NBER WORKING PAPER SERIES

EXCHANGE RATE DETERMINATION, RISK SHARING AND THE ASSET MARKET VIEW

A. Craig Burnside
Jeremy J. Graveline

Working Paper 18646
http://www.nber.org/papers/w18646

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
December 2012

A previous version of this paper was entitled "Exchange Rates and International Risk Sharing." We thank Hengjie Ai, Andrew Ang, Snehal Banerjee, Martin Bodenstein, John Cochrane, Bernard Dumas, Martin Eichenbaum, Martin Evans, Nir Jaimovich, Jonathan Parker, Uday Rajan, Sergio Rebelo, Lucio Sarno, Raj Singh, Gabriel Talmain, Giorgio Valente, Jules Van Binsbergen, seminar participants at Cass Business School, Duke University, Essex Business School, the Federal Reserve Board, and the University of Michigan, and conference participants at Northwestern University, the University of Glasgow, the University of Minnesota, Oxford University, the Chicago Fed, Queen's University, the American Finance Association meetings in San Diego, and the NBER Summer Institute Asset Pricing meeting for helpful comments and conversations. Any errors or omissions are our own. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

Exchange Rate Determination, Risk Sharing and the Asset Market View
A. Craig Burnside and Jeremy J. Graveline
NBER Working Paper No. 18646
December 2012, Revised March 2014
JEL No. F31,G15

ABSTRACT

Recent research in international asset pricing has argued that changes in real exchange rates can be understood using only asset market data and an equation that relates changes in the exchange rate to differences between representative agents' intertemporal marginal rates of substitution (IMRSs). We show that asset market data and this equation, alone, are not sufficient to understand how real exchange rates are determined, nor are they sufficient to economically interpret time-series variation in real exchange rates. Instead, we argue that it is necessary to make specific assumptions about preferences, goods market frictions, and asset markets. We also clarify the connection between agents' IMRSs and reduced-form stochastic discount factors (SDFs) that are identified using only asset market data. We show that reduced-form models of two SDFs that satisfy this equation have exactly the same economic content as an arbitrage-free statistical model of exchange rate dynamics.

A. Craig Burnside
Department of Economics
Duke University
213 Social Sciences Building
Durham, NC 27708-0097
and NBER
craig.burnside@duke.edu

Jeremy J. Graveline
3-129 CSOM
321 19th Ave S
Minneapolis, MN 55410
jeremy@umn.edu

The asset market view of exchange rates is now a dominant theoretical framework in international asset pricing.[1] According to this approach, the change in the real exchange rate between two economies is equal to the difference between the log intertemporal marginal rates of substitution (IMRSs) of representative agents in those economies. To fix ideas, consider the real exchange rate between the United States and the United Kingdom, and let Amy and Bob be representative agents in these two countries. The asset market view of the real U.S./U.K. exchange rate is encapsulated in the simple equation:

$$\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array} = \begin{array}{c}\text{Bob's log IMRS over}\\ \text{his consumption basket}\end{array} - \begin{array}{c}\text{Amy's log IMRS over}\\ \text{her consumption basket}\end{array}. \quad (1)$$

The asset market view in Eq. (1) has been used to gain insights into exchange rate determination, foreign exchange risk premia, and international risk sharing. There are at least three reasons for its widespread use. First, Eq. (1) only relies on consumption aggregates and utility functions, and does not depend on the specific nature of the market for goods and services. For example, the composition of Amy's and Bob's consumption baskets could be the same or different, and they could face the same or different prices for identical goods and services, but Eq. (1) still holds for aggregates. Second, the right hand side of Eq. (1) is often reinterpreted as the difference between log stochastic discount factors (SDFs) denominated in the real currency units of the two economies,

$$\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array} = \text{log SDF in real pounds} - \text{log SDF in real dollars}. \quad (2)$$

Since SDFs can be identified using only asset market data, Eq. (2) appears to offer insights into exchange rates that do not rely on a fully-specified economic model. Third, the connection between Eqs. (1) and (2) is believed to apply in very general asset market settings, including incomplete markets with minimum variance linear projections, as long as there is frictionless trade in assets and no arbitrage opportunities.

In this paper, we argue that the asset market view is not as widely applicable, or useful, as previous literature suggests. In Section 1 we establish that Eq. (1) does not in fact hold when agents' IMRSs are replaced by their minimum variance linear projections onto asset returns

[1] Examples of papers where this approach appears include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Smith and Wickens (2002); Brandt, Cochrane, and Santa-Clara (2006); Lustig and Verdelhan (2006); Brennan and Xia (2006); Lustig and Verdelhan (2007); Bakshi, Carr, and Wu (2008); Verdelhan (2010); Colacito and Croce (2011); Lustig, Roussanov, and Verdelhan (2011); Bansal and Shaliastovich (2013); and Lustig, Roussanov, and Verdelhan (2014). For an overview of this approach, see Lustig and Verdelhan (2012)'s chapter in the recent *Handbook of Exchange Rates*, entitled "Exchange Rates in a Stochastic Discount Factor Framework."

denominated in real currency units of the two economies. Instead, it is only guaranteed to hold in the special case that the available asset returns completely span agents' IMRSs, and all of the returns that are required for spanning are used in the projections. This result directly contradicts previous claims in many papers, including Brandt, Cochrane, and Santa-Clara (2006), Brennan and Xia (2006), Lustig and Verdelhan (2006), Alvarez, Atkeson, and Kehoe (2007), and Lustig and Verdelhan (2012). It has important implications for economic inferences, in these papers and others, that were previously thought to only rely on the weak assumptions of no arbitrage opportunities and frictionless trade in assets, and not require the much stronger assumption of complete markets. Moreover, this result is an example of a broader point that we make in Sections 1 and 4: regardless of asset market completeness, Eqs. (1) and (2) can always be viewed a change of numeraire units (say, from real dollars to real pounds) for a single SDF or the IMRS of a single agent. In general, Eqs. (1) and (2) do not hold true for separately derived SDFs or the IMRSs of different agents.

As an example of the significance of the above result, consider the paper by Brandt, Cochrane, and Santa-Clara (2006). They observe that the volatility of exchange rates is much smaller than the lower bound on the volatility of agents' IMRSs that is necessary to reconcile historical asset returns (Hansen and Jagannathan, 1991). They use this empirical observation, together with Eq. (1), to conclude that Amy's and Bob's IMRSs must be highly correlated, even if the asset market is incomplete. This conclusion—described by Brandt, Cochrane, and Santa-Clara (2006) as logically inescapable—has spawned a new literature that also adopts the asset market view, and seeks models in which IMRSs are both volatile and highly correlated (e.g., see Colacito and Croce, 2011 and Stathopoulos, 2011). The inference that IMRSs must be volatile is due to a well-known result in Hansen and Jagannathan (1991), which does not rely on complete markets. However, the conclusion that IMRSs must be highly correlated requires that Eq. (1) holds, and therefore it can only be drawn in the special case of complete markets. In general, with incomplete markets, as we illustrate with a model in Section 3, the volatility of the exchange rate is not tied to the correlation of agents' IMRSs.

We also argue, in Section 1, that the IMRSs that appear in Eq. (1) are only uniquely identified as Amy's and Bob's if there are frictions in the market for goods and services. If there are no such frictions, then regardless of whether the asset market is complete or incomplete, Eq. (1) can always be rewritten as a change of numeraire units for any single agent,

$$\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array} = \begin{array}{c}\text{any agent's log IMRS}\\ \text{over Bob's basket}\end{array} - \begin{array}{c}\text{that same agent's log IMRS}\\ \text{over Amy's basket}\end{array}. \quad (3)$$

In other words, with frictionless trade in goods, only the different consumption baskets are relevant in Eq. (1), not the different agents. Yet virtually all recent papers that draw economic insights about different agents based solely on Eq. (1) do not model specific frictions in the goods market. Indeed, much of the broad appeal of the asset market view seems to stem from the freedom it affords to abstract from the exact nature of the market for goods and services.

We argue that agents' preferences over, and frictions in the market for, goods and services are central to any explanation or interpretation of time-series variation in real exchange rates each period. The intuition for this point is simple. The real U.S./U.K. exchange rate is defined as the value of a unit of Bob's consumption basket of goods and services, at prices in the U.K., relative to a unit of Amy's consumption basket, valued at prices in the U.S., where both values are expressed in common units. Therefore, the real U.S./U.K. exchange rate can only vary if either the composition of Amy's and Bob's consumption baskets differs, or there are frictions in the market for goods and services, so that Amy and Bob face different prices in that market. These two channels for exchange rate variation have very different economic implications, but in both cases Eq. (1) holds if the asset market is complete.

For example, consider again the paper by Brandt, Cochrane, and Santa-Clara (2006). When Eq. (1) does hold, they interpret non-zero growth in the real exchange rate as evidence of imperfect risk sharing between Amy and Bob. However, as we discuss in Section 2, the validity of this interpretation depends crucially on the specific nature of the market for goods and services. In particular, if the composition of Amy's and Bob's consumption baskets differs, then the real exchange rate can vary simply because it represents the relative price of different baskets. Yet, if the asset market is complete and there are no frictions in the market for goods and services (i.e., there are no frictions in either market), then risk sharing is perfect.

In Section 3 we describe the necessary ingredients in any model that attempts to explain time-series variation in exchange rates and make statements such as: "the real value of the U.S. dollar appreciated this period because U.S. consumption growth fell relative to foreign consumption growth." Many papers that make such statements effectively treat aggregate consumptions in the foreign and domestic economies as exogenous.[2] However, in open (as opposed to closed) economies, each country's aggregate consumption is endogenous and can differ from its aggregate output. Therefore, causal statements that treat aggregate consumptions as exogenous are incongruous with open economy models. Instead, any explanation of the growth in the real exchange rate between two open economies requires a fully-specified model that maps exogenous shocks to agents' endowments or production technologies into

[2]For example, see Verdelhan (2010), Colacito and Croce (2011), and Bansal and Shaliastovich (2013).

equilibrium consumptions and the real exchange rate.[3]

In Section 3.2 we provide an example of a simple two-country endowment economy, based closely on Backus and Smith (1993). We assume that agents in the two countries have standard preferences over two goods. One good is frictionlessly traded. We alternately assume that the second good is not traded, or is frictionlessly traded. In this model, real exchange rate fluctuations arise out of variation in countries' endowments of the two goods. But the exact nature of these fluctuations—the mapping from endowments to consumptions and the real exchange rate—depends crucially on the precise nature of the assumed preference differences, imperfections in goods markets, and imperfections in asset markets.

Our model also illustrates that the theoretical link between imperfect risk sharing and real exchange rate variation (as argued by Brandt, Cochrane, and Santa-Clara, 2006) is tenuous. We construct equilibria with lots of real exchange rate variation combined with perfect or no risk sharing. Similarly, we construct equilibria with no real exchange rate variation combined with perfect or no risk sharing. The weak connection between risk sharing and exchange rate behavior is readily understood if we consider a single economy with two agents, in which all goods have the same prices in all markets. The fact that these agents face the same prices, and use the same numeraire to denominate those prices, does not tell us what the economy's overall risk sharing characteristics are. It only tells us that, if risk is not perfectly shared, goods market imperfections play no role. More generally, in a competitive equilibrium, agents must agree on the relative prices of all goods and assets that they can frictionlessly trade with each other. Therefore, no model-free economic inferences about different agents can be made using only these relative prices, regardless of whether the asset market is complete or incomplete.

Finally, in Section 4 we consider the large literature that uses no-arbitrage models of two (or more) SDFs to characterize and interpret exchange rate growth via Eq. (2). The models in these papers only require that there are no-arbitrage opportunities, and thus are subject to all the critiques and limitations that we described earlier, as well as others that we discuss. First, we show that Eq. (2) always represents a simple change of numeraire units for a single SDF rather than two different SDFs. Therefore, any no-arbitrage model with two SDFs that exploits Eq. (2) is always isomorphic to a model with the exchange rate and a single SDF. Second, we examine the common assumption of complete markets in this no-arbitrage literature. Many papers use this assumption as justification to interpret the SDFs in their no-arbitrage model as the IMRSs of representative agents in the two countries. However, the point that we previously highlighted around Eq. (3) also applies to these no-arbitrage

[3]As we make clear in Section 3, we are not suggesting that every interesting question in international economics must be addressed with a fully-specified economic model.

models: regardless of asset market completeness, it is impossible to distinguish between the IMRSs of different agents using only the returns on assets, including currencies, that they can frictionlessly trade with each other. Moreover, since there are not any agents in no-arbitrage models, the assumption of complete markets only ensures that every contingent claim on the assets in a no-arbitrage model can be exactly replicated by a self-financing trading strategy. Importantly, complete markets in a no-arbitrage model does not imply that the SDF can be interpreted as the IMRS of a representative agent. Instead, that economic interpretation carries the additional implication that the SDF in the model must also correctly price the return on every asset that agents can invest in (including assets that are outside the model). Third, we show that many papers that model multiple SDFs actually introduce additional assumptions beyond complete markets or the mere absence of arbitrage opportunities. For example, we show that a number of these papers assume that a common set of shocks drive both currencies and interest rates.[4] We also demonstrate that the model of two SDFs in Brandt and Santa-Clara (2002) is not arbitrage-free.

1 Real Exchange Rates and the Asset Market View

In this section we formally develop the asset market view of real exchange rates in Eq. (1).

1.1 Defining the Real Exchange Rate

We begin with the definition of the real exchange rate between two agents, Amy and Bob. They could be located anywhere in the world, including places that use the same nominal currency to denominate prices (e.g., different countries within the eurozone, or different states in the U.S.). To be concrete, we assume that they are representative agents in the United States and the United Kingdom. We make the standard assumption that there is frictionless trade in nominal currencies so that, without loss of generality, we use U.S. dollars to denominate the prices of all goods and assets, regardless of their location. If the price of a good or asset is denominated in a different nominal currency, then it can be converted to the U.S. dollar equivalent at the relevant nominal exchange rate.

Let P be the dollar value today of one unit of Amy's consumption basket of goods and services, at prices in the U.S., and let P' be its (uncertain) dollar value next period. Similarly, let $\tilde{P}$ be the U.S. dollar value today of one unit of Bob's consumption basket, at prices in the U.K., and let $\tilde{P}'$ be its (uncertain) dollar value next period. The real U.S./U.K. exchange

[4] A few examples of papers that make this assumption include: Backus, Foresi, and Telmer (2001); Brennan and Xia (2006); Backus, Gavazzoni, Telmer, and Zin (2010); Lustig, Roussanov, and Verdelhan (2011); and Gavazzoni, Sambalaibat, and Telmer (2013).

rate, e, is defined as the value of a unit of Bob's consumption basket relative to a unit of Amy's consumption basket,

$$e \equiv \tilde{P}/P \quad \text{and} \quad e' \equiv \tilde{P}'/P' \,. \tag{4}$$

Empirically, P is usually measured as the dollar value of the basket of consumer goods and services that is used to compute the consumer price index (CPI) in the U.S. Likewise, let $\tilde{P}^*$ denote the U.K. pound value, measured at U.K. prices, of the basket of consumer goods and services that is used to compute the CPI in the U.K. If S is the nominal dollar/pound exchange rate (i.e., the U.S. dollar price of one U.K. pound), then $\tilde{P} \equiv S\tilde{P}^*$ is the U.S. dollar value of this U.K. basket. Therefore, the growth in the real U.S./U.K. exchange rate is typically measured as

$$\ln(e'/e) \;=\; \ln(\tilde{P}^{*\prime}/\tilde{P}^*) \;+\; \ln(S'/S) \;-\; \ln(P'/P)\,. \tag{5}$$

The real U.S./U.K. exchange rate is always constant if the composition of Amy's and Bob's consumption baskets is the same and they face identical prices (measured in common units such as U.S. dollars) for the goods and services in their baskets. Therefore, the real U.S./U.K. exchange rate can only vary over time if:

1. The composition of Amy's and Bob's consumption baskets is different; and/or

2. Amy and Bob face different prices, in the U.S. and the U.K., for identical goods in their baskets (where, for comparison, prices are measured in common units).

Hence, to understand why real exchange rates vary over time, at a minimum it is necessary to understand why different agents may face different prices for identical goods and services, and/or why the composition of their consumption baskets may differ. We elaborate on this point in Section 3.

From an empirical standpoint, both of these necessary conditions for a variable real exchange rate are satisfied for virtually every country pair in the world. Different countries use different baskets of consumer goods and services to compute their respective consumer price indices. Also, identical goods and services frequently have different prices in different countries (i.e., purchasing power parity does not typically hold across countries, or even in different locations within the same country).

1.2 Euler Equations

Let λ be Amy's marginal utility today over units of her consumption basket, and let λ' denote the discounted value of her (uncertain) marginal utility next period. Amy's intertemporal marginal rate of substitution (IMRS), or discounted marginal utility growth, over units of her consumption basket is $m \equiv \lambda'/\lambda$. Today, with one dollar, Amy can purchase $1/P$ units of her consumption basket in the U.S. goods market. Therefore, $\Lambda = \lambda/P$ is her indirect marginal utility today over dollars, and $\Lambda' = \lambda'/P'$ is the discounted value of her indirect marginal utility tomorrow over dollars. It follows that

$$M \equiv \frac{\Lambda'}{\Lambda} = \frac{\lambda'}{\lambda}\frac{P}{P'} = m\frac{P}{P'} \tag{6}$$

is her IMRS over dollars.

Analogous to Amy, let $\tilde{\lambda}$ denote Bob's marginal utility today over a unit of his consumption basket and let $\tilde{\lambda}'$ denote the discounted value of his (uncertain) marginal utility next period. Then $\tilde{m} \equiv \tilde{\lambda}'/\tilde{\lambda}$ is Bob's IMRS over units of his consumption basket. Letting $\tilde{\Lambda} = \tilde{\lambda}/\tilde{P}$ and $\tilde{\Lambda}' = \tilde{\lambda}'/\tilde{P}'$, it follows that his IMRS over dollars is

$$\tilde{M} \equiv \frac{\tilde{\Lambda}'}{\tilde{\Lambda}} = \frac{\tilde{\lambda}'}{\tilde{\lambda}}\frac{\tilde{P}}{\tilde{P}'} = \tilde{m}\frac{\tilde{P}}{\tilde{P}'}\,. \tag{7}$$

As its name suggests, the asset market (or SDF) view of exchange rates emphasizes using asset markets to improve our understanding of exchange rates. The standard assumption in the literature is that trade in assets and currencies is frictionless. Therefore, we assume that Amy and Bob can trade a set of k assets at the same dollar-denominated prices. Let $\boldsymbol{X}$ be the k-dimensional vector of uncertain asset payoffs next period, and let $\boldsymbol{P_X}$ be the k-dimensional vector of asset prices today, with both being measured in U.S. dollars. For notational convenience, denote the vector of uncertain dollar-denominated returns on the assets by $\boldsymbol{R} \equiv \boldsymbol{X}/\boldsymbol{P_X}$. Then $\boldsymbol{r} \equiv \boldsymbol{R}P/P'$ is the vector of those asset returns denominated in units of Amy's consumption basket in the U.S. (real dollars), and $\tilde{\boldsymbol{r}} \equiv \boldsymbol{R}\tilde{P}/\tilde{P}' \equiv \boldsymbol{r}e/e'$ is the vector of the asset returns denominated in units of Bob's consumption basket in the U.K (real pounds).

In any standard model—for example, the one we outline in Section 3.2—if Amy maximizes her expected discounted lifetime utility, then her first order condition for optimality (i.e., her Euler equation) implies that

$$\boldsymbol{P_X}\Lambda = \mathbb{E}\left[\boldsymbol{X}\Lambda'\right], \quad \text{or equivalently,} \quad \mathbf{1} = \mathbb{E}\left[\boldsymbol{R}M\right]. \tag{8}$$

In Eq. (8), $\mathbf{1}$ denotes a k-dimensional vector of 1's and $\mathbb{E}[\cdot]$ denotes the expectations operator conditional on Amy's current information. Likewise, Bob's Euler equation is

$$\mathbf{1} = \mathbb{E}[\boldsymbol{R}\tilde{M}]\,, \tag{9}$$

under the standard assumption that Amy and Bob have the same information.

1.3 Complete Asset Markets and the Asset Market View

Together, Eqs. (8) and (9) do not imply that Amy and Bob always equate IMRSs over dollars (i.e., in general, $M \neq \tilde{M}$). However, Eqs. (8) and (9) do imply that the linear projections of their IMRSs over dollars, onto the dollar-denominated asset returns $\boldsymbol{R}$, must agree. In other words,

$$\mathbb{E}[\boldsymbol{R}M] = \mathbf{1} = \mathbb{E}[\boldsymbol{R}\tilde{M}] \quad \Rightarrow \quad \text{proj}\,[M\,|\boldsymbol{R}] = \text{proj}[\tilde{M}\,|\boldsymbol{R}]\,. \tag{10}$$

When the dollar-denominated asset returns, $\boldsymbol{R}$, completely span Amy's and Bob's IMRSs over dollars in every possible state of the world next period, we get $\text{proj}\,[M \mid \boldsymbol{R}] = M$ and $\text{proj}[\tilde{M} \mid \boldsymbol{R}] = \tilde{M}$, so that by Eq. (10) we have

$$M = \tilde{M}\,. \tag{11}$$

That is, in the special case of complete asset markets, Amy and Bob equate IMRSs over dollars in every state of the world next period.

Since, via a change of units, we always have $\tilde{M}/M = (\tilde{m}/m) \times (e/e')$, Eq. (11) holds if and only if the asset market view of exchange rates in Eq. (1) holds,

$$\underbrace{\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array}}_{\ln e' \;-\; \ln e} = \underbrace{\begin{array}{c}\text{Bob's log IMRS over}\\ \text{his consumption basket}\end{array}}_{\ln \tilde{m}} - \underbrace{\begin{array}{c}\text{Amy's log IMRS over}\\ \text{her consumption basket}\end{array}}_{\ln m}\,. \tag{1}$$

Eq. (1)—or, equivalently, Eq. (11)—is an equilibrium condition in any model with frictionless trade in assets whose returns completely span all possible states of the world next period. It is important to emphasize that Eq. (1) is not a moment condition (such as an Euler equation) that relates the expected (or average) growth in the real exchange rate to the expected (or average) difference between Bob's and Amy's log IMRSs. Instead, when Eq. (1) applies, it holds in every period and state of the world.

To gain some intuition for Eq. (1), note that states of the world in which an agent's IMRS is high are typically labeled as "bad times" for that agent, while states of the world where the agent's IMRS is low are referred to as "good times." Therefore, roughly speaking, when Eq. (1) holds it says that the real pound always depreciates against the real dollar whenever times improve more for Bob than for Amy, and it always appreciates whenever times improve more for Amy than for Bob. Again, since Eq. (1) applies in every period and state of the world, we use the term "always" rather than "tends to", "on average," or "is expected to." Also, note that we have been careful not to imply causality. Eq. (1) is only an equilibrium condition: the right hand side of Eq. (1) does not determine the left hand side any more than the left determines the right. Finally, there is an important caveat to our rough economic interpretation of Eq. (1). From standard microeconomics, it is only meaningful to compare Amy's and Bob's IMRSs if they are measured over identical consumption baskets. We elaborate on this point in Section 2.

1.4 Uniqueness of Agents in the Asset Market View

The IMRSs in Eq. (1) are only uniquely identified as Amy's and Bob's if there are frictions in the market for goods and services so that prices in the U.S. differ from those in the U.K. To appreciate this point, it is first helpful to understand that a version of Eq. (1) can always be viewed as a change of numeraire units from any agent's IMRS over Amy's basket to that same agent's IMRS over Bob's basket (and vice versa). To illustrate, suppose that there are no frictions in the goods market, so that Amy and Bob always face the same prices in the U.S. and the U.K. for identical goods and services in their baskets. In that case, in the U.S. Amy can trade a unit of Bob's basket today for e units of her basket, and next period she can trade one for e' units. Therefore, her marginal utility today over units of Bob's basket is λe and her IMRS over units of his basket is $m\, e'/e$. Thus, regardless of whether Eq. (1) holds, we can always rewrite this change of numeraire units for Amy (or any other agent) as

$$\underbrace{\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array}}_{\ln e' \;-\; \ln e} = \underbrace{\begin{array}{c}\text{Amy's log IMRS}\\ \text{over Bob's basket}\end{array}}_{\ln(m\, e'/e)} - \underbrace{\begin{array}{c}\text{Amy's log IMRS}\\ \text{over Amy's basket}\end{array}}_{\ln m} . \tag{12}$$

Eq. (12) is simply the formal version of Eq. (3) in the introduction (applied to Amy).

Likewise, if today Bob can trade a unit of Amy's basket for $1/e$ units of his basket in the U.K., then his marginal utility today over units of Amy's basket is $\tilde{\lambda}/e$, and his IMRS over units of her basket is $\tilde{m}\, e/e'$ (again, this change of numeraire units always applies, regardless of asset market completeness). In that case, if Eq. (1) holds then so too does

$$\underbrace{\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array}}_{\ln e' \;-\; \ln e} = \underbrace{\begin{array}{c}\text{Amy's log IMRS}\\ \text{over Bob's basket}\end{array}}_{\ln(m\, e'/e)} - \underbrace{\begin{array}{c}\text{Bob's log IMRS}\\ \text{over Amy's basket}\end{array}}_{\ln(\tilde{m}\, e/e')} . \tag{13}$$

Intuitively, if Amy and Bob can frictionlessly trade the goods and services in their consumption baskets, then they must agree on the relative price of those baskets, and therefore it is impossible to distinguish between them using only that relative price.

There are (at least) two important implications of the fact that Amy and Bob are only uniquely identified in Eq. (1) by frictions they face in the market for goods and services. First, since trade in currencies (and derivatives on currencies) is almost always assumed to be frictionless, the asset market view in Eq. (1) cannot be used as the sole basis to infer differences between Bob and Amy from the returns to currency investments (including the prices and returns on currency derivatives). Second, and related, Eq. (1) cannot be used as the sole basis to explain returns on currency investments, or growth in the real exchange rate. To elaborate on this second point, note that if there are no frictions in the market for goods and services, then Amy and Bob are not uniquely identified in Eq. (1), and therefore it cannot serve as the sole basis for an explanation (an analogous argument applies to nominal currencies with frictionless trade). On the other hand, if there are frictions in the goods market, then any explanation of growth in the real exchange rate also requires an understanding of the specific impact of those frictions. We elaborate on these points in Sections 2, 3, and 4.

1.5 Projections of IMRSs onto Asset Returns

Many papers claim that Eq. (1) holds more broadly, in incomplete markets, when agents' IMRSs—m and $\tilde{m}$—are replaced by their minimum variance linear projections onto asset returns—$\mathrm{proj}[m|\boldsymbol{r}]$ and $\mathrm{proj}[\tilde{m}|\tilde{\boldsymbol{r}}]$—denominated in real currency units of the two economies. Papers that make this claim include Brandt, Cochrane, and Santa-Clara (2006, p. 675), Brennan and Xia (2006, p. 759), Lustig and Verdelhan (2006, p. 648), Alvarez, Atkeson, and Kehoe (2007, p. 342), and Lustig and Verdelhan (2012, p. 395).[5]

[5]For example, Brandt, Cochrane, and Santa-Clara (2006, p. 675) states that:

> These discount factors are the projections of any possible domestic and foreign discount factors onto the relevant spaces of asset payoffs, and they are also the minimum-variance discount factors. We show that Eq. (1) continues to hold with this particular choice of discount factors.

This claim is incorrect. In general,

$$\ln e' - \ln e \;\neq\; \ln\left(\text{proj}[\tilde{m} \mid \tilde{\boldsymbol{r}}]\right) \;-\; \ln\left(\text{proj}[m \mid \boldsymbol{r}]\right) . \tag{14}$$

Eq. (14) follows immediately from the observation that, in general,

$$\text{proj}[m \mid \boldsymbol{r}] \;\times\; e'/e \neq \text{proj}[\tilde{m} \mid \tilde{\boldsymbol{r}}] . \tag{15}$$

To illustrate, suppose that $\boldsymbol{r}$ is a k-dimensional vector. The left hand side of Eq. (15) is linear in $\boldsymbol{r}e'/e$, so that

$$\text{proj}[m \mid \boldsymbol{r}] \times\; e'/e \;= \boldsymbol{\beta} \cdot \boldsymbol{r}\, e'/e\,, \tag{16}$$

for a k-dimensional vector $\boldsymbol{\beta}$ (where $\cdot$ denotes the dot product of two vectors). The right hand side of Eq. (15) is not linear in $\boldsymbol{r}e'/e$, but instead is linear in $\tilde{\boldsymbol{r}} = \boldsymbol{r}e/e'$. In general, $\boldsymbol{\beta} \cdot \boldsymbol{r}e'/e$ is not in the linear span of $\tilde{\boldsymbol{r}} = \boldsymbol{r}e/e'$.[6] To be more specific, suppose that there are $n > k$ states of the world next period, indexed by $\omega = 1, \ldots, n$. For example, the k asset returns could be log-normally distributed in discrete time, in which case the number of states of world next period is infinite (i.e., $n = \infty > k$). Let $\boldsymbol{r}\,(\omega)$ denote the k-dimensional vector of asset returns in state ω and let $e'\,(\omega)$ denote the real U.S./U.K. exchange rate in that state. In general, there does not exist a k-dimensional vector $\tilde{\boldsymbol{\beta}}$ that satisfies the necessary $n > k$ equations:

$$\tilde{\boldsymbol{\beta}} \cdot \boldsymbol{r}\,(\omega)\;\, e/e'\,(\omega) = \boldsymbol{\beta} \cdot \boldsymbol{r}\,(\omega)\;\, e'\,(\omega)\,/e\,, \quad \omega = 1, \ldots, n\,. \tag{17}$$

So, an equation that plays a prominent role in the asset market view of exchange rates does not hold. Instead, it only holds in the special case that the available asset returns completely span agents' IMRSs, and all of the returns that are required for spanning are used in the projections. In other words, Eq. (1) is not guaranteed to hold if the asset market is incomplete, and therefore it cannot serve as the basis for economic insights in that case. For example, in the introduction we highlighted that Brandt, Cochrane, and Santa-Clara

[6]Brandt, Cochrane, and Santa-Clara (2006), and others, incorrectly claim otherwise. For example, Brandt, Cochrane, and Santa-Clara (2006, p. 675) state that:

> If we start with $m^d_{t+1} \in X$, form $m^f_{t+1} = m^d_{t+1} \times e_{t+1}/e_t$ to satisfy (1), we can quickly see that m^f_{t+1} is in the payoff space available to the foreign investor.

The error in this statement is that if $m^d_{t+1} \in X$ is in the payoff space denominated in real domestic currency units, then $m^d_{t+1} \times e_t/e_{t+1}$ is always in the payoff space denominated in real foreign currency units, but, in general, $m^d_{t+1} \times e_{t+t}/e_t$ is not in that space. Brandt, Cochrane, and Santa-Clara (2006) also prove this claim in Appendix A of their paper. The error in their proof is that they show the result in a continuous-time diffusion setting, which is actually a complete market setting (for example, see Harrison and Pliska, 1981, 1983), and therefore cannot be used to prove that the result holds, in general, with incomplete markets.

(2006)'s inference about international risk sharing, based on Eq. (1), does not apply if the asset market is incomplete (we expand on this point in Section 2). As another specific example of the significance of the result in Eq. (14), consider the paper by Alvarez, Atkeson, and Kehoe (2007) who use the asset market view of exchange rates in Eq. (1) to argue that changes in nominal interest rates by central banks must have a large impact on the difference between the conditional variances of representative agents' IMRSs. Again, to generalize their argument beyond the special case of complete markets, they suggest that Eq. (1) continues to hold for minimum variance linear projections.[7]

Our paper is not about whether the asset market is complete or incomplete. Instead, in Section 2, we question the premise of Brandt, Cochrane, and Santa-Clara (2006) that Eq. (1) can be used to measure risk sharing without making important assumptions about the underlying economic model.[8] We show that any inference about risk sharing that is drawn from Eq. (1) requires strong assumptions. In Section 3 we argue that even when Eq. (1) does hold, it is not sufficient to explain or interpret time-series variation in real exchange rates. That objective requires a more fully specified model in order to understand the mapping between exogenous shocks and exchange rates. In Section 3.2 we provide a model to illustrate these points. Finally, in Section 4 we discuss limitations of no-arbitrage models that are used to understand and explain exchange rates via Eq. (2).

2 Correlated IMRSs and International Risk Sharing

In this section we provide a more detailed analysis of Brandt, Cochrane, and Santa-Clara (2006), and the subsequent literature that builds on their work.

As their starting point, Brandt, Cochrane, and Santa-Clara (2006) calculate the variance of both sides of Eq. (1) and rearrange the result as:

$$\operatorname{cov}(\ln \tilde{m}, \ln m) = \tfrac{1}{2}\{\operatorname{var}(\ln \tilde{m}) + \operatorname{var}(\ln m) - \operatorname{var}(\ln[e'/e])\}. \tag{18}$$

[7] To be clear, we take no stand on whether changes in nominal interest rates by central banks have a large or small impact on the difference between the conditional variances of representative agents' IMRSs. Rather, our point is that Eq. (1) is only guaranteed to hold if the asset market is complete, and therefore it cannot be used to generalize arguments beyond this special case.

[8] For example, on page 673 of Brandt, Cochrane, and Santa-Clara (2006) they state:

> Yet the conclusion is hard to escape. Our calculation uses only price data, and no quantity data or economic modeling (utility functions, income or productivity shock processes, and so forth). A large degree of international risk sharing is an inescapable logical conclusion of Eq. (1), a reasonably high equity premium (over 1%, as we show below), and the basic economic proposition that price ratios measure marginal rates of substitution.

From the abstract of Brandt, Cochrane, and Santa-Clara (2006):

> Exchange rates depreciate by the difference between domestic and foreign marginal utility growth or discount factors. Exchange rates vary a lot, as much as 15% per year. However, equity premia imply that marginal utility growth varies much more, by at least 50% per year. Therefore, marginal utility growth must be highly correlated across countries.

The point of the example in their abstract is that the data inform us that $\mathrm{var}(\ln \tilde{m})$ and $\mathrm{var}(\ln m)$ are much larger than $\mathrm{var}(\ln[e'/e])$. This empirical observation leads to their inference, from Eq. (18), that $\mathrm{cov}(\ln \tilde{m}, \ln m)$ is large so that Amy's and Bob's IMRSs are highly correlated.

Eq. (18) has spawned a new literature that also adopts the asset market view, and seeks models in which IMRSs are both volatile and highly correlated (e.g., see Colacito and Croce, 2011 and Stathopoulos, 2011). For example, from Colacito and Croce (2011, p. 154):

> We like to view this as an *international equity premium puzzle*. In a one-country model, consumption growth does not vary enough to explain the excess return over the risk-free rate. In a two-country model, consumption growth does not *covary* enough to track movements in the exchange rate and returns. This dichotomy of prices and quantities strikes us as an important unresolved puzzle in international finance.

The assumption that IMRSs must be volatile follows from Hansen and Jagannathan (1991)'s well-known results, which do not rely on complete markets. However, Eq. (18) is an implication of Eq. (1), which is only guaranteed to hold in complete markets. Therefore, the conclusion that highly correlated IMRSs are necessary for exchange rates to vary much less than IMRSs can only be drawn in the special case of complete markets. In general, with incomplete markets, as we illustrate with a model in Section 3.2, the volatility of the exchange rate is not tied to the correlation of agents' IMRSs.

2.1 What Can We Learn from Projections?

As we discussed in Section 1.5, Brandt, Cochrane, and Santa-Clara (2006) claim that Eq. (1) continues to hold for the linear projections, $\mathrm{proj}[m|\boldsymbol{r}]$ and $\mathrm{proj}[\tilde{m}|\tilde{\boldsymbol{r}}]$. They argue that this result makes their inference robust to the case in which asset markets do not completely span agents' IMRSs. In Section 1.5, we showed that Eq. (1) does not hold for these linear projections. However, irrespective of that result, we argue that it is impossible to draw conclusions about the correlation of m and $\tilde{m}$ from the correlation of $\mathrm{proj}[m|\boldsymbol{r}]$ and $\mathrm{proj}[\tilde{m}|\tilde{\boldsymbol{r}}]$.

More generally, projections can only be informative about commonalities of m and $\tilde{m}$, not about differences.

To better understand this point, it is helpful to imagine a case where the projections are done onto returns denominated in the same units. For example, it might be natural to project Amy's IMRS, M, and her neighbor Amelia's IMRS, $\check{M}$, onto a vector of asset returns, $\boldsymbol{R}$, measured in dollars. Both of these projections are equal to $\boldsymbol{R}^{\top}\mathbb{E}\left[\boldsymbol{R}\boldsymbol{R}^{\top}\right]^{-1}\mathbf{1}$, and are therefore perfectly correlated, but we cannot infer that Amy and Amelia's IMRSs are perfectly correlated. Although we learn that their IMRSs share the component $\boldsymbol{R}^{\top}\mathbb{E}\left[\boldsymbol{R}\boldsymbol{R}^{\top}\right]^{-1}\mathbf{1}$, the projection exercise sheds no light on the size of the unshared components.[9] The unshared components include all of Amy's and her neighbor's risks that go unshared due to asset market incompleteness, as well as any shared risk that happens to not be spanned by the specific asset returns used in the empirically-implemented projections.

A similar argument applies for the projections of Amy's and Bob's respective IMRSs onto asset returns measured in real dollars and pounds:

$$m = \text{proj}\left[m \mid \mathbf{r}\right] + \varepsilon\,, \quad \text{where} \quad \mathbb{E}\left[\mathbf{r}\varepsilon\right] = \mathbf{0}\,, \tag{19}$$

and

$$\tilde{m} = \text{proj}[\tilde{m} \mid \tilde{\mathbf{r}}] + \tilde{\varepsilon}\,, \quad \text{where} \quad \mathbb{E}\left[\tilde{\mathbf{r}}\tilde{\varepsilon}\right] = \mathbf{0}\,. \tag{20}$$

The projections, themselves, are uninformative about ε and $\tilde{\varepsilon}$, the components of the IMRSs that are orthogonal to the available asset returns.

More broadly, in any competitive equilibrium, agents must agree on the relative prices of all goods and assets that they can frictionlessly trade with each other. Therefore, no model-free economic inferences about different agents can be made using only these relative prices, regardless of whether the asset market is complete or incomplete.

In Section 4 we discuss no-arbitrage (statistical) models of asset returns that also provide a stochastic discount factor (SDF) for those returns. In the no-arbitrage literature, SDFs are frequently modeled (or can be expressed) as functions of the asset returns themselves. Since the projections in Eq. (14) are linear in the asset returns, the inequality in Eq. (14) obviously remains if the functional form of the SDF is linear in the asset returns. However, in Section 4.5 we provide two examples of functional forms for an SDF where the inequality in Eq. (14) becomes an equality: if the inverse of the SDF is linear in the asset returns, or if the log of the SDF is linear in the log of the asset returns.[10] Nevertheless, the broader

[9] Putting it differently, the projections put a lower bound on the variance of every agent's IMRS over dollars (Hansen and Jagannathan, 1991). We can't infer anything about the correlation of IMRSs from this lower bound unless we also have an upper bound on the variance of the agents' IMRSs.

[10] Affine models of asset returns often assume that the log of the SDF is linear in the log of the asset

economic point—that the returns on frictionlessly traded assets can only be informative about commonalities of m and $\tilde{m}$, but not about differences—always applies, irrespective of the functional form that a no-arbitrage model assumes for an SDF, or whether the inequality in Eq. (14) becomes an equality for that particular functional form.

2.2 Inference About International Risk Sharing

As we mentioned in the introduction, Brandt, Cochrane, and Santa-Clara (2006) interpret non-zero growth in the real exchange rate as evidence of imperfect risk sharing between Amy and Bob. They use Eq. (18) to compute $\mathrm{cov}(\ln\tilde{m}, \ln m)$, which they interpret as a measure of the degree of international risk sharing. Using this measure, and fixing $\mathrm{var}(\ln m)$ and $\mathrm{var}(\ln\tilde{m})$, they argue that a high volatility of the exchange rate indicates a low degree of international risk sharing, and vice versa. In fact, if $\mathrm{var}(\ln[e'/e]) = 0$, then Brandt, Cochrane, and Santa-Clara (2006) would infer that risk sharing is perfect between foreign and domestic investors. What exactly constitutes perfect risk sharing, and how is it related to the difference in agents' IMRSs (or discounted marginal utility growths)? To consider these issues, we return to the example of Amy and Bob.

Amy and Bob share risk perfectly if they equate IMRSs over all individual goods and services, and all common baskets of goods and services, in every state of the world next period. As a helpful example, consider again the setup in Section 1.4. Suppose that Amy and Bob face the same prices for identical goods and services in their baskets. Then Amy can trade one unit of Bob's basket for $\tilde{P}/P \equiv e$ units of her basket. Therefore, her marginal utility over units of Bob's basket is λe and her IMRS over units of his basket is me'/e. Likewise, Bob's marginal utility over units of Amy's basket is $\tilde{\lambda}/e$ (since he can trade a unit of Amy's basket for $P/\tilde{P} \equiv 1/e$ units of his basket), so his IMRS over units of her basket is $\tilde{m}e/e'$. If, in addition, we assume that asset markets are complete, then in every state of the world next period,

$$\underbrace{\text{Amy's IMRS over Amy's basket}}_{m} = \underbrace{\text{Bob's IMRS over Amy's basket}}_{\tilde{m}e/e'}, \tag{21}$$

or equivalently,

$$\underbrace{\text{Amy's IMRS over Bob's basket}}_{me'/e} = \underbrace{\text{Bob's IMRS over Bob's basket}}_{\tilde{m}}. \tag{22}$$

returns. For example, see Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Brennan and Xia (2006); and Lustig, Roussanov, and Verdelhan (2011).

So, in this example, Amy and Bob equate IMRSs over each others' baskets in every possible state of the world next period. It is straightforward to extend this analysis to show that, when asset markets are complete, Amy and Bob equate IMRSs over any individual good, or basket of goods and services, that they can frictionlessly trade with each other (i.e., for which they face identical prices). Therefore, Amy and Bob share risk perfectly in this example with complete markets and identical prices for the same goods and services. However, note that even if Amy and Bob share risk perfectly, their IMRSs in Eq. (1) can still differ when they are measured over different baskets of goods and services. In this case, the real exchange rate can also still vary because it too reflects the relative price of different baskets.[11]

With frictionless trade in assets, risk sharing can only be imperfect if asset markets are incomplete and/or agents face different prices for identical goods and services. If asset markets are incomplete, then Amy's and Bob's IMRSs can differ across states of the world that are not spanned by the available assets. If they face different prices for identical goods and services, then there must be a friction in that market that prevents these prices from being equal in different locations, and that friction can also prevent perfect risk sharing.

Contrary to the premise of Brandt, Cochrane, and Santa-Clara (2006), the conditions required for imperfect risk sharing do not completely overlap with the conditions for a variable real exchange rate. For example, as we noted earlier, if the composition of Amy's and Bob's consumption baskets is the same and they face identical prices for the goods and services in their baskets, then the real exchange rate is constant. Yet, risk sharing can still be imperfect if asset markets are incomplete. Here, Brandt, Cochrane, and Santa-Clara would draw the wrong inference because Eq. (1) does not hold in this setting, and therefore neither does Eq. (18). Conversely, if the asset market is complete and agents face identical prices for the goods and services in their baskets, then risk sharing is perfect (since they always equate IMRSs over any common basket of goods and services). Yet the exchange rate can still vary

[11] This point is certainly not new, as it is contained in most standard texts on microeconomics. The point is also not new to the international economics literature. For example, from Colacito and Croce (2011, p. 156–157):

> *Is correlation the same as risk sharing?*—Not necessarily, as this crucially depends on the model's assumptions. For example, consider an economy populated by two consumers with standard time-additive preferences defined over an aggregate of two tradable goods (e.g., one domestic and one foreign good). Risk sharing implies that the intertemporal marginal rates of substitution (IMRS) with respect to each good are equalized and therefore perfectly correlated across agents. If agents' preferences are biased toward the consumption of opposite goods (e.g., there is consumption home bias), however, their IMRS with respect to the consumption bundle will be less than perfectly correlated, even if in equilibrium all risk-sharing opportunities have been exhausted.

if the composition of Amy's and Bob's consumption baskets differs. Here, Brandt, Cochrane and Santa-Clara would draw the wrong inference because the covariance between $\ln \tilde{m}$ and $\ln m$ is not informative about risk sharing when Amy's and Bob's baskets differ.

	Composition of Consumption Baskets	
Asset Markets	Identical	Different
Complete	Yes	No
Incomplete	No	No

Table 1: Does a variable real exchange rate directly reflect imperfect risk sharing?

Table 1 provides the necessary conditions such that variation in the real exchange rate is a direct reflection of imperfect risk sharing. When the composition of Amy's and Bob's consumption baskets is identical, the exchange rate can only vary if they face different prices for identical goods and services in their baskets. Likewise, if asset markets are complete, then risk sharing can only be imperfect if there are frictions in the market for goods and services that prevent prices from being the same in different locations. In other words, under these specific assumptions, both imperfect risk sharing and variation in the real exchange rate can only occur if there are frictions in the market for goods and services.[12] Unfortunately, as Table 1 indicates, the link between risk sharing and a volatile real exchange rate only holds in this one special case. As previously noted, the difference in composition of Amy's and Bob's consumption baskets can contribute to variation in the real exchange rate, but it need not affect risk sharing. Likewise, incomplete asset markets can contribute to imperfect risk sharing, without affecting the volatility of the real exchange rate.

To emphasize, if we only observe variation in the real exchange rate, and nothing more, then we cannot be sure of the extent to which that variation reflects the relative prices of different baskets of goods and services, versus different prices for identical goods and services in those baskets. Similarly, incomplete asset markets are a source of imperfect risk sharing, but one cannot learn the extent of market incompleteness (i.e., the extent to which agents' IMRSs are not spanned by asset returns) from asset returns alone (including the returns on currencies). Therefore, very specific assumptions are required to make any inferences about international risk sharing using only observations of asset returns and variation in the real exchange rate. Any such inferences are not robust to different assumptions.

[12] Backus and Smith (1993) develop a model with complete markets and identical consumption baskets in order to study the impact of non-traded goods (i.e., frictions in the market for goods and services) on real exchange rates. In Section 3.2 we extend Backus and Smith (1993) to allow for incomplete markets, different preferences over goods across countries, and the possibility of frictionless trade in all goods. The model in Section 3.2 serves as a specific example to illustrate our points in Table 1.

3 Explaining Real Exchange Rates with a Model

In this section we discuss models of real exchange rate growth. In Section 3.1 we provide necessary ingredients for any model that is designed to explain real exchange rate growth. We also discuss recent papers that attempt to do so with models that treat aggregate consumptions in each country as exogenous. In Section 3.2 we provide an example of a model to illustrate these necessary ingredients.

3.1 Necessary Ingredients to Explain Exchange Rate Growth

A number of recent papers use Eq. (1) to bring insights about the nature of preferences from the asset pricing literature to the study of exchange rates. For example, Verdelhan (2010) uses representative agents with external habit preferences (see Campbell and Cochrane, 1999), while Colacito and Croce (2011) and Bansal and Shaliastovich (2013) use representative agents with Epstein-Zin recursive utility functions (see Epstein and Zin, 1989) who also face long-run risk (see Bansal and Yaron, 2004). Each of these papers use Eq. (1) to argue that shocks to the real exchange rate can be explained (or, are determined) by exogenous shocks to aggregate consumption growth in country.[13] For example, the introduction of Verdelhan (2010, p. 124) states that:

> When markets are complete, the real exchange rate, measured in units of domestic goods per foreign good, equals the ratio of foreign to domestic pricing kernels. Exchange rates thus depend on foreign and domestic consumption growth shocks. If the conditional variance of the domestic stochastic discount factor (SDF) is large relative to its foreign counterpart, then domestic consumption growth shocks determine variation in exchange rates.

What are the necessary ingredients of a model that attempts to explain real exchange rates? By "explain" we mean that the model provides a mapping from exogenous structural shocks (such as those to endowments or the production technology) into outcomes for the real exchange rate and agents' IMRSs. Without such a mapping, it is impossible to make

[13] Colacito and Croce (2011) treat aggregate endowments as exogenous, but they assume complete home bias in preferences so that aggregate consumptions equal aggregate endowments in each country. With complete home bias, the real exchange rate is not uniquely determined. Any real exchange rate clears the market since agents are assumed to have no desire to consume goods that they are not endowed with, regardless of the exchange rate. However, Colacito and Croce (2011) use the asset market view of real exchange rates in Eq. (1) to uniquely characterize the growth in the real exchange rate, even though it is not uniquely determined in their setup. Similarly, Gourio, Siemer, and Verdelhan (2013) assume that goods market frictions make trade in goods impossible, yet the real exchange rate between two countries is determined by Eq. (1).

a statement—such as the one above from Verdelhan (2010)—that "the real exchange rate appreciated or depreciated because X occurred."

As we argued in Section 1, real exchange rates don't vary over time unless agents have different consumption baskets and/or they face different prices for identical goods and services. Therefore, a model that is designed to explain real exchange rate growth must either specify the source of differences in basket composition (such as differences between agents' preferences over goods), or it must make specific assumptions about goods market frictions (such as tradability of different goods, or the magnitude of trade costs) that cause prices to differ across locations.

All of the papers that we referenced above are missing key ingredients that are necessary to explain real exchange rates. The models in all of these papers treat aggregate consumption growth in the foreign and domestic economies as exogenous structural shocks. In a closed endowment economy without physical investment or government purchases, aggregate consumption growth can be treated as exogenous because it must always equal the growth in aggregate endowment. However, there is a fundamental distinction between open and closed endowment economies: in an open economy, each country's aggregate consumption can differ from its aggregate output. To explain how the real exchange rate between two open economies is determined, such a model must map both countries' endowments into both countries' aggregate consumptions and the real exchange rate between them. Any model of open economies that treats aggregate consumption in each country as exogenous is silent about this map and, hence, is silent on the economic mechanism that determines real exchange rates. There may be asset markets, endowments (production technology), goods market frictions, and preferences over goods that could generate the assumed aggregate consumption processes of each country. Our point is that these elements are required to understand how aggregate consumptions and the real exchange rate are jointly determined in open economies. In contrast to the papers that we referenced above, in a recent paper, Colacito, Croce, Ho, and Howard (2013) provide a fully-specified model that maps exogenous shocks into IMRSs and exchange rates. This model has a richer set of implications for the joint behavior of consumptions, outputs, and the real exchange rate.

To be clear, we are not suggesting that every interesting question in international economics must be addressed with a fully-specified economic model. For example, there is no issue in open economies with the standard empirical exercise in consumption-based asset pricing that uses data on aggregate consumption and asset returns to directly test Euler equations (e.g., Eqs. 8 and 9 for Amy and Bob). As Hansen and Singleton (1982) illustrate, that approach can be fruitful for understanding the average (or expected) returns on different investments based on how the returns on those investments covary with a representative agent's IMRS. Additionally, there is no issue with closed-economy models that map trivially from endowments to consumptions and then to returns. Our point is simply that open economy models that treat shocks to consumption in each country as exogenous are not particularly useful for understanding or interpreting changes in real exchange rates in each period (rather than average, or expected, changes).

To illustrate our point with a specific example from the recent literature, consider the paper by Lustig and Verdelhan (2007). In Section I.D ("US Investor's Euler Equation") they specify a utility function over aggregate consumption growth for the representative agent in the U.S. (i.e., Amy). In Section II ("Does Consumption Risk Explain Foreign Currency Excess Returns?") they test Amy's first order condition in Eq. (8) using aggregate U.S. consumption growth and returns on portfolios of foreign currencies that are formed based on short-term interest rates. For this exercise, the exogeneity or endogeneity of aggregate U.S. consumption growth is irrelevant, as the question is simply whether the average, or expected, returns on these currency portfolios can be rationalized by their covariance with Amy's IMRS. Put differently, the specific economic mechanism that generates the joint distribution of Amy's IMRS with the returns on currency portfolios is not central to the question that is addressed in Section II of Lustig and Verdelhan (2007).

In Section III ("Mechanism") of Lustig and Verdelhan (2007) they aim to explain the covariance between the currency returns and Amy's IMRS, which is a much more ambitious objective. Their explanation for this covariance appeals to the asset market view of exchange rates in Eq. (1). From Lustig and Verdelhan (2007, p. 104) in Section III.B ("Where Do Consumption Betas of Currencies Come From?"):

> The answer is time variation in the conditional distribution of the foreign stochastic discount factor m^i. Investing in foreign currency is like betting on the difference between your own and your neighbor's IMRS. These bets are very risky if your IMRS is not correlated with that of your neighbor, but they provide a hedge when her IMRS is highly correlated and more volatile.

Here is exactly where we disagree. As we argued above, a model that treats aggregate consumption growth (or the IMRSs of representative agents) in each economy as exogenous is not useful for understanding the economic mechanism that determines the joint distribution of those aggregate consumptions with the growth in the real exchange rate.

The explanation in Lustig and Verdelhan (2007) actually just restates the change of numeraire units that we described in Eqs. (3) and (12). For example, take the common assumption in this literature that there is frictionless trade in assets and currencies. Analogous to the change of numeraire analysis in Section 1.4, if M is Amy's IMRS over U.S. dollars, and she can purchase S dollars with a U.K. pound, then MS'/S is her IMRS over pounds. We can always rewrite this change of numeraire as

$$\ln S' - \ln S = \underbrace{\text{Amy's log IMRS over pounds}}_{\ln(MS'/S)} - \underbrace{\text{Amy's log IMRS over dollars}}_{\ln M}. \tag{23}$$

Following Lustig and Verdelhan (2007), one could use Eq. (23) to argue that investing in U.K. pounds is like betting on the difference between Amy's log IMRS over pounds and her log IMRS over U.S. dollars (or, more generally, the difference between any agent's log IMRS over pounds and that same agent's log IMRS over dollars, where the agent could live anywhere in the world). This reasoning simply restates Amy's first order condition that must hold in any equilibrium. If the objective is to explain growth in the dollar/pound exchange rate, then obviously Amy's IMRS over dollars and her IMRS over pounds cannot be treated as exogenous. If the asset market is complete then, as Eq. (13) illustrates, one can switch Bob for Amy in either or both places in Eq. (23), but the same logic still applies.

One final point is worth highlighting. We have argued that models with exogenous consumption growth in each country are not useful for explaining real exchange rate growth via Eq. (1). Nevertheless, for any given model, it could still be the case that Eq. (1) provides a good empirical fit to the data. However, to date, we are not aware of any recent papers that treat consumption growth in each country as exogenous and empirically test whether Eq. (1) holds in every period and state of the world (rather than simply on average, or in expectation).[14]

Again, to be clear, we are not suggesting that any paper that assumes complete markets must empirically test whether Eq. (1) holds in every period. Models of exchange rates that contain the necessary ingredients we described above often assume complete markets for clarity and tractability. For example, Backus and Smith (1993) assume complete markets (i.e., completely frictionless asset markets) so as to focus exclusively on the role of nontradable goods (i.e., frictions in the goods market). In those papers, it may well be more appropriate

[14] Backus and Smith (1993) develop a model of open endowment economies with complete markets and non-tradable goods. They provide empirical evidence that Eq. (1) for their model does not hold in every period. From Backus and Smith (1993, p. 312-313):

> Further implications of the theory are that the growth rates of consumption ratios and of real exchange rates should have identical dynamics and be perfectly correlated. Fig. 2 shows that the growth rates of all 28 bilateral real exchange rates are positively autocorrelated, while 27 of the growth rates of consumption ratios are negatively autocorrelated. In addition, the cross-correlation between the growth rate of the consumption ratio and the growth rate of the real exchange rate, averaged across countries, is 0.045, with a range of [-0.08, 0.17]. Thus there is little evidence in favor of either of these implications of the theory.

to empirically test the central mechanism in the model, rather than the specific assumption of complete markets that is made to emphasize that mechanism. However, as we have argued above, papers that treat consumption growths as exogenous, do not completely specify the economic mechanism by which exchange rates are determined in open economies.

3.2 An Example of an Endowment Economy

In this section we provide an example of a full-fledged model to illustrate how aggregate consumptions and the real exchange rate are *jointly* determined in equilibrium. The model is a generalization of Backus and Smith (1993) along three dimensions. First, we allow for preference differences across countries. Second, we allow for incomplete markets. Third, like Backus and Smith (1993) we have two goods, but we explicitly compare the case where the second good is non-traded to the case in which it is frictionlessly traded. We don't view this model as a solution to existing exchange rate puzzles. Rather, it is merely illustrative of our point about the joint determination of consumptions and the real exchange rate.[15]

We describe an endowment economy with two countries ("home" and "foreign") and representative households within each country. Utility is defined over two goods, A and B. All goods are perishable and households live for two periods.

The representative household in the home economy has an instantaneous utility function

$$U(c_A, c_B) = u[c(c_A, c_B)], \tag{24}$$

where c_A and c_B denote, respectively, the consumption of goods A and B by the home household, $c(\cdot)$ is a homogeneous of degree one quasi-concave function of its arguments, and u is a monotonic function with standard properties. Similarly, the representative household in the foreign economy has the instantaneous utility function

$$\tilde{U}(\tilde{c}_A, \tilde{c}_B) = u[\tilde{c}(\tilde{c}_A, \tilde{c}_B)], \tag{25}$$

where $\tilde{c}_A$ and $\tilde{c}_B$ denote, respectively, the consumption of goods A and B by the foreign household, and $\tilde{c}(\cdot)$ is a homogeneous of degree one quasi-concave function of its arguments.

Both economies are cashless and use good A as the numeraire. Our model would have

[15] Dating back, at least, to the contributions of Stockman (1980) and Lucas (1982), a large literature has developed explicit equilibrium models of the exchange rate. To name but a few, Cole and Obstfeld (1991); Backus, Kehoe, and Kydland (1992); Dumas (1992); Backus and Smith (1993); Baxter and Crucini (1995); Obstfeld and Rogoff (1995); Sercu, Uppal, and Van Hulle (1995); Stockman and Tesar (1995); Bekaert (1996); Betts and Devereux (1996); Chari, Kehoe, and McGrattan (2002); Apte, Sercu, and Uppal (2004); Bacchetta and Wincoop (2006); Kocherlakota and Pistaferri (2007); Pavlova and Rigobon (2007); Bodenstein (2008); Benigno and Thoenissen (2008); Corsetti, Dedola, and Leduc (2008); and Corsetti, Dedola, and Viani (2011).

the same implications for the real exchange rate if we chose different numeraires. Goods markets meet sequentially. Good A is frictionlessly tradable. We alternately assume that good B is frictionlessly tradable or non-tradable. We let P_B and P_B' denote the prices of good B in the home economy in the first and second periods. Similarly, we let $\tilde{P}_B$ and $\tilde{P}_B'$ denote the prices of good B in the foreign economy in the first and second periods. When good B is frictionlessly tradable, its price must be the same in both countries,

$$P_B = \tilde{P}_B \quad \text{and} \quad P_B' = \tilde{P}_B' . \tag{26}$$

The natural definition of the consumer price index (CPI) in the home country is a variable P such that $c_A + P_B c_B = P\, c(c_A, c_B)$. Since $c(\cdot)$ and $\tilde{c}(\cdot)$ are homogeneous of degree one functions, it can be shown that there are homogeneous of degree one functions $H(\cdot)$ and $\tilde{H}(\cdot)$ whose form depends on $c(\cdot)$ and $\tilde{c}(\cdot)$, such that the home and foreign CPIs are:[16]

$$P = H(1, P_B) \quad \text{and} \quad \tilde{P} = \tilde{H}(1, \tilde{P}_B) . \tag{27}$$

Similarly the CPIs in period two are

$$P = H(1, P_B') \quad \text{and} \quad \tilde{P}' = \tilde{H}(1, \tilde{P}_B') . \tag{28}$$

Identical to Eq. (4) in Section 1, the real exchange rates in periods one and two are

$$e \equiv \tilde{P}/P \quad \text{and} \quad e' \equiv \tilde{P}'/P'. \tag{29}$$

In the special case where preferences are identical in the two countries, we have $H(\cdot) = \tilde{H}(\cdot)$. If, additionally, both goods are traded, $e = 1 = e'$, regardless of the asset market structure. If preferences differ across countries and both goods are traded, variation in the real exchange rate *can* arise even though $\tilde{P}_B = P_B$. All that is needed is variation in P_B. We can make these statements even though we've said nothing about asset markets. This is one concrete sense in which the link between exchange rates and asset markets is tenuous.

As was the case in Section 1, we assume that there are k assets with $k \times 1$ random payoff vector $\boldsymbol{X}$. The $k \times 1$ price vector today for these assets is $\boldsymbol{P_X}$. The payoffs and prices of the assets are measured in units of good A. The asset payoffs, and all variables in period two, depend on the state of the world in period two. For notational simplicity, however, we suppress the dependence of period two variables on the state of the world.

The household in the home country chooses c_A, c_B, c_A', c_B', and the $k \times 1$ vector $\boldsymbol{a}$, to

[16] For details, see the section on price aggregation in the appendix.

maximize

$$u[c(c_A, c_B)] + \beta\,\mathbb{E}\left\{u[c(c'_A, c'_B)\right\}\,, \tag{30}$$

subject to

$$c_A + P_B c_B + \boldsymbol{P_X}\cdot\boldsymbol{a} = y_A + P_B y_B \quad \text{and} \quad c'_A + P'_B c'_B = y'_A + P'_B y'_B + \boldsymbol{X}\cdot\boldsymbol{a}\,. \tag{31}$$

Here $0 < \beta < 1$, c_A and c_B are the household's current consumption of the two goods, c'_A and c'_B are the household's plans for future consumption of the two goods (in every possible state of the world), the jth element of $\boldsymbol{a}$ is the household's net purchases of asset j, and y_A, y_B, y'_A and y'_B are the household's current and future endowments of the two goods.

Similarly, the foreign household chooses $\tilde{c}_A$, $\tilde{c}_B$, $\tilde{c}'_A$, $\tilde{c}'_B$, and $\tilde{\boldsymbol{a}}$ to maximize

$$u[\tilde{c}(\tilde{c}_A, \tilde{c}_B)] + \beta\,\mathbb{E}\left\{u[\tilde{c}(\tilde{c}'_A, \tilde{c}'_B)]\right\}\,, \tag{32}$$

subject to

$$\tilde{c}_A + \tilde{P}_B \tilde{c}_B + \boldsymbol{P_X}\cdot\tilde{\boldsymbol{a}} = \tilde{y}_A + \tilde{P}_B \tilde{y}_B \quad \text{and} \quad \tilde{c}'_A + \tilde{P}'_B \tilde{c}'_B = \tilde{y}'_A + \tilde{P}'_B \tilde{y}'_B + \boldsymbol{X}\cdot\tilde{\boldsymbol{a}}\,. \tag{33}$$

Here $\tilde{c}_A$ and $\tilde{c}_B$ are the household's current consumption of the two goods, $\tilde{c}'_A$ and $\tilde{c}'_B$ are the household's plans for future consumption of the two goods (in every possible state of the world), $\tilde{\boldsymbol{a}}$ is a $k \times 1$ vector whose jth element is the household's net purchases of asset j, and $\tilde{y}_A$, $\tilde{y}_B$, $\tilde{y}'_A$ and $\tilde{y}'_B$ are the household's current and future endowments of the two goods.

The market clearing conditions for good A are

$$c_A + \tilde{c}_A = y_A + \tilde{y}_A \quad \text{and} \quad c'_A + \tilde{c}'_A = y'_A + \tilde{y}'_A\,. \tag{34}$$

When good B is tradable we have the following market clearing conditions

$$c_B + \tilde{c}_B = y_B + \tilde{y}_B \quad \text{and} \quad c'_B + \tilde{c}'_B = y'_B + \tilde{y}'_B\,. \tag{35}$$

When it is non-tradable, instead, we have

$$c_B = y_B\,, \quad \tilde{c}_B = \tilde{y}_B\,, \quad c'_B = y'_B \quad \text{and} \quad \tilde{c}'_B = \tilde{y}'_B\,. \tag{36}$$

The market clearing condition in asset markets is

$$\boldsymbol{a} + \tilde{\boldsymbol{a}} = \boldsymbol{0}\,. \tag{37}$$

Definition. A competitive equilibrium is values of the quantities c_A, c_B, c'_A, c'_B, $\boldsymbol{a}$, $\tilde{c}_A$, $\tilde{c}_B$, $\tilde{c}'_A$, $\tilde{c}'_B$, $\tilde{\boldsymbol{a}}$ and prices, P_B, P'_B, $\tilde{P}_B$, $\tilde{P}'_B$, and $\boldsymbol{P_X}$ such that the quantities solve the home and foreign country optimization problems (taking the prices as given), and such that the market clearing conditions are satisfied. When good B is frictionlessly traded, Eq. (26) must also be satisfied.

3.3 Risk Sharing and IMRSs

The two countries' discounted marginal utility growths, or IMRSs, defined over aggregate consumption are

$$m \equiv \beta u_c(c')/u_c(c) \quad \text{and} \quad \tilde{m} \equiv \beta u_{\tilde{c}}(\tilde{c}')/u_{\tilde{c}}(\tilde{c})\,. \tag{38}$$

Similarly, we can define the two countries' IMRSs over goods A and B:

$$m_A \equiv \beta u_{c_A}(c')/u_{c_A}(c)\,, \quad \tilde{m}_A \equiv \beta u_{\tilde{c}_A}(\tilde{c}')/u_{\tilde{c}_A}(\tilde{c})\,, \tag{39}$$

$$m_B \equiv \beta u_{c_B}(c')/u_{c_B}(c)\,, \quad \tilde{m}_B \equiv \beta u_{\tilde{c}_B}(\tilde{c}')/u_{\tilde{c}_B}(\tilde{c})\,. \tag{40}$$

Definition. Perfect risk sharing describes any competitive equilibrium in which $\tilde{m}_A = m_A$ and $\tilde{m}_B = m_B$ in every possible state of the world next period.

Our definition of perfect risk sharing is the same as the one in Section 1. For any individual good, the IMRSs are equated across agents. For any identical basket of goods, suitably defined, the same is true.

As we show in the Appendix, equilibrium in the goods market always produces the intuitive result that

$$\frac{u_c(c)}{u_{c_A}(c)} = P\,, \quad \frac{u_c(c')}{u_{c_A}(c')} = P'\,, \quad \frac{u_{\tilde{c}}(\tilde{c})}{u_{\tilde{c}_A}(\tilde{c})} = \tilde{P}\,, \quad \text{and} \quad \frac{u_{\tilde{c}}(\tilde{c}')}{u_{\tilde{c}_A}(\tilde{c}')} = \tilde{P}'\,. \tag{41}$$

Combining Eq. (41) with the definitions of IMRSs in Eqs. (38) and (39), produces

$$\frac{m}{m_A} = \frac{P'}{P} \quad \text{and} \quad \frac{\tilde{m}}{\tilde{m}_A} = \frac{\tilde{P}'}{\tilde{P}}\,, \tag{42}$$

so that

$$\frac{\tilde{m}}{m} = \frac{e'}{e}\Xi\,, \quad \text{with} \quad \Xi \equiv \frac{\tilde{m}_A}{m_A}\,. \tag{43}$$

In Eq. (43), $\Xi = 1$ whenever risk sharing is perfect in frictionlessly traded goods, and $\Xi \neq 1$ when risk sharing in those goods is imperfect. For example, when asset markets are complete, agents equate IMRSs across frictionlessly traded goods, and so $m_A = \tilde{m}_A$ and therefore

$\Xi = 1$. But in any incomplete asset markets setting, in general, $m_A \neq \tilde{m}_A$ and thus $\Xi \neq 1$. Also, note that Ξ is the same for any frictionlessly traded good (or basket of goods). Thus, $\tilde{m}_B/m_B = \tilde{m}_A/m_A \equiv \Xi$ whenever good B is frictionlessly traded.

3.4 Four Specific Examples

This section discusses four specific examples of our model, which combine different assumptions about financial markets (complete markets vs. financial autarky) and goods market frictions (good B is frictionlessly traded vs. good B is non-traded). By explicitly solving for the equilibrium in these four cases, we demonstrate that real exchange rates and agents' IMRSs are jointly determined by the laws of motion of the endowments, together with our assumptions about preferences, goods market frictions, and asset markets. We also illustrate a point we made in Section 1: The conditions under which risk sharing is imperfect, and those under which the real exchange rate varies, are different.

We adopt the assumption that $u(c) = \ln c$, and the consumption aggregates in the two countries are $c = c_A^{\theta} c_B^{1-\theta}$, and $\tilde{c} = \tilde{c}_A^{\tilde{\theta}} c_B^{1-\tilde{\theta}}$. These assumptions are useful because equilibrium prices and quantities can be worked out with pencil and paper. They imply that the CPIs in the two countries, measured in units of good A, are

$$P = \rho\, P_B^{1-\theta}\,, \qquad \text{and} \qquad \tilde{P} = \tilde{\rho}\, \tilde{P}_B^{1-\tilde{\theta}}\,, \tag{44}$$

with $\rho = \theta^{-\theta}(1-\theta)^{\theta-1}$, and $\tilde{\rho} = \tilde{\theta}^{-\tilde{\theta}}(1-\tilde{\theta})^{\tilde{\theta}-1}$. The real exchange rates in periods one and two are

$$e = (\tilde{\rho}/\rho)\tilde{P}_B^{1-\tilde{\theta}}/P_B^{1-\theta} \quad \text{and} \quad e' = (\tilde{\rho}/\rho)\tilde{P}_B'^{\,1-\tilde{\theta}}/P_B'^{\,1-\theta}\,. \tag{45}$$

We derive all of the solutions in detail in the Appendix. We use some notation in what follows. The global endowment of good A in period one is $Y_A = y_A + \tilde{y}_A$, while in period two it is $Y_A' = y_A' + \tilde{y}_A'$. Analogously, for good B we have $Y_B = y_B + \tilde{y}_B$, and $Y_B' = y_B' + \tilde{y}_B'$. The growth rates of the global endowments are $G_A = Y_A'/Y_A$ and $G_B = Y_B'/Y_B$. We also define $g_A = y_A'/y_A$, $g_B = y_B'/y_B$, $\tilde{g}_A = \tilde{y}_A'/\tilde{y}_A$ and $\tilde{g}_B = \tilde{y}_B'/\tilde{y}_B$. The home country's shares of the global endowment of good A are $s_A = y_A/Y_A$ and $s_A' = y_A'/Y_A'$, in periods one and two, respectively. Similarly, $s_B = y_B/Y_B$ and $s_B' = y_B'/Y_B'$. We let $\bar{s}_A' = \mathbb{E}[s_A']$ and $\bar{s}_B' = \mathbb{E}[s_B']$ denote the home country's average shares of the global endowments in period two.

3.4.1 Complete Markets, No Goods Market Frictions

When asset markets are complete internationally and there are no goods market frictions (i.e., good B is frictionlessly traded), then $P_B = \tilde{P}_B$ and $P_B' = \tilde{P}_B'$, and IMRSs in the individual goods are always equated across countries. As we show in the Appendix, in good A the IMRS is β/G_A. In good B the IMRS is β/G_B. Risk is shared perfectly, regardless of preferences.

In the case where preferences are identical, $e = 1$ and $e' = 1$. When preferences differ across countries the expressions in Eq. (45) simplify to $e = (\tilde{\rho}/\rho)P_B^{\theta-\tilde{\theta}}$ and $e' = (\tilde{\rho}/\rho){P'_B}^{\theta-\tilde{\theta}}$, where $P_B = \kappa Y_A/Y_B$, $P'_B = \kappa Y'_A/Y'_B$ and

$$\kappa = \frac{(1-\tilde{\theta})(1+\beta) + (\tilde{\theta}-\theta)(s_A + \beta\bar{s}'_A)}{\tilde{\theta}(1+\beta) + (\theta-\tilde{\theta})(s_B + \beta\bar{s}'_B)} \,. \tag{46}$$

Hence,

$$\ln(e'/e) = (\theta - \tilde{\theta})\ln(P'_B/P_B) = (\theta - \tilde{\theta})\ln(G_A/G_B)\,. \tag{47}$$

Real exchange rate fluctuations are driven by differences in the global growth rates of the endowments of goods A and B. We see that if the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If the foreign country's preferences put more weight on good B than home country preferences (i.e., $\tilde{\theta} < \theta$), then the foreign basket becomes relatively more expensive (the foreign country's real exchange rate appreciates).

3.4.2 Complete Markets, Good B is Non-traded

Now consider the case where asset markets are complete internationally, but good B is non-traded. In this case, in general, $P_B \neq \tilde{P}_B$. IMRSs in good A are always equated across countries: $m_A = \tilde{m}_A = \beta/G_A$. IMRSs in good B are, respectively, $m_B = \beta/g_B$ and $\tilde{m}_B = \beta/\tilde{g}_B$, so risk is not shared perfectly unless $g_B = \tilde{g}_B$ in every possible state of the world next period.

When preferences differ across countries the real exchange rates are given by Eq. (45), with prices given by

$$P_B = \kappa\frac{Y_A}{y_B}\,, \quad \tilde{P}_B = \tilde{\kappa}\frac{Y_A}{\tilde{y}_B}\,, \quad P'_B = \kappa\frac{Y'_A}{y'_B}\,, \quad \tilde{P}'_B = \tilde{\kappa}\frac{Y'_A}{\tilde{y}'_B}\,, \tag{48}$$

and

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_A + \beta\bar{s}'_A)\,, \quad \tilde{\kappa} = \frac{1-\tilde{\theta}}{(1+\beta)\tilde{\theta}}\left[1 - s_A + \beta(1-\bar{s}'_A)\right]\,. \tag{49}$$

This implies that

$$\ln(e'/e) = (1-\theta)\ln g_B - (1-\tilde{\theta})\ln \tilde{g}_B + (\theta - \tilde{\theta})\ln G_A\,. \tag{50}$$

Here, the real exchange rate depends on the relative growth rates of the endowment of good B in the two countries, but the two growth rates matter to different extents due to preference differences. Additionally, as was the case when good B was traded, if the foreign country's preferences put more weight on good B than home country preferences ($\tilde{\theta} < \theta$) then, other things being equal, the foreign county's real exchange rate appreciates when the global endowment of good A grows.

If preferences are identical, then the real exchange rate in Eq. (45) simplifies to $e = (\tilde{P}_B/P_B)^{1-\theta}$ and $e' = (\tilde{P}'_B/P'_B)^{1-\theta}$ with prices still given by Eq. (48), but Eq. (49) becomes

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_A + \beta \bar{s}'_A)\,, \quad \tilde{\kappa} = \frac{1-\theta}{(1+\beta)\theta}[1 - s_A + \beta(1-\bar{s}'_A)]\,. \tag{51}$$

This means that

$$\ln(e'/e) = (1-\theta)\ln(g_B/\tilde{g}_B)\,. \tag{52}$$

Here, the real exchange rate depends entirely on the relative growth rates of the endowment of good B in the two countries. If the endowment grows more slowly in the foreign country, its basket becomes relatively more expensive and its real exchange rate appreciates.

3.4.3 Financial Autarky, No Goods Market Frictions

The third case we consider is where no assets are traded internationally, but goods markets are frictionless. In this case, $P_B = \tilde{P}_B$ and $P'_B = \tilde{P}'_B$ in every possible state of the world next period. Risk sharing, in general, is imperfect. As we show in the Appendix, the ratio of IMRSs in the two countries is the same in goods A and B. That is

$$\frac{\tilde{m}_A}{m_A} = \frac{\tilde{m}_B}{m_B} = \Xi = \frac{\theta(1-s_A) + (1-\theta)(1-s_B)}{\tilde{\theta}s_A + (1-\tilde{\theta})s_B} \times \frac{\tilde{\theta}s'_A + (1-\tilde{\theta})s'_B}{\theta(1-s'_A) + (1-\theta)(1-s'_B)}. \tag{53}$$

This expression is the same when preferences are identical, except that $\theta = \tilde{\theta}$.

In the case where preferences are identical, $e = 1$ and $e' = 1$ in every possible state of the world next period. Risk sharing, on the other hand, can be good or bad. Suppose, for example, that the home country's shares of the global endowments vary and comove positively. In this case, Ξ deviates from one a lot, implying that risk sharing is limited.

On the other hand, suppose that business cycles are strongly correlated across countries, so that the home country's shares of the global endowments do not change very much across different states of the world next period. In this case, Ξ will be close to one in all states, implying a high degree of risk sharing.

When preferences differ across countries then $e = (\tilde{\rho}/\rho)P_B^{\theta-\tilde{\theta}}$ and $e' = (\tilde{\rho}/\rho){P'_B}^{\theta-\tilde{\theta}}$, where $P_B = \kappa Y_A/Y_B$, $P'_B = \kappa' Y'_A/Y'_B$, and

$$\kappa = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B}\,, \qquad \kappa' = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B}\,. \tag{54}$$

Hence,

$$\ln\left(e'/e\right) = (\theta-\tilde{\theta})\left[\ln(G_A/G_B) + \ln\left(\kappa'/\kappa\right)\right]. \tag{55}$$

As in the case of complete markets, real exchange rate fluctuations are driven by differences in the growth rates of the two endowments. If the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If the foreign country's preferences put more weight on good B than home country preferences ($\tilde{\theta} < \theta$) then the foreign basket becomes relatively more expensive (the foreign country's real exchange rate appreciates). But the way in which the countries' shares of the global endowments fluctuate also matters for the real exchange rate. In the example we just described, the real exchange rate rises more in states of the world where $\kappa' > \kappa$. This could reflect, for example, a rise in the foreign country's share of the global endowment of good A (a drop of s'_A) at the same time as the global endowment of A rises relative to the global endowment of B.

3.4.4 Financial Autarky, Good B is Non-traded

The final case we consider combines financial autarky with the assumption that good B is non-traded. In this case, each country simply consumes its own endowments. IMRSs in the individual goods are determined by the country-specific endowment growth rates. For good A they are $m_A = \beta/g_A$ and $\tilde{m}_A = \beta/\tilde{g}_A$. In good B they are $m_B = \beta/g_B$ and $m_B = \beta/\tilde{g}_B$. Risk is not shared unless growth rates happen to coincide. The real exchange rates in the two periods are given by Eq. (45), with

$$P_B = \frac{(1-\theta)}{\theta}\frac{y_A}{y_B}\,, \quad \tilde{P}_B = \frac{(1-\tilde{\theta})}{\tilde{\theta}}\frac{\tilde{y}_A}{\tilde{y}_B}\,, \quad P'_B = \frac{(1-\theta)}{\theta}\frac{y'_A}{y'_B}\,, \quad \text{and} \quad \tilde{P}'_B = \frac{(1-\tilde{\theta})}{\tilde{\theta}}\frac{\tilde{y}'_A}{\tilde{y}'_B}. \tag{56}$$

Hence

$$\ln(e'/e) = (1-\tilde{\theta})\ln(\tilde{g}_A/\tilde{g}_B) - (1-\theta)\ln(g_A/g_B). \tag{57}$$

Suppose endowment growth rates are identical across goods; i.e., $g_A = g_B$ and $\tilde{g}_A = \tilde{g}_B$. Notice that this implies $e = e'$. There is no variation in the real exchange rate. The extent of risk sharing, in contrast, depends only on whether $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. It could be good or bad. Suppose, on the other hand, that risk sharing is perfect; i.e., $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. We only get the result that $e = e'$ if $\theta = \tilde{\theta}$.

3.4.5 Discussion

Consider Table 1 from Section 2.2. It states that under complete markets, the observation that real exchange rates are variable only implies imperfect risk sharing when the two countries have the same consumption basket. In our model, the countries have identically-composed consumption baskets if and only if $\theta = \tilde{\theta}$, because θ and $\tilde{\theta}$ are the constant expenditure shares of good A in the two countries.

So suppose that $\theta = \tilde{\theta}$. Under complete markets, we saw that $\ln(e'/e) = 0$ and risk sharing is perfect if trade in both goods is frictionless. On the other hand, $\ln(e'/e) = (1-\theta)\ln(g_B/\tilde{g}_B)$ and $\tilde{m}_B/m_B = g_B/\tilde{g}_B$ if good B is non-traded. If one is willing to assume that markets are complete, and that countries have identical preferences, risk sharing and exchange rate changes are intimately linked in our model.

Under incomplete markets, however, there is no link, in general, between risk sharing and exchange rates, even when $\theta = \tilde{\theta}$. When $\theta = \tilde{\theta}$, and trade in both goods is frictionless, $\ln(e'/e) = 0$ yet Ξ can depart arbitrarily from one, and therefore risk sharing can be arbitrarily imperfect. When $\theta = \tilde{\theta}$, and good B is non-traded, $\ln(e'/e) = 0$ when risk sharing happens to be perfect (i.e., when $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$ in every possible state of the world next period), but we also have $\ln(e'/e) = 0$ when risk sharing is "poor" and $g_A = g_B \neq \tilde{g}_A = \tilde{g}_B$.

More generally, our model illustrates that there is no direct link between the degree of risk sharing and real exchange rate variability.

4 No-Arbitrage Models of Asset Returns

In this section we discuss the large literature that uses no-arbitrage models of two (or more) stochastic discount factors (SDFs) to characterize and interpret exchange rate growth via Eq. (2).[17] Papers in this literature typically assume that there are two distinct SDFs in

[17]Examples of papers that pursue this modeling approach include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Brennan and Xia (2006); Brandt, Cochrane, and Santa-Clara (2006); Bakshi, Carr, and Wu (2008); Lustig, Roussanov, and Verdelhan (2011); and Lustig, Roussanov, and Verdelhan (2014).

Eq. (2). However, we show that Eq. (2) actually characterizes the change of numeraire units for any single SDF. In other words, the two SDFs in Eq. (2) actually represent the same SDF simply denominated in different units. This result is analogous to the change of numeraire units for any single agent's IMRS, which we illustrated in Eqs. (3), (12), and (23). The flip side of this result is that Eq. (2) does not hold for two different SDFs. We are not aware of any benefit of modeling more than one SDF in a no-arbitrage model. However, we illustrate at least point potential pitfall: we show that Brandt and Santa-Clara (2002)'s model of two distinct SDFs is not arbitrage-free because it assigns different prices to the same zero-coupon bond.

Papers in this literature also frequently assume complete markets. This assumption is often used as justification to economically interpret the SDF denominated in a country's (real or nominal) currency as the IMRS of a representative agent in that country. However, since there are not any agents in no-arbitrage models, complete markets only implies that every contingent claim on the assets in the model can be exactly replicated by a self-financing trading strategy. The additional assumption that the asset returns in the model completely span agents' IMRSs implies that the SDF in the model must also be consistent with the returns on all of the assets that agents can invest in, including assets that are outside of the model. Moreover, even if agents' IMRSs are completely spanned, it is impossible to distinguish between those agents using only the returns on assets, including currencies, that they can frictionlessly trade with each other. We made this same point in Section 1.4, Eq. (13), and in Section 1.5.

Finally, we show that many papers that model two (or more) SDFs also assume that currency returns are driven by the same shocks as the returns on other assets.[18] This relationship is not an implication of either no-arbitrage or complete markets, and existing empirical evidence strongly suggests that it does not hold in the data. We also show that it is challenging to relax this assumption in a model of two SDFs, but it is trivial to do so in a model where exchange rate growth or currency returns are modeled directly.

4.1 Stochastic Discount Factors

We begin by formalizing the notion of a stochastic discount factor (SDF). Consider again the setup in Section 1. Let $\boldsymbol{R}$ denote a k-dimensional random vector of asset returns denominated in nominal U.S. dollars. An SDF for these dollar-denominated asset returns is any strictly

[18]For instance, examples of papers that assume that currencies and interest rates are driven by a common set of shocks include: Backus, Foresi, and Telmer (2001); Brennan and Xia (2006); Backus, Gavazzoni, Telmer, and Zin (2010); Lustig, Roussanov, and Verdelhan (2011); and Gavazzoni, Sambalaibat, and Telmer (2013).

positive random variable $M > 0$ such that

$$\mathbf{1} = \mathbb{E}\left[\boldsymbol{R}M\right] . \tag{58}$$

From Eqs. (8) and (9) in Section 1.2, Amy's and Bob's IMRSs over nominal U.S. dollars are both examples of SDFs for the asset returns denominated in dollars. By the Fundamental Theorem of Asset Pricing (e.g., see Dybvig and Ross, 2003 or Dybvig and Ross, 2008), there are no arbitrage opportunities within a set of asset returns $\boldsymbol{R}$ if and only if there exists an SDF that satisfies Eq. (58) for those returns.

An SDF effectively assigns a strictly positive dollar value to each state of the world next period. To illustrate, suppose that the returns on the k assets vary over $n \geq k$ states of the world next period, indexed by $\omega = 1, 2, \ldots, n$. Let $\pi(\omega)$ be the probability that state ω occurs next period, so that Eq. (58) can be written more explicitly as

$$\mathbf{1} = \mathbb{E}\left[\boldsymbol{R}M\right] \equiv \sum_{\omega=1}^{n} \boldsymbol{R}(\omega) M\left(\omega\right) \pi\left(\omega\right) . \tag{59}$$

In Eq. (59), $M\left(\omega\right) \pi\left(\omega\right)$ is commonly interpreted as the dollar price today of a claim that pays one dollar next period when state ω occurs (i.e., the price of an Arrow-Debreu state contingent claim), whether or not such claims are available in asset markets.

In a reduced-form statistical model of arbitrage-free asset returns, it is not necessary to explicitly construct, or model, an SDF for those returns.[19] However, most recent asset pricing papers provide an SDF because it serves (at least) two convenient purposes. First, an SDF that satisfies Eq. (58) guarantees—by the Fundamental Theorem of Asset Pricing—that there are no arbitrage opportunities within the set of asset returns in the model. Second, an SDF can be used to conveniently compute arbitrage-free prices of contingent claims on those assets (e.g., see Harrison and Kreps, 1979; and Harrison and Pliska, 1981, 1983).

4.2 Change of Numeraire for an SDF

Suppose that we denominate the asset returns in a different numeraire with dollar-price ζ today and $\zeta'\left(\omega\right)$ in state ω next period. What price today does an SDF, M, for the asset returns denominated in nominal U.S. dollars, assign to a claim that pays one unit of this different numeraire in state ω next period?

[19] For instance, early examples of papers that provide arbitrage-free models of asset returns, but do not explicitly model an SDF, include Black and Scholes (1973), Merton (1973), and Vasicek (1977). Of course, since these models are arbitrage-free, by the Fundamental Theorem of Asset Pricing there always exists an SDF for the asset returns. Our point is simply that, even though it may be convenient to explicitly model an SDF, it is not strictly necessary to do so.

A claim that pays one unit of the new numeraire in state ω next period is equivalent to $\zeta'(\omega)$ units of a claim that pays one U.S. dollar in state ω next period. From Eq. (59), one unit of this U.S. dollar state contingent claim is worth $M(\omega)\,\pi(\omega)$ dollars today. Therefore, $\zeta'(\omega)$ units of it are worth $M(\omega)\,\pi(\omega)\,\zeta'(\omega)$ dollars today, or equivalently, $M(\omega)\,\pi(\omega)\,\zeta'(\omega)/\zeta$ units of the new numeraire today (since one dollar today is worth $1/\zeta$ units of the new numeraire). Thus, if M is an SDF for the dollar-denominated asset returns, $\boldsymbol{R}$, then $M\zeta'/\zeta$ is that same SDF when the same asset returns, $\boldsymbol{R}\,\zeta/\zeta'$, are instead denominated in the new numeraire with dollar price ζ. It is straightforward to verify that this mapping corresponds to the change of numeraire units for an SDF, since

$$\boldsymbol{R}\tfrac{\zeta}{\zeta'}M\tfrac{\zeta'}{\zeta}=\boldsymbol{R}M \quad \text{implies that} \quad \mathbf{1}=\mathbb{E}\left[\boldsymbol{R}M\right] \quad \Leftrightarrow \quad \mathbf{1}=\mathbb{E}\left[\boldsymbol{R}\tfrac{\zeta}{\zeta'}M\tfrac{\zeta'}{\zeta}\right]. \tag{60}$$

As an example of this change of numeraire, if M is an SDF for the asset returns denominated in nominal U.S. dollars, then MS'/S is that same SDF when the same asset returns, $\boldsymbol{R}S/S'$, are instead denominated in nominal U.K. pounds. If the same asset returns, $\boldsymbol{R}P/P'$, are instead denominated in units of Amy's consumption basket in the U.S., then MP'/P is that same SDF. If the same asset returns, $\boldsymbol{R}\tilde{P}/\tilde{P}'$, are denominated in units of Bob's consumption basket in the U.K., then $M\tilde{P}'/\tilde{P}\equiv(MP'/P)\,e'/e$ is that same SDF. If G is the dollar price of an ounce of gold, then $\boldsymbol{R}G/G'$ are the same asset returns denominated in ounces of gold, and MG'/G is that same SDF for those gold-denominated returns. And so on. Analogous to Eqs. (3), (12), and (23), this change of numeraire units for an SDF can always be written as

$$\underbrace{\begin{array}{c}\text{growth in nominal}\\ \text{dollar/pound}\\ \text{exchange rate}\end{array}}_{\ln S'\;-\;\ln S}=\underbrace{\begin{array}{c}\text{log SDF for asset}\\ \text{returns denominated}\\ \text{in nominal pounds}\end{array}}_{\ln(MS'/S)}-\underbrace{\begin{array}{c}\text{log of same SDF for same}\\ \text{asset returns denominated}\\ \text{in nominal dollars}\end{array}}_{\ln\mathrm{M}}, \tag{61}$$

or, equivalently,

$$\underbrace{\begin{array}{c}\text{growth in real}\\ \text{dollar/pound}\\ \text{exchange rate}\end{array}}_{\ln e'\;-\;\ln e}=\underbrace{\begin{array}{c}\text{log SDF for asset}\\ \text{returns denominated}\\ \text{in real pounds}\end{array}}_{\ln(\mathrm{M}\tilde{P}'/\tilde{P})}-\underbrace{\begin{array}{c}\text{log of same SDF for same}\\ \text{asset returns denominated}\\ \text{in real dollars}\end{array}}_{\ln(MP'/P)}. \tag{62}$$

4.3 Alternative Modeling Approach

As we mentioned in Section 4.1, an SDF is a convenient, but not a necessary, element of a no-arbitrage model of asset returns. However, SDFs play a central role in the formulation of many no-arbitrage models in which the asset returns include bank accounts denominated in more than one currency. We'll use a simple example to illustrate this alternative modeling approach that is common in the recent international asset pricing literature.

Let R_B be the certain dollar-denominated gross return from today to next period on a dollar-denominated default-free bank account (or, equivalently, a one-period default-free dollar-denominated bond). Similarly, let R_B^* be the certain pound-denominated gross return over that period on a pound-denominated default-free bank account. Let $\boldsymbol{R}_Y$ be the uncertain dollar-denominated gross returns from today to next period on a set Y of assets, and let $\boldsymbol{R}_Z^*$ be the uncertain pound-denominated gross returns on a different set Z of assets over that same period.[20] Let S and S' be the dollar/pound exchange rate today and next period. Then $R_B^* S'/S$ is the uncertain dollar-denominated gross return on the pound-denominated default-free bank account and $\boldsymbol{R}_Z^* S'/S$ are the uncertain dollar-denominated gross returns on the set Z of assets. The vector of dollar-denominated gross returns on all these assets stacked together is

$$\boldsymbol{R} = \left[\; R_B \,, \quad \boldsymbol{R}_Y \,, \quad \boldsymbol{R}_Z^* S'/S \,, \quad R_B^* S'/S \; \right]. \tag{63}$$

First, consider the standard approach that is typically employed in the broader asset pricing literature that develops reduced-form statistical models of arbitrage-free asset returns. In this approach, the joint distribution of the asset returns—in this example, $\boldsymbol{R}_Y$, $\boldsymbol{R}_Z^*$, and S'/S—are modeled directly.[21] A model might also explicitly construct an SDF for the asset returns in the model. In this particular example, any such SDF $M > 0$, must satisfy

$$1 = R_B \mathbb{E}\big[M\big]\,, \quad \mathbf{1} = \mathbb{E}\big[\boldsymbol{R}_Y M\big]\,, \quad \mathbf{1} = \mathbb{E}\big[\boldsymbol{R}_Z^* \tfrac{S'}{S} M\big]\,, \qquad \text{and} \qquad 1 = R_B^* \mathbb{E}\big[\tfrac{S'}{S} M\big]\,, \tag{64}$$

where $\mathbf{1}$ denotes a vector of 1's with the appropriate dimension in each equation. As we highlighted above, it is not necessary to explicitly formulate an SDF for the asset returns, but one is often included as a means to demonstrate the absence of arbitrage opportunities in the model, and/or to conveniently compute arbitrage-free prices of contingent claims on

[20] In many international asset pricing papers, $\boldsymbol{R}_Y$ are the dollar-denominated returns on dollar-denominated long-term bonds and, similarly, $\boldsymbol{R}_Z^*$ are the pound-denominated returns on pound-denominated long-term bonds. For example, see: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); and Brennan and Xia (2006).

[21] Note that the joint distribution of $\boldsymbol{R}_Y$, $\boldsymbol{R}_Z^*$, and S'/S completely characterizes the joint distribution of the vector of dollar-denominated asset returns, $\boldsymbol{R}$, in Eq. (63), and vice versa.

the assets in the model. As we also highlighted above, if the same asset returns are instead denominated in pounds, then MS'/S is the same SDF for those pound-denominated returns.

Many papers in the international asset pricing literature do not directly model the growth in the dollar/pound exchange rate, but instead model it indirectly. These papers model the joint distribution of the asset returns, $\boldsymbol{R}_Y$ and $\boldsymbol{R}_Z^*$, together with the distribution of an SDF, M, for the dollar-denominated returns on the assets, and an SDF, M^*, for the pound-denominated asset returns. If M and M^* represent the same SDF, simply denominated in different units, then from Eq. (61), the dollar/pound exchange rate must be

$$\ln S' - \ln S = \ln M^* - \ln M \,. \tag{65}$$

Likewise, papers that work with real (rather than nominal) asset returns model an SDF, m, for the asset returns denominated in units of Amy's consumption basket, and an SDF, m^*, for the same asset returns denominated instead in units of Bob's consumption basket. Again, if m and m^* represent the same SDF, then from the change of numeraire units in Eq. (62), we have

$$\ln e' - \ln e = \ln m^* - \ln m \,. \tag{66}$$

In theory, these two modeling approaches are isomorphic. Models that rely on Eq. (65) or Eq. (66) produce a joint distribution of the arbitrage-free asset returns $\boldsymbol{R}_Y$, $\boldsymbol{R}_Z^*$, and $S'/S = M^*/M$. Going in the other direction, if the asset returns are modeled in an arbitrage-free fashion then the Fundamental Theorem of Asset Pricing ensures that there exists a strictly positive SDF, $M > 0$, that satisfies Eq. (58). The change of numeraire in Eqs. (61) and (62) immediately implies Eqs. (65) and (66), with M^* defined as $M^* \equiv MS'/S$ and m^* defined as $m^* \equiv me'/e$.

In general, there is not a unique SDF that satisfies Eq. (58), yet Eq. (65) or Eq. (66) still holds for any single SDF expressed in different numeraire units. Therefore, if Eq. (65) does not hold, i.e.,

$$M^* S/S' \neq M \,, \tag{67}$$

then M and $M^* S/S'$ are simply different SDFs for the same dollar-denominated asset returns $\boldsymbol{R}$. Although there can be more than one SDF that satisfies Eq. (58), we are not aware of any benefit of modeling more than one SDF in a no-arbitrage model of asset returns.

Brandt and Santa-Clara (2002) model more than one SDF. In particular, they provide models of M and M^*, and assume that

$$MS'/S = M^* O \,, \quad \text{or equivalently,} \quad \frac{S'}{S} = \frac{M^*}{M} O \,, \tag{68}$$

where $\mathbb{E}[O] = 1$ and O is independent of M, M^*, and all assets.[22] Since Eq. (68) implies Eq. (67), Brandt and Santa-Clara (2002) model two different SDFs. However, those two SDFs cannot both satisfy Eq. (58) for all of the asset returns in their model. To illustrate, note that Brandt and Santa-Clara (2002) use their model of M and M^* to price foreign and domestic zero-coupon bonds with certain returns, R_B^* and R_B, denominated in the local currency. If M and $M^* S/S'$ both satisfy Eq. (58) for the dollar-denominated zero-coupon bond, then

$$1 = R_B \mathbb{E}[M] \quad \text{and} \quad 1 = R_B \mathbb{E}[M^* \tfrac{S}{S'}] \,. \tag{69}$$

However, Eq. (68) is inconsistent with Eq. (69) since, by Jensen's inequality,

$$1/R_B = \mathbb{E}[M^* \tfrac{S}{S'}] = \mathbb{E}[M/O] = \mathbb{E}[M]\,\mathbb{E}[1/O] > \mathbb{E}[M]/\mathbb{E}[O] = \mathbb{E}[M] = 1/R_B \,. \tag{70}$$

In other words, the model in Brandt and Santa-Clara (2002) is not free of arbitrage opportunities (i.e., it is not internally consistent), since it assigns two different prices to the same dollar-denominated zero-coupon bond.[23]

4.4 Complete Markets in No-Arbitrage Models

In the international asset pricing literature, many papers that model the growth in the exchange rate via Eq. (65) or (66) assume that there is a unique SDF that satisfies Eq. (58). The motivation for this assumption is twofold. First, if there is a unique SDF that satisfies Eq. (58) then Eqs. (65) and (66) must hold. However, as we illustrated in Section 4.2, Eqs. (61) and (62) always hold as a change of numeraire for any single SDF. In other words, uniqueness of the SDF in Eq. (58) is a sufficient, but not a necessary condition for Eqs. (65) and (66) to hold. Moreover, as we mentioned in Section 4.3, even if there is more than one SDF that satisfies Eq. (58), we are not aware of any benefit of modeling more than one SDF in a no-arbitrage model of asset returns (and the arbitrage opportunity in Brandt and Santa-Clara (2002) demonstrates at least one potential pitfall).

The second motivation for assuming uniqueness of the SDF in Eq. (58) is the desire to economically interpret the SDFs in Eq. (65) or (66) as the IMRSs of representative agents

[22] See Eq. (24) in Brandt and Santa-Clara (2002, p. 176). They state that "the key insight of our model is that when markets are incomplete, the volatility of the exchange rate is not uniquely determined by the domestic and foreign stochastic discount factors. ... If markets are incomplete, the volatility of the exchange rate can contain an element that is orthogonal to the priced sources of risk in both countries. ... To capture this excess volatility, we specify a stochastic process for the degree of market incompleteness."

[23] Similarly, Anderson, Hammond, and Ramezani (2010) show that, in the special case of an affine setting, the assumptions in Brandt and Santa-Clara (2002) are infeasible. Eq. (68) illustrates that the internal inconsistency (i.e., the arbitrage opportunity) applies more generally, beyond the specific affine structure.

in the two countries. For example, the introduction of Bakshi, Carr, and Wu (2008, p. 133) states that:

> In particular, because the ratio of the stochastic discount factors in two economies governs the exchange rate between them, the exchange rate market offers a direct information source for assessing the relative risk-taking behavior of investors in international economies.

As another example, from Lustig, Roussanov, and Verdelhan (2011, p. 26):

> We derive new restrictions on the stochastic discount factors (at home and abroad) that need to be satisfied in order to reproduce the carry trade risk premium that we have documented in the data.

In Section 1.4, Eq. (13), and in Section 1.5 we argued that it is impossible to distinguish between agents using only the returns on assets, including currencies, that they can frictionlessly trade with each other. That same point also applies to no-arbitrage models, regardless of whether there is a unique SDF that satisfies Eq. (58). Furthermore, as we discuss below, the notion of complete markets, or a unique SDF, has a different connotation in no-arbitrage models.

Following Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), there is a unique SDF in a no-arbitrage model of asset returns if the payoff on every square-integrable contingent claim on the assets can be exactly replicated by a self-financing trading strategy in those assets (see also Section 6I in Duffie, 2001).[24] For example, there is a unique SDF in the simple binomial tree model of Cox, Ross, and Rubinstein (1979) that is frequently used to illustrate the pricing of contingent claims (e.g., options) on a single asset such as a stock. However, it is important to recognize that uniqueness of this SDF does not imply that it is equal to the IMRS of a representative agent. Instead, if there is a unique SDF in a no-arbitrage model of asset returns, then as Eq. (71) illustrates, we can only say that it is equal to the conditional expectation of any agent's IMRS given the returns on those specific assets. To go a step further, and assume that the unique SDF in a no-arbitrage model is equal to a representative agent's IMRS (rather than just the conditional expectation of their IMRS), one must also assume that the specific set of asset returns in the model completely span the agent's IMRS. Moreover, even if there is some set of assets that completely span the agent's IMRS, if a no-arbitrage model only contains a subset of those assets that are necessary for

[24] In Eq. (71) we highlighted that in no-arbitrage models of asset returns alone, it is only possible to learn about the conditional expectation, $\mathbb{E}[M|\boldsymbol{R}]$, of an SDF given those returns. Therefore, in those models, uniqueness of an SDF is defined over the space of stochastic processes that are adapted to the filtration generated by the asset returns themselves.

spanning, then the SDF in that model cannot be interpreted as a representative agent's IMRS. Yet papers in this literature that economically interpret an SDF as the representative agent's IMRS typically model only a very small subset of the assets that are available for agents to invest in. Therefore, this interpretation carries with it the the implicit assumption that the small set of assets in the model completely spans agents' IMRSs (i.e., spans all of the risks that agents care about).

Since there are not any agents in a no-arbitrage model of asset returns, it is impossible to test whether the IMRS of a representative agent is completely spanned by the asset returns in a no-arbitrage model. Nevertheless, that assumption and the economic interpretation of an SDF as a representative agent's IMRS has an important testable implication; namely, the SDF must be also be consistent with the expected returns on all assets—including all (foreign and domestic) equities, bonds, currencies, commodities, and derivatives on those assets. Put differently, the returns on any assets outside of the model must only depend on their exposure to (i.e., covariance with) the returns on the specific set of assets in the model. To be clear, this additional implication does not apply to every no-arbitrage model of asset returns. Rather, it only applies to models in which the SDF is economically interpreted as the IMRS of a representative agent. For example, consider a no-arbitrage model for the term structure of U.S. dollar interest rates. An SDF in that model does not necessarily price the returns on other assets, such as equities, that are not completely spanned by those bond returns. However, if the SDF in that model is economically interpreted as the IMRS of a representative agent, then the implicit assumption is that any risks that are independent of U.S. bond returns do not carry a risk premium. Thus, with this economic interpretation, the SDF must also be consistent with the expected returns on all assets (i.e., not just the dollar-denominated bonds in the model).

4.5 SDFs Conditional on Asset Returns

Note that if M is an SDF that satisfies Eq. (58) for a vector of asset returns $\boldsymbol{R}$, then so too is $\mathbb{E}\big[M\big|\boldsymbol{R}\big]\xi$, where $\mathbb{E}\big[M\big|\boldsymbol{R}\big]$ is the conditional expectation of M given $\boldsymbol{R}$, and ξ is any random variable that is independent of $\boldsymbol{R}$, with $\xi > 0$ and $\mathbb{E}[\xi] = 1$.[25] To be more explicit,

$$\mathbf{1} = \mathbb{E}[\boldsymbol{R}M] = \mathbb{E}\left[\boldsymbol{R}\,\mathbb{E}\big[M\big|\boldsymbol{R}\big]\right] = \mathbb{E}\left[\boldsymbol{R}\,\mathbb{E}\big[M\big|\boldsymbol{R}\big]\right]\mathbb{E}[\xi] = \mathbb{E}\left[\boldsymbol{R}\,\mathbb{E}\big[M\big|\boldsymbol{R}\big]\xi\right] . \tag{71}$$

No-arbitrage models of asset returns are necessarily silent about random variables, such as ξ above, that are independent of those returns. Therefore, in a no-arbitrage model of asset

[25]It is important to note that the conditional expectation of M given $\boldsymbol{R}$, $\mathbb{E}\big[M\big|\boldsymbol{R}\big]$, is not the same as the linear projection of M onto $\boldsymbol{R}$.

returns alone, it is only possible learn about the conditional expectation of an SDF, M, given those returns (i.e., $\mathbb{E}\big[M\big|\boldsymbol{R}\big]$).

Many (if not most) no-arbitrage models provide an SDF that can be written explicitly as a function of the asset returns themselves, so that $\mathbb{E}\big[M\big|\boldsymbol{R}\big] = M$, and the conditioning argument above is trivially satisfied. For example, the minimum variance SDF, which is the unique SDF that is linear in the asset returns, is given by

$$M = \boldsymbol{\beta} \cdot \boldsymbol{R}\,, \quad \text{where} \quad \mathbb{E}\left[\boldsymbol{R}M\right] = \mathbf{1} \quad \Rightarrow \quad \boldsymbol{\beta} = \left(\mathbb{E}\left[\boldsymbol{R}\boldsymbol{R}^{\top}\right]\right)^{-1}\mathbf{1}\,. \tag{72}$$

From Long (1990) we know that another example of an SDF that can be formed from the asset returns is

$$M = \left(\boldsymbol{\Theta} \cdot \boldsymbol{R}\right)^{-1}\,, \quad \text{where} \quad \boldsymbol{\Theta} = \arg\max_{\boldsymbol{\Theta}\cdot\mathbf{1}=1} \mathbb{E}\left[\ln\left(\boldsymbol{\Theta} \cdot \boldsymbol{R}\right)\right]\,. \tag{73}$$

It is the unique SDF who's inverse is linear in the asset returns. Finally, we have the SDF that is used in many no-arbitrage models that assume log-normally distributed asset returns,

$$M = \frac{\exp\left(-\boldsymbol{\Gamma} \cdot \ln \boldsymbol{R}\right)}{\frac{1}{k}\mathbf{1} \cdot \mathbb{E}\left[\boldsymbol{R}\exp\left(-\boldsymbol{\Gamma} \cdot \ln \boldsymbol{R}\right)\right]}\,, \tag{74}$$

where k is the number of assets and $\boldsymbol{\Gamma}$ is the unique k-dimensional vector such that $\mathbb{E}\left[\boldsymbol{R}M\right] = \mathbf{1}$ and $\boldsymbol{\Gamma} \cdot \mathbf{1} = 1$.[26] It is the unique SDF whose log is affine in the log asset returns.[27] Note that if there is a unique SDF that satisfies Eq. (58) then the SDFs in Eqs. (72), (73), and (74) must all be equal.

So, as Eq. (71) illustrates, SDFs in no-arbitrage models are effectively functions of the asset returns that they are constructed to price.[28] This fact has an important implication for papers that model two SDFs, say M and M^*, and use Eq. (65) or (66) to characterize the growth in the exchange rate. The asset returns that M and M^* are constructed to price, depend on the growth in the exchange rate, which therefore appears on both the left hand side and the right hand side of Eq. (65) or (66). For example, consider a model of M and M^* that are assumed to satisfy Eq. (65) and price the asset returns in Eq. (63). Intuitively, an SDF must price the dollar-denominated returns, $\boldsymbol{R}_Y$, on the set Y of assets, but it must

[26] Note that if $\boldsymbol{R}$ is strictly positive, then M in Eq. (74) is also strictly positive, but M in Eqs. (72) and (73) is not always guaranteed to be strictly positive.

[27] A few examples of papers that use the SDF in Eq. (74) include: Backus, Foresi, and Telmer (2001); Brennan and Xia (2006); and Lustig, Roussanov, and Verdelhan (2011). More generally, it is commonly used in affine models.

[28] More formally, $\mathbb{E}\big[M\big|\boldsymbol{R}\big] \in \sigma\left(\boldsymbol{R}\right)$ is measurable with respect to the σ-algebra, $\sigma\left(\boldsymbol{R}\right)$, generated by the asset returns $\boldsymbol{R}$.

also price the dollar-denominated returns, $R_B S/S'$ and $\boldsymbol{R}_Z^* S/S'$, on the pound-denominated bank account and set Z of assets. In this case, to emphasize the dependence of M and M^* on the asset returns that they are constructed to price, Eq. (65) can be written more explicitly as

$$\ln S' - \ln S = \ln \mathbb{E}\big[M^* \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big] - \ln \mathbb{E}\big[M \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big]\,, \tag{75}$$

or equivalently,

$$\mathbb{E}\big[M^* \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big] = \tfrac{S'}{S}\mathbb{E}\big[M \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big] \equiv \mathbb{E}\big[M \tfrac{S'}{S} \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big]\,. \tag{76}$$

As Eqs. (75) and (76) emphasize, the growth in the (nominal or real) exchange rate is a necessary input to the right hand side of Eqs. (65) and (66), and therefore it cannot be treated as an output on the left hand side of these equations.

Finally, as a brief mathematical aside, we return to the reduced-form SDFs in Eqs. (72), (73), and (74) and consider the following question: for which SDFs does the change of numeraire units maintain the same functional form? If we use a different numeraire with dollar price ζ, then the asset returns denominated in this numeraire are $\boldsymbol{R}\,\zeta/\zeta'$. The corresponding change of numeraire units for the linear (i.e., minimum variance) SDF, $M = \boldsymbol{\beta}\cdot\boldsymbol{R}$, is not linear in those returns, since $M\zeta'/\zeta = \boldsymbol{\beta}\cdot\boldsymbol{R}\,\zeta'/\zeta$. In other words, if M is the minimum variance (i.e., linear) SDF for a set of dollar-denominated asset returns, then $M\zeta'/\zeta$ is not the minimum variance SDF when the same set of asset returns are denominated in a different numeraire with dollar price ζ (since, in general, $M\zeta'/\zeta = \boldsymbol{\beta}\cdot\boldsymbol{R}\,\zeta'/\zeta$ is not linear in the asset returns, $\boldsymbol{R}\,\zeta/\zeta'$, denominated in that numeraire). We made this same point in Eq. (15).

Although the change of numeraire units does not maintain the same functional form for the linear SDF in Eq. (72), it does maintain the same functional form for the SDFs in Eqs. (73) and (74). In particular, for the SDF in Eq. (73) we have

$$M\zeta'/\zeta = (\boldsymbol{\Theta}\cdot\boldsymbol{R})^{-1}\,\zeta'/\zeta \equiv (\boldsymbol{\Theta}\cdot\boldsymbol{R}\,\zeta/\zeta')^{-1}\ . \tag{77}$$

Similarly, for the SDF in Eq. (74) we have

$$M\zeta'/\zeta = \frac{\exp(-\boldsymbol{\Gamma}\cdot\ln\boldsymbol{R})}{\frac{1}{k}\mathbf{1}\cdot\mathbb{E}\left[\boldsymbol{R}\exp(-\boldsymbol{\Gamma}\cdot\ln\boldsymbol{R})\right]}\zeta'/\zeta \equiv \frac{\exp\big(-\boldsymbol{\Gamma}\cdot\ln\big(\boldsymbol{R}\,\zeta/\zeta'\big)\big)}{\frac{1}{k}\mathbf{1}\cdot\mathbb{E}\left[(\boldsymbol{R}\,\zeta/\zeta')\exp\left(-\boldsymbol{\Gamma}\cdot\ln\left(\boldsymbol{R}\,\zeta/\zeta'\right)\right)\right]}\,, \tag{78}$$

since

$$\boldsymbol{\Gamma}\cdot\mathbf{1} = 1 \quad \Rightarrow \quad \exp\big(-\boldsymbol{\Gamma}\cdot\ln\boldsymbol{R}\big)\zeta'/\zeta = \exp\big(-\boldsymbol{\Gamma}\cdot\ln\big(\boldsymbol{R}\,\zeta/\zeta'\big)\big)\,. \tag{79}$$

As we emphasized in Section 2.1, it is only a matter of mathematics, and not economics, whether the change of numeraire units for a particular specification of a a reduced-form SDF maintains the same functional form.

4.6 Where Two Equivalent Modeling Approaches Diverge

In the alternative modeling approach that we described in Section 4.3, the joint distribution of asset returns, $\boldsymbol{R}_Y$ and $\boldsymbol{R}_Z^*$, is modeled together with the distribution of an SDF, M, for the asset returns denominated in dollars, and an SDF, M^*, for the same asset returns denominated in pounds. We showed that this alternative approach is isomorphic to a direct model of the arbitrage-free distribution of asset returns, together with the growth in the exchange rate. Since these two modeling approaches are exactly equivalent, in theory it shouldn't matter which approach a particular paper employs. However, in practice, many (if not most) papers that directly model M and M^* assume that the growth in the exchange rate, S'/S, is perfectly known given the returns on the two sets of assets, $\boldsymbol{R}_Y$ and $\boldsymbol{R}_Z^*$. This additional assumption is not an implication of no-arbitrage or complete markets, and it represents an important distinction that leads these models to differ along a critical dimension.

As a concrete example, many papers assume that the exchange rate between two currencies is driven by exactly the same shocks that drive interest rates (i.e., the yield curve) in those currencies (e.g., see Backus, Foresi, and Telmer, 2001; Brandt and Santa-Clara, 2002; Brennan and Xia, 2006; Backus, Gavazzoni, Telmer, and Zin, 2010; Lustig, Roussanov, and Verdelhan, 2011; and Gavazzoni, Sambalaibat, and Telmer, 2013). In other words, these papers assume that the growth in the exchange rate between two currencies can be expressed as a function of the change in the yield curves in those currencies.[29] More formally, these papers assume that the growth in the exchange rate is measurable with respect to the σ-algebra generated by the other asset returns, so that $S'/S \in \sigma\big(\boldsymbol{R}_Y, \boldsymbol{R}_Z^*\big)$. In that case, $\mathbb{E}\left[\cdot \middle| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \frac{S'}{S}\right] = \mathbb{E}\left[\cdot \middle| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*\right]$ and therefore Eq. (75) can be equivalently written as

$$\ln S' - \ln S = \ln \mathbb{E}\big[M^* \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*\big] - \ln \mathbb{E}\big[M \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*\big]\,. \tag{80}$$

Brennan and Xia (2006) make an even stronger assumption than Eq. (80). They empirically test whether Eq. (80) holds for an SDF, $\mathbb{E}\big[M\big|\boldsymbol{R}_Y\big]$, that they estimate using only the dollar-denominated returns on long-term bonds, and a separate SDF, $\mathbb{E}\big[M^*\big|\boldsymbol{R}_Z^*\big]$, that they estimate using only the returns on long-term bonds in other currencies (denominated in those currencies).

Eq. (80) requires an additional assumption over and above complete markets or the ab-

[29]In these papers, the other assets are bonds in the two currencies. In that case, $\boldsymbol{R}_Y$ are the dollar-denominated returns on dollar-denominated long-term bonds, and similarly, $\boldsymbol{R}_Z^*$ are the pound-denominated returns on pound-denominated long-term bonds.

sence of arbitrage opportunities. Moreover, it is a crucial assumption in papers, such Backus, Foresi, and Telmer (2001), that use no-arbitrage models to connect currency returns to the returns on other assets (such as the returns on long-term bonds denominated in the two currencies). Alone, the absence of arbitrage does not provide strong restrictions on the joint distribution of currency returns with the returns on other assets. Instead, no-arbitrage only restricts the joint distribution of asset returns that are exposed to a common (small) set of risks, or shocks.[30] In Appendix B we consider a specific example of a no-arbitrage model with log-normal asset returns to explicitly illustrate the restrictions that the additional assumptions in Eq. (80) impose, over and above complete markets or the absence of arbitrage opportunities.

To be clear, our point is not that Eq. (80) is necessarily wrong, or that it violates a fundamental economic principal (such as the absence of arbitrage). Ultimately, it is an empirical question whether the assumption in Eq. (80) holds in the data. In other words, it is an empirical question whether currency returns are completely spanned by the returns on other assets such as bonds in the two currencies. If Eq. (80) does indeed hold, then it is a useful starting point for deriving restrictions implied by the absence of arbitrage. On the other hand, if Eq. (80) does not hold in the data, then there is not reason to expect that any no-arbitrage restrictions derived from Eq. (80) should hold in the data either (though they might). Put differently, if Eq. (80) does not hold in the data, then it is not puzzling that no-arbitrage restrictions derived from (or that rely on) Eq. (80) also do not hold in the data.

To date, the existing empirical evidence suggests that the assumption in Eq. (80) does not in fact hold in the data. Brandt and Santa-Clara (2002) provide empirical evidence that currency returns are not well-spanned by bond returns. Burnside (2012) shows that factors that price the cross-section of equity returns do not price the cross-section of currency returns. In their empirical section, Lustig, Roussanov, and Verdelhan (2011) argue that a separate currency factor is necessary to understand that the cross-section of returns on portfolios of currencies.[31] All of this existing empirical evidence does not imply that there cannot be some set of asset returns for which the assumption in Eq. (80) holds in the data. However, one of the major puzzles in the economics of exchange rates is that, empirically, time-series variation in exchange rates is not tightly related to time-series variation in any

[30]For example, no-arbitrage models have been particularly fruitful for understanding the relationship between prices of options with different strikes and maturities, as well as the term structure of yields on bonds with different maturities (e.g., see Black and Scholes, 1973; Merton, 1973; and Vasicek, 1977). In both of these cases, the absence of arbitrage is useful for understanding the relative prices of many different securities that are jointly driven by a smaller set of underlying risks, or shocks.

[31]Lustig, Roussanov, and Verdelhan (2011) provide empirical evidence that equity market volatility has some explanatory power for the cross-section of currency returns. However, in a horse race they find that their currency-specific factor drives out the equity volatility factor.

other variables. Indeed, for this reason, currencies are often considered to be a separate asset class.

If the assumption in Eq. (80) does not hold in the data, then it may be necessary for SDFs in no-arbitrage models of currency returns to depend on the currency returns themselves. As the SDFs in Eqs. (72)—(74) make clear, this feature is trivial to incorporate into no-arbitrage models that take the standard approach and treat the returns on currency investments the same as the returns on any other asset.[32] However, it is much more difficult (but not impossible) to incorporate this feature into models in which the exchange rate is indirectly characterized as the ratio of an SDF denominated in the two currencies. To appreciate this challenge, substitute $M^*/M = S'/S$ into Eq. (76), which then becomes

$$\mathbb{E}\big[M^* \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{M^*}{M}\big] = \tfrac{M^*}{M}\mathbb{E}\big[M \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{M^*}{M}\big]\,, \tag{81}$$

or equivalently,

$$M\,\mathbb{E}\big[M^* \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{M^*}{M}\big] = M^*\mathbb{E}\big[M \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{M^*}{M}\big]\,. \tag{82}$$

It is not a trivial exercise to directly parameterize a no-arbitrage model of two (or more) SDFs that satisfies Eq. (81), with the additional feature that the ratio of the two SDFs (i.e., the indirect model of the growth in the exchange rate) is not completely spanned by the other assets in the model. For example, as we illustrated in Eq. (70), Brandt and Santa-Clara (2002) tried to relax the assumption in Eq. (80), but in process they introduced an arbitrage opportunity into their model. By contrast, in any no-arbitrage model where the exchange rate is modeled directly together with a single SDF for the asset returns in the model, it always trivially holds that

$$\mathbb{E}\big[M\tfrac{S'}{S} \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big] = \tfrac{S'}{S}\mathbb{E}\big[M \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big]\,, \tag{83}$$

or equivalently,

$$\ln S'/S = \ln \mathbb{E}\big[M\tfrac{S'}{S} \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big] - \ln \mathbb{E}\big[M \big| \boldsymbol{R}_Y, \boldsymbol{R}_Z^*, \tfrac{S'}{S}\big]\,. \tag{84}$$

5 Conclusion

The recent literature in international finance has used the asset market view of real exchange rates, encapsulated by Eqs. (1) and (2), to explain and interpret time-series variation in real and nominal exchange rates and the returns to speculation in currencies. In this paper we

[32]We provide a specific example in Appendix B.

showed that the asset market view is not as widely applicable, or useful, as the previous literature suggests.

We have shown, via the arguments in Sections 1, 2 and 3, that in order to explain how real exchange rates are determined, or economically interpret time-series variation in exchange rates, it is necessary to make specific assumptions about preferences, frictions in the market for goods and services, the assets agents can trade, and the nature of endowments or production.

Additionally, in Sections 1, 2 and 4 we have pointed out some misconceptions, and clarified assumptions made, in the literature that models reduced-form SDFs for different numeraires. Most importantly, we have argued that when Eq. (2) holds, a model of reduced-form SDFs for different numeraires is isomorphic to a direct statistical model of arbitrage-free exchange rate dynamics. As such, the economic content in these two equivalent modeling approaches is exactly the same.

References

Alvarez, Fernando, Andrew Atkeson, and Patrick J. Kehoe, 2007, If exchange rates are random walks, then almost everything we say about monetary policy is wrong, *American Economic Review* 97, 339–345.

Anderson, Bing, Peter J. Hammond, and Cyrus A. Ramezani, 2010, Affine models of the joint dynamics of exchange rates and interest rates, *Journal of Financial and Quantitative Analysis* 45, 1341–1365.

Apte, Prakash, Piet Sercu, and Raman Uppal, 2004, The exchange rate and purchasing power parity: extending the theory and tests, *Journal of International Money and Finance* 23, 553–571.

Bacchetta, Philippe, and Eric van Wincoop, 2006, Can information heterogeneity explain the exchange rate determination puzzle?, *The American Economic Review* 96, 552–576.

Backus, David K., Silverio Foresi, and Chris I. Telmer, 2001, Affine term structure models and the forward premium anomaly, *Journal of Finance* 56.

Backus, David K., Federico Gavazzoni, Christopher Telmer, and Stanley E. Zin, 2010, Monetary policy and the uncovered interest parity puzzle, Working Paper 16218 National Bureau of Economic Research.

Backus, David K., Patrick J. Kehoe, and Finn E. Kydland, 1992, International real business cycles, *Journal of Political Economy* 100, 745–775.

Backus, David K., and Gregor W. Smith, 1993, Consumption and real exchange rates in dynamic economies with non-traded goods, *Journal of International Economics* 35, 297 – 316.

Bakshi, Gurdip, Peter Carr, and Liuren Wu, 2008, Stochastic risk premiums, stochastic skewness in currency options, and stochastic discount factors in international economies, *Journal of Financial Economics* 87, 132–156.

Bansal, Ravi, 1997, An exploration of the forward premium puzzle in currency markets, *The Review of Financial Studies* 10, 369–403.

——— , and Ivan Shaliastovich, 2013, A long-run risks explanation of predictability puzzles in bond and currency markets, *Review of Financial Studies* 26, 1–33.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *The Journal of Finance* 59, 1481–1509.

Baxter, Marianne, and Mario J. Crucini, 1995, Business cycles and the asset structure of foreign trade, *International Economic Review* 36, 821–854.

Bekaert, Geert, 1996, The time variation of risk and return in foreign exchange markets: A general equilibrium perspective, *The Review of Financial Studies* 9, 427–470.

Benigno, Gianluca, and Christoph Thoenissen, 2008, Consumption and real exchange rates with incomplete markets and non-traded goods, *Journal of International Money and Finance* 27, 926–948.

Betts, Caroline, and Michael B. Devereux, 1996, The exchange rate in a model of pricing-to-market, *European Economic Review* 40, 1007–1021.

Black, Fischer, and Myron Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–654.

Bodenstein, Martin, 2008, International asset markets and real exchange rate volatility, *Review of Economic Dynamics* 11, 688–705.

Brandt, Michael W., John H. Cochrane, and Pedro Santa-Clara, 2006, International risk sharing is better than you think, or exchange rates are too smooth, *Journal of Monetary Economics* 53, 671–698.

Brandt, Michael W., and Pedro Santa-Clara, 2002, Simulated likelihood estimation of diffusions with an application to exchange rate dynamics in incomplete markets, *Journal of Financial Economics* 63, 161–210.

Brennan, Michael J., and Yihong Xia, 2006, International capital markets and foreign exchange risk, *The Review of Financial Studies* 19, 753–795.

Burnside, Craig, 2012, Carry trades and risk, in Jessica James, Ian Marsh, and Lucio Sarno, ed.: *Handbook of Exchange Rates*Wiley Handbooks in Financial Engineering and Econometrics . chap. 10, pp. 283–312 (Wiley).

Campbell, John Y., and John H. Cochrane, 1999, By force of habit: A consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy* 107, 205–251.

Chari, V. V., Patrick J. Kehoe, and Ellen R. McGrattan, 2002, Can sticky price models generate volatile and persistent real exchange rates?, *The Review of Economic Studies* 69, 533–563.

Colacito, Riccardo, and Mariano M. Croce, 2011, Risks for the long run and the real exchange rate, *Journal of Political Economy* 119, 153–181.

——— , Steven Ho, and Philip Howard, 2013, Bkk the ez way: An international production economy with recursive preferences, SSRN 2177231.

Cole, Harold L., and Maurice Obstfeld, 1991, Commodity trade and international risk sharing: How much do financial markets matter?, *Journal of Monetary Economics* 28, 3–24.

Corsetti, Giancarlo, Luca Dedola, and Sylvain Leduc, 2008, International risk sharing and the transmission of productivity shocks, *The Review of Economic Studies* 75, 443–473.

Corsetti, Giancarlo, Luca Dedola, and Francesca Viani, 2011, Traded and nontraded goods prices, and international risk sharing: An empirical investigation, in *NBER International Seminar on Macroeconomics 2011*NBER Chapters . pp. 403–466 (National Bureau of Economic Research, Inc).

Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option pricing: A simplified approach, *Journal of Financial Economics* 7, 229 – 263.

Dai, Qiang, and Kenneth J. Singleton, 2003, Term structure dynamics in theory and reality, *Review of Financial Studies* 16, 631–678.

Duffie, Darrell, 2001, *Dynamic Asset Pricing Theory* (Princeton Univ. Press: Princeton, NJ) third edition edn.

Dumas, Bernard, 1992, Dynamic equilibrium and the real exchange rate in a spatially separated world, *Review of Financial Studies* 5, 153–180.

Dybvig, Philip H., and Stephen A. Ross, 2003, Arbitrage, state prices and portfolio theory, in M. Harris G.M. Constantinides, and R.M. Stulz, ed.: *Financial Markets and Asset Pricing*vol. 1 of *Handbook of the Economics of Finance* . chap. 10, pp. 605 – 637 (Elsevier).

——— , 2008, Arbitrage, in Steven N. Durlauf, and Lawrence E. Blume, ed.: *The New Palgrave Dictionary of Economics* (Palgrave Macmillan: Basingstoke).

Epstein, Larry G., and Stanley E. Zin, 1989, Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Gavazzoni, Federico, Batchimeg Sambalaibat, and Chris Telmer, 2013, Currency risk and pricing kernel volatility, Working paper, INSEAD.

Gourio, François, Michael Siemer, and Adrien Verdelhan, 2013, International risk cycles, *Journal of International Economics* 89, 471–484.

Hansen, Lars Peter, and Ravi Jagannathan, 1991, Implications of security market data for models of dynamic economies, *Journal of Political Economy* 99, 225–262.

Hansen, Lars Peter, and Kenneth J. Singleton, 1982, Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica* 50, 1269–1286.

Harrison, J.Michael, and David M Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory* 20, 381–408.

Harrison, J.Michael, and Stanley R. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications* 11, 215–260.

——— , 1983, A stochastic calculus model of continuous trading: Complete markets, *Stochastic Processes and their Applications* 15, 313–316.

Jarrow, Robert, and Dilip Madan, 1995, Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns, *Mathematical Finance* 5, 311–336.

Jarrow, Robert, and Dilip B. Madan, 1999, Hedging contingent claims on semimartingales, *Finance and Stochastics* 3, 111–134.

Kocherlakota, Narayana R., and Luigi Pistaferri, 2007, Household heterogeneity and real exchange rates, *The Economic Journal* 117, C1–C25.

Long, John B., 1990, The numeraire portfolio, *Journal of Financial Economics* 26, 29–69.

Lucas Jr., Robert E., 1982, Interest rates and currency prices in a two-country world, *Journal of Monetary Economics* 10, 335–359.

Lustig, Hanno, and Adrien Verdelhan, 2006, Investing in foreign currency is like betting on your intertemporal marginal rate of substitution, *Journal of the European Economic Association* 4, 644–655.

——— , 2007, The cross section of foreign currency risk premia and consumption growth risk, *The American Economic Review* 97, 89–117.

Lustig, Hanno N., Nikolai L. Roussanov, and Adrien Verdelhan, 2011, Common risk factors in currency markets, *Review of Financial Studies* 24, 3731–3777.

——— , 2014, Countercyclical currency risk premia, *Journal of Financial Economics forthcoming.*

Lustig, Hanno N., and Adrien Verdelhan, 2012, Exchange rates in a stochastic discount factor framework, in Jessica James, Ian Marsh, and Lucio Sarno, ed.: *Handbook of Exchange Rates*Wiley Handbooks in Financial Engineering and Econometrics . chap. 14, pp. 391–420 (Wiley).

Merton, Robert C., 1973, Theory of rational option pricing, *The Bell Journal of Economics and Management Science* 4, 141–183.

Obstfeld, Maurice, and Kenneth Rogoff, 1995, Exchange rate dynamics redux, *Journal of Political Economy* 103, 624–660.

Pavlova, Anna, and Roberto Rigobon, 2007, Asset prices and exchange rates, *The Review of Financial Studies* 20, 1139–1181.

Sercu, Piet, Raman Uppal, and Cynthia Van Hulle, 1995, The exchange rate in the presence of transaction costs: Implications for tests of purchasing power parity, *The Journal of Finance* 50, 1309–1319.

Smith, Peter, and Michael Wickens, 2002, Asset pricing with observable stochastic discount factors, *Journal of Economic Surveys* 16, 397–446.

Stathopoulos, Andreas, 2011, Asset Prices and Risk Sharing in Open Economies, *SSRN eLibrary*.

Stockman, Alan C., 1980, A theory of exchange rate determination, *Journal of Political Economy* 88, 673–698.

——— , and Linda L. Tesar, 1995, Tastes and technology in a two-country model of the business cycle: Explaining international comovements, *The American Economic Review* 85, 168–185.

Varian, Hal R., 1984, *Microeconomic Analysis* (W.W. Norton and Company) 2nd edn.

Vasicek, Oldrich, 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177–188.

Verdelhan, Adrien, 2010, A habit-based explanation of the exchange rate risk premium, *The Journal of Finance* 65, 123–146.

A Section 3.2 Appendix

In this appendix we provide detailed solutions for the model in Section 3.2. It is only for completeness and is not intended for publication.

A.1 Aggregate Prices

The overall consumption aggregate for the domestic household is $c(c_A, c_B)$. Given a particular set of prices (in an arbitrary numeraire) for the individual goods, we can solve the household's static expenditure minimization problem

$$\min_{c_A, c_B} P_A c_A + P_B c_B \quad \text{subject to} \quad c = c(c_A, c_B)\,. \tag{85}$$

Because $c(\cdot)$ is a homogenous of degree one function, minimized expenditure is equal to Pc where $P = H(P_A, P_B)$. The function $H(\cdot)$ is also homogenous of degree one in its arguments, and is related to the function $c(\cdot)$ [see Varian (1984)]. The aggregate price index has the interpretation of being the Lagrange multiplier on the constraint at the optimum. To see this, notice that the first order conditions for the expenditure minimization problem are

$$P_A = \theta c_{c_A}(c_A, c_B) \quad P_B = \theta c_{c_B}(c_A, c_B)\,. \tag{86}$$

Multiplying these through these conditions by c_A and c_B and adding up you get $P_A c_A + P_B c_B = \theta c$ hence $P = \theta$. We also have

$$H(P_A, P_B) = H[P c_{c_A}(\cdot), P c_{c_B}(\cdot)] = P H[c_{c_A}(\cdot), c_{c_B}(\cdot)]\,,$$

establishing that at the optimum, $H[c_{c_A}(\cdot), c_{c_B}(\cdot)] = 1$.

Of course, a similar approach may be used for the foreign household.

A.2 Overall Marginal Utility

The asset payoffs, and all variables in period two, depend on the state of the world in period two. For concreteness, in this appendix we assume that the state of the world is indexed by $z \in \mathcal{Z} = \{1, 2, \ldots, n\}$, with n finite. The assumption that the number possible states of the world in period two is finite, or even countable, is not important and is only for ease of exposition.

By nesting the expenditure minimization problem described in Section A.1 within the domestic household's problem, we can rewrite the latter as follows. The household in the home country chooses c, $c'(z)$, and $\mathbf{a}$ to maximize

$$u(c) + \beta \sum_{z=1}^{n} u\left[c'(z)\right] \pi(z) \tag{87}$$

subject to

$$Pc + \mathbf{P_X} \cdot \mathbf{a} = y_A + P_B y_B\,, \tag{88}$$

$$P'(z)c'(z) = y'_A(z) + P'_B(z)y'_B(z) + \mathbf{X}\,(z) \cdot \mathbf{a}\,, \quad z = 1, \ldots, n. \tag{89}$$

The first order conditions for c, $c'(z)$, and $\mathbf{a}$ are

$$u_c(c) = P\lambda\,, \tag{90}$$

$$\beta u_c[c'(z)]\pi(z) = P'(z)\mu(z)\,, \quad z = 1, \ldots, n, \tag{91}$$

$$\mathbf{P_X}\lambda = \sum_{z=1}^{n} \mu(z)\mathbf{X}(z)\,. \tag{92}$$

Here λ is the Lagrange multiplier on the constraint (90), and $\mu(z)$ is the Lagrange multiplier on the constraint (91). So, combining (90) and (91), we get the following expression for the home household's discounted marginal utility growth, or intertemporal marginal rate of substitution, defined over its basket:

$$m(z) = \frac{\beta u_c[c'(z)]}{u_c(c)} = \frac{P'(z)}{P}\frac{\mu(z)}{\lambda\pi(z)}\,. \tag{93}$$

The household in the foreign country chooses $\tilde{c}$, $\{\tilde{c}'(z)\}_{z=1}^{n}$, and $\tilde{\mathbf{a}}$ to maximize

$$u(\tilde{c}) + \beta \sum_{z=1}^{n} u\left[\tilde{c}'(z)\right] \pi(z) \tag{94}$$

subject to

$$\tilde{P}\tilde{c} + \mathbf{P_X} \cdot \tilde{\mathbf{a}} = \tilde{y}_A + \tilde{P}_B \tilde{y}_B\,, \tag{95}$$

$$\tilde{P}'(z)\tilde{c}'(z) = \tilde{y}'_A(z) + \tilde{P}'_B(z)\tilde{y}'_B(z) + \mathbf{X}\,(z) \cdot \tilde{\mathbf{a}}\,, \quad z = 1, \ldots, n. \tag{96}$$

The first order conditions for $\tilde{c}$, $\{\tilde{c}'(z)\}_{z=1}^{n}$, and $\tilde{\mathbf{a}}$ are

$$u_c(\tilde{c}) = \tilde{P}\tilde{\lambda}\,, \tag{97}$$

$$\beta u_c[\tilde{c}'(z)]\pi(z) = \tilde{P}'(z)\tilde{\mu}(z)\,, \quad z = 1, \ldots, n, \tag{98}$$

$$\mathbf{P_X}\tilde{\lambda} = \sum_{z=1}^{n} \tilde{\mu}(z)\mathbf{X}(z)\,. \tag{99}$$

Here $\tilde{\lambda}$ is the Lagrange multiplier on the constraint (97), and $\tilde{\mu}(z)$ is the Lagrange multiplier on the constraint (98). So, combining (97) and (98), we get the following expression for the home household's discounted marginal utility growth, or intertemporal marginal rate of substitution, defined over its basket:

$$\tilde{m}(z) = \frac{\beta u_c[\tilde{c}'(z)]}{u_c(\tilde{c})} = \frac{\tilde{P}'(z)}{\tilde{P}}\frac{\tilde{\mu}(z)}{\tilde{\lambda}\pi(z)}\,. \tag{100}$$

Notice that m is an SDF for payoffs and prices measured in home country basket units. This is because Eqs. (92) and (93) combined imply

$$\frac{\mathbf{P_X}}{P} = \sum_{z=1}^{n} m(z)\frac{\mathbf{X}(z)}{P'(z)}\pi(z)\,. \tag{101}$$

Similarly, $\tilde{m}$ is an SDF for payoffs and prices measured in foreign country basket units. This is because Eqs. (99) and (100) combined imply

$$\frac{\mathbf{P_X}}{\tilde{P}} = \sum_{z=1}^{n} \tilde{m}(z)\frac{\mathbf{X}(z)}{\tilde{P}'(z)}\pi(z)\,. \tag{102}$$

From (93) and (100), the ratio of $\tilde{m}$ to m is

$$\frac{\tilde{m}(z)}{m(z)} = \left[\frac{\tilde{P}'(z)}{\tilde{P}}\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \Big/ \left[\frac{P'(z)}{P}\frac{\mu(z)}{\lambda}\right] = \left[\frac{e'(z)}{e}\right]\cdot\left[\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \Big/ \left[\frac{\mu(z)}{\lambda}\right]. \tag{103}$$

We define

$$\Xi(z) = \left[\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \Big/ \left[\frac{\mu(z)}{\lambda}\right].$$

Notice that since good A is the numeraire, the first order conditions for c_A and $\tilde{c}_A$, given in Eq. (86), along with Eqs. (90) and (97) imply that the time one marginal utilities of good A in the two countries are

$$u_c(c)c_{c_A}(c_A, c_B) = \lambda \quad u_c(\tilde{c})\tilde{c}_{c_A}(\tilde{c}_A, c_B) = \tilde{\lambda}\,. \tag{104}$$

Similarly, when these first order conditions are combined with Eqs. (91) and (98), we get expressions for the time two discounted marginal utilities of good A in the two countries:

$$\beta u_c[c'(z)]c_{c_A}[c'_A(z), c'_B(z)] = \mu(z)/\pi(z)\,, \quad z = 1, \ldots, n, \tag{105}$$

$$\beta u_c[\tilde{c}'(z)]\tilde{c}_{c_A}[\tilde{c}'_A(z), \tilde{c}'_B(z)] = \tilde{\mu}(z)/\pi(z)\,, \quad z = 1, \ldots, n. \tag{106}$$

Thus, $m_A(z) = \mu(z)/[\lambda\pi(z)]$ and $\tilde{m}_A(z) = \tilde{\mu}(z)/[\tilde{\lambda}\pi(z)]$ are the discounted marginal utility growths of good A in the two countries. Consequently,

$$\frac{\tilde{m}(z)}{m(z)} = \left[\frac{e'(z)}{e}\right] \cdot \Xi(z)\,, \tag{107}$$

with $\Xi(z) = \tilde{m}_A(z)/m_A(z)$ being a measure of risk sharing in the frictionlessly traded good (good A).

The first order conditions for c_B and $\tilde{c}_B$, given in Eq. (86), along with Eqs. (90) and (97) imply that the time one marginal utilities of good B in the two countries are

$$u_c(c)c_{c_B}(c_A, c_B) = \lambda P_B \quad u_c(\tilde{c})\tilde{c}_{c_B}(\tilde{c}_A, c_B) = \tilde{\lambda}\tilde{P}_B\,. \tag{108}$$

Similarly, when these first order conditions are combined with Eqs. (91) and (98), we get expressions for the time two discounted marginal utilities of good B in the two countries:

$$\beta u_c[c'(z)]c_{c_B}[c'_A(z), c'_B(z)] = \mu(z)P'_B(z)/\pi(z)\,, \quad z = 1, \ldots, n, \tag{109}$$

$$\beta u_c[\tilde{c}'(z)]\tilde{c}_{c_B}[\tilde{c}'_A(z), \tilde{c}'_B(z)] = \tilde{\mu}(z)\tilde{P}'_B(z)/\pi(z)\,, \quad z = 1, \ldots, n. \tag{110}$$

Thus, $m_B(z) = m_A(z)P'_B(z)/P_B$ and $\tilde{m}_B(z) = \tilde{m}_A(z)\tilde{P}'_B(z)/\tilde{P}_B$ are the discounted marginal utility growths of good B in the two countries. Consequently, $\Xi(z)[\tilde{P}'_B(z)/\tilde{P}_B]/[P'_B(z)/P_B]$ is a measure of how well risk is shared in good B. If good B is frictionlessly traded the price terms in this expression cancel out and the measure of risk sharing in good B is also $\Xi(z)$.

When the securities span variation in households' marginal utilities (i.e., if financial markets are complete) the first order conditions for a and a become equivalent to

$$\boldsymbol{\psi}\lambda = \boldsymbol{\mu}\,, \qquad \boldsymbol{\psi}\tilde{\lambda} = \tilde{\boldsymbol{\mu}}\,, \tag{111}$$

where $\boldsymbol{\mu}$ is an $n \times 1$ vector whose zth element is $\mu(z)$, $\tilde{\boldsymbol{\mu}}$ is an $n \times 1$ vector whose zth element is $\tilde{\mu}(z)$ and $\boldsymbol{\psi}$ is an $n \times 1$ vector whose zth element is $\psi(z)$, the price of a claim that pays one unit of good A in state z. Notice that when financial markets are complete, this implies $m_A(z) = \tilde{m}_A(z) = \psi(z)/\pi(z)$ and $\Xi(z) = 1$.

A.3 Equilibrium in the Special Cases

To solve the model in the special cases we assume from the start that there is a complete set of state contingent securities indexed by z. Security z pays one unit of good A in state z and zero otherwise. It's price is $\psi(z)$ in the home country and $\tilde{\psi}(z)$ in the foreign country. If there is international trade in these assets (the complete markets case), we have $\psi(z) = \tilde{\psi}(z)$. Under financial autarky, the prices can be different.

The first order conditions for the individual consumption goods and holdings of the securities are

$$\theta c_A^{-1} = \lambda, \tag{112}$$

$$(1-\theta)c_B^{-1} = P_B \lambda, \tag{113}$$

$$\beta\theta c_A'(z)^{-1}\pi(z) = \mu(z), \quad z = 1, \ldots, n, \tag{114}$$

$$\beta(1-\theta)c_B'(z)^{-1}\pi(z) = P_B'(z)\mu(z), \quad z = 1, \ldots, n, \tag{115}$$

$$\psi(z)\lambda = \mu(z), \quad z = 1, \ldots, n. \tag{116}$$

$$\tilde{\theta}\tilde{c}_A^{-1} = \tilde{\lambda}, \tag{117}$$

$$(1-\tilde{\theta})\tilde{c}_B^{-1} = \tilde{P}_B \tilde{\lambda}, \tag{118}$$

$$\beta\tilde{\theta}\tilde{c}_A'(z)^{-1}\pi(z) = \tilde{\mu}(z), \quad z = 1, \ldots, n, \tag{119}$$

$$\beta(1-\tilde{\theta})\tilde{c}_B'(z)^{-1}\pi(z) = \tilde{P}_B'(z)\tilde{\mu}(z), \quad z = 1, \ldots, n, \tag{120}$$

$$\tilde{\psi}(z)\tilde{\lambda} = \tilde{\mu}(z), \quad z = 1, \ldots, n. \tag{121}$$

We can rewrite the first order conditions for the consumptions, using the first order conditions for the securities, as:

$$\theta = \lambda c_A \tag{122}$$

$$1-\theta = \lambda c_B P_B \tag{123}$$

$$\beta\theta = \frac{\psi(z)\lambda}{\pi(z)} c_A'(z) \tag{124}$$

$$\beta(1-\theta) = \frac{\psi(z)\lambda}{\pi(z)} c_B'(z) P_B'(z) \tag{125}$$

$$\tilde{\theta} = \tilde{\lambda}\tilde{c}_A \tag{126}$$

$$1-\tilde{\theta} = \tilde{\lambda}\tilde{P}_B\tilde{c}_B \tag{127}$$

$$\beta\tilde{\theta} = \frac{\tilde{\psi}(z)\tilde{\lambda}}{\pi(z)}\tilde{c}'_A(z) \tag{128}$$

$$\beta(1-\tilde{\theta}) = \frac{\tilde{\psi}(z)\tilde{\lambda}}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{129}$$

Here, we have dropped the $z = 1, \dots, n$, from the equations for convenience.

In what follows we will use the notation $L = \lambda^{-1}$, $\tilde{L} = \tilde{\lambda}^{-1}$. From Eqs. (122), (122), (122) and (122), we see that L and $\tilde{L}$ are the households' respective total expenditures on goods in period one. We also define the global endowments: $Y_A = y_A + \tilde{y}_A$, $Y_B = y_B + \tilde{y}_B$, $Y'_A(z) = y'_A(z) + \tilde{y}'_A(z)$, $Y'_B(z) = y'_B(z) + \tilde{y}'_B(z)$. Additionally we define $G_A(z) = Y'_A(z)/Y_A$, $G_B(z) = Y'_B(z)/Y_B$, $g_A(z) = y'_A(z)/y_A$, $g_B(z) = y'_B(z)/y_B$, $\tilde{g}_A(z) = \tilde{y}'_A(z)/\tilde{y}_A$, $\tilde{g}_B(z) = \tilde{y}'_B(z)/\tilde{y}_B$, We also use the following notation

$$s_A = y_A/Y_A \quad s_B = y_B/Y_B \quad s'_A(z) = y'_A(z)/Y'_A(z) \quad s'_B(z) \equiv y'_B(z)/Y'_B(z)$$

$$\bar{s}'_A = \sum_{z=1}^{n} s'_A(z)\pi(z) \quad \bar{s}'_B = \sum_{z=1}^{n} s'_B(z)\pi(z)$$

A.3.1 When International Asset Markets are Complete

Here we have $\psi(z) = \tilde{\psi}(z)$, which allows us to rewrite the first order conditions for the consumptions as

$$\theta L = c_A \tag{130}$$

$$(1-\theta)L = c_B P_B \tag{131}$$

$$\beta\theta L = \frac{\psi(z)}{\pi(z)}c'_A(z) \tag{132}$$

$$\beta(1-\theta)L = \frac{\psi(z)}{\pi(z)}c'_B(z)P'_B(z) \tag{133}$$

$$\tilde{\theta}\tilde{L} = \tilde{c}_A \tag{134}$$

$$(1-\tilde{\theta})\tilde{L} = \tilde{P}_B\tilde{c}_B \tag{135}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\psi(z)}{\pi(z)}\tilde{c}'_A(z) \tag{136}$$

$$\beta(1-\tilde{\theta})\tilde{L} = \frac{\psi(z)}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{137}$$

The home country household's lifetime budget constraint is

$$c_A + P_B c_B + \sum_{z=1}^{n} \psi(z) \left[c'_A(z) + P'_B(z) c'_B(z)\right] = y_A + P_B y_B + \sum_{z=1}^{n} \psi(z) \left[y'_A(z) + P'_B(z) y'_B(z)\right] \tag{138}$$

From Eqs. (130), (132), (130), and (132) we see that discounted marginal utility growth in good A in the two countries are equated:

$$m_A(z) = \beta \frac{c_A}{c'_A(z)} = \frac{\psi(z)}{\pi(z)} \qquad \tilde{m}_A(z) = \beta \frac{\tilde{c}_A}{\tilde{c}'_A(z)} = \frac{\psi(z)}{\pi(z)} \tag{139}$$

From Eqs. (131), (133), (131), and (133), discounted marginal utility growths in good B are

$$m_B(z) = \beta \frac{c_B}{c'_B(z)} = \frac{\psi(z)}{\pi(z)} \frac{P'_B(z)}{P_B} \qquad \beta \frac{\tilde{c}_B}{\tilde{c}'_B(z)} = \frac{\psi(z)}{\pi(z)} \frac{\tilde{P}'_B(z)}{\tilde{P}_B} \tag{140}$$

A.3.2 When Good B is Traded

The market clearing conditions for good A are

$$c_A + \tilde{c}_A = Y_A \tag{141}$$

$$c'_A(z) + \tilde{c}'_A(z) = Y'_A(z) \tag{142}$$

$$c_B + \tilde{c}_B = Y_B \tag{143}$$

$$c'_B(z) + \tilde{c}'_B(z) = Y'_B(z) \tag{144}$$

These market clearing conditions, together with the first order conditions, (130)–(137), imply

$$\theta L + \tilde{\theta} \tilde{L} = Y_A \tag{145}$$

$$\beta(\theta L + \tilde{\theta} \tilde{L}) = \frac{\psi(z)}{\pi(z)} Y'_A(z) \tag{146}$$

$$(1-\theta)L + (1-\tilde{\theta})\tilde{L} = P_B Y_B \tag{147}$$

$$\beta[(1-\theta)L + (1-\tilde{\theta})\tilde{L}] = \frac{\psi(z)}{\pi(z)} P'_B(z) Y'_B(z) \tag{148}$$

Given a value of L we can solve the Eqs. (145) and (147) for L and P_B:

$$\tilde{L} = \frac{Y_A}{\tilde{\theta}} - \frac{\theta}{\tilde{\theta}} L \tag{149}$$

$$P_B = \frac{\frac{\tilde{\theta}-\theta}{\tilde{\theta}} L + (\frac{1-\tilde{\theta}}{\tilde{\theta}}) Y_A}{Y_B} \tag{150}$$

If you combine Eqs. (145) and (146) you get

$$m_A(z) = \beta G_A(z)^{-1} = \frac{\psi(z)}{\pi(z)} \tag{151}$$

If you combine Eqs. (147) and (148) and previous results you get

$$P'_B(z)/P_B = G_A(z)/G_B(z) \tag{152}$$

Marginal utility growth in good B is

$$m_B(z) = m_A(z) P'_B(z)/P_B = \beta G_B(z)^{-1} \tag{153}$$

Marginal utility growths are across countries in both goods (but not across goods) are equated: $m_A(z) = \tilde{m}_A(z)$ and $m_B(z) = \tilde{m}_B(z)$. This is true regardless of preferences.

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that Eqs. (150) becomes

$$P_B = \frac{1-\theta}{\theta} \frac{Y_A}{Y_B} \tag{154}$$

and Eq. (152) implies

$$P'_B(z) = \frac{1-\theta}{\theta} \frac{Y'_A(z)}{Y'_B(z)} \tag{155}$$

Since trade is frictionless and preferences are identical $e = e'(z) = 1$.

We can solve for allocations by solving for L. To do this we consider the lifetime budget constraint, (138), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_A + \beta \bar{s}'_A + (\frac{1-\theta}{\theta})(s_B + \beta \bar{s}'_B) \right] Y_A \tag{156}$$

This implies

$$L = \frac{\theta(s_A + \beta \bar{s}'_A) + (1-\theta)\,(s_B + \beta \bar{s}'_B)}{\theta(1+\beta)} Y_A \tag{157}$$

Eq. (149) then implies that

$$\tilde{L} = \frac{\theta[(1-s_A) + \beta(1-\bar{s}'_A)] + (1-\theta)[(1-s_B) + \beta(1-\bar{s}'_B)]}{\theta(1+\beta)} Y_A \tag{158}$$

Different Preferences With different preferences we need to solve for L. To do this we consider the lifetime budget constraint, (138), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_A + \beta \bar{s}'_A + (\frac{1-\tilde{\theta}}{\tilde{\theta}})(s_B + \beta \bar{s}'_B) \right] Y_A + \frac{\tilde{\theta}-\theta}{\tilde{\theta}}(s_B + \beta \bar{s}'_B)L \tag{159}$$

This implies

$$L = \frac{\tilde{\theta}(s_A + \beta \bar{s}'_A) + (1-\tilde{\theta})\,(s_B + \beta \bar{s}'_B)}{\tilde{\theta}(1+\beta) + (\theta - \tilde{\theta})\,(s_B + \beta \bar{s}'_B)} Y_A \tag{160}$$

Eq. (149) then implies that

$$\tilde{L} = \frac{\theta[(1-s_A) + \beta(1-\bar{s}'_A)] + (1-\theta)[(1-s_B) + \beta(1-\bar{s}'_B)]}{\tilde{\theta}(1+\beta) + (\theta - \tilde{\theta})\,(s_B + \beta \bar{s}'_B)} Y_A \tag{161}$$

and (150) implies that

$$P_B = \frac{(1-\tilde{\theta})\,(1+\beta) + \left(\tilde{\theta} - \theta\right)(s_A + \beta \bar{s}'_A)}{\tilde{\theta}\,(1+\beta) + \left(\theta - \tilde{\theta}\right)(s_B + \beta \bar{s}'_B)} \frac{Y_A}{Y_B}\,, \tag{162}$$

Given Eq. (152) we have

$$P'_B(z) = \frac{G_A(z)}{G_B(z)} P_B = \frac{(1-\tilde{\theta})\,(1+\beta) + \left(\tilde{\theta} - \theta\right)(s_A + \beta \bar{s}'_A)}{\tilde{\theta}\,(1+\beta) + \left(\theta - \tilde{\theta}\right)(s_B + \beta \bar{s}'_B)} \frac{Y'_A(z)}{Y'_B(z)}\,. \tag{163}$$

Since good B is traded, $\tilde{P}_B = P_B$ and $\tilde{P}'_B(z) = P'_B(z)$ so

$$e = (\tilde{\rho}/\rho) P_B^{\theta - \tilde{\theta}} \qquad e'(z) = (\tilde{\rho}/\rho) P'_B(z)^{\theta - \tilde{\theta}}$$

But this means

$$\ln\left[e'(z)/e\right] = (\theta - \tilde{\theta}) \ln\left[P'_B(z)/P_B\right] = (\theta - \tilde{\theta}) \ln\left[G_A(z)/G_B(z)\right]$$

A.3.3 When Good B is Non-traded

The market clearing conditions for good A are (141) and (142). For good B they are

$$c_B = y_B\,, \quad \tilde{c}_B = \tilde{y}_B \tag{164}$$

$$c_B'(z) = y_B'(z)\,, \quad \tilde{c}_B'(z) = \tilde{y}_B'(z) \tag{165}$$

The market clearing conditions and the first order conditions together imply that Eqs. (145) and (146) hold along with

$$(1-\theta)L = P_B y_B \qquad (1-\tilde{\theta})\tilde{L} = \tilde{P}_B \tilde{y}_B \tag{166}$$

$$\beta(1-\theta)L = \frac{\psi(z)}{\pi(z)} P_B'(z) y_B'(z)\,, \quad \beta(1-\tilde{\theta})\tilde{L} = \frac{\psi(z)}{\pi(z)} \tilde{P}_B'(z)\tilde{y}_B'(z) \tag{167}$$

Given the results so far, the lifetime budget constraint of the home household, (138), becomes:

$$(1+\beta)\theta L = \kappa Y_A,$$

where $\kappa = s_A + \beta \bar{s}_A'$, and implies

$$L = \frac{\kappa}{\theta(1+\beta)} Y_A. \tag{168}$$

If we combine Eqs. (145) and (168) we have

$$\tilde{L} = \frac{\tilde{\kappa}}{\tilde{\theta}(1+\beta)} Y_A \tag{169}$$

where $\tilde{\kappa} = 1 - s_A + \beta(1 - \bar{s}_A')$.

Combining Eqs. (145) and (146) you get

$$m_A(z) = \beta G_A(z)^{-1} = \frac{\psi(z)}{\pi(z)} \tag{170}$$

Combining Eqs. (166), (168) and (169) we have

$$P_B = \frac{(1-\theta)\kappa}{\theta(1+\beta)} \frac{Y_A}{y_B} \qquad \tilde{P}_B = \frac{(1-\tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1+\beta)} \frac{Y_A}{\tilde{y}_B}. \tag{171}$$

If we combine (166) and (167), and make use of (170) we get

$$\frac{P_B'(z)}{P_B} = \frac{G_A(z)}{g_B(z)} \qquad \frac{\tilde{P}_B'(z)}{\tilde{P}_B} = \frac{G_A(z)}{\tilde{g}_B(z)}. \tag{172}$$

Therefore, we can write

$$P_B'(z) = \frac{(1-\theta)\kappa}{\theta(1+\beta)} \frac{Y_A'(z)}{y_B'(z)} \qquad \tilde{P}_B'(z) = \frac{(1-\tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1+\beta)} \frac{Y_A'(z)}{\tilde{y}_B'(z)}. \tag{173}$$

Discounted marginal utility growth in good B is

$$m_B(z) = \frac{\psi(z)}{\pi(z)} \frac{P'_B(z)}{P_B} = \beta / g'_B(z) \qquad \tilde{m}_B(z) = \frac{\psi(z)}{\pi(z)} \frac{\tilde{P}'_B(z)}{\tilde{P}_B} = \beta / \tilde{g}_B(z) \tag{174}$$

Identical Preferences We have

$$e = \left(\tilde{P}_B / P_B\right)^{1-\theta} = \left(\frac{\tilde{\kappa}}{\kappa} \frac{y_B}{\tilde{y}_B}\right)^{1-\theta}.$$

$$e'(z) = \left(\tilde{P}'_B(z) / P'_B(z)\right)^{1-\theta} = \left[\frac{\tilde{\kappa}}{\kappa} \frac{y'_B(z)}{\tilde{y}'_B(z)}\right]^{1-\theta}.$$

And this means

$$\ln[e'(z)/e] = (1-\theta) \ln\left[g'_B(z)/\tilde{g}'_B(z)\right]$$

Different Preferences

$$e = (\frac{\tilde{\rho}}{\rho}) \frac{\tilde{P}_B^{1-\tilde{\theta}}}{P_B^{1-\theta}} = (\frac{\tilde{\rho}}{\rho}) \frac{\left[\frac{(1-\tilde{\theta})}{(1+\beta)\tilde{\theta}} \tilde{\kappa} \frac{Y_A}{\tilde{y}_B}\right]^{1-\tilde{\theta}}}{\left[\frac{(1-\theta)}{(1+\beta)\theta} \kappa \frac{Y_A}{y_B}\right]^{1-\theta}} = (\frac{\theta}{\tilde{\theta}}) \frac{\left(\tilde{\kappa} \tilde{y}_B^{-1}\right)^{1-\tilde{\theta}}}{\left(\kappa y_B^{-1}\right)^{1-\theta}} \left(\frac{Y_A}{1+\beta}\right)^{\theta-\tilde{\theta}}.$$

$$e'(z) = (\frac{\tilde{\rho}}{\rho}) \frac{\tilde{P}'_B(z)^{1-\tilde{\theta}}}{P'_B(z)^{1-\theta}} = (\frac{\theta}{\tilde{\theta}}) \frac{\left[\tilde{\kappa} \tilde{y}'_B(z)^{-1}\right]^{1-\tilde{\theta}}}{\left[\kappa y'_B(z)^{-1}\right]^{1-\theta}} \left[\frac{Y'_A(z)}{1+\beta}\right]^{\theta-\tilde{\theta}}$$

So

$$\ln[e'(z)/e] = (1-\theta) \ln g'_B(z) - (1-\tilde{\theta}) \ln \tilde{g}'_B(z) + (\theta - \tilde{\theta}) \ln G'_A(z)$$

A.3.4 Financial Autarky

Since the countries are in financial autarky, we no longer have $\psi(z) = \tilde{\psi}(z)$, so the rearranged first order conditions for the consumptions are

$$\theta L = c_A \tag{175}$$

$$(1-\theta) L = c_B P_B \tag{176}$$

$$\beta \theta L = \frac{\psi(z)}{\pi(z)} c'_A(z) \tag{177}$$

$$\beta(1-\theta) L = \frac{\psi(z)}{\pi(z)} c'_B(z) P'_B(z) \tag{178}$$

$$\tilde{\theta}\tilde{L} = \tilde{c}_A \tag{179}$$

$$(1-\tilde{\theta})\tilde{L} = \tilde{P}_B\tilde{c}_B \tag{180}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\tilde{\psi}(z)}{\pi(z)}\tilde{c}'_A(z) \tag{181}$$

$$\beta(1-\tilde{\theta})\tilde{L} = \frac{\tilde{\psi}(z)}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{182}$$

The home country's flow budget constraints must be satisfied with no asset holdings so we have

$$c_A + P_B c_B = y_A + P_B y_B \tag{183}$$

$$c'_A(z) + P'_B(z)c'_B(z) = y'_A(z) + P'_B(z)y'_B(z). \tag{184}$$

Using the first order conditions for the consumptions we get expressions for discounted marginal utility growth:

$$m_A(z) = \beta\frac{c_A}{c'_A(z)} = \frac{\psi(z)}{\pi(z)} \qquad \tilde{m}_A(z) = \beta\frac{\tilde{c}_A}{\tilde{c}'_A(z)} = \frac{\tilde{\psi}(z)}{\pi(z)} \tag{185}$$

Discounted marginal utility growth in good B is

$$m_B(z) = \beta\frac{c_B}{c'_B(z)} = \frac{\psi(z)}{\pi(z)}\frac{P'_B(z)}{P_B} \qquad \tilde{m}_B(z) = \beta\frac{\tilde{c}_B}{\tilde{c}'_B(z)} = \frac{\tilde{\psi}(z)}{\pi(z)}\frac{\tilde{P}'_B(z)}{\tilde{P}_B} \tag{186}$$

A.3.5 When Good B is Traded

The market clearing conditions for goods are Eqs. (141)–(144). The market clearing conditions and the first order conditions together imply

$$\theta L + \tilde{\theta}\tilde{L} = Y_A \tag{187}$$

$$\frac{\theta}{\psi(z)}L + \frac{\tilde{\theta}}{\tilde{\psi}(z)}\tilde{L} = \frac{1}{\beta\pi(z)}Y'_A(z) \tag{188}$$

$$(1-\theta)L + (1-\tilde{\theta})\tilde{L} = P_B Y_B \tag{189}$$

$$(1-\theta)\frac{L}{\psi(z)} + (1-\tilde{\theta})\frac{\tilde{L}}{\tilde{\psi}(z)} = \frac{1}{\beta\pi(z)}P'_B(z)Y'_B(z)\,, \quad z = 1,\ldots,n, \tag{190}$$

We can rearrange Eqs. (187) and (189) to get:

$$\tilde{L} = \frac{1}{\tilde{\tilde{\theta}}}(Y_A - \theta L) \tag{191}$$

$$P_B = \frac{1}{\tilde{\tilde{\theta}}}\frac{(\tilde{\theta}-\theta)L + (1-\tilde{\theta})Y_A}{Y_B}\,, \tag{192}$$

We can rearrange Eqs. (188) and (190) to get:

$$\frac{\beta\pi(z)}{\tilde{\psi}(z)}\tilde{L} = \frac{1}{\tilde{\tilde{\theta}}}\left[Y_A'(z) - \theta\frac{\beta\pi(z)}{\psi(z)}L\right]. \tag{193}$$

$$P_B'(z) = \frac{1}{\tilde{\tilde{\theta}}}\frac{(\tilde{\theta}-\theta)\frac{\beta\pi(z)}{\psi(z)}L + (1-\tilde{\theta})Y_A'(z)}{Y_B'(z)} \tag{194}$$

The flow budget constraint, (183), and Eq. (192) imply that

$$L = y_A + \frac{\frac{\tilde{\theta}-\theta}{\tilde{\tilde{\theta}}}L + (1-\tilde{\theta})\frac{Y_A}{\tilde{\tilde{\theta}}}}{Y_B}y_B$$

or

$$L = \frac{\tilde{\theta}s_A + (1-\tilde{\theta})s_B}{\tilde{\theta} + (\theta-\tilde{\theta})s_B}Y_A \tag{195}$$

The flow budget constraint, (184), and Eq. (194) imply that

$$\frac{\beta\pi(z)}{\psi(z)}L = \frac{\tilde{\theta}s_A'(z) + (1-\tilde{\theta})s_B'(z)}{\tilde{\theta} + (\theta-\tilde{\theta})s_B'(z)}Y_A'(z). \tag{196}$$

Using (195) we then have

$$m_A(z) = \frac{\psi(z)}{\pi(z)} = \beta\frac{\xi_A(z)}{G_A(z)} \quad \text{with} \quad \xi_A(z) = \frac{\frac{\tilde{\theta}s_A + (1-\tilde{\theta})s_B}{\tilde{\theta}+(\theta-\tilde{\theta})s_B}}{\frac{\tilde{\theta}s_A'(z) + (1-\tilde{\theta})s_B'(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s_B'(z)}} \tag{197}$$

Substituting (195) into (191) we get

$$\tilde{L} = \frac{\theta(1-s_A) + (1-\theta)\,(1-s_B)}{\theta + (\tilde{\theta}-\theta)(1-s_B)}Y_A \tag{198}$$

Substituting (196) into (193) we get

$$\frac{\beta\pi(z)}{\tilde{\psi}(z)}\tilde{L} = \frac{\theta[1-s_A'(z)] + (1-\theta)[1-s_B'(z)]}{\theta + (\tilde{\theta}-\theta)[1-s_B'(z)]}Y_A'(z) \tag{199}$$

Given these results, discounted marginal utility growth in good A in the foreign country is

$$\tilde{m}_A(z) = \frac{\tilde{\psi}(z)}{\pi(z)} = \beta \frac{\tilde{\xi}_A(z)}{G_A(z)} \quad \text{with} \quad \tilde{\xi}_A(z) = \frac{\frac{\theta(1-s_A)+(1-\theta)(1-s_B)}{\theta+(\tilde{\theta}-\theta)(1-s_B)}}{\frac{\theta[1-s'_A(z)]+(1-\theta)[1-s'_B(z)]}{\theta+(\tilde{\theta}-\theta)[1-s'_B(z)]}} \tag{200}$$

Substituting (195) into (192)

$$P_B = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B} \frac{Y_A}{Y_B} \tag{201}$$

Substituting (196) into (194)

$$P'_B(z) = \left[\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}\right] \frac{Y'_A(z)}{Y'_B(z)} \tag{202}$$

Discounted marginal utility growth in good B in the two countries is

$$m_B(z) = \beta \frac{\xi_A(z)}{G_A(z)} \frac{P'_B(z)}{P_B} = \beta \frac{\xi_A(z)}{G_B(z)} \xi_B(z) \quad \text{with} \quad \xi_B(z) = \frac{\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}}{\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B}} \tag{203}$$

$$\tilde{m}_B(z) = \beta \frac{\tilde{\xi}_A(z)}{G_A(z)} \frac{P'_B(z)}{P_B} = \beta \frac{\tilde{\xi}_A(z)}{G_B(z)} \xi_B(z) \tag{204}$$

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that Eqs. (201) and (202) simplify to

$$P_B = \frac{1-\theta}{\theta} \frac{Y_A}{Y_B} \tag{205}$$

$$P'_B(z) = \frac{1-\theta}{\theta} \frac{Y'_A(z)}{Y'_B(z)} \tag{206}$$

Since both goods are frictionlessly traded and preferences are identical $e = e'(z) = 1$.

The expressions for ξ_A and $\tilde{\xi}_A$ in Eqs. (197) and (200) simplify to

$$\xi_A(z) = \frac{\theta s_A + (1-\theta)s_B}{\theta s'_A(z) + (1-\theta)s'_B(z)} \tag{207}$$

$$\tilde{\xi}_A(z) = \frac{\theta(1-s_A) + (1-\theta)(1-s_B)}{\theta[1-s'_A(z)] + (1-\theta)[1-s'_B(z)]} \tag{208}$$

The expression for ξ_B in Eq. (203) simplifies to $\xi_B(z) = 1$, implying that

$$m_B(z) = \beta \frac{\xi_A(z)}{G_B(z)} \quad \tilde{m}_B(z) = \beta \frac{\tilde{\xi}_A(z)}{G_B(z)} \tag{209}$$

The wedge between marginal utility growths in good A, good B, and in terms of aggregate consumption is

$$\tilde{m}_A(z)/m_A(z) = \tilde{m}_B(z)/m_B(z) = \tilde{m}(z)/m(z) = \tilde{\xi}_A(z)/\xi_A(z).$$

Different Preferences Given the expressions for prices, above,

$$e = (\tilde{\rho}/\rho) P_B^{\theta - \tilde{\theta}} = (\tilde{\rho}/\rho) \left(\frac{1 - \tilde{\theta} + \left(\tilde{\theta} - \theta\right) s_A}{\tilde{\theta} + \left(\theta - \tilde{\theta}\right) s_B} \frac{Y_A}{Y_B} \right)^{\theta - \tilde{\theta}}$$

and

$$e'(z) = (\tilde{\rho}/\rho) P_B'(z)^{\theta - \tilde{\theta}} = (\tilde{\rho}/\rho) \left(\frac{1 - \tilde{\theta} + (\tilde{\theta} - \theta) s_A'(z)}{\tilde{\theta} + (\theta - \tilde{\theta}) s_B'(z)} \frac{Y_A'(z)}{Y_B'(z)} \right)^{\theta - \tilde{\theta}}$$

A.3.6 When Good B is Non-traded

Because the B good cannot be traded the goods market clearing conditions and the home household budget constraints together imply,

$$c_A = y_A\,, \quad \tilde{c}_A = \tilde{y}_A \tag{210}$$

$$c_A'(z) = y_A'(z)\,, \quad \tilde{c}_A'(z) = \tilde{y}_A'(z) \tag{211}$$

$$c_B = y_B\,, \qquad \tilde{c}_B = \tilde{y}_B\,, \tag{212}$$

$$c_B'(z) = y_B'(z)\,, \quad \tilde{c}_B'(z) = \tilde{y}_B'(z)\,, \tag{213}$$

So

$$L = y_A/\theta, \tag{214}$$

$$P_B = \frac{1-\theta}{\theta} \frac{y_A}{y_B} \tag{215}$$

$$\frac{\psi(z)}{\pi(z)} = \beta/g_A(z) \tag{216}$$

$$P_B'(z) = \frac{1-\theta}{\theta}\frac{y_A'(z)}{y_B'(z)} \tag{217}$$

$$\tilde{L} = \tilde{y}_A/\tilde{\theta} \tag{218}$$

$$\tilde{P}_B = \frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_A}{\tilde{y}_B}, \tag{219}$$

$$\frac{\tilde{\psi}(z)}{\pi(z)} = \beta/\tilde{g}_A(z) \tag{220}$$

$$\tilde{P}_B'(z) = \frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_A'(z)}{\tilde{y}_B'(z)} \tag{221}$$

Discounted marginal utility growths in goods A and B are

$$m_A(z) = \beta/g_A(z) \qquad \tilde{m}_A(z) = \beta/\tilde{g}_A(z)$$

$$m_B(z) = \beta/g_B(z) \qquad \tilde{m}_B(z) = \beta/\tilde{g}_B(z)$$

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that

$$e = \left(\frac{\tilde{y}_A/\tilde{y}_B}{y_A/y_B}\right)^{1-\theta} = \left(\frac{(1-s_A)/s_A}{(1-s_B)/s_B}\right)^{1-\theta}.$$

$$e'(z) = \left(\frac{\tilde{y}_A'(z)/\tilde{y}_B'(z)}{y_A(z)/y_B(z)}\right)^{1-\theta} = \left(\frac{[1-s_A'(z)]/s_A'(z)}{[1-s_B'(z)]/s_B'(z)}\right)^{1-\theta}.$$

Different Preferences

$$e = (\tilde{\rho}/\rho)[\frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_A}{\tilde{y}_B}]^{1-\tilde{\theta}}/[\frac{1-\theta}{\theta}\frac{y_A}{y_B}]^{1-\theta}.$$

$$e'(z) = (\tilde{\rho}/\rho)[\frac{1-\tilde{\theta}}{\tilde{\theta}}\frac{\tilde{y}_A'(z)}{\tilde{y}_B'(z)}]^{1-\tilde{\theta}}/[\frac{1-\theta}{\theta}\frac{y_A'(z)}{y_B'(z)}]^{1-\theta}.$$

B Section Appendix

As we mentioned earlier, many papers in this literature use the SDF in Eq. (74) together with the assumption of log-normal asset returns.[33] Here we include an illustrative example of

[33] A few examples include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002) ; Brennan and Xia (2006); and Lustig, Roussanov, and Verdelhan (2011).

this setup to highlight the role of the asset market view in Eq. (65), or equivalently Eq. (66), in these models.

For simplicity, consider an arbitrage-free model of the returns on four assets. Two assets are default-free bank accounts.One default-free bank account is denominated in U.S. dollars and pays a continuously-compounded interest rate i from time t to $t+\Delta t$, so that the certain dollar-denominated gross return on this bank account is $\exp(i\,\Delta t)$. The other default-free bank account is denominated in U.K. dollars and pays a continuously-compounded interest rate i^*. Let S and S' denoted the spot dollar/pound exchange rate at time t and $t+\Delta t$. Then the dollar-denominated gross return on this bank account from time t to $t+\Delta t$ is $\exp(i^*\Delta t)\,S'/S$. The other two assets are not bank accounts. They could be long-term bonds, stocks, or any other assets located anywhere in the world. Let Y and Y' be the dollar-denominated prices at time t and $t+\Delta t$ of an asset that pays a continuously-compounded dividend δ over that period. Similarly, let Z^* and $Z^{*\prime}$ be the pound-denominated prices at time t and $t+\Delta t$ of different asset that pays a continuously-compounded dividend δ^* over that period. The vector of dollar-denominated gross returns on these four assets is

$$\boldsymbol{R} = \left[\ \exp\big(i\,\Delta t\big)\,, \quad \exp\big(i^*\Delta t\big)\,\tfrac{S'}{S}\,, \quad \exp\big(\delta\Delta t\big)\,\tfrac{Y'}{Y}\,, \quad \exp\big(\delta^*\Delta t\big)\,\tfrac{S'Z^{*\prime}}{SZ^*}\ \right]. \tag{222}$$

A common assumption in this literature (and asset pricing in general) is that the asset returns are log-normally distributed, with

$$\left[\ \ln S'/S\,, \quad \ln Y'/Y\,, \quad \ln Z^{*\prime}/Z^*\ \right] \sim \mathcal{N}\left(\boldsymbol{\mu}\Delta t, \boldsymbol{\Omega}\Delta t\right)\,. \tag{223a}$$

We'll decompose the covariance matrix, $\boldsymbol{\Omega}$, as

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}\boldsymbol{P}\boldsymbol{\Sigma} \quad \text{with} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma & & \\ & \sigma_d & \\ & & \sigma_f \end{bmatrix} \quad \text{and} \quad \boldsymbol{P} = \begin{bmatrix} 1 & \rho_d & \rho_f \\ \rho_d & 1 & \rho \\ \rho_f & \rho & 1 \end{bmatrix}. \tag{223b}$$

To ensure that there are no arbitrage opportunities within the asset returns (as we'll discuss below), it is convenient to parameterize the mean vector, $\boldsymbol{\mu}$, as

$$\boldsymbol{\mu} = \left[\ i - i^* + \varphi - \tfrac{1}{2}\sigma^2\,, \quad i - \delta + \varphi_d - \tfrac{1}{2}\sigma_d^2\,, \quad i^* - \delta^* + \varphi_f - \tfrac{1}{2}\sigma_f^2 - \sigma_f\,\sigma\rho_f\ \right], \tag{223c}$$

where

$$\left[\ \varphi\,, \quad \varphi_d\,, \quad \varphi_f\ \right] = \boldsymbol{\Omega}\,\boldsymbol{\Upsilon} \quad \text{and} \quad \boldsymbol{\Upsilon} = \left[\ \gamma + \gamma_f\,, \quad \gamma_d\,, \quad \gamma_f\ \right], \tag{223d}$$

for parameters γ, γ_d, and γ_f. In Section B.1 we use this illustrative no-arbitrage model to analyze specific papers in this literature, but first we briefly discuss its relevant features.

In the general case, this simple model of log-normal asset returns has 13 free parameters. The continuously-compounded dividend yields on the four assets are described by $\{i, i^*, \delta, \delta^*\}$. There are three relative asset returns to consider (once we arbitrarily choose one of the assets, or a portfolio of them, as the numeraire to denominate returns). The volatility of the three asset returns are described by $\{\sigma, \sigma_d, \sigma_f\}$, which are contained within the matrix $\boldsymbol{\Sigma}$. The correlation of the three asset returns are described by $\{\rho_d, \rho_f, \rho\}$, which are contained within the matrix $\boldsymbol{P}$. Finally, given the other parameters, the mean of the three log asset returns are characterized by the parameters $\{\gamma, \gamma_d, \gamma_f\}$, which are contained within the vector $\boldsymbol{\Upsilon}$.

For this example, we'll focus on the specific SDF in Eq. (74), since it is used by much of the literature that works with log-normal models of asset returns. It is straightforward to verify that $\mathbb{E}(\boldsymbol{R}m = \mathbf{1})$ for the SDF in Eq. (74) when

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1-\gamma-\gamma_d-\gamma_f & \gamma & \gamma_d & \gamma_f \end{bmatrix}, \tag{224}$$

for γ, γ_d, and γ_f in Eq. (223d). By the Fundamental Theorem of Asset Pricing (e.g., see Dybvig and Ross, 2003 or Dybvig and Ross, 2008), there are no arbitrage opportunities within a set of returns $\boldsymbol{R}$ if there is a strictly positive SDF, M, that satisfies Eq. (58) for those returns. Thus, the parameterization in Eq. (223) ensures that there are no arbitrage opportunities within the asset returns, $\boldsymbol{R}$, in Eq. (222). Put differently, the absence of arbitrage opportunities does not impose any restrictions on the 13 parameters, $\{i, i^*, \delta, \delta^*, \sigma, \sigma_d, \sigma_f, \rho_d, \rho_f, \rho, \gamma, \gamma_d, \gamma_f\}$, in Eq. (223).

It is important to recognize that, in general, the SDF in Eq. (74) is not the unique SDF consistent with the returns. For example, the SDF in Eq. (73) is also consistent with the returns, and it differs from the SDF in Eq. (74) when there is a continuous state space in discrete time with a finite number of asset returns. However, the continuous-time counterpart of this model with log-normal returns does have a unique SDF (e.g., see Harrison and Pliska, 1981, 1983). That continuous-time counterpart of Eq. (223) is given by,

$$d\ln S_t = \left(i - i^* + \varphi - \tfrac{1}{2}\sigma^2\right) dt + \sigma\, dW_t\,, \tag{225a}$$

$$d\ln X_t = \left(i - \delta + \varphi_d - \tfrac{1}{2}\sigma_d^2\right) dt + \sigma_d\, dW_t^d\,, \tag{225b}$$

$$d\ln Z_t^* = \left(i^* - \delta^* + \varphi_f - \tfrac{1}{2}\sigma_f^2 - \sigma_f\, \sigma\rho_f\right) dt + \sigma_f\, dW_t^f\,, \tag{225c}$$

where W, W^d, and W^f are Brownian motions with correlation matrix $\boldsymbol{P}$ in Eq. (223b). The continuous-time dynamics of the unique dollar-denominated SDF for the asset returns in

Eq. (225) are

$$dM_t = -M_t\big[i\,dt + \big(\gamma+\gamma_f\big)\sigma\,dW_t + \gamma_d\,\sigma_d\,dW_t^d + \gamma_f\,\sigma_f\,dW_t^f\big]\,, \tag{226a}$$

and the continuous-time dynamics of hat same unique SDF when the returns are instead denominated in pounds are

$$d\big(M_tS_t\big) = -\big(M_tS_t\big)\big[i^*dt + \big(\gamma+\gamma_f-1\big)\sigma\,dW_t + \gamma_d\,\sigma_d\,dW_t^d + \gamma_f\,\sigma_f\,dW_t^f\big]\,. \tag{226b}$$

It is important to recognize that uniqueness of the SDF in this case requires both continuous-time and continuous sample paths (i.e., a continuous diffusion without any jumps).[34]

In many examples in this literature, the vector of mean log asset returns, $\boldsymbol{\mu}$, and the covariance matrix, $\boldsymbol{\Omega}$, in Eq. (223) are state dependent (for example, they are allowed to depend on the short-term interest rate in each currency). We have omitted this state dependence purely for notational simplicity. For the same reason, we have also omitted the dynamics of the short-term interest rates, i and i^*, in the two currencies because they are not central to our analysis. In most dynamic term structure models (including the models that are used in this literature), short-term interest rates and long-term yields in a currency are driven by the same shocks. In this case, dynamic term structure models are useful for modeling and understanding the no-arbitrage relationship between yields with different maturities.[35] Our analysis below does not depend on the specific relationship between short-term interest rates and long-term yields in a currency, so for simplicity we omit those details. Instead, when the two asset that are not bank accounts are long-term bonds in dollars and pounds, we focus directly on the relationship between the exchange rate and long-term yields in the two currencies, which is the object of interest.

As we highlighted in Section 4.3, much (if not most) of the international asset pricing literature uses an alternative, but equivalent, parameterization of the log-normal asset return dynamics in Eq. (223). Rather than model the three asset returns directly, these papers instead model the returns on the two assets that are not bank accounts, together with an SDF denominated in both dollars and pounds. In Section 4.3 we argued that these two modeling approaches are isomorphic to each other. In particular, using Eq. (223) and the SDF in Eq. (74) we can write

$$\big[\;\ln MS'/S\,,\quad \ln M\,,\quad \ln X'/X\,,\quad \ln Z^{*\prime}/Z^*\;\big] \sim \mathcal{N}\left(\boldsymbol{\mu}_M\Delta t, \boldsymbol{\Omega}_M\Delta t\right)\,, \tag{227a}$$

[34]Jarrow and Madan (1995, 1999) highlight that SDFs are not unique in continuous-time models with a finite number of a securities and jumps that have a continuous distribution.

[35]See Dai and Singleton (2003) for a review of this literature

with

$$\boldsymbol{\mu}_M^\top = \begin{bmatrix} -i^* - \frac{1}{2}\boldsymbol{\Upsilon}_* \boldsymbol{\Omega}\, \boldsymbol{\Upsilon}_*^\top \\ -i - \frac{1}{2}\boldsymbol{\Upsilon}\, \boldsymbol{\Omega}\, \boldsymbol{\Upsilon}^\top \\ i - \delta + \varphi_d - \frac{1}{2}\sigma_d^2 \\ i^* - \delta + \varphi_f^* - \frac{1}{2}\sigma_f^2 \end{bmatrix}, \tag{227b}$$

and

$$\boldsymbol{\Omega}_M = \begin{bmatrix} \boldsymbol{\Upsilon}_* \boldsymbol{\Omega}\, \boldsymbol{\Upsilon}_*^\top & \boldsymbol{\Upsilon}_* \boldsymbol{\Omega}\, \boldsymbol{\Upsilon}^\top & -\varphi_d^* & -\varphi_f^* \\ \boldsymbol{\Upsilon}\, \boldsymbol{\Omega}\, \boldsymbol{\Upsilon}_*^\top & \boldsymbol{\Upsilon}\, \boldsymbol{\Omega}\, \boldsymbol{\Upsilon}^\top & -\varphi_d & -\varphi_f \\ -\varphi_d^* & -\varphi_d & \sigma_d & \sigma_d \sigma_f \rho \\ -\varphi_f^* & -\varphi_f & \sigma_d \sigma_f \rho & \sigma_f^2 \end{bmatrix}, \tag{227c}$$

where, for notational convenience, we've defined the analog of Eq. (223d) as

$$\boldsymbol{\Upsilon}_* = \begin{bmatrix} \gamma + \gamma_f - 1\,, & \gamma_d\,, & \gamma_f \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \varphi^*\,, & \varphi_d^*\,, & \varphi_f^* \end{bmatrix} = \boldsymbol{\Omega}\, \boldsymbol{\Upsilon}_*\,. \tag{227d}$$

Note that, with these definitions,

$$\begin{bmatrix} \varphi - \varphi^*\,, & \varphi_d - \varphi_d^*\,, & \varphi_f - \varphi_f^* \end{bmatrix} = \begin{bmatrix} \sigma\sigma\,, & \sigma_d\, \sigma\rho_d\,, & \sigma_f\, \sigma\rho_f \end{bmatrix}. \tag{228}$$

Papers that parameterize the model using Eq. (227) instead of Eq. (223) often attach different labels to the variables and parameters. For example, MS'/S is often labeled as M^* or $\tilde{M}$. Other parameters that are often given different labels include

$$\lambda = \sigma\big(\gamma + \gamma_f\big)\,, \qquad \lambda_d = \sigma_d\, \gamma_d\,, \qquad \lambda_f = \sigma_f\, \gamma_f\,, \tag{229a}$$

$$\lambda^* = \sigma\big(\gamma + \gamma_f - 1\big) = \lambda - \sigma\,, \qquad \lambda_d^* = \lambda_d - \sigma\rho_d\,, \qquad \lambda_f^* = \lambda_f - \sigma\rho_f\,. \tag{229b}$$

(Sometimes a $\sim$ on top of the parameter is used instead of a superscript $*$.) These two parameterizations are exactly equivalent, since one can always recover the original parameters in Eq. (223) as

$$\sigma = \lambda - \lambda^*\,, \qquad \rho_d = \tfrac{\lambda_d - \lambda_d^*}{\lambda - \lambda^*}\,, \qquad \rho_f = \tfrac{\lambda_f - \lambda_f^*}{\lambda - \lambda^*}\,, \tag{230a}$$

$$\gamma + \gamma_f = \tfrac{\lambda}{\lambda - \lambda^*}\,, \qquad \gamma_d = \sigma_d^{-1}\lambda_d\,, \qquad \gamma_f = \sigma_f^{-1}\lambda_f\,. \tag{230b}$$

Obviously different parameterizations of the same no-arbitrage model are innocuous. However, much of this literature attaches different economic interpretations to this alternative modeling approach and parameterization. For example, even though there are not any agents in no-arbitrage models, λ, λ_d, and λ_f are often interpreted as market prices of risk that apply to domestic (U.S.) investors, while λ^*, λ_d^*, and λ_f^* are viewed as market prices of risk that apply to foreign (U.K.) investors. Moreover, many papers use language which suggests that M and MS'/S are different SDFs that are associated with different economies. For example, Bakshi, Carr, and Wu (2008) provide a no-arbitrage model using this alternative approach and on page 135 they write:

> In complete markets, the stochastic discount factor for each economy is unique. Hence, the ratio of two stochastic discount factors uniquely determines the exchange rate dynamics between the two economies.

Similarly, Brandt and Santa-Clara (2002) also provide a no-arbitrage model and on page 173 they write:

> The key insight of our model is that when markets are incomplete, the volatility of the exchange rate is not uniquely determined by the domestic and foreign stochastic discount factors.

Again, to emphasize, the model in Eq. (223) only imposes that there are no arbitrage opportunities between these four assets. There are no agents or separate economies. There is no requirement that the asset market is complete. There is no economic mechanism in the no-arbitrage model that determines the exchange rate. Eqs. (227) and (229) provide an alternative parameterization of this same no-arbitrage model, but an alternative parameterization does not alter any of these statements.

Without loss of generality, for the remainder of this section we'll work with the model formulation in Eq. (223) because our view is that it affords the most transparent analysis.

B.1 Literature Discussion

Backus, Foresi, and Telmer (2001) was one of the first papers to consider a model of the form in Eq. (223). In their setup, Y is the price of a dollar-denominated long-term bond and Z^* is the price of a pound-denominated long-term bond. They argue that the forward premium anomaly for currencies, together with the restriction of no arbitrage, has "strong implications for the structure and parameter values of affine models." The forward premium anomaly pertains to the mean of the change in the log exchange rate. In particular,

$$\varphi - \tfrac{1}{2}\sigma^2 = \alpha + \beta\big(i - i^*\big)\,. \tag{231}$$

To understand the source of the restrictions that Backus, Foresi, and Telmer (2001) derive, it is first important to recognize that if the covariance matrix, $\mathbf{\Omega}$, in Eq. (223) is invertible, then the absence of arbitrage does not impose any restrictions on the model. In that case, the vector of mean log asset returns, $\boldsymbol{\mu}$, could literally be anything and one can still solve for γ, γ_d, and γ_f in Eq. (223) as

$$\begin{aligned}&\left[\ \gamma+\gamma_f\,,\qquad \gamma_d\,,\qquad \gamma_f\ \right] \\ &\qquad = \boldsymbol{\Omega}^{-1}\boldsymbol{\mu} - \boldsymbol{\Omega}^{-1}\left[\ i-i^*-\tfrac{1}{2}\sigma^2\,,\qquad i-\delta-\tfrac{1}{2}\sigma_d^2\,,\qquad i^*-\delta^*-\tfrac{1}{2}\sigma_f^2-\sigma_f\,\sigma\rho_f\ \right].\end{aligned} \tag{232}$$

The intuition behind the lack of no arbitrage restrictions is straightforward. If the covariance matrix is invertible, then the three asset returns are driven by three linearly independent shocks. The absence of arbitrage only restricts the returns on assets that are exposed to the same shocks.

Thus, if the covariance matrix, $\boldsymbol{\Omega}$, in Eq. (223) is invertible (i.e., nonsingular) then the model is completely free to match the forward premium anomaly (i.e., no-arbitrage does not impose any restrictions on the model that prevent it from matching the forward premium anomaly). Backus, Foresi, and Telmer (2001) assume that the covariance matrix, $\boldsymbol{\Omega}$, is singular so that the three asset returns are driven by only two sources of uncertainty. The restrictions they derive are primarily driven by this assumption. Intuitively, a singular covariance matrix implies that the return on any of the four assets can be exactly replicated by trading in the other three (i.e., one of the four assets is redundant). For example, if the covariance matrix is singular, then the pound-denominated bank account could be exactly replicated with a portfolio of the two non-bank account assets and the dollar-denominated bank account. Therefore, the return on the pound-denominated bank account must exactly match the return on the portfolio that replicates it.

To illustrate, if

$$1-\rho_d^2-\rho_f^2-\rho^2+2\rho_d\rho_f\rho=0\,, \tag{233}$$

then the correlation matrix, $\boldsymbol{P}$, in Eq. (223) is singular and can be written as

$$\boldsymbol{P} = \begin{bmatrix} \frac{\rho_d-\rho_f\rho}{1-\rho^2} & \frac{\rho_f-\rho_d\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \frac{\rho_d-\rho_f\rho}{1-\rho^2} & \frac{\rho_f-\rho_d\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{\top}. \tag{234}$$

Therefore, in the continuous-time limit,[36] one can replicate the pound-denominated bank account using a portfolio with weights ω_d in the dollar-denominated asset (that is not a bank account), ω_f in the pound-denominated asset (that is not a bank account), and $1-\omega_d-\omega_f$

[36]Technically speaking, the covariance matrix of the log of the returns is singular, not the covariance matrix of the gross returns. Therefore, there is only exact replication in the continuous-time limit of the model (i.e., as $\Delta t \to 0$).

in the dollar-denominated bank account, where

$$\omega_d = \frac{\big(\rho_d - \rho_f \rho\big)\sigma\sigma_f}{\big[\big(\rho_f - \rho_d\rho\big)\sigma + \big(1-\rho^2\big)\sigma_f\big]\sigma_d} \quad \text{and} \quad \omega_f = \frac{\big(\rho_f - \rho_d\rho\big)\sigma}{\big(\rho_f - \rho_d\rho\big)\sigma + \big(1-\rho^2\big)\sigma_f}\,. \tag{235}$$

With some algebra, one can verify ω_d and ω_f solve the replicating problem since

$$\begin{aligned} \begin{bmatrix} \omega_f\,, & \omega_d\,, & \omega_f \end{bmatrix} \mathbf{\Sigma} \begin{bmatrix} \frac{\rho_d - \rho_f\rho}{1-\rho^2} & \frac{\rho_f-\rho_d\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \\ = \begin{bmatrix} 1\,, & 0\,, & 0 \end{bmatrix} \mathbf{\Sigma} & \begin{bmatrix} \frac{\rho_d - \rho_f\rho}{1-\rho^2} & \frac{\rho_f-\rho_d\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{236}$$

The model in Eq. (225), which is the continuous-time limit of the model in Eq. (223), naturally incorporates this no-arbitrage restriction. In particular, Eq. (236) implies that

$$\begin{bmatrix} \omega_f\,, & \omega_d\,, & \omega_f \end{bmatrix} \mathbf{\Omega} = \begin{bmatrix} 1\,, & 0\,, & 0 \end{bmatrix} \mathbf{\Omega}\,, \tag{237}$$

and therefore

$$\begin{aligned} \mathbb{E}_t \left[(1-\omega_d-\omega_f)\, i\, dt + \omega_d \left(\frac{dX_t}{X_t} + \delta\, dt \right) + \omega_f \left(\frac{d\,(S_t Z_t^*)}{(S_t Z_t^*)} + \delta^*\, dt \right) \right] & \\ = \big(i + \begin{bmatrix} \omega_f\,, & \omega_d\,, & \omega_f \end{bmatrix} \mathbf{\Omega}\mathbf{\Upsilon} \big) dt\,, & \qquad (238a) \\ = \big(i + \begin{bmatrix} 1\,, & 0\,, & 0 \end{bmatrix} \mathbf{\Omega}\mathbf{\Upsilon} \big) dt = \mathbb{E}_t \left[\frac{dS_t}{S_t} + i^*\, dt \right]. & \qquad (238b) \end{aligned}$$

That is, the return on the pound-denominated bank account exactly matches the return on the portfolio that replicates it. Equivalently, the singular covariance matrix in Eq. (234) implies that one of the elements (or a linear combination of the elements) in $\mathbf{\Upsilon}$ is redundant and can be set to zero. For example, if Eq. (234) holds then the first element of $\mathbf{\Upsilon}$ could be set to zero, since

$$\begin{bmatrix} \frac{\rho_d - \rho_f\rho}{1-\rho^2} & \frac{\rho_f-\rho_d\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^\top \mathbf{\Sigma}\mathbf{\Upsilon} = \begin{bmatrix} \frac{\rho_d - \rho_f\rho}{1-\rho^2} & \frac{\rho_f-\rho_d\rho}{1-\rho^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^\top \mathbf{\Sigma} \begin{bmatrix} 0 \\ \frac{(\rho_d-\rho_f\rho)\sigma}{(1-\rho^2)\sigma_d}\,(\gamma+\gamma_f)+\gamma_d \\ \frac{(\rho_f-\rho_d\rho)\sigma}{(1-\rho^2)\sigma_f}\,(\gamma+\gamma_f)+\gamma_d \end{bmatrix}. \tag{239}$$

Eqs. (234) and (239) effectively reduce the number of free parameters in the general model of Eq. (225) from 13 down to 11.

There are a couple of points worth emphasizing. First, Backus, Foresi, and Telmer (2001) assume that currency returns are completely spanned by long-term bond returns in the two currencies (i.e., currencies and interest rates are driven by the same shocks). This spanning assumption (i.e., a singular covariance matrix) is the primary source of the no-arbitrage restrictions that they derive, but it is not an implication of the absence of arbitrage opportunities.[37] When the dollar/pound exchange rate is indirectly modeled via an SDF denominated in both dollars and pounds, this assumption, and its importance, may be less transparent. Second, as we illustrated above, an SDF in not necessary to understand or derive the no-arbitrage restrictions in Backus, Foresi, and Telmer (2001), because those restrictions follows immediately from a simple static replication problem in Eq. (236). Put differently, given the assumption of a singular covariance matrix, one can derive the restrictions implied by no-arbitrage, without specifying, or explicitly solving for, an SDF. Third, as we have shown, the singular correlation matrix in Eq. (234) immediately implies (via no arbitrage) that the expected return on the exchange rate is directly related to the expected return on long-term bonds in the two currencies according to Eq. (238). Backus, Foresi, and Telmer (2001) find that the restriction in Eq. (238) has undesirable features that do not match the data well. It seems most natural to start by testing whether correlation matrix is close to singular (i.e., test whether exchange rates are spanned by movements in the two long-term bonds denominated in their respective currencies). If the covariance matrix is not singular then one should not expect the restriction in Eq. (238) to hold (since it relies on a singular covariance matrix).

Brandt and Santa-Clara (2002) provide empirical evidence that exchange rates are not spanned by movements in long-term bonds in both currencies. They interpret this empirical evidence as an indication that the asset market is incomplete, so that there is not a unique SDF. In that case, Eqs. (65) and (66) do not necessarily hold for any two SDFs denominated in different units. As we described in Section 4.4, Brandt and Santa-Clara (2002) model two separate SDFs, M and M^*, together with a third stochastic process, O, that they claim captures the degree of market incompleteness. On page 164 they state that the stochastic process, O, in their model captures the notion that "if markets are incomplete, the volatility of the exchange rate can contain an element that is orthogonal to the priced sources of risk in both countries."

In Section 4.4 we illustrated that the model in Brandt and Santa-Clara (2002) is not arbitrage-free. Here we describe the restrictions that their model imposes relative to the general model that we provided in Eq. (223). First, based on their empirical evidence that

[37] Lustig, Roussanov, and Verdelhan (2011) also assume that currency returns are completely spanned by long-term bond returns (i.e., they assume that currencies and interest rates are driven by the same shocks) .

exchange rates are not spanned by interest rates, they relax Eq. (233) and allow for a nonsingular correlation/covariance matrix. However, as we illustrated in Eqs. (225) and (226), there is a unique SDF in the continuous-time diffusion counterpart of the model in Eq. (223), even when the correlation/covariance matrix is nonsingular. In other words, the fact that currency returns are not well-spanned by the returns on other assets does not imply that the market is incomplete or that there is not a unique SDF in a no-arbitrage model.

Second, regardless of whether the market is complete or incomplete, the exchange rate cannot contain an element that is orthogonal to an SDF denominated in both dollars and pounds (in the language of Brandt and Santa-Clara (2002), the exchange rate cannot contain an element that is orthogonal to the priced sources of risk in both countries). This assumption in Brandt and Santa-Clara (2002) is the source of the arbitrage opportunity that we demonstrated in Eq. (70). As an alternative proof of that result (and a less general proof than the one we provided in Section 4.4), note that Brandt and Santa-Clara (2002) use the alternative parametrization in Eq. (229). They argue that if the market is incomplete, then it can be the case that $\lambda = 0 = \lambda^*$ and the growth in the exchange rate can contain an element that is orthogonal to both M and M^*. As Eqs. (70) and (230) illustrate, λ and λ^* can only be equal if $\sigma = 0$. It is true that it can be the case that either $\lambda = 0$ or $\lambda^* = 0$ (i.e., either $\sigma(\gamma + \gamma_f) = 0$ or $\sigma(\gamma + \gamma_f - 1) = 0$), but Jensen's inequality ensures that both conditions cannot hold simultaneously unless $\sigma = 0$. If we overlook this error in their model, Brandt and Santa-Clara (2002) effectively relax Backus, Foresi, and Telmer (2001)'s assumption of a singular covariance matrix in Eq. (234). However, they still restrict the general model in Eq. (223) by assuming that shocks to exchange rates that are independent of shocks to other assets must also be independent of SDFs that are consistent with the returns on those assets. Again, as the the general model in Eq. (223) illustrates, this restriction is not an implication of no-arbitrage or (in)complete markets.

As a third and final example we consider the paper by Brennan and Xia (2006). They estimate an SDF, M, that is consistent with the dollar-denominated returns on the dollar-denominated default-free bank account and long-term dollar-denominated bonds. Separately, they estimate an SDF, M^*, for the pound-denominated returns on the pound-denominated default-free bank account and long-term pound-denominated bonds. Then they test whether the asset market view in Eq. (65) holds for these two, separately identified, SDFs.

For ease of exposition, we'll translate the exercise in Brennan and Xia (2006) to the continuous-time counterpart in Eq. (225) of the general model in Eq. (223).[38] Let Y denote the dollar price of a dollar-denominated long-term bond, and let Z^* denote the pound price of a pound-denominated long-term bond. Let $\check{M}$ denote the SDF is that is consistent with the

[38]Brennan and Xia, 2006 also use a continuous-time model.

dollar-denominated returns on the dollar-denominated bank account and long-term bond. Then the dynamics of $\check{M}$ are given by

$$d\check{M}_t = -\check{M}_t \left\{ i\, dt + \left[\rho_d \sigma \left(\gamma + \gamma_f \right) + \sigma_d \gamma_d + \rho \sigma_f \gamma_f \right] dW_t^d \right\} . \tag{240}$$

Similarly, let $\check{M}^*$ denote the SDF is that is consistent with the pound-denominated returns on the pound-denominated bank account and long-term bond. Then the dynamics of $\check{M}^*$ are given by

$$d\check{M}_t^* = -\check{M}_t^* \left\{ i^*\, dt + \left[\rho_f \sigma \left(\gamma + \gamma_f - 1 \right) + \rho \sigma_d \gamma_d + \sigma_f \gamma_f \right] dW_t^f \right\} . \tag{241}$$

Therefore, Brennan and Xia (2006) interpret Eq. (65), or equivalently Eq. (66), as restricting exchange rate dynamics as follows

$$d \ln S_t = d \ln \check{M}_t^* - d \ln \check{M}_t \, . \tag{242}$$

They claim that Eq. (242) is an implication of integrated capital markets.

From Eqs. (240–242), it is clear that Brennan and Xia (2006) inherit Backus, Foresi, and Telmer (2001)'s assumption that shocks to currencies are completely spanned by shocks to interest rates in each currency. However, Brennan and Xia (2006) make a much stronger assumption: they assume that the SDFs in Eq. (65), or equivalently Eq. (66), can be identified using distinct sets assets. From Eqs. (240–242), this additional assumption reduces the number of free parameters from 11 in Backus, Foresi, and Telmer (2001) down to 4 (5 if we also include the unspecified correlation between W^d and W^f) in Brennan and Xia (2006). Instead, as we illustrated in Section 4, Eq. (66) only holds if M and M^* are the same SDF (that price the same assets) denominated in different units. In general, Eqs. (65) and (66) do not need to hold for SDFs derived from distinct sets of assets, even if capital markets are completely integrated.