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ABSTRACT

We offer a critique of the popular notion that the log-change of the real exchange rate equals the log-difference between the IMRSs of economically distinct agents in two economies. Contrary to existing claims, we show that this interpretation does not hold true in reduced-form SDF models that only rely on the absence of arbitrage in asset markets. In structural models, we show that this economic interpretation requires much stronger assumptions that emphasize the importance of goods markets rather than asset markets. We demonstrate the significance of our results for a broad range of topics in the international asset pricing literature.

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If there are no arbitrage opportunities then the log-change in the real exchange rate between two economies (say, the United States and the United Kingdom) can always be expressed as the log difference between a stochastic discount factor (SDF) denominated in the real currency units of those economies:

$$\begin{matrix}\text{log change in the real} \\ \text{U.S./U.K. exchange rate}\end{matrix} = \begin{matrix}\text{log SDF in} \\ \text{real pounds}\end{matrix} - \begin{matrix}\text{log SDF in} \\ \text{real dollars}\end{matrix}. \tag{1}$$

The recent international asset pricing literature frequently interprets the right hand side of Eq. (1) as the difference between the log intertemporal marginal rates of substitution (IMRSs) of distinct representative agents in the two economies.[1] With this interpretation, the literature argues that the returns on foreign currency investments (including various carry trade strategies) reflect differences between the log IMRSs of distinct agents in the two economies. Following Brandt, Cochrane, and Santa-Clara (2006), we refer to this economic interpretation of Eq. (1) as the asset market view of exchange rates. It is now a dominant theoretical framework in international asset pricing and has been used to gain insights into exchange rate determination, foreign exchange risk premia, and international risk sharing.[2]

We offer a critique of the fundamental premise of this literature. Our main point is that the returns on currency investments, alone, cannot possibly be informative about differences between the log IMRSs of distinct agents in the two economies (expressed in common units). The intuition for our main critique is quite simple: all agents face the same relative prices for any assets and goods that they can frictionlessly trade with each other. Therefore,

[1]The SDF that is linear in a set of asset returns can be interpreted as the linear projection of an agent's IMRS down onto those returns. Thus, if a set of asset returns completely span the agent's IMRS then the SDF that is linear in those returns must equal the agent's IMRS.

[2]Examples of papers where this approach appears include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Smith and Wickens (2002); Brandt et al. (2006); Lustig and Verdelhan (2006); Brennan and Xia (2006); Lustig and Verdelhan (2007); Bakshi, Carr, and Wu (2008); Verdelhan (2010); Colacito and Croce (2011); Lustig, Roussanov, and Verdelhan (2011); Bansal and Shaliastovich (2013); and Lustig, Roussanov, and Verdelhan (2014). For an overview of this approach, see Lustig and Verdelhan (2012)'s chapter in the recent *Handbook of Exchange Rates*, entitled "Exchange Rates in a Stochastic Discount Factor Framework."

these prices alone can only be informative about the common component of agents' IMRSs (measured in the same units), not about differences.

To be clear, we do not question Eq. (1). As we highlighted at the onset, it is guaranteed to hold if there are no arbitrage opportunities. Instead, we argue that the economic content in Eq. (1) is much weaker than previous literature suggests. Central to our argument is the observation that, in general, there are not two economically distinct SDFs on the right hand side of Eq. (1). Rather, it is the same SDF simply denominated in different units. In Section 2 we show that this change of numeraire units is the analog of denominating a set of asset returns in real pounds rather than real dollars: the assets are the same, only the units that are used to denominate the returns on those assets are different. Likewise, in general, there is a single SDF on the right hand side of Eq. (1), and only the numeraire units differ.

The real U.S./U.K. exchange rate is the ratio of the values of the consumption baskets of representative agents in the U.S. and the U.K. It is an intratemporal price, not the real return on a foreign currency investment. One aspect of our critique is that much of the recent literature on real exchange rates focuses exclusively on asset markets, but is silent about the specific nature of preferences and frictions in the goods market. Of course, asset markets are important, but asset prices and the returns on currency investments don't provide the whole picture.

Our main critique is quite simple, but it applies very broadly. In Section 2 we examine the large literature that uses the asset market view as the basis for reduced-form no-arbitrage models of foreign currency returns. In Section 3 we consider structural models of real exchange rates. In Section 4 we discuss inferences that have been made about international risk sharing and the correlation of agents' IMRSs. In each case, our main message is that the economic insights afforded by the asset market view of exchange rates are much weaker than the previous literature suggests.

Although our paper is primarily critical, we offer several positive takeaways. First, reduced-form models of currency returns only need to specify an SDF denominated in a single numeraire, rather than multiple numeraire units. In Section 2 we show that models of the same SDF denominated in different units offer no additional insights, but are much less transparent, and papers that use this modeling approach often inadvertently introduce additional assumptions or even arbitrage opportunities. Modeling an SDF denominated in a single numeraire is more straightforward and exploits the same rich modeling toolset that has been developed in the broader asset pricing literature. Second, Eq. (1), expressed in terms of IMRSs instead of SDFs, offers a useful empirical diagnostic for any structural model in which markets are complete, and preferences are specified as functions of observable data. This diagnostic could be particularly beneficial for future research because it allows specific assumptions to be quickly ruled out without solving a social planning problem.[3] Third, we suggest that progress in understanding the determination of exchange rates would benefit from more work on fully specified models.

Roughly 40 years ago, a consensus emerged that exchange rate models should not ignore the influence of asset markets on exchange rates.[4] This insight was the original "asset market view," and we agree with it. Our critique, however, says that while the literature has developed in interesting directions, it has gone too far towards the notion that exchange rates can be understood *only* with reference to financial markets.

1 Notation

Throughout the paper we use the following notation. We let P_t^* be the time-t U.K. pound cost of a basket of goods purchased at U.K. market prices. Empirically, we think of this basket as the one corresponding to the construction of the U.K. consumer price index (CPI). When we discuss structural models, we think of it as a unit of the consumption basket of

[3]This diagnostic is in the spirit of Backus and Smith (1993) and Kollmann (1995).
[4]See, for example, Dornbusch (1976), Frenkel (1976), Kouri (1976), and Mussa (1976).

a representative British agent who we refer to as Bob. We let $\tilde{P}_t$ be the time-t U.S. dollar cost of a basket of goods purchased at U.S. market prices. Empirically, this basket is the one used to construct the U.S. CPI. In structural models, it is the consumption basket of a representative American agent who we refer to as Amy. In general, we drop the distinction between the empirical and model concepts by referring to the basket purchased at U.K. prices as Bob's basket, and the one purchased at U.S. prices as Amy's basket.

We let S_t be the time-t U.S. dollar per U.K. pound nominal exchange rate. Finally, because it is useful to compare the costs of the two baskets measured in common currency units, we let $P_t \equiv P_t^* S_t$ denote the U.S. dollar cost of Bob's basket (still purchased at U.K. market prices).[5] Our definition of the real U.S./U.K. exchange rate, e, is standard and compares the cost of Bob's and Amy's baskets when measured in common currency units:

$$e_t \equiv P_t/\tilde{P}_t \equiv P_t^* S_t/\tilde{P}_t \,. \tag{2}$$

The change in the real exchange rate is defined as $X_t \equiv e_t/e_{t-1}$.

2 Reduced-Form Models

In this section we consider the large literature that uses reduced-form models of two (or more) SDFs to characterize and interpret changes in the exchange rates via Eq. (1).[6] To analyze this framework, in Section 2.1 we begin by developing formal notation for asset returns and SDFs.

[5] Although our notation is somewhat unusual, it helps us maintain consistency throughout the paper. We use a superscript asterisk (*) to distinguish between numeraires. So the cost of Bob's basket in two different numeraires is either P_t (U.S. dollars) or P_t^* (U.K. pounds). We use a tilde ($\tilde{}$) to distinguish between (potentially) economically distinct objects measured in a common numeraire. So the cost of Bob's basket is P_t and the cost of Amy's basket is $\tilde{P}_t$, when both are measured in U.S. dollars.

[6] Examples of papers that pursue this modeling approach include: Bansal (1997); Backus et al. (2001); Brandt and Santa-Clara (2002); Brennan and Xia (2006); Brandt et al. (2006); Bakshi et al. (2008); Lustig et al. (2011); and Lustig et al. (2014).

Papers in this literature typically assume that there are two distinct SDFs in Eq. (1). However, in Section 2.2 we show that Eq. (1) actually characterizes the change of numeraire units for any single SDF. In other words, the two SDFs in Eq. (1) are actually the same SDF simply denominated in different units. In Sections 2.2.1 and 2.2.2 we use two examples to explicitly demonstrate that the change of numeraire for an SDF in a reduced-form model is always the flip side of the change of numeraire for the asset returns that it prices.

Many international asset pricing papers interpret an SDF denominated in a country's (real or nominal) currency as the IMRS (or discounted marginal utility growth) of a representative agent in that country or, more generally, as the projection of a representative agent's IMRS down onto the asset returns. With this interpretation, when Eq. (1) holds, changes in the exchange rate appear to offer insights into the difference between the IMRSs of agents in the two countries. In Section 2.3 we show that this logic is fundamentally flawed. In Section 2.3.1 we show that Eq. (1) does not hold for the minimum variance (i.e., linear) SDFs denominated in each currency. This result directly contradicts previous claims in many papers, including Brandt, Cochrane, and Santa-Clara (2006), Brennan and Xia (2006), Lustig and Verdelhan (2006), Alvarez, Atkeson, and Kehoe (2007), and Lustig and Verdelhan (2012). In Section 2.3.2 we discuss the implications of this result for the international asset pricing literature. In particular, we make the point that, in any equilibrium, agents must agree on the prices of goods and assets that they can frictionlessly trade with each other. Therefore, alone, these prices can only be informative about the common component of agents' IMRSs (measured in the same units), not about differences. Relatedly, in Section 2.3.3 we show that this argument always applies, regardless of whether there is a unique SDF in a particular reduced-form model.

Finally, in Section 2.4 we discuss a few well-known examples of papers that claim that Eq. (1) imposes important restrictions. In Section 2.4.1 we show that Eq. (1) does not hold for SDFs that price distinct sets of assets denominated in different currencies. In Section

2.4.2 we show that Eq. (1) does not impose distinct restrictions on an exponential affine SDF denominated in different units. In Section 2.4.3 we show that the restrictions derived in the well-known paper by Backus et al. (2001) are not in fact due to Eq. (1), but instead reflect their implicit assumption that the returns on currencies are completely spanned by the returns on long-term bonds. We also show that Brandt and Santa-Clara's model of two distinct SDFs that don't satisfy Eq. (1) is not arbitrage-free because it assigns different prices to the same zero-coupon bond.

2.1 Notation

To begin, we fix notation.

2.1.1 Asset Returns. Consider a set of k assets that can be located anywhere in the world, and assume that there are no arbitrage opportunities within these assets. Let $\boldsymbol{R}_t$ denote the vector of cum-dividend returns on these assets from time $t-1$ to t, denominated in units of Amy's consumption basket (i.e., real U.S. dollars). Then $\boldsymbol{R}_t/X_t$ are these are asset returns denominated in units of Bob's consumption basket (i.e., real U.K. pounds).

2.1.2 Stochastic Discount Factors. An SDF for these asset returns, $\boldsymbol{R}_t$, denominated in real dollars is any strictly positive (scalar) random variable, $M_t > 0$, such that

$$\mathbf{1} = \mathbb{E}\left[M_t \boldsymbol{R}_t \mid \mathcal{F}_{t-1}\right] , \tag{3a}$$

where $\mathcal{F}_{t-1}$ is the information filtration available at time $t-1$ and $\mathbf{1}$ denotes a vector of 1's. If we instead use real pounds as the numeraire, then an SDF for these asset returns, $\boldsymbol{R}_t/X_t$, is any strictly positive (scalar) random variable, $M_t X_t > 0$, such that

$$\mathbf{1} = \mathbb{E}\left[M_t X_t \boldsymbol{R}_t / X_t \mid \mathcal{F}_{t-1}\right] . \tag{3b}$$

2.2 Change of Numeraire Units for an SDF

Eqs. (3a) and (3b) are identical, since

$$M_t X_t \boldsymbol{R}_t / X_t = M_t \boldsymbol{R}_t \,. \tag{4}$$

That is, M_t is an SDF for the these asset returns denominated in real dollars, $\boldsymbol{R}_t$, if and only if $M_t X_t$ is that same SDF for the same asset returns, $\boldsymbol{R}_t / X_t$, denominated instead in real pounds. Put differently, $\boldsymbol{R}_t \to \boldsymbol{R}_t / X_t$ characterizes the change of units mapping from real dollars to real pounds for a set of asset returns, if and only if $M_t \to M_t X_t$ characterizes the corresponding change of units mapping for any SDF that prices those asset returns.

We can always write this change of numeraire units in words as

$$\underbrace{\begin{array}{c}\text{log SDF for asset}\\ \text{returns in real}\\ \text{U.K. pounds}\end{array}}_{\ln M_t X_t} = \underbrace{\begin{array}{c}\text{log SDF for asset}\\ \text{returns in real}\\ \text{U.S. dollars}\end{array}}_{\ln M_t} + \underbrace{\begin{array}{c}\text{log change in}\\ \text{real U.S./U.K.}\\ \text{exchange rate}\end{array}}_{\ln X_t} , \tag{5a}$$

or equivalently, akin to Eq. (1),

$$\underbrace{\begin{array}{c}\text{log change in}\\ \text{real U.S./U.K.}\\ \text{exchange rate}\end{array}}_{\ln X_t} = \underbrace{\begin{array}{c}\text{log SDF for asset}\\ \text{returns in real}\\ \text{U.K. pounds}\end{array}}_{\ln M_t X_t} - \underbrace{\begin{array}{c}\text{log SDF for asset}\\ \text{returns in real}\\ \text{U.S. dollars}\end{array}}_{\ln M_t} , \tag{5b}$$

Importantly, as Eq. (4) illustrates, this change of numeraire units is a tautology that always holds, and therefore it does not impose any restrictions (economic or otherwise) on exchange rates, asset returns, or SDFs that price those asset returns. The change of numeraire units in Eq. (5) also applies between real and nominal units, and between different nominal units.

To avoid repetition, we focus exclusively on the real case.

The change of numeraire in Eq. (5) for an SDF has a natural economic interpretation. An SDF effectively assigns a value to each state of the world. To illustrate, let $\pi(\omega)$ be the probability at time $t-1$ that state ω occurs at time t. A claim that pays one unit of Amy's basket in the U.S. at time t when state ω occurs (i.e., an Arrow-Debreu state contingent claim) is worth $\pi(\omega)\, M_t(\omega)$ units of her basket at time $t-1$. At time t in state ω, one unit of Bob's basket in the U.K. is worth $e_t(\omega)$ units of Amy's basket in the U.S. Therefore, a claim that pays one unit of Bob's basket in the U.K. at time t when state ω occurs is worth $\pi(\omega)\, M_t(\omega)\, e_t(\omega)$ units of Amy's basket at time $t-1$, or equivalently, $\pi(\omega)\, M_t(\omega)\, X_t(\omega)$ units of Bob's basket.

2.2.1 Example for Exponential Affine SDF. In a reduced-form model, the change of numeraire for an SDF is always the flip side of the change of numeraire for the asset returns that it prices. To emphasize this connection with an example, consider an exponential affine SDF, which is used in many papers in the asset pricing literature due to the tractability it provides. The log of an exponential affine SDF is affine in the log of the asset returns. Therefore, an exponential SDF for the asset returns denominated in real dollars is given by,

$$\ln M_t = -\boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t - c\,, \quad \text{for some } \boldsymbol{\Upsilon} \in \mathbb{R}^k \text{ such that } \boldsymbol{\Upsilon} \cdot \mathbf{1} = 1\,, \tag{6a}$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and c is such that Eq. (3) is satisfied,

$$\mathbf{1} = \mathbb{E}\big[\underbrace{e^{-c-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t}}_{M_t} \boldsymbol{R}_t \mid \mathcal{F}_{t-1}\big] \quad \Longrightarrow \quad c = \ln \tfrac{1}{k}\mathbf{1} \cdot \mathbb{E}\big[e^{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t} \boldsymbol{R}_t \mid \mathcal{F}_{t-1}\big]\,. \tag{6b}$$

If we instead denominate the same asset returns in real pounds rather than real dollars, then the same exponential affine function of those returns is an SDF for the returns, since

$\boldsymbol{\Upsilon} \cdot \mathbf{1} = 1$ implies that Eq. (6) can be written equivalently as

$$\ln M_t X_t = -\boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t / X_t - c\,, \quad \text{where} \;\; \boldsymbol{\Upsilon} \cdot \mathbf{1} = 1\,. \tag{7}$$

To emphasize, $\boldsymbol{\Upsilon}$ is the same in Eqs. (6) and (7): only the units that are used to denominate the asset returns differ. Moreover, the tautology in Eq. (5) obviously holds for this SDF since

$$\ln M_t X_t - \ln M_t = \boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t + c - \ln \boldsymbol{\Upsilon} \cdot \ln \boldsymbol{R}_t / X_t - c = \ln X_t\,, \tag{8}$$

for any $\boldsymbol{\Upsilon}$ such that $\boldsymbol{\Upsilon} \cdot \mathbf{1} = 1$.

2.2.2 Example for Long's Numeraire Portfolio.

As another concrete example, consider the SDF associated with the inverse of the return on Long's numeraire portfolio. Long (1990) showed that if there are no arbitrage opportunities within a set of asset returns, then there always exists a portfolio of those assets such that the inverse of the return on that portfolio is an SDF for the returns on those assets (see also Karatzas and Kardaras, 2007). That is, under some mild regularity conditions, there always exists an SDF of the form

$$M_t = (\boldsymbol{\theta} \cdot \boldsymbol{R}_t)^{-1}\,, \quad \text{where} \quad \boldsymbol{\theta} \cdot \mathbf{1} = 1\,. \tag{9}$$

Therefore, the inverse of the return on a portfolio denominated in real dollars, $(\boldsymbol{\theta} \cdot \boldsymbol{R}_t)^{-1} = M_t$, is an SDF for those asset returns denominated in real dollars. Likewise, if the same asset returns are instead denominated in real pounds, then the inverse of the (scalar) return on the same portfolio denominated instead in real pounds, $(\boldsymbol{\theta} \cdot \boldsymbol{R}_t / X_t)^{-1} = M_t X_t$, is that same SDF for those returns. Again, to emphasize, the portfolio weights $\boldsymbol{\theta}$ are the same: only the units that are used to denominate the returns differ. Also, the tautology in Eq. (5) obviously holds for this SDF since

$$\ln M_t X_t - \ln M_t = \ln \boldsymbol{\theta} \cdot \boldsymbol{R}_t - \ln \boldsymbol{\theta} \cdot \boldsymbol{R}_t / X_t = \ln X_t \,. \tag{10}$$

2.3 Projections

In general, there is not a unique SDF that satisfies Eq. (3). For example, if M_t is an SDF that satisfies Eq. (3) then so too is $M_t \, \xi_t$ for any random variable $\xi_t > 0$ that is independent of $M_t \boldsymbol{R}_t$ with $\mathbb{E}\left[\xi_t \mid \mathcal{F}_{t-1}\right] = 1$. Therefore,

$$\mathbb{E}\left[M_t \boldsymbol{R}_t \mid \mathcal{F}_{t-1}\right] = \mathbf{1} = \mathbb{E}\big[\tilde{M}_t \boldsymbol{R}_t \mid \mathcal{F}_{t-1}\big] \quad \not\Rightarrow \quad M_t = \tilde{M}_t \,. \tag{11}$$

However, if none of the asset returns are redundant (i.e., they are linearly independent) then there is a unique SDF that is linear in those returns (assuming that it is strictly positive almost surely). Moreover, this unique linear SDF is equal to the linear projection of any SDF down onto the asset returns. For example, if M_t and $\tilde{M}_t$ are SDFs that satisfy Eq. (3) for the asset returns denominated in real dollars, $\boldsymbol{R}_t$, then the linear projections of these SDFs down onto those returns must agree,

$$\operatorname{proj}\left[M_t \parallel \boldsymbol{R}_t\right] = \boldsymbol{\beta} \cdot \boldsymbol{R}_t = \operatorname{proj}\big[\tilde{M}_t \parallel \boldsymbol{R}_t\big] \,, \tag{12}$$

where $\boldsymbol{\beta} = \mathbf{1}\big(\mathbb{E}\big[\boldsymbol{R}_t^\top \boldsymbol{R}_t \mid \mathcal{F}_{t-1}\big]\big)^{-1} \in \mathbb{R}^k$.

2.3.1 Change of Numeraire Units for Linear SDFs. Analogous to Eq. (12), we can linearly project the same SDFs, $M_t X_t$ and $\tilde{M}_t X_t$, down onto the same asset returns denominated instead in real pounds, $\boldsymbol{R}_t / X_t$,

$$\operatorname{proj}\left[M_t X_t \parallel \boldsymbol{R}_t / X_t\right] = \boldsymbol{\beta}^* \cdot \boldsymbol{R}_t / X_t = \operatorname{proj}\big[\tilde{M}_t X_t \parallel \boldsymbol{R}_t / X_t\big] \,, \tag{13}$$

where $\boldsymbol{\beta}^* = \mathbf{1}\big(\mathbb{E}\big[\boldsymbol{R}_t^\top \boldsymbol{R}_t X_t^{-2} \mid \mathcal{F}_{t-1}\big]\big)^{-1} \in \mathbb{R}^k$. In general, the linear SDFs in Eqs. (12) and (13) are not the same. To see this result, we first need to express these SDFs in common units for comparison. If $\boldsymbol{\beta} \cdot \boldsymbol{R}_t$ in Eq. (12) is an SDF for the asset returns denominated in real dollars, then $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$ is that same SDF for the same asset returns denominated instead in real pounds. However, $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$ is not linear in the asset returns denominated in real pounds, $\boldsymbol{R}_t / X_t$, but instead it is linear in $\boldsymbol{R}_t X_t$. Thus, in general,

$$\boldsymbol{\beta}^* \cdot \boldsymbol{R}_t / X_t \;\neq\; \boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t \,, \tag{14a}$$

or equivalently,

$$\ln X_t \;\neq\; \ln \boldsymbol{\beta}^* \cdot \boldsymbol{R}_t / X_t \;-\; \ln \boldsymbol{\beta} \cdot \boldsymbol{R}_t \,. \tag{14b}$$

Eq. (14) is an example of a much broader point that we make in this section: in general, Eq. (1) does not hold for separately identified SDFs.

The proof of the inequality in Eq. (14) is quite simple. If there are $n > k$ states of the world next period, then for any $\boldsymbol{\beta} \in \mathbb{R}^k$, equality in Eq. (14) would represent n equations with $k < n$ unknowns (i.e., $\boldsymbol{\beta}^* \in \mathbb{R}^k$), which, in general, does not have a solution.[7]

2.3.2 Inferences About Different Agents. The inequality in Eq. (14) has important implications for the international finance literature because many papers incorrectly claim that it always holds with equality (e.g., see Brandt et al., 2006, p. 675; Brennan and Xia, 2006, p. 759; Lustig and Verdelhan, 2006, p. 648; Alvarez et al., 2007, p. 342; and Lustig and Verdelhan, 2012, p. 395).[8]

[7]More formally, assume that there are $n > k$ states of the world at time t, indexed by $\omega = 1, \ldots, n$. For any $\boldsymbol{\beta} \in \mathbb{R}^k$, in general,

$$\nexists\, \boldsymbol{\beta}^* \in \mathbb{R}^k \quad \text{such that} \quad \boldsymbol{\beta}^* \cdot \boldsymbol{R}_t(\omega) / X_t(\omega) = \boldsymbol{\beta} \cdot \boldsymbol{R}_t(\omega)\, X_t(\omega)\,, \quad \forall\, \omega = 1, \ldots, n\,.$$

[8]For example, Brandt et al. (2006, p. 675) states that:

> These discount factors are the projections of any possible domestic and foreign discount factors

The reason for the significance of the inequality in Eq. (14) is as follows. It is well-known that the IMRS of any maximizing agent who can frictionlessly trade a set of assets is an SDF for the returns on those assets.[9] For example, consider again the U.K. and U.S. representative agents Bob and Amy. Suppose that M_t^* is Bob's IMRS over his consumption basket in the U.K. (i.e., real pounds). To be notationally consistent with Eq. (13), define $M_t \equiv M_t^*/X_t$, which is Bob's IMRS, indirectly expressed using Amy's basket as the numeraire (and note that $M_t^* \equiv M_t X_t$). Then $\text{proj}\left[M_t X_t \parallel \boldsymbol{R}_t/X_t\right] = \boldsymbol{\beta}^* \cdot \boldsymbol{R}_t/X_t$ in Eq. (13) can be economically interpreted as the part of Bob's IMRS that is in the linear span of these particular asset returns denominated in real pounds. Likewise, suppose that $\tilde{M}_t$ is Amy's IMRS over her consumption basket in the U.S. (i.e., real dollars). Then $\text{proj}\left[\tilde{M}_t \parallel \boldsymbol{R}_t\right] = \boldsymbol{\beta} \cdot \boldsymbol{R}_t$ in Eq. (12) can be economically interpreted as the part of Amy's IMRS that is in the linear span of the returns on these particular assets denominated in real dollars. Thus, if Eq. (14) always holds with equality (which it does not), then the change in the real exchange rate would appear to offer model-free insights into the differences between the IMRSs of agents in the two countries. Indeed, the papers referenced above make this exact claim.

As an example of how Eq. (14) is used in the literature, Alvarez et al. (2007) claim that it always holds with equality and they use it to argue that changes in nominal interest rates by central banks must have a large impact on the difference between the conditional variances of representative agents' IMRSs. To be clear, we take no stand on this claim. Rather, our point is that Eq. (14) cannot be used as the basis for model-free insights into the differences between IMRSs of agents in two countries.

As another example, Brandt et al. (2006) also incorrectly claim that Eq. (14) always holds with equality, and they use it to draw model-free inferences about the correlation of the marginal utility growths of agents in different countries. On page 672 they state that:

onto the relevant spaces of asset payoffs, and they are also the minimum-variance discount factors. We show that Eq. (1) continues to hold with this particular choice of discount factors.

[9] We elaborate on this point in Section 3.

> Hansen and Jagannathan showed that marginal utility growths must be highly volatile in order to explain the equity premium. We show that marginal utility growths must also be highly correlated across countries in order to explain the relative smoothness of exchange rates.

This conclusion—described by Brandt et al. (2006) as logically inescapable—has motivated a new literature that seeks models in which IMRSs are both volatile and highly correlated (e.g., see Colacito and Croce, 2011 and Stathopoulos, 2011). For example, from Colacito and Croce (2011, p. 154):

> We like to view this as an *international equity premium puzzle*. In a one-country model, consumption growth does not vary enough to explain the excess return over the risk-free rate. In a two-country model, consumption growth does not *covary* enough to track movements in the exchange rate and returns. This dichotomy of prices and quantities strikes us as an important unresolved puzzle in international finance.

Again, to be clear, there is nothing inherently wrong with models in which IMRSs are highly correlated. Rather, our point is that these models do not solve a puzzle: it is impossible to know from price data alone whether agents' IMRSs are in fact highly correlated.

Moreover, even if Eq. (14) did hold with equality, it still would not offer model-free insights into the differences between the IMRSs of agents in the two countries. The reason is simple. It is impossible to distinguish between different agents' IMRSs using only the returns on assets, including currencies, that they can frictionlessly trade with each other: as the projections in Eqs. (12) and (13) illustrate, agents are observationally equivalent along that dimension. Moreover, if we only observe the returns on these particular assets then we cannot say anything about the elements of Bob's and Amy's IMRSs that are orthogonal to (i.e., not in the linear span of) those returns. Instead, the returns on assets that Bob and

Amy can frictionlessly trade with each other are only informative, alone, about the common component of their IMRSs (measured in the same units), not about differences.

2.3.3 Complete Markets in Reduced-Form Models. There are two illustrative cases in which Eq. (14) holds with equality. First, if the change in the exchange rate is deterministic (or, more generally, the squared change in the exchange rate is deterministic). Second, if the exchange rate and k asset returns only vary over k distinct events. In that case, there is a unique SDF within the space of SDFs that only vary over the same events as the returns on these particular assets.[10] Since the SDFs in Eqs. (6), (9), (12), and (13) all only vary over the same events as the asset returns that they are constructed from, they must all be equal if there is a unique SDF in that space. In the no-arbitrage asset pricing literature, this second case is typically referred to as a complete market. It implies that any contingent claim on these particular assets must have a unique no-arbitrage price because it can be exactly replicated by trading in those assets.

Many papers in international asset pricing that develop reduced-form models assume that there is a unique SDF. When that unique SDF is denominated in a country's (real or nominal) currency, they economically interpret it as the IMRS of a representative agent in that country. However, regardless of whether there is a unique SDF in a particular reduced-form model, the argument in Section 2.3.2 still applies: alone, the returns on assets that agents can frictionlessly trade with each other can only be informative about the common component of their IMRSs, not about differences.

Moreover, it is important to recognize that even if these k assets only vary over k distinct events next period, Amy's and Bob's IMRSs can still vary over more than those k events. In other words, a complete market in a reduced-form model (e.g., k asset returns that only vary over k distinct events next period) does not imply that the returns on those particular

[10]More formally, there is a unique SDF within the space of SDFs that are adapted to the minimum filtration generated by the returns on these particular assets.

assets completely span agents' IMRSs. Of course, one is always free to make that additional assumption; but it is an assumption, not an implication. Moreover, it is an assumption that cannot be fully tested in a reduced-form model without any agents. Nevertheless, one testable implication of this assumption is that the expected return on any asset, including those outside the model, only depends on its covariance with the returns on these k assets. However, to the best of our knowledge, we are not currently aware of any reduced-form model of a set of assets that empirically satisfies this criteria.

Perhaps the best known example of a reduced-form model with a complete market is a binomial tree model for the return on a single asset and a default-free bond (e.g., see Cox, Ross, and Rubinstein, 1979). In that simple case, there is a unique SDF (or risk-neutral measure) within the space of SDFs that only vary over two distinct events in the model. However, although this simple model is useful for valuing contingent claims on that particular asset, it does not imply that agents' IMRSs also only vary over those two events. Another well-known example of a reduced-form model with a complete market is the Black-Scholes-Merton model (e.g., see Black and Scholes, 1973 and Merton, 1973), which is the continuous-time limit of the binomial tree model.

Brandt et al. (2006) show that Eq. (14) holds in a continuous-time diffusion setting that is similar to Black and Scholes (1973) and Merton (1973), which they claim is proof that Eq. (14) always holds with equality, even if the asset market is incomplete so that there is not a unique SDF. However, they fail to recognize that their continuous-time diffusion setting is actually a complete market (e.g., see Harrison and Pliska, 1981, 1983), and therefore it cannot serve as the basis for a proof that equality in Eq. (14) holds more generally in incomplete markets. Moreover, to be clear, the inequality in Eq. (14) persists, in general, even in continuous time.[11] Regardless of whether time is discrete or continuous, the fundamental reason for the inequality in Eq. (14) is that $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$ is not, in general, in the linear space of

[11] Jarrow and Madan (1995, 1999) highlight that SDFs are not unique in continuous-time models with a finite number of a securities and jumps that have a continuous distribution.

the asset returns, $\boldsymbol{R}_t/X_t$, denominated in real pounds.[12]

2.4 Restrictions Imposed by a Change of Numeraire

Many papers in the international asset pricing literature model two (or more) reduced-form SDFs that are assumed to satisfy Eq. (1).[13] In Section 2.2 we showed that Eq. (1) actually characterizes the change of numeraire for a single SDF. Therefore, these papers do not actually model different SDFs, but instead they model the same SDF simply denominated in two different units (e.g., dollars and pounds). There is nothing technically wrong with this modeling approach: it is always equivalent to model, X_t and M_t, or X_t and M_tX_t, or M_t and M_tX_t. However, as we illustrated in Section 2.2, it is important recognize that the change of numeraire units in Eq. (1) does not impose any restrictions (economic or otherwise) on an SDF in a reduced-form model. In other words, Eq. (1) is a tautology in a reduced-form model, not a testable restriction. Nevertheless, many papers argue that Eq. (1) imposes important restrictions. For example, in Bakshi et al. (2008, p. 133) they state that:

> In particular, because the ratio of the stochastic discount factors in two economies governs the exchange rate between them, the exchange rate market offers a direct information source for assessing the relative risk-taking behavior of investors in international economies.

Below we discuss examples in some well-known papers in this literature. For clarity, we consider simple cases with bank accounts in two currencies. A more general treatment is provided in Appendix A.

[12]Brandt et al. (2006, p. 675) incorrectly claim that $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$ is always in the linear space of the asset returns denominated in real pounds, $\boldsymbol{R}_t/X_t$. Instead, it is $\boldsymbol{\beta} \cdot \boldsymbol{R}_t/X_t$, not $\boldsymbol{\beta} \cdot \boldsymbol{R}_t X_t$, that is always in this linear space.

[13]Examples of papers that pursue this modeling approach include: Bansal (1997); Backus et al. (2001); Brandt and Santa-Clara (2002); Brennan and Xia (2006); Brandt et al. (2006); Bakshi et al. (2008); Lustig et al. (2011); and Lustig et al. (2014).

2.4.1 Different Assets. Suppose that we partition the set of assets into two distinct sets. Let $\tilde{M}_t$ be an SDF that prices the returns on the first subset of assets, $\boldsymbol{R}_{1,t}$, denominated in real dollars. Similarly, let $M_t X_t$ be an SDF that prices the returns on the second subset of assets, $\boldsymbol{R}_{2,t}/X_t$, denominated in real pounds (so that M_t prices those same asset returns, $\boldsymbol{R}_{2,t}$, denominated in real dollars). In general, $M_t \neq \tilde{M}_t$ since they price different sets of assets, and therefore

$$\ln X_t \neq \ln M_t X_t - \ln \tilde{M}_t \,. \tag{15}$$

Brennan and Xia (2006) argue that Eq. (15) should hold with equality for an SDF, $M_t X_t$, that prices one set of bond returns denominated in pounds, and an SDF, $\tilde{M}_t$, that prices a different set of bond returns denominated in dollars.

To illustrate the inequality in Eq. (15) with a simple example, suppose that there are two assets (i.e., $k = 2$), which are one-period default-free bank accounts denominated in real dollars and pounds, that pay continuously-compounded interest rates r and r^*, respectively, from time $t-1$ to t.[14] Then

$$\boldsymbol{R}_{1,t} = e^{r} \quad \text{and} \quad \boldsymbol{R}_{2,t}/X_t = e^{r^*} \,. \tag{16}$$

Therefore, $\tilde{M}_t = e^{-r}$ is an SDF that only prices the return on the dollar-denominated bank account, and $M_t = e^{-r^*}/X_t$ is an SDF that only prices the return on the pound-denominated bank account. However, if the real exchange rate at time t isn't known at time $t-1$ (i.e., it is stochastic), then

$$\ln X_t \neq \ln M_t X_t - \ln \tilde{M}_t = \ln e^{-r^*} - \ln e^{-r} = r - r^* \,. \tag{17}$$

Therefore, as this simple example illustrates, theory does not imply that Eq. (15) should

[14]For notational convenience, we omit the time $t-1$ subscript on the interest rates, so that $r \equiv r_{t-1}$ and $r^* \equiv r^*_{t-1}$.

hold with equality for SDFs that price different sets of assets.

2.4.2 Exponential Affine Structure. Many papers in the international asset pricing literature work within an exponential affine structure because of the tractability it provides (e.g., see Backus et al., 2001; Brennan and Xia, 2006; and Lustig et al., 2011). In Section 2.2.1 we showed that the change of units for an exponential affine SDF is also exponential affine with exactly the same parameters. Nevertheless, many papers in this literature suggest that exchange rate dynamics impose distinct restrictions on exponential affine models of M_t and $M_t X_t$. For example, from Lustig et al. (2011, p. 26):

> We derive new restrictions on the stochastic discount factors (at home and abroad) that need to be satisfied in order to reproduce the carry trade risk premium that we have documented in the data.

In this section we explicitly demonstrate that an exponential affine structure does not impose distinct restrictions on M_t and $M_t X_t$.

To illustrate our point with an example, consider again the simple setup in Section 2.4.1 with two assets (i.e., $k = 2$) that are one-period default-free bank accounts denominated in real dollars and pounds, that pay continuously-compounded interest rates r and r^*, respectively. In that case, the log asset returns are

$$\ln \boldsymbol{R}_t = [r\,,\ r^* + \ln X_t]\ , \tag{18a}$$

or equivalently,

$$\ln \boldsymbol{R}_t / X_t = [r - \ln X_t\,,\ r^*]\ . \tag{18b}$$

Assume that the change in the real U.S./U.K. exchange rate is lognormal, so that

$$\ln X_t + r^* - r = \mu + \nu\, \varepsilon_t\,, \quad \text{where} \quad \varepsilon_t \sim \mathcal{N}(0, 1)\ , \tag{19}$$

and $|\nu| > 0$. Much of the international asset pricing literature focuses on the drift of exchange rates. For example, one could set $\mu = \alpha + \beta\big(r - r^*\big)$ in order to incorporate the forward premium anomaly for this exchange rate (or, more generally, it is straightforward to incorporate any dependence on interest rates). We suppress any time $t-1$ subscripts on the drift and volatility of the exchange rate purely for ease of exposition below.

Given the default-free interest rates, r and r^*, and the exchange rate dynamics in Eq. (18), we can construct the SDFs in Eqs. (6), (9), (12), and (13). For this example with lognormal asset returns, the exponential affine SDF in Eq. (6) is particularly convenient because $\boldsymbol{\Upsilon}$ and c can be characterized in closed form as

$$\boldsymbol{\Upsilon} = \left[\tfrac{1}{2} - \tfrac{\mu}{\nu^2}\,,\ \tfrac{1}{2} + \tfrac{\mu}{\nu^2}\right] \quad \text{and} \quad c = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{\mu}{\nu^2}\right)\nu^2\left(\tfrac{1}{2} - \tfrac{\mu}{\nu^2}\right). \tag{20}$$

For notational convenience, define

$$\gamma = \tfrac{1}{2} + \tfrac{\mu}{\nu^2}\,, \quad \text{or equivalently,} \quad 1 - \gamma = \tfrac{1}{2} - \tfrac{\mu}{\nu^2}\,. \tag{21}$$

Then the exponential affine SDF for this example is given by

$$\ln M_t \;=\; \underbrace{-\,(1-\gamma)\,r - \gamma\,[r^* + \ln X_t]}_{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t} \;-\; \underbrace{\tfrac{1}{2}\,\gamma\,\nu^2\,(1-\gamma)}_{c}\,, \tag{22a}$$

or equivalently,

$$\ln M_t X_t \;=\; \underbrace{-\,(1-\gamma)\,[r - \ln X_t] - \gamma\,r^*}_{-\boldsymbol{\Upsilon}\cdot\ln \boldsymbol{R}_t/X_t} \;-\; \underbrace{\tfrac{1}{2}\,\gamma\,\nu^2\,(1-\gamma)}_{c}\,, \tag{22b}$$

Much of the international asset pricing literature models the simple lognormal change in the real exchange rate in Eq. (19) by instead parameterizing M_t and $M_t X_t$ in Eq. (22). To illustrate this approach, define

$$\lambda = \tfrac{\mu}{\nu} + \tfrac{1}{2}\nu \quad \text{and} \quad \lambda^* = \tfrac{\mu}{\nu} - \tfrac{1}{2}\nu \,, \tag{23}$$

so that $\lambda = \nu\,\gamma$ and $\lambda^* = -\nu\,(1-\gamma)$. With these alternative labels, Eq. (22) simplifies to

$$\ln M_t = -r - \lambda\,\varepsilon_t - \tfrac{1}{2}\lambda^2 \,, \tag{24a}$$

or equivalently,

$$\ln M_t X_t = -r^* - \lambda^* \varepsilon_t - \tfrac{1}{2}\left(\lambda^*\right)^2 \,, \tag{24b}$$

and the lognormal exchange rate dynamics in Eq. (19) can be written as

$$\ln X_t + r^* - r = \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\left(\lambda^*\right)^2 + \left(\lambda - \lambda^*\right)\varepsilon_t \,, \tag{25}$$

where, again, $\varepsilon_t \sim \mathcal{N}\left(0,1\right)$ according to Eq. (19).

At first glance, it might appear that this alternative parameterization in Eq. (24) does in fact impose separate restrictions on M_t and $M_t X_t$ because they depend on different parameters. However, as Eq. (23) illustrates, this alternative approach just relabels parameters: λ and λ^* in Eq. (24) are simply different ways to specify the mean and volatility of the change in the exchange rate.

To be clear, our point here is not that the change in the real exchange rate in Eq. (19) does not impose any restrictions on an SDF: the weights, $\gamma = \frac{1}{2} + \frac{\mu}{\nu^2}$ and $1 - \gamma = \frac{1}{2} - \frac{\mu}{\nu^2}$, in Eq. (22) are functions of both the drift and volatility, μ and ν, of the change in the exchange rate.[15] Rather, our point is that Eq. (19) does not impose separate restrictions because the

[15]Many papers in the international asset pricing literature document the first moment of the return on a currency investment strategy and then construct an SDF that is consistent with that moment. However, the first moment (or the higher order moments) of the change in the exchange rate, alone, is not sufficient to completely characterize an SDF. For example, the drift and volatility, μ and ν, are both necessary to characterize the SDF in Eq. (22). Indeed, perhaps the main purpose of an SDF is to help shed light on the

logs of both M_t and $M_t X_t$ in Eq. (22) are exactly the same affine function of the log asset returns in Eq. (18): only the numeraire units that are used to the denominate those asset returns differs. Put differently, M_t and $M_t X_t$ are the same SDF denominated in different units, so the same asset returns (including currency returns) can't possibly impose separate restrictions on a single SDF.

2.4.3 Spanning of Exchange Rates with Other Assets. In a well-known paper, Backus et al. (2001) "characterize the (forward premium) anomaly in the context of affine models of the term structure of interest rates." On page 280 of Backus et al. (2001) they state that:

> We formulate our models as discrete time processes for currency-specific pricing kernels—essentially, processes for prices of state-contingent claims—and translate Fama's (1984) conditions for risk premiums into restrictions on pricing kernels. We show that these restrictions have strong implicates for the structure and parameter values of affine models, and then consider several specific examples. We find, based on both theory and estimation, that affine models have difficulty accounting for the anomaly.

In Section 2.4.2 we explicitly demonstrated that one can always construct an exponential affine SDF from a lognormal exchange rate, even if the forward premium puzzle holds.[16] Backus et al. (2001) assume that the change in the exchange rate is lognormally distributed, so how exactly do they find "that affine models have difficulty accounting for the anomaly"?

The answer is simple: Backus et al. (2001) formulate an affine model of M_t and $M_t X_t$, but in the process of doing so, they implicitly assume that shocks to exchange rates are completely spanned by shocks to interest rates. More specifically, in the context of our examples above,

relationship between the first moment and the higher order moments of asset returns.

[16] More general, given any set of lognormal asset returns, one can always construct (i.e., reverse engineer) an exponential affine SDF from the returns themselves.

they assume that changes in the dollar and pound default-free interest rates, r and r^*, are driven by two correlated normal shocks. However, they also assume that M_t and $M_t X_t$ only depend on these same two shocks, and therefore they implicitly assume that the change in the exchange rate, $X_t = M_t X_t / M_t$, is driven by exactly the same two shocks as the interest rates. Many other papers in this literature also make this same implicit assumption (e.g., see Brandt and Santa-Clara, 2002; Brennan and Xia, 2006; Backus, Gavazzoni, Telmer, and Zin, 2010; Lustig, Roussanov, and Verdelhan, 2011; and Gavazzoni, Sambalaibat, and Telmer, 2013).

It is straightforward to understand the source of the restrictions that Backus et al. (2001) derive without referring to the specific details of their model. Any dynamic no-arbitrage model of the term structure of interest rates in two currencies must have at least four assets: short- and long-term bonds in both currencies. Therefore, there are at least three prices to consider once we choose one of these assets as the numeraire (e.g., a default-free dollar or pound bank account). However, if we assume that those three prices are driven by only two shocks, then one of the three must be redundant, since it can be replicated by a combination of the other two. It is this redundancy, not Eq. (1) or the forward premium anomaly, that is in fact the primary source of the restrictions derived in Backus et al. (2001) and other papers referenced above.[17] Put differently, if none of these assets are redundant then the forward premium anomaly is entirely consistent with an exponential affine SDF.

To be clear, the assumption that changes in exchange rates are completely spanned by the returns on other assets such as bonds, does not violate a fundamental economic principal (such as the absence of arbitrage). Therefore, a priori, it is not necessarily wrong to make

[17]Alone, the absence of arbitrage does not provide strong restrictions on the joint distribution of currency returns with the returns on other assets. Instead, no-arbitrage only restricts the joint distribution of asset returns that are exposed to a common (small) set of risks, or shocks. For example, no-arbitrage models have been particularly fruitful for understanding the relationship between the prices of options with different strikes and maturities, as well as the term structure of yields on bonds with different maturities (e.g., see Black and Scholes, 1973; Merton, 1973; and Vasicek, 1977). In both of these cases, the absence of arbitrage is useful for understanding the relative prices of many different securities that are jointly driven by a smaller set of underlying risks, or shocks.

this assumption. Our point is simply that it is this assumption, not Eq. (1) or the forward premium anomaly, that is the primary source of any restrictions that are derived. Ultimately, it is an empirical question whether currency returns are completely spanned by the returns on other assets such as bonds in the two currencies. If they are, then it can be useful to derive no-arbitrage restrictions implied by this spanning. However, if they're not, then there is no reason to expect that these no arbitrage restrictions should hold in the data either (though they might).

To date, the existing empirical evidence suggests that the returns on currency investments are not completely spanned by the returns on other assets. Brandt and Santa-Clara (2002) provide empirical evidence that currency returns are not well-spanned by bond returns. In their empirical section, Lustig et al. (2011) argue that a separate currency factor is necessary to understand that the cross-section of returns on portfolios of currencies.[18] Burnside (2012) shows that factors that price the cross-section of equity returns do not price the cross-section of currency returns. Of course, all of this existing empirical evidence does not imply that there cannot be some, as yet undiscovered, set of asset returns that completely span currency returns. However, one of the major puzzles in the economics of exchange rates is that, empirically, time-series variation in exchange rates is not tightly related to contemporaneous time-series variation in any other variables. Indeed, for this reason, currencies are often considered to be a separate asset class.

Finally, as noted above, Brandt and Santa-Clara (2002) provide empirical evidence that currency returns are not well-spanned by bond returns. They interpret their empirical results as evidence that Eq. (1) must not be satisfied. To capture this feature, they model two

[18]Lustig et al. (2011) provide empirical evidence that equity market volatility has some explanatory power for the cross-section of currency returns. However, in a horse race they find that their currency-specific factor drives out the equity volatility factor.

distinct SDFs, M_t^* and $\tilde{M}_t$, for the returns on the same assets. They assume that

$$X_t = \frac{M_t^*}{\tilde{M}_t} O_t \,, \tag{26}$$

where O_t is independent of M_t^*, $\tilde{M}_t$, and all assets, and $\mathbb{E}\left[O_t \mid \mathcal{F}_{t-1}\right] = 1$.[19]

The model in Brandt and Santa-Clara (2002) violates no arbitrage. To illustrate, note that M_t^* and $\tilde{M}_t$ must both price a default-free dollar bank account, so that

$$\mathbb{E}\big[M_t^* X_t^{-1} e^r \mid \mathcal{F}_{t-1}\big] = 1 = \mathbb{E}\big[\tilde{M}_t e^r \mid \mathcal{F}_{t-1}\big] \,. \tag{27}$$

However, Eq. (26) together with the assumptions below it are inconsistent with Eq. (27) since, by Jensen's inequality,

$$e^{-r} = \mathbb{E}\big[M_t^* X_t^{-1} \mid \mathcal{F}_{t-1}\big] = \mathbb{E}\big[\tilde{M}_t O_t^{-1} \mid \mathcal{F}_{t-1}\big] \,, \tag{28a}$$

$$= \mathbb{E}\big[\tilde{M}_t \mid \mathcal{F}_{t-1}\big] \mathbb{E}\big[O_t^{-1} \mid \mathcal{F}_{t-1}\big] \,, \tag{28b}$$

$$> \mathbb{E}\big[\tilde{M}_t \mid \mathcal{F}_{t-1}\big] / \mathbb{E}\big[O_t \mid \mathcal{F}_{t-1}\big] \,, \tag{28c}$$

$$= \mathbb{E}\big[\tilde{M}_t \mid \mathcal{F}_{t-1}\big] = e^{-r} \,. \tag{28d}$$

In other words, the model in Brandt and Santa-Clara (2002) is not free of arbitrage opportunities (i.e., it is not internally consistent), since it assigns two different returns to the same dollar-denominated default-free bank account.[20]

[19] See Eq. (24) in Brandt and Santa-Clara (2002, p. 176). They state that "the key insight of our model is that when markets are incomplete, the volatility of the exchange rate is not uniquely determined by the domestic and foreign stochastic discount factors. ... If markets are incomplete, the volatility of the exchange rate can contain an element that is orthogonal to the priced sources of risk in both countries. ... To capture this excess volatility, we specify a stochastic process for the degree of market incompleteness."

[20] Similarly, Anderson, Hammond, and Ramezani (2010) show that, in the special case of an affine setting, the assumptions in Brandt and Santa-Clara (2002) are infeasible. Eq. (26) illustrates that the internal inconsistency (i.e., the arbitrage opportunity) applies more generally, beyond the specific affine structure.

3 Structural Models

In Section 2.3.2 we highlighted that in a reduced-form asset pricing model it is impossible to say anything about differences between agents' IMRSs (measured in the same units). Models with more structure, possibly combined with data on quantities, have the potential to allow us to say something about different agents.

As in Section 2.3.2, let $\tilde{M}_t$ be Amy's IMRS, defined over units of her basket, and let M_t^* be Bob's IMRS, defined over units of his basket, between times $t-1$ and t. Because it is useful to compare IMRSs over common units, we again use $M_t \equiv M_t^*/X_t$ to denote Bob's IMRS, indirectly expressed using Amy's basket as the numeraire.

Some models assume or imply that Bob and Amy always equate IMRSs (direct or indirect) measured in units of individual goods or currencies that they can frictionlessly trade. In this case, we have

$$M_t = \tilde{M}_t \,, \qquad \text{or equivalently,} \qquad M_t^* = X_t \tilde{M}_t \,. \tag{29}$$

What modeling assumptions lead to this result? One might assume that there is a complete set of state contingent claims denominated in a numeraire that Bob and Amy can frictionlessly trade. In standard models, as long as Bob and Amy can also frictionlessly trade at least one consumption good, Eq. (29) emerges from the equilibrium conditions of the model under this assumption of complete asset markets. Alternatively, there might not be complete state contingent claims, but the existing asset markets might nonetheless span the variation in M_t and $\tilde{M}_t$ in equilibrium. A trivial example would be an endowment economy, with no asset markets, but perfect correlation of the agents' endowments.

Eq. (29), or the assumptions that lead to it, can be useful in a variety of ways. First, it acts as a simple diagnostic. In many cases, M^* and $\tilde{M}$ are analytic functions of observable variables. Since X_t is also observable, one can test whether Eq. (29) holds for any values of the model parameters. This simple diagnostic can be useful for ruling out specific assumptions about preferences or market completeness, without first solving the full model. Curiously, however, we are unaware of any use of this simple diagnostic other than by Backus and Smith (1993) and Kollmann (1995).

Second, when deriving the model's equilibrium mapping from state variables and shocks to prices and quantities, the complete asset markets assumption is useful because it allows the mapping to be derived from a tractable social planning problem.[21] But many researchers adopt the complete markets assumption without using it in this way. For example, Verdelhan (2010), Colacito and Croce (2011), Bansal and Shaliastovich (2013) and Gourio, Siemer, and Verdelhan (2013), model equilibrium consumption realizations in the two economies as exogenous processes, which effectively makes $\tilde{M}_t$ and M_t^* exogenous. They then assume that Eq. (29) holds and solve for the implicit change in the real exchange rate.

On the positive side, we view this approach as a diagnostic test, though a less direct one than a test of Eq. (29), for the models in these papers. In particular, if the consumption processes used to model $\tilde{M}_t$ and M_t^* are empirically realistic, and Eq. (29) is assumed to hold, then this approach can be used to ask if the implied X_t is empirically realistic.

Although we recognize the potential value of this approach for model diagnosis, we would argue that it has serious limitations. For example, because consumption processes and exchange rates are actually jointly determined in these models, this diagnostic approach is uninformative about how exchange rates are determined, and why they behave as they do. To illustrate this point, in Appendix B we outline a simple two country model with exogenous endowments of two goods. In the complete markets version of that model, with both goods being frictionlessly traded, exchange rates move in response to variation in the relative global endowments of the two goods, but only if agents have different preferences

[21] See, to name but a few examples, Cole and Obstfeld (1991), Backus, Kehoe, and Kydland (1992), Devereux, Gregory, and Smith (1992), Backus and Smith (1993), Tesar (1993), Baxter and Crucini (1995), Stockman and Tesar (1995), Backus and Crucini (2000), Kose and Yi (2006), Bodenstein (2008), Burstein, Kurz, and Tesar (2008), Stathopoulos (2011) and Colacito, Croce, Ho, and Howard (2013).

for the two goods. On the other hand, if one of the goods is nontraded, but agents have identical preferences, exchange rates move in response to differences in the growth rates of the local endowments of the nontraded good. The point is that the mechanism via which exchange rates move is entirely different in these two cases, and Eq. (29) alone is incapable of distinguishing them. Instead, one must solve the social planning problem.[22]

Additionally, in some cases, Eq. (29) is not actually an implication of the model. For example, Colacito and Croce (2011) use a two good model with complete home bias in preferences. In this setting, the real exchange rate is indeterminate, since households in the two economies consume their endowments regardless of their relative price. Verdelhan (2010) and Gourio, Siemer, and Verdelhan (2013) use single good models, but rule out the possibility of trade between the two economies. If trade is not ruled out, of course, the real exchange rate in these economies is always 1, but if it is ruled out, again, the real exchange rate is indeterminate.[23]

In many international asset pricing exercises, we are interested in understanding the expected excess returns to currency investments. For these exercises, the good news is that it actually doesn't matter whether or not Eq. (29) holds, because one is more interested in Eq. (3). When Eq. (29) fails, this could either be because the model of preferences is incorrect, or because asset markets are incomplete. If it is the latter, then $\tilde{M}_t$ and M_t^* may still correctly price the returns to currency investments measured, respectively, in units of Amy's and Bob's baskets. In other words, Eq. (3) can still hold for both M_t and $\tilde{M}_t$ even if

[22]Stathopoulos (2011) solves a complete model analogous to our first example, except that agents' preferences display habit formation over an aggregate that displays home bias. Colacito et al. (2013) solve a model of a production economy in which agents have Epstein-Zin preferences that display home bias, and the levels of technology follow processes in the spirit of the long-run risk literature. In both cases, these authors' preference specifications are guided by Hansen and Jagannathan's evidence that IMRSs must be volatile to be consistent with asset prices.

[23]The internet appendix to Verdelhan (2010) is revealing in this regard, because endowments rather than consumptions are assumed to be exogenous. In this case, proportional and quadratic trade costs are introduced to force the real exchange rate away from 1, as in Dumas (1992). When these costs are sufficiently big, the real exchange rate is pinned down by a static trade equilibrium. When they are small, the real exchange rate is, again, indeterminate, although Verdelhan sets it, arbitrarily, equal to the ratio of marginal utilities of consumption in the two economies.

Eq. (29) does not hold (which is simply a restatement of Eq. (11) in Section 2.3).

4 Risk Sharing

In this section we discuss the claim in Brandt et al. (2006) that the behavior of exchange rates reveals the degree of risk sharing across countries. Before we discuss risk sharing, we must first settle on a definition. Like Colacito and Croce (2011, p. 156), we think a natural definition of perfect risk sharing is that Amy and Bob equate IMRSs over all individual goods and services in every state of the world and at every point in time.[24] This definition of perfect risk sharing is not equivalent to asset markets being complete, nor, similarly, is it equivalent to an allocation that coincides with the solution to a social planner's problem that respects market frictions. In either of these situations, risk sharing will be as good as it can be, but frictions may prevent IMRSs over some goods (e.g., nontraded goods) being equated. Our definition, instead, means that risk sharing is perfect in a model with complete asset markets and frictionless trade in all goods.[25] Our definition also means that risk sharing is likely to be imperfect in a model with incomplete asset markets and/or goods market frictions.

To measure the extent of risk sharing, Brandt et al. (2006) consider the variance of the difference between $\ln M^*$ and $\ln \tilde{M}$, relative to the sum of the variances of $\ln M^*$ and $\ln \tilde{M}$:[26]

$$\text{RSI} = 1 - \frac{\text{var}(\ln M^* - \ln \tilde{M})}{\text{var}(\ln M^*) + \text{var}(\ln \tilde{M})}. \tag{30}$$

[24] To be consistent with the literature that we are critiquing, we focus exclusively on the notion of risk sharing in a purely theoretical sense. Essentially, we're interested in how much of the total risk can agents theoretically share. One might also be interested in the empirical question of whether agents do in fact share as much risk as theoretically possible, but we do not consider that issue.

[25] The importance of defining terminology is highlighted by the fact that ours is not universally accepted. For example, Stathopoulos (2011) defines perfect risk sharing between Amy and Bob as the equality of $\tilde{M}$ and M^*. His model assumes complete markets and frictionless trade, so we would describe it as a model with perfect risk sharing. Instead, he would describe it as a model with imperfect risk sharing arising from agents' home bias.

[26] The index is the same as the correlation between $\ln M^*$ and $\ln \tilde{M}$ when they have the same variance.

If Eq. (29) holds, then the difference between $\ln \tilde{M}$ and $\ln M^*$ measured in different units must have the same volatility as the change of units, i.e., $\text{var}(\ln \tilde{M} - \ln M^*) = \text{var}(\ln X)$. Real exchange rate data tell us that $\text{var}(\ln X)$ is on the order of 0.01 to 0.02 for major currency pairs. To measure the denominator in RSI, Brandt et al. (2006) use lower bounds on $\text{var}(\ln \tilde{M})$ and $\text{var}(\ln M^*)$ from Hansen and Jagannathan (1991), where IMRSs are projected onto vectors of asset returns. In this sense, their calculated RSI is a lower bound for the actual RSI. In practice, for projections onto risk free assets and broad portfolios of equities, the implied bounds imply that $\text{var(IMRS)} \geq 0.25$ at an annual frequency. Thus, empirically, RSI in Eq. (29) is close to 1 and Brandt et al. (2006) conclude that international risk sharing must be quite high. On page 673 of Brandt et al. (2006) they state:

> Yet the conclusion is hard to escape. Our calculation uses only price data, and no quantity data or economic modeling (utility functions, income or productivity shock processes, and so forth). A large degree of international risk sharing is an inescapable logical conclusion of Eq. (1), a reasonably high equity premium (over 1%, as we show below), and the basic economic proposition that price ratios measure marginal rates of substitution.

We take issue with the analysis in Brandt et al. (2006) on several grounds, although all of these objections center on the question of whether price data, alone, can be informative about risk sharing. First, Eq. (29) holds whenever the available asset returns completely span agents' IMRSs. However, if the available asset returns completely span Amy's and Bob's IMRSs and they face identical prices for the goods and services in their baskets, then risk sharing is perfect (since they can equate IMRSs over all goods and services). Yet the exchange rate can still vary if they have different preferences over goods, and, therefore, different consumption baskets. So, unless one assumes that Amy and Bob's consumption baskets are identical, it is impossible to infer anything about risk sharing, as we have defined it.

Second, Eq. (29) is not guaranteed to hold if the available asset returns do not completely span agents' IMRSs. Brandt et al. (2006) claim that in this case it holds for the minimum variance projections of Amy's and Bob's IMRSs down onto the space of asset returns denominated in real dollars and pounds respectively. In Section 2.3.1 we showed that this claim is incorrect. Therefore, if markets are incomplete, exchange rates alone are not informative about the difference between Amy's and Bob's IMRSs. For example, if the composition of Amy's and Bob's consumption baskets is the same and they face identical prices for the goods and services in their baskets, then the real exchange rate is always constant. Yet, risk sharing can still be arbitrarily imperfect if the available asset returns returns do not completely span agents' IMRSs (i.e., asset markets are incomplete).

Consistent with our main message in the paper, asset returns alone can provide useful information about how much risk is shared, but any measure of risk sharing also requires a measure of the amount of unshared risk. Of course, one is always free to introspect, as Brandt et al. (2006) do in Sec. 3.5 of their paper, on how much unshared risk seems "reasonable". Our view is that this introspection is merely speculative if additional data and theory are not brought into the picture. Additionally, this introspection has nothing to do with, and is not informed by, exchange rates. We know, from Hansen and Jagannathan's lower bounds, that the representative agents of Minnesota and North Carolina have highly volatile IMRSs. Brandt et al.'s introspection about incomplete markets applies equally well to these agents, but it involves imposing an *a priori* upper limit on the volatility of these IMRSs as in Cochrane and Saa-Requejo (2000).

Table 1 summarizes our critique of Brandt et al.'s inference about international risk sharing. Contrary to the premise of this paper, the interpretation of RSI in Eq. (30) as a measure of international risk sharing actually requires strong assumptions about the underlying economic model. In particular, it only holds if agents choose identical consumption baskets and the available asset returns completely span their IMRSs. In that one special case, frictions in the goods market (not the asset market) are the only source of both imperfect risk sharing and variation in the real exchange rate.

	Composition of Consumption Baskets	
Asset Markets	Identical	Different
Complete	Yes	No
Incomplete	No	No

Table 1: Does a variable real exchange rate directly reflect imperfect risk sharing?

How might one address our critique? First and foremost, an economic model is required to measure M^* and $\tilde{M}$ in Eq. (29). As we argued, in general, prices alone are not informative about the difference between these two objects. However, this exercise certainly does not require a fully-specified multi-country model. For example, it is perfectly reasonable to measure M^* and $\tilde{M}$ using a utility function for the representative agent in each country measured over aggregate consumption in that country.[27] In fact, a fully-specified multi-country model could easily be considered as over-kill if the task is simply to measure M^* and $\tilde{M}$.

Second, to consider the question of risk sharing, agent's IMRS must be measured and compared in common units. For example, RSI in Eq. (29) could be adjusted to either

$$\mathrm{RSI} = 1 - \frac{\mathrm{var}(\ln M^* - \ln \tilde{M} X)}{\mathrm{var}(\ln M^*) + \mathrm{var}(\ln \tilde{M} X)}\,, \tag{31}$$

or

$$\mathrm{RSI} = 1 - \frac{\mathrm{var}(\ln M^*/X - \ln \tilde{M})}{\mathrm{var}(\ln M^*/X) + \mathrm{var}(\ln \tilde{M})}\,. \tag{32}$$

Even still, Eqs. (31) and (32) are not ideal measures of risk sharing. For example, in Eq. (31), M^*/X is Bob's IMRS, indirectly expressed using Amy's basket as the numeraire. However,

[27]We agree with Brandt et al. (2006) that the model of the IMRS adopted for this purpose ought to be, at least, consistent with Hansen and Jagannathan's lower bounds on volatility.

it is only Bob's IMRS over units of Amy's basket in the U.K. if there are no goods market frictions so that Amy's basket has the same price in both countries. Simply put, unless Amy's and Bob's consumption baskets are identical, Eqs. (31) and (32) ignore the extent of imperfect risk sharing that is due to goods market frictions.[28] Furthermore, we feel that it is important to be cognizant of that fact that, in general, Eqs. (31) and (32) are not the same. In other words, RSI as a measure of risk sharing depends on the the numeraire that is used to measure and compare agents' IMRSs. The numerator in the fractions in Eqs. (31) and (32) does not depend on the choice of numeraire, since

$$\ln M^* - \ln \tilde{M}X \equiv \ln M^*/X - \ln \tilde{M}\,. \tag{33}$$

However, in general, the denominators in those fractions differ, and therefore so do Eqs. (31) and (32).

5 Conclusion

The log-change in the exchange rate between two economies can always be expressed as the log difference between an SDF denominated in the currency units of those economies. In a reduced-form model, this relationship merely characterizes the change of units for an SDF. In structural models with complete asset markets, the log-change of the real exchange rate is equal to the log-difference between IMRSs that may be economically distinct, but only if there are goods market frictions.

We argue that reduced-form no-arbitrage models of the exchange rate are more transparent if they specify a single SDF and a model of exchange rate dynamics. Structural modelers

[28]One could assess the extent of imperfect risk sharing due to goods market frictions by digging down to the micro level on prices and studying the construction of the CPIs in detail. Otherwise, a model of preferences in the goods market is necessary because it's impossible to tell from aggregate price indices whether differences across countries reflect preference differences (i.e., different composition of baskets) or price differences at the micro level. Models pin this decomposition down (by assumption).

who adopt the assumption of complete asset markets can use Eq. (29) as a diagnostic test that does not require a completely specified model (including the details of any goods market frictions). However, if one wishes to make further progress towards understanding how exchange rates are determined and why they vary, this diagnostic alone is not adequate: one must solve a fully specified model.

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A Section 2.4 Appendix

In Section 2.4 we discussed restrictions on reduced-form no-arbitrage model that are imposed by a change of numeraire. In that section, for simplicity, we focused on simple examples with default-free bank accounts in two currencies. Here, we extend that analysis to consider more assets.

To begin, we augment the returns vector in Eq. (18) with two additional assets that are not default-free bank accounts. They could be long-term bonds, stocks, or any other assets located anywhere in the world. Let Y_t be the dollar-denominated price of the first additional asset, and assume that it pays a continuously-compounded dividend δ^y. Similarly, let Z_t^* be the pound-denominated price of the second additional asset, and assume that it pays a continuously-compounded dividend δ^z. The vector of dollar-denominated gross return son these four assets is

$$\ln \boldsymbol{R}_t = \left[r\,,\ r^* + \Delta \ln S_t\,,\ \delta_y + \Delta \ln Y_t\,,\ \delta_z + \Delta \ln Z_t^* + \Delta \ln S_t\right]\,, \tag{34}$$

where, for notational convenience, $\Delta \ln S_t \equiv \ln S_t - \ln S_{t-1} \equiv \ln X_t$, $\Delta \ln Y_t \equiv \ln Y_t - \ln Y_{t-1}$, and $\Delta \ln Z_t^* \equiv \ln Z_t^* - \ln Z_{t-1}^*$.

A common assumption in this literature (and asset pricing in general) is that the asset returns are log-normally distributed, with

$$\left[\Delta \ln S_t\,,\ \Delta \ln Y_t\,,\ \Delta \ln Z_t^*\right] \sim \mathcal{N}\left(\boldsymbol{\mu}\,,\boldsymbol{\Omega}\right)\,. \tag{35a}$$

We'll decompose the covariance matrix as $\boldsymbol{\Omega} \equiv \boldsymbol{\Sigma}\boldsymbol{P}\boldsymbol{\Sigma}$, with

$$\boldsymbol{\Sigma} \equiv \begin{bmatrix} \sigma & & \\ & \sigma_y & \\ & & \sigma_z \end{bmatrix} \quad \text{and} \quad \boldsymbol{P} \equiv \begin{bmatrix} 1 & \rho_y & \rho_z \\ \rho_y & 1 & \rho \\ \rho_z & \rho & 1 \end{bmatrix}\,. \tag{35b}$$

To ensure that there are no arbitrage opportunities within the asset returns (as we'll discuss below), it is convenient to parameterize the mean vector, $\boldsymbol{\mu}$, as

$$\boldsymbol{\mu}^\top \equiv \begin{bmatrix} r - r^* + \varphi \ - \frac{1}{2}\sigma^2 \\ r - \delta_y + \varphi_y - \frac{1}{2}\sigma_y^2 \\ r^* - \delta_z + \varphi_z - \frac{1}{2}\sigma_z^2 - \sigma_z \sigma \rho_z \end{bmatrix} \tag{35c}$$

where

$$[\varphi\,,\ \varphi_y\,,\ \varphi_z] \equiv \boldsymbol{\Gamma}\,\boldsymbol{\Omega} \quad \text{and} \quad \boldsymbol{\Gamma} \equiv [\gamma + \gamma_z\,,\ \gamma_y\,,\ \gamma_z]\ , \tag{35d}$$

for parameters γ, γ_y, and γ_z. In Section A.1 we use this illustrative no-arbitrage model to analyze specific papers in this literature, but first we briefly discuss its relevant features.

In the general case, this simple model of log-normal asset returns has 13 free parameters. The continuously-compounded dividend yields on the four assets are described by $\{r, r^*, \delta_y, \delta_z\}$. There are three relative asset returns to consider (once we arbitrarily choose one of the assets, or a portfolio of them, as the numeraire to denominate returns). The volatility of the three asset returns are described by $\left\{\sigma, \sigma_y, \sigma_z\right\}$, which are contained within the matrix $\boldsymbol{\Sigma}$. The correlation of the three asset returns are described by $\left\{\rho_y, \rho_z, \rho\right\}$, which are contained within the matrix $\boldsymbol{P}$. Finally, given the other parameters, the mean of the three log asset returns are characterized by the parameters $\left\{\gamma, \gamma_y, \gamma_z\right\}$, which are contained within the vector $\boldsymbol{\Gamma}$.

For this example, we'll focus on the specific SDF in Eq. (6), since it is used by much of the literature that works with log-normal models of asset returns. It is straightforward to verify that $\mathbf{1} = \mathbb{E}\left[M_t \boldsymbol{R}_t \mid \mathcal{F}_{t-1}\right]$ for the SDF in Eq. (6) when

$$\boldsymbol{\Upsilon} \equiv [1 - \gamma - \gamma_y - \gamma_z\,,\ \gamma\,,\ \gamma_y\,,\ \gamma_z]\ , \tag{36}$$

for γ, γ_y, and γ_z in Eq. (35d). By the Fundamental Theorem of Asset Pricing (e.g., see Dybvig and Ross, 2003 or Dybvig and Ross, 2008), there are no arbitrage opportunities within a set of returns $\boldsymbol{R}$ if there is a strictly positive SDF, M, that satisfies Eq. (1) for those returns. Thus, the parameterization in Eq. (35) ensures that there are no arbitrage opportunities within the asset returns, $\boldsymbol{R}$, in Eq. (34). Put differently, the absence of arbitrage opportunities does not impose any restrictions on the 13 parameters, $\{r, r^*, \delta_y, \delta_z, \sigma, \sigma_y, \sigma_z, \rho_y, \rho_z, \rho, \gamma, \gamma_y, \gamma_z\}$, in Eq. (35).

It is important to recognize that, in general, the SDF in Eq. (6) is not the unique SDF consistent with the returns. For example, the SDF in Eq. (9) is also consistent with the returns, and it differs from the SDF in Eq. (6) when there is a continuous state space in discrete time with a finite number of asset returns. However, the continuous-time counterpart of this model with log-normal returns does have a unique SDF (e.g., see Harrison and Pliska, 1981, 1983). That continuous-time counterpart of Eq. (35) is given by,

$$d\ln S_t = \left(r - r^* + \varphi - \tfrac{1}{2}\sigma^2\right) dt + \sigma\, dW_t\,, \tag{37a}$$

$$d\ln Y_t = \left(r - \delta_y + \varphi_y - \tfrac{1}{2}\sigma_y^2\right) dt + \sigma_y\, dW_t^y\,, \tag{37b}$$

$$d\ln Z_t^* = \left(r^* - \delta_z + \varphi_z - \tfrac{1}{2}\sigma_z^2 - \sigma_z\sigma\rho_z\right) dt + \sigma_z\, dW_t^z\,, \tag{37c}$$

where W, W^y, and W^z are Brownian motions with correlation matrix $\boldsymbol{P}$ in Eq. (35b). The continuous-time dynamics of the unique dollar-denominated SDF for the asset returns in Eq. (37) are

$$dM_t = -M_t\left[r\, dt + \left(\gamma + \gamma_z\right)\sigma\, dW_t + \gamma_y\,\sigma_y\, dW_t^y + \gamma_z\,\sigma_z\, dW_t^z\right], \tag{38a}$$

and the continuous-time dynamics of that same unique SDF when the returns are instead denominated in pounds are

$$\begin{aligned} &d\left(M_t S_t\right) \\ &= -\left(M_t S_t\right)\left[r^* dt + \left(\gamma + \gamma_z - 1\right)\sigma\, dW_t + \gamma_y\,\sigma_y\, dW_t^y + \gamma_z\,\sigma_z\, dW_t^z\right]. \end{aligned} \tag{38b}$$

It is important to recognize that uniqueness of the SDF in this case requires both continuous-time and continuous sample paths (i.e., a continuous diffusion without any jumps).[29]

As we highlighted in Section 2.4.2, much (if not most) of the international asset pricing literature uses an alternative, but equivalent, parameterization of the log-normal asset return dynamics in Eq. (35). Rather than model the three asset returns directly, these papers instead model the returns on the two assets that are not bank accounts, together with an SDF denominated in both dollars and pounds. In Section 2.4.2 we argued that these two modeling approaches are isomorphic to each other. In particular, using Eq. (35) and the SDF in Eq. (6) we can write

$$\left[\ln M_t\,,\ \ln M_t + \Delta\ln S_t\,,\ \Delta\ln Y_t\,,\ \Delta\ln Z_t^*\right] \sim \mathcal{N}\left(\boldsymbol{\mu}_M\,, \boldsymbol{\Omega}_M\right)\,, \tag{39a}$$

[29] Jarrow and Madan (1995, 1999) highlight that SDFs are not unique in continuous-time models with a finite number of a securities and jumps that have a continuous distribution.

with

$$\boldsymbol{\mu}_M^\top \equiv \begin{bmatrix} -r - \frac{1}{2}\boldsymbol{\Gamma}\,\boldsymbol{\Omega}\,\boldsymbol{\Gamma}^\top \\ -r^* - \frac{1}{2}\boldsymbol{\Gamma}^*\,\boldsymbol{\Omega}\,\boldsymbol{\Gamma}^{*\top} \\ r - \delta_y + \varphi_y - \frac{1}{2}\sigma_y^2 \\ r^* - \delta_z + \varphi_z - \frac{1}{2}\sigma_z^2 - \sigma_z\sigma\rho_z \end{bmatrix}, \tag{39b}$$

and

$$\boldsymbol{\Omega}_M \equiv \begin{bmatrix} \boldsymbol{\Gamma}\,\boldsymbol{\Omega}\,\boldsymbol{\Gamma}^\top & \boldsymbol{\Gamma}\,\boldsymbol{\Omega}\,\boldsymbol{\Gamma}^{*\top} & -\varphi_y & -\varphi_z \\ \boldsymbol{\Gamma}^*\,\boldsymbol{\Omega}\,\boldsymbol{\Gamma}^\top & \boldsymbol{\Gamma}^*\,\boldsymbol{\Omega}\,\boldsymbol{\Gamma}^{*\top} & -\varphi_y^* & -\varphi_z^* \\ -\varphi_y & -\varphi_y^* & \sigma_y & \sigma_y\sigma_z\rho \\ -\varphi_z & -\varphi_z^* & \sigma_y\sigma_z\rho & \sigma_z^2 \end{bmatrix}, \tag{39c}$$

where, for notational convenience, we've defined the analog of Eq. (35d) as

$$\boldsymbol{\Gamma}^* \equiv [\gamma + \gamma_z - 1\,,\ \gamma_y\,,\ \gamma_z] \quad \text{and} \quad \left[\varphi^*\,,\ \varphi_y^*\,,\ \varphi_z^*\right] \equiv \boldsymbol{\Gamma}^*\,\boldsymbol{\Omega}\,. \tag{39d}$$

Note that, with these definitions,

$$\left[\varphi - \varphi^*\,,\ \varphi_y - \varphi_y^*\,,\ \varphi_z - \varphi_z^*\right] \equiv \left[\sigma^2\,,\ \sigma_y\,\sigma\rho_y\,,\ \sigma_z\,\sigma\rho_z\right]\ . \tag{40}$$

Papers that parameterize the model using Eq. (39) instead of Eq. (35) often attach different labels to the variables and parameters. For example, $M_t S_t / S_{t-1} \equiv M_t X_t$ is often labeled as M^* or . Other parameters that are often given different labels include

$$\lambda = \sigma\big(\gamma + \gamma_z\big)\,, \qquad \lambda_y = \sigma_y\,\gamma_y\,, \qquad \lambda_z = \sigma_z\,\gamma_z\,, \tag{41a}$$

$$\lambda^* = \sigma\big(\gamma + \gamma_z - 1\big) = \lambda - \sigma\,, \qquad \lambda_y^* = \lambda_y - \sigma\rho_d\,, \qquad \lambda_z^* = \lambda_z - \sigma\rho_z\,. \tag{41b}$$

These two parameterizations are exactly equivalent, since one can always recover the original parameters in Eq. (35) as

$$\sigma = \lambda - \lambda^*\,, \qquad \rho_y = \tfrac{\lambda_y - \lambda_y^*}{\lambda - \lambda^*}\,, \qquad \rho_z = \tfrac{\lambda_z - \lambda_z^*}{\lambda - \lambda^*}\,, \tag{42a}$$

$$\gamma + \gamma_z = \tfrac{\lambda}{\lambda - \lambda^*}\,, \qquad \gamma_y = \sigma_y^{-1}\lambda_y\,, \qquad \gamma_z = \sigma_z^{-1}\lambda_z\,. \tag{42b}$$

Obviously different parameterizations of the same no-arbitrage model are innocuous. However, much of this literature attaches different economic interpretations to this alternative modeling approach and parameterization. For example, even though there are not any agents in no-arbitrage models, λ, λ_y, and λ_z are often interpreted as market prices of risk that apply to domestic (U.S.) investors, while λ^*, λ_y^*, and λ_z^* are viewed as market prices of risk that apply to foreign (U.K.) investors. Moreover, many papers use language which suggests that M and $M_t S_t / S_{t-1} \equiv M_t X_t$ are different SDFs that are associated with different economies. For example, Bakshi et al. (2008) provide a no-arbitrage model using this alternative approach and on page 135 they write:

> In complete markets, the stochastic discount factor for each economy is unique. Hence, the ratio of two stochastic discount factors uniquely determines the exchange rate dynamics between the two economies.

Similarly, Brandt and Santa-Clara (2002) also provide a no-arbitrage model and on page 173 they write:

> The key insight of our model is that when markets are incomplete, the volatility of the exchange rate is not uniquely determined by the domestic and foreign stochastic discount factors.

Again, to emphasize, the model in Eq. (35) only imposes that there are no arbitrage opportunities between these four assets. There are no agents or separate economies. There is no requirement that the asset market is complete. There is no economic mechanism in the no-arbitrage model that determines the exchange rate. Eqs. (39) and (23) provide an alternative parameterization of this same no-arbitrage model, but an alternative parameterization does not alter any of these statements.

Without loss of generality, for the remainder of this section we'll work with the model formulation in Eq. (35) because our view is that it affords the most transparent analysis.

A.1 Literature Discussion

Backus et al. (2001) was one of the first papers to consider a model of the form in Eq. (35). In their setup, Y is the price of a dollar-denominated long-term bond and Z^* is the price of a pound-denominated long-term bond. They argue that the forward premium anomaly for currencies, together with the restriction of no arbitrage, has "strong implications for the structure and parameter values of affine models." The forward premium anomaly pertains to the mean of the change in the log exchange rate. In particular,

$$\varphi - \tfrac{1}{2}\sigma^2 = \alpha + \beta\big(r - r^*\big)\,. \tag{43}$$

To understand the source of the restrictions that Backus et al. (2001) derive, it is first important to recognize that if the covariance matrix, $\boldsymbol{\Omega}$, in Eq. (35) is invertible, then the absence of arbitrage does not impose any restrictions on the model. In that case, the vector of mean log asset returns, $\boldsymbol{\mu}$, could literally be anything and one can still solve for γ, γ_d, and γ_f in Eq. (35) as

$$
\begin{aligned}
&[\gamma + \gamma_z\, ,\ \gamma_y\, ,\ \gamma_z] \\
&\quad = \left(\boldsymbol{\mu} - \left[r - r^* - \tfrac{1}{2}\sigma^2\, ,\ r - \delta_y - \tfrac{1}{2}\sigma_y^2\, ,\ r^* - \delta_z - \tfrac{1}{2}\sigma_z^2 - \sigma_z\sigma\rho_z\right]\right)\boldsymbol{\Omega}^{-1}\, .
\end{aligned}
\tag{44}
$$

The intuition behind the lack of no arbitrage restrictions is straightforward. If the covariance matrix is invertible, then the three asset returns are driven by three linearly independent shocks. The absence of arbitrage only restricts the returns on assets that are exposed to the same shocks.

Thus, if the covariance matrix, $\boldsymbol{\Omega}$, in Eq. (35) is invertible (i.e., nonsingular) then the model is completely free to match the forward premium anomaly (i.e., no-arbitrage does not impose any restrictions on the model that prevent it from matching the forward premium anomaly). Backus et al. (2001) assume that the covariance matrix, $\boldsymbol{\Omega}$, is singular so that the three asset returns are driven by only two sources of uncertainty. The restrictions they derive are primarily driven by this assumption. Intuitively, a singular covariance matrix implies that the return on any of the four assets can be exactly replicated by trading in the other three (i.e., one of the four assets is redundant). For example, if the covariance matrix is singular, then the pound-denominated bank account could be exactly replicated with a portfolio of the two non-bank account assets and the dollar-denominated bank account. Therefore, the return on the pound-denominated bank account must exactly match the return on the portfolio that replicates it.

To illustrate, if

$$
1 - \rho_y^2 - \rho_z^2 - \rho^2 + 2\rho_y\rho_z\rho = 0\, ,
\tag{45}
$$

then the correlation matrix, $\boldsymbol{P}$, in Eq. (35) is singular and can be written as

$$
\boldsymbol{P} = \begin{bmatrix} \frac{\rho_y - \rho_z\rho}{1-\rho^2} & \frac{\rho_y - \rho_z\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \frac{\rho_y - \rho_z\rho}{1-\rho^2} & \frac{\rho_z - \rho_y\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{\top} .
\tag{46}
$$

Therefore, in the continuous-time limit,[30] one can replicate the pound-denominated bank

[30] Technically speaking, the covariance matrix of the log of the returns is singular, not the covariance

account using a portfolio with weights ω_y in the dollar-denominated asset (that is not a bank account), ω_z in the pound-denominated asset (that is not a bank account), and $1-\omega_y-\omega_z$ in the dollar-denominated bank account, where

$$\omega_y = \frac{\big(\rho_y - \rho_z\rho\big)\sigma\sigma_z}{\big[\big(\rho_z - \rho_y\rho\big)\sigma + \big(1-\rho^2\big)\sigma_z\big]\sigma_y}\,, \quad \text{and} \tag{47a}$$

$$\omega_z = \frac{\big(\rho_z - \rho_y\rho\big)\sigma}{\big(\rho_z - \rho_y\rho\big)\sigma + \big(1-\rho^2\big)\sigma_z}\,. \tag{47b}$$

With some algebra, one can verify that ω_y and ω_z solve the replicating problem since

$$\begin{aligned} \big[\,\omega_z\,, \quad \omega_y\,, \quad \omega_z\,\big]\boldsymbol{\Sigma} \begin{bmatrix} \frac{\rho_y-\rho_z\rho}{1-\rho^2} & \frac{\rho_y-\rho_z\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ = \big[\,1\,, \quad 0\,, \quad 0\,\big]\boldsymbol{\Sigma} \begin{bmatrix} \frac{\rho_y-\rho_z\rho}{1-\rho^2} & \frac{\rho_z-\rho_y\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & . \end{aligned} \tag{48}$$

The model in Eq. (37), which is the continuous-time limit of the model in Eq. (35), naturally incorporates this no-arbitrage restriction. In particular, Eq. (48) implies that

$$[\omega_z\,,\ \omega_y\,,\ \omega_z]\,\boldsymbol{\Omega} = [1\,,\ 0\,,\ 0]\,\boldsymbol{\Omega}\,, \tag{49}$$

and therefore

$$\mathbb{E}\left[(1-\omega_y-\omega_z)\,r\,dt + \omega_y\left(\frac{dY_t}{Y_t} + \delta_y\,dt\right) + \omega_z\left(\frac{d\,(Z_t^* S_t)}{(Z_t^* S_t)} + \delta_z\,dt\right) \mid \mathcal{F}_t\right]\,,$$

$$= \Big(r + \big[\omega_z\,,\ \omega_y\,,\ \omega_z\big]\boldsymbol{\Omega}\boldsymbol{\Gamma}^\top\Big)dt\,, \tag{50a}$$

$$= \Big(r + \big[[1\,,\ 0\,,\ 0]\big]\boldsymbol{\Omega}\boldsymbol{\Gamma}^\top\Big)dt = \mathbb{E}\left[\frac{dS_t}{S_t} + r^*\,dt \mid \mathcal{F}_t\right]\,. \tag{50b}$$

That is, the return on the pound-denominated bank account exactly matches the return

matrix of the gross returns. Therefore, there is only exact replication in the continuous-time limit of the model (i.e., as $\Delta t \to 0$).

on the portfolio that replicates it. Equivalently, the singular covariance matrix in Eq. (46) implies that one of the elements (or a linear combination of the elements) in $\mathbf{\Gamma}$ is redundant and can be set to zero. For example, if Eq. (46) holds then the first element of $\mathbf{\Gamma}$ could be set to zero, since

$$\begin{aligned}
&\begin{bmatrix} \frac{\rho_y-\rho_z\rho}{1-\rho^2} & 1 & 0 \\ \frac{\rho_z-\rho_y\rho}{1-\rho^2} & 0 & 1 \end{bmatrix} \mathbf{\Sigma}\mathbf{\Gamma}^\top \\
&\qquad = \begin{bmatrix} \frac{\rho_y-\rho_z\rho}{1-\rho^2} & 1 & 0 \\ \frac{\rho_z-\rho_y\rho}{1-\rho^2} & 0 & 1 \end{bmatrix} \mathbf{\Sigma} \begin{bmatrix} 0 \\ \frac{(\rho_y-\rho_z\rho)\sigma}{(1-\rho^2)\sigma_y}(\gamma+\gamma_z)+\gamma_y \\ \frac{(\rho_z-\rho_y\rho)\sigma}{(1-\rho^2)\sigma_z}(\gamma+\gamma_z)+\gamma_y \end{bmatrix}.
\end{aligned} \tag{51}$$

Eqs. (46) and (51) effectively reduce the number of free parameters in the general model of Eq. (37) from 13 down to 11.

There are a couple of points worth emphasizing. First, Backus et al. (2001) assume that currency returns are completely spanned by long-term bond returns in the two currencies (i.e., currencies and interest rates are driven by the same shocks). This spanning assumption (i.e., a singular covariance matrix) is the primary source of the no-arbitrage restrictions that they derive, but it is not an implication of the absence of arbitrage opportunities.[31] When the dollar/pound exchange rate is indirectly modeled via an SDF denominated in both dollars and pounds, this assumption, and its importance, may be less transparent. Second, as we illustrated above, an SDF in not necessary to understand or derive the no-arbitrage restrictions in Backus et al. (2001), because those restrictions follows immediately from a simple static replication problem in Eq. (48). Put differently, given the assumption of a singular covariance matrix, one can derive the restrictions implied by no-arbitrage, without specifying, or explicitly solving for, an SDF. Third, as we have shown, the singular correlation matrix in Eq. (46) immediately implies (via no arbitrage) that the expected return on the exchange rate is directly related to the expected return on long-term bonds in the two currencies according to Eq. (50). Backus et al. (2001) find that the restriction in Eq. (50) has undesirable features that do not match the data well. It seems most natural to start by testing whether correlation matrix is close to singular (i.e., test whether exchange rates are spanned by movements in the two long-term bonds denominated in their respective currencies). If the covariance matrix is not singular then one should not expect the restriction in Eq. (50) to hold (since it relies on a singular covariance matrix).

Brandt and Santa-Clara (2002) provide empirical evidence that exchange rates are not

[31] Lustig et al. (2011) also assume that currency returns are completely spanned by long-term bond returns (i.e., they assume that currencies and interest rates are driven by the same shocks) .

spanned by movements in long-term bonds in both currencies. They interpret this empirical evidence as an indication that the asset market is incomplete, so that there is not a unique SDF. As we described in Section 2.4.3, Brandt and Santa-Clara (2002) model two separate SDFs, $\tilde{M}$ and M^*, together with a third stochastic process, O, that they claim captures the degree of market incompleteness. On page 164 they state that the stochastic process, O, in their model captures the notion that "if markets are incomplete, the volatility of the exchange rate can contain an element that is orthogonal to the priced sources of risk in both countries."

In Section 2.4.3 we illustrated that the model in Brandt and Santa-Clara (2002) is not arbitrage-free. Here we describe the restrictions that their model imposes relative to the general model that we provided in Eq. (35). First, based on their empirical evidence that exchange rates are not spanned by interest rates, they relax Eq. (45) and allow for a nonsingular correlation/covariance matrix. However, as we illustrated in Eqs. (37) and (38), there is a unique SDF in the continuous-time diffusion counterpart of the model in Eq. (35), even when the correlation/covariance matrix is nonsingular. In other words, the fact that currency returns are not well-spanned by the returns on other assets does not imply that the market is incomplete or that there is not a unique SDF in a no-arbitrage model.

Second, regardless of whether the market is complete or incomplete, the exchange rate cannot contain an element that is orthogonal to an SDF denominated in both dollars and pounds (in the language of Brandt and Santa-Clara (2002), the exchange rate cannot contain an element that is orthogonal to the priced sources of risk in both countries). This assumption in Brandt and Santa-Clara (2002) is the source of the arbitrage opportunity that we demonstrated in Eq. (28). As an alternative proof of that result (and a less general proof than the one we provided in Section 2.4.3), note that Brandt and Santa-Clara (2002) use the alternative parametrization in Eq. (23). They argue that if the market is incomplete, then it can be the case that $\lambda = 0 = \lambda^*$ and the change in the exchange rate can contain an element that is orthogonal to both M^* and $\tilde{M}$. As Eqs. (28) and (42) illustrate, λ and λ^* can only be equal if $\sigma = 0$. It is true that it can be the case that either $\lambda = 0$ or $\lambda^* = 0$ (i.e., either $\sigma(\gamma + \gamma_z) = 0$ or $\sigma(\gamma + \gamma_z - 1) = 0$), but Jensen's inequality ensures that both conditions cannot hold simultaneously unless $\sigma = 0$. If we overlook this error in their model, Brandt and Santa-Clara (2002) effectively relax Backus et al. (2001)'s assumption of a singular covariance matrix in Eq. (46). However, they still restrict the general model in Eq. (35) by assuming that shocks to exchange rates that are independent of shocks to other assets must also be independent of SDFs that are consistent with the returns on those assets. Again, as the the general model in Eq. (35) illustrates, this restriction is not an implication of no-arbitrage or (in)complete markets.

As a third and final example we consider the paper by Brennan and Xia (2006). They estimate an SDF, $\tilde{M}$, that is consistent with the dollar-denominated returns on the dollar-denominated default-free bank account and long-term dollar-denominated bonds. Separately, they estimate an SDF, $MX \equiv M^*$, for the pound-denominated returns on the pound-denominated default-free bank account and long-term pound-denominated bonds. Then they test whether Eq. (17) holds with equality for these two, separately identified, SDFs.

For ease of exposition, we'll translate the exercise in Brennan and Xia (2006) to the continuous-time counterpart in Eq. (37) of the general model in Eq. (35).[32] Let Y denote the dollar price of a dollar-denominated long-term bond, and let Z^* denote the pound price of a pound-denominated long-term bond. Let $\tilde{M}$ denote the SDF is that is consistent with the dollar-denominated returns on the dollar-denominated bank account and long-term bond. Then the dynamics of $\tilde{M}$ are given by

$$d\tilde{M}_t = -\tilde{M}_t \left\{ r\, dt + \left[\rho_y\, \sigma \left(\gamma + \gamma_z \right) + \sigma_y \gamma_y + \rho\, \sigma_z \gamma_z \right] dW_t^y \right\} . \tag{52}$$

Similarly, let M^* denote the SDF is that is consistent with the pound-denominated returns on the pound-denominated bank account and long-term bond. Then the dynamics of M^* are given by

$$dM_t^* = -M_t^* \left\{ r^*\, dt + \left[\rho_z\, \sigma \left(\gamma + \gamma_z - 1 \right) + \rho\, \sigma_y \gamma_y + \sigma_z \gamma_z \right] dW_t^z \right\} . \tag{53}$$

Brennan and Xia (2006) claim that if capital markets are integrated then exchange rate dynamics should follow

$$d \ln S_t = d \ln M_t^* - d \ln \tilde{M}_t \,. \tag{54}$$

From Eqs. (52–54), it is clear that Brennan and Xia (2006) inherit Backus et al. (2001)'s assumption that shocks to currencies are completely spanned by shocks to interest rates in each currency. However, Brennan and Xia (2006) make a much stronger assumption: they assume that Eq. (1) holds for SDFs that can be identified using distinct sets assets. From Eqs. (52–54), this additional assumption reduces the number of free parameters from 11 in Backus et al. (2001) down to 4 (5 if we also include the unspecified correlation between W^y and W^z) in Brennan and Xia (2006). Instead, as we illustrated in Section 2, Eq. (1) simply characterizes the change of numeraire units for an SDF that prices the same assets denominated in different units. In general, it does not need to hold for SDFs derived from distinct sets of assets, even if capital markets are completely integrated.

[32]Brennan and Xia, 2006 also use a continuous-time model.

B Structural Models

We provide an example of a full-fledged model to illustrate how aggregate consumptions and the real exchange rate are *jointly* determined in equilibrium. The model is a generalization of Backus and Smith (1993) along three dimensions. First, we allow for preference differences across countries. Second, we allow for incomplete markets. Third, like Backus and Smith (1993) we have two goods, but we explicitly compare the case where the second good is non-traded to the case in which it is frictionlessly traded. We don't view this model as a solution to existing exchange rate puzzles. Rather, it is merely illustrative of our point about the joint determination of consumptions and the real exchange rate.

We describe an endowment economy with two countries ("U.S." and "U.K.") and representative households within each country, Amy (American agent) and Bob (British agent). Utility is defined over two goods, A and B. All goods are perishable and households live for two periods (0 and 1). We suppress date subscripts unless strictly necessary.

Bob has the instantaneous utility function

$$U(c_A, c_B) = u[c(c_A, c_B)], \tag{55}$$

where c_A and c_B denote, respectively, his consumption of goods A and B, $c(\cdot)$ is a homogeneous of degree one quasi-concave function of its arguments, and u is a monotonic function with standard properties. Amy has the instantaneous utility function

$$\tilde{U}(\tilde{c}_A, \tilde{c}_B) = u[\tilde{c}(\tilde{c}_A, \tilde{c}_B)], \tag{56}$$

where $\tilde{c}_A$ and $\tilde{c}_B$ denote, respectively, her consumption of goods A and B, and $\tilde{c}(\cdot)$ is a homogeneous of degree one quasi-concave function of its arguments.

Both economies are cashless and use good A as the numeraire. Our model would have the same implications for the real exchange rate if we chose different numeraires. Goods markets meet sequentially. Good A is frictionlessly tradable. We alternately assume that good B is frictionlessly tradable or non-tradable. Bob faces the price P_B for good B. Amy faces the price $\tilde{P}_B$ for good B. When good B is frictionlessly traded, its price must be the same in both countries,

$$P_B = \tilde{P}_B. \tag{57}$$

The natural definition of the consumer price index (CPI) in the U.S. (U.K.) is a variable $\tilde{P}$ (P) such that $\tilde{c}_A + \tilde{P}_B\tilde{c}_B = \tilde{P}\,\tilde{c}(\tilde{c}_A, \tilde{c}_B)$ $[c_A + P_B c_B = P\,c(c_A, c_B)]$. Since $c(\cdot)$ and $\tilde{c}(\cdot)$ are homogeneous of degree one functions, it can be shown that there are homogeneous of degree one functions $H(\cdot)$ and $\tilde{H}(\cdot)$ whose form depends on $c(\cdot)$ and $\tilde{c}(\cdot)$, such that the U.K. and U.S. CPIs are:

$$P = H(1, P_B) \quad \text{and} \quad \tilde{P} = \tilde{H}(1, \tilde{P}_B)\,. \tag{58}$$

To see that this is true consider Bob's consumption aggregate, $c(c_A, c_B)$. Given a particular set of prices (in an arbitrary numeraire) for the individual goods, we can solve his static expenditure minimization problem

$$\min_{c_A, c_B} P_A c_A + P_B c_B \quad \text{subject to} \quad c = c(c_A, c_B)\,. \tag{59}$$

Because $c(\cdot)$ is a homogenous of degree one function, minimized expenditure is equal to Pc where $P = H(P_A, P_B)$. The function $H(\cdot)$ is also homogenous of degree one in its arguments, and is related to the function $c(\cdot)$ [see Varian (1984)]. The aggregate price index has the interpretation of being the Lagrange multiplier on the constraint at the optimum. To see this, notice that the first order conditions (FOCs) for the expenditure minimization problem are

$$P_A = \theta c_{c_A}(c_A, c_B) \quad P_B = \theta c_{c_B}(c_A, c_B)\,. \tag{60}$$

Multiplying these through these conditions by c_A and c_B and adding up you get $P_A c_A + P_B c_B = \theta c$ hence $P = \theta$. We also have

$$H(P_A, P_B) = H[P c_{c_A}(\cdot), P c_{c_B}(\cdot)] = P H[c_{c_A}(\cdot), c_{c_B}(\cdot)]\,,$$

establishing that at the optimum, $H[c_{c_A}(\cdot), c_{c_B}(\cdot)] = 1$. The same argument applies to Amy. With good A being the numeraire, we get Eq. (58).

The real exchange rate is given by Eq. (2); i.e. $e \equiv P/\tilde{P}$. In the special case where preferences are identical in the two countries, we have $H(\cdot) = \tilde{H}(\cdot)$. If, additionally, both goods are traded, $e = 1$, regardless of the asset market structure. If preferences differ across countries and both goods are traded, variation in the real exchange rate *can* arise even though $\tilde{P}_B = P_B$. All that is needed is variation in P_B.

We use two versions of the model. The first version of the model assumes that asset markets are incomplete and that the assets that are traded are common to both countries. This allows us to derive general results on the relationship between the IMRSs of agents in the two economies. Preferences are assumed to be time separable over the consumption aggregate, but take an otherwise general form. The results we are interested in, with this version of the model, extend to more general preference specifications.

The second version of the model is used to outline four specific examples where we can derive the mapping from agents' endowments to the equilibrium real exchange rate. Here we assume that agents have log utility over a Cobb-Douglas basket of the two goods. In this version of the version of the model we assume that there exists a complete set of state-contingent claims. We solve two extreme versions of the model. In one case, these assets are traded in a single global market. In the second case, the two countries are in financial autarky, so these assets can have different prices in the two locations. This is a different example of incomplete markets.

B.1 Model 1

We assume that there are k assets with $k \times 1$ random payoff vector $\boldsymbol{Z}(\omega)$ in period 1, where $\omega \in \Omega$ represents the state of the world in period 1, which has probability $\pi(\omega)$. We assume that k is smaller than the number of states of the world, which is countable (for simplicity). This means that, in general, asset markets are incomplete. The $k \times 1$ price vector for these assets in period 0 is $\boldsymbol{P_Z}$. The payoffs and prices of the assets are measured in units of good A.

Bob chooses c_{A0}, c_{B0}, $\{c_{A1}(\omega), c_{B1}(\omega)\}_{\omega\in\Omega}$, and $\boldsymbol{a}$ to maximize

$$u[c(c_{A0}, c_{B0})] + \beta \sum_{\omega\in\Omega} u\{c[c_{A1}(\omega), c_{B1}(\omega)]\}\pi(\omega)\,, \tag{61}$$

subject to

$$c_{A0} + P_{B0}c_{B0} + \boldsymbol{P_Z} \cdot \boldsymbol{a} = y_{A0} + P_{B0}y_{B0}\,, \tag{62}$$

$$c_{A1}(\omega) + P_{B1}(\omega)c_{B1}(\omega) = y_{A1}(\omega) + P_{B1}(\omega)y_{B1}(\omega) + \boldsymbol{Z}(\omega) \cdot \boldsymbol{a}\,, \quad \omega \in \Omega. \tag{63}$$

Here $0 < \beta < 1$, $c_{A1}(\omega)$ and $c_{B1}(\omega)$ are Bob's plans for future consumption (in every possible state of the world), the jth element of $\boldsymbol{a}$ is Bob's net purchases of asset j, and y_A, y_B are Bob's endowments of the two goods, which, in period 1, depend on the state of the world.

Similarly, Amy chooses $\tilde{c}_{A0}$, $\tilde{c}_{B0}$, $\{\tilde{c}_{A1}(\omega), \tilde{c}_{B1}(\omega)\}_{\omega\in\Omega}$, and $\tilde{\boldsymbol{a}}$ to maximize

$$u[\tilde{c}(\tilde{c}_{A0}, \tilde{c}_{B0})] + \beta \sum_{\omega\in\Omega} u\{\tilde{c}[\tilde{c}_{A1}(\omega), \tilde{c}_{B1}(\omega)]\}\pi(\omega)\,, \tag{64}$$

subject to

$$\tilde{c}_{A0} + \tilde{P}_{B0}\tilde{c}_{B0} + \boldsymbol{P_Z} \cdot \tilde{\boldsymbol{a}} = \tilde{y}_{A0} + \tilde{P}_{B0}\tilde{y}_{B0}\,. \tag{65}$$

$$\tilde{c}_{A1}(\omega) + \tilde{P}_{B1}(\omega)\tilde{c}_{B1}(\omega) = \tilde{y}_{A1}(\omega) + \tilde{P}_{B1}(\omega)\tilde{y}_{B1}(\omega) + \boldsymbol{Z}(\omega) \cdot \tilde{\boldsymbol{a}}\,, \quad \omega \in \Omega. \tag{66}$$

The market clearing conditions for good A are

$$
\begin{aligned}
c_{A0} + \tilde{c}_{A0} &= y_{A0} + \tilde{y}_{A0}\,, && (67)\\
c_{A1}(\omega) + \tilde{c}_{A1}(\omega) &= y_{A1}(\omega) + \tilde{y}_{A1}(\omega)\,, \quad \omega \in \Omega. && (68)
\end{aligned}
$$

When good B is tradable we have the following market clearing conditions

$$
\begin{aligned}
c_{B0} + \tilde{c}_{B0} &= y_{B0} + \tilde{y}_{B0}\,, && (69)\\
c_{B1}(\omega) + \tilde{c}_{B1}(\omega) &= y_{B1}(\omega) + \tilde{y}_{B1}(\omega)\,, \quad \omega \in \Omega. && (70)
\end{aligned}
$$

When it is non-tradable, instead, we have

$$
\begin{aligned}
c_{B0} &= y_{B0}\,, \quad \tilde{c}_{B0} = \tilde{y}_{B0}\,, && (71)\\
c_{B1}(\omega) &= y_{B1}(\omega)\,, \quad \tilde{c}_{B1}(\omega) = \tilde{y}_{B1}(\omega)\,, \quad \omega \in \Omega. && (72)
\end{aligned}
$$

The market clearing condition in the asset market is

$$\boldsymbol{a} + \tilde{\boldsymbol{a}} = \boldsymbol{0}\,. \tag{73}$$

Definition. A competitive equilibrium is values of the quantities c_{A0}, c_{B0}, $\tilde{c}_{A0}$, $\tilde{c}_{B0}$, $\{c_{A1}(\omega), c_{B1}(\omega)\}_{\omega\in\Omega}$, $\{\tilde{c}_{A1}(\omega), \tilde{c}_{B1}(\omega)\}_{\omega\in\Omega}$, $\boldsymbol{a}$, $\tilde{\boldsymbol{a}}$ and prices, P_{B0}, $\tilde{P}_{B0}$, $\{P_{B1}(\omega), \tilde{P}_{B1}(\omega)\}_{\omega\in\Omega}$, $\boldsymbol{P_Z}$ such that the quantities solve Bob and Amy's optimization problems (taking the prices as given), and such that the market clearing conditions are satisfied. When good B is frictionlessly traded, Eq. (57) must also be satisfied.

B.1.1 Risk Sharing and IMRSs. The two agents' IMRSs, defined over units of their respective consumption baskets are

$$M^*(\omega) \equiv \beta u_c[c_1(\omega)]/u_c(c_0) \quad \text{and} \quad \tilde{M}(\omega) \equiv \beta u_{\tilde{c}}[\tilde{c}_1(\omega)]/u_{\tilde{c}}(\tilde{c}_0)\,, \tag{74}$$

where c_0 and $\tilde{c}_0$ are shorthand for $c(c_{A0}, c_{B0})$ and $\tilde{c}(\tilde{c}_{A0}, \tilde{c}_{B0})$, and $c_1(\omega)$ and $\tilde{c}_1(\omega)$ are shorthand for $c[c_{A1}(\omega), c_{B1}(\omega)]$ and $\tilde{c}[\tilde{c}_{A1}(\omega), \tilde{c}_{B1}(\omega)]$.

We can also define the agents' IMRSs over goods A and B:

$$M_A(\omega) \equiv \beta u_{c_A}[c_1(\omega)]/u_{c_A}(c_0)\,, \quad \tilde{M}_A(\omega) \equiv \beta u_{\tilde{c}_A}[\tilde{c}_1(\omega)]/u_{\tilde{c}_A}(\tilde{c}_0)\,, \tag{75}$$

$$M_B(\omega) \equiv \beta u_{c_B}[c_1(\omega)]/u_{c_B}(c_0)\,, \quad \tilde{M}_B(\omega) \equiv \beta u_{\tilde{c}_B}[\tilde{c}_1(\omega)]/u_{\tilde{c}_B}(\tilde{c}_0)\,. \tag{76}$$

Definition. Perfect risk sharing describes any competitive equilibrium in which $\tilde{M}_A(\omega) = M_A(\omega)$ and $\tilde{M}_B(\omega) = M_B(\omega)$ for all $\omega \in \Omega$.

B.1.2 Solving the Model. By nesting the expenditure minimization problem, described above, within Bob's problem, we can rewrite the latter as follows. Bob chooses c_0, $\{c_1(\omega)\}_{\omega\in\Omega}$ and $\boldsymbol{a}$ to maximize

$$u(c_0) + \beta \sum_{\omega\in\Omega} u\left[c_1(\omega)\right]\pi(\omega) \tag{77}$$

subject to

$$P_0 c_0 + \boldsymbol{P_Z} \cdot \boldsymbol{a} = y_{A0} + P_{B0} y_{B0}\,, \tag{78}$$

$$P_1(\omega)c_1(\omega) = y_{A1}(\omega) + P_{B1}(\omega)y_{B1}(\omega) + \boldsymbol{Z}\,(\omega) \cdot \boldsymbol{a}\,, \quad \omega \in \Omega. \tag{79}$$

The FOCs for c_0, $c_1(\omega)$, and $\boldsymbol{a}$ are

$$u_c(c_0) = P_0 \lambda\,, \tag{80}$$

$$\beta u_c[c_1(\omega)]\pi(\omega) = P_1(\omega)\mu(\omega)\,, \quad \omega \in \Omega. \tag{81}$$

$$\boldsymbol{P_Z}\lambda = \sum_{\omega\in\Omega} \mu(\omega)\boldsymbol{Z}\,(\omega)\ . \tag{82}$$

Here λ is the Lagrange multiplier on the constraint (78), and $\mu(\omega)$ is the Lagrange multiplier on the constraint (79). So, combining (80) and (81), we get the following expression for Bob's IMRS, defined over his basket:

$$M^*(\omega) = \frac{\beta u_c[c_1(\omega)]}{u_c(c_0)} = \frac{P_1(\omega)}{P_0}\frac{\mu(\omega)}{\lambda\pi(\omega)}\,. \tag{83}$$

Amy chooses $\tilde{c}_0$, $\{\tilde{c}_1(\omega)\}_{\omega\in\Omega}$, and $\tilde{\boldsymbol{a}}$ to maximize

$$u(\tilde{c}_0) + \beta \sum_{\omega\in\Omega} u\left[\tilde{c}_1(\omega)\right]\pi(\omega) \tag{84}$$

subject to

$$\tilde{P}_0 \tilde{c}_0 + \boldsymbol{P_Z} \cdot \tilde{\boldsymbol{a}} = \tilde{y}_{A0} + \tilde{P}_{B0} \tilde{y}_{B0}\,, \tag{85}$$

$$\tilde{P}_1(\omega)\tilde{c}_1(\omega) = \tilde{y}_{A1}(\omega) + \tilde{P}_{B1}(\omega)\tilde{y}_{B1}(\omega) + \boldsymbol{Z}(\omega) \cdot \tilde{\boldsymbol{a}}\,, \quad \omega \in \Omega. \tag{86}$$

The FOCs for $\tilde{c}_0$, $\{\tilde{c}_1(\omega)\}_{\omega\in\Omega}$, and $\tilde{\boldsymbol{a}}$ are

$$u_c(\tilde{c}_0) = \tilde{P}_0\tilde{\lambda}\,, \tag{87}$$

$$\beta u_c[\tilde{c}_1(\omega)]\pi(\omega) = \tilde{P}_1(\omega)\tilde{\mu}(\omega)\,, \quad \omega \in \Omega, \tag{88}$$

$$\boldsymbol{P_Z}\tilde{\lambda} = \sum_{\omega\in\Omega} \tilde{\mu}(\omega)\boldsymbol{Z}(\omega)\ . \tag{89}$$

Here $\tilde{\lambda}$ is the Lagrange multiplier on the constraint (85), and $\tilde{\mu}(\omega)$ is the Lagrange multiplier on the constraint (86). So, combining (87) and (88), we get the following expression for Amy's IMRS, defined over her basket:

$$\tilde{M}(\omega) = \frac{\beta u_c[\tilde{c}_1(\omega)]}{u_c(\tilde{c}_0)} = \frac{\tilde{P}_1(\omega)}{\tilde{P}_0}\frac{\tilde{\mu}(\omega)}{\tilde{\lambda}\pi(\omega)}\,. \tag{90}$$

Notice that M^* is an SDF for payoffs and prices measured in Bob basket units. This is because Eqs. (82) and (83) combined imply

$$\frac{\boldsymbol{P_Z}}{P_0} = \sum_{\omega\in\Omega} M^*(\omega)\frac{\boldsymbol{Z}(\omega)}{P_1(\omega)}\pi(\omega)\,. \tag{91}$$

Similarly, $\tilde{M}$ is an SDF for payoffs and prices measured in Amy basket units. This is because Eqs. (89) and (90) combined imply

$$\frac{\boldsymbol{P_Z}}{\tilde{P}_0} = \sum_{\omega\in\Omega} \tilde{M}(\omega)\frac{\boldsymbol{Z}(\omega)}{\tilde{P}_1(\omega)}\pi(\omega)\,. \tag{92}$$

From (83) and (90), the ratio of M^* to $\tilde{M}$ is

$$\frac{M^*(\omega)}{\tilde{M}(\omega)} = \left[\frac{P_1(\omega)}{P_0}\frac{\mu(\omega)}{\lambda}\right] \Big/ \left[\frac{\tilde{P}_1(\omega)}{\tilde{P}_0}\frac{\tilde{\mu}(\omega)}{\tilde{\lambda}}\right] = \frac{e_1(\omega)}{e_0}\cdot\left[\frac{\mu(\omega)}{\lambda}\Big/\frac{\tilde{\mu}(\omega)}{\tilde{\lambda}}\right]\,. \tag{93}$$

Notice that since good A is the numeraire, the FOCs for c_{A0} and $\tilde{c}_{A0}$, given in Eq. (60), along with Eqs. (80) and (87) imply that the time-0 marginal utilities of good A are

$$u_c(c_0)c_{c_A}(c_0) = \lambda \quad u_c(\tilde{c}_0)\tilde{c}_{c_A}(\tilde{c}_0) = \tilde{\lambda}\,. \tag{94}$$

Similarly, when these FOCs are combined with Eqs. (81) and (88), we get expressions for the time-1 discounted marginal utilities of good A:

$$\beta u_c[c_1(\omega)]c_{c_A}[c_1(\omega)] = \mu(\omega)/\pi(\omega)\,, \quad \omega \in \Omega, \tag{95}$$

$$\beta u_c[\tilde{c}_1(\omega)]\tilde{c}_{c_A}[\tilde{c}_1(\omega)] = \tilde{\mu}(\omega)/\pi(\omega)\,, \quad \omega \in \Omega. \tag{96}$$

Thus, $M_A(\omega) = \mu(\omega)/[\lambda\pi(\omega)]$ and $\tilde{M}_A(\omega) = \tilde{\mu}(\omega)/[\tilde{\lambda}\pi(\omega)]$ are the IMRSs over good A. Consequently, we can define

$$\Xi(\omega) = \left[\frac{\mu(\omega)}{\lambda}\right] \Big/ \left[\frac{\tilde{\mu}(\omega)}{\tilde{\lambda}}\right] = M_A(\omega)/\tilde{M}_A(\omega).$$

Since we have $X(\omega) = e_1(\omega)/e_0$ we can rewrite Eq. (93) as

$$\frac{M^*(\omega)}{\tilde{M}(\omega)} = X(\omega) \cdot \Xi(\omega)\,, \tag{97}$$

with $\Xi(\omega)$ being a measure of risk sharing in the frictionlessly traded good (good A). We can also define $M(\omega) \equiv M^*(\omega)/X(\omega)$, which is Bob's IMRS expressed indirectly using Amy's basket as the numeraire. Notice that this means

$$\frac{M(\omega)}{\tilde{M}(\omega)} = \Xi(\omega)\,. \tag{98}$$

The FOCs for c_{B0} and $\tilde{c}_{B0}$, given in Eq. (60), along with Eqs. (80) and (87) imply that the time-0 marginal utilities of good B are

$$u_c(c_0)c_{c_B}(c_0) = \lambda P_{B0} \quad u_c(\tilde{c}_0)\tilde{c}_{c_B}(\tilde{c}_0) = \tilde{\lambda}\tilde{P}_{B0}\,. \tag{99}$$

Similarly, when these FOCs are combined with Eqs. (81) and (88), we get expressions for the time-1 discounted marginal utilities of good B:

$$\beta u_c[c_1(\omega)]c_{c_B}[c_1(\omega)] = \mu(\omega)P_{B1}(\omega)/\pi(\omega)\,, \quad \omega \in \Omega, \tag{100}$$

$$\beta u_c[\tilde{c}_1(\omega)]\tilde{c}_{c_B}[\tilde{c}_1(\omega)] = \tilde{\mu}(\omega)\tilde{P}_{B1}(\omega)/\pi(\omega)\,, \quad \omega \in \Omega. \tag{101}$$

Thus, $M_B(\omega) = M_A(\omega)P_{B1}(\omega)/P_{B0}$ and $\tilde{M}_B(\omega) = \tilde{M}_A(\omega)\tilde{P}_{B1}(\omega)/\tilde{P}_{B0}$ are the IMRSs over good B. Consequently, $\Xi(\omega)[\tilde{P}_{B1}(\omega)/\tilde{P}_{B0}]/[P_{B1}(\omega)/P_{B0}]$ is a measure of how well risk is shared in good B. If good B is frictionlessly traded the price terms in this expression cancel out and the measure of risk sharing in good B, as for good A, is $\Xi(\omega)$.

As we see in the next section, when the securities span variation in households' marginal utilities (i.e., if financial markets are complete) the FOCs for $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ become equivalent to

$$\psi(\omega)\lambda = \mu(\omega)\,, \qquad \psi(\omega)\tilde{\lambda} = \tilde{\mu}(\omega)\,, \tag{102}$$

for all ω, where $\psi(\omega)$ is the price of a claim that pays one unit of good A in state ω. Notice that when financial markets are complete, this implies $M_A(\omega) = \tilde{M}_A(\omega) = \psi(\omega)/\pi(\omega)$ and $\Xi(\omega) = 1$.

B.2 Model 2

In the second version of the model we make two changes. First, we assume that there is a complete set of state contingent securities indexed by ω. Security ω pays one unit of good A in state ω and zero otherwise. It's price is $\psi(\omega)$ in the U.K. and $\tilde{\psi}(\omega)$ in the U.S. If there is frictionless international trade in these assets (the complete markets case), we have $\psi(\omega) = \tilde{\psi}(\omega)$. Under financial autarky these prices can be different.

We adopt the assumptions that $u(c) = \ln c$, $c = c_A^{\theta} c_B^{1-\theta}$, and $\tilde{c} = \tilde{c}_A^{\tilde{\theta}} c_B^{1-\tilde{\theta}}$. These assumptions are useful because equilibrium prices and quantities can be worked out with pencil and paper. They imply that the CPIs in the two countries, measured in units of good A, are

$$P = \rho\, P_B^{1-\theta}\,, \qquad \text{and} \qquad \tilde{P} = \tilde{\rho}\, \tilde{P}_B^{1-\tilde{\theta}}\,, \tag{103}$$

with $\rho = \theta^{-\theta}(1-\theta)^{\theta-1}$, and $\tilde{\rho} = \tilde{\theta}^{-\tilde{\theta}}(1-\tilde{\theta})^{\tilde{\theta}-1}$. The real exchange rate is

$$e = (\rho/\tilde{\rho})(P_B^{1-\theta}/\tilde{P}_B^{1-\tilde{\theta}})\,. \tag{104}$$

We discuss four specific examples of our model, which combine different assumptions about financial markets (complete markets vs. financial autarky) and goods market frictions (good B is frictionlessly traded vs. good B is non-traded). By explicitly solving for the equilibrium in these four cases, we demonstrate that real exchange rates and agents' IMRSs are jointly determined by the laws of motion of the endowments, together with our assumptions about preferences, goods market frictions, and asset markets. We also illustrate that the conditions under which risk sharing is imperfect, and those under which the real exchange rate varies, are different.

With the different asset market setup, Bob's budgets constraints are

$$c_{A0} + P_{B0}c_{B0} + \sum_{\omega} \psi(\omega)a(\omega) = y_{A0} + P_{B0}y_{B0}\,, \tag{105}$$

$$c_{A1}(\omega) + P_{B1}(\omega)c_{B1}(\omega) = y_{A1}(\omega) + P_{B1}(\omega)y_{B1}(\omega) + a(\omega)\,, \quad \omega \in \Omega. \tag{106}$$

while Amy's are

$$\tilde{c}_{A0} + \tilde{P}_{B0}\tilde{c}_{B0} + \sum_{\omega} \tilde{\psi}(\omega)\tilde{a}(\omega) = \tilde{y}_{A0} + \tilde{P}_{B0}\tilde{y}_{B0}\,, \tag{107}$$

$$\tilde{c}_{A1}(\omega) + \tilde{P}_{B1}(\omega)\tilde{c}_{B1}(\omega) = \tilde{y}_{A1}(\omega) + \tilde{P}_{B1}(\omega)\tilde{y}_{B1}(\omega) + \tilde{a}(\omega)\,, \quad \omega \in \Omega. \tag{108}$$

We work out the full solutions in four special cases below. But, first, we summarize the features of the competitive equilibria in each case. We use the following notation, and henceforth drop the notational dependence of time 1 variables on ω unless it is needed. The global endowment of good A in period t is $Y_{At} = y_{At} + \tilde{y}_{At}$. Analogously, for good B we have $Y_{Bt} = y_{Bt} + \tilde{y}_{Bt}$. The growth rates of the global endowments are $G_A = Y_{A1}/Y_{A0}$ and $G_B = Y_{B1}/Y_{B0}$. We also define $g_A = y_{A1}/y_{A0}$, $g_B = y_{B1}/y_{B0}$, $\tilde{g}_A = \tilde{y}_{A1}/\tilde{y}_{A0}$ and $\tilde{g}_B = \tilde{y}_{B1}/\tilde{y}_{B0}$. Bob's time t share of the global endowment of good A is $s_{At} = y_{At}/Y_{At}$. Similarly, $s_{Bt} = y_{Bt}/Y_{Bt}$. We let $\bar{s}_{A1} = \sum_{\omega} s_{A1}(\omega)\pi(\omega)$ and $\bar{s}_{B1} = \sum_{\omega} s_{B1}(\omega)\pi(\omega)$ denote Bob average share of the global endowments in period 1.

B.2.1 Complete Markets, No Goods Market Frictions. When asset markets are complete internationally and there are no goods market frictions (i.e., good B is frictionlessly traded), then $P_{Bt} = \tilde{P}_{Bt}$ for all t and IMRSs in the individual goods are always equated across countries. As we show in detail below, in good A the IMRS is β/G_A. In good B the IMRS is β/G_B. Risk is shared perfectly, regardless of preferences.

In the case where preferences are identical, $e_t = 1$ for all t. When preferences differ across countries the expressions in Eq. (104) simplify to $e_t = (\rho/\tilde{\rho})P_{Bt}^{\tilde{\theta}-\theta}$ with $P_{Bt} = \kappa Y_{At}/Y_{Bt}$ and

$$\kappa = \frac{(1-\tilde{\theta})(1+\beta) + (\tilde{\theta}-\theta)(s_{A0} + \beta \bar{s}'_{A1})}{\tilde{\theta}(1+\beta) + (\theta-\tilde{\theta})(s_{B0} + \beta \bar{s}'_{B1})} \,. \tag{109}$$

Hence,

$$\ln X = (\tilde{\theta}-\theta)\ln(P_{B1}/P_{B0}) = (\tilde{\theta}-\theta)\ln(G_A/G_B)\,. \tag{110}$$

Real exchange rate fluctuations are driven by differences in the global growth rates of the endowments of goods A and B. We see that if the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If Amy's preferences put more weight on good B than Bob's preferences (i.e., $\tilde{\theta} < \theta$), then her basket becomes relatively more expensive (the U.S. real exchange rate appreciates).

B.2.2 Complete Markets, Good B is Non-traded. Now consider the case where asset markets are complete internationally, but good B is non-traded. In this case, in general, $P_B \neq \tilde{P}_B$. IMRSs in good A are always equated across countries: $M_A = \tilde{M}_A = \beta/G_A$. IMRSs in good B are, respectively, $M_B = \beta/g_B$ and $\tilde{M}_B = \beta/\tilde{g}_B$, so risk is not shared perfectly unless $g_B = \tilde{g}_B$ in every state of the world in period 1.

When preferences differ across countries the real exchange rates are given by Eq. (104), with prices given by

$$P_{Bt} = \kappa\frac{Y_{At}}{y_{Bt}}\,, \quad \tilde{P}_{Bt} = \tilde{\kappa}\frac{Y_{At}}{\tilde{y}_{Bt}}\,, \tag{111}$$

and

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_{A0} + \beta\bar{s}'_{A1})\,, \quad \tilde{\kappa} = \frac{1-\tilde{\theta}}{(1+\beta)\tilde{\theta}}\left[1 - s_{A0} + \beta(1-\bar{s}'_{A1})\right]\,. \tag{112}$$

This implies that

$$\ln X = (1-\tilde{\theta})\ln\tilde{g}_B - (1-\theta)\ln g_B + (\tilde{\theta}-\theta)\ln G_A\,. \tag{113}$$

Here, the real exchange rate depends on the relative growth rates of the endowment of good B in the two countries, but the two growth rates matter to different extents due to preference differences. Additionally, as was the case when good B was traded, if Amy's preferences put more weight on good B than Bob's preferences ($\tilde{\theta} < \theta$) then, other things being equal, the U.S. real exchange rate appreciates when the global endowment of good A grows.

If preferences are identical, then the real exchange rate in Eq. (104) simplifies to $e_t =$

$(P_{Bt}/\tilde{P}_{Bt})^{1-\theta}$ with prices still given by Eq. (111), but Eq. (112) becomes

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_{A0} + \beta \bar{s}'_{A1}) \,, \quad \tilde{\kappa} = \frac{1-\theta}{(1+\beta)\theta}[1 - s_{A0} + \beta(1 - \bar{s}'_{A1})] \,. \tag{114}$$

This means that

$$\ln X = (1-\theta)\ln(\tilde{g}_B/g_B) \,. \tag{115}$$

Here, the real exchange rate depends entirely on the relative growth rates of the endowment of good B in the two countries. If the endowment grows more slowly in the U.S. ($\tilde{g}_B < g_B$), its basket becomes relatively more expensive and its real exchange rate appreciates.

B.2.3 Financial Autarky, No Goods Market Frictions. The third case we consider is where no assets are traded internationally, but goods markets are frictionless. In this case, $P_{Bt} = \tilde{P}_{Bt}$ for all t. Risk sharing, in general, is imperfect. The ratio of IMRSs in the two countries is the same in goods A and B. That is

$$\frac{M_A}{\tilde{M}_A} = \frac{M_B}{\tilde{M}_B} = \Xi = \frac{\tilde{\theta} s_{A0} + (1-\tilde{\theta}) s_{B0}}{\theta(1-s_{A0}) + (1-\theta)(1-s_{B0})} \times \frac{\theta(1-s_{A1}) + (1-\theta)(1-s_{B1})}{\tilde{\theta} s_{A1} + (1-\tilde{\theta}) s_{B1}} . \tag{116}$$

This expression is the same when preferences are identical, except that $\theta = \tilde{\theta}$.

In the case where preferences are identical, $e_0 = e_1 = 1$. Risk sharing, on the other hand, can be good or bad. Suppose, for example, that Bob's shares of the global endowments vary and comove positively. In this case, Ξ deviates from one a lot, implying that risk sharing is limited. On the other hand, suppose that business cycles are strongly correlated across countries, so that Bob's shares of the global endowments do not change very much across different states of the world next period. In this case, Ξ will be close to one in all states, implying a high degree of risk sharing.

When preferences differ across countries then $e_t = (\rho/\tilde{\rho}) P_{Bt}^{\tilde{\theta}-\theta}$ where $P_{Bt} = \kappa_t Y_{At}/Y_{Bt}$, and

$$\kappa_t = \frac{1 - \tilde{\theta} + (\tilde{\theta} - \theta) s_{At}}{\tilde{\theta} + (\theta - \tilde{\theta}) s_{Bt}} \,. \tag{117}$$

Hence,

$$\ln X = (\tilde{\theta} - \theta)\left[\ln(G_A/G_B) + \ln(\kappa_1/\kappa_0)\right] . \tag{118}$$

As in the case of complete markets, real exchange rate fluctuations are driven by differences in the growth rates of the two endowments. If the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If Amy's preferences put more weight on good B than Bob's preferences ($\tilde{\theta} < \theta$) then Amy's basket becomes relatively more expensive (the U.S. real exchange rate appreciates). But the way in which the agents' shares of the global endowments fluctuate also matters for the real exchange rate. In the example we just described, the U.S. real exchange rate rises more in states of the world where $\kappa_1 > \kappa_0$. This could reflect, for example, a rise in Amy's share of the global endowment of good A (a drop of s_{At}) at the same time as the global endowment of A rises relative to the global endowment of B.

B.2.4 Financial Autarky, Good B is Non-traded. The final case we consider combines financial autarky with the assumption that good B is non-traded. In this case, each agent simply consumes its own endowments. IMRSs in the individual goods are determined by the country-specific endowment growth rates. For good A they are $M_A = \beta/g_A$ and $\tilde{M}_A = \beta/\tilde{g}_A$. In good B they are $M_B = \beta/g_B$ and $\tilde{M}_B = \beta/\tilde{g}_B$. Risk is not shared unless growth rates happen to coincide. The real exchange rates in the two periods are given by Eq. (104), with

$$P_{Bt} = \frac{(1-\theta)}{\theta}\frac{y_{At}}{y_{Bt}}, \quad \tilde{P}_{Bt} = \frac{(1-\tilde{\theta})}{\tilde{\theta}}\frac{\tilde{y}_{At}}{\tilde{y}_{Bt}}. \tag{119}$$

Hence

$$\ln X = (1-\theta)\ln(g_A/g_B) - (1-\tilde{\theta})\ln(\tilde{g}_A/\tilde{g}_B). \tag{120}$$

Suppose endowment growth rates are identical across goods; i.e., $g_A = g_B$ and $\tilde{g}_A = \tilde{g}_B$. Notice that this implies $\ln X = 0$. There is no variation in the real exchange rate. The extent of risk sharing, in contrast, depends only on whether $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. It could be good or bad. Suppose, on the other hand, that risk sharing is perfect; i.e., $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. We only get the result that $\ln X = 0$ if $\theta = \tilde{\theta}$.

B.2.5 Discussion. Consider Table 1 from Section 4. It states that under complete markets, the observation that real exchange rates are variable only implies imperfect risk sharing when the two agents have the same consumption basket. In our model, the agents have identically-composed consumption baskets if and only if $\theta = \tilde{\theta}$, because θ and $\tilde{\theta}$ are the constant expenditure shares of good A in the two countries.

So suppose that $\theta = \tilde{\theta}$. Under complete markets, we saw that $\ln X = 0$ and risk sharing is perfect if trade in both goods is frictionless. On the other hand, $\ln X = (1-\theta)\ln(\tilde{g}_B/g_B)$ and $M_B/\tilde{M}_B = \tilde{g}_B/g_B$ if good B is non-traded. If one is willing to assume that markets are complete, and that countries have identical preferences, risk sharing and exchange rate changes are intimately linked in our model.

Under incomplete markets, however, there is no link, in general, between risk sharing and exchange rates, even when $\theta = \tilde{\theta}$. When $\theta = \tilde{\theta}$, and trade in both goods is frictionless, $\ln X = 0$ yet Ξ can depart arbitrarily from one, and therefore risk sharing can be arbitrarily imperfect. When $\theta = \tilde{\theta}$, and good B is non-traded, $\ln X = 0$ when risk sharing happens to be perfect (i.e., when $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$ in every possible state of the world next period), but we also have $\ln X = 0$ when risk sharing is "poor" and $g_A = g_B \neq \tilde{g}_A = \tilde{g}_B$.

More generally, our model illustrates that there is no direct link between the degree of risk sharing and real exchange rate variability.

B.2.6 Solving Model 2 (Details). The FOCs for the individual consumption goods and holdings of the securities are

$$\theta c_{A0}^{-1} = \lambda, \tag{121}$$

$$(1-\theta) c_{B0}^{-1} = P_{B0}\lambda, \tag{122}$$

$$\beta\theta c_{A1}(\omega)^{-1}\pi(\omega) = \mu(\omega), \quad \omega \in \Omega, \tag{123}$$

$$\beta(1-\theta) c_{B1}(\omega)^{-1}\pi(\omega) = P_{B1}(\omega)\mu(\omega), \quad \omega \in \Omega, \tag{124}$$

$$\psi(\omega)\lambda = \mu(\omega), \quad \omega \in \Omega. \tag{125}$$

$$\tilde{\theta}\tilde{c}_{A0}^{-1} = \tilde{\lambda}, \tag{126}$$

$$(1-\tilde{\theta})\tilde{c}_{B0}^{-1} = \tilde{P}_{B0}\tilde{\lambda}, \tag{127}$$

$$\beta\tilde{\theta}\tilde{c}_{A1}(\omega)^{-1}\pi(\omega) = \tilde{\mu}(\omega), \quad \omega \in \Omega, \tag{128}$$

$$\beta(1-\tilde{\theta})\tilde{c}_{B1}(\omega)^{-1}\pi(\omega) = \tilde{P}_{B1}(\omega)\tilde{\mu}(\omega), \quad \omega \in \Omega, \tag{129}$$

$$\tilde{\psi}(\omega)\tilde{\lambda} = \tilde{\mu}(\omega), \quad \omega \in \Omega. \tag{130}$$

We can rewrite (121)–(124) and (126)–(129) using (125) and (130), as:

$$\theta = \lambda c_{A0} \tag{131}$$

$$1-\theta = \lambda c_{B0} P_{B0} \tag{132}$$

$$\beta\theta = \frac{\psi(\omega)\lambda}{\pi(\omega)} c_{A1}(\omega) \tag{133}$$

$$\beta(1-\theta) = \frac{\psi(\omega)\lambda}{\pi(\omega)} c_{B1}(\omega) P_{B1}(\omega) \tag{134}$$

$$\tilde{\theta} = \tilde{\lambda}\tilde{c}_{A0} \tag{135}$$

$$1 - \tilde{\theta} = \tilde{\lambda}\tilde{P}_{B0}\tilde{c}_{B0} \tag{136}$$

$$\beta\tilde{\theta} = \frac{\tilde{\psi}(\omega)\tilde{\lambda}}{\pi(\omega)}\tilde{c}_{A1}(\omega) \tag{137}$$

$$\beta(1 - \tilde{\theta}) = \frac{\tilde{\psi}(\omega)\tilde{\lambda}}{\pi(\omega)}\tilde{P}_{B1}(\omega)\tilde{c}_{B1}(\omega) \tag{138}$$

Here, we have dropped "$\omega \in \Omega$" from the equations for convenience.

In what follows we use the notation $L = \lambda^{-1}$, $\tilde{L} = \tilde{\lambda}^{-1}$. Notice that L and $\tilde{L}$ are the households' total expenditures on goods in period zero.

When International Asset Markets are Complete Here we have $\psi(\omega) = \tilde{\psi}(\omega)$, which allows us to rewrite Eqs. (131)–(138) as

$$\theta L = c_{A0} \tag{139}$$

$$(1 - \theta)L = c_{B0}P_{B0} \tag{140}$$

$$\beta\theta L = \frac{\psi(\omega)}{\pi(\omega)}c_{A1}(\omega) \tag{141}$$

$$\beta(1 - \theta)L = \frac{\psi(\omega)}{\pi(\omega)}c_{B1}(\omega)P_{B1}(\omega) \tag{142}$$

$$\tilde{\theta}\tilde{L} = \tilde{c}_{A0} \tag{143}$$

$$(1 - \tilde{\theta})\tilde{L} = \tilde{P}_{B0}\tilde{c}_{B0} \tag{144}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\psi(\omega)}{\pi(\omega)}\tilde{c}_{A1}(\omega) \tag{145}$$

$$\beta(1 - \tilde{\theta})\tilde{L} = \frac{\psi(\omega)}{\pi(\omega)}\tilde{P}_{B1}(\omega)\tilde{c}_{B1}(\omega) \tag{146}$$

Bob's lifetime budget constraint is

$$c_{A0} + P_{B0}c_{B0} + \sum_{\omega} \psi(\omega)\left[c_{A1}(\omega) + P_{B1}(\omega)c_{B1}(\omega)\right] =$$

$$y_{A0} + P_{B0} y_{B0} + \sum_{\omega} \psi(\omega) \left[y_{A1}(\omega) + P_{B1}(\omega) y_{B1}(\omega) \right] \tag{147}$$

From (139), (141), (143), and (145) we see that IMRSs in good A in the two countries are equated:

$$M_A(\omega) = \beta \frac{c_{A0}}{c_{A1}(\omega)} = \frac{\psi(\omega)}{\pi(\omega)} \qquad \tilde{M}_A(\omega) = \beta \frac{\tilde{c}_{A0}}{\tilde{c}_{A1}(\omega)} = \frac{\psi(\omega)}{\pi(\omega)} \tag{148}$$

From (140), (142), (144), and (146), IMRSs in good B are

$$M_B(\omega) = \beta \frac{c_{B0}}{c_{B1}(\omega)} = \frac{\psi(\omega)}{\pi(\omega)} \frac{P_{B1}(\omega)}{P_{B0}} \tag{149}$$

$$\tilde{M}_B(\omega) = \beta \frac{\tilde{c}_{B0}}{\tilde{c}_{B1}(\omega)} = \frac{\psi(\omega)}{\pi(\omega)} \frac{\tilde{P}_{B1}(\omega)}{\tilde{P}_{B0}} \tag{150}$$

When Good B is Traded The market clearing conditions for good A are

$$c_{A0} + \tilde{c}_{A0} = Y_{A0} \tag{151}$$

$$c_{A1}(\omega) + \tilde{c}_{A1}(\omega) = Y_{A1}(\omega) \tag{152}$$

$$c_{B0} + \tilde{c}_{B0} = Y_{B0} \tag{153}$$

$$c_{B1}(\omega) + \tilde{c}_{B1}(\omega) = Y_{B1}(\omega) \tag{154}$$

These conditions, together with (139)–(146), imply

$$\theta L + \tilde{\theta} \tilde{L} = Y_{A0} \tag{155}$$

$$\beta(\theta L + \tilde{\theta} \tilde{L}) = \frac{\psi(\omega)}{\pi(\omega)} Y_{A1}(\omega) \tag{156}$$

$$(1 - \theta) L + (1 - \tilde{\theta}) \tilde{L} = P_{B0} Y_{B0} \tag{157}$$

$$\beta[(1 - \theta) L + (1 - \tilde{\theta}) \tilde{L}] = \frac{\psi(\omega)}{\pi(\omega)} P_{B1}(\omega) Y_{B1}(\omega) \tag{158}$$

Given a value of L we can solve the Eqs. (155) and (157) for L and P_{B0}:

$$\tilde{L} = \frac{Y_{A0}}{\tilde{\theta}} - \frac{\theta}{\tilde{\theta}} L \tag{159}$$

$$P_{B0} = \frac{\frac{\tilde{\theta}-\theta}{\tilde{\theta}} L + (\frac{1-\tilde{\theta}}{\tilde{\theta}}) Y_{A0}}{Y_{B0}} \tag{160}$$

Combining Eqs. (155) and (156) we get

$$M_A(\omega) = \beta G_A(\omega)^{-1} = \frac{\psi(\omega)}{\pi(\omega)} \tag{161}$$

Combining Eqs. (157) and (158) and previous results we get

$$P_{B1}(\omega)/P_{B0} = G_A(\omega)/G_B(\omega) \tag{162}$$

The IMRS in good B is

$$M_B(\omega) = M_A(\omega) P_{B1}(\omega)/P_{B0} = \beta G_B(\omega)^{-1} \tag{163}$$

IMRSs are equated across countries in both goods (but not across goods) are equated: $M_A(\omega) = \tilde{M}_A(\omega)$ and $M_B(\omega) = \tilde{M}_B(\omega)$. This is true regardless of preferences.

Identical Preferences

If preferences are identical we have $\theta = \tilde{\theta}$ so that Eq. (160) becomes

$$P_{B0} = \frac{1-\theta}{\theta} \frac{Y_{A0}}{Y_{B0}} \tag{164}$$

and Eq. (162) implies

$$P_{B1}(\omega) = \frac{1-\theta}{\theta} \frac{Y_{A1}(\omega)}{Y_{B1}(\omega)} \tag{165}$$

Since trade is frictionless and preferences are identical $e_0 = e_1(\omega) = 1$.

We can solve for allocations by solving for L. To do this we consider the lifetime budget constraint, (147), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_{A0} + \beta\bar{s}_{A1} + (\frac{1-\theta}{\theta})(s_{B0} + \beta\bar{s}_{B1})\right] Y_{A0} \tag{166}$$

This implies

$$L = \frac{\theta(s_{A0} + \beta\bar{s}_{A1}) + (1-\theta)\,(s_{B0} + \beta\bar{s}_{B0})}{\theta(1+\beta)} Y_{A0} \tag{167}$$

Eq. (159) then implies that

$$\tilde{L} = \frac{\theta[(1-s_{A0}) + \beta(1-\bar{s}_{A1})] + (1-\theta)[(1-s_{B0}) + \beta(1-\bar{s}_{B1})]}{\theta(1+\beta)} Y_{A0} \tag{168}$$

Different Preferences

With different preferences we need to solve for L. To do this we consider the lifetime budget constraint, (147), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_{A0} + \beta\bar{s}_{A1} + (\frac{1-\tilde{\theta}}{\tilde{\theta}})(s_{B0} + \beta\bar{s}_{B1})\right] Y_{A0} + \frac{\tilde{\theta}-\theta}{\tilde{\theta}}(s_{B0} + \beta\bar{s}_{B1})L \tag{169}$$

This implies

$$L = \frac{\tilde{\theta}(s_{A0} + \beta\bar{s}_{A1}) + (1-\tilde{\theta})\,(s_{B0} + \beta\bar{s}_{B1})}{\tilde{\theta}(1+\beta) + (\theta-\tilde{\theta})\,(s_{B0} + \beta\bar{s}_{B1})} Y_{A0} \tag{170}$$

Eq. (159) then implies that

$$\tilde{L} = \frac{\theta[(1-s_{A0}) + \beta(1-\bar{s}_{A1})] + (1-\theta)[(1-s_{B0}) + \beta(1-\bar{s}_{B1})]}{\tilde{\theta}(1+\beta) + (\theta-\tilde{\theta})\,(s_{B0} + \beta\bar{s}_{B1})} Y_{A0} \tag{171}$$

and (160) implies that

$$P_{B0} = \frac{(1-\tilde{\theta})\,(1+\beta) + \left(\tilde{\theta}-\theta\right)(s_{A0} + \beta\bar{s}_{A1})}{\tilde{\theta}\,(1+\beta) + \left(\theta-\tilde{\theta}\right)(s_{B0} + \beta\bar{s}_{B1})} \frac{Y_{A0}}{Y_{B0}}\,, \tag{172}$$

Given Eq. (162) we have

$$P_{B1}(\omega) = \frac{G_A(\omega)}{G_B(\omega)} P_{B0} = \frac{(1-\tilde{\theta})\,(1+\beta) + \left(\tilde{\theta}-\theta\right)(s_{A0}+\beta\bar{s}_{A1})}{\tilde{\theta}\,(1+\beta) + \left(\theta-\tilde{\theta}\right)(s_{B0}+\beta\bar{s}_{B1})} \frac{Y_{A1}(\omega)}{Y_{B1}(\omega)}\,. \tag{173}$$

Since good B is traded, $\tilde{P}_{B0} = P_{B0}$ and $\tilde{P}_{B1}(\omega) = P_{B1}(\omega)$ so

$$e_0 = (\rho/\tilde{\rho}) P_{B0}^{\tilde{\theta}-\theta} \qquad e_1(\omega) = (\rho/\tilde{\rho}) P_{B1}(\omega)^{\tilde{\theta}-\theta}$$

But this means

$$\ln\left[e_1(\omega)/e_0\right] = (\tilde{\theta}-\theta)\ln\left[P_{B1}(\omega)/P_{B0}\right] = (\tilde{\theta}-\theta)\ln\left[G_A(\omega)/G_B(\omega)\right]$$

When Good B is Non-traded The market clearing conditions for good A are (151) and (152). For good B they are

$$c_{B0} = y_{B0}\,, \quad \tilde{c}_{B0} = \tilde{y}_{B0} \tag{174}$$

$$c_{B1}(\omega) = y_{B1}(\omega)\,, \quad \tilde{c}_{B1}(\omega) = \tilde{y}_{B1}(\omega) \tag{175}$$

The market clearing conditions and the FOCs together imply that Eqs. (155) and (156) hold along with

$$(1-\theta)L = P_{B0}y_{B0} \qquad (1-\tilde{\theta})\tilde{L} = \tilde{P}_{B0}\tilde{y}_{B0} \tag{176}$$

$$\beta(1-\theta)L = \frac{\psi(\omega)}{\pi(\omega)} P_{B1}(\omega)y_{B1}(\omega) \quad \beta(1-\tilde{\theta})\tilde{L} = \frac{\psi(\omega)}{\pi(\omega)}\tilde{P}_{B1}(\omega)\tilde{y}_{B1}(\omega) \tag{177}$$

Given the results so far, the lifetime budget constraint of the home household, (147), becomes:

$$(1+\beta)\theta L = \kappa Y_{A0},$$

where $\kappa = s_{A0} + \beta\bar{s}_{A1}$, and implies

$$L = \frac{\kappa}{\theta(1+\beta)} Y_{A0}. \tag{178}$$

If we combine Eqs. (155) and (178) we have

$$\tilde{L} = \frac{\tilde{\kappa}}{\tilde{\theta}(1+\beta)} Y_{A0} \tag{179}$$

where $\tilde{\kappa} = 1 - s_{A0} + \beta(1 - \bar{s}_{A1})$.

Combining Eqs. (155) and (156) we have

$$M_A(\omega) = \beta G_A(\omega)^{-1} = \frac{\psi(\omega)}{\pi(\omega)} \tag{180}$$

Combining Eqs. (176), (178) and (179) we have

$$P_{B0} = \frac{(1-\theta)\kappa}{\theta(1+\beta)} \frac{Y_{A0}}{y_{B0}} \qquad \tilde{P}_{B0} = \frac{(1-\tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1+\beta)} \frac{Y_{A0}}{\tilde{y}_{B0}}. \tag{181}$$

If we combine (176) and (177), and make use of (180) we get

$$\frac{P_{B1}(\omega)}{P_{B0}} = \frac{G_A(\omega)}{g_B(\omega)} \qquad \frac{\tilde{P}_{B1}(\omega)}{\tilde{P}_{B0}} = \frac{G_A(\omega)}{\tilde{g}_B(\omega)}. \tag{182}$$

Therefore, we can write

$$P_{B1}(\omega) = \frac{(1-\theta)\kappa}{\theta(1+\beta)} \frac{Y_{A1}(\omega)}{y_{B1}(\omega)} \qquad \tilde{P}_{B1}(\omega) = \frac{(1-\tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1+\beta)} \frac{Y_{A1}(\omega)}{\tilde{y}_{B1}(\omega)}. \tag{183}$$

The IMRSs in good B are

$$M_B(\omega) = \frac{\psi(\omega)}{\pi(\omega)} \frac{P_{B1}(\omega)}{P_{B0}} = \frac{\beta}{g_B(\omega)} \quad \tilde{M}_B(\omega) = \frac{\psi(\omega)}{\pi(\omega)} \frac{\tilde{P}_{B1}(\omega)}{\tilde{P}_{B0}} = \frac{\beta}{\tilde{g}_B(\omega)}. \tag{184}$$

Identical Preferences

We have

$$e_0 = \left(P_{B0}/\tilde{P}_{B0}\right)^{1-\theta} = \left(\frac{\kappa}{\tilde{\kappa}} \frac{\tilde{y}_{B0}}{y_{B0}}\right)^{1-\theta}.$$

$$e_1(\omega) = \left(P_{B1}(\omega)/\tilde{P}_{B1}(\omega)\right)^{1-\theta} = \left[\frac{\kappa}{\tilde{\kappa}} \frac{\tilde{y}_{B1}(\omega)}{y_{B1}(\omega)}\right]^{1-\theta}.$$

And this means

$$\ln[e_1(\omega)/e_0] = (1-\theta)\ln\left[\tilde{g}_B(\omega)/g_B(\omega)\right]$$

Different Preferences

$$e_0 = (\frac{\rho}{\tilde{\rho}})\frac{P_{B0}^{1-\theta}}{\tilde{P}_{B0}^{1-\tilde{\theta}}} = (\frac{\rho}{\tilde{\rho}})\frac{\left[\frac{(1-\theta)}{(1+\beta)\theta}\kappa\frac{Y_{A0}}{y_{B0}}\right]^{1-\theta}}{\left[\frac{(1-\tilde{\theta})}{(1+\beta)\tilde{\theta}}\tilde{\kappa}\frac{Y_{A0}}{\tilde{y}_{B0}}\right]^{1-\tilde{\theta}}} = (\frac{\tilde{\theta}}{\theta})\frac{\left(\kappa y_{B0}^{-1}\right)^{1-\theta}}{\left(\tilde{\kappa}\tilde{y}_{B0}^{-1}\right)^{1-\tilde{\theta}}}\left(\frac{Y_{A0}}{1+\beta}\right)^{\tilde{\theta}-\theta}.$$

$$e_1(\omega) = (\frac{\rho}{\tilde{\rho}})\frac{P_{B1}(\omega)^{1-\theta}}{\tilde{P}_{B1}(\omega)^{1-\tilde{\theta}}} = (\frac{\tilde{\theta}}{\theta})\frac{\left[\kappa y_{B1}(\omega)^{-1}\right]^{1-\theta}}{\left[\tilde{\kappa}\tilde{y}_{B1}(\omega)^{-1}\right]^{1-\tilde{\theta}}}\left[\frac{Y_{A1}(\omega)}{1+\beta}\right]^{\tilde{\theta}-\theta}$$

So

$$\ln[e_1(\omega)/e_0] = (1-\tilde{\theta})\ln\tilde{g}_B(\omega) - (1-\theta)\ln g_B(\omega) + (\tilde{\theta}-\theta)\ln G_A(\omega).$$

Financial Autarky Since the countries are in financial autarky, we no longer have $\psi(\omega) = \tilde{\psi}(\omega)$, so the rearranged FOCs for consumption are

$$\theta L = c_{A0} \tag{185}$$

$$(1-\theta)L = c_{B0}P_{B0} \tag{186}$$

$$\beta\theta L = \frac{\psi(\omega)}{\pi(\omega)}c_{A1}(\omega) \tag{187}$$

$$\beta(1-\theta)L = \frac{\psi(\omega)}{\pi(\omega)}c_{B1}(\omega)P_{B1}(\omega) \tag{188}$$

$$\tilde{\theta}\tilde{L} = \tilde{c}_{A0} \tag{189}$$

$$(1-\tilde{\theta})\tilde{L} = \tilde{P}_{B0}\tilde{c}_{B0} \tag{190}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\tilde{\psi}(\omega)}{\pi(\omega)}\tilde{c}_{A1}(\omega) \tag{191}$$

$$\beta(1-\tilde{\theta})\tilde{L} = \frac{\tilde{\psi}(\omega)}{\pi(\omega)}\tilde{P}_{B1}(\omega)\tilde{c}_{B1}(\omega) \tag{192}$$

Bob's flow budget constraints must be satisfied with no asset holdings so we have

$$c_{A0} + P_{B0} c_{B0} = y_{A0} + P_{B0} y_{B0} \tag{193}$$

$$c_{A1}(\omega) + P_{B1}(\omega) c_{B1}(\omega) = y_{A1}(\omega) + P_{B1}(\omega) y_{B1}(\omega). \tag{194}$$

The expressions for the IMRSs in goods A and B:

$$M_A(\omega) = \beta \frac{c_{A0}}{c_{A1}(\omega)} = \frac{\psi(\omega)}{\pi(\omega)} \qquad \tilde{M}_A(\omega) = \beta \frac{\tilde{c}_{A0}}{\tilde{c}_{A1}(\omega)} = \frac{\tilde{\psi}(\omega)}{\pi(\omega)} \tag{195}$$

$$M_B(\omega) = \beta \frac{c_{B0}}{c_{B1}(\omega)} = \frac{\psi(\omega)}{\pi(\omega)} \frac{P_{B1}(\omega)}{P_{B0}} \quad \tilde{M}_B(\omega) = \beta \frac{\tilde{c}_{B0}}{\tilde{c}_{B1}(\omega)} = \frac{\tilde{\psi}(\omega)}{\pi(\omega)} \frac{\tilde{P}_{B1}(\omega)}{\tilde{P}_{B0}} \tag{196}$$

When Good B is Traded The market clearing conditions for goods are Eqs. (151)–(154), which, together with the FOCs, imply

$$\theta L + \tilde{\theta} \tilde{L} = Y_{A0} \tag{197}$$

$$\frac{\theta}{\psi(\omega)} L + \frac{\tilde{\theta}}{\tilde{\psi}(\omega)} \tilde{L} = \frac{1}{\beta \pi(\omega)} Y_{A1}(\omega) \tag{198}$$

$$(1-\theta) L + (1-\tilde{\theta}) \tilde{L} = P_{B0} Y_{B0} \tag{199}$$

$$(1-\theta) \frac{L}{\psi(\omega)} + (1-\tilde{\theta}) \frac{\tilde{L}}{\tilde{\psi}(\omega)} = \frac{1}{\beta \pi(\omega)} P_{B1}(\omega) Y_{B1}(\omega)\,. \tag{200}$$

We can rearrange Eqs. (197) and (199) to get:

$$\tilde{L} = \frac{1}{\tilde{\theta}} (Y_{A0} - \theta L) \tag{201}$$

$$P_{B0} = \frac{1}{\tilde{\theta}} \frac{(\tilde{\theta} - \theta) L + (1-\tilde{\theta}) Y_{A0}}{Y_{B0}}\,, \tag{202}$$

We can rearrange Eqs. (198) and (200) to get:

$$\frac{\beta \pi(\omega)}{\tilde{\psi}(\omega)} \tilde{L} = \frac{1}{\tilde{\theta}} \left[Y_{A1}(\omega) - \theta \frac{\beta \pi(\omega)}{\psi(\omega)} L \right]. \tag{203}$$

$$P_{B1}(\omega) = \frac{1}{\tilde{\theta}} \frac{(\tilde{\theta} - \theta)\frac{\beta\pi(\omega)}{\psi(\omega)} L + (1 - \tilde{\theta}) Y_{A1}(\omega)}{Y_{B1}(\omega)} \tag{204}$$

Eqs. (193) and (202) imply that

$$L = y_{A0} + \frac{\frac{\tilde{\theta}-\theta}{\tilde{\theta}} L + (1 - \tilde{\theta})\frac{Y_{A0}}{\tilde{\theta}}}{Y_{B0}} y_{B0}$$

or

$$L = \frac{\tilde{\theta} s_{A0} + (1 - \tilde{\theta}) s_{B0}}{\tilde{\theta} + (\theta - \tilde{\theta}) s_{B0}} Y_{A0} \tag{205}$$

Eqs. (194) and (204) imply that

$$\frac{\beta\pi(\omega)}{\psi(\omega)} L = \frac{\tilde{\theta} s_{A1}(\omega) + (1 - \tilde{\theta}) s_{B1}(\omega)}{\tilde{\theta} + (\theta - \tilde{\theta}) s_{B1}(\omega)} Y_{A1}(\omega). \tag{206}$$

Using (205) we then have Bob's IMRS in good A:

$$M_A(\omega) = \frac{\psi(\omega)}{\pi(\omega)} = \beta \frac{\xi_A(\omega)}{G_A(\omega)} \quad \text{with} \quad \xi_A(\omega) = \frac{\frac{\tilde{\theta} s_{A0} + (1-\tilde{\theta}) s_{B0}}{\tilde{\theta} + (\theta - \tilde{\theta}) s_{B0}}}{\frac{\tilde{\theta} s_{A1}(\omega) + (1-\tilde{\theta}) s_{B1}(\omega)}{\tilde{\theta} + (\theta - \tilde{\theta}) s_{B1}(\omega)}} \tag{207}$$

Substituting (205) into (201) we get

$$\tilde{L} = \frac{\theta(1 - s_{A0}) + (1 - \theta)\,(1 - s_{B0})}{\theta + (\tilde{\theta} - \theta)(1 - s_{B0})} Y_{A0}. \tag{208}$$

Substituting (206) into (203) we get

$$\frac{\beta\pi(\omega)}{\tilde{\psi}(\omega)} \tilde{L} = \frac{\theta[1 - s_{A1}(\omega)] + (1 - \theta)[1 - s_{B1}(\omega)]}{\theta + (\tilde{\theta} - \theta)[1 - s_{B1}(\omega)]} Y_{A1}(\omega). \tag{209}$$

Given these results, Amy's IMRS in good A is

$$\tilde{M}_A(\omega) = \frac{\tilde{\psi}(\omega)}{\pi(\omega)} = \beta \frac{\tilde{\xi}_A(\omega)}{G_A(\omega)}, \quad \tilde{\xi}_A(\omega) = \frac{\frac{\theta(1-s_{A0}) + (1-\theta)(1-s_{B0})}{\theta + (\tilde{\theta}-\theta)(1-s_{B0})}}{\frac{\theta[1-s_{A1}(\omega)] + (1-\theta)[1-s_{B1}(\omega)]}{\theta + (\tilde{\theta}-\theta)[1-s_{B1}(\omega)]}}. \tag{210}$$

Substituting (205) into (202)

$$P_{B0} = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_{A0}}{\tilde{\theta}+(\theta-\tilde{\theta})s_{B0}}\frac{Y_{A0}}{Y_{B0}} \tag{211}$$

Substituting (206) into (204)

$$P_{B1}(\omega) = \left[\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_{A1}(\omega)}{\tilde{\theta}+(\theta-\tilde{\theta})s_{B1}(\omega)}\right]\frac{Y_{A1}(\omega)}{Y_{B1}(\omega)} \tag{212}$$

IMRSs in good B are

$$M_B(\omega) = \beta\frac{\xi_A(\omega)}{G_A(\omega)}\frac{P_{B1}(\omega)}{P_{B0}} = \beta\frac{\xi_A(\omega)}{G_B(\omega)}\xi_B(\omega), \tag{213}$$

$$\tilde{M}_B(\omega) = \beta\frac{\tilde{\xi}_A(\omega)}{G_A(\omega)}\frac{P_{B1}(\omega)}{P_{B0}} = \beta\frac{\tilde{\xi}_A(\omega)}{G_B(\omega)}\xi_B(\omega) \tag{214}$$

with

$$\xi_B(\omega) = \left[\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_{A1}(\omega)}{\tilde{\theta}+(\theta-\tilde{\theta})s_{B1}(\omega)}\right] \bigg/ \left[\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_{A0}}{\tilde{\theta}+(\theta-\tilde{\theta})s_{B0}}\right]. \tag{215}$$

Identical Preferences

If preferences are identical Eqs. (211) and (212) simplify to

$$P_{B0} = \frac{1-\theta}{\theta}\frac{Y_{A0}}{Y_{B0}} \tag{216}$$

$$P_{B1}(\omega) = \frac{1-\theta}{\theta}\frac{Y_{A1}(\omega)}{Y_{B1}(\omega)} \tag{217}$$

Since both goods are frictionlessly traded and preferences are identical $e_0 = e_1(\omega) = 1$.

The expressions for ξ_A and $\tilde{\xi}_A$ in Eqs. (207) and (210) simplify to

$$\xi_A(\omega) = \frac{\theta s_{A0}+(1-\theta)s_{B0}}{\theta s_{A1}(\omega)+(1-\theta)s_{B1}(\omega)} \tag{218}$$

$$\tilde{\xi}_A(\omega) = \frac{\theta(1 - s_{A0}) + (1-\theta)\,(1 - s_{B0})}{\theta[1 - s_{A1}(\omega)] + (1-\theta)[1 - s_{B1}(\omega)]} \tag{219}$$

The expression for ξ_B in Eq. (213) simplifies to $\xi_B(\omega) = 1$, implying that

$$M_B(\omega) = \beta\frac{\xi_A(\omega)}{G_B(\omega)} \quad \tilde{M}_B(\omega) = \beta\frac{\tilde{\xi}_A(\omega)}{G_B(\omega)} \tag{220}$$

The wedge between IMRSs in goods A and B, and in terms of aggregate consumption, is

$$\tilde{M}_A(\omega)/M_A(\omega) = \tilde{M}_B(\omega)/M_B(\omega) = \tilde{M}(\omega)/M(\omega) = \tilde{\xi}_A(\omega)/\xi_A(\omega).$$

Different Preferences

Given the expressions for prices, above,

$$e_0 = (\rho/\tilde{\rho})P_{B0}^{\tilde{\theta}-\theta} = (\rho/\tilde{\rho})\left(\frac{1 - \tilde{\theta} + \left(\tilde{\theta} - \theta\right)s_{A0}}{\tilde{\theta} + \left(\theta - \tilde{\theta}\right)s_{B0}}\frac{Y_{A0}}{Y_{B0}}\right)^{\tilde{\theta}-\theta}$$

and

$$e_1(\omega) = (\rho/\tilde{\rho})P_{B1}(\omega)^{\tilde{\theta}-\theta} = (\rho/\tilde{\rho})\left(\frac{1 - \tilde{\theta} + (\tilde{\theta} - \theta)s_{A1}(\omega)}{\tilde{\theta} + (\theta - \tilde{\theta})s_{B1}(\omega)}\frac{Y_{A1}(\omega)}{Y_{B1}(\omega)}\right)^{\tilde{\theta}-\theta}$$

When Good B is Non-traded Because good B is non-traded the market clearing conditions and Bob's budget constraints together imply

$$c_{A0} = y_{A0}\,, \quad \tilde{c}_{A0} = \tilde{y}_{A0} \tag{221}$$

$$c_{A1}(\omega) = y_{A1}(\omega)\,, \quad \tilde{c}_{A1}(\omega) = \tilde{y}_{A1}(\omega) \tag{222}$$

$$c_{B0} = y_{B0}\,, \qquad \tilde{c}_{B0} = \tilde{y}_{B0}\,, \tag{223}$$

$$c_{B1}(\omega) = y_{B1}(\omega)\,, \quad \tilde{c}_{B1}(\omega) = \tilde{y}_{B1}(\omega)\,. \tag{224}$$

So

$$L = y_{A0}/\theta \tag{225}$$

$$P_{B0} = \frac{1-\theta}{\theta}\frac{y_{A0}}{y_{B0}} \tag{226}$$

$$\frac{\psi(\omega)}{\pi(\omega)} = \beta/g_A(\omega) \tag{227}$$

$$P_{B1}(\omega) = \frac{1-\theta}{\theta}\frac{y_{A1}(\omega)}{y_{B1}(\omega)} \tag{228}$$

$$\tilde{L} = \tilde{y}_{A0}/\tilde{\theta} \tag{229}$$

$$\tilde{P}_{B0} = \frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_{A0}}{\tilde{y}_{B0}}, \tag{230}$$

$$\frac{\tilde{\psi}(\omega)}{\pi(\omega)} = \beta/\tilde{g}_A(\omega) \tag{231}$$

$$\tilde{P}_{B1}(\omega) = \frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_{A1}(\omega)}{\tilde{y}_{B1}(\omega)} \tag{232}$$

IMRSs in goods A and B are

$$M_A(\omega) = \beta/g_A(\omega) \qquad \tilde{M}_A(\omega) = \beta/\tilde{g}_A(\omega)$$

$$M_B(\omega) = \beta/g_B(\omega) \qquad \tilde{M}_B(\omega) = \beta/\tilde{g}_B(\omega)$$

If preferences are identical we have

$$e_0 = \left(\frac{y_{A0}/y_{B0}}{\tilde{y}_{A0}/\tilde{y}_{B0}}\right)^{1-\theta} = \left(\frac{(1-s_{B0})/s_{B0}}{(1-s_{A0})/s_{A0}}\right)^{1-\theta}.$$

$$e_1(\omega) = \left(\frac{y_{A1}(\omega)/y_{B1}(\omega)}{\tilde{y}_{A1}(\omega)/\tilde{y}_{B1}(\omega)}\right)^{1-\theta} = \left(\frac{[1-s_{B1}(\omega)]/s_{B1}(\omega)}{[1-s_{A1}(\omega)]/s_{A1}(\omega)}\right)^{1-\theta}.$$

otherwise

$$e_0 = (\rho/\tilde{\rho})[\frac{1-\theta}{\theta}\frac{y_{A0}}{y_{B0}}]^{1-\theta}/[\frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_{A0}}{\tilde{y}_{B0}}]^{1-\tilde{\theta}}.$$

$$e_1(\omega) = (\rho/\tilde{\rho})[\frac{1-\theta}{\theta}\frac{y_{A1}(\omega)}{y_{B1}(\omega)}]^{1-\theta}/[\frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_{A1}(\omega)}{\tilde{y}_{B1}(\omega)}]^{1-\tilde{\theta}}.$$