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ABSTRACT

We consider demand systems for utility-maximizing consumers facing general budget constraints whose utilities are perturbed by additive linear shifts in marginal utilities. Budgets are required to be compact but are not required to be convex. We define demand generating functions (DGF) whose subgradients with respect to these perturbations are convex hulls of the utility-maximizing demands. We give necessary as well as sufficient conditions for DGF to be consistent with utility maximization, and establish under quite general conditions that utility-maximizing demands are almost everywhere single-valued and smooth in their arguments. We also give sufficient conditions for integrability of perturbed demand. Our analysis provides a foundation for applications of consumer theory to problems with nonlinear budget constraints.

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1 Introduction

Economic consumer demand analysis traditionally starts from a utility function that characterizes preferences, maximizes it subject to a budget linear in income and prices, and obtains demands as functions of these budget parameters. A useful extension starts from a probability over a field of preferences and obtains a conditional distribution of demands given the budget parameters. A crowning achievement of neoclassical consumer theory is characterization in terms of expenditure and indirect utility functions of demands obtained from utility maximization. This allows development of utility-consistent demand systems for applications without requiring constructive solution of the constrained optimization problem.[1] However, this characterization depends critically on the envelope/duality properties of optimization subject to a *linear* constraint, and is not immediately applicable to non-linear budget sets. We tackle this problem by introducing utility fields with linear additive marginal utility perturbations of a base utility function.[2] Through this device, we are able to establish conditions for consistency with utility maximization of demand systems for nonlinear budget sets, with the perturbation parameters playing a role analogous to prices in classical consumer theory.

Section 2 specifies our assumptions and gives an elementary envelope result that provides a framework for specifying *demand generating functions* (DGF) for a general consumer problem. This section also provides some further properties of DGF and demand that follow from utility maximization. In section 3, sufficient conditions are given for a candidate DGF to be consistent with utility

[1] See Mas-Colell et al. (1995, Chap. 3D-H) and Varian (1993, Chap. 7.3-7.5). Diewert (1974) gives mathematical details and historical references. In a nutshell, if $v(\mathbf{t}, \mathbf{p}, y)$ is the indirect utility function and $e(\mathbf{t}, \mathbf{p}, u)$ is the expenditure function for locally non-satiated utility with prices $\mathbf{p}$, income y, and environment $\mathbf{t}$, then market demand is given by Roy's identity, $X(\mathbf{t}, \mathbf{p}, y) = -\partial_p v(\mathbf{t}, \mathbf{p}, y)/\partial_y v(\mathbf{t}, \mathbf{p}, y)$, and the net welfare gain from a move from $(\mathbf{t}', \mathbf{p}', y')$ to $(\mathbf{t}'', \mathbf{p}'', y'')$ in money-metric utility at prices $\mathbf{p}$ and environment $\mathbf{t}$ is $e(\mathbf{t}, \mathbf{p}, v(\mathbf{t}', \mathbf{p}', y')) - e(\mathbf{t}, \mathbf{p}, v(\mathbf{t}'', \mathbf{p}'', y''))$.

[2] Our approach follows Matzkin and McFadden (2011), where payoff tremble induced by perturbations is used to establish existence and regularity of Bayes-Nash solutions to continuous games of imperfect information. These perturbations can be interpreted as translations of the gradients of indifference curves, which may be of independent interest as the dual of the location shifts of indifference curves associated with Stone-Geary and Gorman demand systems. For related use of perturbation methods in mathematical programming; see Rockafellar (1970), Birge and Murty (1994), and Luderer et al. (2002). Where possible, we reference Aliprantis and Border (2006), hereafter AB, for results that we use from mathematical analysis.

maximization, and some results are provided to assist the derivation of DGF for applications. Section 4 provides conditions for integrability of demand. Section 5 considers computation of the DGF in the case when utility has the Gorman polar form. Section 6 concludes. All proofs are given in the Appendix.

Acronym	Definition
ACCM	Axiom of Congruent Cyclic Monotonicity
DGF	Demand Generating Function
GFFI	Generating Function Fundamental Inequality

Table 1: Acronyms

2 An Envelope Theorem for Perturbed Consumers

Let S denote a convex body (a compact convex set with a non-empty interior) in $\mathbb{R}^n_+$ that includes all possible consumption vectors and let T be a compact metric space that indexes tastes, beliefs, and other attributes of the consumer's environment that are not necessarily finite-dimensional. Consider a base utility function $U(\mathbf{x};\mathbf{t})$, where $\mathbf{x} \in S$ is a consumption vector and $\mathbf{t} \in T$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$, and let $\eta(\mathbf{t},\mathbf{t}')$ denote the metric on T.

We form a utility field by adding a linear perturbation to base utility, $U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}$, where $\mathbf{m}$ is a vector of perturbation parameters that can be interpreted as shifts in the marginal utilities of the commodities. The effect of this perturbation on demand is <u>not</u> invariant under increasing transformations of U, so that further analysis is conditioned on a particular ordinal representation of base preferences. We will assume that $\mathbf{m}$ is contained in a closed ball $M = \{\mathbf{m} \in \mathbb{R}^n | \|\mathbf{m}\| \leq c\}$ for some $c > 0$. We do not assume that $U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}$ is quasiconcave in $\mathbf{x}$, but when it is and c is sufficiently large, we show in the Appendix that as a consequence $U(\mathbf{x};\mathbf{t})$, and hence $U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}$, are concave in $\mathbf{x}$.

Let $\mathcal{B}$ denote a subset of the space of non-empty compact sets in S, interpreted as the family of possible budget sets. Let

$$h(B',B'') = \max_{\mathbf{x}''\in B''} \min_{\mathbf{x}'\in B'} \|\mathbf{x}'-\mathbf{x}''\| + \max_{\mathbf{x}'\in B'} \min_{\mathbf{x}''\in B''} \|\mathbf{x}'-\mathbf{x}''\|$$

denote the Hausdorff set metric on $\mathcal{B}$; see AB 3.17. Assume that $\mathcal{B}$ is compact in the Hausdorff set metric topology. We do not require the $B \in \mathcal{B}$ to be convex sets or linear budget sets. Let $[B]$ denote the convex hull of a set B.

Consider the problem of maximizing $U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}$ in $\mathbf{x}$ over a budget set $B \in \mathcal{B}$. Term $X(\mathbf{m};\mathbf{t},B) = \mathrm{argmax}_{\mathbf{x}\in B}(U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x})$ the *demand mapping*, or when it is non-empty, the *demand correspondence*, and term $V(\mathbf{m};\mathbf{t},B) = \max_{\mathbf{x}\in B}(U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x})$ the *demand generating function* (DGF). Term $E(\mathbf{t}) = \bigcup_{(\mathbf{m},B)\in M\times\mathcal{B}} X(\mathbf{m};\mathbf{t},B)$ the *exposed set*; it is the set of $\mathbf{x} \in S$ that are chosen for some budget and some perturbation $\mathbf{m}$. We call $E(\mathbf{t})$ *full* if its closure is S. In the classical case of linear budget sets $B(p,y) = \{\mathbf{x} \in S | \mathbf{p}\cdot\mathbf{x} \le y\}$ with strictly positive prices $\mathbf{p}$ and incomes y satisfying $\min_{x\in S}\mathbf{p}\cdot\mathbf{x} \le y \le \max_{\mathbf{x}\in S}\mathbf{p}\cdot\mathbf{x}$ let $v(\mathbf{m},\mathbf{t},p,y) \equiv V(\mathbf{m},\mathbf{t},B(p,y))$, and note that v is a classical indirect utility function in (p,y). When $\mathcal{B}$ contains the classical linear budget sets and utility is quasiconcave, continuous, and locally non-satiated in the interior of S, $E(t)$ is full. It may also be full for sufficiently rich families $\mathcal{B}$ even if utility is not quasiconcave; e.g., the extreme case where $\mathcal{B}$ contains all singleton budget sets in S.

As a consequence of maximization of perturbed utility, if $\mathbf{x}^0 \in X(\mathbf{m}^0;\mathbf{t},B^0)$ and $\mathbf{x}^0 \in B^1$, then $V(\mathbf{m}^0;\mathbf{t},B^0) + (\mathbf{m}^1 - \mathbf{m}^0)\cdot\mathbf{x}^0 \equiv U(\mathbf{x}^0;\mathbf{t}) + \mathbf{m}^1\cdot\mathbf{x}^0 \le \max_{x\in B^1}(U(\mathbf{x};\mathbf{t}) + \mathbf{m}^1\cdot\mathbf{x}) \equiv V(\mathbf{m}^1;\mathbf{t},B^1)$. Rearranging gives

Definition 1 *Generating Function Fundamental Inequality (GFFI): If* $\mathbf{m}^0,\mathbf{m}^1 \in M, B^0, B^1 \in \mathcal{B}, \mathbf{t} \in T$, *and* $\mathbf{x}^0 \in B^1 \cap X(\mathbf{m}^0;\mathbf{t},B^0)$, *then* $(\mathbf{m}^1 - \mathbf{m}^0)\cdot\mathbf{x}^0 \le V(\mathbf{m}^1;\mathbf{t},B^1) - V(\mathbf{m}^0;\mathbf{t},B^0)$, *with equality if and only if* $\mathbf{x}^0 \in X(\mathbf{m}^1;\mathbf{t},B^1)$.

Summing the GFFI over a cycle of perturbations and budget sets gives an analog for perturbed utility maximization of the conventional axiom of revealed preference:

Definition 2 *Axiom of Congruent Cyclic Monotonicity (ACCM): If* $\mathbf{m}^j \in M, B^j \in \mathcal{B}$, *and* $\mathbf{x}^j \in B^{j+1} \cap X(\mathbf{m}^j;\mathbf{t},B^j)$ *for* $j = 1,...,J$, *where* $J \ge 2$ *is an arbitrary integer and* $(\mathbf{m}^{J+1}, B^{J+1}, \mathbf{x}^{J+1}) = (\mathbf{m}^1, B^1, \mathbf{x}^1)$, *then* $\sum_{j=1}^{J}(\mathbf{m}^{j+1} - \mathbf{m}^j)\cdot\mathbf{x}^j \le 0$, *with the inequality strict unless* $\mathbf{x}^{j+1} \in X(\mathbf{m}^j;\mathbf{t},B^j)$ *for* $j = 1,...,J$.

When the B^j are all the same, the ACCM implies that $X(\mathbf{m}^j; \mathbf{t}, B)$ satisfies a cyclic monotonicity condition (Rockafellar, 1970, chap. 24) that guarantees that it is contained in the subdifferential of a convex function. When the $\mathbf{m}^j$ are all the same, the ACCM reduces to the Congruence Axiom of Revealed Preference (Richter, 1966; Matzkin, 1991; McFadden and Richter, 1990).

The starting point for our analysis is an elementary theorem that summarizes properties of perturbed utility maximization; all proofs are given in an appendix.

Theorem 1 *(Envelope Theorem) Suppose $U : S \times T \to \mathbb{R}$ is continuous, M is a closed ball, and budget sets B are contained in a compact (in the Hausdorff set metric topology) subset $\mathcal{B}$ of the space of non-empty compact subsets of S. Then*

[Cond. X] The demand mapping $X(\mathbf{m}; \mathbf{t}, B)$ is a compact-valued, upper hemicontinuous correspondence from $M \times T \times \mathcal{B}$ into S that satisfies the ACCM. The exposed set $E(\mathbf{t})$ is a compact-valued, upper hemicontinuous correspondence from T into S.

[Cond. V] The demand generating function $V(\mathbf{m}; \mathbf{t}, B)$ from $M \times T \times \mathcal{B}$ into $\mathbb{R}$ is continuous in its arguments; and convex, closed, and non-decreasing in $\mathbf{m}$ for each $(\mathbf{t}, B)$. Further, $V(\mathbf{m}; \mathbf{t}, B)$ and $X(\mathbf{m}; \mathbf{t}, B)$ satisfy the generating function fundamental inequality (GFFI).

[Cond. SD] The subdifferential

$$\partial_m V(\mathbf{m}; \mathbf{t}, B) = \{\mathbf{x} \in \mathbb{R}^n | \forall \mathbf{m}' \in M, V(\mathbf{m}'; \mathbf{t}, B) - V(\mathbf{m}; \mathbf{t}, B) \geq \mathbf{x} \cdot (\mathbf{m}' - \mathbf{m})\}$$

exists for all $(\mathbf{m}, \mathbf{t}, B) \in M \times T \times \mathcal{B}$, and is a convex-valued, compact-valued, upper hemicontinuous correspondence satisfying $[X(\mathbf{m}; \mathbf{t}, B)] = \partial_m V(\mathbf{m}; \mathbf{t}, B)$ and consequently $X(\mathbf{m}; \mathbf{t}, B) \subseteq B \cap \partial_m V(\mathbf{m}; \mathbf{t}, B)$. Then, all extreme points of $\partial_m V(\mathbf{m}; \mathbf{t}, B)$ are in $X(\mathbf{m}; \mathbf{t}, B)$. If, further, U is quasiconcave, then $X(\mathbf{m}; \mathbf{t}, B) = B \cap \partial_m V(\mathbf{m}; \mathbf{t}, B)$, and if in addition, B is convex, then $X(\mathbf{m}; \mathbf{t}, B) = \partial_m V(\mathbf{m}; \mathbf{t}, B)$. For each $(\mathbf{t}, B) \in T \times \mathcal{B}$, the subdifferential is almost everywhere (w.r.t. $\mathbf{m}$) a singleton, and where it is a singleton, it is continuous and continuously differentiable, and $\partial_{mm} V(\mathbf{m}; \mathbf{t}, B) = \partial_m X(\mathbf{m}; \mathbf{t}, B)$ is symmetric and positive semidefinite.

Remark [Cond X] is the standard result that the demand correspondence coming from continuous utility maximization in a compact budget set is upper hemicontinuous, but here continuity with respect to the budget is characterized in terms of the Hausdorff distance between general budgets rather than Euclidean distance between price and income parameters determining classical linear budgets. The condition does not require that $U(\mathbf{x})+\mathbf{m}\cdot\mathbf{x}$ be nondecreasing in $\mathbf{x}$ and locally nonsatiated, but if it is, then $X(\mathbf{m};\mathbf{t},B)$ is *northeast exposed*; i.e., if $\mathbf{x} \in X(\mathbf{m};\mathbf{t},B)$, then $\mathbf{x}' \geq \mathbf{x}$ and $\mathbf{x}' \neq \mathbf{x}$ imply $\mathbf{x}' \notin B$.

[Cond. V] corresponds to the definition of an indirect utility function, but now the dependence of this function on the perturbation vector $\mathbf{m}$ is emphasized, and V is written as a function of the budget set B itself rather than as a function of parameters that determine B.

[Cond. SD] establishes that for almost all $\mathbf{m}$, the utility-maximizing demand is a singleton that is given by the gradient of V with respect to $\mathbf{m}$. Compare this for a classical linear budget constraint with Shepard's lemma giving Hicksian demands as a gradient of the expenditure function with respect to price, and Roy's identity, giving market demands as a scaled gradient of the indirect utility function with respect to price.

If continuity of $U(\mathbf{x};\mathbf{t})$ is strengthened to Lipschitz continuity, the following corollary establishes that the demand generating function is Lipschitz in its arguments:

Corollary 1 *Suppose S, U, M, and $\mathcal{B}$ satisfy the assumptions of Theorem 1, and $U(\mathbf{x};\mathbf{t})$ is Lipschitz on $S \times T$. Then $V(\mathbf{m};\mathbf{t},B)$ is Lipschitz on $M \times T \times \mathcal{B}$.*

Remark The usual way to define a (nonlinear) budget set B is to specify the expenditure required to purchase $\mathbf{x}$. Then $B = \{\mathbf{x} \in S | R(\mathbf{x};\pi) \leq y\}$, where y is income and π is a vector characterizing costs that varies with the economic environment. The neoclassical case is $R(\mathbf{x};\pi) = \mathbf{p}\cdot\mathbf{x}$, with $\pi = \mathbf{p}$. An important application with nonlinear budget sets is study of progressive income taxation and labor supply; see Burtless and Hausman (1978), Diamond and Mirrlees (1971), Gan and Ju (2011), Hausman (1985), Moffitt (1990), and tacitly Milgrom and Segal (2002). In this case, y is the value of endowment income and entitlements when the consumer does not work, $R(\mathbf{x};\pi)$ gives expenditure on goods less net after-tax wages, and π describes goods prices, the wage rate, and the income tax schedule. Figure 1 illustrates a typical budget set for this application, with leisure on the horizontal axis, a consumption good on the vertical axis, and a finite number of income tax brackets so that $R(\mathbf{x};\pi)$ is piecewise linear and B is a (non-convex) polytope. Similar budget sets arise under block-rate utility tariffs (McFadden et al., 1978; Nauges and Blundell, 2010), and under tax-qualified savings programs (Beshears et al., 2010; McFadden, 2010); in the last case, a discrete choice of owning or renting induces distinct nonlinear conditional budget sets, so B is not necessarily connected. A generalization of the framework above considers budget sets of the form $B(\pi, \mathbf{q}; y) = \{\mathbf{x} \in S | R(\mathbf{x};\pi) + \mathbf{q}\cdot\mathbf{x} \le y\}$, where $\mathbf{q}$ is a vector of perturbations that behave like prices and can be interpreted as net commodity-specific excise taxes. The Lagrangian for utility maximization is $L = U(\mathbf{x},\mathbf{t}) + \mathbf{m}\cdot\mathbf{x} + \mu[y - \mathbf{q}\cdot\mathbf{x} - R(\mathbf{x};\pi)]$, with a first-order condition (for an interior maximum when U is differentiable) $\partial_{\mathbf{x}}U - \mu\partial_{\mathbf{x}}R = \mu\mathbf{q} - \mathbf{m}$. The demand generating function $V(\mathbf{m};\mathbf{t};\mathbf{q};y;\pi)$ is the optimized value of this Lagrangian. Then singleton market demands satisfy $X = \partial_{\mathbf{m}}V = -\partial_q V/\partial_y V$, and V satisfies Roy's identity in $(y;\mathbf{q})$ even though there are additional nonlinear expenditure elements in the budget. Also, from Theorem 1, X continues to satisfy the subgradient property of V with respect to $\mathbf{m}$.

Theorem 1 does not establish regularity properties for the demand correspondence $X(\mathbf{m};\mathbf{t},B)$ other than upper hemicontinuity in $(\mathbf{m},\mathbf{t},B)$, the ACCM, and almost everywhere single-valuedness and continuous differentiability in m. It is known from the work of Katzner (1968) and Rader (1973a,b) that strong differentiability and strict concavity conditions on utility are needed in the classical case to ensure that X is everywhere continuously differentiable in the budget parameters. However, we establish that $X(\mathbf{m};\mathbf{t},B)$ is almost everywhere Lipschitz in $\mathbf{t}$ and B under much weaker conditions - smooth utility and a class of polytope budget sets, defined next, that include economic applications such as labor supply with progressive income taxes and demand given utility block rate tariffs that have piecewise linear budget frontiers.

Definition 3 *A polytope budget set is a finite union $B = \cup_{k=1}^{K} C^k$, where each C^k is a convex polytope; i.e., a non-empty compact set contained in S that is formed by the intersection of an affine subspace (which may be the whole space) and a finite number of half-spaces. Let $W^k \equiv \{\mathbf{x} \in \mathbb{R}^n | Q^k\mathbf{x} = w^k\}$, where Q^k is $i_k \times n$,*

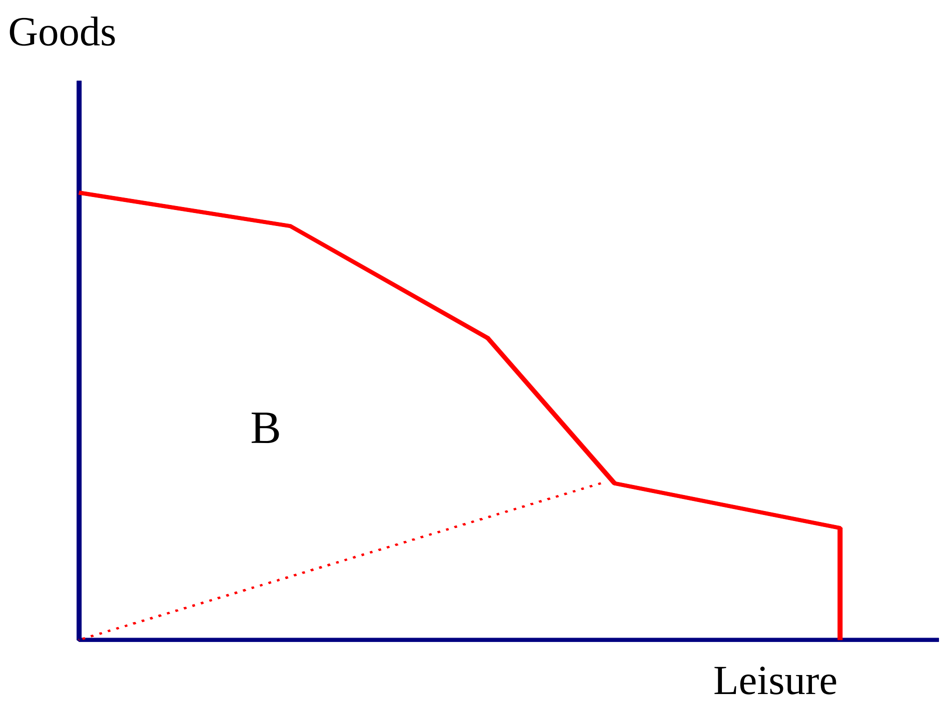

Figure 1: A polytope budget set

denote the affine subspace spanned by C^k, and assume that redundant equality constraints are eliminated so that Q^k is of full rank i_k. Let $\{\mathbf{x} \in \mathbb{R}^n | P^k\mathbf{x} \leq y^k\}$, where P^k is $j_k \times n$, denote the intersection of half-spaces that complete the definition of C^kCk, and exclude from P^k any column that is redundant because the associated half-space contains W^k. Then C^k has a non-empty interior relative to W^k; i.e., there exists an $\mathbf{x} \in W^k$ such that $P^k\mathbf{x} < y^k$. Let $\beta = (\beta^1, ..., \beta^K)$ denote the finite-dimensional vector with components $\beta^k = (Q^k, w^k, P^k, y^k)$; then the polytope budget set B is finitely parameterized by β.

In the classical case of a single linear budget constraint, B is simply the non-empty intersection of the non-negative orthant and a half-space in which positive income determines the inequality constraint bound. Hence, in the inequality constraints $P^k\mathbf{x} \leq y^k$ in the definition of C^k, y^k can be interpreted as a vector of "generalized income" bounds. Assume that if the inequality constraint bounds y^k define a non-empty polytope C^k, then strictly larger vectors y'^k are also economically possible. Then, the set Y^k of economically possible generalized income bounds has a non-empty interior in $\mathbb{R}^{j_k}$. The role of the assumption that Y^k has a non-empty interior will be to ensure that the constraints $Q^k\mathbf{x} = w^k$ and $P^k\mathbf{x} \leq y^k$ fail to satisfy a linear independence constraint qualification for y^k on a set of at most Lebesgue measure zero. In specific applications, it will often be possible to satisfy this requirement with some components of y^k fixed, such as non-negativity constraints $x \geq 0$.

Polytope budget sets B need not be convex, and if some commodities are discrete, they need not be connected. The convex polytopes C^k that are elements of B need not be disjoint.

Any compact budget set B can be approximated arbitrarily closely (in the Hausdorff set metric topology) by a polytope budget set B'. The argument is trivial. A compact set has a finite covering by open neighborhoods of arbitrarily small diameter. Just take these neighborhoods to be hypercubes. Then, their closures are convex polytopes, and the closure of the covering is then a polytope budget set B' that contains B.

Corollary 2 *Suppose S, U, M, and B satisfy the assumptions of Theorem 1, $U(\mathbf{x}; \mathbf{t})$ is twice continuously differentiable in $\mathbf{x}$, $\nabla_\mathbf{x} U(\mathbf{x}; \mathbf{t})$ is Lipschitz in $\mathbf{t}$ and $\nabla_{\mathbf{xx}} U(\mathbf{x}; \mathbf{t})$*

is continuous in $\mathbf{t}$. *Suppose the* $B \in \mathcal{B}$ *are polytope budget sets,* $B = \cup_{k=1}^{K} C^k$, *with* C^k *a non-empty convex polytope spanning* $W^k \equiv \{\mathbf{x} \in \mathbb{R}^n | Q^k \mathbf{x} = w^k\}$, *where* Q^k *is* $i_k \times n$ *matrix of rank* i_k, *and* $P^k \mathbf{x} \leq y^k$ *with* y^k *in a set* $Y^k \subseteq \mathbb{R}^{j_k}$ *that has a non-empty interior, and let* β *denote the finite parameterization of* B. *Then for almost all* $(\mathbf{m}, y^1, ..., y^K) \in M \times Y^1 \times \cdots \times Y^K$, *the demand correspondence* $X(\mathbf{m}, \mathbf{t}, \beta)$ *is a singleton that is continuously differentiable in* $(\mathbf{m}, \beta)$ *and locally Lipschitz in* $\mathbf{t}$.

Remark We give an elementary direct proof of this corollary in the Appendix. When $U(x, t)$ is concave in x, a stronger assumption than we have used, the methods of convex analysis can be applied to establish generic, but not almost everywhere, regularity without our restriction to polytope budget sets; see Aubin (1984), Dontchev (1998), Dontchev and Rockafellar (1996), Henrion (1992), Levy (2001), Klatte and Tammer (1990), Magaril-Il'aev (1978), Robinson (1980), and Scholtes and Stohr (2001).

Hausman (1985) discusses the stochastic specification and estimation of econometric models with non-convex budgets, in particular those where the budget can be represented as a finite union of convex sets. Blomquist and Newey (2002) propose a nonparametric estimation approach for the case of convex preferences and convex and piecewise linear budget. Milgrom and Segal (2002) prove a general envelope theorem and relate that to a range of economic models.

3 Demand generating functions

To guarantee that a demand system derived from a candidate demand generating function is consistent with utility maximization, one must verify that the candidate satisfies sufficient conditions to be derivable from maximization of some utility function. The following result provides such conditions.

Theorem 2 *(Converse envelope) Let* $V(\mathbf{m}; \mathbf{t}, B)$ *be a continuous function on* $M \times T \times \mathcal{B}$, *that is convex, closed, and nondecreasing in* $\mathbf{m}$, *where* M *is a compact ball and* $\mathcal{B}$ *is a compact subset of the space of non-empty compact sets in a convex body* $S \subseteq \mathbb{R}^n_+$. *Define* $X(\mathbf{m}; \mathbf{t}, B) = B \cap \partial_{\mathbf{m}} V(\mathbf{m}; \mathbf{t}, B)$ *and*

$E(\mathbf{t}) = \cup_{(\mathbf{m},B)\in M\times\mathcal{B}} X(\mathbf{m};\mathbf{t},B)$. *Assume that* $B \cap \partial_{\mathbf{m}} V(\mathbf{m};\mathbf{t},B)$ *is non-empty, and that* $X(\mathbf{m};\mathbf{t},B)$ *and* $V(\mathbf{m};\mathbf{t},B)$ *satisfy the generating function fundamental inequality (GFFI).*

Then the function $U(\mathbf{x};\mathbf{t}) = \inf\{V(\mathbf{m};\mathbf{t},B) - \mathbf{m}\cdot\mathbf{x} | (\mathbf{m},B) \in M\times\mathcal{B}, \mathbf{x}\in B\}$ *for* $\mathbf{x}\in E(\mathbf{t})$, *and* $U(\mathbf{x};\mathbf{t}) = -\infty$ *for* $\mathbf{x}\notin E(\mathbf{t})$, *is defined on* $S\times T$ *and continuous on* $\{(\mathbf{x},\mathbf{t})|\mathbf{t}\in T, \mathbf{x}\in E(\mathbf{t})\}$, *and satisfies* $V(\mathbf{m};\mathbf{t},B) = \max_{\mathbf{x}\in B}(U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x})$, *with* $X(\mathbf{m};\mathbf{t},B)$ *as its demand correspondence. If* $E(\mathbf{t})$ *is full, then* $U(\mathbf{x};\mathbf{t})$ *is continuous on* $S\times T$.

Remark The assumptions of Theorem 2 are conclusions of Theorem 1 when U is quasiconcave. However, Theorem 2 does not guarantee that $E(\mathbf{t})$ is full or that the utility it constructs from a candidate demand generating function is quasiconcave, so these assumptions are not quite necessary and sufficient.

Lack of a palette of known functions V and composition rules for these functions are a practical limitation to the use of this theorem to construct utility-consistent demand systems for nonlinear budget sets. For the current general case, the following results will be somewhat useful in applications such as demand for goods with block rate tariffs, and labor supply in response to progressive tax rates. For $B\in\mathcal{B}$, define the support function $s(\mathbf{p},B) = \max_{\mathbf{x}\in B}\mathbf{p}\cdot\mathbf{x}$. Then for $0\neq\mathbf{p}\in\mathbb{R}^n$, $\{\mathbf{x}\in\mathbb{R}^n|\mathbf{p}\cdot\mathbf{x}\le s(\mathbf{p},B)\}$ is a supporting half-space. If B is convex, then it is the intersection of its supporting half-spaces. Let $w(\mathbf{t},\mathbf{p},y) = \max_x\{U(\mathbf{x};\mathbf{t})|\mathbf{p}\cdot\mathbf{x}\le y\}$ denote the indirect utility function of the base utility function $U(\mathbf{x};\mathbf{t})$. The following corollaries relate demand generating functions to conventional indirect utility functions; establish that V exhibits a quasiconvexity property with respect to unions of sets in B which in the case of linear budget constraints implies the conventional quasiconvexity of the indirect utility function in income and prices; and give further composition rules.

Corollary 3 *Suppose* $U(\mathbf{x};\mathbf{t})$ *is non-decreasing and locally nonsatiated in* $\mathbf{x}$, *and* $w(\mathbf{t},\mathbf{p},y)$ *is its indirect utility function. The indirect utility function of* $U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}$ *for* $m\ge 0$ *is*

$$v(\mathbf{m},\mathbf{t},\mathbf{p},y) = \max\left\{\min_{\mathbf{p}'} w\left(\mathbf{t},\mathbf{p}', y\cdot\sum_j \theta_j p'_j/p_j\right) + y\cdot\sum_j \theta_j m_j/p_j \middle| \theta_j\ge 0, \sum_j\theta_j = 1\right\}.$$

If U is homogeneous of degree one, and $e(\mathbf{t},\mathbf{p}) = \min\{\mathbf{p}\cdot\mathbf{x}|U(\mathbf{t},\mathbf{x}) \geq 1\}$ is its unit expenditure function, then $v(\mathbf{m},\mathbf{t},\mathbf{p},y) = y\cdot\min_{\mathbf{p}'}\max_j(p'_j/e(\mathbf{p}')+m_j)/\mathbf{p}_j$.

Corollary 4 *If $B \in \mathcal{B}$ is convex with support function $s(\mathbf{p}, B)$, then for $\mathbf{m} \geq 0$, $V(\mathbf{m};\mathbf{t},B) \leq \min_{\mathbf{p}\in\mathbb{R}^n_+} v(\mathbf{m},\mathbf{t},\mathbf{p},s(\mathbf{p};B))$. If in addition, $U(\mathbf{x};\mathbf{t})$ is non-decreasing, locally nonsatiated, and quasiconcave in $\mathbf{x}$, then*

$$V(\mathbf{m};\mathbf{t},B) = \min_{\mathbf{p}\in\mathbb{R}^n_+} v(\mathbf{m},\mathbf{t},\mathbf{p},s(\mathbf{p};B)).$$

Corollary 5 *(Composition) Demand generating functions combine under the following rules.*

(1) If $B, B', B'' \in \mathcal{B}$ and $B \subseteq B' \cup B''$, then $V(\mathbf{m},\mathbf{t},B) \leq \max\{V(\mathbf{m},\mathbf{t},B'), V(\mathbf{m},\mathbf{t},B'')\}$.

(2) If $B^j \in \mathcal{B}$ for j in an arbitrary index set J, and $B = \cup_{j\in J} B^j \in \mathcal{B}$, then $V(\mathbf{m},\mathbf{t},B) = \sup_{j\in J} V(\mathbf{m},\mathbf{t},B^j)$.

(3) Suppose M_i is compact ball in $\mathbb{R}^{n_i}_+$ and $\mathcal{B}_i$ is a subset of the family of non-empty compact sets in $\mathbb{R}^{n_i}$ for i = 1,2. Suppose that $V_i : M_i \times T \times B_i \to \mathbb{R}$ are demand generating functions. Then, $V(\mathbf{m}_1,\mathbf{m}_2;\mathbf{t},B_1 \times B_2) = V_1(\mathbf{m}_1,\mathbf{t},B_1) + V_2(\mathbf{m}_2,\mathbf{t},B_2)$ is a demand generating function on $M_1 \times M_2 \times T \times \mathcal{B}_1 \times \mathcal{B}_2$.

(4) If $B \in \mathcal{B}$ is written as a union over an index set J of convex subsets B_j that are also in $\mathcal{B}$ and if U is non-decreasing, locally non-satiated, and quasiconvex, then $V(\mathbf{m};\mathbf{t},B) = \sup_{j\in J} V(\mathbf{m};\mathbf{t},B_j) = \sup_{j\in J}\min_{\mathbf{p}} v(\mathbf{m},\mathbf{t},\mathbf{p},s(\mathbf{p};B_j))$.

Remark Corollary 3 gives the relationship between the indirect utility functions of the base utility and the perturbed utility. While this is not in general useful for construction, it reduces to a practical formula in the special case of homothetic preferences with a utility representation that is homogeneous of degree one. In Corollary 4, if B is a convex polytope, then it is sufficient in the definition of $V(\mathbf{m};\mathbf{t},B)$ to consider only convex combinations of the vectors $\mathbf{p}$ that are outward normals to maximal facets of B. An example of a demand generating function of closed form that might be utilized in Corollary 4 is $v(\mathbf{m},\mathbf{t},\mathbf{p},y) = \frac{1}{2}\mathbf{m}^\top C\mathbf{m} - \frac{1}{2}(\max(0,\mathbf{p}^\top C\mathbf{m} - y))^2/\mathbf{p}^\top C\mathbf{p}$, where C is a positive definite matrix; this is associated with the quadratic utility $\mathbf{m}\cdot\mathbf{x} - \frac{1}{2}\mathbf{x}^\top C^{-1}\mathbf{x}$, and range restrictions on $\mathbf{m}$ and y are required for the utility function to be nondecreasing and locally nonsatiated and for demands to be non-negative.

Remark The demand generating function can be used in welfare calculus to determine the net benefit from a policy change, with the net welfare gain (in utiles) from a move from $(\mathbf{t}', B')$ to $(\mathbf{t}'', B'')$ equal to $V(\mathbf{m}, \mathbf{t}'', B'') - V(\mathbf{m}, \mathbf{t}', B')$. Normalizing the utile difference by a marginal utility of money gives a measure that corresponds to Hicksian consumer surplus.

4 Integrability of demand

We now seek conditions on the demand correspondence $X(\mathbf{m}; \mathbf{t}, B)$ that guarantee that it can be rationalized by utility $U(\mathbf{x}; \mathbf{t}) + \mathbf{m} \cdot \mathbf{x}$ maximized over the set B.

If X is the demand correspondence of a utility maximizing consumer and $X(\mathbf{m}; t, B) \cap B' \neq \emptyset$ then budget B' may be revealed preferred to budget B. Define then for any given demand correspondence a set L consisting of all finite sequences of $(\mathbf{m}, \mathbf{x})$ that connect in a finite number of steps from a given pair $(\mathbf{m}^a, B^a)$ to another pair $(\mathbf{m}^b, B^b)$ with $\mathbf{x}^j \in X(\mathbf{m}^j; t, B^j) \cap B^{j+1}$ in each step.

$$
\begin{aligned}
& L\left(\mathbf{m}^a, B^a, \mathbf{m}^b, B^b\right) \\
= & \left\{ \begin{array}{c} \left(\mathbf{x}^0, ..., \mathbf{x}^{J-1}, \mathbf{m}^0, ..., \mathbf{m}^{J-1}\right) | 1 \leq J < \infty, \mathbf{m}^0 = \mathbf{m}^a, \mathbf{m}^J = \mathbf{m}^b, \\ B^0 = B^a, B^J = B^b, \forall j < J : \mathbf{x}^j \in X\left(\mathbf{m}^j; t, B^j\right) \cap B^{j+1} \end{array} \right\}
\end{aligned}
$$

The sets L are convenient in the formulation of the following result.

Theorem 3 *(Integrability of demand) Let $X(\mathbf{m}; t, B) : S \times T \times \mathcal{B}$ be an upper hemicontinuous compact-valued correspondence satisfying the ACCM and with $X(\mathbf{m}; t, B) \subseteq B$. Assume there exists $(\mathbf{m}^*, B^*) \in M \times \mathcal{B}$ such that $B = \{B \in \mathcal{B} | \exists \mathbf{m} \in M : L(\mathbf{m}^*, B^*, \mathbf{m}, B) \cup L(\mathbf{m}, B, \mathbf{m}^*, B^*) \neq \emptyset\}$. Define*

$$
\begin{aligned}
V^1(\mathbf{m}; t, B) &= \sup \left\{ \sum_{j=0}^{J-1} \left(\mathbf{m}^{j+1} - \mathbf{m}^j\right) \cdot \mathbf{x}^j | \left(\mathbf{x}^0, ..., \mathbf{x}^{J-1}, \mathbf{m}^0, ..., \mathbf{m}^{J-1}\right) \in L(\mathbf{m}^*, B^*, \mathbf{m}, B) \right\}, \\
V^2(\mathbf{m}; t, B) &= \sup \left\{ \sum_{j=0}^{J-1} \left(\mathbf{m}^{j+1} - \mathbf{m}^j\right) \cdot \mathbf{x}^j | \left(\mathbf{x}^0, ..., \mathbf{x}^{J-1}, \mathbf{m}^0, ..., \mathbf{m}^{J-1}\right) \in L(\mathbf{m}, B, \mathbf{m}^*, B^*) \right\}
\end{aligned}
$$

and $V(\mathbf{m}; t, B) = \max\{V^1(\mathbf{m}; t, B), -V^2(\mathbf{m}; t, B)\}$*. Then* $V(\mathbf{m}; t, B) > -\infty$*, and* $X(\mathbf{m}; t, B) \subseteq B \cap \partial_{\mathbf{m}} V(\mathbf{m}; t, B)$*. Assume further that* $X(\mathbf{m}; t, B) = B \cap \partial_{\mathbf{m}} V(\mathbf{m}; t, B)$ *and that* V *is continuous on* $M \times T \times B$*. Then* V *satisfies the conditions of Theorem 2.*

Remark Recall that Theorem 2 establishes the existence of perturbed utility having V is its DGF and hence rationalizing the demand X. As was also remarked after Theorem 2, the condition that $X(\mathbf{m}; t, B) = B \cap \partial_{\mathbf{m}} V(\mathbf{m}; t, B)$ is a conclusion of Theorem 1 when U is quasiconcave.

Theorem 3 supposes that for any $B \in \mathcal{B}$ there is an $\mathbf{m} \in M$ and a path in L connecting $(\mathbf{m}, B)$ to $(\mathbf{m}^*, B^*)$. This condition may be relaxed, for example if $\mathcal{B}$ can be divided into subsets where the budgets only overlap within subsets and where the subsets may be separated by open sets. In such a case, the Theorem could be used to construct DGF separately for each subset with corresponding utilities for each of these islands.

5 Example - Gorman polar form

Suppose base preferences are described by a Gorman (1961) polar form: $U(\mathbf{x}; \mathbf{t})$ is obtained from a neoclassical indirect utility function

$$v(\mathbf{p}, y, \mathbf{t}) = \frac{y - C(\mathbf{p}; \mathbf{t})}{A(\mathbf{p}; \mathbf{t})},$$

where committed expenditure $C(\mathbf{p}; \mathbf{t})$ and the price index $A(\mathbf{p}; \mathbf{t})$ are concave, non-decreasing, and linear homogeneous in $\mathbf{p}$. Then,

$$U(\mathbf{x}; \mathbf{t}) = \min_{\mathbf{p}} \left\{ \frac{\mathbf{x} \cdot \mathbf{p} - C(\mathbf{p}; \mathbf{t})}{A(\mathbf{p}; \mathbf{t})} \right\}.$$

From this base utility, and for a convex compact budget B the perturbed consumer DGF is

$$\begin{aligned} V(\mathbf{m};\mathbf{t},B) &= \max_{\mathbf{x}\in B}\{U(\mathbf{x};\mathbf{t})+\mathbf{m}\cdot\mathbf{x}\} \\ &= \max_{\mathbf{x}\in B}\min_{\mathbf{p}}\left\{\mathbf{x}\cdot\left(\mathbf{m}+\frac{\mathbf{p}}{A(\mathbf{p};\mathbf{t})}\right)-\frac{C\left(\mathbf{p};t\right)}{A(\mathbf{p};\mathbf{t})}\right\}. \end{aligned}$$

But the function $\mathbf{x}\cdot\left(\mathbf{m}+\frac{\mathbf{p}}{A(\mathbf{p};\mathbf{t})}\right)-\frac{C(\mathbf{p};\mathbf{t})}{A(\mathbf{p};\mathbf{t})}$ is linear, hence concave, in $\mathbf{x}$ and so long as $\mathbf{p}\cdot\mathbf{x}>C(\mathbf{p};\mathbf{t})$, quasi-convex in p. Then by the Sion (1958) saddle point theorem, the max and min can be reversed, giving

$$V(\mathbf{m};\mathbf{t},B) = \min_{\mathbf{p}}\left\{R\left(\mathbf{m}+\frac{\mathbf{p}}{A(\mathbf{p};\mathbf{t})};B\right)-\frac{C\left(\mathbf{p};\mathbf{t}\right)}{A(\mathbf{p};\mathbf{t})}\right\},$$

where $R(\mathbf{p};B)=\max_{\mathbf{x}\in B}\{\mathbf{p}\cdot\mathbf{x}\}$ is the convex linear homogeneous maximum revenue function for B.

Then, a practical way to generate the demand systems from nonlinear budgets is to first specify the Gorman committed expenditure function $C(\mathbf{p};\mathbf{t})$, and second to calculate the revenue function $R(\mathbf{p};B)$. The next step is to compute $V(\mathbf{m};\mathbf{t},B)$ by minimizing $R(\mathbf{m}+\mathbf{p}/A(\mathbf{p};\mathbf{t});B)-C(\mathbf{p}/A(\mathbf{p};\mathbf{t});\mathbf{t})$ in $\mathbf{p}$. If the committed expenditure function is piecewise linear, the price index $A(\mathbf{p})$ is linear, and B is a convex polytope budget set, then the third stage is a finite (linear programming) calculation. Finally, for budgets that are unions of convex budgets, $B=\cup_j B_j$, the DGF is the maximum of the DGF over the convex budgets $V(\mathbf{m};\mathbf{t},B)=\max_j V(\mathbf{m};\mathbf{t},B_j)$ (Corollary 5).

6 Conclusion

This paper has contributed by characterizing demands and demand generating functions that are consistent with maximization of linearly perturbed utility under general compact, not necessarily convex budgets. Results are provided to assist the development of such models for applications.

The demand generating function defined and characterized in this section takes a particularly simple form in applications with discrete budget sets. Suppose S is the set of vertices of the unit simplex in $\mathbb{R}^n$, and B is any non-empty subset of these points. Suppose $U(\mathbf{x};\mathbf{t}) : S \times T \to \mathbb{R}$ is a continuous utility function that is random, reflecting unobserved heterogeneity across consumers. Consider the demand generating function $V(\mathbf{m};\mathbf{t},B) = \max_{\mathbf{x}\in B}\{U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}\}$. If the distribution of utilities is absolutely continuous, so that the probability of ties is zero, then with probability one, its subdifferential $\partial_{\mathbf{m}}V(\mathbf{m};\mathbf{t},B)$ is a unique utility-maximizing vertex of B. Hence, the expected demand generating function $EV(\mathbf{m};\mathbf{t},B)$ has a differential $\partial_{\mathbf{m}}EV(\mathbf{m};\mathbf{t},B)$ whose components are the probabilities that each of the alternatives in B will be chosen. Fosgerau et al. (2010) elaborates the properties of such choice probability generating functions.

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A Proofs

Lemma 1 *Let $\mathbf{m}$ be contained in the closed ball $M = \{\mathbf{m} \in \mathbb{R}^n | \|\mathbf{m}\| \leq c\}$ for some $c > 0$. If $U(\mathbf{x};\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}$ is quasiconcave in x for all $\mathbf{m}$ and c is sufficiently large, then $U(\mathbf{x};\mathbf{t})$ and $U(\mathbf{x};\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}$ are concave in $\mathbf{x}$.*

Proof. Suppose $U(\mathbf{x};\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}$ is quasiconcave in x for all $\mathbf{m} \in M$. For $\mathbf{x}' \neq \mathbf{x}''$ and $\theta \in (0, 1)$, define $\mathbf{x}^\theta = \theta\mathbf{x}' + (1 - \theta)\mathbf{x}''$. Then

$$U(\mathbf{x}^\theta, \mathbf{t}) + \mathbf{m} \cdot \mathbf{x}^\theta \geq \min\{U(\mathbf{x}';\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}', U(\mathbf{x}'';\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}''\},$$

implying

$$\begin{aligned} & U(\mathbf{x}^\theta, \mathbf{t}) - \theta U(\mathbf{x}', \mathbf{t}) - (1-\theta)U(\mathbf{x}'', \mathbf{t}) \\ \geq\ & \min\{(1-\theta)(U(\mathbf{x}', \mathbf{t}) - U(\mathbf{x}'', \mathbf{t}) + \mathbf{m} \cdot (\mathbf{x}' - \mathbf{x}'')), \theta(U(\mathbf{x}'', \mathbf{t}) - U(\mathbf{x}', \mathbf{t}) + \mathbf{m} \cdot (\mathbf{x}'' - \mathbf{x}'))\}. \end{aligned}$$

Take $\mathbf{m} = -(\mathbf{x}' - \mathbf{x}'')(U(\mathbf{x}', \mathbf{t}) - U(\mathbf{x}'', \mathbf{t}))/\|\mathbf{x}' - \mathbf{x}''\|^2$ to obtain $U(\mathbf{x}^\theta, \mathbf{t}) - \theta U(\mathbf{x}', \mathbf{t}) - (1 - \theta)U(\mathbf{x}'', \mathbf{t}) \geq 0$. When the ball M is of sufficient radius to contain all such $\mathbf{m}$, the inequality proves that $U(\mathbf{x}, \mathbf{t})$ is concave in $\mathbf{x}$. Since the sum of concave functions is concave, $U(\mathbf{x}, \mathbf{t}) + \mathbf{m} \cdot \mathbf{x}$ is concave in $\mathbf{x}$. ■

For more general theory of quasiconcave sums, see Crouziex and Lindberg (1986).

Proof of Theorem 1. The conclusions of this theorem are well-known with standard proofs in the linear budget case, and the extension to demand generating functions in the case of nonlinear budgets is straightforward. However, for completeness we give a full proof. It has already been shown, in the text preceding the statement of this theorem, the [Cond. V] claim that V and X satisfy GFFI, and hence the [Cond. X] claim that X satisfies the ACCM.

Since $\mathcal{B}$ is a metric space with the Hausdorff set metric, the Berge maximum theorem (AB 17.31) and AB 17.35(2) establish that $X(\mathbf{m};\mathbf{t},B)$ is a compact-valued, upper hemicontinuous correspondence from $M \times T \times \mathcal{B}$ into S and that V is continuous in its arguments. The correspondence $E(\mathbf{t})$ is compact-valued (AB 17.8) and upper hemicontinuous.

Suppose $\mathbf{m}^0 = \theta\mathbf{m}' + (1-\theta)\mathbf{m}''$ for some $0 < \theta < 1, \mathbf{x}_0 \in X(\mathbf{m}_0;\mathbf{t},B)$. Then $V(\mathbf{m}';\mathbf{t},B) \geq U(\mathbf{x}_0;\mathbf{t}) + \mathbf{m}' \cdot \mathbf{x}_0$ and $V(\mathbf{m}'';\mathbf{t},B) \geq U(\mathbf{x}_0;\mathbf{t}) + \mathbf{m}'' \cdot \mathbf{x}_0$, implying $\theta V(\mathbf{m}';\mathbf{t},B) + (1-\theta)V(\mathbf{m}'';\mathbf{t},B) \geq U(\mathbf{x}_0;\mathbf{t}) + \mathbf{m}^0 \cdot \mathbf{x}^0 = V(\mathbf{m}^0;\mathbf{t},B)$, proving convexity. If $\mathbf{m}^j \to \mathbf{m}^0$ and $\mathbf{x}^j \in X(\mathbf{m}^j;\mathbf{t},B)$, by upper hemicontinuity there exists a subsequence (retain notation) $\mathbf{x}^j \to \mathbf{x}^0 \in X(\mathbf{m}^0;\mathbf{t},B)$. Then $V(\mathbf{m}^j;\mathbf{t},B) \geq U(\mathbf{x}^0;\mathbf{t},B) + \mathbf{m}^j \cdot \mathbf{x}^0 \to V(\mathbf{m}^0;\mathbf{t},B)$, proving that the epigraph of V is closed. If $\mathbf{m}' \geq \mathbf{m}''$ and $\mathbf{x}'' \in X(\mathbf{m}'';\mathbf{t},B)$, then $V(\mathbf{m}',\mathbf{t},B) \geq U(\mathbf{x}'';\mathbf{t}) + \mathbf{m}' \cdot \mathbf{x}'' \geq U(\mathbf{x}'';\mathbf{t}) + \mathbf{m}'' \cdot \mathbf{x}'' = V(\mathbf{m}'';\mathbf{t},B)$, proving that V is nondecreasing in $\mathbf{m}$, completing the proof of [Cond. V].

The convexity and closure of V in $\mathbf{m}$ implies that the subdifferential $\partial_m V(\mathbf{m};\mathbf{t},B)$ exists and is non-empty, convex-valued, and compact-valued (AB 7.13). If $(\mathbf{m}^j;\mathbf{t}^j,B^j) \to (\mathbf{m}^0;\mathbf{t}^0,B^0), \mathbf{x}^j \in \partial_m V(\mathbf{m}^j;\mathbf{t}^j,B^j)$, and $\mathbf{x}^j \to \mathbf{x}^0$, then for each $\mathbf{m}'$, $V(\mathbf{m}';\mathbf{t}^j,B^j) - V(\mathbf{m}^j;\mathbf{t}^j,B^j) \geq \mathbf{x}^j \cdot (\mathbf{m}' - \mathbf{m}^j)$, implying by continuity that $V(\mathbf{m}';\mathbf{t}^0,B^0) - V(\mathbf{m}^0;\mathbf{t}^0,B^0) \geq \mathbf{x}^0 \cdot (\mathbf{m}' - \mathbf{m}^0)$; then $\partial_m V$ is upper hemicontinuous. If $\mathbf{x} \in X(\mathbf{m};\mathbf{t},B)$, then $V(\mathbf{m}';\mathbf{t},B) \geq U(\mathbf{x};\mathbf{t}) + \mathbf{m}' \cdot \mathbf{x}$ and $V(\mathbf{m};\mathbf{t},B) = U(\mathbf{x};\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}$ imply $V(\mathbf{m}';t,B) - V(\mathbf{m};\mathbf{t},B) \geq \mathbf{x} \cdot (\mathbf{m}' - m)$, and hence $X(\mathbf{m};\mathbf{t},B) \subseteq \partial_m V(\mathbf{m};\mathbf{t},B)$. Since $\partial_m V(\mathbf{m};\mathbf{t},B)$ is convex, $[X(\mathbf{m};\mathbf{t},B)] \subseteq \partial_m V(\mathbf{m};\mathbf{t},B)$.

Suppose $\mathbf{x}^* \in \partial_m V(\mathbf{m};\mathbf{t},B) \setminus [X(\mathbf{m};\mathbf{t},B)]$. Then there exists a separating hyperplane with normal $\mathbf{q}$ satisfying $\mathbf{q} \cdot \mathbf{x}^* > \mathbf{q} \cdot \mathbf{x}$ for all $\mathbf{x} \in X(\mathbf{m};\mathbf{t},B)$. Also, $V(\mathbf{m} + \lambda\mathbf{q};\mathbf{t},B) - V(\mathbf{m};\mathbf{t},B) \geq \lambda\mathbf{q} \cdot \mathbf{x}^*$. Consider $\lambda \downarrow 0$, and $\mathbf{x}^\lambda \in X(\mathbf{m}+\lambda\mathbf{q};\mathbf{t},B)$. Then, $V(\mathbf{m};\mathbf{t},B) - V(\mathbf{m}+\lambda\mathbf{q};\mathbf{t},B) \geq -\lambda\mathbf{q} \cdot \mathbf{x}^\lambda$. Adding these inequalities, $0 \geq \lambda\mathbf{q} \cdot (\mathbf{x}^* - \mathbf{x}^\lambda)$. There exists a subsequence (retain notation) of $\mathbf{x}^\lambda$ that converges to a point $\mathbf{x}^0$, which by upper hemicontinuity of X is contained in $X(\mathbf{m};\mathbf{t},B)$. Then, $\mathbf{q} \cdot \mathbf{x}^* \leq \mathbf{q} \cdot \mathbf{x}^0$, a contradiction. Hence, $\partial_m V(\mathbf{m};\mathbf{t},B) = [X(\mathbf{m};\mathbf{t},B)]$. Then $\mathbf{x} \in [X(\mathbf{m};\mathbf{t},B)] \cap B$ can be written as a convex combination $\mathbf{x} = \sum_{j=0}^n \theta_j \mathbf{x}^j$ of points $\mathbf{x}^j \in X(\mathbf{m};\mathbf{t},B)$, so $U(\mathbf{x}^j;\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}^j$ is the same for all j. If U is quasiconcave, then $U(\mathbf{x};\mathbf{t}) + \mathbf{m} \cdot \mathbf{x} \geq \min_j(U(\mathbf{x}^j;\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}^j) \equiv \sum_{j=0}^n \theta_j(U(\mathbf{x}^j;\mathbf{t}) + \mathbf{m} \cdot \mathbf{x}^j)$, implying $\mathbf{x} \in X(\mathbf{m};\mathbf{t},B)$. Finally if, in addition to quasiconvexity of U, B is convex, then $[X(\mathbf{m};\mathbf{t},B)] \subseteq B$, implying that $[X(\mathbf{m};\mathbf{t},B)] = X(\mathbf{m};\mathbf{t},B)) = \partial_m V(\mathbf{m};\mathbf{t},B)$.

The almost everywhere twice continuous differentiability of V in $\mathbf{m}$, with a symmetric positive semidefinite second derivative, is established by Alexandroff (1939); see also Rockafellar (1970), Howard (1998, Theorem 7.1), and AB 7.25 & 7.28. Then, $X(\mathbf{m};\mathbf{t},B)$ is almost everywhere (in $\mathbf{m}$) a singleton that is continuous and continuously differentiable with $\nabla_m X(\mathbf{m};\mathbf{t},B)$ symmetric and positive semidefinite. ■

Proof of Corollary 1. Let λ be the Lipschitz constant for U, and let it also bound S. If $B^k \to B^0, \mathbf{t}^k \to \mathbf{t}^0, \mathbf{m}^k \to \mathbf{m}^0$, $\mathbf{x}^k$ is a maximand of $U(\mathbf{x},\mathbf{t}^k) + \mathbf{m}^k \cdot \mathbf{x}$ on B^k, $\mathbf{x}^{k0}$ is a point in B^k closest to $\mathbf{x}^0$, and x^{0k} is a point in B^0 closest to $\mathbf{x}^k$, then $V(\mathbf{m}^k;\mathbf{t}^k,B^k) = U(\mathbf{x}^k;\mathbf{t}^k) + \mathbf{m}^k \cdot \mathbf{x}^k \geq U(\mathbf{x}^{k0};\mathbf{t}^k) + \mathbf{m}^k \cdot \mathbf{x}^{k0} \geq U(\mathbf{x}^0;\mathbf{t}^0) + \mathbf{m}^0 \cdot \mathbf{x}^0 - \lambda(\eta(\mathbf{t}^k,\mathbf{t}^0) + h(B^k,B^0) + \|\mathbf{m}^k - \mathbf{m}^0\|) = V(\mathbf{m}^0,\mathbf{t}^0,B^0) - \lambda(\eta(\mathbf{t}^k,\mathbf{t}^0) + h(B^k,B^0) + \|\mathbf{m}^k - \mathbf{m}^0\|)$ and $V(\mathbf{m}0;\mathbf{t}^0,B^0) = U(\mathbf{x}^0,\mathbf{t}^0) + \mathbf{m}^0 \cdot \mathbf{x}^0 \geq U(\mathbf{x}^{0k},\mathbf{t}^0) + \mathbf{m}^0 \cdot \mathbf{x}^{0k} \geq V(\mathbf{m}^k;\mathbf{t}^k,B^k) - \lambda(\eta(\mathbf{t}^k,\mathbf{t}^0) + h(B^k,B^0) + \|\mathbf{m}^k - \mathbf{m}^0\|)$. Then V is Lipschitz in its arguments. ■

Proof of Corollary 2. For each $(\mathbf{t},\beta)$, Theorem 1 applied to a single convex polytope C^k in the union that forms B establishes that the maximand correspondence $X^k(\mathbf{m},\mathbf{t},\beta^k)$ is for almost every $\mathbf{m} \in M$, say $\mathbf{m}^0$, a singleton that is continuous and continuously differentiable in $\mathbf{m}$ at $\mathbf{m}^0$. Define the Lagrangian

$$L(\mathbf{x},\mathbf{t},\mathbf{m},\beta^k,\lambda,\mu) = U(\mathbf{x},\mathbf{t}) + \mathbf{m}^0 \cdot \mathbf{x} + (w^k - Q^k\mathbf{x})^\top \lambda^k + (y^k - P^k\mathbf{x})^\top \mu^k.$$

By the Karush-Kuhn-Tucker Theorem applied to maximization of a continuously differentiable function subject to linear constraints, at $(\mathbf{x}^0,\mathbf{m}^0,\mathbf{t},\beta^k)$ there exist multipliers λ^{k0} and μ^{k0} such that the following Lagrangian necessary (local KKT) conditions hold; see Eustaquiro et al. (2007, Theorem 3.3), for a simple polar cone proof of this standard result.

$$
\begin{aligned}
\nabla_{\mathbf{x}} U(\mathbf{x}^0, \mathbf{t}) + \mathbf{m}^0 - Q^{k\top}\lambda^{k0} - P^{k\top}\mu^{k0} &= 0 \\
w^k - Q^k\mathbf{x}^0 &= 0 \\
y^k - P^k\mathbf{x}^0 &\geq 0 \\
\mu^{k0} &\geq 0 \\
(y^k - P^k\mathbf{x}^0)^\top \mu^{k0} &= 0
\end{aligned}
$$

The convex polytope C^k is the union of a finite number of facets, each defined as the set of $\mathbf{x}$ satisfying $Q^k\mathbf{x} = w^k$, $P_1^k\mathbf{x} = y_1^k$, and $P_2^k\mathbf{x} < y_2^k$, where $P^k = [P_1^k \vdots P_2^k]$ is a partition of the columns of P^k, y^k is partitioned commensurately, Q^k is $i_k n$, and P_1^k is $r_k \times n$. By construction, $Q^{k\top}$ is of full column rank. If $[Q^{k\top} \vdots P_1^{k\top}]$ is not of full column rank, then there exists a further partition $[Q^{k\top} \vdots P_{11}^{k\top} \vdots P_{12}^{k\top}]$ such that $[Q^{k\top} \vdots P_{11}^{k\top}]$ is of full column rank and $P_{12}^{k\top} = Q^{k\top}\Lambda + P_{11}^{k\top}\Gamma$ for some linear combinations Λ, Γ. Pre-multiply this equality by $\mathbf{x}^\top$ for any $\mathbf{x}$ in the relative interior of the facet to obtain $y_{12}^{k\top} = w_{12}^{k\top}\Lambda + y_{11}^{k\top}\Gamma$. Then, the set of $y^k \in Y^k$ such that $[Q^{k\top} \vdots P_1^{k\top}]$ is not of full column rank is contained in an affine subspace of $\mathbb{R}^{j_k}$ of dimension less than j_k, and hence of Lebesgue measure zero. This is true for each facet, implying that except for a set of $y^k \in Y^k$ of Lebesgue measure zero, the active constraints $Q^k\mathbf{x} = w^k$ and $P_1^k\mathbf{x} = y_1^k$ at any $\mathbf{x} \in C^k$ will satisfy the *linear independence constraint qualification* that $[Q^{k\top} \vdots P_1^{k\top}]$ is of full column rank $i_k + r_k$. Then the system of equations $\nabla_{\mathbf{x}} U(\mathbf{x}^0; \mathbf{t}) + \mathbf{m}^0 = [Q^{k\top} \vdots P^{k\top}][\lambda^{k0\top} \vdots \mu^{k0\top}]^\top$ has a unique solution $\lambda^{k0} = \Lambda^k(\mathbf{m}; \mathbf{t}, \beta^k)$ and $\mu^{k0} = \Gamma^k(\mathbf{m}; \mathbf{t}, \beta^k)$, with $\mu_2^{k0} = 0$, in a neighborhood of $\mathbf{m}^0$, and this solution is locally continuously differentiable in $\mathbf{m}$. Then, the KKT conditions are differentiable in $\mathbf{m}$ at $\mathbf{m}^0$ and satisfy

$$
\begin{bmatrix}
\nabla_{\mathbf{xx}} U(X^k(\mathbf{m}; \mathbf{t}, \beta^k); \mathbf{t}) & -Q^{k\top} & -P_1^{k\top} \\
-Q^k & \mathbf{0}_{i_k i_k} & \mathbf{0}_{i_k r_k} \\
-P_1^k & \mathbf{0}_{r_k i_k} & \mathbf{0}_{r_k r_k}
\end{bmatrix}
\begin{bmatrix}
\nabla_{\mathbf{m}} X^k(\mathbf{m}; \mathbf{t}, \beta^k) \\
\nabla_{\mathbf{m}} \Lambda^k(\mathbf{m}; \mathbf{t}, \beta^k) \\
\nabla_{\mathbf{m}} \Gamma_1^k(\mathbf{m}; \mathbf{t}, \beta^k)
\end{bmatrix}
=
\begin{bmatrix}
-\mathbf{I}_{nn} \\
\mathbf{0}_{\imath_k n} \\
\mathbf{0}_{r_k n}
\end{bmatrix},
$$

where i_k is the number of rows of Q^k, and r_k is the number of rows of P_1^k. The existence of the derivatives then implies that the bordered hessian on the left-hand-side of these linear equalities is non-singular. Therefore, differentiating the KKT conditions with respect to y_j^k for an active constraint j gives

$$\begin{bmatrix} \nabla_{\mathbf{xx}} U(X^k(\mathbf{m};\mathbf{t},\beta^k);\mathbf{t}) & -Q^{k\top} & -P_1^{k\top} \\ -Q^k & \mathbf{0}_{i_k i_k} & \mathbf{0}_{i_k r_k} \\ -P_1^k & \mathbf{0}_{r_k i_k} & \mathbf{0}_{r_k r_k} \end{bmatrix} \begin{bmatrix} \nabla_{y_j^k} X^k(\mathbf{m};\mathbf{t},\beta^k) \\ \nabla_{y_j^k} \Lambda^k(\mathbf{m};\mathbf{t},\beta^k) \\ \nabla_{y_j^k} \Gamma_1^k(\mathbf{m};\mathbf{t},\beta^k) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n1} \\ \mathbf{0}_{1_k 1} \\ -\mathbf{1}_{r_k,j} \end{bmatrix},$$

where $\mathbf{1}_{r_k,j}$ is an $r_k \times 1$ unit vector with a one in the jth component, and differentiating with respect to the associated column j in P_1^k gives

$$\begin{bmatrix} \nabla_{\mathbf{xx}} U(X^k(\mathbf{m};\mathbf{t},\beta^k);\mathbf{t}) & -Q^{k\top} & -P_1^{k\top} \\ -Q^k & \mathbf{0}_{i_k i_k} & \mathbf{0}_{i_k r_k} \\ -P_1^k & \mathbf{0}_{r_k i_k} & \mathbf{0}_{r_k r_k} \end{bmatrix} \begin{bmatrix} \nabla_{P_{1\,j}^k} X^k(\mathbf{m};\mathbf{t},\beta^k) \\ \nabla_{P_{1\,j}^k} \Lambda^k(\mathbf{m};\mathbf{t},\beta^k) \\ \nabla_{P_{1\,j}^k} \Gamma_1^k(\mathbf{m};\mathbf{t},\beta^k) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n,j} \Gamma_1^k(\ \mathbf{t},\beta^k)^\top \\ \mathbf{0}_{1_k r_k} \\ X_j^k(\mathbf{m};\mathbf{t},\beta^k)\mathbf{I}_{r_k r_k} \end{bmatrix}$$

Hence, all the derivatives in the linear equalities above exist and are continuous at $\mathbf{m}^0$. Analogous formulas hold for derivatives with respect to w^k and columns of Q^k. The inactive constraints have $\nabla_{P_{2\,j}^k} X^k(\mathbf{m};\mathbf{t},\beta^k) \equiv 0$.

Finally, consider a shift from $\mathbf{t}$ to $\mathbf{t}'$. The KKT conditions in the two cases are

$\nabla_{\mathbf{x}} U(\mathbf{x}^0;\mathbf{t}) + \mathbf{m}^0 - Q^{k\top}\lambda^{k0} - P_1^{k\top}\mu_1^{k0} = 0$	$\nabla_{\mathbf{x}} U(\mathbf{x}';\mathbf{t}') + \mathbf{m}^0 - Q^{k\top}\lambda^{k\prime} - P_1^{k\top}\mu_1^{k\prime} = 0$
$w^k - Q^k\mathbf{x}^0 = 0$	$w^k - Q^k\mathbf{x}' = 0$
$y_1^k - P_1^k\mathbf{x}^0 = 0$	$y_1^k - P_1^k\mathbf{x}' \geq 0$
$\mu_1^{k0} \geq 0$	$\mu_1^{k\prime} \geq 0$
$y_2^k - P_2^k\mathbf{x}^0 < 0$	$(y_1^k - P_1^k\mathbf{x}')^\top\mu_1^{k\prime} = 0$

where P_1^k corresponds to the active constraints at $\mathbf{x}^0$. For small changes in $\mathbf{t}$, no previously inactive constraints can become active. Some previously active constraints can become inactive, but that is accommodated in the KKT conditions by making the corresponding components of $\mu_1^{k\prime}$ zero. The assumption that $\nabla_x U$ is Lipschitz in $\mathbf{t}$ implies $\nabla_x U(\mathbf{x};\mathbf{t}') - \nabla_x U(\mathbf{x};\mathbf{t}) = O(\eta(\mathbf{t}',\mathbf{t}))$, where $\eta(\mathbf{t}',\mathbf{t})$ is the metric distance between $\mathbf{t}$ and $\mathbf{t}'$. Then, differencing the KKT conditions,

$$\begin{bmatrix} \nabla \mathbf{x}_x U(\mathbf{x}'';\mathbf{t}) & -Q^{k\top} & -P_1^{k\top} \\ -Q^k & \mathbf{0}_{i_k i_k} & \mathbf{0}_{i_k r_k} \\ -P_1^k & \mathbf{0}_{r_k i_k} & \mathbf{0}_{r_k r_k} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x} \\ \Delta\lambda \\ \Delta\mu_1 \end{bmatrix} = \begin{bmatrix} O(\eta(\mathbf{t}',\mathbf{t})) \\ \mathbf{0}_{i_k 1} \\ \mathbf{0}_{r_k 1} \end{bmatrix}$$

where $\mathbf{x}''$ is between $\mathbf{x}$ and $\mathbf{x}'$. As $\mathbf{t}' \to \mathbf{t}$, the upper hemicontinuity of $X^k(\mathbf{m};\mathbf{t},\beta^k)$ implies $\mathbf{x}' \to \mathbf{x}$, and hence the bordered hessian remains non-singular for $\mathbf{t}'$ in a neighborhood of $\mathbf{t}$. Therefore, $\Delta\mathbf{x} = O(\eta(\mathbf{t}',\mathbf{t}))$, and X^k is Lipschitz in $\mathbf{t}$.

Now consider $B = \cup_{k=1}^{K} C^k$, and $X(\mathbf{m};\mathbf{t},\beta) = \mathrm{argmax}_k\{U(X^k(\mathbf{m};\mathbf{t},\beta^k),\mathbf{t}) + \mathbf{m}\cdot X^k(\mathbf{m};\mathbf{t},\beta^k)\}$. From the argument above, the linear independence constraint qualification is satisfied at each $\mathbf{x} \in C^k$ on a set $y^k \in Y^k$ of full measure. Applying Theorem 1, to B and to each C^k separately, the maximizers $X^k(\mathbf{m},\mathbf{t},\beta^k)$ and $X(\mathbf{m},\mathbf{t},\beta)$ are all singletons on a set of $\mathbf{m} \in M$ of full measure. Suppose $\mathbf{m}^0$ is a point where all these maximizers are singletons. For each k, either $U(X^k(\mathbf{m}^0;\mathbf{t},\beta^k);\mathbf{t}) + \mathbf{m}^0 \cdot X^k(\mathbf{m}^0,\mathbf{t},\beta^k) < U(X(\mathbf{m}^0;\mathbf{t},\beta);\mathbf{t}) + \mathbf{m}^0 \cdot X(\mathbf{m}^0;\mathbf{t},\beta)$, in which case $\nabla_{\beta^k} X(\mathbf{m}^0;\mathbf{t},\beta) \equiv 0$, or else $U(X^k(\mathbf{m}^0;\mathbf{t},\beta^k);\mathbf{t}) + \mathbf{m}^0 \cdot X^k(\mathbf{m}^0;\mathbf{t},\beta^k) = U(X(\mathbf{m}^0;\mathbf{t},\beta);\mathbf{t}) + \mathbf{m}^0 \cdot X(\mathbf{m}^0;\mathbf{t},\beta)$, in which case $X^k(\mathbf{m}^0;\mathbf{t},\beta^k) = X(\mathbf{m}^0,\mathbf{t},\beta)$, since $X(\mathbf{m}^0,\mathbf{t},\beta)$ is a singleton. The earlier argument establishes that each $X^k(\mathbf{m}^0;\mathbf{t},\beta^k)$ satisfying the second case is Lipschitz in β^k and $\mathbf{t}$. Therefore $X(\mathbf{m}^0;\mathbf{t},\beta)$ is Lipschitz in β^k and $\mathbf{t}$, and, by Rademacher (1919) (see Howard, 1998, Theorem 7.1), is therefore almost everywhere differentiable in β. This completes the proof that the maximand correspondence is continuously differentiable in β and Lipschitz in $\mathbf{t}$, almost everywhere (w.r.t. $M \times Y^1 \times \cdots \times Y^K$), in the case of a function $U(\mathbf{x};\mathbf{t})$ that is twice continuously differentiable in $\mathbf{x}$, and a polytope budget set.

■

Proof of Theorem 2. From the convexity of V, $\partial_m V(\mathbf{m};\mathbf{t},B)$ is a convex-valued upper hemicontinuous correspondence, implying that $B \cap \partial_{\mathbf{m}} V(\mathbf{m};\mathbf{t},B)$ is an upper hemicontinuous mapping, assumed non-empty, that admits a selection $X(\mathbf{m};\mathbf{t},B)$ that is an upper hemicontinuous correspondence. The compactness of $M \times B$ implies that $E(t)$ is a compact-valued upper hemicontinuous correspondence. Define $U(\mathbf{x};\mathbf{t}) = \min\{V(\mathbf{m};\mathbf{t},B) - \mathbf{m}\cdot\mathbf{x} | (\mathbf{m},B) \in M \times \mathcal{B} : \mathbf{x} \in B\}$ for $\mathbf{x} \in E(\mathbf{t})$, and $U(\mathbf{x};\mathbf{t}) = -\infty$ for $x \notin E(\mathbf{t})$. By the Berge maximum theorem (AB 17.31), U is continuous for $\mathbf{x} \in E(t)$.

If $\mathbf{x} \in E(\mathbf{t})$, then by construction there exist $(\mathbf{m},B) \in M \times \mathcal{B}$ with $\mathbf{x} \in B$ such that $U(\mathbf{x};\mathbf{t}) = V(\mathbf{m};\mathbf{t},B)\mathbf{m}\cdot\mathbf{x} \le V(\mathbf{m}'';t,B'') - \mathbf{m}''\cdot\mathbf{x}$ for all $\mathbf{m}'' \in M$ and all $B'' \in \mathcal{B}$ that have $\mathbf{x} \in B''$. Taking $B'' = B$, this inequality implies that

$\mathbf{x} \in B \cap \partial_{\mathbf{m}} V(\mathbf{m};\mathbf{t},B) = X(\mathbf{m};\mathbf{t},B)$. Suppose $\mathbf{x}'' \in B \cap E(\mathbf{t})$. Then there exist $(\mathbf{m}'',B'') \in M \times \mathcal{B}$ such that $\mathbf{x}'' \in X(\mathbf{m}'';\mathbf{t},B'')$. By Assumption (ii), if $\mathbf{x}'' \in X(\mathbf{m};\mathbf{t},B)$, then $U(\mathbf{x}'';\mathbf{t}) = V(\mathbf{m}'';\mathbf{t},B'') - \mathbf{m}''\cdot\mathbf{x}'' = V(\mathbf{m};\mathbf{t},B) - \mathbf{m}\cdot\mathbf{x} = U(\mathbf{x};\mathbf{t})$, and if $\mathbf{x}'' \notin X(\mathbf{m};\mathbf{t},B)$, then $U(\mathbf{x}'';\mathbf{t}) = V(\mathbf{m}'';\mathbf{t},B'') - \mathbf{m}''\cdot\mathbf{x}'' < V(\mathbf{m};\mathbf{t},B) - \mathbf{m}\cdot\mathbf{x} = U(\mathbf{x};\mathbf{t})$. Finally, if $\mathbf{x}'' \in B$ and $\mathbf{x}'' \notin E(\mathbf{t})$, then $U(\mathbf{x};\mathbf{t}) > U(\mathbf{x}'';\mathbf{t}) = -\infty$. Then, $U(\mathbf{x}';\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}'$ is maximized in B at $\mathbf{x}' \in B$ if and only if $\mathbf{x}' \in X(\mathbf{m};\mathbf{t},B)$, and $\max_{\mathbf{x}'\in B}(U(\mathbf{x}';t) + \mathbf{m}\cdot\mathbf{x}') = U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x} = V(\mathbf{m};\mathbf{t},B)$ for $\mathbf{x} \in X(\mathbf{m};\mathbf{t},B)$. Then, V is the demand generating function for U, and $X(\mathbf{m};\mathbf{t},B)$ is the associated demand correspondence.

Suppose $\mathbf{t}^j \to \mathbf{t}^0$, $\mathbf{x}^j \in E(\mathbf{t}^j)$, and $\mathbf{x}^j \to \mathbf{x}^0$. The definition and upper hemicontinuity of E imply there exist $(\mathbf{m}^j, B^j) \in M \times \mathcal{B}$ with $\mathbf{x}^j \in X(\mathbf{m}^j;\mathbf{t}^j,B^j)$ that have a convergent subsequence (retain notation) to $(\mathbf{m}^0, B^0) \in M \times \mathcal{B}$, and $\mathbf{x}^0 \in X(\mathbf{m}^0;\mathbf{t}^0,B^0)$. Then $U(\mathbf{x}^j;\mathbf{t}^j) = V(\mathbf{m}^j;\mathbf{t}^j,B^j) - \mathbf{m}^j\cdot\mathbf{x}^j \to V(\mathbf{m}^0;\mathbf{t}^0,B^0) - \mathbf{m}^0\cdot\mathbf{x}^0 = U(\mathbf{x}^0;\mathbf{t}^0)$ by the continuity of V, and U is continuous on $S \times T$. ■

Proof of Corollary 3. Define $U^*(\mathbf{x};\mathbf{t}) = \inf_{\mathbf{p}\neq 0} w(\mathbf{t},\mathbf{p},\mathbf{p}\cdot\mathbf{x})$; then U^* is the quasiconvex hull of U; e.g., the upper contour sets of U^* are the convex hulls of the upper contour sets of U, and $U^*(\mathbf{x};\mathbf{t}) \geq U(\mathbf{x};\mathbf{t})$. Define $v^*(\mathbf{m};\mathbf{t},\mathbf{p},y) = \max_x\{U^*(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x} | \mathbf{p}\cdot\mathbf{x} \leq y\}$, and let x^* denote a maximizer. Then $v^*(\mathbf{m};\mathbf{t},\mathbf{p},y) \geq v(\mathbf{m};\mathbf{t},\mathbf{p},y)$, and there exists a linear combination $\mathbf{x}^* = \sum_{j=0}^n \theta_j \mathbf{x}^j$ of points $\mathbf{x}^j$ (not necessarily all distinct) such that $\theta_j > 0$, $\sum_{j=0}^n \theta_j = 1$, and $U^*(\mathbf{x}^*;\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}^* = U(\mathbf{x}^j;\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}^j$. For at least one j, $\mathbf{p}\cdot\mathbf{x}^j \leq y$. Then, $v(\mathbf{m};\mathbf{t},\mathbf{p},y) \geq U(\mathbf{x}^j;\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}^j = v^*(\mathbf{m};\mathbf{t},\mathbf{p},y)$. Then $v(\mathbf{m};\mathbf{t},\mathbf{p},y) = \max_x\{\inf_{\mathbf{p}\neq 0} w(\mathbf{t},\mathbf{p}',\mathbf{p}'\cdot\mathbf{x}) + \mathbf{m}\cdot\mathbf{x} | \mathbf{p}\cdot\mathbf{x} \leq y\}$. Rewrite points in the simplex $\{\mathbf{x} \geq 0 | \mathbf{p}\cdot\mathbf{x} = y\}$ as linear combinations of the vertices y/p_j to obtain the final form $v(\mathbf{m},\mathbf{t},\mathbf{p},y) = \max\{min_{p'} w(\mathbf{t},\mathbf{p}', y\cdot\sum_j \theta_j p'_j/p_j) + y\cdot\mathbf{m}_j\theta_j m_j/p_j | \theta_j \geq 0, \sum_j \theta_j = 1\}$. When U is homogeneous of degree one with unit expenditure function $e(\mathbf{t},\mathbf{p})$, $w(\mathbf{t},\mathbf{p},y) = y/e(\mathbf{t},\mathbf{p})$, and one obtains the simplified form $v(\mathbf{m},\mathbf{t},\mathbf{p},y) = y\cdot\min_{\mathbf{p}'}\max_j(p'_j/e(\mathbf{p}') + \mathbf{m}_j)/p_j$, where the order of minimum and maximum can be reversed since the function is convex in $\mathbf{p}'$ and concave in $\mathbf{x}$. ■

Proof of corollary 4. $B \subseteq \{\mathbf{x} \in \mathbb{R}^n | \mathbf{p}\cdot\mathbf{x} \leq s(\mathbf{p},B)\}$ implies $V(\mathbf{m};\mathbf{t},B) \leq v(\mathbf{m};\mathbf{t},\mathbf{p},s(\mathbf{p},B))$. If U is locally non-satiated and quasiconcave, any maximizer

$\mathbf{x}^*$ in B of $U(\mathbf{x};\mathbf{t})+\mathbf{m}\cdot\mathbf{x}$ is exposed, and there exists a hyperplane containing $\mathbf{x}^*$ that separates B and the upper contour set of $U(\mathbf{x};\mathbf{t})+\mathbf{m}\cdot\mathbf{x}$. If $\mathbf{p}^* \geq 0$ is a normal to this hyperplane, $V(\mathbf{m};\mathbf{t},B) = U(\mathbf{x}^*;\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}^* = v(\mathbf{m};\mathbf{t},\mathbf{p}^*,s(\mathbf{p}^*,B))$. ∎

Proof of Corollary 5. (1) If $\mathbf{x} \in X(\mathbf{m},\mathbf{t},B)$ and $B \subseteq B' \cup B''$, then $\mathbf{x}$ is in either B' or B'', implying either $V(\mathbf{m},\mathbf{t},B') \geq U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x} = V(\mathbf{m},\mathbf{t},B)$ or $V(\mathbf{m},\mathbf{t},B'') \geq U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x} = V(\mathbf{m},\mathbf{t},B)$.

(2) Evidently, $V(\mathbf{m};\mathbf{t},B^j) \leq V(\mathbf{m};\mathbf{t},B)$ for all j. The set B is compact such that $V(\mathbf{m};\mathbf{t},B) = U(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x}$ for some $\mathbf{x} \in B$. Then there exists j such that $V(\mathbf{m};\mathbf{t},B^j) \geq Ut(\mathbf{x};\mathbf{t}) + \mathbf{m}\cdot\mathbf{x} = V(\mathbf{m},\mathbf{t},B)$.

(3)

$$\begin{aligned} & \sup\{U_1(\mathbf{x}_1;\mathbf{t}) + \mathbf{m}_1\cdot\mathbf{x}_1 + U_2(\mathbf{x}_2;\mathbf{t}) + \mathbf{m}_2\cdot\mathbf{x}_2 | (x_1,x_2) \in B_1 \times B_2\} \\ = \; & \sup\{U_1(\mathbf{x}_1,\mathbf{t}) + \mathbf{m}_1\cdot\mathbf{x}_1 | \mathbf{x}_1 \in B_1\} + \sup\{U_2(\mathbf{x}_2,\mathbf{t}) + \mathbf{m}_2\cdot\mathbf{x}_2 | x_2 \in B_2\}. \end{aligned}$$

(4) This result is an immediate consequence of result (2) and Corollary 4. ∎

Corollary 6 *Assume the conditions of Theorem 1. Let utility be written as* $U(\mathbf{x};\mathbf{t}) = f_{\mathbf{t}}(U_0(\mathbf{x};\mathbf{t}))$*, where for each* $\mathbf{t}$*,* $f_{\mathbf{t}}$ *is differentiable and strictly increasing. Then* U_0 *rationalizes choices at* $\mathbf{m} = 0$ *and*

$$\begin{aligned} (1) \qquad 0 \;=\; & f'_{\mathbf{t}}(U_0(X(\mathbf{m};\mathbf{t},B)))\nabla_x U_0(X(\mathbf{m};\mathbf{t},B);\mathbf{t})\cdot\nabla_{\mathbf{m}}X(\mathbf{m};\mathbf{t},B) \\ & +\mathbf{m}\cdot\nabla_{\mathbf{m}}X(\mathbf{m};\mathbf{t},B) \text{ a.e.}(\mathbf{m}). \end{aligned}$$

Proof. For almost all $\mathbf{m}$, $X(\mathbf{m};\mathbf{t},B)$ is a singleton, $\partial_{\mathbf{m}}V(\mathbf{m};\mathbf{t},B) = X(\mathbf{m};\mathbf{t},B)$ and $V(\mathbf{m};\mathbf{t},B)$ is differentiable at $\mathbf{m}$. Differentiate the identity

$$V(\mathbf{m};\mathbf{t},B) = f_{\mathbf{t}}(U_0(X(\mathbf{m};\mathbf{t},B);\mathbf{t})) + \mathbf{m}\cdot X(\mathbf{m};\mathbf{t},B),$$

to find that a.e. $(\mathbf{m})$

$$\begin{aligned} X(\mathbf{m};\mathbf{t},B) \;=\; & f'_{\mathbf{t}}(U_0(X(\mathbf{m};\mathbf{t},B);\mathbf{t}))\nabla_x U_0(X(\mathbf{m};\mathbf{t},B);\mathbf{t})\cdot\nabla_{\mathbf{m}}X(\mathbf{m};\mathbf{t},B) \\ & +X(\mathbf{m};\mathbf{t},B) + \mathbf{m}\cdot\nabla_{\mathbf{m}}X(\mathbf{m};\mathbf{t},B). \end{aligned}$$

The conclusion follows. ■

Proof of Theorem 3. Let $\mathbf{x} \in X(\mathbf{m}; t, B)$ and consider any $\mathbf{m}' \in M$; then $V(\mathbf{m}'; t, B) \geq (\mathbf{m}' - \mathbf{m}) \cdot \mathbf{x} + V(\mathbf{m}; t, B)$ and hence $X(\mathbf{m}; t, B) \subseteq \partial_{\mathbf{m}} V(\mathbf{m}; t, B)$. It also follows that $V(\mathbf{m}'; t, B) > -\infty$ if $V(\mathbf{m}; t, B) > -\infty$ and by assumption there exists such an $\mathbf{m}$. By Rockafellar (1970, Thm. 24.8) and the fact that $S \subset \mathbb{R}^n_+$, it follows that V is convex, closed, and nondecreasing in $\mathbf{m}$. It remains to show that $X(\mathbf{m}; t, B)$ and $V(\mathbf{m}; t, B)$ satisfy the GFFI. Consider then $\mathbf{x}^0 \in B^1 \cap X(\mathbf{m}^0; t, B^0)$. It must be shown that

$$V\left(\mathbf{m}^1; t, B^1\right) - V\left(\mathbf{m}^0; t, B^0\right) \geq \left(\mathbf{m}^1 - \mathbf{m}^0\right) \cdot \mathbf{x}^0$$

with equality if $\mathbf{x}^0 \in X(\mathbf{m}^1; t, B^1)$. But if $\mathbf{x}^0 \in B^1 \cap X(\mathbf{m}^0; t, B^0)$ then

$$\begin{aligned}
& V\left(\mathbf{m}^1; t, B^1\right) - V\left(\mathbf{m}^0; t, B^0\right) \\
= \ & \left[V\left(\mathbf{m}^1; t, B^1\right) - V\left(\mathbf{m}^1; t, B^0\right)\right] + \left[V\left(\mathbf{m}^1; t, B^0\right) - V\left(\mathbf{m}^0; t, B^0\right)\right] \\
\geq \ & \left(\mathbf{m}^1 - \mathbf{m}^1\right) \cdot \mathbf{x}^0 + \left(\mathbf{m}^1 - \mathbf{m}^0\right) \cdot \mathbf{x}^0.
\end{aligned}$$

If also $\mathbf{x}^0 \in X(\mathbf{m}^1; t, B^1)$ then we obtain the reverse inequality and the desired conclusion follows. ■