NBER WORKING PAPER SERIES

ON THE ASSET MARKET VIEW OF EXCHANGE RATES

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Working Paper 18646
http://www.nber.org/papers/w18646

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
December 2012, Revised November 2018

Previous versions of this paper were entitled “Exchange Rates and International Risk Sharing” and “Exchange Rate Determination, Risk Sharing, and the Asset Market View.” We thank three anonymous referees, Hengjie Ai, Andrew Ang, Snehal Banerjee, Martin Bodenstein, John Cochrane, Bernard Dumas, Martin Eichenbaum, Martin Evans, Nir Jaimovich, Jonathan Parker, Uday Rajan, Sergio Rebelo, Lucio Sarno, Raj Singh, Gabriel Talmain, Giorgio Valente, Jules Van Binsbergen, seminar participants at Cass Business School, Duke University, Essex Business School, the Federal Reserve Board, and the University of Michigan, and conference participants at Northwestern University, the University of Glasgow, the University of Minnesota, Oxford University, the Chicago Fed, Queen's University, the American Finance Association meetings in San Diego, the NBER Summer Institute Asset Pricing meeting, and the NBER International Finance and Macroeconomics meeting for helpful comments and conversations. Any errors or omissions are our own. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 18646
December 2012, Revised November 2018
JEL No. F31,G15

ABSTRACT

If the asset market is complete then the log change in the real exchange rate equals the log difference between foreign and domestic agents' intertemporal marginal rates of substitution (IMRSs). This equation is frequently used to argue that changes in real exchange rates reflect differences between agents in the compensation for risk they require to own various assets. We show that the relative returns on frictionlessly traded assets are only reflected in the common component of agents' IMRSs, not differences. Instead, when this equation does offer insights, frictions in the goods market are the source of economic distinction between agents.

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If the asset market is complete then the log change in the real exchange rate equals the log difference between foreign and domestic agents' intertemporal marginal rates of substitution (IMRSs):

$$\begin{matrix}\text{log change in the real} \\ \text{domestic/foreign exchange rate}\end{matrix} = \begin{matrix}\text{foreign agent's} \\ \text{log IMRS}\end{matrix} - \begin{matrix}\text{domestic agent's} \\ \text{log IMRS}\end{matrix} . \quad (1)$$

Brandt, Cochrane, and Santa-Clara (2006) refer to Eq. (1) as the asset market view of exchange rates.[1] The central insight is that changes in real exchange rates reflect differences between agents in the compensation for risk they require to own various assets. It is now a dominant theoretical framework in the recent international asset pricing literature and has been used to understand exchange rate determination, foreign exchange risk premia, and international risk sharing.[2]

We offer a simple and intuitive critique of the theoretical basis for this literature. The returns on frictionlessly traded assets (including currencies) can only reveal a common component of agents' IMRSs, not differences. When Eq. (1) holds, we show that the foreign and domestic agents actually require the same compensation for exposure to these asset returns. Instead, any difference between their IMRSs only reflects the units – different consumption baskets of goods and services, and/or different prices for the components of those baskets – used to denominate the asset returns.

Therefore, when Eq. (1) holds, structural assumptions about preferences and goods market frictions are necessary to interpret variation in real exchange rates as differences between agents' IMRSs. For example, suppose that the asset market is complete and agents in different economies have the same consumption aggregators over individual goods. With these assumptions, frictions in the goods market are the only source of economic distinction between agents, and the amount of variation in the real exchange rate reflects the degree to which risk is not shared across these economies. However, there is an important observational equivalence problem. We show that the same real exchange rate behavior can result

[1] In a distinct earlier literature, the "asset market view of exchange rates" referred to the role of asset markets and capital mobility in exchange rate determination. This literature emphasized the importance of the fact that nominal exchange rates and asset markets adjust much more quickly than goods markets. See, for example, Dornbusch (1976), Frenkel (1976), Kouri (1976), and Mussa (1976).

[2] Examples of papers where this approach appears include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Smith and Wickens (2002); Ahn (2004); Brandt, Cochrane, and Santa-Clara (2006); Lustig and Verdelhan (2006); Brennan and Xia (2006); Lustig and Verdelhan (2007); Bakshi, Carr, and Wu (2008); Verdelhan (2010); Colacito and Croce (2011); Lustig, Roussanov, and Verdelhan (2011); Bansal and Shaliastovich (2013); and Lustig, Roussanov, and Verdelhan (2014). For an overview of this approach, see Lustig and Verdelhan (2012)'s chapter in the *Handbook of Exchange Rates*, entitled "Exchange Rates in a Stochastic Discount Factor Framework."

if the asset market is complete and goods markets are frictionless, but agents have different preferences over individual goods. In this case, agents share risk perfectly. Thus, armed only with asset returns, exchange rates, and aggregate consumption data, it is impossible to differentiate between models in which agents share risk perfectly and ones where they don't. Although our paper is a critique, a positive message from it is that structural modeling is helpful if it is specific about goods markets and preferences over disaggregated consumption, and brings relevant evidence to bear.

1 The Asset Market View of Exchange Rates

In this section we develop notation and formalize our main critique.

1.1 Frictionlessly Traded Assets and Numeraire

Consider a set of k assets that can be located anywhere in the world and can include currencies. Assume that all assets can be frictionlessly traded across economies and there are no arbitrage opportunities. With these assumptions, it is convenient to use a portfolio of the assets as a numeraire to denominate the asset returns, as well as the prices of consumption baskets of goods and services in different economies. To this end, let $\boldsymbol{\eta}$ be a vector of weights for our numeraire portfolio. The particular choice is irrelevant, since any frictionlessly traded basket of assets and/or goods serves the same purpose equally well.[3]

1.2 Stochastic Discount Factor for Relative Asset Returns

Let $\boldsymbol{R}_t^{\eta}$ denote the vector of cum-dividend asset returns from time $t-1$ to t, relative to (i.e., denominated or normalized by) the cum-dividend returns on our numeraire portfolio. With k assets there are $k-1$ relative asset returns to consider, which can be seen by noting that

[3] The reciprocal (or inverse) of the return on our numeraire portfolio is an example of what Duffie (2001) refers to as a *numeraire deflator.* In Chapter 6 Section B (entitled "Numeraire Invariance") of his Dynamic Asset Pricing Theory textbook he states that:

> It is often convenient to renormalize all security prices, sometimes relative to a particular price process. This section shows that such a renormalization has essentially no economic effects.

$\boldsymbol{R}_t^\eta \cdot \boldsymbol{\eta} = 1$ (where $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$).[4] A stochastic discount factor (SDF) for the relative asset returns, $\boldsymbol{R}_t^\eta$, is any (almost surely) strictly positive M_t^η such that

$$\mathbf{1} = \mathbb{E}\left[M_t^\eta \boldsymbol{R}_t^\eta\right] . \tag{2}$$

In Eq. (2), $\mathbf{1}$ is a vector of 1's, and $\mathbb{E}[\cdot]$ denotes expectation conditional on information available at time $t-1$ (for notational convenience, we drop the explicit dependence on the time $t-1$ information set).[5]

1.3 Consumption Baskets and the Real Exchange Rate

We also use our numeraire portfolio to denominate and compare the prices of consumption baskets across economies. Let $P_{d,t}$ be the number of (cum-dividend) units of the portfolio that can be traded for one unit of the representative agent's consumption basket of goods and services in the domestic economy. Let $P_{f,t}$ be the counterpart for one unit of the representative agent's consumption basket in the foreign economy. The real domestic/foreign exchange rate, $e_t = P_{f,t}/P_{d,t}$, is the price of a unit of the foreign consumption basket, expressed in units of the domestic consumption basket.[6] Importantly, any numeraire that is frictionlessly traded across economies yields the same real exchange rate. Finally, let $X_t = e_t/e_{t-1}$ denote the gross change in the real domestic/foreign exchange rate from time $t-1$ to t.[7]

[4] In footnote 8 below, we show that if $\boldsymbol{R}_t$ and $\boldsymbol{R}_t^*$ denote the vector of these asset returns denominated in units of the domestic and foreign consumption baskets, respectively, then

$$\frac{\boldsymbol{R}_t}{\boldsymbol{R}_t \cdot \boldsymbol{\eta}} = \boldsymbol{R}_t^\eta = \frac{\boldsymbol{R}_t^*}{\boldsymbol{R}_t^* \cdot \boldsymbol{\eta}} .$$

[5] In Eq. (2), M_t^η defines an equivalent martingale pricing measure $\mathbb{Q}$ (via $d\mathbb{Q}/d\mathbb{P} = M_t^\eta$) with respect to our numeraire portfolio. To verify that $\mathbb{Q}$ is a valid probability measure, note that M_t^η is almost surely strictly positive and

$$\mathbb{E}\left[M_t^\eta\right] = \mathbb{E}\left[M_t^\eta \boldsymbol{R}_t^\eta \cdot \boldsymbol{\eta}\right] = \boldsymbol{\eta} \cdot \mathbb{E}\left[M_t^\eta \boldsymbol{R}_t^\eta\right] = \boldsymbol{\eta} \cdot \mathbf{1} = 1 .$$

[6] One unit of the foreign consumption basket is worth $P_{f,t}$ units of our numeraire portfolio, which in turn is worth $P_{f,t}/P_{d,t}$ units of the domestic consumption basket.

[7] The change in the real exchange rate is the change in the relative price of (identical or different) consumption baskets of goods and services in two different economies (where prices are measured using a common numeraire that is frictionlessly traded between the economies). It differs from the real return on a foreign currency investment, which is the return on a default-free foreign currency bank account, denominated in units of the domestic consumption basket. The distinction between these two concepts is most evident when the prices of domestic and foreign consumption baskets are naturally measured in the same nominal currency. For example, there is a real exchange rate between Finland and Spain, but both countries use the Euro as

1.4 Change of Units for Asset Returns and SDFs

For notational convenience, define $\Pi_{d,t} = P_{d,t}/P_{d,t-1}$ and $\Pi_{f,t} = P_{f,t}/P_{f,t-1}$, and note that $X_t = \Pi_{f,t}/\Pi_{d,t}$. The asset returns denominated in units of the domestic and foreign consumption baskets are given by $\boldsymbol{R}_t = \Pi_{d,t}^{-1}\boldsymbol{R}_t^\eta$ and $\boldsymbol{R}_t^* = \Pi_{f,t}^{-1}\boldsymbol{R}_t^\eta$, respectively.[8]

SDFs M_t and M_t^* for $\boldsymbol{R}_t$ and $\boldsymbol{R}_t^*$ must satisfy

$$\mathbf{1} = \mathbb{E}\left[M_t \boldsymbol{R}_t\right] \qquad \text{and} \qquad \mathbf{1} = \mathbb{E}\left[M_t^* \boldsymbol{R}_t^*\right] . \tag{4}$$

Since $M_t^\eta \Pi_{d,t}\boldsymbol{R}_t = M_t^\eta \boldsymbol{R}_t^\eta = M_t^\eta \Pi_{f,t}\boldsymbol{R}_t^*$, the change of numeraire units for an SDF mirrors the change of units for the asset returns it prices. If M_t^η is an SDF for $\boldsymbol{R}_t^\eta$, then $M_t^\eta \Pi_{d,t}$ and $M_t^\eta \Pi_{f,t}$ are always SDFs for $\boldsymbol{R}_t$ and $\boldsymbol{R}_t^*$, respectively. This change of numeraire units for an SDF, M_t^η, is often expressed as

$$\underbrace{\begin{array}{c}\text{log change in the real}\\ \text{domestic/foreign exchange rate}\end{array}}_{\ln X_t = \ln \Pi_{f,t} - \ln \Pi_{d,t}} = \underbrace{\text{log SDF for } \boldsymbol{R}_t^*}_{\ln \Pi_{f,t} + \ln M_t^\eta} - \underbrace{\text{log SDF for } \boldsymbol{R}_t}_{\ln \Pi_{d,t} + \ln M_t^\eta} . \tag{5}$$

Conversely, given SDFs M_t for $\boldsymbol{R}_t$ and M_t^* for $\boldsymbol{R}_t^*$, then $M_t \Pi_{d,t}^{-1}$ and $M_t^* \Pi_{f,t}^{-1}$ are always SDFs for $\boldsymbol{R}_t^\eta$, and the change of numeraire units in Eq. (5) holds if $M_t \Pi_{d,t}^{-1} = M_t^* \Pi_{f,t}^{-1}$.

1.5 Critique of the Asset Market View of Exchange Rates

Agents' IMRSs are examples of SDFs, so the asset market view in Eq. (1) is a special case of the change of numeraire units in Eq. (5). It holds if and only if there is an SDF, M_t^η for $\boldsymbol{R}_t^\eta$, such that $M_t^\eta \Pi_{f,t}$ and $M_t^\eta \Pi_{d,t}$ equal the IMRSs of the foreign and domestic representative

their nominal currency, so a Spaniard cannot invest in a foreign Finnish currency.

[8]Note that

$$\left(\boldsymbol{R}_t \cdot \boldsymbol{\eta}\right)^{-1} = \left(\Pi_{d,t}^{-1}\boldsymbol{R}_t^\eta \cdot \boldsymbol{\eta}\right)^{-1} = \Pi_{d,t}\left(\boldsymbol{R}_t^\eta \cdot \boldsymbol{\eta}\right)^{-1} = \Pi_{d,t} . \tag{3}$$

Likewise, $\left(\boldsymbol{R}_t^* \cdot \boldsymbol{\eta}\right)^{-1} = \Pi_{f,t}$. Therefore, regardless of their consumption baskets, agents in the domestic and foreign economies both face the same relative (or normalized) asset returns, since

$$\frac{\boldsymbol{R}_t}{\boldsymbol{R}_t \cdot \boldsymbol{\eta}} = \boldsymbol{R}_t^\eta = \frac{\boldsymbol{R}_t^*}{\boldsymbol{R}_t^* \cdot \boldsymbol{\eta}} .$$

agents,

$$\underbrace{\begin{array}{c}\text{log change in the real}\\ \text{domestic/foreign exchange rate}\end{array}}_{\ln X_t = \ln \Pi_{f,t} - \ln \Pi_{d,t}} = \underbrace{\begin{array}{c}\text{foreign agent's}\\ \text{log IMRS}\end{array}}_{\ln \Pi_{f,t} + \ln M_t^\eta} - \underbrace{\begin{array}{c}\text{domestic agent's}\\ \text{log IMRS}\end{array}}_{\ln \Pi_{d,t} + \ln M_t^\eta} . \tag{6}$$

The literature we critique argues from the asset market view equation that changes in the real exchange rate reflect differences between agents in the prices they assign to the asset return risks in $\boldsymbol{R}_t^\eta$. However, M_t^η in Eq. (6) is common to both the foreign and domestic agents' IMRSs, so they actually require the same, not different, compensation for exposure to these risks. Instead, any difference between their IMRSs only reflects the units – different consumption baskets of goods and services, and/or different prices for the components of those baskets – used to denominate the asset returns.

Thus, to draw conclusions about meaningful economic distinctions between agents, we must make further assumptions about the underlying economic environment beyond the asset market structure. In particular, as we show in Section 4, exchange rate data, together with Eq. (1), can be used to draw important distinctions between domestic and foreign agents if we further assume that they have the same consumption aggregators over individual goods and there are goods market frictions. However, regardless of any additional assumptions about goods markets, it still remains that the relative returns to frictionlessly traded assets are only informative about the common component of agents' IMRSs, not about differences.

1.6 Complete Markets

Before closing out this section, we demonstrate how our critique applies to the special case in which the returns, $\boldsymbol{R}_t^\eta$, are a complete asset market for the foreign and domestic agents.

In general, there can be events (i.e., states of the world) over which M_t^η in Eq. (6) varies, but the asset returns do not. If there are no such events, then M_t^η is in the set of SDFs that are measurable with respect to the σ-algebra, $\sigma(\boldsymbol{R}_t^\eta)$, generated by those asset returns. If it is the unique SDF in this set, then the returns, $\boldsymbol{R}_t^\eta$, are a complete asset market for these agents.[9] In this case, M_t^η must equal the inverse return on the growth-optimal portfolio.

To illustrate, let $\boldsymbol{\theta}^\star$ be the vector of growth-optimal portfolio weights that maximize

[9] There can be a unique SDF within the space of SDFs that are measurable with respect to $\sigma(\boldsymbol{R}_t^\eta)$, but M_t^η may not be in this space, in which case the asset market is not complete for these agents. We discuss this issue in more depth in Section 3.4.

expected log wealth,

$$\boldsymbol{\theta}^{\star} = \arg\max_{\boldsymbol{\theta}\cdot\mathbf{1}=1} \mathbb{E}\left[\ln\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}\right)\right] . \tag{7}$$

Long (1990) was the first to recognize from the first order (Euler) equations,

$$\mathbf{1} = \mathbb{E}\left[\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}^{\star}\right)^{-1}\boldsymbol{R}_t^{\eta}\right] , \tag{8}$$

that $\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}^{\star}\right)^{-1}$ is an SDF for $\boldsymbol{R}_t^{\eta}$. This SDF is measurable with respect to $\sigma\left(\boldsymbol{R}_t^{\eta}\right)$, and Karatzas and Kardaras (2007) prove that it always exists (whenever there is an SDF that satisfies Eq. 2).

Therefore, if the returns, $\boldsymbol{R}_t^{\eta}$, are a complete asset market for the foreign and domestic agents, then their log IMRSs must equal

$$\ln M_t^{*} = \ln\Pi_{f,t}\underbrace{-\ln\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}^{\star}\right)}_{\ln M_t^{\eta}} \qquad \text{and} \qquad \ln M_t = \ln\Pi_{d,t}\underbrace{-\ln\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}^{\star}\right)}_{\ln M_t^{\eta}} , \tag{9}$$

respectively.[10] As this example explicitly demonstrates, the relative asset returns are only reflected in the common component of agents' log IMRSs, which in this case is $\ln M_t^{\eta} = -\ln\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}^{\star}\right)$. Instead, any difference, $\ln\Pi_{f,t} - \ln\Pi_{d,t} = \ln X_t$, reflects different consumption baskets of goods and services, and/or different prices for the items in those baskets.

2 Economic Interpretation of SDFs in Affine Models

In this section we demonstrate how our critique applies to the large literature that uses reduced-form affine models of two (or more) log SDFs to characterize and economically interpret currency returns as differences between agents' IMRSs.[11]

To begin, we extend our earlier notation to a setting with $n \geq 2$ different economies. Let X_t^i denote the change in the ith real exchange rate (expressed in units of the domestic consumption basket per unit of the consumption basket in the ith foreign economy). Let M_t^i

[10] Note that

$$M_t^{*} = \Pi_{f,t}\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}^{\star}\right)^{-1} = \left(\boldsymbol{R}_t^{*}\cdot\boldsymbol{\theta}^{\star}\right)^{-1} \qquad \text{and} \qquad M_t = \Pi_{d,t}\left(\boldsymbol{R}_t^{\eta}\cdot\boldsymbol{\theta}^{\star}\right)^{-1} = \left(\boldsymbol{R}_t\cdot\boldsymbol{\theta}^{\star}\right)^{-1} .$$

[11] A few examples of papers that pursue this modeling approach include: Brandt and Santa-Clara (2002); Bakshi, Carr, and Wu (2008); Lustig, Roussanov, and Verdelhan (2011); and Lustig, Roussanov, and Verdelhan (2014).

be an SDF that prices the asset returns denominated in units of the ith foreign consumption basket. Let r_t and r_t^i denote the one-period continuously-compounded real interest rates in the domestic and ith foreign economy, respectively. Finally, let $\boldsymbol{X}_t$, $\boldsymbol{M}_t^*$, and $\boldsymbol{r}_t^*$ denote $(n-1)\times 1$ vectors with ith elements X_t^i, M_t^i, and r_t^i respectively.

In this literature, the dynamics of SDFs for asset returns denominated in units of the n different economies are frequently modeled as

$$\ln M_t = -r_{t-1} - \boldsymbol{\lambda}\,\boldsymbol{\varepsilon}_t - \tfrac{1}{2}\boldsymbol{\lambda}\boldsymbol{\lambda}^\top \qquad \text{and} \qquad \ln \boldsymbol{M}_t^* = -\boldsymbol{r}_{t-1}^* - \Lambda^*\,\boldsymbol{\varepsilon}_t - \tfrac{1}{2}\mathrm{diag}\left(\Lambda^*\Lambda^{*\top}\right)\,, \tag{10}$$

where superscript $^\top$ denotes the transpose of a matrix or vector, and $\mathrm{diag}\,(\cdot)$ denotes main diagonal vector a matrix. In Eq. (10), $\boldsymbol{\varepsilon}_t \sim \mathcal{N}\left(\mathbf{0},\mathcal{I}\right)$ is an $\ell\times 1$ vector of independent standard normals, $\boldsymbol{\lambda}$ is a $1\times\ell$ vector, and Λ^* is an $(n-1)\times\ell$ matrix. For notational convenience, we suppress any time/state dependence of the parameters (i.e., $\boldsymbol{\lambda}\equiv\boldsymbol{\lambda_t}$ and $\Lambda^*\equiv\Lambda_t^*$). If the change of units, $\ln X_t^i = \ln M_t^i - \ln M_t$, is assumed to hold for each pair of SDFs then

$$\ln \boldsymbol{X}_t \;=\; \mathbf{1}r_{t-1} - \boldsymbol{r}_{t-1}^* + \left(\mathbf{1}\boldsymbol{\lambda}-\Lambda^*\right)\boldsymbol{\varepsilon}_t + \underbrace{\left(\mathbf{1}\boldsymbol{\lambda}-\Lambda^*\right)\boldsymbol{\lambda}^\top - \tfrac{1}{2}\mathrm{diag}\left(\left[\mathbf{1}\boldsymbol{\lambda}-\Lambda^*\right]\left[\mathbf{1}\boldsymbol{\lambda}-\Lambda^*\right]^\top\right)}_{\frac{1}{2}\left[\mathbf{1}\boldsymbol{\lambda}\boldsymbol{\lambda}^\top - \mathrm{diag}\left(\Lambda^*\Lambda^{*\top}\right)\right]}\,. \tag{11}$$

In the literature we critique, the reduced-form SDFs in Eq. (10) are often assumed to equal the IMRSs of representative agents in the domestic and foreign economies. $\boldsymbol{\lambda}$ is economically interpreted as the compensation for risk that agents in the domestic economy require for exposure to the vector of shocks, $\boldsymbol{\varepsilon}_t$. Likewise, the ith row of Λ^* is interpreted as the prices of risk that agents in the ith foreign economy assign to these shocks. According to this economic interpretation of Eqs. (10) and (11), both the volatility and expected return on foreign currency investments reflect heterogeneity across agents in different economies in the compensation for risk they require for exposure to these asset market shocks.

2.1 Example: Lustig, Roussanov, and Verdelhan (2014)

Lustig, Roussanov, and Verdelhan (2014) is a recent and representative example in the literature that we critique. They document that currency returns can be explained by a small set of currency factors (i.e., dynamic long/short portfolios of currencies). To interpret these empirical results, they provide a reduced-form affine model of agents' IMRSs that is driven by a vector of $n+2$ latent shocks, $\boldsymbol{\varepsilon}_t$. Their model is of the form in Eqs. (10) and

(11), with

$$\Lambda^* = \begin{bmatrix} \lambda_1^1 & 0 & \cdots & 0 & 0 & \lambda_{n+1}^1 & \lambda_{n+2}^1 \\ 0 & \lambda_2^2 & \ddots & \vdots & \vdots & \lambda_{n+1}^2 & \lambda_{n+2}^2 \\ \vdots & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1}^{n-1} & 0 & \lambda_{n+1}^{n-1} & \lambda_{n+2}^{n-1} \end{bmatrix} \tag{12a}$$

and

$$\boldsymbol{\lambda} = \left[\begin{array}{cccccc} 0 & \cdots & 0 & \lambda_n & \lambda_{n+1} & \lambda_{n+2} \end{array}\right]. \tag{12b}$$

Lustig, Roussanov, and Verdelhan (2014) interpret the last two latent shocks in their model, ε_t^{n+1} and ε_t^{n+2}, as global shocks because agents in all economies require (possibly different) compensation for exposure to these risks. By contrast, the first n latent shocks – $\varepsilon_t^1, \ldots, \varepsilon_t^n$ – are viewed as country-specific because only agents in a single economy require compensation for bearing each of these risks. They argue that the particular properties of currency (portfolio) returns that they study in their paper reflect heterogeneity across agents in different economies in the prices of risk they assign to these local and global shocks.[12]

2.2 Critique Applied to Affine Models

To see how our critique applies to this literature, note that the vector of log asset returns that the SDFs price are also affine functions of the latent shocks, $\boldsymbol{\varepsilon}_t$, in Eq. (10).[13] Let $\boldsymbol{R}_t^i$ denote a vector of non-redundant asset returns, denominated in consumption units of the ith foreign economy, that completely span the vector of shocks, $\boldsymbol{\varepsilon}_t$. The log SDFs in Eq. (10) can be expressed as affine functions of these log returns,

$$\ln M_t \;=\; c - \boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t \qquad \text{and} \qquad \ln M_t^i \;=\; c_i - \boldsymbol{\Upsilon}_i \cdot \ln \boldsymbol{R}_t^i\,, \quad \forall i \in \{1, \ldots, n-1\}\,, \tag{13}$$

where $\boldsymbol{R}_t = X_t^i \boldsymbol{R}_t^i$ are the asset returns denominated in domestic consumption units.

The change of units, $\ln X_t^i = \ln M_t^i - \ln M_t$, for each pair of SDFs in Eq. (13) implies that

[12]For example, Lustig, Roussanov, and Verdelhan (2014, p. 537) state that "Accounting for the variation in expected currency excess returns across different currencies requires variation in the SDFs' exposures to the common innovation. ... Lustig, Roussanov, and Verdelhan (2011) show that permanent heterogeneity in loadings ... is necessary to explain the variation in unconditional expected returns (why high interest rate currencies tend not to depreciate on average), whereas the transitory heterogeneity in loadings ... is necessary to match the variation in conditional expected returns (why currencies with currently high interest rates tend to appreciate)."

[13]In the case of Lustig, Roussanov, and Verdelhan (2014), the asset returns include short- and long-term default-free zero-coupon bonds denominated in the domestic and foreign currencies. Other assets, such as domestic and foreign stocks can also be included if they are assumed to be priced by the SDFs in the model.

$$\ln M_t^i = \ln M_t + \ln X_t^i = c - \boldsymbol{\Upsilon} \cdot \underbrace{\left(\ln \boldsymbol{R}_t - \mathbf{1} \ln X_t^i\right)}_{\ln \boldsymbol{R}_t^i} + (1 - \boldsymbol{\Upsilon} \cdot \mathbf{1}) \ln X_t^i \,. \tag{14}$$

Therefore,

$$c_i = c\,, \qquad \boldsymbol{\Upsilon}_i = \boldsymbol{\Upsilon}\,, \qquad \text{and} \qquad \boldsymbol{\Upsilon} \cdot \mathbf{1} = 1\,, \qquad \forall i \in \{1, \ldots, n-1\}\,, \tag{15}$$

so the SDFs in Eqs. (10) and (13) can be written equivalently as

$$\ln M_t \;=\; \underbrace{c \overbrace{- \boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t^\eta}_{\ln M_t^\eta} + \ln \Pi_{d,t}^\eta}^{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t} \qquad \text{and} \qquad \ln M_t^i \;=\; \underbrace{c \overbrace{- \boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t^\eta}_{\ln M_t^\eta} + \ln \Pi_{i,t}^\eta}^{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t^i}\,, \tag{16}$$

with $\boldsymbol{\Upsilon} \cdot \mathbf{1} = 1$.[14] Thus, contrary to the claims in this literature, the relative asset returns (including currency returns) are only reflected in the common component of agents' log IMRSs, which is $\ln M_t^\eta = c - \boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t^\eta$. Instead, the difference, $\ln \Pi_{i,t}^\eta - \ln \Pi_{d,t}^\eta = \ln X_t$, reflects different consumption baskets and/or different prices for the items in those baskets.

2.3 Implications of No-Arbitrage

The dynamics of short-term interest rates can also be included in models of the form in Eq. (10), in which case the model will have no-arbitrage implications for the term structure of interest rates in the domestic and foreign economies.[15] However, the asset market view

[14]Also,

$$\mathbf{1} = \mathbb{E}\big[\underbrace{e^{c - \boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t^\eta}}_{M_t^\eta} \boldsymbol{R}_t^\eta\big] \quad \Longrightarrow \quad c = -\ln \tfrac{1}{k}\mathbf{1} \cdot \mathbb{E}\big[e^{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t^\eta} \boldsymbol{R}_t^\eta\big]\,.$$

Moreover, the characterization of c can be symmetrically expressed using the asset returns denominated in the domestic and foreign consumption baskets, since

$$\begin{aligned} \boldsymbol{\Upsilon} \cdot \mathbf{1} = 1 \quad \Longrightarrow \quad e^{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t} \boldsymbol{R}_t = e^{-\boldsymbol{\Upsilon}\cdot \Pi_{d,t} \boldsymbol{R}_t} \Pi_{d,t} \boldsymbol{R}_t &= e^{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t^\eta} \boldsymbol{R}_t^\eta\,, \\ &= e^{-\boldsymbol{\Upsilon}\cdot \Pi_{i,t} \boldsymbol{R}_t^i} \Pi_{i,t} \boldsymbol{R}_t^i = e^{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t^i} \boldsymbol{R}_t^i\,, \quad \forall i\,. \end{aligned}$$

In Appendix A we explicitly solve for c and $\boldsymbol{\Upsilon}$ in Eq. (16) for the specific model in Lustig, Roussanov, and Verdelhan (2014).

[15]For example, in Lustig, Roussanov, and Verdelhan (2014), the short-term interest rate in the ith foreign economy, r_t^i, is driven by ε_t^i and ε_t^{n+1}, while the interest rate in the domestic economy, r_t, is driven by ε_t^n and ε_t^{n+1}. The shocks ε_t^i and ε_t^n also drive time-variation in λ_i^i and λ_n, while ε_t^{n+1} also drives variation in λ_{n+1}, λ_{n+2}, λ_{n+1}^i, and λ_{n+2}^i. Given this dependence, the term structure of interest rates in each economy is determined by no-arbitrage, and is driven by the same two shocks that drive short-term interest rates in that economy.

itself does not impose any such overidentifying restrictions.

A particularly well-known paper in this literature is Backus, Foresi, and Telmer (2001) who "characterize the (forward premium) anomaly in the context of affine models of the term structure of interest rates."[16] The restrictions they derive are not actually due to Eqs. (10) and (11), but instead reflect an implicit spanning assumption. The intuition can be understood as follows. Any dynamic no-arbitrage model of the term structure of interest rates in two economies must have at least four assets: short- and long-term bonds in both economies. With four assets, there are three relative returns to consider. However, Backus, Foresi, and Telmer (2001) assume that those three relative returns are driven by only two shocks, in which case one (or a portfolio) of the three assets must be redundant, since it can be replicated by a combination of the other two. It is this redundancy, not the forward premium anomaly or Eqs. (10) and (11), that is the source of the restrictions they derive.

It is important to recognize that the asset market view equation does not imply that a foreign currency return is a redundant asset that can be replicated by a combination of other assets. For example, Ahn (2004) provides a model in which shocks to real exchange rates are not completely spanned by the shocks that drive bonds returns denominated in local units. The model in Lustig, Roussanov, and Verdelhan (2014) also has this feature.[17]

3 Restrictions in Reduced-Form Models

Beyond the economic interpretation of SDFs, some papers in the literature we critique suggest that the change of units in Eq. (5) also imposes restrictions on reduced-form models of currency returns. In this section we show that the restrictions in these papers are actually errors. The change of units does not impose any such restrictions because it can always be applied to any SDF, regardless of whether asset markets are complete.

3.1 SDFs for Different Sets of Assets

Brennan and Xia (2006) test whether the change of units, $\ln \boldsymbol{M}_t^* = \mathbf{1} \ln M_t + \ln \boldsymbol{X}_t$, holds for SDFs, M_t and $\boldsymbol{M}_t^*$, that they estimate to only price domestic and foreign bonds, respectively. However, this test is not motivated by theory, since the change of units only applies to SDFs

[16] In contrast to much of the literature, Backus, Foresi, and Telmer (2001) are careful not to interpret the SDFs in their model as the IMRSs of domestic and foreign representative agents.

[17] In Lustig, Roussanov, and Verdelhan (2014), a single shock, ε_t^{n+2}, drives all of the variation in the $n-1$ exchange rates that is independent of changes in the term structure of interest rates in those economies.

that price the same set of assets. To prove this point with a simple counterexample, note that $\ln M_t = -r_{t-1}$ and $\ln \boldsymbol{M}_t^* = -\boldsymbol{r}_{t-1}^*$ are log SDFs that only price domestic and foreign one-period default-free bank account returns denominated in local units. However, if the exchange rate is not determined prior to time t, then

$$\ln \boldsymbol{X}_t \neq \ln \boldsymbol{M}_t^* - \mathbf{1} \ln M_t = \mathbf{1} r_{t-1} - \boldsymbol{r}_{t-1}^* \,. \tag{17}$$

3.2 Minimum Variance Projections

Let $\boldsymbol{\beta} \cdot \boldsymbol{R}_t$ be the linear projection of the domestic representative agent's IMRS onto the asset returns denominated in domestic consumption units. Likewise, let $\boldsymbol{\beta}^* \cdot \boldsymbol{R}_t^*$ be the linear projection of the foreign representative agent's IMRS onto the asset returns in foreign consumption units. Brandt, Cochrane, and Santa-Clara (2006, p. 675) claim that $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$ is always in the linear span of $\boldsymbol{R}_t^*$, which implies that Eq. (1) generalizes to incomplete markets if we replace agents' IMRSs with these minimum variance projections.[18] In general, this claim is not correct. Instead, it is $\boldsymbol{\beta} \cdot \boldsymbol{R}_t / X_t$, not $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$, that is always in the linear span of $\boldsymbol{R}_t^* = \boldsymbol{R}_t / X_t$.[19] If the asset market is incomplete then, for any given $\boldsymbol{\beta}$ there does not generally exist $\boldsymbol{\beta}^*$ such that $\boldsymbol{\beta^*} \cdot \boldsymbol{R}_t^* = \boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$, and therefore,

$$\ln X_t \;\neq\; \ln \boldsymbol{\beta}^* \cdot \boldsymbol{R}_t^* \;-\; \ln \boldsymbol{\beta} \cdot \boldsymbol{R}_t \,. \tag{18}$$

3.3 Incomplete Markets

Consider SDFs, M_t and M_t^*, for $\boldsymbol{R}_t$ and $\boldsymbol{R}_t^*$. By the change of numeraire units, $M_t \Pi_{d,t}^{-1}$ and $M_t^* \Pi_{f,t}^{-1}$ are SDFs for the relative asset returns, $\boldsymbol{R}_t^\eta$. If these SDFs are different, then $M_t \Pi_{d,t}^{-1} \neq M_t^* \Pi_{f,t}^{-1}$. In that case, define

$$\ln \Xi_t = \ln \left(M_t^* \Pi_{f,t}^{-1} \right) - \ln \left(M_t \Pi_{d,t}^{-1} \right) \quad \text{so that} \quad \ln X_t = \ln M_t^* - \ln M_t - \ln \Xi_t \,. \tag{19}$$

Brandt and Santa-Clara (2002) provide a reduced-form model of different SDFs, M_t and M_t^*, such that $M_t \Pi_{d,t}^{-1} \neq M_t^* \Pi_{f,t}^{-1}$. However, they claim that Ξ_t is independent of M_t and M_t^*,

[18] Lustig and Verdelhan (2012, p. 395) also make this claim in their survey chapter.

[19] In the appendix of their paper, Brandt, Cochrane, and Santa-Clara (2006) provide a specific example with complete asset markets in which $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$ is in the linear span of $\boldsymbol{R}_t^*$. However, this result does not hold true in general if markets are incomplete (which is the relevant case required to prove their desired result).

with $\mathbb{E}\left[\Xi_t^{-1}\right] = 1$.[20]

This claim violates no arbitrage because it implies that their model assigns two different returns to the same default-free domestic currency bank account. To prove this result, note that $M_t^* \boldsymbol{R}_t^* = M_t \Xi_t \boldsymbol{R}_t$. Therefore M_t and $M_t \Xi_t$ must both price the asset returns, $\boldsymbol{R}_t$, including a default-free domestic currency bank account,

$$\mathbb{E}\left[M_t\right] = e^{-r_{t-1}} = \mathbb{E}\left[M_t \Xi_t\right]. \tag{20}$$

However, if Ξ_t is independent of M_t and M_t^*, then by Jensen's inequality,

$$\mathbb{E}\left[M_t \Xi_t\right] > \mathbb{E}\left[M_t\right] / \mathbb{E}\left[\Xi_t^{-1}\right] = \mathbb{E}\left[M_t\right].^{21} \tag{21}$$

3.4 Market Completeness in Reduced-Form Models

In reduced-form models of asset returns, there is a unique SDF (or, equivalently, a unique pricing measure) if and only if any contingent claim on the assets can be exactly replicated by a (dynamic) portfolio of the assets.[22] Since reduced-form models focus exclusively on asset returns, the SDFs in these models can only be unique within the space of SDFs that are measurable with respect to the σ-algebra generated by these returns. Perhaps the best known examples of reduced-form models with unique SDFs are the binomial tree and Black-Scholes-Merton model.[23]

Although these models are useful for pricing and hedging contingent claims (e.g., options) on a single asset such as a stock or foreign currency, the unique SDFs in these models do not

[20] See Eq. (24) in Brandt and Santa-Clara (2002, p. 176). They state that "the key insight of our model is that when markets are incomplete, the volatility of the exchange rate is not uniquely determined by the domestic and foreign stochastic discount factors. ... If markets are incomplete, the volatility of the exchange rate can contain an element that is orthogonal to the priced sources of risk in both countries. ... To capture this excess volatility, we specify a stochastic process for the degree of market incompleteness."

[21] Similarly, Anderson, Hammond, and Ramezani (2010) show that, in the special case of an affine setting, the assumptions in Brandt and Santa-Clara (2002) are infeasible. Eq. (21) illustrates that the internal inconsistency (i.e., the arbitrage opportunity) applies more generally, beyond the specific affine structure.

[22] See the seminal work by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983).

[23] For details on the binomial tree model, see Cox, Ross, and Rubinstein (1979). For the Black-Scholes-Merton model, see Black and Scholes (1973) and Merton (1973). The discrete-time (continuous state space) model in Eqs. (10) and (11) is an example of a reduced-form model that does not have a unique SDF; for example, Long's SDF in Eq. (8) is another, different SDF that is also consistent with the asset return dynamics. However, the continuous-time limit of Eq. (10) is a continuous diffusion and in that case, $\boldsymbol{\Upsilon}$ in Eq. (16) is equal to $\boldsymbol{\theta}^\star$ in Eq. (8).

necessarily equal the IMRS of a representative agent in Eq. (1). For example, in both of these models, the unique SDF can be expressed as $(\boldsymbol{R}_t^\eta \cdot \boldsymbol{\theta}^\star)^{-1}$, but the domestic agent's IMRSs could be $M_t^\eta \Pi_{d,t}$, where $M_t^\eta = \zeta_t (\boldsymbol{R}_t^\eta \cdot \boldsymbol{\theta}^\star)^{-1}$ and ζ_t is a random variable that is independent of $\boldsymbol{R}_t^\eta$ with $\mathbb{E}[\zeta_t] = 1$. In this case, M_t^η is an SDF for $\boldsymbol{R}_t^\eta$, but the agent's IMRS varies over states of the world, captured by ζ_t, that the asset returns do not.[24]

Thus, an SDF in a reduced-form model can only be economically interpreted as the IMRS of a representative agent under the additional assumption that there are no events (i.e., states of the world) over which the agent's IMRS varies, but the asset returns in the model do not. This additional assumption is common in more structural models (with agents) because it makes them much easier to solve.[25] However, it is not a common assumption in reduced-form models (outside of the international asset pricing literature). In fact, the primary appeal of these models is that one is free to model the relative returns on a subset of the assets available to trade, and can be agnostic about any variation in agents' IMRSs that is independent of the returns on those (or any other) assets.

4 Models with Agents and Risk Sharing

We now turn to a discussion of models in which there are two agents, who reside, respectively, in the domestic and foreign economies. As the basis for our discussion, we provide a simple two period model that generalizes Backus and Smith (1993). In Section 4.1 we highlight the elements of this model that are particularly relevant for our subsequent discussions. In Section 4.2 we characterize the conditions under which variation in the real exchange rate directly reflects imperfect risk sharing. In Section 4.3 we show that asset returns can be informative about the amount of shared risk between agents, but not about the amount of unshared risk. Finally, in Section 4.4 we show that fundamentally different models of real exchange rates can be observationally equivalent at the level of aggregate consumption data.

4.1 Model Highlights

To be as concrete as possible, we discuss the predictions of a simple two period model that generalizes Backus and Smith (1993) in three dimensions: It allows for the possibility of financial market incompleteness, it allows for preferences across individual goods to potentially vary across agents in the two economies, and it allows for more generalized goods market

[24]Using formal notation, $M_t^\eta \notin \sigma(\boldsymbol{R}_t^\eta)$.

[25]We discuss these models in Section 4 below.

trading frictions. The complete model, which assumes that both agents' utility functions are defined over the same set of goods, is worked out in Appendix B.

Let $M_{d,t}^{j}$ denote the domestic agent's IMRS, and $M_{f,t}^{j}$ denote the foreign agent's IMRS, defined over units of an individual good, j. In the general setup of the model, the following equilibrium condition holds:

$$\frac{M_{f,t}^{j}}{M_{d,t}^{j}} = \Xi_t \frac{(P_{f,t}^{j}/P_{f,t-1}^{j})}{(P_{d,t}^{j}/P_{d,t-1}^{j})}\,. \tag{22}$$

Here $P_{d,t}^{j}$ and $P_{f,t}^{j}$ are the time-t prices of good j in the domestic and foreign countries, measured in a common numeraire, and Ξ_t is a wedge that reflects financial market incompleteness. If financial markets are complete, or if, by chance, perfect risk sharing is nonetheless possible, then $\Xi_t = 1$ for all t. Importantly, Ξ_t is not indexed by j. If trade in good j is frictionless, then $P_{d,t}^{j} = P_{f,t}^{j}$ for all t. That is, purchasing power parity (PPP) holds for good j, and the price change terms drop out of the equation.

Definition. *The domestic and foreign agents share risk perfectly if* $M_{f,t}^{j} = M_{d,t}^{j}$ *for all* j, t.

Colacito and Croce (2011, p. 156) also adopt this definition of perfect risk sharing, which states that the domestic and foreign agents equate IMRSs over all individual goods and services at every point in time. It is not equivalent to asset markets being complete, which would only imply that $\Xi_t = 1$ for all t. Nor is it equivalent to an allocation that coincides with the solution to a social planner's problem that respects goods market frictions. In either of these situations, risk sharing will be as good as it can be, but any market frictions that result in a wedge between $P_{d,t}^{j}$ and $P_{f,t}^{j}$, for some j, prevent IMRSs over some goods (e.g., nontraded goods) being equated. Our definition, instead, means that risk sharing is perfect in a model with complete asset markets and frictionless trade in all goods.[26]

IMRSs may also be defined over consumption baskets rather than individual consumption goods. We let $M_{d,t}$ denote the domestic agent's IMRS defined over units of her consumption basket, and $M_{f,t}^{*}$ denote the foreign agent's IMRS, defined over units of his consumption basket. We use an asterisk to denote the foreign agent's IMRS because the foreign agent may have different preferences over individual goods than the domestic agent. This would imply that the two IMRSs are expressed in different units. We assume that the agents' baskets are homogeneous of degree one aggregates of their consumption of individual goods. This implies that the prices of these baskets (in some common numeraire) may be written as

[26] We focus exclusively on the notion of risk sharing in a purely theoretical sense. Essentially, we're interested in how much risk can agents theoretically share. One might also be interested in the empirical question of whether agents do in fact share as much risk as theoretically possible, but we do not consider that issue.

homogenous of degree one functions of the individual prices (in the same numeraire) faced by these agents: $P_{d,t} = H_d(P^1_{d,t}, P^2_{d,t}, \cdots)$ and $P_{f,t} = H_f(P^1_{f,t}, P^2_{f,t}, \cdots)$. The subscripts on these functions allow for the possibility that the agents' preferences differ. At the level of aggregate consumption the equation that corresponds to Eq. (22) is

$$\frac{M^*_{f,t}}{M_{d,t}} = \Xi_t X_t \,, \tag{23}$$

where $X_t = e_t/e_{t-1}$ and $e_t = P_{f,t}/P_{d,t}$.

4.2 Risk Sharing and Variation in Real Exchange Rates

When agents have the same consumption aggregator, $H_d = H_f = H$, the real exchange rate is $e_t = H(P^1_{f,t}, P^2_{f,t}, \cdots)/\ H(P^1_{d,t}, P^2_{d,t}, \cdots)$. If goods markets are frictionless then $P^j_{d,t} = P^j_{f,t}$ for all j, t. Therefore, if agents have the same consumption aggregator and goods markets are frictionless, $e_t = 1$ and $X_t = 1$ for all t. If we observe real exchange rate variation, it necessarily implies that either markets for some goods are not frictionless or agents have different consumption aggregators (or both).

Does a variable real exchange rate, by itself, directly reflect imperfect risk sharing? Brandt, Cochrane, and Santa-Clara (2006) argue very strongly that the answer is always yes. Table 1 summarizes our answer to this question, and as the table indicates, it depends on underlying assumptions about the economic environment. The answer is "yes" if we assume that asset markets are complete, and that agents have identical consumption aggregators. With these assumptions, $\Xi_t = 1$ and variation in X_t (due to variation in the real exchange rate, e_t) can only happen if there are goods market frictions that drive wedges between $P^j_{d,t}/P^j_{d,t-1}$ and $P^j_{f,t}/P^j_{f,t-1}$ for some j. The existence of such wedges, in turn, necessarily implies imperfect risk sharing.

Under any of the other combinations of assumptions shown in Table 1, however, the answer is "no". For example, the "northeast" corner of the table indicates that when asset markets are assumed to be complete, but preferences over goods differ ($H_d \neq H_f$), perfect risk sharing and real exchange rate variation are compatible. This is because we can have $M^j_{d,t} = M^j_{f,t}$ for all j, t, even when $M_{d,t} \neq M^*_{f,t}$.

In the bottom row of the table the answer is also "no". When asset markets are incomplete, or, more precisely, when $\Xi_t \neq 1$ for some t, then risk sharing is imperfect regardless of the behavior of the real exchange rate. For example, goods markets could be frictionless

	Nature of Consumption Aggregators	
Asset Markets	Identical	Different
Complete	Yes	No
Incomplete	No	No

Table 1: Does a variable real exchange rate directly reflect imperfect risk sharing?

(implying that $P^j_{d,t} = P^j_{f,t}$ for all j, t) and preferences over goods could be identical (with the further implication that $e_t = 1$, for all t) but we would nonetheless have $M^j_{f,t}/M^j_{d,t} = \Xi_t \neq 1$ for some t.

To summarize, there is no direct connection between risk sharing and real exchange rate variation except in models where asset markets are complete and agents have identical consumption aggregators. In that case, goods market frictions are the only source of difference between agents' IMRSs.

4.3 Asset Prices and Unshared Risks

In contrast to our discussion in the previous section, Brandt, Cochrane, and Santa-Clara (2006) argue very strongly that real exchange rate variation is always informative about risk sharing. In fact, they conclude that because real exchange rate variation is small compared to Hansen and Jagannathan (1991) bounds on the volatility of domestic and foreign SDFs, domestic and foreign marginal utility growths must be highly correlated, and risk sharing must be "better than you think". On page 673, Brandt, Cochrane, and Santa-Clara (2006) emphasize that their conclusion is drawn by only using data on asset returns, and data on real exchange rates:

> Yet the conclusion is hard to escape. Our calculation uses only price data, and no quantity data or economic modeling (utility functions, income or productivity shock processes, and so forth). A large degree of international risk sharing is an inescapable logical conclusion of Eq. (1), a reasonably high equity premium (over 1%, as we show below), and the basic economic proposition that price ratios measure marginal rates of substitution.

To understand why Brandt, Cochrane, and Santa-Clara (2006) draw this different conclusion, consider a version of their quantitative risk sharing index, applied to the logarithms of the IMRSs in our structural model ($m^*_{f,t} = \ln M^*_{f,t}$ and $m_{d,t} = \ln M_{d,t}$). The index is based on the variance of $m^*_{f,t} - m_{d,t}$ relative to the sum of the variances of $m^*_{f,t}$ and $m_{d,t}$:[27]

$$\mathrm{RSI} = 1 - \frac{\mathrm{var}(m^*_{f,t} - m_{d,t})}{\mathrm{var}(m^*_{f,t}) + \mathrm{var}(m_{d,t})}. \tag{24}$$

Given our discussion in Section 4.2, it is clear that to base a risk sharing measure on RSI, one must assume that preferences across goods are identical across locations. In other words, one has to rule out the second column of Table 1 to make the calculation meaningful. This is an unstated but implicit assumption in their calculation.

Brandt, Cochrane, and Santa-Clara (2006) appeal to Hansen-Jagannathan bounds in order to avoid structurally modeling $m^*_{f,t}$ and $m_{d,t}$. In particular, they note that using asset market data they can project $M^*_{f,t}$ and $M_{d,t}$ onto common vectors of asset returns (denominated in real foreign and domestic currency, respectively). We denote the logs of these projections as $\hat{m}^*_{f,t} = \ln \hat{M}^*_{f,t}$ and $\hat{m}_{d,t} = \ln \hat{M}_{d,t}$. If we assume that $\mathrm{var}(m^*_{f,t}) \geq \mathrm{var}(\hat{m}^*_{f,t})$ and $\mathrm{var}(m_{d,t}) \geq \mathrm{var}(\hat{m}_{d,t})$ then we can put a lower bound on RSI:[28]

$$\mathrm{RSI} \geq \underline{\mathrm{RSI}} = 1 - \frac{\mathrm{var}(m^*_{f,t} - m_{d,t})}{\mathrm{var}(\hat{m}^*_{ft}) + \mathrm{var}(\hat{m}_{d,t})}. \tag{25}$$

Of course, $m^*_{f,t}$ and $m_{d,t}$ remain in the numerator in Eq. (25). In our model $m^*_{f,t} - m_{d,t} = \xi_t + x_t$, where $\xi_t = \ln \Xi_t$ and $x_t = \ln X_t$. We can treat x_t as measurable without a model, given data on consumer price indices and the nominal exchange rate. But ξ_t is not directly observable. To get around this issue Brandt, Cochrane, and Santa-Clara (2006) again appeal to the projections, $\hat{m}^*_{f,t}$ and $\hat{m}_{d,t}$, and argue that $\hat{m}^*_{ft} - \hat{m}_{d,t} = x_t$ always holds, even when $m^*_{f,t} - m_{d,t} \neq x_t$.[29] Therefore, they use the following risk sharing measure:

$$\mathrm{RSI}_{\mathrm{BCS}} = 1 - \frac{\mathrm{var}(x_t)}{\mathrm{var}(\hat{m}^*_{f,t}) + \mathrm{var}(\hat{m}_{d,t})}. \tag{26}$$

The advantage of Brandt, Cochrane, and Santa-Clara's measure is that $\hat{m}^*_{f,t}$, $\hat{m}_{d,t}$ and x_t can be constructed using nothing more than data on asset returns and real exchange rates and a

[27]The index is the same as the correlation between m^*_{ft} and m_{dt} when they have the same variance, because $\mathrm{RSI} = 2\,\mathrm{cov}(m^*_{ft}, m_{dt})/[\mathrm{var}(m^*_{ft}) + \mathrm{var}(m_{dt})]$.

[28]While $\mathrm{var}(M^*_{f,t}) \geq \mathrm{var}(\hat{M}^*_{f,t})$ and $\mathrm{var}(M_{d,t}) \geq \mathrm{var}(\hat{M}_{d,t})$, it does not necessarily follow that $\mathrm{var}(m^*_{f,t}) \geq \mathrm{var}(\hat{m}^*_{f,t})$ and $\mathrm{var}(m_{d,t}) \geq \mathrm{var}(\hat{m}_{d,t})$. This assumption is useful for intuition, but our argument does not hinge on it.

[29]As we showed in Section 3.2, in general $\hat{m}^*_{f,t} - \hat{m}_{d,t} \neq x_t$.

minimal appeal to asset pricing theory in forming the projections.[30] The problem, however, is that, unlike $\underline{\text{RSI}}$, RSI_{BCS} is not a lower bound for RSI. To make RSI_{BCS} meaningful as a lower bound for RSI we have to assume that asset markets are complete, so that $\xi_t = 0$. In other words, we have to rule out the second column *and* the second row of Table 1.

One interpretation of RSI_{BCS} is that $\text{var}(x_t)/[\text{var}(\hat{m}^*_{f,t})+\text{var}(\hat{m}_{d,t})] = 1-\text{RSI}_{\text{BCS}}$ provides an upper bound on the amount of risk that is not shared due only to goods market frictions. Of course, this calculation still requires that we assume away preference differences across the agents. It also abstracts from any covariance there might be between x_t and ξ_t.

Clearly, however, RSI_{BCS} is silent on the amount of risk that is not shared due to asset market incompleteness (the ξ_t component). This issue brings us back to a problem we highlighted in the section on reduced form models. These models are silent about differences between agents. Consistent with our main message in Sections 1–3, asset returns, alone, can provide useful information about how much risk is shared (via Hansen-Jagannathan bounds), but any measure of the degree of risk sharing also requires a measure of the amount of unshared risk.[31] Of course, one is always free to introspect, as Brandt, Cochrane, and Santa-Clara (2006) do in Sec. 3.5 of their paper, on how much unshared risk seems "reasonable". Our view is that this introspection is merely speculative if additional data and theory are not brought into the picture. Additionally, this introspection is not informed by exchange rates and applies equally well to agents living in the same economy, who face the same prices.

4.4 Observational Equivalence

In this section we highlight the issue that fundamentally different models of real exchange rates can be observationally equivalent at the level of aggregate consumption data. A version of our model in the appendix is illustrative. When the utility function in that model is logarithmic over a Cobb-Douglas aggregate of two individual goods (A and B), it is straightforward to solve for the dynamics of the real exchange rate for a number of different variants of the model. When assets markets are complete, representative agents in the two economies have different preferences over two individual goods, and trade in goods is frictionless, we

[30]Real exchange rate data tell us that $\text{var}(x_t)$ is on the order of 0.01 to 0.02 for major currency pairs, at an annual frequency. In practice, when the projections are formed using data on risk free assets and broad portfolios of equities, the implied values of $\text{var}(\hat{m}^*_{ft})$ and $\text{var}(\hat{m}_{dt})$ are approximately 0.25 at an annual frequency. Thus, empirically, RSI_{BCS} is close to 1 and Brandt, Cochrane, and Santa-Clara (2006) conclude that international risk sharing must be quite high.

[31]Likewise, any measure of the correlation between agents' IMRSs also requires a measure of ξ_t.

have

$$\ln(C_{d,t}/C_{d,t-1}) = \theta_d \ln G_t^A + (1-\theta_d) \ln G_t^B\,, \quad \text{and} \tag{27a}$$
$$\ln(C_{f,t}/C_{f,t-1}) = \theta_f \ln G_t^A + (1-\theta_f) \ln G_t^B\,. \tag{27b}$$

Here θ_d and θ_f are, respectively, the weights that the domestic and foreign agents have on good A in their utility functions, $C_{d,t}$ and $C_{f,t}$ are the aggregate consumption levels in the domestic and foreign countries, and G_t^A and G_t^B are, respectively, the growth rates of the global endowments of goods A and B. Models of this type, which emphasize the role of different preferences over individual goods, have been important workhorses in the modern exchange rate literature.[32]

Consider an alternative model, in which assets markets are complete, representative agents in two economies have identical preferences over the two goods, A being frictionlessly traded, but B being nontraded. Then we have

$$\ln(\tilde{C}_{d,t}/\tilde{C}_{d,t-1}) = \theta \ln \tilde{G}_t^A + (1-\theta) \ln \tilde{g}_{dt}^B\,, \quad \text{and} \tag{28a}$$
$$\ln(\tilde{C}_{f,t}/\tilde{C}_{f,t-1}) = \theta \ln \tilde{G}_t^A + (1-\theta) \ln \tilde{g}_{ft}^B\,. \tag{28b}$$

Here θ is the weight that the domestic and foreign agents have on good A in their utility functions, $\tilde{C}_{d,t}$ and $\tilde{C}_{f,t}$ are the aggregate consumption levels in the domestic and foreign countries, $\tilde{G}_t^A$ is the growth rate of the global endowment of good A, and $\tilde{g}_{dt}^B$ and $\tilde{g}_{ft}^B$ are, respectively, the growth rates of the endowments of good B in the domestic and foreign economies. Models which emphasize simple trade frictions have also played a prominent role in the modern exchange rate literature.[33]

The two models are observationally equivalent for the real exchange rate when they are observationally equivalent with respect to aggregate consumption. As an example, we can specify the stochastic processes for the endowments so that

$$\tilde{G}_t^A = G_t^A\,, \tag{29a}$$
$$\ln \tilde{g}_{d,t}^B = \left[(\theta_d - \theta) \ln G_t^A + (1-\theta_d) \ln G_t^B\right]/(1-\theta)\,, \quad \text{and} \tag{29b}$$
$$\ln \tilde{g}_{f,t}^B = \left[(\theta_f - \theta) \ln G_t^A + (1-\theta_f) \ln G_t^B\right]/(1-\theta)\,. \tag{29c}$$

With these assumptions, the two models have the same consumption growth rates and real

[32]See, for example, Stockman (1980), Bekaert (1996), Stathopoulos (2017), or Colacito, Croce, Ho, and Howard (2018 forthcoming).

[33]See, for example, for example, Backus and Smith (1993), Tesar (1993), Stockman and Tesar (1995) or Ready, Roussanov, and Ward (2017).

exchange rate:

$$\begin{aligned}\ln X_t &= \ln(C_{d,t}/C_{d,t-1}) - \ln(C_{f,t}/C_{f,t-1})\,,\\ &= (\theta_d - \theta_f)(\ln G_t^A - \ln G_t^B) = (1-\theta)(\ln \tilde{g}_{d,t}^B - \ln \tilde{g}_{f,t}^B)\,.\end{aligned} \tag{30}$$

The implication of this example is that an econometrician would not be able to discern which mechanism is more relevant without looking at consumption and endowments at the level of individual goods. The two models are also very different in terms of risk sharing. In one model, risk is shared perfectly, and only global endowments matter. When the global endowment of good A rises faster than that of good B and the foreign agent puts more weight on good A, its relative abundance means the real exchange rate falls.[34] In the other model, risk sharing is imperfect when the endowment growth rates for good B differ across countries. If the foreign agent's endowment of good B grows faster, this makes his basket relatively abundant, and the real exchange rate falls.

It should also be clear, of course, that if one were to estimate these structural models using only the pricing equations expressed in terms of consumption aggregates, it would be impossible to distinguish between model variants in which (i) asset markets are complete or incomplete, (ii) agents have the same or different preferences over individual goods, and (iii) trade in goods is frictionless or not. A direct test of the model based on Eq. (23), à la Backus and Smith (1993) would face the same problem.

This is not to say that it is impossible to distinguish between models with imperfect or perfect risk sharing. We simply have to look beyond their implications for asset pricing equations and the joint behavior of aggregate consumption and exchange rates to find their predictions for individual goods or categories of goods. While this is not possible in cases where only the behavior of the consumption aggregate is modeled—for example, Verdelhan (2010), Colacito and Croce (2011), or Bansal and Shaliastovich (2013)—it is possible for the models cited above, in which preference differences are specified over individual tradable goods, or in which there are specific trade frictions over some goods.

5 Conclusion

Hansen and Jagannathan (1991) offer a powerful tool for constructing a lower bound on the

[34] In fact, this model is equivalent to a model of a single representative agent whose preference weight is a weighted average of θ_d and θ_f. The equilibrium price of good B is identical. The real exchange rate can be replicated by defining it as the relative price of two different baskets of good A and good B. See Appendix B.

variation of all agents' IMRSs. Their lower bound is constructed from a reduced-form SDF, which can be thought of as a common component of all agents' IMRSs, and relies only on asset return data and the assumption of no-arbitrage. Unfortunately, reduced-form SDFs, combined with asset return and exchange rate data, cannot be used in the same way to identify economically meaningful differences between agents' IMRSs.

Our positive message is that structural models – with explicit assumptions about preferences over goods, goods market frictions, or asset market imperfections – are necessary, and useful, to address specific questions about real exchange rate determination and risk sharing. We are certainly not the first to recognize this point. An early example is Backus and Smith (1993), who "examine the possibility that non-traded goods may account for several striking features of international macroeconomic data". In order to focus on this friction in the goods market, they assumed away frictions in the asset market (frictionless asset trading and complete markets). Their approach is in line with our main point. When the asset market view of exchange rates holds, variation in the real exchange rate only reflects frictions and/or preferences differences in the goods market. It does not reflect heterogeneity across agents in different economies in the compensation for risk they require to own various assets such as currencies.

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A Lustig, Roussanov, and Verdelhan (2014)

Our critique of papers such Lustig, Roussanov, and Verdelhan (2014) focuses on their claim that differences between agents' IMRSs can be inferred using only asset returns. As Eq. (16) highlights, when the SDFs in Eq. (10) are expressed as functions of the asset returns they price (rather than latent shocks), it's clear that the relative asset returns are only informative about the common component of agents' IMRSs. In this appendix we explicitly solve for c and $\boldsymbol{\Upsilon}$ in Eq. (16) for the specific model in Lustig, Roussanov, and Verdelhan (2014).

Since there are $n+2$ latent shocks in their model, we require $n+3$ non-redundant asset returns (i.e., $n+2$ non-redundant relative asset returns). The default-free bank accounts in the domestic and foreign economies provide n asset returns. The remaining three non-redundant returns are provided by portfolios of long-term zero-coupon bonds in the different economies.[35]

Let G_t denote the return on a long-term zero-coupon bond denominated in domestic consumption units. In their model,

$$\ln G_t = r_{t-1} + \boldsymbol{\psi}\boldsymbol{\lambda}^\top + \boldsymbol{\psi}\,\boldsymbol{\varepsilon}_t - \tfrac{1}{2}\boldsymbol{\psi}\boldsymbol{\psi}^\top\,, \tag{31}$$

where $\boldsymbol{\psi}$ is of the form

$$\boldsymbol{\psi} = \left[\begin{array}{cccccc} 0 & \cdots & 0 & \boldsymbol{\psi}_n & \boldsymbol{\psi}_{n+1} & 0 \end{array}\right]. \tag{32}$$

Analogously, let $\boldsymbol{G}_t^*$ denote the $(n-1)\times 1$ vector of returns on long-term zero-coupon bonds denominated in the local consumption units of the foreign economies. Then

$$\ln \boldsymbol{G}_t^* = \boldsymbol{r}_{t-1}^* + \mathrm{diag}\left(\Psi^*\Lambda^{*\top}\right) + \Psi^*\,\boldsymbol{\varepsilon}_t - \tfrac{1}{2}\mathrm{diag}\left(\Psi^*\Psi^{*\top}\right)\,, \tag{33}$$

where Ψ^* is of the form

$$\Psi^* = \left[\begin{array}{ccccccc} \Psi_1^1 & 0 & \cdots & 0 & 0 & \Psi_{n+1}^1 & 0 \\ 0 & \Psi_2^2 & \ddots & \vdots & \vdots & \Psi_{n+1}^2 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & \Psi_{n-1}^{n-1} & 0 & \Psi_{n+1}^2 & 0 \end{array}\right]. \tag{34}$$

Let Θ be a $3\times n$ matrix of weights for the three portfolios of long-term zero-coupon

[35] In the general case, one could use $n+3$ non-redundant portfolios of short-term and long-term bonds in the n economies. In the model, given the returns on $n+3$ non-redundant (portfolios of) assets, the remaining asset returns are completely determined.

bonds. $\Theta_{j,n}$ and $\Theta_{j,i}$ are the weights in the jth portfolio on the long-term zero-coupons in the domestic and ith foreign economy, respectively. The vector of returns on these three portfolios, denominated in the domestic consumption basket, is given by

$$\ln \boldsymbol{G}_t^{\Theta} = \mathbf{1} r_{t-1} + \Psi\, \boldsymbol{\lambda}^{\top} + \Psi\, \boldsymbol{\varepsilon}_t - \tfrac{1}{2}\mathrm{diag}\left(\Psi\Psi^{\top}\right)\,, \tag{35}$$

where

$$\Psi = \Theta \begin{bmatrix} \Psi^* + \mathbf{1}\boldsymbol{\lambda} - \Lambda^* \\ \boldsymbol{\psi} \end{bmatrix}. \tag{36}$$

The full vector of non-redundant asset returns that we use to express the SDFs in Lustig, Roussanov, and Verdelhan (2014) is

$$\ln \boldsymbol{R}_t = \begin{bmatrix} r_{t-1} \\ \boldsymbol{r}^*_{t-1} + \ln \boldsymbol{X}_t \\ \ln \boldsymbol{G}_t^{\Theta} \end{bmatrix}. \tag{37}$$

With these asset returns, solving for the coefficients c and $\boldsymbol{\Upsilon}$ in Eq. (16) yields

$$\boldsymbol{\Upsilon} = [1 - \boldsymbol{\gamma}\cdot\mathbf{1}\,,\ \boldsymbol{\gamma}] \qquad \text{and} \qquad c \;=\; \tfrac{1}{2}\boldsymbol{\gamma}\Sigma\Sigma^{\top}\boldsymbol{\gamma}^{\top} - \tfrac{1}{2}\boldsymbol{\gamma}\cdot\mathrm{diag}\left(\Sigma\Sigma^{\top}\right)\,, \tag{38}$$

where

$$\Sigma = \begin{bmatrix} \mathbf{1}\boldsymbol{\lambda} - \Lambda^* \\ \Psi \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\gamma} = \boldsymbol{\lambda}\Sigma^{\top}\left(\Sigma\Sigma^{\top}\right)^{-1}. \tag{39}$$

B A Structural Model

We provide an example of a full-fledged model, based on Backus and Smith (1993), to illustrate how aggregate consumption levels and the real exchange rate are jointly determined in equilibrium.

We describe an endowment economy with two countries (domestic and foreign), each with a representative agent. Utility is defined over n individual goods, which are indexed by $j = 1, 2, \ldots, n$. All goods are perishable and agents live for two periods (0 and 1). We suppress date subscripts unless strictly necessary.

The domestic agent has an instantaneous level of utility

$$u[c_d(c_d^1, c_d^2, \cdots, c_d^n)], \tag{40}$$

where c_d^j is her consumption of good j, $c_d(\cdot)$ is a homogeneous of degree one quasi-concave function of its arguments, and u is a monotonic function with standard properties. The foreign agent has an instantaneous level of utility

$$u[c_f(c_f^1, c_f^2, \cdots, c_f^n)], \tag{41}$$

where c_f^j is his consumption of good j, and $c_f(\cdot)$ is a homogeneous of degree one quasi-concave function of its arguments. For convenience we assume that the utility functions over the aggregate are the same, but that the consumption aggregators, c_d and c_f may be different.

Both economies are cashless and use good 1 as the numeraire. Our model would have the same implications for the real exchange rate if we chose different numeraires. Goods markets meet sequentially. Good 1 is frictionlessly traded. The other goods may or may not be frictionlessly traded, so prices for those goods may differ across the economies. The domestic and foreign prices of good j are, respectively, P_d^j and P_f^j. For any good that is frictionlessly traded, its price must be the same in both countries:

$$P_d^j = P_f^j. \tag{42}$$

Since good 1 is frictionlessly traded and is the numeraire, we have $P_d^1 = P_f^1 = 1$.

To make our notation compact we define the $n \times 1$ vectors $\boldsymbol{c}_d = (c_d^1, c_d^2, \cdots, c_d^n)$, $\boldsymbol{c}_f = (c_f^1, c_f^2, \cdots, c_f^n)$, $\boldsymbol{P}_d = (1, P_d^2, \cdots, P_d^n)$ and $\boldsymbol{P}_f = (1, P_f^2, \cdots, P_f^n)$.

We now turn to a dynamic version of the model in which there are two periods, with time indexed by $t = 0$, 1. In the most general version of the model, asset markets are assumed to be incomplete, but we also explore a complete markets version of the model, as well as one with specific assumptions about the utility functions and consumption aggregators.

We assume that there are k assets with $k \times 1$ random payoff vector $\boldsymbol{Z}(\omega)$ in period 1, where $\omega \in \Omega$ represents the state of the world in period 1, which has probability $\pi(\omega)$. We assume that k is smaller than the number of states of the world, which is assumed to be finite. The $k \times 1$ price vector for these assets in period 0 is $\boldsymbol{P_Z}$. The payoffs and prices of the assets are measured in units of good 1.

The domestic agent chooses $\boldsymbol{c}_{d0}$, $\{\boldsymbol{c}_{d1}(\omega)\}_{\omega\in\Omega}$, and $\boldsymbol{a}_d$ to maximize

$$u[c_d(\boldsymbol{c}_{d0})] + \beta \sum_{\omega\in\Omega} u\{c_d[\boldsymbol{c}_{d1}(\omega)]\}\pi(\omega)\,, \tag{43}$$

subject to

$$\boldsymbol{P}_{d0} \cdot \boldsymbol{c}_{d0} + \boldsymbol{P}_{\boldsymbol{Z}} \cdot \boldsymbol{a}_d = \boldsymbol{P}_{d0} \cdot \boldsymbol{y}_{d0}\,, \tag{44}$$

$$\boldsymbol{P}_{d1}(\omega) \cdot \boldsymbol{c}_{d1}(\omega) = \boldsymbol{P}_{d1}(\omega) \cdot \boldsymbol{y}_{d1}(\omega) + \boldsymbol{Z}(\omega) \cdot \boldsymbol{a}_d\,, \quad \omega \in \Omega. \tag{45}$$

Here $0 < \beta < 1$, $\boldsymbol{c}_{d0}$ is the agent's consumption vector at time 0, $\boldsymbol{y}_{d0}$ is an $n \times 1$ vector of her endowments of the goods at time 0, $\boldsymbol{c}_{d1}(\omega)$ are her plans for future consumption (in every possible state of the world), $\boldsymbol{y}_{d1}(\omega)$ are her future endowments (in every possible state of the world), and $\boldsymbol{a}_d$ is a $k \times 1$ vector whose ith element is her net purchases of asset i.

Similarly, the foreign agent chooses $\boldsymbol{c}_{f0}$, $\{\boldsymbol{c}_{f1}(\omega)\}_{\omega\in\Omega}$, and $\boldsymbol{a}_f$ to maximize

$$u[c_f(\boldsymbol{c}_{f0})] + \beta \sum_{\omega\in\Omega} u\{c_f[\boldsymbol{c}_{f1}(\omega)]\}\pi(\omega)\,, \tag{46}$$

subject to

$$\boldsymbol{P}_{f0} \cdot \boldsymbol{c}_{f0} + \boldsymbol{P}_{\boldsymbol{Z}} \cdot \boldsymbol{a}_f = \boldsymbol{P}_{f0} \cdot \boldsymbol{y}_{f0}\,. \tag{47}$$

$$\boldsymbol{P}_{f1}(\omega) \cdot \boldsymbol{c}_{f1}(\omega) = \boldsymbol{P}_{f1}(\omega) \cdot \boldsymbol{y}_{f1}(\omega) + \boldsymbol{Z}(\omega) \cdot \boldsymbol{a}_f\,, \quad \omega \in \Omega. \tag{48}$$

For simplicity, we assume that each good is either frictionlessly traded across economies or is not traded across economies. The market clearing conditions for any frictionlessly traded good, j, are

$$\begin{aligned} c^j_{d0} + c^j_{f0} &= y^j_{d0} + y^j_{f0}\,, && (49)\\ c^j_{d1}(\omega) + c^j_{f1}(\omega) &= y^j_{d1}(\omega) + y^j_{f1}(\omega)\,, \quad \omega \in \Omega. && (50) \end{aligned}$$

For a nontraded good, j, we have

$$\begin{aligned} c^j_{d0} &= y^j_{d0}\,, \quad c^j_{f0} = y^j_{f0}\,, && (51)\\ c^j_{d1}(\omega) &= y^j_{d1}(\omega)\,, \quad c^j_{f1}(\omega) = y^j_{f1}(\omega)\,, \quad \omega \in \Omega. && (52) \end{aligned}$$

The market clearing condition in the asset market is

$$\boldsymbol{a}_d + \boldsymbol{a}_f = \boldsymbol{0}\,. \tag{53}$$

Definition. A competitive equilibrium is a set of vectors of quantities $\boldsymbol{c}_{d0}$, $\boldsymbol{c}_{f0}$, $\{\boldsymbol{c}_{d1}(\omega)\}_{\omega\in\Omega}$, $\{\boldsymbol{c}_{f1}(\omega)\}_{\omega\in\Omega}$, $\boldsymbol{a}_d$, $\boldsymbol{a}_f$, and prices $\boldsymbol{P}_{d0}$, $\boldsymbol{P}_{f0}$, $\{\boldsymbol{P}_{d1}(\omega)\}_{\omega\in\Omega}$, $\{\boldsymbol{P}_{f1}(\omega)\}_{\omega\in\Omega}$, $\boldsymbol{P}_{\boldsymbol{Z}}$ such that the quantities solve the agents' optimization problems (taking the prices as given), and such that the market clearing conditions are satisfied. The law of one price must hold for any good that is frictionlessly traded.

The first order conditions for the domestic agent are

$$u_c(c_d(\boldsymbol{c}_{d0}))\frac{dc_d(\boldsymbol{c}_{d0})}{d\boldsymbol{c}_{d0}} = \boldsymbol{P}_{d0}\lambda_d\,, \tag{54}$$

$$\beta u_c\{c_d[\boldsymbol{c}_{d1}(\omega)]\}\frac{dc_d[\boldsymbol{c}_{d1}(\omega)]}{d\boldsymbol{c}_{d1}(\omega)}\pi(\omega) = \boldsymbol{P}_{d1}(\omega)\mu_d(\omega)\,, \quad \omega \in \Omega. \tag{55}$$

$$\boldsymbol{P}_{\boldsymbol{Z}}\lambda_d = \sum_{\omega\in\Omega}\mu_d(\omega)\boldsymbol{Z}(\omega)\ . \tag{56}$$

Here λ_d is the Lagrange multiplier on the constraint (44), and $\mu_d(\omega)$ is the Lagrange multiplier on the constraint (45).

The first order conditions for the foreign agent are

$$u_c(c_f(\boldsymbol{c}_{f0}))\frac{dc_f(\boldsymbol{c}_{f0})}{d\boldsymbol{c}_{f0}} = \boldsymbol{P}_{f0}\lambda_f\,, \tag{57}$$

$$\beta u_c\{c_f[\boldsymbol{c}_{f1}(\omega)]\}\frac{dc_f[\boldsymbol{c}_{f1}(\omega)]}{d\boldsymbol{c}_{f1}(\omega)}\pi(\omega) = \boldsymbol{P}_{f1}(\omega)\mu_f(\omega)\,, \quad \omega \in \Omega. \tag{58}$$

$$\boldsymbol{P}_{\boldsymbol{Z}}\lambda_f = \sum_{\omega\in\Omega}\mu_f(\omega)\boldsymbol{Z}(\omega)\ . \tag{59}$$

Here λ_f is the Lagrange multiplier on the constraint (47), and $\mu_f(\omega)$ is the Lagrange multiplier on the constraint (48).

Consider the first order conditions for the numeraire good in periods 0 and 1. If we combine these we get an expression for the IMRSs in the numeraire good::

$$M_d^1(\omega) \equiv \frac{\beta u_c\{c_d[\boldsymbol{c}_{d1}(\omega)]\}\frac{dc_d[\boldsymbol{c}_{d1}(\omega)]}{dc_{d1}^1(\omega)}}{u_c(c_d(\boldsymbol{c}_{d0}))\frac{dc_d(\boldsymbol{c}_{d0})}{dc_{d0}^1}} = \frac{\mu_d(\omega)}{\lambda_d\pi(\omega)}\,, \quad \omega \in \Omega, \tag{60}$$

$$M_f^1(\omega) \equiv \frac{\beta u_c\{c_f[\boldsymbol{c}_{f1}(\omega)]\}\frac{dc_f[\boldsymbol{c}_{f1}(\omega)]}{dc_{f1}^1(\omega)}}{u_c(c_f(\boldsymbol{c}_{f0}))\frac{dc_f(\boldsymbol{c}_{f0})}{dc_{f0}^1}} = \frac{\mu_f(\omega)}{\lambda_f\pi(\omega)}\,, \quad \omega \in \Omega. \tag{61}$$

We define the ratio between these IMRSs as

$$\Xi(\omega) \equiv \frac{M_f^1(\omega)}{M_d^1(\omega)} = \left[\frac{\mu_f(\omega)}{\lambda_f}\right] / \left[\frac{\mu_d(\omega)}{\lambda_d}\right]. \tag{62}$$

For any other good we have the IMRSs:

$$M_d^j(\omega) \equiv \frac{\beta u_c\{c_d[\boldsymbol{c}_{d1}(\omega)]\}\frac{dc_d[\boldsymbol{c}_{d1}(\omega)]}{dc_{d1}^j(\omega)}}{u_c(c_d(\boldsymbol{c}_{d0}))\frac{dc_d(\boldsymbol{c}_{d0})}{dc_{d0}^j}} = \frac{P_{d1}^j(\omega)}{P_{d0}^j}\frac{\mu_d(\omega)}{\lambda_d \pi(\omega)}, \quad \omega \in \Omega, \tag{63}$$

$$M_f^j(\omega) \equiv \frac{\beta u_c\{c_f[\boldsymbol{c}_{f1}(\omega)]\}\frac{dc_f[\boldsymbol{c}_{f1}(\omega)]}{dc_{f1}^j(\omega)}}{u_c(c_f(\boldsymbol{c}_{f0}))\frac{dc_f(\boldsymbol{c}_{f0})}{dc_{f0}^j}} = \frac{P_{f1}^j(\omega)}{P_{f0}^j}\frac{\mu_f(\omega)}{\lambda_f \pi(\omega)}, \quad \omega \in \Omega. \tag{64}$$

For any frictionlessly traded good, because the law of one price holds, we have

$$\frac{M_f^j(\omega)}{M_d^j(\omega)} = \Xi(\omega). \tag{65}$$

For nontraded goods we have

$$\frac{M_f^j(\omega)}{M_d^j(\omega)} = \Xi(\omega)\frac{P_{f1}^j(\omega)/P_{f0}^j}{P_{d1}^j(\omega)/P_{d0}^j}. \tag{66}$$

Complete Asset Markets

When asset markets are complete we can assume that there is a complete set of state contingent claims. That is, we can assume that there are k states of the world (just as there are k assets) and that the payoff on asset k in state $\omega = k$ is 1 and is zero otherwise. This means that Eqs. (56) and (59) are equivalent to

$$\boldsymbol{P_Z}\lambda_d = \boldsymbol{\mu}_d\,, \tag{67}$$

$$\boldsymbol{P_Z}\lambda_f = \boldsymbol{\mu}_f\,, \tag{68}$$

where $\boldsymbol{\mu}_d = [\mu_d(1), \ldots, \mu_d(k)]$ and $\boldsymbol{\mu}_f = [\mu_f(1), \ldots, \mu_f(k)]$. This means that when asset markets are complete we have

$$\frac{\boldsymbol{\mu}_d}{\lambda_d} = \frac{\boldsymbol{\mu}_f}{\lambda_f}\,. \tag{69}$$

Consequently, when asset markets are complete, we have the result that $\Xi(\omega) = 1$ for all ω,

$$\frac{M_f^j(\omega)}{M_d^j(\omega)} = \Xi(\omega) = 1, \tag{70}$$

for any frictionlessly traded good, and

$$\frac{M_f^j(\omega)}{M_d^j(\omega)} = \frac{P_{f1}^j(\omega)/P_{f0}^j}{P_{d1}^j(\omega)/P_{d0}^j}, \tag{71}$$

for any nontraded good.

Price Indices and the Real Exchange Rate

Given a particular set of prices for the individual goods, we can solve the domestic agent's static expenditure minimization problem

$$\min_{\boldsymbol{c}_{d1}} \boldsymbol{P}_d \cdot \boldsymbol{c}_d \quad \text{subject to} \quad c_d = c_d(\boldsymbol{c}_d)\,. \tag{72}$$

Because $c_d(\cdot)$ is a homogenous of degree one function, minimized expenditure is equal to $P_d c_d$ where $P_d = H_d(\boldsymbol{P}_d)$, with $H_d(\cdot)$ also being homogenous of degree one in its arguments, and having a form related to the function $c_d(\cdot)$ [see Varian (1984)]. Similarly, the foreign price index is $P_f = H_f(\boldsymbol{P}_f)$. Since all prices are measured in the same numeraire, the real exchange rate is $e \equiv P_f/P_d$.

In the special case where $c_d(\cdot) = c_f(\cdot)$, we have $H_d(\cdot) = H_f(\cdot)$. If, additionally, all goods are frictionlessly traded, $e = 1$. If preferences differ across countries and all goods are frictionlessly traded, variation in the real exchange rate *can* arise even though $P_d^j = P_f^j$ for all j.

Aggregate IMRSs

Eqs. (54) and (55) imply

$$u_c(c_d(\boldsymbol{c}_{d0}))[\frac{dc_d(\boldsymbol{c}_{d0})}{d\boldsymbol{c}_{d0}} \cdot \boldsymbol{c}_{d0}] = \lambda_d[\boldsymbol{c}_{d0} \cdot \boldsymbol{P}_{d0}],$$

$$\beta u_c\{c_d[\boldsymbol{c}_{d1}(\omega)]\}[\frac{dc_d[\boldsymbol{c}_{d1}(\omega)]}{d\boldsymbol{c}_{d1}(\omega)} \cdot \boldsymbol{c}_{d1}(\omega)]\pi(\omega) = [\boldsymbol{c}_{d1}(\omega) \cdot \boldsymbol{P}_{d1}(\omega)]\mu_d(\omega)\,, \quad \omega \in \Omega.$$

Because $\boldsymbol{c}_{d0} \cdot \boldsymbol{P}_{d0} = c_{d0}P_{d0}$, $\boldsymbol{c}_{d1}(\omega) \cdot \boldsymbol{P}_{d1}(\omega) = c_{d1}(\omega) \cdot P_{d1}(\omega)$, and $c_d(\cdot)$ is homogenous of degree 1, this means we can rewrite these two Eqs. as

$$u_c(c_d(\boldsymbol{c}_{d0})) = P_{d0}\lambda_d,$$

$$\beta u_c\{c_d[\boldsymbol{c}_{d1}(\omega)]\}\pi(\omega) = P_{d1}(\omega)\mu_d(\omega)\,, \quad \omega \in \Omega.$$

It follows that the IMRS over aggregate consumption for the domestic agent is

$$M_d(\omega) \equiv \beta\frac{u_c\{c_d[\boldsymbol{c}_{d1}(\omega)]\}}{u_c(c_d(\boldsymbol{c}_{d0}))} = \frac{P_{d1}(\omega)}{P_{d0}}\frac{\mu_d(\omega)}{\lambda_d\pi(\omega)}\,, \quad \omega \in \Omega. \tag{73}$$

Similarly, for the foreign agent, the IMRS over aggregate consumption is

$$M_f^*(\omega) \equiv \beta\frac{u_c\{c_f[\boldsymbol{c}_{f1}(\omega)]\}}{u_c(c_f(\boldsymbol{c}_{f0}))} = \frac{P_{f1}(\omega)}{P_{f0}}\frac{\mu_f(\omega)}{\lambda_f\pi(\omega)}\,, \quad \omega \in \Omega. \tag{74}$$

We use the $*$ notation for the foreign agent to emphasize that the numeraires are different for the two agents' aggregate consumptions.

Together. Eqs. (73) and (74) imply that

$$\frac{M_f^*(\omega)}{M_d(\omega)} = \left[\frac{P_{f1}(\omega)}{P_{f0}}\frac{\mu_f(\omega)}{\lambda_f}\right] \Big/ \left[\frac{P_{d1}(\omega)}{P_{d0}}\frac{\mu_d(\omega)}{\lambda_d}\right] = \frac{e_1(\omega)}{e_0}\Xi(\omega)\,. \tag{75}$$

Letting $X(\omega) = e_1(\omega)/e_0$ we can rewrite Eq. (75) as

$$\frac{M_f^*(\omega)}{M_d(\omega)} = X(\omega)\Xi(\omega)\,. \tag{76}$$

A Specific Model Under Complete Markets

We adopt the following assumptions: (1) There are two goods, A and B, with good A being the numeraire, and being frictionlessly traded. (2) $u(\cdot) = \ln(\cdot)$, (3) $c_d(c_d^A, c_d^B) = (c_d^A)^{\theta_d}(c_d^B)^{1-\theta_d}$ and $c_f(c_f^A, c_f^B) = (c_f^A)^{\theta_f}(c_f^B)^{1-\theta_f}$. These assumptions imply that the CPIs in the two countries, measured in units of good A, are

$$P_d = \rho_d\,(P_d^B)^{1-\theta_d}\,, \qquad \text{and} \qquad P_f = \rho_f\,(P_f^B)^{1-\theta_f}\,, \tag{77}$$

with $\rho_d = \theta_d^{-\theta_d}(1-\theta_d)^{\theta_d-1}$, and $\rho_f = \theta_f^{-\theta_f}(1-\theta_f)^{\theta_f-1}$. The real exchange rate is

$$e = (\rho_f/\rho_d)[(P_f^B)^{1-\theta_f}/(P_d^B)^{1-\theta_d}]\,. \tag{78}$$

We discuss two specific examples of our model, which assume, alternatively, that good B is frictionlessly traded or nontraded. We use the following notation, and henceforth drop the notational dependence of time 1 variables on ω unless it is needed. The global endowment of good j in period t is $Y_t^j = y_{dt}^j + y_{ft}^j$, $j = A, B$. The growth rates of the global endowments of good j is $G^j = Y_1^j / Y_0^j$, $j = A, B$. We also define $g_d^j = y_{d1}^j / y_{d0}^j$ and $g_f^j = y_{f1}^j / y_{f0}^j$, $j = A, B$. The domestic agent's share of the global endowment of good j at time t is $s_t^j = y_{dt}^j / Y_t^j$, $j = A, B$. We let $\bar{s}_1^j = \sum_\omega s_1^j(\omega)\pi(\omega)$, $j = A, B$, denote the domestic agent's expected shares of the global endowments in period 1.

With the above assumptions the first order conditions for the two agents can be written as

$$\theta_d / c_{d0}^A = \lambda_d \,, \tag{79}$$

$$(1 - \theta_d)/c_{d0}^B = P_{d0}^B \lambda_d \,, \tag{80}$$

$$\beta \left[\theta_d / c_{d1}^A(\omega)\right] \pi(\omega) = \mu_d(\omega) \,, \quad \omega \in \Omega. \tag{81}$$

$$\beta \left[(1 - \theta_d)/c_{d1}^B(\omega)\right] \pi(\omega) = P_{d1}^B(\omega)\mu_d(\omega) \,, \quad \omega \in \Omega. \tag{82}$$

$$P_Z(\omega)\lambda_d = \mu_d(\omega) \,, \quad \omega \in \Omega. \tag{83}$$

$$\theta_f / c_{f0}^A = \lambda_f \,, \tag{84}$$

$$(1 - \theta_f)/c_{f0}^B = P_{f0}^B \lambda_f \,, \tag{85}$$

$$\beta \left[\theta_f / c_{f1}^A(\omega)\right] \pi(\omega) = \mu_f(\omega) \,, \quad \omega \in \Omega. \tag{86}$$

$$\beta \left[(1 - \theta_f)/c_{f1}^B(\omega)\right] \pi(\omega) = P_{f1}^B(\omega)\mu_f(\omega) \,, \quad \omega \in \Omega. \tag{87}$$

$$P_Z(\omega)\lambda_f = \mu_f(\omega) \,, \quad \omega \in \Omega. \tag{88}$$

To solve the model we also use the domestic agent's lifetime budget constraint:

$$\begin{aligned} c_{d0}^A + P_{d0}^B c_{d0}^B + \sum_\omega P_Z(\omega)\left[c_{d1}^A(\omega) + P_{d1}^B(\omega) c_{d1}^B(\omega)\right] = \\ y_{d0}^A + P_{d0}^B y_{d0}^B + \sum_\omega P_Z(\omega)\left[y_{d1}^A(\omega) + P_{d1}^B(\omega) y_{d1}^B(\omega)\right] \end{aligned} \tag{89}$$

Both Goods are Frictionlessly Traded. When both goods are frictionlessly traded, we drop location subscripts from the price of good B, and we write the market clearing conditions for goods as

$$c_{d0}^A + c_{f0}^A = Y_0^A \tag{90}$$

$$c_{d1}^A(\omega) + c_{f1}^A(\omega) = Y_1^A(\omega) \tag{91}$$

$$c_{d0}^B + c_{f0}^B = Y_0^B \tag{92}$$

$$c_{d1}^B(\omega) + c_{f1}^B(\omega) = Y_1^B(\omega) \tag{93}$$

To solve the model we let eliminate unknowns by solving for them in terms of the domestic agent's expenditure at time 0, which we denote $F_d = c_{d0}^A + P_0^B c_{d0}^B$. We denote the foreign agent's expenditure at time 0 as $F_f = c_{f0}^A + P_0^B c_{f0}^B$. Eqs. (79) and (80) imply that $F_d = \lambda_d^{-1}$, while Eqs. (84) and (85) imply that $F_f = \lambda_d^{-1}$. These results together with Eqs. (83) and (83) allow us to rewrite Eqs. (79)–(82) and (84)–(87) as

$$c_{d0}^A = \theta_d F_d. \tag{94}$$

$$c_{d0}^B P_0^B = (1-\theta_d) F_d\,, \tag{95}$$

$$c_{d1}^A(\omega) = \beta\theta_d \frac{\pi(\omega)}{P_Z(\omega)} F_d\,, \quad \omega \in \Omega. \tag{96}$$

$$c_{d1}^B(\omega) P_1^B(\omega) = \beta(1-\theta_d) \frac{\pi(\omega)}{P_Z(\omega)} F_d\,, \quad \omega \in \Omega. \tag{97}$$

$$c_{f0}^A = \theta_f F_f. \tag{98}$$

$$c_{f0}^B P_0^B = (1-\theta_f) F_f\,, \tag{99}$$

$$c_{f1}^A(\omega) = \beta\theta_f \frac{\pi(\omega)}{P_Z(\omega)} F_f\,, \quad \omega \in \Omega. \tag{100}$$

$$c_{f1}^B(\omega) P_1^B(\omega) = \beta(1-\theta_f) \frac{\pi(\omega)}{P_Z(\omega)} F_f\,, \quad \omega \in \Omega. \tag{101}$$

If we substitute these results into the market clearing conditions we get

$$\theta_d F_d + \theta_f F_f = Y_0^A \tag{102}$$

$$\beta \frac{\pi(\omega)}{P_Z(\omega)} \left(\theta_d F_d + \theta_f F_f\right) = Y_1^A(\omega) \tag{103}$$

$$[(1-\theta_d)F_d + (1-\theta_f)F_f]/P_0^B = Y_0^B \tag{104}$$

$$\beta \frac{\pi(\omega)}{P_Z(\omega)} [(1-\theta_d)F_d + (1-\theta_f)F_f]/P_1^B(\omega) \ = Y_1^B(\omega) \tag{105}$$

Eq. (102) implies

$$F_f = \frac{1}{\theta_f}\left(Y_0^A - \theta_d F_d\right) \tag{106}$$

Eqs. (104) and (106)

$$P_0^B = \frac{(1-\theta_d)F_d + (1-\theta_f)F_f}{Y_0^B} = \frac{\frac{\theta_f-\theta_d}{\theta_f}F_d + \frac{1-\theta_f}{\theta_f}Y_0^A}{Y_0^B} \tag{107}$$

Eqs. (102) and (103) together imply

$$P_Z(\omega) = \beta\frac{\pi(\omega)}{G^A(\omega)}. \tag{108}$$

Eq. (105), (104) and (108) imply

$$P_1^B(\omega) = \beta\frac{\pi(\omega)}{P_Z(\omega)}\frac{Y_0^B}{Y_1^B(\omega)}P_0^B = \frac{G^A(\omega)}{G^B(\omega)}P_0^B. \tag{109}$$

Because the price of good 2 is the same in both economies, from Eq. (78) we can see that the logarithm of the real exchange rate in each period is just $\ln e = \ln(\rho_f/\rho_d) + (\theta_d - \theta_f)\ln(P^B)$. Hence, from Eq. (109), the log change in the real exchange rate is

$$\ln X(\omega) = (\theta_f - \theta_d)\ln[G^B(\omega)/G^A(\omega)]\,. \tag{110}$$

All that remains is to solve for F_d. We can do this by substituting Eqs. (94)–(97) into Eq. (89) while using the notation $s_t^j = y_{dt}^j/Y_t^j$ to get

$$(1+\beta)F_d = s_0^A Y_0^A + P_0^B s_0^B Y_0^B + \sum_\omega P_Z(\omega)\left[s_1^A(\omega)Y_1^A(\omega) + P_1^B(\omega)s_1^B(\omega)Y_1^B(\omega)\right] \tag{111}$$

Using Eqs. (107), (108) and (109), we end up with

$$F_d = \frac{\theta_f(s_0^A + \beta\bar{s}_1^A) + (1-\theta_f)(s_0^B + \beta\bar{s}_1^B)}{\theta_f(1+\beta) + (\theta_d - \theta_f)(s_0^B + \beta\bar{s}_1^B)}Y_0^A \tag{112}$$

By substitution of this result into Eqs. (106) and (107), we get

$$F_f = \frac{\theta_d[(1-s_0^A) + \beta(1-\bar{s}_1^A)] + (1-\theta_d)[(1-s_0^B) + \beta(1-\bar{s}_1^B)]}{\theta_f(1+\beta) + (\theta_d - \theta_f)(s_0^B + \beta\bar{s}_1^B)}Y_0^A \tag{113}$$

$$P_0^B = \frac{(1-\theta_f)(1+\beta) + (\theta_f - \theta_d)(s_0^A + \beta\bar{s}_1^A)}{\theta_f(1+\beta) + (\theta_d - \theta_f)(s_0^B + \beta\bar{s}_1^B)}\frac{Y_0^A}{Y_0^B} \tag{114}$$

Given the form of the utility function, domestic aggregate consumption growth is

$$\ln[C_{d1}(\omega)/C_{d0}] = \theta_d \ln[c^A_{d1}(\omega)/c^A_{d0}] + (1-\theta_d)\ln[c^B_{d1}(\omega)/c^B_{d0}]. \tag{115}$$

Given Eqs. (94)–(97) and (107)–(109) we have

$$c^A_{d1}(\omega)/c^A_{d0} = G^A(\omega)\,, \quad \omega \in \Omega. \tag{116}$$

$$c^B_{d1}(\omega)/c^B_{d0} = G^B(\omega)\,, \quad \omega \in \Omega. \tag{117}$$

Hence, Eq. (115) can be rewritten as

$$\ln[C_{d1}(\omega)/C_{d0}] = \theta_d \ln G^A(\omega) + (1-\theta_d)\ln G^B(\omega). \tag{118}$$

Similarly, for the foreign agent, aggregate consumption growth is

$$\ln[C_{f1}(\omega)/C_{f0}] = \theta_f \ln G^A(\omega) + (1-\theta_f)\ln G^B(\omega). \tag{119}$$

One Good is Not Traded. We now assume that it is not possible to trade good B between the two economies. Thus, we replace the market clearing conditions for good B, (92) and (93), with the following equations:

$$c^B_{d0} = y^B_{d0}\,, \quad c^B_{f0} = y^B_{f0} \tag{120}$$

$$c^B_{d1}(\omega) = y^B_{d1}(\omega)\,, \quad c^B_{f1}(\omega) = y^B_{f1}(\omega) \tag{121}$$

We also assume, as in the main text, that $\theta_d = \theta_f = \theta$. After noting that the price of good B now requires a location subscript, if we substitute Eqs. (94)–(101) into the market clearing conditions we get

$$\theta(F_d + F_f) = Y^A_0 \tag{122}$$

$$\beta\frac{\pi(\omega)}{P_Z(\omega)}\theta(F_d + F_f) = Y^A_1(\omega) \tag{123}$$

$$(1-\theta)F_d/P^B_{d0} = y^B_{d0} \tag{124}$$

$$(1-\theta)F_f/P^B_{f0} = y^B_{f0} \tag{125}$$

$$\beta\frac{\pi(\omega)}{P_Z(\omega)}(1-\theta)F_d/P^B_{d1}(\omega) = y^B_{d1}(\omega) \tag{126}$$

$$\beta \frac{\pi(\omega)}{P_Z(\omega)}(1-\theta)F_f/P_{f1}^B(\omega) = y_{f1}^B(\omega) \tag{127}$$

Eq. (122) implies

$$F_f = Y_0^A/\theta - F_d \tag{128}$$

Eq. (124) implies that

$$P_{d0}^B = \frac{(1-\theta_d)F_d}{y_{d0}^B} \tag{129}$$

while Eqs. (125) and (128) imply that

$$P_{f0}^B = \frac{(1-\theta)F_f}{y_{f0}^B} = \frac{(1-\theta)[Y_0^A/\theta - F_d]}{y_{f0}^B} \tag{130}$$

Eqs. (122) and (123) together imply

$$P_Z(\omega) = \beta \frac{\pi(\omega)}{G^A(\omega)}. \tag{131}$$

Eqs. (126), (124) and (131) imply

$$P_{d1}^B(\omega) = \beta \frac{\pi(\omega)}{P_Z(\omega)} \frac{y_{d0}^B}{y_{d1}^B(\omega)} P_{d0}^B = \frac{G^A(\omega)}{g_d^B(\omega)} P_{d0}^B. \tag{132}$$

Eqs. (127), (125) and (131) imply

$$P_{f1}^B(\omega) = \beta \frac{\pi(\omega)}{P_Z(\omega)} \frac{y_{f0}^B}{y_{f1}^B(\omega)} P_{f0}^B = \frac{G^A(\omega)}{g_f^B(\omega)} P_{f0}^B. \tag{133}$$

Because agents have the same preferences, from Eq. (78) we can see that the logarithm of the real exchange rate in each period is $\ln e = (1-\theta)\ln(P_f^B/P_d^B)$. Hence, from Eqs. (132) and (133) the log change in the real exchange rate is

$$\ln X(\omega) = (1-\theta)[\ln g_d^B(\omega) - \ln g_f^B(\omega)]\,. \tag{134}$$

All that remains is to solve for F_d. In the lifetime budget constraint, Eq. (89), if we use Eqs. (120) and (121), the terms involving the nontraded good cancel out from either side of the equation. We can then use Eqs. (94), (96), and the notation $s_t^A = y_{dt}^A/Y_t^A$, to rewrite Eq. (89) as

$$\theta(1+\beta)F_d = s_0^A Y_0^A + \sum_\omega P_Z(\omega)\left[s_1^A(\omega)Y_1^A(\omega)\right] \tag{135}$$

Using Eq. (131), we end up with

$$F_d = \frac{s_0^A + \beta \bar{s}_1^A}{\theta(1+\beta)} Y_0^A \tag{136}$$

By substitution of this result into Eqs. (128), (129) and (130) we get

$$F_f = \frac{(1 - s_0^A) + \beta(1 - \bar{s}_1^A)}{\theta(1+\beta)} Y_0^A \tag{137}$$

$$P_{d0}^B = \frac{1-\theta}{\theta} \frac{s_0^A + \beta \bar{s}_1^A}{1+\beta} \frac{Y_0^A}{y_{d0}^B} \tag{138}$$

$$P_{f0}^B = \frac{1-\theta}{\theta} \frac{(1 - s_0^A) + \beta(1 - \bar{s}_1^A)}{1+\beta} \frac{Y_0^A}{y_{f0}^B} \tag{139}$$

Given the form of the utility function, domestic aggregate consumption growth is

$$\ln[C_{d1}(\omega)/C_{d0}] = \theta \ln[c_{d1}^A(\omega)/c_{d0}^A] + (1-\theta)\ln[c_{d1}^B(\omega)/c_{d0}^B]. \tag{140}$$

Given Eqs. (94), (96), (131), (120), and (121) we have

$$c_{d1}^A(\omega)/c_{d0}^A = G^A(\omega)\,, \quad \omega \in \Omega. \tag{141}$$

$$c_{d1}^B(\omega)/c_{d0}^B = g_d^B(\omega)\,, \quad \omega \in \Omega. \tag{142}$$

Hence, Eq. (140) can be rewritten as

$$\ln[C_{d1}(\omega)/C_{d0}] = \theta \ln G^A(\omega) + (1-\theta)\ln g_d^B(\omega). \tag{143}$$

Similarly, for the foreign agent, aggregate consumption growth is

$$\ln[C_{f1}(\omega)/C_{f0}] = \theta \ln G^A(\omega) + (1-\theta)\ln g_f^B(\omega). \tag{144}$$

A Representative Agent Model. We now consider a model with a single representative agent whose preferences have the same functional form as in the previous examples, and whose endowments correspond to the global endowments of the two goods. The agent also has access to a complete set of state contingent claims. Letting F be the agent's expenditure on consumption in period 0, and letting θ be the weight on good A in the utility function, we can write the first order conditions as

$$c_0^A = \theta F. \tag{145}$$

$$c_0^B P_0^B = (1-\theta)F\,, \tag{146}$$

$$c_1^A(\omega) = \beta\theta\frac{\pi(\omega)}{P_Z(\omega)}F\,, \quad \omega \in \Omega. \tag{147}$$

$$c_1^B(\omega)P_1^B(\omega) = \beta(1-\theta)\frac{\pi(\omega)}{P_Z(\omega)}F\,, \quad \omega \in \Omega. \tag{148}$$

Given that in equilibrium the agent eats the global endowments of the two goods we have $c_0^A = Y_0^A$, $c_0^B = Y_0^B$, $c_1^A(\omega) = Y_1^A(\omega)$, and $c_1^B(\omega) = Y_1^B(\omega)$, so it follows immediately from the first order conditions that $F = Y_0^A/\theta$, $P_0^B = (1-\theta)Y_0^A/(\theta Y_0^B)$, $P_1^B(\omega) = (1-\theta)Y_1^A/(\theta Y_1^B)$, and $P_Z(\omega) = \beta\pi(\omega)/G^A(\omega)$. The price of a state contingent claim against state ω is the same as in the model with two agents. So is the rate of change of the price of good B:

$$\frac{P_1^B(\omega)}{P_0^B} = \frac{G^A(\omega)}{G^B(\omega)}. \tag{149}$$

The level of the price of good B at time 0 is also the same as in the two agent model if the parameter weight of the single agent satisfies

$$\frac{1-\theta}{\theta} = \frac{(1-\theta_f)(1+\beta) + (\theta_f - \theta_d)(s_0^A + \beta\bar{s}_1^A)}{\theta_f(1+\beta) + (\theta_d - \theta_f)(s_0^B + \beta\bar{s}_1^B)}. \tag{150}$$

The agent's IMRSs over the individual goods are

$$\beta\frac{c_0^A}{c_1^A(\omega)} = \frac{\beta}{G^A(\omega)} \qquad \text{and} \qquad \beta\frac{c_0^B}{c_1^B(\omega)} = \frac{\beta}{G^B(\omega)}, \tag{151}$$

which are the same expressions we had for both agents in the two agent model.