t\right) \equiv \alpha_{1,t}\left(S^t\right),$$

as *the* asset ratio, since from the asset market clearing equation, the financial assets of type 1 agents must equal the financial liabilities of type 2 agents, and

$$\alpha_{2,t}\left(S^t\right) = -\alpha_t\left(S^t\right). \tag{33}$$

Corresponding to the definition of $\lambda_t\left(S_t\right)$ as the share of income *consumed* by type 1 agents, we will define μ,

$$\mu = \frac{\omega_{1,t}}{\omega_t}, \quad 1-\mu = \frac{\omega_{2,t}}{\omega_t}, \tag{34}$$

to be the share of income *owned* by type 1 agents.

Using these newly defined terms, we have the following definition of a competitive equilibrium.

Definition 1. A competitive equilibrium is a set of sequences for the consumption share, $\{\lambda_t(S^t)\}$, the asset ratio $\{\alpha_t(S^t)\}$, and the human wealth ratios $\{z_{1,t}(S^t)\}$ and $\{z_t(S^t)\}$ and a sequence of Arrow security prices $\left\{Q_t^{t+1}(S^{t+1})\right\}$ such that each household of each generation maximizes expected utility, taking their budget constraint and the sequence of Arrow security prices as given and the goods and asset markets clear. An equilibrium is *admissible* if $\{\lambda_t(S^t)\} \in (0,1)$ for all S^t.

In the remainder of the paper, we drop the explicit dependence of λ_t, α_t, ω_t, c_t, $c_{1,t}$ and Q_t on S^t to make the notation more readable.

V.2. **Equilibria with intrinsic uncertainty.** In their paper, 'Do Sunspots Matter?' Cass and Shell (1983) distinguish between intrinsic uncertainty and extrinsic uncertainty. Intrinsic uncertainty in our model is captured by endowment fluctuations. In this section, we study the case where this is the only kind of uncertainty to influence the economy. Before characterizing equilibrium sequences, we prove the following lemma.

Lemma 1. Let $z_a = 1/B_2$ and $z_b = 1/B_1$ and recall that $B_2 > B_1$. There exists an increasing affine function $\zeta : \hat{Z} \equiv [z_a, z_b] \to [0,1]$ such that for all values of the aggregate human wealth ratio, $z \in \hat{Z}$ the equilibrium consumption share $\lambda \in [0,1]$ is given by the expression

$$\lambda = \zeta(z) \equiv \frac{B_1 B_2}{B_2 - B_1}\left(z - \frac{1}{B_2}\right). \tag{35}$$

Define the real number

$$\theta_0 = B_1(B_2 - B_1). \tag{36}$$

Then, in a competitive equilibrium, the aggregate human wealth ratio z_t, the human wealth ratio of type 1, $z_{1,t}$ and the asset share α_t, are related by the affine function,

$$\theta_0 z_{1,t} - B_1 B_2 z_t + \theta_0 \alpha_t + B_1 = 0. \tag{37}$$

Proof. See Appendix B. □

Using Lemma 1, we establish the following Proposition which characterizes the fundamental equilibrium.

Proposition 2. Define the real numbers,

$$\begin{aligned}
\theta_1 &= B_2 - \pi(1 - B_1) + \mu(1 - \pi)(B_1 - B_2), \\
\theta_2 &= -B_2(1 - \pi) - \pi B_1 B_2 < 0, \\
\theta_3 &= (B_2 - B_1)(1 - \pi) > 0, \\
\theta_4 &= B_2 - \pi\,(1 - B_1) = \theta_1 + \mu\theta_3.
\end{aligned} \tag{38}$$

In the case when all uncertainty is intrinsic, the following pair of non-stochastic difference equations describes the evolution of the human wealth ratio of type 1, $z_{1,t}$, and of the human wealth ratio, z_t, in a competitive equilibrium,

$$z_{1,t+1} = \frac{\mu - z_{1,t}}{\theta_1 + \theta_2 z_t + \theta_3 z_{1,t}}, \tag{39}$$

$$z_{t+1} = \frac{1 - z_t}{\theta_1 + \theta_2 z_t + \theta_3 z_{1,t}}. \tag{40}$$

In period 0, $z_{1,0}$ and z_0 are linked by the initial condition,

$$\theta_0 z_{1,0} - B_1 B_2 z_0 + \theta_0 \alpha_0 + B_1 = 0, \tag{41}$$

where

$$\alpha_0 = \bar{\alpha}_0, \tag{42}$$

is the initial asset ratio. The normalized pricing kernel is related to z_t and $z_{1,t}$ by the expression

$$\tilde{Q}_t^{t+1} = \frac{-1}{\pi}\left(\frac{\theta_4 + \theta_2 z_t}{1 + \theta_3 z_{1,t+1}}\right). \tag{43}$$

The consumption share λ_t and the asset ratio, α_t are given by equations (44) and (45),

$$\lambda_t = \frac{B_1 B_2}{B_2 - B_1}\left(z_t - \frac{1}{B_2}\right), \tag{44}$$

$$\alpha_t = -\frac{B_1}{\theta_0} + \frac{(B_1 B_2 - \mu\theta_0)}{\theta_0} z_t. \tag{45}$$

Proof. See Appendix C. □

Equations (39) and (40) constitute a two-dimensional system in two variables with a single initial condition, represented by Equation (41). These equations are non-stochastic, even when the economy is hit by fundamental shocks, because we have normalized z_t, $z_{1,t}$ and $\tilde{Q}_t^{t+1}$ by the random endowment. Although h_t and $h_{1,t}$ fluctuate in response to random shocks, z_t and $z_{1,t}$ do not.

Removing the time subscripts from equations (39) and (40) we define a *steady state equilibrium* to be a solution to the equations

$$z_1\left(\theta_1+\theta_2 z+\theta_3 z_1\right) = \mu - z_1, \tag{46}$$

$$z\left(\theta_1+\theta_2 z+\theta_3 z_1\right) = 1 - z. \tag{47}$$

The following proposition characterizes the properties of a steady state equilibrium and finds two equivalent representations of an equilibrium sequence; one using z_t as a state variable and one using $\tilde{Q}_t^{t+1}$.

Proposition 3. Equations (46) and (47) have a unique admissible steady state equilibrium, $\{z^*, z_1^*\}$ such that $z^* \in (z_a, z_b)$ and $z_1^* = \mu z^*$. The Jacobian of the system (39) and (40), evaluated at $\{z^*, \mu z^*\}$, has two real roots, one less than 1 in absolute value and one greater than 1. It follows that $\{z^*, \mu z^*\}$ is a saddle. The stable branch of this saddle is described by a set $\hat{Z} \equiv [z_a, z_b]$ and a function $g(\cdot) : \hat{Z} \to \hat{Z}$ such that the first order difference equation

$$z_{t+1} = g\left(z_t\right), \tag{48}$$

where

$$g(z_t) \equiv \frac{1-z_t}{\theta_1+(\theta_2+\mu\theta_3)z_t}, \tag{49}$$

defines a competitive equilibrium sequence for $\{z_{t+1}\}$. For all initial values of z_0 and $z_{1,0}$ where

$$\begin{aligned} z_0 &\in \hat{Z}, \\ z_{1,0} &= \mu z_0, \end{aligned} \tag{50}$$

z_t converges to z^*. There is an equivalent representation of equilibrium as a difference equation in $\tilde{Q}$. In this representation there exists a set $\hat{Q} = [q_a, q_b]$ and a function $h(\cdot) : \hat{Q} \to \hat{Q}$, such that any sequence $\left\{\tilde{Q}_t^{t+1}\right\}$ generated by the difference equation

$$\tilde{Q}_{t+1}^{t+2} = h\left(\tilde{Q}_t^{t+1}\right), \quad Q_0^1 \in \hat{Q}, \tag{51}$$

is a competitive equilibrium sequence. The set $\hat{Q}$ and the function $h(\cdot)$ are defined by equations (52) and (53),

$$q_a = -\frac{\theta_1+(\theta_2+\mu\theta_3)\,z_a}{\pi}, \quad q_b = -\frac{\theta_1+(\theta_2+\mu\theta_3)\,z_b}{\pi}, \tag{52}$$

$$h\left(\tilde{Q}_t^{t+1}\right) \equiv \frac{1-\theta_1}{\pi} - \frac{(1-B_1)(1-B_2)}{\pi\tilde{Q}_t^{t+1}}. \tag{53}$$

Every sequence generated by Equation (51), converges to the steady state $\tilde{Q}^*$, where $\tilde{Q}^*$ is defined in Equation (54),

$$\tilde{Q}^* \equiv -\frac{\theta_1 + (\theta_2 + \mu\theta_3)\, z^*}{\pi} = \frac{1 - \theta_1 + \sqrt{(1+\theta_1)^2 + 4(\theta_2 + \mu\theta_3)}}{2\pi}. \tag{54}$$

Proof. See Appendix D. □

Proposition 3 implies that the two-dimensional dynamical system in $\{z_t, z_{1,t}\}$ can be reduced to a one-dimensional difference equation, represented by Equation (49), which describes the dynamics of the system on the saddle path.

In Figure 1 we have plotted $z_{t+1} - z_t$ on the vertical axis and z_t on the horizontal axis. This figure illustrates the dynamics of z_t, the human wealth ratio, for a parameterized example. To construct this figure, we set the survival probability to 0.98, which implies that the expected lifetime, conditional on being alive today, is 50 years. The discount factor of type 1 agents is 0.98, the discount factor of type 2 agents is 0.9 and there are equal shares of each type in the population.

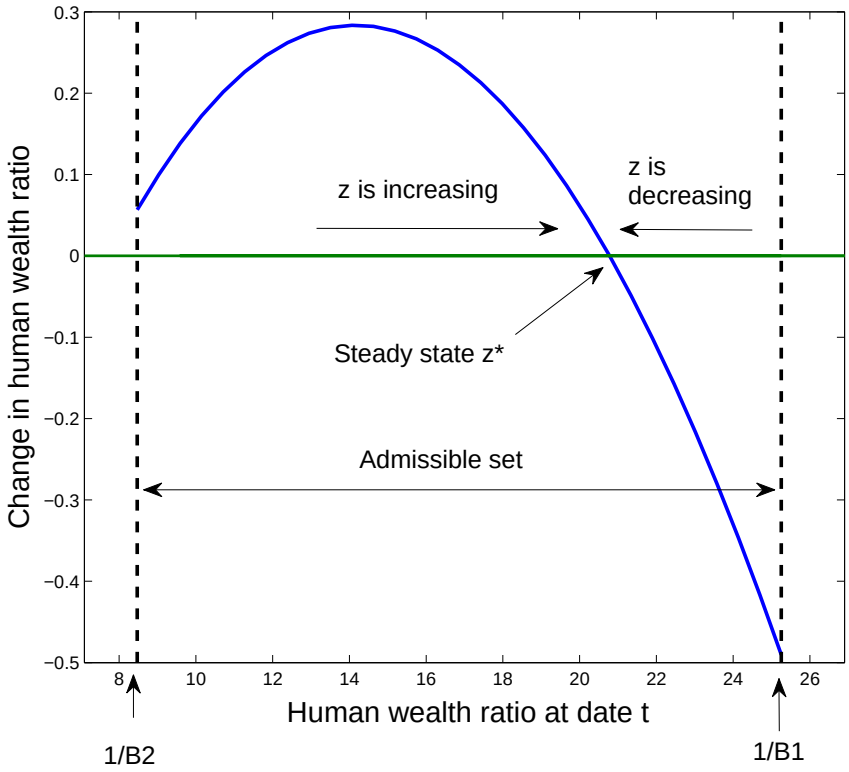

Figure 1: The dynamics of the human wealth equation

We refer to the area between the two vertical dashed lines, one at $z_a \equiv B_2^{-1}$ and one at $z_b \equiv B_1^{-1}$ as $\hat{Z} \equiv [z_a, z_b]$; this is the set of admissible initial conditions for an

interior equilibrium. When z_0 is equal to B_2^{-1}, $\lambda_0 = 0$, and type 2 agents consume the entire endowment. When $z_0 = B_1^{-1}$, $\lambda_0 = 1$, and type 1 agents consume the entire endowment.

V.3. **Equilibria with extrinsic uncertainty.** Although we assume that agents are able to trade a complete set of Arrow securities, that assumption does not insulate the economy from extrinsic uncertainty. In our model, the human wealth ratio z_t and the normalized pricing kernel $\tilde{Q}_t^{t+1}$ can fluctuate simply as a consequence of self-fulfilling beliefs. That idea is formalized in the following proposition.

Proposition 4. Let $\varepsilon_{s,t}$ be a sunspot random variable with support $\epsilon \equiv [\varepsilon_a, \varepsilon_b]$ where

$$\varepsilon_a = \frac{-[\theta_4(1-\mu\theta_3)+2\theta_2] + \sqrt{[\theta_4(1-\mu\theta_3)+2\theta_2]^2 - \theta_4^2(1+\mu\theta_3)^2}}{(1+\mu\theta_3)^2}, \tag{55}$$

$$\varepsilon_b = \frac{z_b(\theta_4+\theta_2 z_b)}{(1-z_b)(1+\mu\theta_3 z_b)}, \tag{56}$$

and $E_t[\varepsilon_{s,t}] = 1$. Then there exist sets $Z \equiv [z_1, z_b]$, and $\epsilon \equiv [\varepsilon_a, \varepsilon_b]$, a function $g(\cdot) : \epsilon \times Z \to Z$ and a stochastic process defined by the equation

$$z_{t+1} = g(z_t, \varepsilon_{s,t+1}), \tag{57}$$

where

$$g(z_t, \varepsilon_{s,t+1}) \equiv \frac{(1-z_t)\varepsilon_{s,t+1}}{(\theta_4 - \mu\theta_3\varepsilon_{s,t+1}) + (\theta_2 + \mu\theta_3\varepsilon_{s,t+1})z_t}, \tag{58}$$

such that any sequence $\{z_t\}$ generated by (58) for $z_0 \in Z$, is a competitive equilibrium sequence. Further, there is an equivalent representation of equilibrium as the solution to a stochastic difference equation in Q. In this representation, there exists a function $h(Q, \varepsilon', \varepsilon)$, such that

$$\tilde{Q}_t^{t+1} = h\left(\tilde{Q}_{t-1}^t, \varepsilon_{s,t+1}, \varepsilon_{s,t}\right), \tag{59}$$

where,

$$h\left(\tilde{Q}_{t-1}^t, \varepsilon_{s,t+1}, \varepsilon_{s,t}\right) \equiv a(\varepsilon_{s,t}, \varepsilon_{s,t+1}) + \frac{b(\varepsilon_{s,t}, \varepsilon_{s,t+1})}{\tilde{Q}_{t-1}^t}, \tag{60}$$

$$a(\varepsilon_{s,t}, \varepsilon_{s,t+1}) \equiv \frac{1}{\pi}\left\{\frac{\varepsilon_{s,t}(\theta_2 + \mu\theta_3\varepsilon_{s,t+1})}{(\theta_2+\mu\theta_3\varepsilon_{s,t})} - \theta_4 + \mu\theta_3\varepsilon_{s,t+1}\right\}, \tag{61}$$

and

$$b(\varepsilon_{s,t}, \varepsilon_{s,t+1}) \equiv \frac{1}{\pi^2}\left\{\frac{\varepsilon_{s,t}(\theta_2+\theta_4)(\theta_2+\mu\theta_3\varepsilon_{s,t+1})}{(\theta_2+\mu\theta_3\varepsilon_{s,t})}\right\}. \tag{62}$$

The sequence $\left\{\tilde{Q}_t^{t+1}\right\}$, generated by a solution to Equation (59), is a competitive equilibrium sequence for Q.

Proof. See Appendix E. □

Figure 2 illustrates the method used to construct sunspot equilibria. The solid curve represents the function $g(z,1)$ and the upper and lower dashed curves represent the functions $g(z,\varepsilon_b)$ and $g(z,\varepsilon_a)$. We have exaggerated the curvature of the function $g(\cdot)$ by choosing a value $\beta_1 = 0.98$ and $\beta_2 = 0.3$. The large discrepancy between β_1 and β_2 causes the slopes of these curves to be steeper for low values of z and flatter for high values, thereby making the graph easier to read.

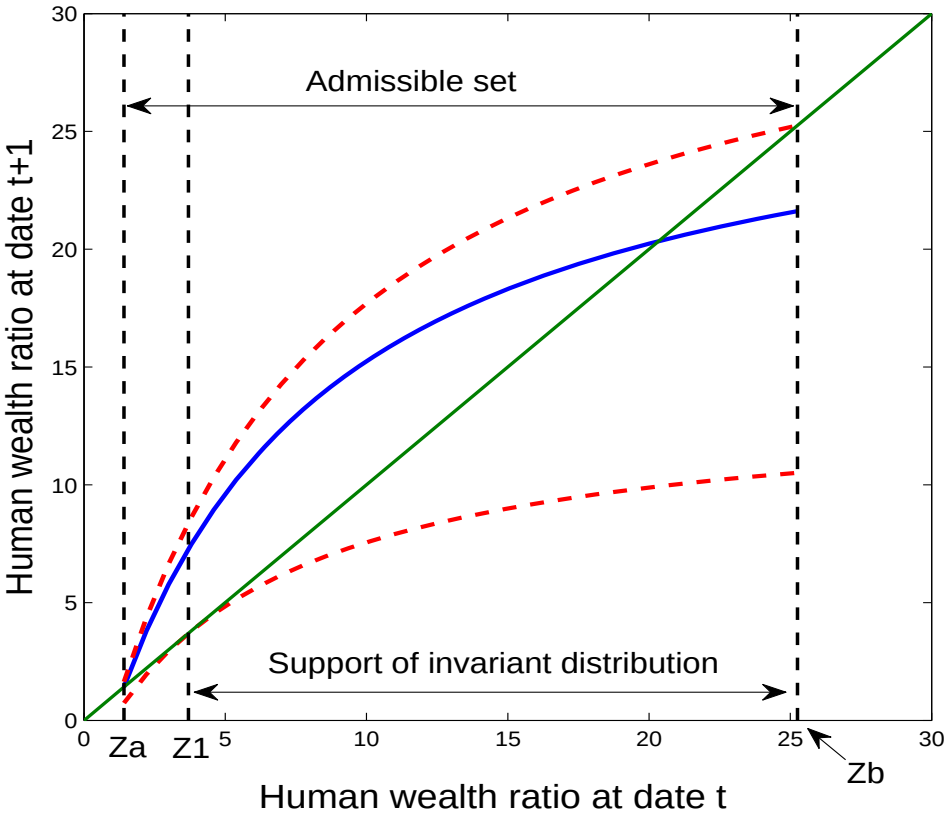

Figure 2: The dynamics of sunspot equilibria

Figure 2 also contrasts the admissible set $\hat{Z} \equiv [z_a, z_b]$ with the support of the invariant distribution $Z \equiv [z_1, z_b]$. The three vertical dashed lines represent the values z_a, z_1 and z_b. The lower bound of the largest possible invariant distribution, z_1 is defined as the point where $g(z,\varepsilon_a)$ is tangent to the 45° line. Recall that the admissible set is the set of values of z for which the consumption of both types is non-negative and notice that z_1 is to the right of z_a, the lower bound of the admissible set. Figure

2 illustrates that the support of the largest possible invariant distribution is a subset of the admissible set. It follows from the results of Futia (1982) that, as ε fluctuates in the set $[\varepsilon_a, \varepsilon_b]$, z converges to an invariant distribution with support $Z \equiv [z_1, z_b]$.[2]

VI. Some Important Implications of our model for asset pricing

In addition to the excess volatility puzzle, the frictionless Arrow Debreu model has trouble explaining why the return on risky assets is so much higher than the safe return in U.S. data. This anomaly, first identified by Rajnish Mehra and Edward Prescott in (1985), is known in the literature as the equity premium puzzle. A common way of measuring the equity premium is to take the ratio of the mean excess return in a finite sample of time series data and divide it by the volatility of the risky rate. That statistic is known as the Sharpe ratio.

Part of the problem faced by representative agents models when confronting asset market data is the inability of those models to generate a sufficiently volatile pricing kernel (Cochrane, 2011). Since our model does not suffer from this defect, it is plausible that our work could offer a more satisfactory explanation of the equity premium. When sunspots are uncorrelated with fundamentals and preferences are logarithmic, our model does not deliver a large equity premium in infinite samples. However, we show that discount rate volatility is so large in our model that there is a significant probability of observing a high Sharpe ratio.

To make these points we show, first, using a local approximation argument, that sunspot volatility does not affect the asymptotic value of the equity premium when sunspots are uncorrelated with fundamentals. Then we compute the invariant distribution of the safe and risky returns for a calibrated example, and we compute the Sharpe ratio in $10,000$ draws of 60 years of data. We show, in our calibrated example, that there is 12% probability of observing a Sharpe ratio of at least 0.2

To construct a local approximation, we first provide a model of consumption growth as a first order autocorrelated process,

$$\log \gamma_{t+1} = \eta \log \gamma_t + (1-\eta) \log \mu_\gamma + \log \varepsilon_{f,t}. \tag{63}$$

Here, η is an autocorrelation parameter and $\log \varepsilon_{f,t}$ is a fundamental, normally distributed i.i.d. shock,

$$\log \varepsilon_{f,t} \sim N\left(-\frac{\sigma_f^2}{2}, \sigma_f^2\right), \tag{64}$$

[2]The proof, which is available upon request, is based on various Definitions and Theorems provided in Futia (1982) (see Theorem 4.6, Proposition 4.4, Definition 2.1, Theorem 3.3 and Theorem 2.9).

with $E(\varepsilon_{f,t}) = 1$. These assumptions imply that the unconditional distribution of γ_t is log-normal with mean m_γ and variance σ_γ^2, where[3]

$$m_\gamma = \log \mu_\gamma - \frac{\sigma_f^2}{2(1-\eta)}, \tag{65}$$

$$\sigma_\gamma^2 = \frac{\sigma_f^2}{1-\eta^2}. \tag{66}$$

From the properties of the log-normal distribution we also have that,

$$E(\gamma_t) = \mu_\gamma e^{-\frac{\eta\sigma_\gamma^2}{2}}. \tag{67}$$

To describe extrinsic uncertainty, we assume that $\varepsilon_{s,t}$ is a log normal random variable, with mean $-\sigma_s^2/2$ and variance σ_s^2:

$$\log \varepsilon_{s,t} \sim N\left(\frac{-\sigma_s^2}{2}, \sigma_s^2\right), \tag{68}$$

so that $E[\varepsilon_{s,t}] = 1$. Although this process is not bounded, by making the variance of $\varepsilon_{s,t}$ small we can make the probability that $\varepsilon_{s,t}$ lies in a bounded interval arbitrarily close to 1. In the following analysis we assume that $\varepsilon_{f,t}$ and $\varepsilon_{s,t}$ are log-normal and we use this assumption to derive approximations to the value of the equity premium. We also assume here that the log-normal distributions of these two shocks are independent.

To get an approximation to the importance of extrinsic volatility, we log linearize Equation (59) around the non-stochastic steady state, $\tilde{Q}^*$. The following lemma derives a log-normal approximation to the distribution of $\tilde{Q}_t^{t+1}$ that is valid for small noise.

Lemma 2. The stochastic process described by Equation (59) has the following approximate representation

$$\log \tilde{Q}_t^{t+1} = \chi \log \tilde{Q}_{t-1}^t + (1-\chi)\log \tilde{Q}^* + \eta_1 \log \varepsilon_{s,t+1} + \eta_2 \log \varepsilon_{s,t} + O\left(x^2\right), \tag{69}$$

[3]Using Equations (63) and (64), it follows that

$$E(\log \gamma_t) = \log \mu_\gamma + \frac{E(\log \varepsilon_{f,t})}{1-\eta} = \log \mu_\gamma - \frac{\sigma_f^2}{2(1-\eta)} \equiv m_\gamma,$$

and

$$Var(\log \gamma_t) = \frac{Var(\log \varepsilon_{f,t})}{1-\eta^2} = \frac{\sigma_f^2}{1-\eta^2} \equiv \sigma_\gamma^2.$$

Further, $\log \gamma_t$ is normally distributed,

$$\log \gamma_t \sim N\left(m_\gamma, \sigma_\gamma^2\right),$$

where

$$\chi = \frac{(1-B_1)(1-B_2)}{\pi(\tilde{Q}^*)^2}, \tag{70}$$

and η_1 and η_2 are computed as the logarithmic derivatives with respect to $\varepsilon_{s,t+1}$ and $\varepsilon_{s,t}$ of the function h in Equation (59) evaluated at its non-stochastic steady state, $\tilde{Q}^*$. The approximation error is $O(x^2)$ where x is the difference of $\log \tilde{Q}_{t-1}^T$ from $\log \tilde{Q}^*$. It follows from the linearity of (69), that $\tilde{Q}_t^{t+1}$ is approximately log-normal with unconditional mean

$$E\left[\tilde{Q}_t^{t+1}\right] = \tilde{Q}^* \exp\left(\frac{\sigma_{\tilde{Q}}^2}{2}\left[1-(1+\chi)\left(\frac{\eta_1+\eta_2}{\eta_1^2+\eta_2^2}\right)\right]\right), \tag{71}$$

or equivalently

$$\log \tilde{Q}_t^{t+1} \sim N\left(m_{\tilde{Q}}, \sigma_{\tilde{Q}}^2\right). \tag{72}$$

where

$$m_{\tilde{Q}} \equiv \log \tilde{Q}^* - \frac{(\eta_1+\eta_2)\sigma_s^2}{2(1-\chi)}, \tag{73}$$

and

$$\sigma_{\tilde{Q}}^2 \equiv \frac{(\eta_1^2+\eta_2^2)\sigma_s^2}{1-\chi^2}. \tag{74}$$

Proof. See Appendix F. □

Next, we use Lemma 2 to derive approximate formulae for the price of a riskless and a risky security. Consider first, a riskless bond that pays a gross return $R_t^{t+1,S}$ in every state. Here we use the superscript S to denote 'safe'. Using the no arbitrage assumption, the return to this riskless asset is given by the expression,

$$R_t^{t+1,S} = \frac{1}{\sum_{S'} Q_t^{t+1}}. \tag{75}$$

Now consider a risky security that pays $\nu\omega_t$ at date t where ν is a real number between zero and one. This is the model analog of equity since it represents a claim to a fraction of the risky endowment stream. If this security sells for price q_t then the risky return between periods t and $t+1$, $R_t^{t+1,R}$, is given by the expression,

$$R_t^{t+1,R} = \frac{q_{t+1}+\nu\omega_{t+1}}{q_t}, \tag{76}$$

where the superscript R denotes 'risky'. We may then compute approximations to the safe return, the return to a risky security and the equity premium.

Proposition 5. Let

$$R_t^{t+1,S} = \frac{1}{E_t\left[\tilde{Q}_t^{t+1}\gamma_{t+1}^{-1}\right]} \tag{77}$$

be the return to a risk free real bond and let q_t be the price of a security that pays a fraction ν of the endowment ω_t in all future states. Let

$$R_t^{t+1,R} = \frac{q_{t+1} + \nu\omega_{t+1}}{q_t}, \tag{78}$$

be the return to this risky security. Assume further that intrinsic and extrinsic uncertainty are uncorrelated,

$$Cov\left(\varepsilon_{s,t+i}, \varepsilon_{f,t+j}\right) = 0, \quad \text{for all } \; i, j. \tag{79}$$

Then i) the safe return is given by the approximation

$$R^S \simeq \frac{\mu_\gamma}{\tilde{Q}^*} e^{-\frac{\sigma_\gamma^2(2+\eta)}{2} - \frac{\sigma_{\tilde{Q}}^2\left[1-(1+\chi)\left(\frac{\eta_1+\eta_2}{\eta_1^2+\eta_2^2}\right)\right]}{2}}, \tag{80}$$

ii) The risky return is given by the approximation

$$R^R \simeq \frac{\mu_\gamma}{\tilde{Q}^*} e^{\frac{-\eta\sigma_\gamma^2}{2} - \frac{\sigma_{\tilde{Q}}^2\left[1-(1+\chi)\left(\frac{\eta_1+\eta_2}{\eta_1^2+\eta_2^2}\right)\right]}{2}}, \tag{81}$$

iii) The equity premium is given by the expression

$$\frac{R^R}{R^S} \simeq e^{\sigma_\gamma^2}. \tag{82}$$

Proof. See Appendix G. □

Proposition 5 establishes that, in the case where sunspot uncertainty is uncorrelated with fundamental uncertainty, the variance of the sunspot term does not enter the expression for the equity premium. Sunspot uncertainty adds noise to the risky return, but it does not explain why a risky security should pay systematically more than a safe bond.

What if sunspots are correlated with consumption growth? Here there is some hope that sunspots may contribute to an explanation of the equity premium. To illustrate this point, we solved the full non-linear model for a series of calibrated examples. The following section reports the result of this exercise.

VII. Simulating the invariant distribution

To compute moments of the invariant distribution, we used the difference equation (57), to construct an approximation to the transition function

$$T(x, A), \tag{83}$$

where $x \equiv \{z, \gamma\}$ is the value of the state at date t and A is a set that represents possible values that z and γ might take at date $t+1$. For every value of A, $T(\cdot, A)$ is a measurable function and for every value of x, $T(x, \cdot)$ is a probability measure (Stokey, Lucas, and Prescott, 1989, Chaper 8). If $p(x)$ is the probability that the system is in state x at date t then

$$p'(x') = \int_X T(x, x')\, dp(x), \tag{84}$$

is the probability that it is in state x' at date $t+1$. By iterating on this operator equation for arbitrary initial p we arrive at an expression for the invariant measure. This invariant measure, $p(x)$, is the unconditional probability of observing the system in state $x = \{z, \gamma\}$. We now explain how to compute a discrete approximation to this invariant measure.

Let Z be the set $Z \equiv [z_1, z_b]$ and define $\Gamma = [\gamma_1, \gamma_2]$ as the set inhabited by γ_t when γ is generated by Equation (63) and $\varepsilon_{f,t} \in \epsilon$. We divided Z and Γ into n_z and n_γ, equally spaced intervals and we approximated the operator T with a matrix $\tilde{T}$ by breaking the state space into $n_z \times n_\gamma$ intervals. In practice, we found that $n_z = 30$ and $n_\gamma = 10$ leads to a fine enough approximation to give good results while remaining computationally manageable.

We need to compute state transition matrices that are $n^2 \times n^2$ where $n = n_z \times n_\gamma$. For $n_z = 30$ and $n_\gamma = 10$, $n = 300$ and $n^2 = 90,000$. For these values, our code takes approximately 5 seconds in Matlab on an Apple Mac with dual 2.13 GHZ Intel processors.

To compute the invariant distribution in a parameterized example, we used values of $\pi = 0.98$, $\mu = 0.5$, $\beta_1 = 0.98$, $\beta_2 = 0.9$ and we set the autocorrelation parameter $\gamma = 0$, and the standard deviation of the shocks at $\sigma_f = 0.05$ and $\sigma_s = 0.06$. We partitioned the space $X = Z \times \Gamma$ into 300 cells and for each pair of cells, $\{k = \{i, j\}, l = \{i', j'\}\}$ we computed the pair of shocks, $\{\varepsilon_{f,k,l}, \varepsilon_{s,k,l}\}$ needed to get from the midpoint of cell $k = \{i, j\}$ to the midpoint of cell $l = \{i', j'\}$. Here i and i', are elements of Z and j and j' are elements of Γ. This process generated a grid, 300×300 and $90,000$ pairs of numbers $\varepsilon_{k,l} = \{\varepsilon_{f,k,l}, \varepsilon_{s,k,l}\}$ associated with each cell of the grid.

To compute the probability of any pair of numbers $\varepsilon_{k,l} = \{\varepsilon_{f,k,l}, \varepsilon_{s,k,l}\}$ we used a discrete approximation to a normal distribution. First we associated a weight

$$w_{k,l} = \exp\left(-\left(\varepsilon_{k,l} - a\right)\Sigma^{-1}\left(\varepsilon_{k,l} - a\right)\right), \tag{85}$$

with each value of $\varepsilon_{k,l} = \{\varepsilon_{f,k,l}, \varepsilon_{s,k,l}\}$. Here

$$\Sigma = \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix}, \tag{86}$$

is the variance covariance matrix of the shocks and $a = [a_f; a_s]$ is a vector of centrality parameters.

In our baseline case we set $\sigma_{sf} = 0$ but we also experimented with the correlated case by choosing values of the correlation coefficient,

$$\rho = \frac{\sigma_{sf}}{\sigma_s \sigma_f}, \tag{87}$$

between -0.95 and $+0.95$.

To compute a discrete approximation to the operator T we constructed a Markov matrix $\tilde{T}$ by normalizing the weights $w_{k,l}$. This process led to a set of n discrete probability measures $p_{k,l}$, where

$$p_{k,l} = \frac{w_{k,l}}{\sum_l w_{k,l}}, \tag{88}$$

is one row of the Markov matrix

$$\tilde{T} = \left[\tilde{T}_{k,l}\right]. \tag{89}$$

Next, we used Newton's method to adjust the centrality parameters, a, to ensure that, for each k,

$$f_k(a) = \sum_l p_{k,l}(a) \begin{bmatrix} \varepsilon_{s,k,l} \\ \varepsilon_{f,k,l} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0. \tag{90}$$

This step ensures that the discrete probability measures p_k for each k are consistent with the assumptions that $E[\varepsilon_s] = 1$ and $E[\varepsilon_f] = 1$. Figure 3 illustrates the invariant joint distribution over γ and z that we computed using this method.

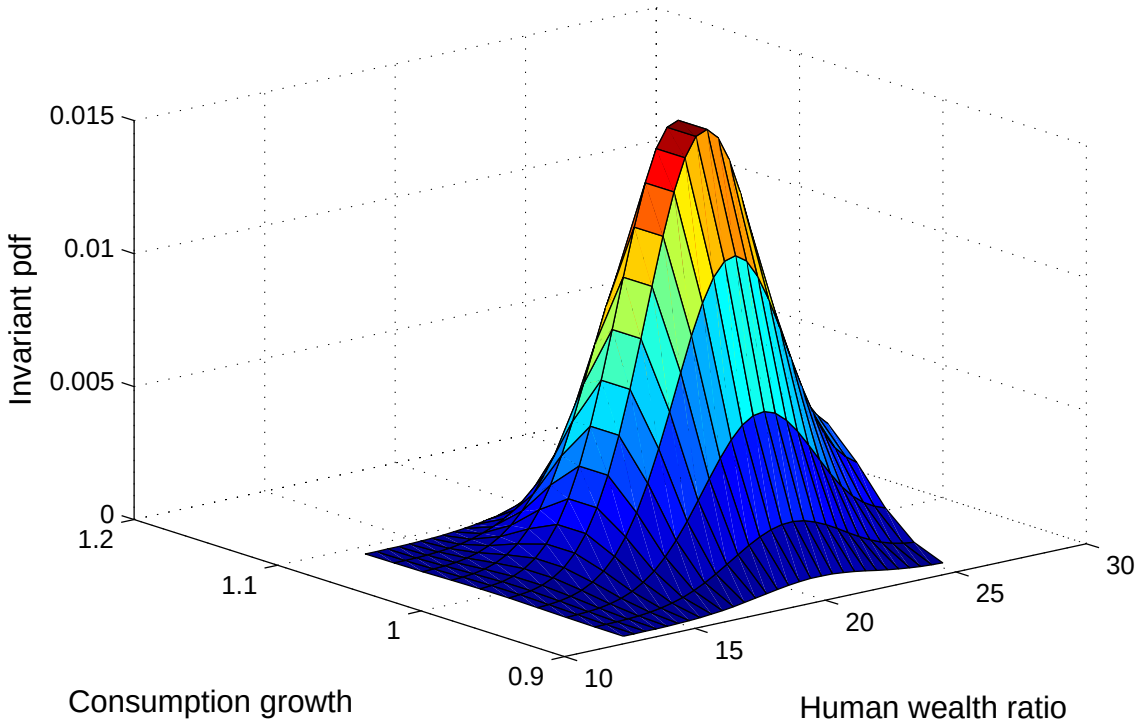

Figure 3: The joint invariant distribution of consumption growth and the human wealth ratio

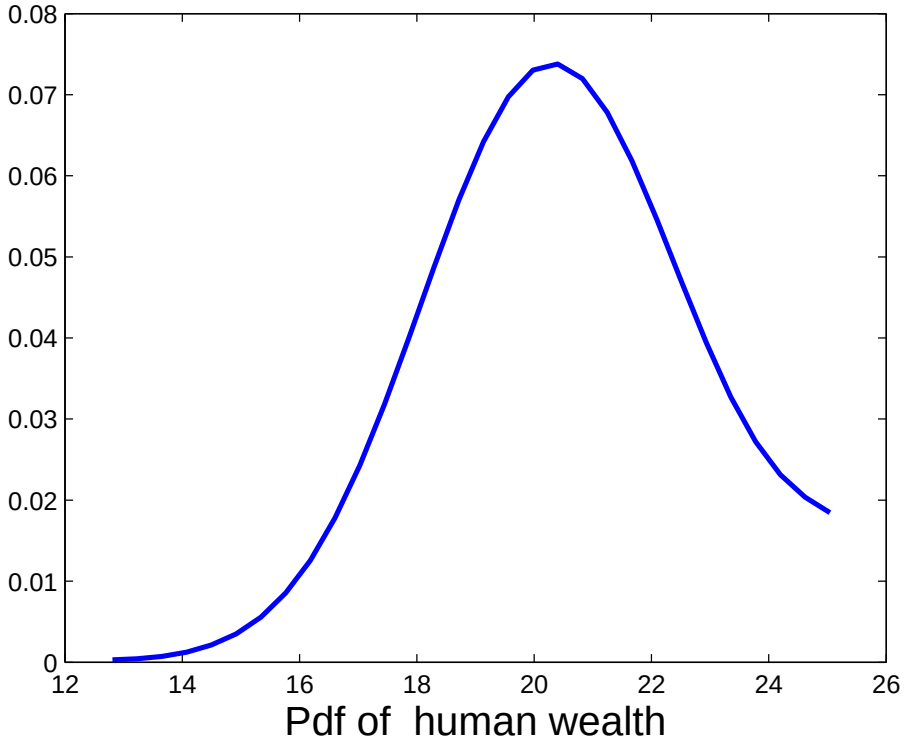

Figure 4: The marginal invariant distribution of the human wealth ratio

We can also compute the marginal distribution of human wealth which we report in Figure 4. The median human wealth ratio is a little over 20, but there is considerable probability mass that this ratio will be less than 18 or greater than 23. That difference represents twenty five percent of the median human wealth. If we define a recession to be a value of the human wealth ratio less than 18 and a boom, a ratio greater than 23, then a person of either type who is born into a recession will have lifetime wealth that is approximately twenty five percent lower than a similar person born in a boom. These are big numbers.

VII.1. **The equity premium in simulated data.** Since the equity premium can always be increased by leverage, a better measure of excess returns is the Sharpe ratio; this is the ratio of the excess return of a risky to a safe security, divided by the standard deviation of the risky security. In U.S. data, the Sharpe ratio has historically been in the range of 0.25 to 0.5, depending on the sample length and the frequency of observation.

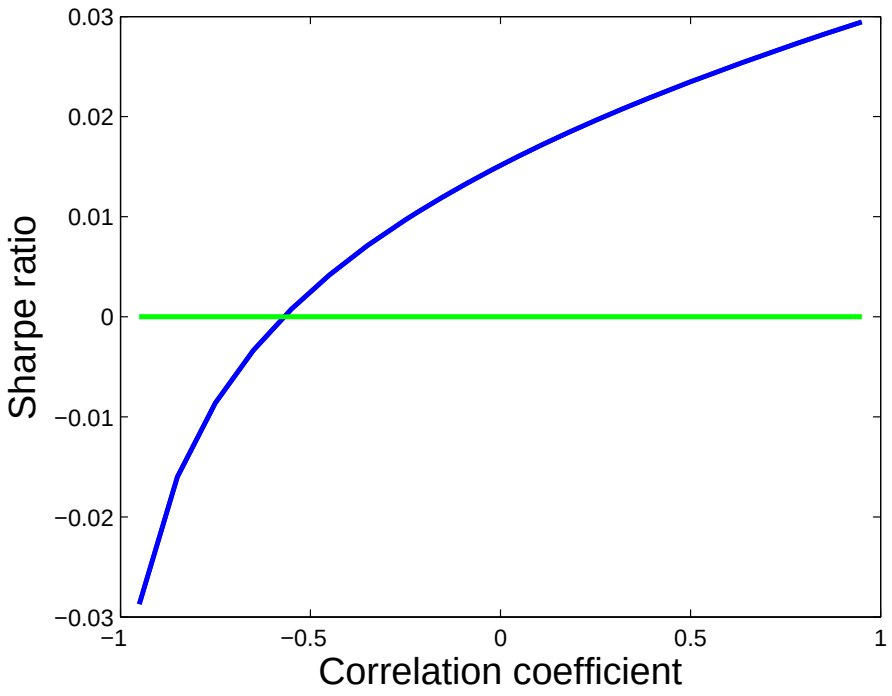

Figure 5: The Sharpe ratio and the correlation coefficient

To explore the role of correlated shocks in determining the Sharpe ratio, we computed the invariant distribution for a sequence of economies in which the shocks to growth, ε_f were correlated with extrinsic uncertainty, ε_s. For each case, we held the parameters of the model constant at the same values used to construct Figure 3, but we allowed the correlation coefficient, ρ, between intrinsic and extrinsic shocks to

vary from $-.095$ to $+0.95$. The results of this exercise are reported in Figure 5 which shows that the value of the Sharpe ratio in the invariant distribution varies from -0.03, when $\rho = -0.95$ to $+0.03$, when $\rho = +0.95$. These numbers do not depend on a local approximation, they are computed using a grid approximation to the entire invariant distribution.

The Sharpe ratio increases as the correlation coefficient increases because a risky security becomes less attractive to hold if its payoff is correlated with consumption. In a representative agent model, consumption and the endowment are the same thing. Figure 4 shows that the same logic holds in the two agent economy where the consumption of the agent holding the risky security is not perfectly correlated with GDP.

But although the model can generate a positive Sharpe ratio, the largest value in our calibrated economy is closer to 0.03 than to 0.5, the value observed in U.S. data. Theoretically, one could make the value of the Sharpe ratio arbitrarily large by increasing the volatility of the sunspot shock. But there is an upper bound on sunspot volatility determined by the difference between the time preference rate of the patient and impatient individuals. This difference determines the range of feasible values of z_t which, in turn, determines a bound on the volatility of ε_s.

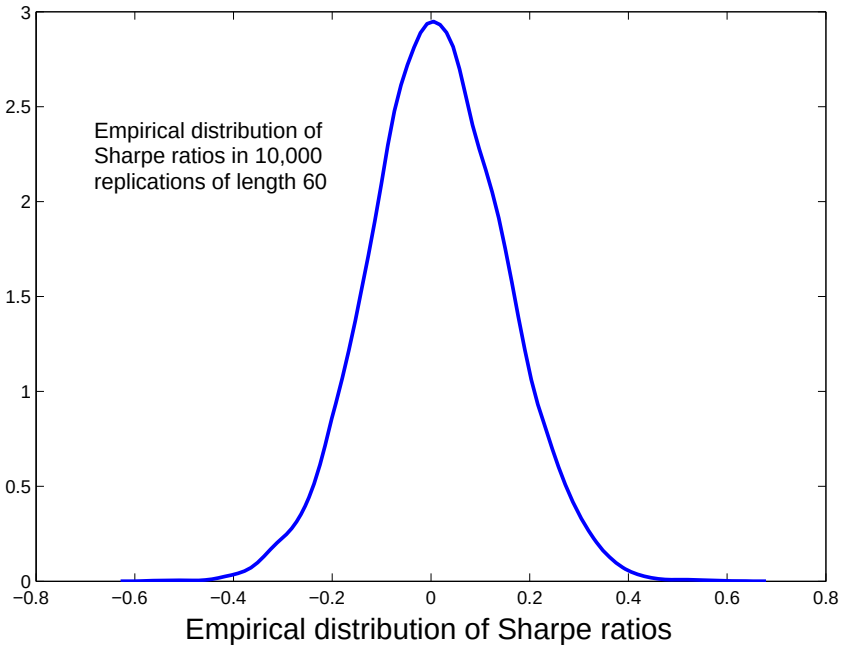

Figure 6: The empirical distribution of Sharpe ratios

Although the theoretical value of the Sharpe ratio is small, the high volatility of equity returns implies that there is a significant probability of observing a high Sharpe ratio in any given sample. To explore the possibility that a high Sharpe ratio is an artifact of sampling variation, we simulated 10,000 samples of length 60. Figure 6 reports the result of that exercise.

We calibrated the model parameters to annual data so each sample represents an observation of 60 years of data. The figure reports the empirical frequency distribution of Sharpe ratios across these 10,000 samples and it is clear from the figure that there is a high probability of observing a Sharpe ratio greater than 0.2. In the empirical distribution, 33% of the observed Sharpe ratios were greater than 0.1, 21% were greater than 0.15 and 12% were 0.2 or larger. There is however, almost no probability of observing a Sharpe ratio of 0.5. In our experiment, only 6 out of 10,000 draws were greater than 0.5.[4]

To give some idea of the time series properties of data generated by our model, Figure 7 reports the values of six time series for a single draw of 60 years of data. The sequence of sunspot shocks used for this simulation were uncorrelated, implying that the theoretical value of the Sharpe ratio is around 1% (see Figure 5). The draw used for this simulation has a Sharpe ratio of 0.24.

The two top left panels of Figure 7 show that the risky return is much more volatile than the safe return and, for this particular draw, the risky return has a higher mean. The bottom left panel illustrates aggregate consumption growth which varies by a little more than plus or minus 4% and has a standard deviation of 0.03, which matches the volatility of consumption growth in post-war U.S. data.

The top two panels on the right of the figure illustrate the share of consumption going to type 1 agents, and the value of the pricing kernel. These panels show that even though aggregate consumption growth is relatively smooth, the consumption growth of type 1 agents can vary by as much as 30% over the business cycle. It is the

[4]Remember, however, that we are maintaining the assumption of logarithmic preferences and we conjecture that by relaxing that assumption and allowing for constant relative risk aversion preferences, we will multiply the range of observed Sharpe ratios by a factor proportional to the coefficient of relative risk aversion. That exercise is feasible using the results we report in Farmer, Nourry, and Venditti (2011), but the exercise of computing the invariant distribution is significantly more complicated since it involves solving for an additional functional equation. We explore that extension in work in progress.

volatility of individual consumption growth rates that allows this model to generate a volatile pricing kernel.[5]

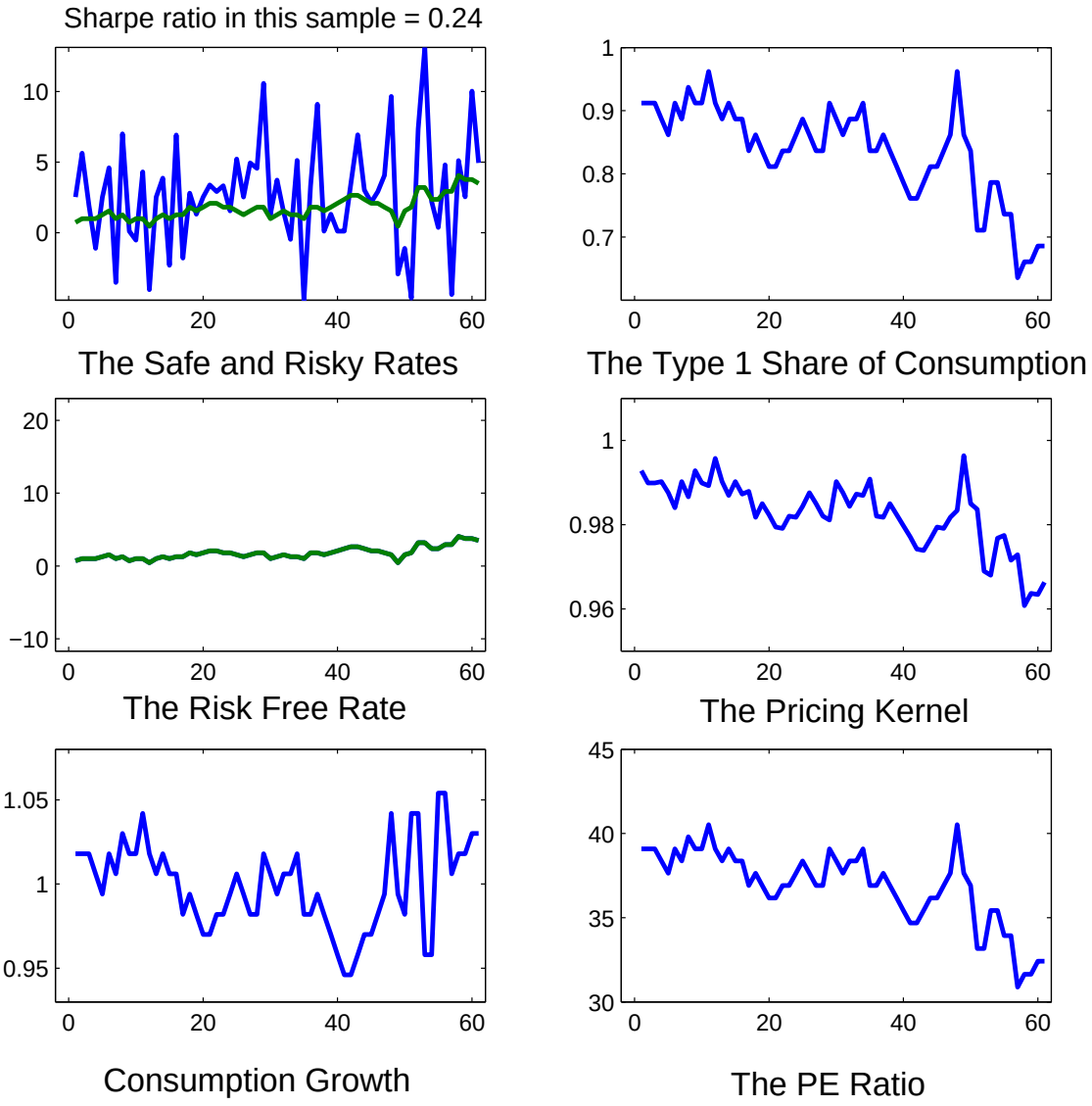

Figure 7: Six time series generated by simulating sixty years of data

The bottom right panel shows what's happening to the price-earnings ratio which takes long slow swings varying over a range from 30 to 40. Although this range is smaller than observed price earnings ratios in U.S. data, which vary from 10 to 45, our price earning statistic does not allow for leverage which is both high and variable in the data.

[5]Although 30% is high relative to aggregate consumption volatility, there is evidence that high income individuals who own stock have consumption growth that is four times as volatile as average consumption growth (Malloy, Moskowitz, and Vissing-Jorgensen, 2009).

VIII. Conclusion

The first welfare theorem of general equilibrium theory asserts that every competitive equilibrium is Pareto optimal. When financial markets are complete, and when all agents are able to participate in financial markets, this theorem implies that unrestricted trade in financial assets will lead to the efficient allocation of risk to those who are best able to bear it.

We have shown, in this paper, that unregulated financial markets do not lead to Pareto efficient outcomes and that the failure of complete financial markets to deliver socially efficient allocations has nothing to do with financial constraints, transactions costs or artificial barriers to trade. The first welfare theorem fails because participation in financial markets is necessarily incomplete as a consequence of the fact that agents cannot trade risk in financial markets that open before they are born. For this reason, financial markets do not work well in the real world.

The Ramsey-Cass-Koopmans model (Ramsey, 1928; Koopmans, 1965; Cass, 1965) underpins not only all of modern macroeconomics, but also all of modern finance theory. Existing literature modifies this model by adding frictions of one kind or another to explain why free trade in competitive markets cannot achieve an efficient allocation of risk. It has not, as yet, offered a convincing explanation for the volatility of the stochastic discount factor in real world data. The most surprising feature of our work is how close it is to the Ramsey-Cass-Koopmans model; yet we do not need to assume frictions of any kind. We have shown that financial markets cannot be Pareto efficient, except by chance. Although individuals, in our model, are rational; markets are not.

Appendix

Appendix A.

Proof of Proposition 1. If we sum Equation (23) over all agents of type i who were alive at date t, we arrive at the equation,

$$Q_t^{t+1}\left(S^{t+1}\right) = \frac{\psi\left(S^{t+1}\right)\beta_i \sum_{j\in\mathcal{A}_t} c_{i,t}^j\left(S^t\right)}{\sum_{j\in\mathcal{A}_t} c_{i,t+1}^j\left(S^{t+1}\right)}. \tag{A.1}$$

The consumption at date $t+1$ of everyone of type i who was alive at date t, is equal to the consumption of all agents of type i minus the consumption of the new borns. For any date $t+1$ and any variable x let x_t^j be the quantity of that variable held by a household of generation j and let x_t be the aggregate quantity. Let $\mathcal{A}_t$ be the index set of all agents alive at date t and let $\mathcal{N}_{t+1}$ denote the set of newborns at period $t+1$. Then,

$$\pi \sum_{j\in\mathcal{A}_t} x_{t+1}^j + \sum_{j\in\mathcal{N}_{t+1}} x_{t+1}^j = \sum_{j\in\mathcal{A}_{t+1}} x_{t+1}^j = x_{t+1}. \tag{A.2}$$

Using Equation (A.2) we can write the denominator of Equation (A.1) as,

$$\sum_{j\in\mathcal{A}_t} c_{i,t+1}^j\left(S^{t+1}\right) = \frac{1}{\pi}\left(\sum_{j\in\mathcal{A}_{t+1}} c_{i,t+1}^j\left(S^{t+1}\right) - \sum_{j\in\mathcal{N}_{t+1}} c_{i,t+1}^j\left(S^{t+1}\right)\right). \tag{A.3}$$

The first term on the right-hand-side of this equation is aggregate consumption of type i, which we define as

$$c_{i,t+1}\left(S^{t+1}\right) \equiv \sum_{j\in\mathcal{A}_{t+1}} c_{i,t+1}^j\left(S^{t+1}\right). \tag{A.4}$$

The second term is the consumption of the newborns of type i. Because these agents are born with no financial assets, their consumption is proportional to their human wealth. This leads to the expression

$$\sum_{j\in\mathcal{N}_{t+1}} c_{i,t+1}^j\left(S^{t+1}\right) = B_i\left(1-\pi\right) h_{i,t+1}\left(S^{t+1}\right), \tag{A.5}$$

where the coefficient $(1-\pi)$, is the fraction of newborns of type i. Using equations (A.3)-(A.5) we can rewrite (A.1) as

$$Q_t^{t+1}\left(S^{t+1}\right) = \frac{\psi\left(S^{t+1}\right)\left(1-B_i\right) c_{i,t}\left(S^t\right)}{c_{i,t+1}\left(S^{t+1}\right) - B_i\left(1-\pi\right) h_{i,t+1}\left(S^{t+1}\right)}, \tag{A.6}$$

which is Equation (28), the expression we seek. □

Appendix B.

Proof of Lemma 1. From Equation(29), evaluated for types 1 and 2, we get

$$\alpha_t = \frac{\lambda_t}{B_1} - z_{1,t}, \tag{B.1}$$

$$-\alpha_t = \frac{1-\lambda_t}{B_2} - z_{2,t}. \tag{B.2}$$

Summing (B.1) and (B.2) and rearranging leads to Equation (35). The fact that ζ is increasing in z follows from the assumption, $B_2 > B_1$. The domain of ζ is found by evaluating ζ^{-1} for values of $\lambda = 0$ and $\lambda = 1$.

From (35) we have that,

$$\lambda_t = \frac{B_1 B_2}{B_2 - B_1}\left(z_t - \frac{1}{B_2}\right), \tag{B.3}$$

which expresses the consumption share as a function of the aggregate human wealth ratio. From the consumption function of type 1 agents, Equation (29), we have the following expression linking the consumption share with the asset share, and with the type 1 human wealth ratio,

$$\lambda_t = B_1\left(\alpha_t + z_{1,t}\right). \tag{B.4}$$

Combing equations (B.3) and (B.4) gives

$$B_1\left(\alpha_t + z_{1,t}\right) = \frac{B_1 B_2}{B_2 - B_1}\left(z_t - \frac{1}{B_2}\right). \tag{B.5}$$

Using definition, (36), this leads to Equation (37), the expression we seek. □

Appendix C.

Proof of Proposition 2. From Proposition 1 and the restriction to perfect foresight equilibria, we have that

$$\lambda_t = \frac{B_1 B_2}{B_2 - B_1}\left(z_t - \frac{1}{B_2}\right). \tag{C.1}$$

Leading Equation (C.1) one period gives,

$$\lambda_{t+1} = \frac{B_1 B_2}{B_2 - B_1}\left(z_{t+1} - \frac{1}{B_2}\right). \tag{C.2}$$

Dividing the numerator of Equation (28), by ω_t and dividing the denominator by ω_{t+1} leads to the following expressions that relate the normalized pricing kernel to the consumption share and to the human wealth ratio of each type,

$$\tilde{Q}_t^{t+1} = \frac{\left(1 - B_1\right)\lambda_t}{\lambda_{t+1} - B_1\left(1-\pi\right)z_{1,t+1}}, \tag{C.3}$$

$$\tilde{Q}_t^{t+1} = \frac{(1-B_2)(1-\lambda_t)}{(1-\lambda_{t+1}) - B_2(1-\pi)z_{2,t+1}}. \tag{C.4}$$

These expressions follow from the fact that agents of each type equate the marginal rate of substitution to the pricing kernel, state by state.

Next, we divide the human wealth equation, (24), by the aggregate endowment. That leads to the following difference equation in the human wealth ratio for type i,

$$z_{i,t} = \mu_i + \pi E_t\left[\tilde{Q}_t^{t+1} z_{i,t+1}\right]. \tag{C.5}$$

Adding up Equation (C.5) over both types leads to the following expression for the aggregate human wealth ratio,

$$z_t = 1 + \pi E_t\left[\tilde{Q}_t^{t+1} z_{t+1}\right]. \tag{C.6}$$

From equations (C.3) and (C.4),

$$\tilde{Q}_t^{t+1} = \frac{(1-B_2)(1-\lambda_t)}{1-\lambda_{t+1} - B_2(1-\pi)z_{2,t+1}}, \tag{C.7}$$

$$\tilde{Q}_t^{t+1} = \frac{(1-B_1)\lambda_t}{\lambda_{t+1} - B_1(1-\pi)z_{1,t+1}}. \tag{C.8}$$

Substituting for λ_t and λ_{t+1} from equations (C.1) and (C.2) into (C.4), we obtain the following expression for $z_{2,t+1}$ as a function of z_t and $z_{1,t}$[6]

$$z_{2,t+1} = -\frac{(\pi B_1 - B_1 - \pi B_1 B_2) z_t z_{1,t} + z_t + (B_1 - \pi + \pi B_2) z_{1,t} - 1}{(\pi B_2 - B_2 - \pi B_1 B_2) z_t + (B_2 + \pi(B_1 - 1))}. \tag{C.9}$$

Define the following transformed parameters

$$\begin{aligned} \theta_1 &= B_2 - \pi(1-B_1) + \mu(1-\pi)(B_1 - B_2), \\ \theta_2 &= -B_2(1-\pi) - \pi B_1 B_2 < 0, \\ \theta_3 &= (B_2 - B_1)(1-\pi), \\ \theta_4 &= B_2 - \pi(1-B_1) = \theta_1 + \mu\theta_3. \end{aligned} \tag{C.10}$$

Combining equation (C.1)–(C.3), with (C.5) gives the following expression for the normalized pricing kernel,

$$\pi\tilde{Q}_t^{t+1} = -\frac{\theta_1 + \mu\theta_3 + \theta_2 z_t}{1 + \theta_3 z_{1,t+1}} = -\frac{\theta_4 + \theta_2 z_t}{1 + \theta_3 z_{1,t+1}}. \tag{C.11}$$

Because we have normalized all variables by income, none of the equations of our model contain random variables. Hence we may drop the expectations operator and write the human wealth equations, (C.5) and (C.6) as follows,

$$z_{1,t} = \mu + \pi\tilde{Q}_t^{t+1} z_{1,t+1}, \tag{C.12}$$

[6]The algebra used to derive (C.9) was checked using Maple in Scientific Workplace. The code is available from the authors on request.

$$z_t = 1 + \pi \tilde{Q}_t^{t+1} z_{t+1}. \tag{C.13}$$

Rearranging Equations (C.12) and (C.13), replacing $\tilde{Q}_t^{t+1}$ from (C.11), gives,

$$z_{1,t+1} = \frac{\mu - z_{1,t}}{\theta_1 + \theta_2 z_t + \theta_3 z_{1,t}}, \tag{C.14}$$

$$z_{t+1} = \frac{1 - z_t}{\theta_1 + \theta_2 z_t + \theta_3 z_{1,t}}, \tag{C.15}$$

which are the expressions for equations (39) and (40) that we seek. The initial condition, Equation (41), follows from Proposition 1 and the expression for the normalized pricing kernel. □

Appendix D.

Proof of Proposition 3. Evaluating equations (C.14) and (C.15) evaluated at a steady state, it follows that a steady state equilibrium is a solution of the following second degree polynomial

$$\mathcal{P}(z) = z^2(\theta_2 + \mu\theta_3) + z(1 + \theta_1) - 1 = 0. \tag{D.1}$$

Define the discriminant

$$\Delta = (1 + \theta_1)^2 + 4(\theta_2 + \mu\theta_3). \tag{D.2}$$

Using the fact that $B_2 > B_1$, it follows from the definitions of θ_1, θ_2 and θ_3 that $\theta_1 > B_2 - \pi(1 - B_1)$, $\theta_3 > 0$ and hence $\theta_2 + \mu\theta_3 > \theta_2$. Using these inequalities to replace $(\theta_2 + \mu\theta_3)$ by θ_2 and replacing θ_2 by its definition, it follows that

$$\Delta > [B_2 + 1 - \pi(1 - B_1)]^2 - 4B_2[1 - \pi(1 - B_1)] = [B_2 - 1 + \pi(1 - B_1)]^2 \geq 0. \tag{D.3}$$

Since the discriminant is non-negative, there exist two real solutions to Equation (D.1), z^* and z^{**}, given by the expressions,

$$z^* = -\frac{1 + \theta_1 + \sqrt{\Delta}}{2(\theta_2 + \mu\theta_3)} > z^{**} = -\frac{1 + \theta_1 - \sqrt{\Delta}}{2(\theta_2 + \mu\theta_3)}. \tag{D.4}$$

We next need to check that these two solutions belong to the admissible set (z_a, z_b). Consider first the lower solution z^{**}. From the definition of z^{**} it follows that $z^{**} > z_a \equiv 1/B_2$ if and only if

$$1 + \theta_1 + \frac{2(\theta_2 + \mu\theta_3)}{B_2} < \sqrt{\Delta}. \tag{D.5}$$

Squaring both sides of (D.5) and substituting for Δ from (D.2) implies that, equivalently,

$$1 + \theta_1 - B_2 + \frac{\theta_2 + \mu\theta_3}{B_2} < 0. \tag{D.6}$$

Substituting the expressions for θ_1, θ_2 and θ_3 into this inequality and rearranging terms leads to the following expression for the left-hand side of (D.6),

$$\mu(1-\pi)(B_2-B_1)\frac{1-B_2}{B_2} > 0, \tag{D.7}$$

where the inequality in (D.7) follows since $1 > B_2 > B_1 > 0$ and $\pi < 1$. It follows that $z^{**} < z_a$ and hence z^{**} is not an admissible steady state.

Consider now the larger of the two solutions, z^*. The same computation as previously allows us to conclude that $z^* > z_a$. We must next show that $z^* < z_b \equiv 1/B_1$. Using the definition of z^*, this occurs if and only if

$$1 + \theta_1 + \frac{2(\theta_2 + \mu\theta_3)}{B_1} < -\sqrt{\Delta}. \tag{D.8}$$

A necessary condition for this inequality to hold is that the left-hand side is negative. Using the definitions of θ_1, θ_2 and θ_3, we may write the left side of (D.8) as the following second degree polynomial in B_1,

$$\begin{aligned} \mathcal{G}(B_1) = B_1 &\left[1 + B_2 - \pi(1-B_1) + \mu(1-\pi)(B_1 - B_2)\right] \\ &-2\left[B_2(1-\pi) + \pi B_1 B_2 + \mu(1-\pi)(B_1-B_2)\right] < 0. \end{aligned} \tag{D.9}$$

Notice that $\mathcal{G}(B_1)$ is convex for $B_1 \in (0, B_2)$. Further, we have that that $\mathcal{G}(0) = 0$ and $\mathcal{G}(B_2) = -B_2(1-\pi)(1-B_2) < 0$. It follows that for any $B_1 \in (0, B_2)$, $\mathcal{G}(B_1) < 0$ and thus the left-hand side of (D.8) is negative. It follows that inequality (D.8) holds if

$$1 + \theta_1 - B_1 + \frac{\theta_2 + \mu\theta_3}{B_1} < 0. \tag{D.10}$$

Substituting the expressions for θ_1, θ_2 and θ_3 into this inequality and simplifying the expression yields

$$-(1-\pi)(1-\mu)(1-B_1)(B_2-B_1) < 0. \tag{D.11}$$

This inequality establishes that $z^* \in \hat{Z}$ and hence z^* is an admissible steady state. We study the stability properties of $z^*$by linearizing the dynamical system (39)–(40) around z^*. Using the steady state relationships (46) and (47), we get after some simplification, the following Jacobian matrix

$$\mathcal{J} = \begin{pmatrix} \frac{z^*-1}{z^*}(1+\theta_2 z^*) & (z^*-1)\theta_3 \\ (z^*-1)\mu\theta_2 & \frac{z^*-1}{z^*}(1+\mu\theta_3 z^*) \end{pmatrix}. \tag{D.12}$$

The associated characteristic polynomial is

$$\begin{aligned}\left[\frac{z^*-1}{z^*}(1+\theta_2 z^*)-x\right]\left[\frac{z^*-1}{z^*}(1+\mu\theta_3 z^*)-x\right]-(z^*-1)^2\mu\theta_2\theta_3,\\ =\left(x-\frac{z^*}{z^*-1}\right)\left(x-\frac{1-\theta_1 z^*}{z^*-1}\right)=0,\end{aligned}\tag{D.13}$$

with characteristic roots

$$x_1=\frac{z^*}{z^*-1}>1,\tag{D.14a}$$

$$x_2=\frac{1-\theta_1 z^*}{z^*-1}.\tag{D.14b}$$

Notice that the dynamical system (39) and (40) admits $z_{1,t}=\mu z_t$ for all t as a solution. It follows that the two-dimensional dynamical system in $(z_t, z_{1,t})$ can be reduced to the one-dimensional difference equation defined by Equation (49). We next establish that this system is a saddle and that the one-dimensional difference equation (49) is globally stable. Since, from (D.14a), x_1 is positive and greater than one, we need only establish a general property to guarantee global conclusion and that $-1<x_2<1$. Let us first consider the derivative of $g(z)$ for any z, $g'(z)=(1-\theta_1 z)/(z-1)$. Since $z>1$, we have $g'(z)\geq -1$ for any $z\in(z_a,z_b)$ if $1-\theta_1\geq 0$, which holds since $B_2>B_1$. This property implies that $x_2>-1$. Moreover, we get $g'(z_a)>0$ if and only if $B_2-\theta_1>0$, which again holds as $B_2>B_1$. We then establish that $x_2<1$. From (D.14b), this follows if and only if $2-(1+\theta_1)z^*<0$. Equivalently using the definitions of θ_1 and z^*, together with the fact that $\theta_2+\mu\theta_3<0$, $x_2<1$, if and only if

$$\begin{aligned}4(\theta_2+\mu\theta_3)+(1+\theta_1)\left[1+\theta_1+\sqrt{\Delta}\right]&=\Delta+(1+\theta_1)\sqrt{\Delta}\\ &=\sqrt{\Delta}\left[1+\theta_1+\sqrt{\Delta}\right]>0.\end{aligned}\tag{D.15}$$

We derive from (C.10) that

$$1+\theta_1=B_2\left[1-\mu(1-\pi)\right]+B_1\mu(1-\pi)+1-\pi(1-B_1)>0,\tag{D.16}$$

and thus $x_2<1$. Since $x_1>1$ we conclude that z^* is saddle-point stable. Further, z^* is globally stable for any $z_t\in\hat{Z}$.

Next we turn to an equivalent representation of the system using $\tilde{Q}_t^{t+1}$ as a state variable. Replacing $z_{1,t+1}$ from (39) in Equation (43), and simplifying the resulting expression gives,

$$\tilde{Q}_t^{t+1}=-\frac{\theta_1+\theta_2 z_t+\theta_3 z_{1,t}}{\pi}.\tag{D.17}$$

Substituting the restriction $z_{1,t} = \mu z_t$ into Equation (D.17) and inverting the equation to find z_t as a function of $\tilde{Q}_t^{t+1}$, leads to

$$z_t = -\frac{\pi \tilde{Q}_t^{t+1} + \theta_1}{\theta_2 + \mu\theta_3}. \tag{D.18}$$

Substituting this expression into (49) and rearranging terms leads to the difference equation

$$\tilde{Q}_{t+1}^{t+2} = h\left(\tilde{Q}_t^{t+1}\right), \tag{D.19}$$

where

$$h\left(\tilde{Q}_t^{t+1}\right) \equiv \frac{1-\theta_1}{\pi} + \frac{\theta_1 + \theta_2 + \mu\theta_3}{\pi^2 \tilde{Q}_t^{t+1}} = \frac{1-\theta_1}{\pi} - \frac{(1-B_1)(1-B_2)}{\pi \tilde{Q}_t^{t+1}}, \tag{D.20}$$

which provides an equivalent representation of the equilibrium in terms of $\tilde{Q}$. Using the same arguments as previously, it follows that for all $Q_0^1 \in (q_a, q_b)$ with

$$q_a = -\frac{\theta_1 + (\theta_2 + \mu\theta_3)\, z_a}{\pi}, \quad q_b = -\frac{\theta_1 + (\theta_2 + \mu\theta_3)\, z_b}{\pi}. \tag{D.21}$$

there exists a sequence of equilibrium asset prices described by the difference equation (D.19), that converges to the steady state pricing kernel. □

Appendix E.

Proof of Proposition 4. Consider the definition of human wealth,

$$z_t = 1 + E_t\left[\pi \tilde{Q}_t^{t+1} z_{t+1}\right]. \tag{E.1}$$

In Proposition 2, Equation (C.11), we derived an expression for the pricing kernel $\tilde{Q}_t^{t+1}(z_t, z_{1,t+1})$, that we write below as Equation (E.2).

$$\pi \tilde{Q}_t^{t+1} = -\left(\frac{\theta_4 + \theta_2 z_t}{1 + \theta_3 z_{1,t+1}}\right). \tag{E.2}$$

Replacing (E.2) in (E.1), and restricting attention to the stable branch of the saddle by setting $z_{1,t} = \mu z_t$, we arrive at a the following functional equation,

$$z_t = 1 + E_t\left[-\left(\frac{\theta_4 + \theta_2 z_t}{1 + \mu\theta_3 z_{t+1}}\right) z_{t+1}\right]. \tag{E.3}$$

Since Equation (E.3) characterizes equilibria, it follows that any admissible sequence $\{z_t\}$ that satisfies Equation (E.3) is an equilibrium sequence. We now show how to construct a stochastic process for $\{z_t\}$ that generates admissible solutions to (E.3). Let $\varepsilon_{s,t+1}$, be a bounded, i.i.d. random variable with support $\epsilon \equiv [e_a, e_b]$ such that

$$E_t(\varepsilon_{s,t+1}) = 1, \tag{E.4}$$

and consider sequences for $\{z_t\}$, $z \in Z \equiv (z_1, z_b)$ that satisfy the equation,

$$(z_t - 1)\,\varepsilon_{s,t+1} = -\left(\frac{\theta_4 + \theta_2 z_t}{1 + \theta_3 \mu z_{t+1}}\right) z_{t+1}. \tag{E.5}$$

Rearranging (E.5), using the fact that $\theta_4 - \mu\theta_3 = \theta_1$, we may define a function $g(\cdot) : Z \times \epsilon \to Z$,

$$z_{t+1} = \frac{(1 - z_t)\,\varepsilon_{s,t+1}}{(\theta_4 - \mu\theta_3\varepsilon_{s,t+1}) + (\theta_2 + \mu\theta_3\varepsilon_{s,t+1})\,z_t} = g\left(z_t, \varepsilon_{s,t+1}\right). \tag{E.6}$$

This is the analog of Equation (49) in Proposition 3. Any admissible sequence must lie in the set $\hat{Z} \equiv [z_a, z_b]$ where $z_a \equiv B_2^{-1}$ and $z_b \equiv B_1^{-1}$. We now show how to construct the largest set $Z \subset \hat{Z}$ such that all sequences $\{z_t\}$ generated by (E.6) are admissible. Note first, that

$$\frac{\partial g(z, \varepsilon_s)}{\partial \varepsilon_s} = \frac{(1 - z)\,(\theta_4 + \theta_2 z)}{\left[(\theta_4 - \mu\theta_3\varepsilon_s) + (\theta_2 + \mu\theta_3\varepsilon_s)\,z\right]^2}, \tag{E.7}$$

where $\theta_4 + \theta_2 z_a = B_2(1 - z_a) < 0$ and $\theta_4 + \theta_2 z_b = (1 - z_b)[\pi B_1 + B_2(1 - \pi)] < 0$. Because $z > 1$, it follows that $\partial g(z, \varepsilon_s)/\partial \varepsilon_s > 0$ for any $z \in (z_a, z_b)$. Moreover, $g(1, \varepsilon_s) = 0$ for any ε_s. We conclude that the graph of the function $g(z, \varepsilon_s)$ rotates counter-clockwise around $z = 1$ as ε_s increases. The upper bound ε_b is then obtained as the solution of the equation, $z_b = g\left(z_b, \varepsilon_s\right)$. A straightforward computation yields

$$\varepsilon_b = \frac{z_b(\theta_4 + \theta_2 z_b)}{(1 - z_b)(1 + \mu\theta_3 z_b)}. \tag{E.8}$$

Starting from the upper bound ε_b, ε_s can be decreased down to the point where the graph of the function $g(z, \varepsilon_s)$ becomes tangent with the 45° line. Consider the equation $g(z, \varepsilon_s) = z$ which can be rearranged to give the equivalent second degree polynomial

$$z^2(\theta_2 + \mu\theta_3\varepsilon) + z[\theta_4 + \varepsilon(1 - \mu\theta_3)] - \varepsilon = 0. \tag{E.9}$$

We denote by z_1, the value of z for which the two roots of (E.9) are equal and hence the discriminant of (E.9) is equal to zero. Equation (E.10) defines a polynomial in ε_s such that this discriminant condition is satisfied and ε_a is the larger of the two values of ε_s such that this condition holds;

$$\begin{aligned} & [\theta_4 + \varepsilon(1 - \mu\theta_3)]^2 + 4\varepsilon(\theta_2 + \mu\theta_3\varepsilon) = 0, \\ \Leftrightarrow \quad & \varepsilon^2(1 + \mu\theta_3)^2 + 2\varepsilon[\theta_4(1 - \mu\theta_3) + 2\theta_2] + \theta_4^2 = 0. \end{aligned} \tag{E.10}$$

Using the formula for the roots of a quadratic, we obtain the following explicit expression for ε_a;

$$\varepsilon_a = \frac{-[\theta_4(1-\mu\theta_3)+2\theta_2] + \sqrt{[\theta_4(1-\mu\theta_3)+2\theta_2]^2 - \theta_4^2(1+\mu\theta_3)^2}}{(1+\mu\theta_3)^2}. \tag{E.11}$$

This establishes the first part of Proposition 4.

We now derive an equivalent difference equation in $\tilde{Q}_t^{t+1}$. Here we use (E.2) and (E.3) to give the following expression for $\tilde{Q}_t^{t+1}$,

$$\tilde{Q}_t^{t+1} = \frac{-\left(\theta_4 - \mu\theta_3\varepsilon_{s,t+1}\right) - \left(\theta_2 + \mu\theta_3\varepsilon_{s,t+1}\right) z_t}{\pi}. \tag{E.12}$$

Rearranging (E.12) gives,

$$z_t = -\frac{\left(\theta_4 - \mu\theta_3\varepsilon_{s,t+1}\right)}{\left(\theta_2 + \mu\theta_3\varepsilon_{s,t+1}\right)} - \frac{\pi\tilde{Q}_t^{t+1}}{\left(\theta_2 + \mu\theta_3\varepsilon_{s,t+1}\right)}. \tag{E.13}$$

Substitute (E.12) into (E.13) to give

$$\tilde{Q}_t^{t+1} = a\left(\varepsilon_{s,t}, \varepsilon_{s,t+1}\right) + \frac{b\left(\varepsilon_{s,t}, \varepsilon_{s,t+1}\right)}{\tilde{Q}_{t-1}^t} \equiv h\left(\tilde{Q}_{t-1}^t, \varepsilon_{s,t+1}, \varepsilon_{s,t}\right), \tag{E.14}$$

where, using the fact that $\theta_1 + \theta_2 + \mu\theta_3 \equiv \theta_2 + \theta_4$, we define,

$$a\left(\varepsilon_{s,t}, \varepsilon_{s,t+1}\right) \equiv \frac{1}{\pi}\left\{\frac{\varepsilon_{s,t}\left(\theta_2 + \mu\theta_3\varepsilon_{s,t+1}\right)}{\left(\theta_2 + \mu\theta_3\varepsilon_{s,t}\right)} - \theta_4 - \mu\theta_3\varepsilon_{s,t+1}\right\}, \tag{E.15}$$

$$b\left(\varepsilon_{s,t}, \varepsilon_{s,t+1}\right) = \frac{1}{\pi^2}\left\{\frac{\varepsilon_{s,t}\left(\theta_2 + \theta_4\right)\left(\theta_2 + \mu\theta_3\varepsilon_{s,t+1}\right)}{\left(\theta_2 + \mu\theta_3\varepsilon_{s,t}\right)}\right\}. \tag{E.16}$$

Equation (E.14) is the analog of Equation (51). This establishes the second part of Proposition 4. □

Appendix F.

Proof of Lemma 2. Taking logs of Equation (59), using the equations (60)–(62), leads to the equivalent expression

$$\begin{aligned} \log\tilde{Q}_t^{t+1}(S') &= \log\tfrac{1}{\pi}\left\{\frac{e^{\log\varepsilon_{s,t}}\left(\theta_2+\mu\theta_3 e^{\log\varepsilon_{s,t+1}}\right)}{\left(\theta_2+\mu\theta_3 e^{\log\varepsilon_{s,t}}\right)} - \theta_4 + \mu\theta_3 e^{\log\varepsilon_{s,t+1}}\right. \\ &+ \left.\tfrac{1}{\pi^2}\left\{\frac{e^{\log\varepsilon_{s,t}}(\theta_2+\theta_4)\left(\theta_2+\mu\theta_3 e^{\log\varepsilon_{s,t+1}}\right)}{\left(\theta_2+\mu\theta_3 e^{\log\varepsilon_{s,t}}\right)}\right\}\frac{1}{e^{\log\tilde{Q}_{t-1}^t(S)}}\right\} \end{aligned} \tag{F.1}$$

Log-linearizing this expression gives

$$\begin{aligned} \log\tilde{Q}_t^{t+1}(S') &= \log\tilde{Q}^* + \left.\frac{\partial\log\tilde{Q}_t^{t+1}(S')}{\partial\log\tilde{Q}_{t-1}^t(S)}\right|_{\tilde{Q}}\left[\log\tilde{Q}_{t-1}^t(S) - \log\tilde{Q}^*\right] \\ &+ \left.\frac{\partial\log\tilde{Q}_t^{t+1}(S')}{\partial\log\varepsilon_{s,t+1}}\right|_{\tilde{Q}}\log\varepsilon_{s,t+1} + \left.\frac{\partial\log\tilde{Q}_t^{t+1}(S')}{\partial\log\varepsilon_{s,t}}\right|_{\tilde{Q}}\log\varepsilon_{s,t}, \end{aligned} \tag{F.2}$$

with

$$\text{(F.3)}\qquad \begin{aligned} \left.\frac{\partial \log \tilde{Q}_t^{t+1}(S')}{\partial \log \tilde{Q}_{t-1}^{t}(S)}\right|_{\tilde{Q}} &= -\frac{b(1,1)}{(\tilde{Q}^*)^2} = -\frac{\theta_2+\theta_4}{\pi^2(\tilde{Q}^*)^2} = \frac{(1-B_1)(1-B_2)}{\pi(\tilde{Q}^*)^2} \equiv \chi, \\ \left.\frac{\partial \log \tilde{Q}_t^{t+1}(S')}{\partial \log \varepsilon_{s,t+1}}\right|_{\tilde{Q}} &= \frac{\mu\theta_3}{\theta_2+\mu\theta_3}\frac{1+\theta_2+\mu\theta_3+\frac{\theta_1+\theta_2+\mu\theta_3}{\pi\tilde{Q}^*}}{1-\theta_1+\frac{\theta_1+\theta_2+\mu\theta_3}{\pi\tilde{Q}^*}} \equiv \eta_1, \\ \left.\frac{\partial \log \tilde{Q}_t^{t+1}(S')}{\partial \log \varepsilon_{s,t+1}}\right|_{\tilde{Q}} &= \frac{\theta_2}{\theta_2+\mu\theta_3}\frac{1+\frac{\theta_1+\theta_2+\mu\theta_3}{\pi\tilde{Q}^*}}{1-\theta_1+\frac{\theta_1+\theta_2+\mu\theta_3}{\pi\tilde{Q}^*}} \equiv \eta_2. \end{aligned}$$

It follows that

$$\text{(F.4)}\qquad \log \tilde{Q}_t^{t+1} = \chi \log \tilde{Q}_{t-1}^{t} + (1-\chi)\log \tilde{Q}^* + \eta_1 \log \varepsilon_{s,t+1} + \eta_2 \log \varepsilon_{s,t}.$$

Let us first compute the unconditional expectation of $\log \tilde{Q}_t^{t+1}$. From equation (F.4) we get

$$\text{(F.5)}\qquad E(\log \tilde{Q}_t^{t+1}) = \log \tilde{Q}^* + \frac{(\eta_1+\eta_2)E(\log \varepsilon_{s,t})}{1-\chi}.$$

Since we have assumed that

$$\text{(F.6)}\qquad \log \varepsilon_{s,t} \sim N\left(\frac{-\sigma_s^2}{2}, \sigma_s^2\right),$$

we derive

$$\text{(F.7)}\qquad E(\log \tilde{Q}_t^{t+1}) = \log \tilde{Q}^* - \frac{(\eta_1+\eta_2)\sigma_s^2}{2(1-\chi)}.$$

Let us now compute the unconditional variance of $\log \tilde{Q}_t^{t+1}$. From equation (F.4) we get

$$\text{(F.8)}\qquad Var(\log \tilde{Q}_t^{t+1}) = \frac{(\eta_1^2+\eta_2^2)Var(\log \varepsilon_{s,t})}{1-\chi^2} = \frac{(\eta_1^2+\eta_2^2)\sigma_s^2}{1-\chi^2} \equiv \sigma_{\tilde{Q}}^2.$$

From the properties of the log-normal distribution, we then have that

$$\text{(F.9)}\qquad \log \tilde{Q}_t^{t+1} \sim N\left(\log \tilde{Q}^* - \tfrac{(\eta_1+\eta_2)\sigma_s^2}{2(1-\chi)}, \tfrac{(\eta_1^2+\eta_2^2)\sigma_s^2}{1-\chi^2}\right),$$

and the unconditional mean of $\tilde{Q}_t^{t+1}$ is

$$\text{(F.10)}\qquad E\left[\tilde{Q}_t^{t+1}\right] = \tilde{Q}^* \exp\left(\frac{\sigma_{\tilde{Q}}^2}{2}\left[1-(1+\chi)\left(\frac{\eta_1+\eta_2}{\eta_1^2+\eta_2^2}\right)\right]\right).$$

□

Appendix G.

Proof of Proposition 5. From equations (1) and (75), we get the following expression for the gross safe real rate of return, R^S,

$$R^S = \frac{1}{\sum_{S'} Q_t^{t+1}} = \frac{1}{E_t\left[\tilde{Q}_t^{t+1}\gamma_{t+1}^{-1}\right]}. \tag{G.1}$$

The safe rate of return is a random variable because income growth is autocorrelated and governed by Equation (63) and also because now the pricing kernel is subject to sunspot shocks. Assuming that intrinsic and extrinsic uncertainty are uncorrelated,

$$Cov\left(\varepsilon_{s,t+i}, \varepsilon_{f,t+j}\right) = 0, \quad \text{for all } \ i, j. \tag{G.2}$$

we derive that the unconditional mean of the safe return is

$$R^S = \frac{1}{E\left[\tilde{Q}_t^{t+1}\right] E\left[\gamma_{t+1}^{-1}\right]}. \tag{G.3}$$

Using the properties of the log-normal distribution, we derive from (65) the following expression for the unconditional expectation of γ_{t+1}^{-1},

$$E\left[\gamma_{t+1}^{-1}\right] = \mu_\gamma^{-1} e^{\frac{\sigma_\gamma^2(2+\eta)}{2}}. \tag{G.4}$$

$E\left[\tilde{Q}_t^{t+1}\right]$ is given by equation (71) in Lemma 2. Substituting this into (G.3) gives

$$R^S = \frac{\mu_\gamma}{\tilde{Q}^*} e^{-\frac{\sigma_\gamma^2(2+\eta)}{2} - \frac{\sigma_Q^2\left[1-(1+\chi)\left(\frac{\eta_1+\eta_2}{\eta_1^2+\eta_2^2}\right)\right]}{2}}. \tag{G.5}$$

We then derive Part i) of Proposition 5.

Consider next, a security that pays a fraction ν of the endowment $\omega_{t+1}\left(S'\right)$ in state S'. From no-arbitrage pricing, we have that,

$$q_t = E_t\left\{\frac{\tilde{Q}_t^{t+1}}{\gamma_{t+1}}\left[q_{t+1} + \nu\omega_{t+1}\right]\right\}. \tag{G.6}$$

Using the definitions,

$$\gamma_{t+1}\left(S'\right) = \frac{\omega_{t+1}}{\omega_t}, \qquad \hat{q}_t(S) = \frac{q_t}{\nu\omega_t}, \tag{G.7}$$

we get the equivalent expression

$$\hat{q}_t = E_t\left\{\tilde{Q}_t^{t+1}\left[\hat{q}_{t+1} + 1\right]\right\}. \tag{G.8}$$

The realized return from holding a risky security is then given by the expression

$$R_t^{t+1,R} = \frac{\gamma_{t+1}\left[\hat{q}_{t+1} + 1\right]}{\hat{q}_t}. \tag{G.9}$$

Taking the unconditional expectation of (G.8) using (G.2) yields

$$\frac{1}{E(\tilde{Q}_t^{t+1})} = \frac{E\left[\hat{q}_{t+1}+1\right]}{\hat{q}_t}. \tag{G.10}$$

with $E(\tilde{Q}_t^{t+1})$ as given by equation (71) in Lemma 2. Consider now the return to a risky security (G.10). Using equation (67), we easily derive the unconditional expectation of the risky return

(G.11)

$$R^R = E\left\{\frac{\gamma_{t+1}\left[\hat{q}_{t+1}+1\right]}{\hat{q}_t}\right\} = E\left\{\gamma_{t+1}\right\} E\left\{\frac{\hat{q}_{t+1}+1}{\hat{q}_t}\right\} = \frac{\mu_\gamma}{\tilde{Q}^*} e^{\frac{-\eta\sigma_\gamma^2}{2} - \frac{\sigma_s^2\left[\eta_1^2+\eta_2^2-(1+\chi)(\eta_1+\eta_2)\right]}{2(1-\chi^2)}}.$$

We then derive Part ii) of Proposition 5. Finally, taking the ratio of (G.11) to (G.5) gives

$$\frac{R^R}{R^S} = E\left\{\gamma_{t+1}\right\} E\left\{\gamma_{t+1}^{-1}\right\} = e^{\frac{\sigma_f^2}{1-\eta^2}} = e^{\sigma_\gamma^2}. \tag{G.12}$$

This establishes Part iii) of Proposition 5. □

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