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Andrew Atkeson  
Ariel T. Burstein

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### **ABSTRACT**

We examine the quantitative impact of policy-induced changes in innovative investment by firms on growth in aggregate productivity and output in a fairly general specification of a model of growth through firms' investments in innovation that nests several commonly used models in the literature. We present simple analytical results isolating the specific features and parameters of the model that play the key roles in shaping its quantitative implications for the aggregate impact of policy-induced changes in innovative investment in the short, medium and long-term and for the socially optimal innovation intensity. We find that under the assumption of no social depreciation of innovation expenditures (a common assumption in Neo-Schumpeterian models), the model's implications for the elasticity of aggregate productivity and output over the medium-term horizon (i.e. 20 years) with respect to policy-induced changes in the innovation intensity of the economy are largely disconnected from the parameters that determine the model's long-run implications and the socially optimal innovation intensity of the economy. We find, in contrast, that plausibly calibrated models based on the Expanding Varieties framework can imply substantial social depreciation of innovation expenditures and a tighter link between the model's medium-term and long-term elasticities of aggregate productivity and output with respect to policy-induced changes in the innovation intensity of the economy.

Andrew Atkeson  
Bunche Hall 9381  
Department of Economics  
UCLA  
Box 951477  
Los Angeles, CA 90095-1477  
and NBER  
andy@atkeson.net

Ariel T. Burstein  
Department of Economics  
Bunche Hall 8365  
Box 951477  
UCLA  
Los Angeles, CA 90095-1477  
and NBER  
arielb@econ.ucla.edu

# 1 Introduction

Firms' investments in innovation are large relative to GDP and are likely an important factor in accounting for economic growth over time.<sup>1</sup> Many OECD countries use taxes and subsidies to encourage these investments in the hope of stimulating economic growth.<sup>2</sup> But what impact should we expect changes in firms' investments in innovation induced by changes in innovation policies to have on economic growth at various time horizons? And what are the welfare implications of policy induced changes in firms' investments in innovation?

We examine these questions in a model of growth through firms' investments in innovation that nests several of the important models of the interaction of firms' investments in innovation and aggregate productivity growth that have been developed over the past 25 years. The models that we nest include the aggregate model of Jones (2002), Neo-Schumpeterian models based on the Quality Ladders framework such as those described in Grossman and Helpman (1991b), Aghion and Howitt (1992), Klette and Kortum (2004) and Lentz and Mortensen (2008), and models based on the Expanding Varieties framework of Romer (1990) such as those described in Grossman and Helpman (1991a), Luttmer (2007), Atkeson and Burstein (2010), and Luttmer (2011). As described in Aghion et al. (2013), these are influential models that link micro data on firm dynamics to incumbent and entrant firms' investments in innovation and, in the aggregate, to economic growth in a tractable manner.<sup>3</sup>

One important feature that distinguishes these models of firms' investments in innovation from standard models of capital accumulation by firms is that these models allow for large gaps between the social and the private returns to firms' investments in innovation. Thus, when using these models to study the impact of innovation policy-induced changes in firms' investments in innovation on aggregate growth, one cannot use stan-

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<sup>1</sup>There is a wide range of estimates of the scale of firms' investments in innovation. In the new National Income and Product Accounts for the U.S. as revised in 2013, private sector investments in intellectual property products were 3.8% of GDP in 2012. Of that amount, Private Research and Development was 1.7% of GDP. The remainder of that expenditure was largely on intellectual property that can be sold such as films and other artistic originals. See Aizcorbe et al. (2009) for a discussion of the measurement of firms' investments in innovation in the National Income and Product Accounts. Corrado et al. (2005) and Corrado et al. (2009) propose a broader measure of firms' investments in innovation, which includes non-scientific R&D, brand equity, firm specific resources, and business investment in computerized information. These broader investments in innovation accounted for roughly 13% of non-farm output in the U.S. in 2005.

<sup>2</sup>See, for example, Chapter 2 of OECD's "Economic Policy Reform: Going for Growth", available at <http://www.oecd.org/eco/growth/economicpolicyreformsgoingforgrowth2009.htm>

<sup>3</sup>Several authors (see, for example, Lentz and Mortensen 2008, Akcigit and Kerr 2010, Luttmer 2011, Acemoglu et al. 2013, and Garcia-Macia et al. 2015) have shown that this class of models can provide a good fit to many features of micro data on firms.

dard growth accounting methods based on the assumptions that private and social returns are equated and that private and social depreciation rates are equal.<sup>4</sup>

In this paper, we take a step towards developing alternative methods for using these new growth models to measure the quantitative link between policy-induced changes in firms' investments in innovation and changes in the aggregate growth of productivity and output. In the spirit of growth accounting, our approach is to study directly the model's reduced form for the link between innovative investments by firms to growth in aggregate productivity and output.<sup>5</sup> We present a baseline set of assumptions that allow us to develop simple analytical results approximating the cumulative impulse responses of the logarithm of aggregate productivity and GDP with respect to a policy-induced change in the *innovation intensity of the economy* as measured by the ratio of firms' spending on innovation relative to GDP. Our approach allows us to isolate the specific features and parameters of the model that play the key roles in shaping its quantitative implications for the response of aggregate productivity and output to a policy induced change in the innovation intensity of the economy. We also use these analytical impulse response functions to highlight the features of the model that drive its implications for the socially optimal innovation intensity of the economy.

Specifically, we show that under a set of assumptions on the model's reduced form linking firms' innovative investments to growth in aggregate productivity that are satisfied by the most tractable specifications of the models we nest, the dynamics of aggregate productivity induced by permanent policy-induced changes in the innovation intensity of the economy can be summarized by two sufficient statistics: the *impact elasticity of aggregate productivity growth with respect to an increase in the innovation intensity of the economy*, and the *degree of intertemporal knowledge spillovers in research*. The first of these statistics, the impact elasticity, is the model's implication for the response of aggregate productivity to a policy-induced change in the innovation intensity of the economy in the short run. The second of these statistics, the degree of intertemporal knowledge spillovers, determines the model's implications for the response of the level of aggregate productivity to a policy-induced change in the innovation intensity of the economy in the long run.

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<sup>4</sup>There is a very large literature that seeks to use standard methods from growth accounting to capitalize firms' investments in innovation and to use the dynamics of that intangible capital aggregate to account for the dynamics of aggregate productivity and output. See, for example, [Griliches, ed \(1987\)](#), [Kendrick \(1994\)](#), [Griliches \(1998\)](#), and [Corrado and Hulten \(2013\)](#). The Bureau of Economic Analysis uses these standard growth accounting methods to incorporate a stock of intangible capital induced by firms' investments in innovation in the Fixed Assets Accounts for the U.S. Relatedly, [McGrattan and Prescott \(2012\)](#) use an overlapping generations model augmented to include firms' investments in intangible capital to ask how changes in various tax and transfer policies will impact the accumulation of intangible capital and aggregate productivity and GDP.

<sup>5</sup>In this sense our approach is close to [Jones \(2002\)](#).

These spillovers index the speed with which a permanent increase in the innovation intensity of the economy runs into diminishing returns, leading to an increased price of real research innovations and a reversion of the growth rate of aggregate productivity to an exogenously specified level in the long run. Together, these two statistics play key roles in determining the model's implications for the dynamics of aggregate productivity and output over the medium term. Moreover, we show that the optimal innovation intensity in our model economy is determined by these two statistics as well as the discount factor of consumers.<sup>6</sup>

What determines our model's quantitative implications for the impact elasticity of aggregate productivity growth with respect to the innovation intensity of the economy? We show that the implicit assumption that one makes regarding the social depreciation of innovation expenditures plays a key role in restricting the magnitude of this impact elasticity. We define the *social depreciation rate of innovation expenditures* as the counterfactual growth rate of aggregate productivity that would obtain if all firms in the economy invested nothing in innovation.<sup>7</sup> Under our set of baseline assumptions, the impact elasticity implied by our model is bounded by the gap between the baseline growth rate of aggregate productivity to which the model is calibrated less the social depreciation rate of innovation expenditures. Thus, if one builds in the implicit assumption that there is no social depreciation of innovation expenditures (a common assumption in Neo-Schumpeterian growth models) and applies the model to study advanced economies, then our model's quantitative implications for the impact elasticity of aggregate productivity growth are tightly constrained by the low baseline growth rate of aggregate productivity typically observed in these advanced economies. Under our assumptions, the elasticity of aggregate productivity and output over the medium term horizon (i.e. 20

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<sup>6</sup>The subsidy level that implements in equilibrium the optimal innovation intensity depends on other model details beyond the sufficient statistics that shape the transition dynamics. However, in Appendix F we provide a simple relationship between the *change* in innovation subsidies, fiscal expenditures, and innovation intensity across balanced growth paths.

<sup>7</sup>Pakes and Schankerman (1984) note the potential difference between private and social depreciation of innovation expenditures when measuring the returns to innovation. In Neo-Schumpeterian growth models there is private depreciation of past investments in innovation in terms of their impact on firms' profits — firms gain and lose products and/or profits as they expend resources to innovate but there is no social depreciation — but there is no social depreciation in the sense that the contribution of past innovation expenditures to aggregate production possibilities never dies out over time. Hence, in these models, aggregate productivity is assumed to remain constant over time if firms were to invest nothing in innovation. In contrast, Expanding Varieties models typically assume that there is private and social depreciation of innovation expenditures in the form of product exit and/or reductions in firm productivity or demand, and hence aggregate productivity would shrink over time if firms were to invest nothing in innovation. Corrado and Hulten (2013), Aizcorbe et al. (2009) and Li (2012) discuss comprehensive estimates of the depreciation rates of innovation expenditures without distinguishing between measures of the private and social depreciation of these expenditures.

years) with respect to policy induced changes in the innovation intensity of the economy is not very large and is not very sensitive to changes in the intertemporal knowledge spillovers that determine the long-run implications of the model and the model's implications for the potential welfare gains that might be achieved from a sustained increase in innovation subsidies.

We show that, in contrast, if one makes the alternative assumption that past innovations experience even moderate social depreciation (which is consistent with a plausibly calibrated Expanding Varieties growth model), then the model can produce significantly larger medium term elasticities of aggregate productivity with respect to policy-induced changes in the innovation intensity of the economy and that these medium term elasticities are much more sensitive to changes in the assumed degree of intertemporal knowledge spillovers.

The three key assumptions that we use to develop our results are as follows. Our first assumption is that the reduced form relationship between firms' investments in innovation in any period  $t$  and aggregate productivity growth between periods  $t$  and  $t + 1$  shows diminishing returns either with respect to proportional increases in all firms' innovative investments or with respect to an increase in entry. Our second assumption is that, in the initial baseline equilibrium before the innovation policy change, the social return to innovative investment across different firms is equated in the sense that the allocation of innovative investment across firms maximizes the growth rate of aggregate productivity between periods  $t$  and  $t + 1$  given the aggregate investment in innovation at  $t$ . These same assumptions hold in a standard accounting of the impact of changes in firms' investments in physical capital on the growth rate of labor productivity in the Solow growth model. We show that these two assumptions are sufficient for us to bound the impact elasticity of aggregate productivity growth with respect to an increase in the innovation intensity of the economy even without the standard assumption that the social and private returns to firms' investments in innovation are equal. Our third assumption is that, following a change in innovation policies, the ratio of the induced changes in the growth rate of aggregate productivity and real aggregate innovative investment is constant at each date as is the ratio of the innovation intensity of the economy to the allocation of labor between current production and research. This third assumption allows us to give an analytical first order approximation to the full macroeconomic dynamics associated with any policy-induced change in the time path for the innovation intensity of the economy. We demonstrate the application of our analytical results to five prominent models in the literature that satisfy these assumptions. We view our three assumptions as a minimal departure from the assumptions underlying standard growth accounting methods to accommodate

the possibility that private and social returns to firms' investments in innovation and the private and social depreciation rates of these investments may not be equal.

Our baseline assumptions imply that, while the aggregate level of innovation expenditures may be sub-optimal, there is no misallocation of innovation expenditures across firms in the model economy at the start of the transition following a change in innovation policies. Thus we abstract from the role innovation policies might play in improving the allocation of innovation expenditures across firms and thus raising the growth rate of aggregate productivity without necessarily increasing aggregate innovation expenditures. There is a growing literature examining the possibility that the social returns to investments across firms may be not equated both in the context of standard growth models (e.g. [Hsieh and Klenow 2009](#)) and in the context of growth through firms' investments in innovation (e.g. [Acemoglu et al. 2013](#), [Peters 2013](#), [Buera and Fattal-Jaef 2014](#), [Lentz and Mortensen 2014](#), and [Luttmer 2011](#)). We see our results as a benchmark to which the results from richer models can be compared.<sup>8</sup>

The paper is organized as follows. Section 2 describes the model. Section 3 characterizes a balanced growth path. Section 4 presents analytic results on the impact of changes in innovation policy on aggregate outcomes at different horizons and for welfare. Section 5 characterizes the optimal level of the innovation intensity of the economy. Section 6 discusses the quantitative implications of our analytic results. Section 8 concludes. Section 7 reviews three models for which our baseline assumptions do not hold. The appendix provides some proofs and other details including the calibration and full numerical solution of the model.

## 2 Model

In this section we first describe the environment and then present equilibrium conditions that we use when deriving our analytic results.

### 2.1 Environment

Time is discrete and labeled  $t = 0, 1, 2, \dots$ . There are two final goods, the first which we call the *consumption good* and the second which we call the *research good*. The representative

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<sup>8</sup>We describe in Section 7 and Appendix G a number of models that violate Assumptions 2 and 3 but satisfy in the initial balanced growth path what we define as conditional efficiency: the equilibrium allocation maximizes welfare subject to a given aggregate allocation of labor between production and research. If the equilibrium is conditional efficient, innovation policies can increase welfare by altering aggregate innovation expenditures but not by changing the allocation of these expenditures across firms.

household has preferences over consumption per capita  $C_t/L_t$  given by  $\sum_{t=0}^{\infty} \frac{\beta^t}{1-\eta} L_t (C_t/L_t)^{1-\eta}$ , with  $\beta \leq 1$ ,  $\eta > 0$ , and where  $L_t$  denotes the population that (without loss of generality for our results) is constant and normalized to 1 ( $L_t = 1$ ). The consumption good is produced as a constant elasticity of substitution (CES) aggregate of the output of a continuum of differentiated intermediate goods. These intermediate goods are produced by firms using capital and labor. Labor can be allocated to current production of intermediate goods,  $L_{pt}$ , and to research,  $L_{rt}$ , subject to the resource constraint  $L_{pt} + L_{rt} = L_t$ .

Output of the consumption good,  $Y_t$ , is used for two purposes. First, as consumption by the representative household,  $C_t$ . Second, as gross investment in physical (tangible) capital,  $K_{t+1} - (1 - d_k) K_t$ , where  $K_t$  denotes the aggregate physical capital stock and  $d_k$  denotes the depreciation rate of physical capital. The resource constraint for the final consumption good is given by

$$C_t + K_{t+1} - (1 - d_k) K_t = Y_t. \quad (1)$$

Under the old national income and product accounting (NIPA) convention that expenditures on innovation are expensed, the quantity in the model corresponding to Gross Domestic Product (GDP) as measured in the data under historical measurement procedures is equal to  $Y_t$ .<sup>9</sup>

**Intermediate Goods Producing Firms:** Intermediate goods (which are used to produce the final consumption good) are produced by heterogeneous firms. Production of an intermediate good with productivity index  $z$  is carried out with physical capital,  $k$ , and labor,  $l$ , according to

$$y = \exp(z) k^\alpha l^{1-\alpha}, \quad (2)$$

where  $0 < \alpha < 1$ .

To maintain a consistent notation across a potentially broad class of models, we assume that there is a countable number of types of firms indexed by  $j = 0, 1, 2, \dots$ . As a matter of convention, let  $j = 0$  indicate the type of entering firms, and let  $j = 1, 2, \dots$  indicate the potentially different types of incumbent firms. The type of a firm,  $j$ , records all the information about the firm regarding the number of intermediate goods it produces, the different productivity indices  $z$  with which it can produce these various goods, the

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<sup>9</sup>The treatment of expenditures on innovation in the NIPA in the U.S. has been revised as of the second half of 2013 to include a portion of those expenditures on innovation in measured GDP. If all intangible investments in the model were measured as part of GDP, then measured GDP would be given by  $GDP_t = C_t + K_{t+1} - (1 - d_k) K_t + P_{rt} Y_{rt}$ , where  $P_{rt} Y_{rt}$  denotes intangible investment expenditure, as defined below. We report results on GDP under both measurement procedures.



markups it charges for each of these goods that it produces, and all the relevant information about the technologies the firm has available to it for innovating. Specifically, focusing on the production technology, firm  $j \geq 1$  is the owner of the frontier technology for producing  $n(j)$  intermediate goods with vector of productivities  $(z_1(j), z_2(j), \dots, z_{n(j)}(j))$ . The type of the firm  $j$  also records the equilibrium markups of price over marginal cost  $(\mu_1(j), \mu_2(j), \dots, \mu_{n(j)}(j))$  that this firm charges on the goods that it produces. The firm type  $j$  may also record the technology for innovation available to the firm.

Let  $\{N_t(j)\}_{j \geq 1}$  denote the measure of each type  $j$  of incumbent firm at time  $t$ . The measure of intermediate goods being produced in the economy at date  $t$  is  $\sum_{j \geq 1} n(j)N_t(j)$ . The vector of types of incumbent firms  $\{N_t(j)\}_{j \geq 1}$  is a state variable that evolves over time depending on entry and the investments in innovative activity of the incumbent firms, as described below.

**Production of the final consumption good:** Letting  $y_{it}(j)$  denote the output of the  $i$ 'th product of firm type  $j$  at time  $t$ , then output of the final good is given by

$$Y_t = \left( \sum_{j \geq 1} \sum_{i=1}^{n(j)} y_{it}(j)^{(\rho-1)/\rho} N_t(j) \right)^{\rho/(\rho-1)} \quad (3)$$

with  $\rho \geq 1$ .<sup>10</sup> This technology for producing the consumption final good is operated by competitive firms, with standard demand functions for each of the intermediate goods.

We assume that within each period, capital and labor are freely mobile across products and intermediate goods producing firms. This implies that the marginal cost of producing the  $i$ 'th product of firm type  $j$  at time  $t$  is given by  $MC_t \exp(-z_i(j))$ , where  $MC_t$  is the standard unit cost for the Cobb-Douglas production function (2) with  $z = 0$ . We assume that this firm charges price  $p_{it}(j) = \mu_{it}(j) MC_t \exp(-z_i(j))$  for this product at markup  $\mu_{it}(j)$  over marginal cost. The assumptions that the final good producing firms maximize profits taking input and output prices as given and that the intermediate goods producing firms minimize costs gives us that, in any equilibrium, in any period  $t$ , aggregate output can be written as

$$Y_t = Z_t (K_t)^\alpha (L_{pt})^{1-\alpha}, \quad (4)$$

where  $L_{pt}$  and  $K_t$  are the aggregates across intermediate goods producing firms of labor and capital used in current production and  $Z_t$  corresponds to *aggregate productivity* in the

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<sup>10</sup>The CES aggregator is standard in the growth literature, but our analytic results do not directly depend on it as long as the three key assumptions stated below are satisfied.

production of the final consumption good:<sup>11</sup>

$$Z_t = Z(\{N_t(j)\}_{j \geq 1}) \equiv \frac{\left( \sum_{j \geq 1} \sum_{i=1}^{n(j)} \exp(z_i(j))^{\rho-1} \mu_i(j)^{1-\rho} N_t(j) \right)^{\rho/(\rho-1)}}{\sum_{j \geq 1} \sum_{i=1}^{n(j)} \exp(z_i(j))^{\rho-1} \mu_i(j)^{-\rho} N_t(j)}. \quad (5)$$

Note that  $Z(\{N_t(j)\}_{j \geq 1})$  depends on the distribution of markups charged by incumbent firms and hence is not purely technological. When markups are equal across firms, this expression for aggregate productivity simplifies to

$$Z(\{N_t(j)\}_{j \geq 1}) = \left( \sum_{j \geq 1} \sum_{i=1}^{n(j)} \exp(z_i(j))^{\rho-1} N_t(j) \right)^{1/(\rho-1)}. \quad (6)$$

**The research good:** Intermediate goods producing firms use the second final good, which we call the research good and whose production is described below, to invest in innovative activities. Let  $\{y_{rt}(j)\}_{j \geq 1}$  denote the use of the research good by each type  $j$  of incumbent firms in period  $t$ . The use of the research good by each entering firm is fixed as a parameter at  $\bar{y}_r(0)$ . Given a mass  $N_t(0)$  of ex-ante identical entering firms in period  $t$  (who can start producing in period  $t + 1$ ), the resource constraint for the research good in period  $t$  is given by

$$\sum_{j \geq 1} y_{rt}(j) N_t(j) + \bar{y}_r(0) N_t(0) = Y_{rt}, \quad (7)$$

where  $Y_{rt}$  denotes the aggregate output of the research good.

Production of the research good is carried out using research labor  $L_{rt}$  according to

$$Y_{rt} = Z_t^{\phi-1} A_{rt} L_{rt}, \quad (8)$$

where  $\phi \leq 1$ .<sup>12</sup> The variable  $A_{rt}$  represents the stock of basic scientific knowledge that is freely available for firms to use in innovative activities. Increases in this stock of scientific knowledge improve the productivity of resources devoted to innovative activity. This stock of scientific knowledge is assumed to evolve exogenously, growing at a steady

<sup>11</sup>In general, this model-based measure of aggregate productivity,  $Z_t$ , does not correspond to measured TFP, which is given by  $TFP_t = GDP_t / (K_t^{\tilde{\alpha}} L_t^{1-\tilde{\alpha}})$ , where  $1 - \tilde{\alpha}$  denotes the share of labor compensation in measured GDP. The growth rate of this model-based measure of aggregate productivity, however, is equal to the growth rate of measured TFP on a balanced growth path.

<sup>12</sup>Here, for simplicity, we assume that the research good is produced entirely with labor. In Appendix E we consider an extension in which research production uses both labor and consumption good, as in the lab-equipment model of [Rivera-Batiz and Romer \(1991\)](#).

rate of  $g_{A_r} \geq 0$  so  $A_{rt+1} = \exp(g_{A_r})A_{rt}$ . The determination of this stock of scientific knowledge is outside the scope of our analysis.<sup>13</sup>

We interpret the parameter  $\phi \leq 1$  as indexing the degree of *intertemporal knowledge spillovers*, that is the extent to which further innovations by firms become more difficult as aggregate productivity  $Z_t$  grows relative to the stock of scientific knowledge  $A_{rt}$ . The impact of advances in  $Z_t$  on the cost of further innovations is external to any particular firm and hence we call it a spillover. Note that if  $\phi < 1$ , then more research labor  $L_{rt}$  is required to produce the same quantity of the research good  $Y_{rt}$  as the level of aggregate productivity  $Z_t$  rises relative to the stock of scientific knowledge  $A_{rt}$ . We show below that, in this case, the growth rate of aggregate productivity on a balanced growth path is pinned down by the growth rate of scientific knowledge  $g_{A_r}$  independent of policies as in a semi-endogenous growth model. As  $\phi$  approaches 1, the resource cost of innovating on the frontier technology becomes independent of  $Z_t$ . Standard specifications of models with fully endogenous growth correspond to the case with full spillovers  $\phi = 1$  and  $g_{A_r} = 0$ .

**Innovation by firms:** Aggregate productivity in the model grows as a result of the investment in innovation by firms. In the models that we consider, since the level of aggregate productivity  $Z_t$  is given as a function of the distribution of incumbent firms across types as in (5), the growth rate of aggregate productivity is a function of the evolution of the distribution of incumbent firms across types over time. We model the evolution of the distribution of incumbent firms across types as a function of firms' investments in innovation abstractly as follows. Given a collection of incumbent firms  $\{N_{jt}\}_{j \geq 1}$  and investments in innovation by these firms  $\{y_{rt}(j)\}_{j \geq 1}$  as well as a measure of entering firms  $N_t(0)$  in period  $t$ , the types of all firms are updated giving a new collection of incumbent firms at  $t + 1$ ,  $\{N_{t+1}(j)\}_{j \geq 1}$ , given by<sup>14</sup>

$$\{N_{t+1}(j)\}_{j \geq 1} = T(\{y_{rt}(j)\}_{j \geq 1}, N_t(0); \{N_t(j)\}_{j \geq 1}). \quad (9)$$

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<sup>13</sup>It is common in the theoretical literature on economic growth with innovating firms to assume that all productivity growth is driven entirely by firms' expenditures on R&D (Griliches 1979, p. 93). As noted in Corrado et al. (2011), this view ignores the productivity-enhancing effects of public infrastructure, the climate for business formation, and the fact that private R&D is not all there is to innovation. With the assumption that  $\phi < 1$ , we capture all of these other productivity enhancing effects with  $A_r$ . Relatedly, Akcigit et al. (2013) consider a growth model that distinguishes between basic and applied research and introduces a public research sector.

<sup>14</sup>Note that the  $T$  operator is not indexed by  $t$ , which implies that we have assumed that in any time period, with a fixed distribution of incumbent firms by type, a given level of innovative investment by each type of firm gives rise to the same change in the distribution of firms over types. The technology for producing the real input to firms' innovative investment changes over time due to growth in the stock of basic scientific knowledge and intertemporal knowledge spillovers.

Through their equilibrium mappings (5) and (9), the models we consider thus deliver an equation that gives the growth rate of aggregate productivity as a function of entry and the use of the research good by all types of incumbent firms,

$$g_{zt} \equiv \log(Z_{t+1}) - \log(Z_t) = G(\{y_{rt}(j)\}_{j \geq 1}, N_t(0); \{N_t(j)\}_{j \geq 1}), \quad (10)$$

where  $g_{zt}$  denotes the growth of log aggregate productivity between  $t$  and  $t + 1$ . Throughout we assume that the function  $G$  is differentiable and that its domain is a convex set. We define the *social depreciation rate of innovation expenditures* as the growth rate of aggregate productivity if all firms, both entrants and incumbents, were to set their use of the research good to zero, given a distribution of incumbents by firm type,<sup>15</sup>

$$G^0(\{N_t(j)\}_{j \geq 1}) = G(\{0, 0, \dots\}, 0; \{N_t(j)\}_{j \geq 1}).$$

We characterize the function  $G$  for five prominent models in the literature in Appendix C. We make reference to these model examples in presenting our quantitative results.

**Policies:** In what follows, we consider our model's quantitative implications for the response of aggregate productivity growth at various horizons to a change in innovation policies. The innovation policies that we consider are firm type-specific subsidies  $\tau_t(j)$  to expenditures on innovation. Specifically, a firm of type  $j$  that purchases  $y_{rt}(j)$  units of the research good at time  $t$  pays  $P_{rt}y_{rt}(j)$  to a research good producer for that purchase and then receives a rebate of  $\tau_t(j)P_{rt}y_{rt}(j)$  from the government. Thus fiscal expenditures on these policies are given by  $E_t = \sum_{j \geq 1} \tau_t(j)P_{rt}y_{rt}(j)N_t(j) + \tau_t(0)P_{rt}\bar{y}_r(0)N_t(0)$ . Changes in innovation policies are then assumed to lead to changes in the equilibrium allocation of the research good across firms and hence aggregate productivity growth and the time path for all other macroeconomic variables. In the examples we consider, these changes in innovation policies do not directly affect the form of functions  $Z$ ,  $T$ , and  $G$  defined above in equations (5), (9), and (10).

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<sup>15</sup>We assume that the allocation with  $y_{rt}(j) = 0$  for all  $j \geq 1$  and  $N_t(0) = 0$  is in the domain of the function  $G$ . Our definition of the social depreciation of innovation is analogous to defining the depreciation of physical capital as  $\log(1 - d_k)$ , that is, as equal to the value of  $\log K_{t+1} - \log K_t$  that would obtain if gross physical investment were zero.

## 2.2 Macroeconomic equilibrium conditions

We assume that the representative household owns the incumbent firms and the physical capital stock, facing a sequence of budget constraints given by

$$C_t + K_{t+1} = [R_{kt} + (1 - d_k)] K_t + W_t L_t + D_t - E_t,$$

in each period  $t$ , where  $W_t$ ,  $R_{kt}$ ,  $D_t$ , and  $E_t$  denote the economy-wide wage (assuming that labor is freely mobile across production of intermediate goods and the research good), rental rate of physical capital, aggregate dividends paid by firms, and aggregate fiscal expenditures on policies (which are financed by lump-sum taxes collected from the representative household), respectively.

Production of the research good is undertaken by firms that do not internalize the intertemporal knowledge spillover from innovation in equation (8). We assume that the price of the research good,  $P_{rt}$ , is equal to its marginal cost,

$$P_{rt} = \frac{Z_t^{1-\phi}}{A_{rt}} W_t. \quad (11)$$

We define the *innovation intensity of the economy*,  $s_{rt}$ , as the ratio of innovation expenditure to the sum of expenditure on consumption and physical capital investment, that is  $s_{rt} = P_{rt} Y_{rt} / GDP_t$ . It is typically a challenge to measure real research output  $Y_{rt}$ . Instead, the data that are usually available are data on research spending.

To use our model to compute how production of the research good  $Y_r$  changes with changes in expenditure on innovation relative to GDP,  $s_r$ , we make use of the following results about the division of GDP into payments to various factors of production and the relationship of those factor shares to the innovation intensity of the economy and the allocation of labor between current production of intermediate goods and research.

Aggregate expenditures on the final consumption good,  $Y_t$ , are paid to factors of production as follows. A share  $\frac{\mu_t - 1}{\mu_t}$  of aggregate expenditures on the final consumption good accrues to variable profits from intermediate goods production, equal to their total sales less aggregate wages paid to production labor and aggregate rental payments to physical capital. We define  $\mu_t$  directly this way as a share of aggregate output of the final consumption good and refer to it as the *average markup*. Of the remaining revenues, a share  $\alpha / \mu_t$  is paid to physical capital,  $R_{kt} K_t = \frac{\alpha}{\mu_t} Y_t$ , and a share  $(1 - \alpha) / \mu_t$  is paid as wages to production labor,  $W_t L_{pt} = \frac{(1-\alpha)}{\mu_t} Y_t$ .

Given that the research good is priced at marginal cost, then wage payments to research labor equal revenues from production of the research good:  $W_t L_{rt} = P_{rt} Y_{rt}$ . Using

the factor shares above and the assumption that labor is freely mobile between production and research, the allocation of labor between production and research is related to expenditures on the research good by<sup>16</sup>

$$\frac{L_{pt}}{L_{rt}} = \frac{(1 - \alpha)}{\mu_t} \frac{1}{s_{rt}}. \quad (12)$$

### 3 Balanced growth paths

To develop our analytical results, we consider the impact of changes in innovation policies on macroeconomic dynamics in an economy that starts on a balanced growth path (BGP). We consider BGPs of the following form, output of the final consumption good and the stock of physical capital both grow at a constant rate  $\bar{g}_y$  and aggregate productivity grows at a constant rate  $\bar{g}_z = (1 - \alpha)\bar{g}_y$ . The innovation intensity of the economy  $s_{rt}$ , the allocation of labor between production and research,  $L_{pt}$  and  $L_{rt}$ , and output of the research good  $Y_{rt}$  all remain constant over time at the levels  $\bar{s}_r$ ,  $\bar{L}_p$ ,  $\bar{L}_r$ , and  $\bar{Y}_r$ , respectively.<sup>17</sup> In deriving our analytic results, we assume that such a BGP exists. We then verify this conjecture for the specific model examples we consider.

If such a BGP exists and if  $\phi < 1$ , then our model is a semi-endogenous growth model with the growth rate along the BGP determined by the exogenous growth rate of scientific knowledge  $g_{Ar}$  and other parameter values independently of innovation policies, as in [Kortum \(1997\)](#) and [Jones \(2002\)](#). In this case, it is not possible to have fully endogenous growth because such growth would require growth in innovation expenditure in excess of the growth rate of GDP. Ongoing balanced growth can occur only to the extent that exogenous scientific progress reduces the cost of further innovation as aggregate productivity  $Z$  grows. Given the assumption that real research output is constant on a BGP, these BGP growth rates are given from equation (8) as  $\bar{g}_z = g_{Ar}/(1 - \phi)$ .

If a BGP exists and the knife-edged conditions  $\phi = 1$  and  $g_{Ar} = 0$  hold, then our

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<sup>16</sup>Here we are assuming that there is one wage for labor in both production and research. In [Appendix E](#) we present an extension in which labor is imperfectly substitutable between production and research as in [Jaimovich and Rebelo \(2012\)](#). The assumption of imperfect substitutability reduces the elasticity of the allocation of labor between production and research with respect to a policy-induced change in the innovation intensity of the economy, resulting in even smaller responses of aggregate productivity and GDP to a given change in the innovation intensity of the economy relative to those in our baseline model. This is similar to assuming congestion in the production of the research good (i.e. in which case research labor in the production of the research good has an exponent less than one), as discussed in [Jones \(2005\)](#).

<sup>17</sup>Our choice of units implies that a constant level of output of the research good can generate constant growth on a BGP. This assumption seems at odds with models such as the expanding varieties model in which the mass of firms and entry grow over time. However, as we show in [Appendix C](#), with a simple transformation of variables these models satisfy this assumption.

model is an endogenous growth model with the growth rate along the BGP determined by firms' investments in innovative activity, as in [Grossman and Helpman \(1991b\)](#) and [Klette and Kortum \(2004\)](#).<sup>18</sup> The transition paths of the response of aggregates to policy changes are continuous as  $\phi$  approaches one and hence the quantitative implications of our model for the response of the level of aggregate productivity at any finite horizon to changes in innovation policies is continuous in this parameter.<sup>19</sup>

In our applications, we calibrate the model parameters to match a given BGP per capita growth rate of output,  $\bar{g}_y$ , rather than making assumptions about the growth rate of scientific knowledge,  $g_{Ar}$ , which is hard to measure in practice. Specifically, given a choice of  $\bar{g}_y$  and physical capital share of  $\alpha$ , the growth rate of aggregate productivity in the BGP is  $\bar{g}_z = \bar{g}_y (1 - \alpha)$ . For a given choice of  $\phi$ , we choose the growth rate of scientific knowledge consistent with this productivity growth rate, that is  $g_{Ar} = (1 - \phi) \bar{g}_z$ .

## 4 Aggregate implications of changes in the innovation intensity of the economy: analytic results

In this section, we derive analytic results regarding the impact of policy-driven changes in the innovation intensity of the economy on aggregate outcomes at different time horizons (the proofs are presented in [Appendix A](#)). These analytical results demonstrate what features of our model are key in determining its implications for the aggregate impact of innovation policies. To show how these results relate to standard growth accounting methods, in [Appendix B](#) we derive these same results for the dynamics of labor productivity in a standard growth model augmented to include an externality in the accumulation of physical capital.

In framing the question of how policy-induced changes in the innovation intensity of the economy impact aggregate outcomes at different time horizons, we consider the following thought experiment. Consider an economy that is initially on a BGP with growth rate of aggregate productivity  $\bar{g}_z$ . As a baseline policy experiment, consider a change in innovation policies to new innovation subsidies beginning in period  $t = 0$  and continuing on for all  $t > 0$ . This policy experiment leads to some observed change in the equilibrium path of the innovation intensity of the economy  $\{s'_{rt}\}_{t=0}^{\infty}$  different from the innovation in-

<sup>18</sup>If  $\phi > 1$ , then our model does not have a BGP, as in this case, a constant innovation intensity of the economy leads to an acceleration of the innovation rate as aggregate productivity  $Z$  grows.

<sup>19</sup>This intertemporal knowledge spillover parameter  $\phi$  hence plays the same role as the knowledge spillover parameter  $\phi$  discussed in [Section 5 of Jones \(2005\)](#). He makes the same argument that the specific choice of  $\phi = 1$  or  $\phi < 1$  but close to one does not significantly impact the model's medium term transition dynamics because of continuity of the transition paths in this parameter.



tensity of the economy  $\bar{s}_r$  on the original BGP as well as some reallocation of innovation expenditures across firms  $\left\{ \{y'_{rt}(j)\}_{j \geq 1} \right\}_{t=0}^{\infty}$ , entry levels  $\{N'_t(0)\}_{t=0}^{\infty}$ , and some new evolution of the distribution of incumbent firms across firm types  $\left\{ \{N'_t(j)\}_{j \geq 1} \right\}_{t=1}^{\infty}$ . We seek to analytically approximate the resulting change in the path of aggregate productivity and GDP,  $\{Z'_t\}_{t=1}^{\infty}$  and  $\{GDP'_t\}_{t=0}^{\infty}$ .

In what follows, we make use of the following first order approximations of the equations of our model. At each date  $t \geq 0$ , using (10), the change in the logarithm of the growth rate of aggregate productivity is related to the changes in firms' investments in innovation and the evolution of the distribution of incumbent firms by type by a first-order Taylor expansion of the function  $G$ :

$$g'_{zt} - \bar{g}_z \approx \mathcal{D}_{gt} \equiv \quad (13)$$

$$\sum_{j \geq 1} \frac{\partial G}{\partial y_{rt}(j)} (y'_{rt}(j) - \bar{y}_{rt}(j)) + \frac{\partial G}{\partial N_t(0)} (N'_t(0) - \bar{N}_t(0)) + \sum_{j \geq 1} \frac{\partial G}{\partial N_t(j)} (N'_t(j) - \bar{N}_t(j)),$$

where all of the partial derivatives are evaluated along the initial BGP allocation. The change in the logarithm of aggregate real research output is related to the changes in firms' investments in innovation and the evolution of the distribution of incumbent firms by type by a first-order Taylor expansion of the equation (7)

$$\log Y'_{rt} - \log \bar{Y}_r \approx \mathcal{D}_{Yr} \equiv \quad (14)$$

$$\frac{1}{\bar{Y}_r} \left( \sum_{j \geq 1} \bar{N}_t(j) (y'_{rt}(j) - \bar{y}_{rt}(j)) + \bar{y}_r(0) (N'_t(0) - \bar{N}_t(0)) + \sum_{j \geq 1} \bar{y}_{rt}(j) (N'_t(j) - \bar{N}_t(j)) \right).$$

The change in the logarithm of aggregate real research output is related to the change in the level of aggregate productivity and the change in the allocation of labor between current production and research using a first order approximation to equation (8)

$$\mathcal{D}_{Yrt} = (\log L'_{rt} - \log \bar{L}_r) - (1 - \phi) (\log Z'_t - \log \bar{Z}_t). \quad (15)$$

Likewise, to a first-order approximation  $\log s'_{rt} - \log \bar{s}_r \approx \mathcal{D}_{srt}$ , where  $\mathcal{D}_{srt}$  is implicitly defined using equation (12) and the constraint that the allocation of labor in production and research sum to one, as

$$\mathcal{D}_{srt} = \frac{1}{\bar{L}_p} (\log L'_{rt} - \log \bar{L}_r) - (\log \mu'_t - \log \bar{\mu}_t). \quad (16)$$



## 4.1 Impact elasticities of productivity growth with respect to the innovation intensity of the economy

We now characterize the ratio of the change in the growth rate of aggregate productivity from date  $t = 0$  to date  $t = 1$  to the initial change in the innovation intensity of the economy at date  $t = 0$ ,

$$\frac{g'_{z0} - \bar{g}_z}{\log s'_{r0} - \log \bar{s}_{r0}} \approx \Gamma_0 \equiv \frac{\mathcal{D}_{g^0}}{\mathcal{D}_{sr0}} = \frac{\mathcal{D}_{g^0}}{\mathcal{D}_{Yr0}} \bar{L}_p. \quad (17)$$

We refer to this ratio  $\Gamma_0$  as the *impact elasticity of productivity growth with respect to a change in the research intensity of the economy*. Note that in computing this impact elasticity we are holding the current state variables  $\{N_0(j)\}_{j \geq 1}$  fixed since the distribution of incumbent firms by type at date  $t = 0$  is given as an initial condition. Hence, the initial average markup,  $\mu'_0$ , and the initial productivity,  $Z'_0$ , are both equal to their baseline BGP levels, implying  $\mathcal{D}_{Yr0} = \bar{L}_p \mathcal{D}_{sr0}$  (the second equality in equation (17)).

As is evident from equations (13) and (14), the exact value of this impact elasticity  $\Gamma_0$  depends, in general, on the specifics of the equilibrium responses of all of the endogenous variables to the specific policy change being modeled — that is, how the change in the aggregate production of the research good is allocated across the different types of firms as we move from the baseline set of policies to the new policies. We now take a stand on the specifics of the changes in innovation expenditure across firms so as to develop simple quantitative bounds on this impact elasticity in two salient special cases.

The first case that we consider is a policy change that induces all firms to increase their investments in innovation proportionally. The second case that we consider is a policy change that induces only an increase in entry. We then argue that these bounds apply to the impact elasticity corresponding to any small policy change if the initial allocation of investment on the BGP solves the problem of maximizing the current growth rate of aggregate productivity given the initial BGP aggregate output of the research good  $\bar{Y}_r$ .

*A proportional increase in all firms' investments:* Consider a change in innovation policies that results in a proportional increase in the innovation expenditures of all incumbent firms at date  $t = 0$  and also results in a proportional increase in the entry level so that  $y'_{r0}(j)$  and  $N'_0(0)$  are both scalar multiples of their corresponding baseline values on the initial BGP for some fixed scalar  $\kappa > 1$ . We now bound the impact elasticity  $\Gamma_0$  corresponding to this perturbation given the assumption that the function  $G$  is concave with respect to proportional increases in the innovation expenditures of all types of firms on the initial BGP.

**Assumption 1:** Define the function  $H_t(a)$  over the domain  $a \in [0, 1 + \epsilon)$  for a fixed  $\epsilon > 0$  as

$$H_t(a) = G(\{a\bar{y}_{rt}(j)\}_{j \geq 1}, a\bar{N}_t(0); \{\bar{N}_t(j)\}_{j \geq 1}).$$

Assume that  $H_t(a)$  is concave in  $a$ .

The following proposition uses Assumption 1 to bound the impact elasticity of productivity growth with respect to a change in the research intensity of the economy.

**Proposition 1.** *If Assumption 1 is satisfied on the initial BGP, then the impact elasticity  $\Gamma_0$  of productivity growth with respect to a change in the research intensity of the economy corresponding to a proportional increase in innovative investments by all firms at time  $t = 0$  is bounded by*

$$\Gamma_0 \leq (\bar{g}_z - G_0^0) \bar{L}_p, \quad (18)$$

where  $\bar{g}_z$  is the growth rate of aggregate productivity on the baseline BGP,  $G_t^0 = G^0(\{\bar{N}_t(j)\}_{j \geq 1})$  is the social depreciation rate of innovation expenditures at time  $t$  on the initial BGP, and  $\bar{L}_p$  is the fraction of the labor force engaged in current production of intermediate goods on the initial BGP.

The intuition for the bound in expression (18) is straightforward. In the model, on a BGP, firms' investments in innovation amount to  $\bar{Y}_r$  units of the research good which result in growth of aggregate productivity of  $\bar{g}_z$  relative to the counterfactual of investing zero and having aggregate productivity growth of  $G_0^0$ . Assumption 1 regarding concavity of the  $G$  function implies that the marginal change in the growth rate from further changes in innovation investments (equally allocated across all firms) is smaller than the average contribution to the growth rate from investing  $\bar{Y}_r$ , or

$$\mathcal{D}_{g0} \leq \left( \frac{\bar{g}_z - G_0^0}{\bar{Y}_r} \right) \bar{Y}_r \mathcal{D}_{Yr0},$$

where the term in parentheses is the average contribution to the growth rate from investing  $\bar{Y}_r$  on innovation and the term  $\bar{Y}_r \mathcal{D}_{Yr0}$  (which is also equal to  $\bar{Y}_r \bar{L}_p \mathcal{D}_{sr0}$ ) is the change in the level of output of the research good.

A key implication of Proposition 1 is that we are able to derive an upper bound on the impact elasticity of the growth of aggregate productivity with respect to a change in the research intensity of the economy,  $\Gamma_0$ , that depends on a small number of sufficient statistics. If there is no social depreciation of innovation expenditures (i.e.  $G_0^0 = 0$ ), then since  $\bar{L}_p \leq 1$  the impact elasticity is simply bounded by the initial calibrated growth rate of aggregate productivity,  $\Gamma_0 \leq \bar{g}_z$ . This bound is quite restrictive quantitatively if the baseline growth rate of productivity to which the model is calibrated is low. In contrast, if

there is social depreciation (i.e.  $G_0^0 < 0$ ) then the bound on the impact elasticity is looser.

*An increase in entry:* Consider now a change in innovation policies that results in an increase in firm entry but no change in the innovation investment level of all incumbent firms. We now bound the impact elasticity  $\Gamma_0$  corresponding to this perturbation given the assumption that the function  $G$  is concave with respect to an increase in entry and an assumption regarding the domain of  $G$  on the initial BGP.

**Assumption 1a:** Let  $N_t^0(0)$  be the level of entry such that, at the BGP level of innovation by incumbents, the growth rate of productivity between periods  $t$  and  $t + 1$  is equal to the social depreciation of innovation expenditures, that is,

$$G(\{\bar{y}_{rt}(j)\}_{j \geq 1}, N_t^0(0); \{\bar{N}_t(j)\}_{j \geq 1}) = G_t^0, \quad (19)$$

where  $\{\bar{y}_{rt}(j)\}_{j \geq 1}, N_t^0(0); \{\bar{N}_t(j)\}_{j \geq 1}$  is in the domain of  $G$ . Let  $Y_{rt}^0$  denote the amount of the research good used at this allocation,<sup>20</sup>

$$Y_{rt}^0 = \sum_{j \geq 1} \bar{y}_{rt}(j) \bar{N}_t(j) + \bar{y}_r(0) N_t^0(0). \quad (20)$$

Define  $\tilde{H}_t(a)$  over the domain  $a \in [0, 1 + \epsilon]$  for a fixed  $\epsilon > 0$  as

$$\tilde{H}_t(a) = G\left(\{\bar{y}_{rt}(j)\}_{j \geq 1}, a N_t^0(0) + (1 - a) \bar{N}_t(0); \{\bar{N}_t(j)\}_{j \geq 1}\right).$$

Assume that  $\tilde{H}_t(a)$  is concave in  $a$ .

The following proposition uses Assumption 1a to provide an additional bound on the impact elasticity of aggregate productivity growth with respect to a change in the innovation intensity of the economy.

**Proposition 2.** *If Assumption 1a is satisfied, then the impact elasticity  $\Gamma_0$  of productivity growth with respect to a change in the research intensity of the economy corresponding to an increase in*

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<sup>20</sup>Assumption 1a requires that one can conduct the thought experiment in the model of reducing entry to the extent required to make the growth rate of aggregate productivity fall to the growth rate  $G_t^0$  that would obtain if all firms, both entrants and incumbents, reduced their investments in innovation to zero. This thought experiment requires that  $G$  can be evaluated at negative values of entry  $N_t^0(0)$  while holding the investments of incumbents at their baseline values. This assumption only requires that such a point be in the domain of  $G$ , not that it be feasible in an economic sense in the model. Likewise, the definition (14) may imply  $Y_{rt}^0 < 0$ . This is allowed as long as such a point exists in the domain of  $G$ . It does not have to be feasible in the model to be defined here. We show that this restriction on the domain of  $G$  is satisfied in several of the example models that we consider (typically those with a linear entry margin) and not in others. [Lentz and Mortensen \(2014\)](#) assume that there is a fixed measure of entrants and a variable expenditure of resources by entrants that cannot be negative. Hence, this model cannot satisfy Assumption 1a.

entry is bounded by

$$\Gamma_0 \leq (\bar{g}_z - \bar{G}_0^0) \left( \frac{\bar{Y}_r}{\bar{Y}_r - \bar{Y}_{r0}^0} \right) \bar{L}_p, \quad (21)$$

where  $\bar{Y}_{r0}^0$  is defined as in equation (20).

The bound in (21) is equal the first bound implied by (18) multiplied by the term  $\bar{Y}_r / (\bar{Y}_r - \bar{Y}_{r0}^0)$ . If there is no innovation by incumbent firms, this term is equal to 1, and hence the two bounds are equal. When there is innovation by incumbent firms, then the magnitude of this term is determined by the gap in the average social cost of innovation for incumbent and entering firms as follows. To see this, consider the ratio

$$\frac{\bar{Y}_r - 0}{\bar{g}_z - \bar{G}_0^0}. \quad (22)$$

This is the cost savings in terms of the research good of reducing the growth rate of aggregate productivity from  $\bar{g}_z$  to  $\bar{G}_0^0$  by decreasing all incumbent and entering firms' research expenditures proportionally from the baseline BGP values to zero (this is the variation considered in the definition of  $H_t(a)$  in assumption 1). Consider now the ratio

$$\frac{\bar{Y}_r - \bar{Y}_{r0}^0}{\bar{g}_z - \bar{G}_0^0}. \quad (23)$$

This is the cost savings in terms of the research good of reducing the growth rate of aggregate productivity from  $\bar{g}_z$  to  $\bar{G}_0^0$  by decreasing *only* entering firms' research expenditures from the baseline BGP value to the required value such that the growth rate is  $\bar{G}_0^0$  (this is the variation considered in the definition of  $\tilde{H}_t(a)$  in assumption 1a). If the expression in (22) is lower than (23), then the term  $\bar{Y}_r / (\bar{Y}_r - \bar{Y}_{r0}^0)$  is less than one and the bound implied by (21) is tighter than the bound implied by (18).

In Propositions 1 and 2, we have derived simple bounds on the quantitative implications of our model for the impact elasticity of the growth rate of aggregate productivity with respect to changes in the innovation intensity of the economy in two salient specifications of how the change in innovation spending is allocated across firms. We now make an additional assumption that allows us to apply these bounds to the impact elasticity that would arise with respect to any small change in innovation policies.

**Assumption 2:** The allocation of innovation investments across firms on the baseline BGP,  $\{\bar{y}_{rt}(j)\}_{j \geq 1}$  and  $\bar{N}_t(0)$ , is an interior solution to the problem of choosing these expenditures to maximize  $G$  at date  $t$  subject to the resource constraint (7) when the state

variables  $\{\bar{N}_t(j)\}_{j \geq 1}$  and total production of the research good  $\bar{Y}_{rt}$  are taken as given.

With this assumption, by a standard application of an Envelope theorem, we have the following Lemma:

**Lemma 1.** *If the baseline allocation of innovation expenditures across firms satisfies Assumption 2, then the impact elasticity of productivity growth with respect to changes in the innovation intensity of the economy  $\Gamma_0$  is independent of how the change in the output of the research good is allocated across incumbent and entering firms.*

Lemma 1 implies that under Assumption 2, on impact, the response of aggregate productivity growth to a policy induced change in the innovation intensity of the economy is independent of the details of the change in innovation policies. It also implies that the impact elasticity is bounded by the minimum of the bounds derived in Propositions 1 and 2.

The approach we took to bounding the impact elasticity in Proposition 1 cannot be applied if Assumption 2 fails because this impact elasticity may not be well defined in this case. If Assumption 2 fails, it is theoretically possible to reallocate the research good across firms to increase the growth rate of aggregate productivity without increasing the aggregate production of the research good at all. In this case, the impact elasticity  $\Gamma_0$  defined in equation (17) would be infinite. Assumption 2 allows us to aggregate firms' investments in innovation and hence focus attention on the implications of innovation policies that change the aggregate innovation intensity of the economy.

## 4.2 Dynamics of aggregate productivity and GDP

We now consider the dynamics of aggregate productivity and GDP following a change in innovation policies beyond the impact effects in period  $t = 0$ . We first consider the dynamics of aggregate productivity. In general, for any change in innovation policies that we may consider, as long as the corresponding change  $\mathcal{D}_{Yrt}$  in aggregate production of the research good at each date  $t$  is non-zero, we can approximate the dynamics of aggregate productivity as follows. By the definition (13) we have trivially

$$\log Z'_{t+1} - \log \bar{Z}_{t+1} \approx \log Z'_t - \log \bar{Z}_t + \frac{\mathcal{D}_{gt}}{\mathcal{D}_{Yrt}} \mathcal{D}_{Yrt}. \quad (24)$$

Plugging this into (15) we have

$$\log Z'_{t+1} - \log \bar{Z}_{t+1} \approx \left[ 1 - (1 - \phi) \frac{\mathcal{D}_{gt}}{\mathcal{D}_{Yrt}} \right] (\log Z'_t - \log \bar{Z}_t) + \frac{\mathcal{D}_{gt}}{\mathcal{D}_{Yrt}} (\log L'_{rt} - \log \bar{L}_r), \quad (25)$$

where  $\log L'_{rt} - \log \bar{L}_r$  satisfies equation (16).

Given a transition path for aggregate research labor  $\{\bar{L}'_{rt}\}$ , if the ratio of the change in the growth rate of aggregate productivity (relative to its BGP level) to the change in aggregate output of the research good is constant over time (that is  $\mathcal{D}_{gt}/\mathcal{D}_{Yrt} = \mathcal{D}_{g0}/\mathcal{D}_{Yr0}$  as defined in equations (13) and (14)), then we can provide a simple characterization of the full dynamics of aggregate productivity. By equation (16), to characterize the dynamics of research labor given a transition path for the innovation intensity of the economy,  $\{s'_{rt}\}$ , we must also know the transition path for the average markup  $\{\mu'_t\}$ . In what follows, we assume that this transition path for the average markup is constant at  $\mu'_t = \bar{\mu}$ . These are assumptions regarding endogenous variables that we verify in particular model examples. We summarize this discussion with the following assumption.

**Assumption 3:** Along the transition path following a policy induced change in the innovation intensity of the economy, the deviations of the growth rate of aggregate productivity from its BGP growth rate and the deviations of aggregate output of the research good from the BGP level of production as defined in equations (13) and (14) are proportional at each date, i.e. they satisfy  $\mathcal{D}_{gt}/\mathcal{D}_{Yrt} = \mathcal{D}_{g0}/\mathcal{D}_{Yr0}$ . In addition, the path for the innovation intensity of the economy, aggregate productivity, and aggregate research labor satisfy equation (15) and equation (16) with  $\mu'_t = \bar{\mu}_0$ .

In the following proposition, we characterize the dynamics of aggregate productivity following any change in innovation policy such that the new equilibrium allocation satisfies Assumption 3.

**Proposition 3.** *Suppose an economy is on an initial baseline BGP and, at time  $t = 0$ , a change in innovation policies induces a new path for the innovation intensity of the economy given by  $\{s'_{rt}\}_{t=0}^{\infty}$ . If assumption 3 is satisfied, then the new path for aggregate productivity  $\{Z'_t\}_{t=1}^{\infty}$  to a first-order approximation is given by*

$$\log Z'_{t+1} - \log \bar{Z}_{t+1} = \sum_{k=0}^t \Gamma_k (\log s'_{rt-k} - \log \bar{s}_r), \quad (26)$$

where  $\bar{Z}_t = \exp(t\bar{g}_z)\bar{Z}_0$ , with  $\Gamma_0$  denoting the impact elasticity of aggregate productivity growth from period  $t=0$  to  $t=1$  in response to the change in the innovation intensity of the economy at time  $t = 0$ , and

$$\Gamma_{k+1} = \left[ 1 - (1 - \phi) \frac{\Gamma_0}{\bar{L}_p} \right] \Gamma_k \text{ for } k \geq 0. \quad (27)$$

Proposition 3 gives us an analytical expression for the dynamics of aggregate productivity in the transition to a new BGP as a function of the transition path for the innovation

intensity of the economy that arises in equilibrium as a result of a change in policies. The following corollary to Proposition 3 provides an expression for the long-run change in aggregate productivity that corresponds to a given permanent change in innovation intensity of the economy.

**Corollary 1.** *Consider a permanent change in innovation policies that results such that the economy converges to a new BGP with innovation intensity  $s'_r$ . In the semi-endogenous growth case ( $\phi < 1$ ), the gap in aggregate productivity between the old and new BGP converges, up to a first-order approximation, to*

$$\log Z'_t - \log \bar{Z}_t = \frac{\bar{L}_p}{1 - \phi} (\log s'_r - \log \bar{s}_r). \quad (28)$$

*In the endogenous growth case ( $\phi = 1$ ), the gap in aggregate productivity between the old and new BGP is unbounded. In this case, the new growth rate of aggregate productivity to a first-order approximation is given by*

$$\log Z'_{t+1} - \log Z'_t = \bar{g}_z + \Gamma_0 (\log s'_r - \log \bar{s}_r). \quad (29)$$

From Corollary 1, we have that in the semi-endogenous growth case, the long run elasticity of aggregate productivity with respect to a permanent change in the innovation intensity of the economy given in equation (28) is independent of the impact elasticity  $\Gamma_0$ . This impact elasticity does, however, affect the model's transition dynamics from the initial BGP to the new BGP. The following corollary to Proposition 3 describes this role of the impact elasticity in shaping the transition dynamics of aggregate productivity, which are determined by the coefficients  $\{\Gamma_k\}$ .

**Corollary 2.** *Consider two specifications of our model (model 1 and model 2) that are calibrated to the same parameter  $\phi$  and have the same implications for  $\bar{g}_z$  and  $\bar{L}_p$  on the initial BGP. Assume that the impact elasticity is higher in model 2 than in model 1,  $\Gamma_0^1 < \Gamma_0^2$ . Then the elasticities  $\{\Gamma_k\}$  defined in Proposition 3 are related in the two models as follows. For any finite  $K \geq 1$ ,  $\sum_{k=0}^K \Gamma_k^1 < \sum_{k=0}^K \Gamma_k^2$ . With semi-endogenous growth,  $\phi < 1$ , as  $K \rightarrow \infty$ ,  $\sum_{k=0}^{\infty} \Gamma_k^1 = \sum_{k=0}^{\infty} \Gamma_k^2$ , while with endogenous growth ( $\phi = 1$ ),  $\Gamma_K^1 = \Gamma_0^1 < \Gamma_0^2 = \Gamma_K^2$  for all  $K \geq 0$ .*

What this result implies is that if we consider innovation policy changes in model 1 that produce the same transition path for the innovation intensity of the economy  $\{s'_{rt}\}$  as the innovation policy changes considered in model 2, then model 2 will imply a larger change in aggregate productivity up to any period  $K$  along the transition path, to a first-order approximation. In the long-run, as  $K \rightarrow \infty$ , with semi-endogenous growth, the two



models deliver the same response of aggregate productivity. Likewise, with endogenous growth, for every  $K$ , the response of the growth rate of aggregate productivity will be larger in model 2 than in model 1.

The following Lemma presents the transition path for GDP as a function of the transition paths for aggregate productivity, the innovation intensity of the economy, and the equilibrium rental rate on physical capital.<sup>21</sup>

**Lemma 2.** *If the average markup is constant, then the path of GDP corresponding to a policy change is given, to a first-order approximation, by*

$$\log GDP'_t - \log G\bar{D}P_t = \frac{1}{1-\alpha} (\log Z'_t - \log \bar{Z}_t) - \bar{L}_r (\log s'_{rt} - \log \bar{s}_r) - \frac{\alpha}{1-\alpha} (\log R'_{kt} - \log \bar{R}_k) \quad (30)$$

*under the old measurement system in which innovation is expensed, and is given by the above plus  $\frac{\bar{s}_r}{1+\bar{s}_r} (\log s'_{rt} - \log \bar{s}_r)$  under a measurement system in which all expenditures on innovation are included in measured GDP.*

Given the result in Lemma 2, the long-run response of GDP to a permanent change in the innovation intensity of the economy is as follows. With semi-endogenous growth, in the long run the interest rate and rental rate on physical capital return to their levels on the initial BGP (i.e  $\log R'_{kt} - \log \bar{R}_k = 0$ ), so the long run response of GDP is a simple function of the long run responses of aggregate productivity and the innovation intensity of the economy. In particular, from Corollary 1, we have that with  $\phi < 1$ , the long run response of GDP is given by

$$\log GDP'_t - \log G\bar{D}P_t = \left[ \frac{1}{1-\alpha} \frac{\bar{L}_p}{1-\phi} - \bar{L}_r \right] (\log s'_{rt} - \log \bar{s}_r). \quad (31)$$

## 5 Aggregate dynamics and optimal innovation intensity

In this section, we use our characterization of the macroeconomic dynamics implied by our model to derive an analytical expression for the innovation intensity of the economy on the BGP of the socially optimal allocation (hereafter, the optimal BGP).

We define the *socially optimal allocation* as the solution to the following problem: choose current production plans of all intermediate goods firms, together with the investments

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<sup>21</sup>The magnitude of the change in the rental rate of physical capital in the transition is related to the equilibrium transition path for the interest rate. From the Euler equation for physical capital,  $\log R'_{kt} - \log \bar{R}_k = \frac{\bar{r}}{d_k + \bar{r}} \log (\bar{r}_{t-1} / \bar{r})$  for  $t \geq 1$ . Solving for the path of the interest rate requires fully solving for the model transition, which we do in Appendix D.



in innovation by those firms and all macroeconomic aggregates to maximize the utility of the representative agent subject to constraints (1) through (4) and (6) through (9). If the economy starts on the optimal BGP, then up to a first order, the change in welfare corresponding to any change in the innovation intensity of the economy should be zero.

We summarize the welfare gains or losses associated with a new allocation that deviates from an initial BGP by the *consumption equivalent change in welfare*, defined as the scalar  $\zeta$  multiplying the baseline BGP path for consumption required to implement the same change in welfare as is achieved under the new allocation. Under the assumption that the initial baseline BGP allocation is not distorted by an average markup, the following Lemma provides an expression for the consumption equivalent change in welfare corresponding to a perturbation of the initial BGP allocation.<sup>22</sup>

**Lemma 3.** *If the Euler equation for physical capital is undistorted on the initial BGP, then, up to a first order approximation, the consumption equivalent change in welfare corresponding to the macroeconomic dynamics induced by a perturbation of the initial BGP allocation is given by*

$$\log \zeta = (1 - \tilde{\beta}) \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{G\bar{D}P_t}{\bar{C}_t} \left[ (\log Z'_t - \log \bar{Z}_t) - (1 - \alpha) \frac{\bar{L}_r}{\bar{L}_p} (\log L'_{rt} - \log \bar{L}_r) \right], \quad (32)$$

where  $\tilde{\beta} = \beta \exp((1 - \eta)\bar{g}_y)$  is the ratio of the gross growth rate of output to the gross interest rate on that BGP.

We see in expression (32) that an increase in the innovation intensity of the economy has two effects on welfare. It will initially decrease the resources available for consumption and physical investment through the reallocation of labor from current production to research, lowering welfare, and then increase those resources as aggregate productivity climbs over time in response to firms' increased investments in innovation, raising welfare. In this way, our model's implications for the welfare change corresponding to a change in the innovation intensity of the economy are tightly linked to its implications for the macroeconomic dynamics induced by that change in policies.

For a transition path that satisfies Assumption 3, the consumption equivalent change in welfare induced by a permanent change in innovation intensity of the economy from  $\bar{s}_r$  to  $s'_r$  is given by

$$\log \zeta = \left[ \frac{\tilde{\beta}^{\frac{\Gamma_0}{\bar{L}_p}}}{1 - \tilde{\beta} \left( 1 - (1 - \phi) \frac{\Gamma_0}{\bar{L}_p} \right)} - (1 - \alpha) \frac{\bar{L}_r}{\bar{L}_p} \right] \frac{G\bar{D}P}{\bar{C}} \bar{L}_p (\log s'_r - \log \bar{s}_r), \quad (33)$$

<sup>22</sup>In equilibrium allocations, the distortions to the allocation of physical capital arising from markups by intermediate goods producing firms can be undone with an appropriate production subsidy.

where we have used the expressions for the dynamics of aggregate productivity in Proposition 3. On the socially optimal BGP, the innovation intensity of the economy should be such that this perturbation has no first order impact on welfare (that is,  $\log \zeta = 0$ ). That is, the term in the square bracket in equation (33) should be equal to zero, which implies

$$s_r^* = (1 - \alpha) \frac{L_r^*}{L_p^*} = \frac{\tilde{\beta} \Gamma_0 / \bar{L}_p^*}{1 - \tilde{\beta} \left[ 1 - (1 - \phi) \Gamma_0 / \bar{L}_p^* \right]}. \quad (34)$$

The first equality uses the fact that the implied average markup  $\bar{\mu}$  in equation (12) on the socially optimal BGP is equal to one. In the second equality,  $\Gamma_0$  is the impact elasticity and  $\bar{L}_p^*$  is the share of labor devoted to current production on the socially optimal BGP. In general,  $\Gamma_0$  in equation (34) depends on the specifics of how additional innovative investment expenditure is allocated across firms. If Assumption 2 holds on the optimal BGP, then this impact elasticity  $\Gamma_0$  is independent of the particular allocation of innovative investment considered. Note as well that if the function  $G$  on the optimal BGP satisfies the concavity assumption 1 or 1a, then we can apply our Proposition 1 or 2 to bound  $\Gamma_0 / \bar{L}_p^*$ , which then implies an upper bound on the optimal innovation intensity of the economy in which  $\bar{g}_z - G_0^0$  replaces the term  $\Gamma_0 / \bar{L}_p^*$  in equation (34).

The innovation subsidy rates needed to implement this allocation in the equilibrium depend on additional details of the equilibrium not specified here. In Appendix F we show that under a number of assumptions (satisfied in all of our model examples), changes in innovation subsidies uniformly applied to entering and incumbent firms are related to changes in the innovation intensity of the economy by  $\bar{s}_{rt} (1 - \bar{\tau}) = \bar{s}'_{rt} (1 - \bar{\tau}')$ , and result in the long-run in a change in fiscal expenditures relative to GDP of  $\frac{\bar{E}'}{GDP'} - \frac{\bar{E}}{GDP} = \bar{s}'_r - \bar{s}_r$ . Therefore, given information on  $\bar{s}_r$  and  $\bar{\tau}_r$ , it is straightforward to calculate the uniform innovation subsidy rate and its fiscal costs relative to GDP that implements  $s_r^*$  in the BGP.

## 6 Quantitative implications of analytical results

In the previous two sections, we characterized, up to a first-order approximation, the dynamics of aggregate productivity following a policy induced change in the innovation intensity of the economy in Proposition 3 and Corollary 1, and we characterized the associated dynamics of GDP in Lemma 2 and the innovation intensity of the economy on the optimal BGP in equation (34). In this section, we use these results to examine the quantitative implications of five example specifications of our model for the dynamics of aggregate productivity and output following a permanent policy induced change in the

innovation intensity of the economy and for the innovation intensity of the economy on the optimal BGP.

Two of these example specifications of our model are based on a Neo-Schumpeterian framework that builds in the assumption of no social depreciation of innovation expenditures. One example has innovation only by entering firms as in [Aghion and Howitt \(1992\)](#) and [Grossman and Helpman \(1991b\)](#), and one has innovation by both entering and incumbent firms as in [Klette and Kortum \(2004\)](#). The other three example specifications are based on the Expanding Varieties framework. One example has innovation only by entering firms as in [Luttmer \(2007\)](#), one includes innovation by incumbents to improve their own products as in [Atkeson and Burstein \(2010\)](#), while the other includes innovation by incumbent firms to create new products as in [Luttmer \(2011\)](#). These three specifications allow for social depreciation of innovation expenditures.

In [Appendix C](#) we show that in all five of these specifications of our model, if incumbents face uniform innovation subsidies in any transition ( $\tau_t(j) = \tau_t$  for all incumbents  $j \geq 1$ ), the function  $G$  can be written as

$$G = \frac{1}{\rho - 1} \log \left( a_0 + a_1 N_t(0) + F(y_{rt}^I) \right), \quad (35)$$

where  $F$  is a concave function with  $F(0) = 0$  and  $y_{rt}^I = \sum_{j \geq 1} y_{rt}(j) N_t(j)$  is the use of the research good by incumbents (that is,  $Y_{rt} = \bar{y}_r(0) N_t(0) + y_{rt}^I$ ). The social rate of depreciation of innovation expenditures in these example models is given by  $G_0^0 = \frac{1}{\rho - 1} \log a_0$ . The five examples differ in terms of their implications for the terms  $a_0, a_1 \geq 0$ , and for the function  $F$ .

Given the result that the function  $G$  takes the form in [equation \(35\)](#) it is immediate that both of the concavity assumptions 1 and 1a are satisfied for all five of our examples. Assumption 2 is satisfied as well if, on the initial BGP,  $a_1 / \bar{y}_r(0) = F'(\bar{y}_{rt}^I)$ . In [Appendix D](#) we show that this condition holds in our examples when entrants and incumbents face uniform innovation subsidies on the initial BGP ( $\bar{\tau}(j) = \bar{\tau}$  for all  $j$ , including entrants). Finally, assumption 3 is also satisfied if assumption 2 holds. Hence, we can use our analytical results to characterize quantitatively the aggregate implications of permanent policy induced changes in the innovation intensity of the economy for these five examples.

We calibrate all of these models to share common values of the parameter governing the elasticity of demand for intermediate goods ( $\rho = 4$ ), the share of physical capital in the marginal cost of producing the final consumption good ( $\alpha = 0.37$ ) and to imply common values for the growth rate of aggregate productivity ( $\bar{g}_z = 0.0125$ ) and the allocation of labor to production ( $\bar{L}_p = 0.83$ ) on the initial BGP. For all of our examples, we calibrate

the innovation intensity of the economy on the initial BGP to be  $\bar{s}_r = 0.11$ , similar to the levels estimated by [Corrado et al. \(2009\)](#) for the U.S. over the last few years. An explanation of these parameter choices and calibration procedure is presented in [Appendix D](#). We consider three alternative values of the intertemporal knowledge spillover parameter  $\phi$ . The first case we call the *high spillover* case. In it we set  $\phi = .99$ . This case implies transition paths very close to those found in the endogenous growth case (with  $\phi = 1$ ). The second case we call the *medium spillover* case, and in it we set  $\phi = 0$ . The third case we call the *low spillover* case, and in it we set  $\phi = -2$ .<sup>23</sup> When we analyze the implications of our example models for the innovation intensity on the optimal BGP, we consider two alternative specifications of  $\tilde{\beta}$  which determines the ratio between the growth rate of output and the gross interest rate on the initial BGP:  $\tilde{\beta} = .99$  and  $\tilde{\beta} = 0.96$  corresponding to a gap between the interest rate and the growth rate of 1% and 4% respectively. The results that we discuss in what follows are displayed in [Table 1](#).<sup>24</sup>

**Long Term Elasticities** First consider the quantitative implications of our model, up to a first-order approximation, for the elasticity of aggregate productivity in the long run with respect to a permanent policy-induced change in the innovation intensity of the economy. This elasticity corresponds to  $\sum_{k=0}^{\infty} \Gamma_k$  and we compute it from [Corollary 1](#). As we see from that corollary, for any value of  $\phi < 1$ , the quantitative implications of our model for this long run elasticity are determined entirely by the value of  $\bar{L}_p$  to which the model is calibrated on the initial BGP and the specification of intertemporal knowledge spillovers  $\phi$ . Hence, all five example specifications of our model give the same quantitative implications, up to a first-order approximation, for the elasticity of aggregate productivity in the long run with respect to a permanent policy-induced change in the innovation intensity of the economy. In the high spillover case with  $\phi = .99$ , this elasticity is given by  $\sum_{k=0}^{\infty} \Gamma_k = .83/.01 = 83$ . In the medium spillover case with  $\phi = 0$ , we have  $\sum_{k=0}^{\infty} \Gamma_k = .83/1 = 0.83$ . In the low spillover case with  $\phi = -2$ , we have

<sup>23</sup>Recall that, as we vary  $\phi$ , we change the growth rate of scientific knowledge,  $g_{A_r}$ , to match the same productivity growth rate,  $\bar{g}_z$ .

<sup>24</sup>The elasticities that we report in this section correspond to the values of  $\log Z'_T - \log \bar{Z}_T$  and  $\log GDP'_T - \log \bar{GDP}_T$ , for different horizons  $T = 1, 20$  and  $\infty$  corresponding to a permanent change in the innovation intensity of the economy of  $\log s'_r - \log \bar{s}_r = 1$  (or  $s'_r/\bar{s}_r = 2.72$ ). Because we use linear approximations, we can calculate the approximate responses in the logarithms of aggregate productivity and GDP (relative to BGP) by multiplying these elasticities by the change in the logarithm of the innovation intensity of the economy. Of course, if we keep constant the *percentage point change* in the innovation intensity of the economy but calibrate the model to a lower initial value of  $\bar{s}_r$ , this mechanically implies that this policy experiment results in a larger change in the *log* of the innovation intensity of the economy. Thus, keeping the model elasticities  $\Gamma_k$  unchanged, the magnitude of the response of aggregate productivity and GDP to this policy experiment will be larger.

Intertemporal spillovers	No social depreciation			With social depreciation		
	Low	Medium	High	Low	Medium	High
<b>Aggregate productivity</b>						
Year 1	0.01	0.01	0.01	0.04	0.04	0.04
Year 20	0.15	0.18	0.20	0.27	0.53	0.83
Long run	0.28	0.83	83.30	0.28	0.83	83.30
<b>GDP</b>						
Year 20	0.07	0.12	0.16	0.26	0.69	1.16
Long run	0.28	1.17	133.41	0.28	1.17	133.41
<b>Welfare (equivalent variation)</b>						
Patient consumers, $\tilde{\beta} = 0.99$	0.17	0.52	1.35	0.22	0.85	5.75
Impatient consumers, $\tilde{\beta} = 0.96$	0.04	0.12	0.21	0.15	0.48	1.24
<b>Optimal innovation intensity</b>						
Patient consumers, $\tilde{\beta} = 0.99$	0.26	0.56	1.26	0.31	0.84	4.95
Impatient consumers, $\tilde{\beta} = 0.96$	0.16	0.24	0.31	0.26	0.56	1.27

Table 1: Elasticity of aggregate productivity, GDP and welfare, and optimal innovation intensity

Parameters:  $\bar{g}_z=0.0125$ ,  $G_0^0=0$  (no depr.) or  $-0.04$  (with depr.),  $\alpha=0.37$ ,  $\rho=4$ ,  $\bar{L}_p=0.833$ ,  $\frac{\bar{Y}_r}{\bar{Y}_r - Y_{r0}^0}=1$ ,  $\phi=-2$ , 0 or 0.99

$$\sum_{k=0}^{\infty} \Gamma_k = .83/3 = 0.28.$$

The corresponding values for the elasticity of aggregate output with respect to a permanent policy-induced change in the innovation intensity of the economy can be found from equation (31). For the case of high intertemporal knowledge spillovers, this elasticity is 133.1, for medium spillovers, 1.17, and for low spillovers 0.28.

Clearly, the quantitative implications of the five example specifications of our model for these long-run elasticities of aggregate productivity and output to a permanent policy induced change in the innovation intensity of the economy are very sensitive to the calibration of intertemporal knowledge spillovers. In contrast, the quantitative implications of the five example specifications of our model for these long-run elasticities are not sensitive to other features of the model.

**Neo-Schumpeterian Short and Medium Term Elasticities** Now consider the quantitative implications of our example models for the elasticity of aggregate productivity and output in the short run and the medium run with respect to a permanent policy-

induced change in the innovation intensity of the economy. To be concrete, let the short run elasticity correspond to the impact elasticity  $\Gamma_0$  and the medium run elasticity correspond to the elasticity of aggregate productivity at a horizon of 20 years  $\sum_{k=0}^{19} \Gamma_k$ . Once we find the quantitative implications of our example models for the impact elasticity  $\Gamma_0$ , we use Proposition 3 to compute the medium term elasticity of aggregate productivity and Lemma 2 to compute the corresponding short and medium term elasticities of GDP.

In Appendix C we show that the impact elasticity implied by the form of the function  $G$  in equation (35) for all five of the example models we consider is given by

$$\Gamma_0 = \frac{1}{\rho - 1} \frac{\exp(\bar{g}_z)^{\rho-1} - \exp(G_0^0)^{\rho-1}}{\exp(\bar{g}_z)^{\rho-1}} \frac{\bar{Y}_r}{\bar{Y}_r - Y_{r0}^0} \bar{L}_p. \quad (36)$$

Note that this expression attains the bound in Proposition 2 in the limit as  $\rho \rightarrow 1$  given fixed  $\bar{g}_z$  and  $G_0^0$ . With a larger elasticity  $\rho$ , given fixed  $\bar{g}_z$  and  $G_0^0$ , the impact elasticity  $\Gamma_0$  is smaller and approaches zero as  $\rho \rightarrow \infty$ .

We also show in Appendix C that as a consequence of assumptions 1 and 2 and the specific form of the function  $G$  in our five example models, the term  $\bar{Y}_r / (\bar{Y}_r - Y_{r0}^0)$  in expression (36) is less than or equal to 1. This result follows from the observation that given the concavity of the function  $F$ , if assumption 2 holds, then we must have the average cost of raising the growth rate from  $G_0^0$  to  $\bar{g}_z$  through investment by both entering and incumbent firms given in (22) is lower or equal than the average cost of implementing the same change in productivity growth through investment only by entering firms given in (23).

We first use equation (36) to analyze the quantitative implications of our two Neo-Schumpeterian models for these short and medium term elasticities. Proposition 1 and Lemma 1 together imply that, given this calibration to  $\bar{g}_z = 0.0125$  and  $\bar{L}_p = 0.83$ , our two Neo-Schumpeterian models cannot imply an impact elasticity higher than  $\Gamma_0 = \bar{g}_z \bar{L}_p = .0104$ . This result follows directly from the implicit assumption made in these models of no social depreciation of innovation expenditures ( $G_0^0 = 0$ ). For the example with innovation only by entering firms, we have  $\bar{Y}_r / (\bar{Y}_r - Y_{r0}^0) = 1$  and hence it is possible to calibrate that example to attain this bound by choosing  $\rho$  close to 1. In Appendix C we discuss a calibration of our Neo-Schumpeterian models with innovation by both entering and incumbent firms that imply values of  $\Gamma_0$  close to this bound. For purposes of illustration here, we focus on the quantitative implications of these two Neo-Schumpeterian examples with an impact elasticity for aggregate productivity of  $\Gamma_0 = 0.0102$  (calculated using equation (36) with  $\bar{Y}_r / (\bar{Y}_r - Y_{r0}^0) = 1$ ), very close to its upper bound.

Now consider the implications of these two Neo-Schumpeterian example models for



the medium term elasticities of aggregate productivity and output. Using proposition 3 to compute the medium term elasticity of aggregate productivity, we find in the high spillover case ( $\phi = 0.99$ ), this elasticity is  $\sum_{k=0}^{19} \Gamma_k = 0.20$ . In the medium spillover case ( $\phi = 0$ ), this elasticity is  $\sum_{k=0}^{19} \Gamma_k = 0.18$ . In the low spillover case ( $\phi = -2$ ), this elasticity is  $\sum_{k=0}^{19} \Gamma_k = 0.15$ . Clearly, the difference between these medium term elasticities for different values of the intertemporal knowledge spillovers is quite small in comparison to what is found for the impact of the intertemporal knowledge spillover parameter  $\phi$  on the long run elasticities  $\sum_{k=0}^{\infty} \Gamma_k$ .

To understand this finding, recall from Proposition 3 that when  $\phi < 1$ , we have the terms  $\Gamma_k$  shrinking geometrically over time at a rate given by the term  $1 - (1 - \phi)\Gamma_0/\bar{L}_p$  as shown in equation (27). Within a range of assumptions regarding  $\phi$  that gives starkly different long run implications for aggregate productivity ( $\phi \in [-2, 1]$ ), the rate of decay  $1 - (1 - \phi)\Gamma_0/\bar{L}_p$  of the terms  $\Gamma_k$  does not vary that much (from roughly 0.97 to 1) because the ratio  $\Gamma_0/\bar{L}_p$  is small. This finding for our Neo-Schumpeterian models is independent of other aspects of the specification of the model since, from Proposition 1, given the assumption of no social depreciation on innovation expenditures, this ratio is bounded by  $\Gamma_0/\bar{L}_p \leq \bar{g}_z$ . Here, we have calibrated our Neo-Schumpeterian example models such that the impact elasticity  $\Gamma_0$  is close to its upper bound. If we were to choose parameters leading to a lower value of this impact elasticity  $\Gamma_0$ , the variation of the rate of decay  $1 - (1 - \phi)\Gamma_0/\bar{L}_p$  with  $\phi$  would be even smaller and hence the sensitivity of the medium term elasticity to intertemporal knowledge spillovers  $\phi$  would also be smaller.

We use Lemma 2 to find the corresponding medium-term elasticities of aggregate output, assuming that the capital-output ratio is constant along the transition (in Appendix D we calculate numerically the full transition to a change in innovation policies, taking into account the dynamics in the capital-output ratio, and show that the results do not change substantially). These are 0.16 in the high spillover case, 0.14 in the medium spillover case, and 0.08 in the low spillover case. Again we see that the medium term elasticity of GDP is not that sensitive to the assumed intertemporal knowledge spillovers in comparison to what is found for the long run elasticities. To convert these results to implications for GDP inclusive of innovation expenditure one must multiply the level of GDP exclusive of these expenditures by  $(1 + s_{rt})$ .

**Neo-Schumpeterian Optimal Innovation Intensities** We now consider the implications of our two Neo-Schumpeterian examples for the innovation intensity of the economy on the optimal BGP and for the elasticity of the consumption equivalent variation in welfare corresponding to a permanent policy induced variation in the innovation intensity of the

economy, assuming that the Euler equation for physical capital accumulation is undistorted. To do so, we use equations (33) and (34), and we set the term  $\Gamma_0/\bar{L}_p^*$  very close to its upper bound of  $\bar{g}_z = 0.0125$ . The quantitative implications of the model for this optimal innovation intensity of the economy depend on assumptions regarding consumers' adjusted discount factor  $\tilde{\beta}$  and of the intertemporal knowledge spillover parameters  $\phi$ . Specifically, using our parameter values, equation (34) then implies that with a high ratio of the growth rate to the interest rate  $\tilde{\beta} = .99$  (the interest rate is one percentage point higher than the growth rate) and high intertemporal knowledge spillovers  $\phi = .99$ , the optimal innovation intensity of the economy is  $s_r^* = 1.26$ . That is, in this case, innovation expenditures should exceed expenditure on consumption and physical investment combined by 28%. The elasticity of the consumption equivalent variation in welfare, starting at  $s_r = 0.11$ , is 1.35. In contrast, on the conservative side, with a low ratio of the growth rate to the interest rate  $\tilde{\beta} = .96$  (the interest rate is four percentage points higher than the growth rate) and low intertemporal knowledge spillovers ( $\phi = -2$ ), the optimal innovation intensity of the economy is  $s_r^* = 0.16$ , a figure that is not too far from the measures of investments in intangible capital relative to GDP calculated by [Corrado et al. \(2009\)](#). Moreover, the elasticity of the consumption equivalent variation in welfare is only 0.04.

Clearly, our Neo-Schumpeterian example models' implications for the innovation intensity of the economy on the optimal BGP and for the welfare gains to be had from a policy change inducing a permanently higher innovation intensity of the economy are highly sensitive to assumptions about consumers' patience and intertemporal elasticity of substitution as summarized by  $\tilde{\beta}$  and the level of intertemporal knowledge spillovers  $\phi$ .

To sum up these quantitative findings for our two Neo-Schumpeterian examples, we see that the model's predictions for the response of aggregate productivity and output to a given permanent change in the innovation intensity of the economy 20 years into the transition are not particularly sensitive to choices of the intertemporal knowledge spillover parameter  $\phi$  in comparison to the strong dependence of the model's long run predictions for aggregate productivity and output and for the optimal innovation intensity of the economy on this parameter. These results follow from the implicit assumption of zero social depreciation of innovation expenditures.

**Expanding Varieties Short and Medium term Elasticities** We now consider the quantitative implications of our three example models based on the Expanding Varieties framework. In contrast to the example models based on the Neo-Schumpeterian framework, these models do not directly make assumptions about the social depreciation of innova-



tion expenditures. We show, however, that given the structure of the function  $G$  in these models shown in equation (35), reasonably calibrated versions of these models necessarily imply a high social rate of depreciation of innovation expenditures and a high impact elasticity  $\Gamma_0$  compared to what we found in our Neo-Schumpeterian examples. As a consequence they also imply higher medium term elasticities and that these medium term elasticities are substantially more sensitive to different assumed values of intertemporal knowledge spillovers. Thus, even though our three example models based on the Expanding Varieties framework share the same implications for long run elasticities as we found in our two Neo-Schumpeterian models, they have very different implications for the short and medium term elasticities of aggregate output and GDP and hence also for the innovation intensity of the economy on the optimal BGP and for the welfare gains to be had from increasing the innovation intensity of the economy with innovation policy.

Because our example models based on the Expanding Varieties framework do not make explicit assumptions about the social depreciation of innovation expenditures, the formula (36) (which applies in all of our model examples) is not immediately useful as a guide for using data to discipline the impact elasticity of aggregate productivity implied by these models. Instead, we now demonstrate how the additional structure imposed on the function  $G$  in equation (35) together with the implicit assumption of no business stealing inherent in the Expanding Varieties framework allows us to link our example models' implications for data on the employment share and research expenditure share of entering firms to their quantitative implications for the impact elasticity  $\Gamma_0$ .

To do so, we first differentiate the function  $G$  shown in equation (35) to calculate the impact elasticity,  $\Gamma_0$ . Given that assumption 2 is satisfied, we can compute this elasticity by varying only entry  $N_t(0)$  as we vary the research intensity of the economy  $s_{rt}$  and total output of the research good  $Y_{rt}$ . Computing  $\Gamma_0$  in this manner gives

$$\Gamma_0 = \frac{\partial G}{\partial N_t(0)} \bar{N}_t(0) \frac{d \log N_t(0)}{d \log Y_{rt}} \frac{d \log Y_{rt}}{d \log s_{rt}} = \frac{1}{\rho - 1} \frac{a_1 \bar{N}_t(0)}{\exp(\bar{g}_z)^{\rho-1}} \frac{\bar{Y}_r}{\bar{y}_r(0) \bar{N}_t(0)} \bar{L}_p, \quad (37)$$

where all of the derivatives are evaluated on the initial BGP. All of the terms in this expression correspond directly to commonly calibrated parameters (the elasticity of substitution among varieties  $\rho$ ) or quantities that are measurable in data (the share of labor in current production on the initial BGP,  $\bar{L}_p$ , and the share of research expenditure conducted by entering firms on the initial BGP  $\bar{y}_r(0) \bar{N}_t(0) / \bar{Y}_{rt}$ ) except for the term  $a_1 \bar{N}_t(0) / \exp(\bar{g}_z)^{\rho-1}$ .

Under the implicit assumption made in all three of our Expanding Varieties models of no business stealing by entrants<sup>25</sup>, the term  $a_1 \bar{N}_t(0) / \exp(\bar{g}_z)^{\rho-1}$  is equal to the share of

<sup>25</sup>Specifically, an increase in entry at  $t$ ,  $N_t(0)$ , does not result through the equilibrium transition law  $T$  in

employment in entering firms on the initial BGP. The intuition for this result is straightforward. The aggregate across firms entering at  $t$  of the statistic  $\exp(z)^{\rho-1}/Z_{t+1}^{\rho-1}$  at  $t+1$  is equal to the share of total employment in entering firms at  $t+1$ . Since it is assumed that there is no business stealing, i.e. that an increase in entry is assumed not to result in the exit of any incumbent firms, the employment share of entering firms is equal to their social share in contributing to  $Z_{t+1}^{\rho-1}$ . Therefore, the four terms in equation (37) can be measured with data. In contrast, in Neo-Schumpeterian models the term  $a_1 \bar{N}_t(0) / \exp(\bar{g}_z)^{\rho-1}$  corresponds to the gap between the share of jobs created in entering firms less the share of jobs displaced by entering firms, which may be harder to measure in practice.

In order to use equation (37), we have calibrated  $1/(\rho-1) = 1/3$  and  $\bar{L}_p = 0.83$ . For our three Expanding Varieties examples, we choose parameters such that the share of employment in entering firms on the initial BGP is equal to 0.03, consistent with data on firm dynamics in the U.S. In the first of these examples in which incumbent firms do not innovate, the share of research expenditure carried out by entering firms is equal to one by assumption. Given the clear importance of this term in determining our models' implications for the impact elasticity  $\Gamma_0$ , we focus on the quantitative implications of our example models in which incumbents do innovate either to improve their existing products or to produce new products. In those two examples, we choose parameters such that the share of innovation expenditure carried out by entrants is equal to 0.2. This number is consistent with the view expressed in [Aghion et al. \(2013\)](#) that most investment in innovation is carried out by incumbents. With this calibration, which we discuss in [Appendix D](#), these two example Expanding Varieties models imply that the impact elasticity of aggregate productivity growth with respect to a policy-induced change in the innovation intensity of the economy is  $\Gamma_0 = 0.042$ . This impact elasticity is roughly four times larger than the bound we found for our Neo-Schumpeterian example models.

Now consider the implications of these two example Expanding Varieties models for the medium term elasticities of aggregate productivity and output. From [Corollary 2](#) we have that, given a fixed value of  $\phi$ , these medium term elasticities are larger than those we found in our Neo-Schumpeterian examples due to the higher impact elasticity  $\Gamma_0$ . In the high spillover case ( $\phi = 0.99$ ), this elasticity is  $\sum_{k=0}^{19} \Gamma_k = 0.83$ . In the medium spillover case ( $\phi = 0$ ), this elasticity is  $\sum_{k=0}^{19} \Gamma_k = 0.53$ . In the low spillover case ( $\phi = -2$ ), this elasticity is  $\sum_{k=0}^{19} \Gamma_k = 0.27$ . Clearly, not only are these medium-term elasticities larger than those that we found in our Neo-Schumpeterian examples, but also the difference between these medium term elasticities for different values of the intertemporal knowledge spillovers are now significantly larger than those we found with our Neo-Schumpeterian

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a reduction in the measure of any type of incumbent firm at  $t+1$ ,  $N_{t+1}(j)$ .

examples. This finding follows directly from our finding of a higher ratio  $\Gamma_0/\bar{L}_p$  since, given this higher ratio, the rate at which  $\Gamma_k$  decays is now more sensitive to different assumptions about intertemporal knowledge spillovers  $\phi$ .

We use Lemma 2 to find the corresponding medium-term elasticities of aggregate output. These are 1.16 in the high spillover case, 0.69 in the medium spillover case, and 0.26 in the low spillover case. Now we see that the medium term elasticity of GDP is potentially quite large and is quite sensitive to the assumed intertemporal knowledge spillovers in comparison to what we found with our Neo-Schumpeterian examples.

**Expanding Varieties Optimal Innovation Intensities** We now consider the implications of our two Expanding Varieties examples with innovation by incumbents for the innovation intensity of the economy on the optimal BGP. To do so, we use equation (34) and we set the term  $\Gamma_0/\bar{L}_p^*$  equal to 0.05 which is consistent with the employment and research expenditure shares of entrants the same in our initial calibration as on the optimal BGP. With  $\tilde{\beta} = .99$  and  $\phi = .99$ , the optimal innovation intensity of the economy is  $s_r^* = 4.95$ , that is, innovation expenditures should exceed expenditure on consumption and physical investment by a factor close to five. The corresponding elasticity of the welfare equivalent variation in consumption with respect to a permanent change in the innovation intensity of the economy, starting at  $s_r = 0.11$ , is close to 6. With  $\tilde{\beta} = .96$  and low intertemporal knowledge spillovers ( $\phi = -2$ ), the optimal innovation intensity of the economy is  $s_r^* = 0.26$ . The corresponding elasticity of the welfare equivalent variation in consumption is 0.15.

To sum up these quantitative findings for our two Expanding Varieties examples that include innovation by incumbents, we see that the model's predictions for the impact elasticity of aggregate productivity growth with respect to a policy induced change in the innovation intensity of the economy are roughly four times higher than is found in Neo-Schumpeterian models. As a consequence, the Expanding Varieties examples imply that the medium term elasticities of aggregate productivity and output with respect to a permanent policy-induced change in the innovation intensity of the economy are larger and significantly more sensitive to alternative assumptions regarding the degree of intertemporal knowledge spillovers than we found with our Neo-Schumpeterian examples. Likewise, our Expanding Varieties examples imply higher optimal innovation intensities and higher welfare elasticities than our Neo-Schumpeterian models. But still, our model's implications for the innovation intensity of the economy on the optimal BGP and for the welfare gains from a policy induced increase in the innovation intensity of the economy are highly sensitive to assumptions about consumers' patience and intertempo-

ral elasticity of substitution as summarized by  $\tilde{\beta}$  and the level of intertemporal knowledge spillovers  $\phi$  for all five examples.

**Social Depreciation of Innovation Expenditures and Business Stealing** What accounts for this striking difference in the quantitative implications of our Neo-Schumpeterian and Expanding Varieties examples for the short and medium term macroeconomic dynamics following a permanent policy induced change in the innovation intensity of the economy? It is the different implications that these models have for the social depreciation of innovation expenditures. While in our Neo-Schumpeterian examples, it is assumed that there is no social depreciation of innovation expenditures ( $G_0^0 = 0$ ), in our Expanding Varieties examples, a high social depreciation rate of innovation expenditures is implicitly assumed due to the assumption of no business stealing.

To see this connection between the assumption of no business stealing and social depreciation in an Expanding Varieties model, recall that the impact elasticity we found in equation (36) is equivalent to the one given in equation (37). Using assumption 1 and 2, we have that the ratio  $\bar{Y}_r / (\bar{Y}_r - Y_{r0}^0)$  is bounded above by one. Hence, under assumption 2, these two expressions for the impact elasticity  $\Gamma_0$  provide a lower bound on the absolute value of the social depreciation of innovation expenditures implied by all of our examples. For the Expanding Varieties models this bound implies that the log of aggregate productivity would fall by over 0.04 if all firms were to stop investing in innovation. This bound is higher, the greater is the contribution of entry to growth as measured by the term  $a_1 \bar{N}_t(0)$  and the lower is the share of total research expenditure carried out by entering firms. Intuitively, if on the initial BGP entrants account for a large portion of observed growth of aggregate productivity while spending only a small share of aggregate innovation expenditures and our assumption 2 equating the marginal social returns to innovation by incumbents and entrants holds, then it is necessarily the case that observed innovation expenditure by incumbents is compensating for a high social depreciation rate of innovation expenditures because the average social returns to innovative investment by incumbents must be at least as large as that for entrants.

## 7 Model specifications for which baseline assumptions do not hold

The five example model specifications that we have considered in Section 6 are all fairly special in that their reduced forms satisfy assumptions 1 - 3. In particular, in the mod-

els with innovation by incumbents and entrants (examples 2, 4, and 5) we assumed that all incumbent firms have identical innovation technologies, and that entrants and all incumbents face uniform innovation subsidies on the initial BGP. These assumptions about technology and policies deliver the result that assumption 2 is satisfied on the initial BGP and, hence, up to a first-order approximation, we can focus on aggregate investment in innovation without concern for how that investment is allocated across firms. In models with multiple types of firms engaged in innovative activity it is possible, and even likely, that the baseline allocation of innovative investment on an initial BGP does not satisfy assumption 2 (e.g. if innovation by incumbents is subsidized but entry is not). In this case, the magnitude of the impact elasticity of aggregate productivity depends on the specific details of how the additional units of the research good are allocated across different types of firms in response to a specific policy change. This impact elasticity can be greater or smaller depending on whether the reallocation of the research good induced by the policy change favors or disfavors the firms with the higher marginal social contribution to aggregate productivity growth.

Some recent papers in the literature have developed richer models of innovation and firm dynamics in which our baseline assumptions do not hold. We discuss some of these model specifications here and present additional details in Appendix G.

[Peters \(2013\)](#) presents a Neo-Schumpeterian model in which incumbent firms invest to improve their own products. In his model, an incumbent firm that innovates on its own product charges a higher markup on its product than does an entering firm. As a result of this assumption, the distribution of markups enters into the formula (5) for aggregate productivity. A policy-induced increase in the innovation intensity of the economy has an additional effect on aggregate productivity that depends on whether it is entering or incumbent firms that increase their investment in innovation. Hence, the equilibria of his model do not satisfy assumption 2 and may violate assumption 1a since a sufficiently large increase in entry can have a more than proportional impact on productivity through its impact on reducing markups.

[Lentz and Mortensen \(2014\)](#) and [Luttmer \(2011\)](#) present examples of a Neo-Schumpeterian model and an Expanding Varieties model respectively in which different types of incumbent firms have different technologies for innovating. The equilibria of these models on the initial BGP typically fail to satisfy assumption 2. In addition, assumption 3 typically is not satisfied along a transition path following a change in innovation policies. Under an alternative assumption, however, we can use a straightforward method to bound the welfare gains associated with a change in innovation policies even if we cannot characterize the positive implications of these models for the transition path following a change

in innovation policies. We refer to this assumption as conditional efficiency.

We say that an allocation is *conditionally efficient* if it is the solution to the problem of choosing current production plans of all intermediate goods firms together with the investments in innovation by those firms and all macroeconomic aggregates, except the aggregate allocation of labor between current production and research, to maximize the utility of the representative agent subject to constraints (1) through (4) and (6) through (9) conditional on the allocation of labor between current production and research labor  $L_{rt} = L_r$  fixed as a parameter.

Let  $V^*(\{N(j)\}_{j \geq 1}, K_0; A_r, L_r)$  denote the welfare associated with the conditionally efficient allocation starting with initial distribution of incumbent firms  $\{N(j)\}_{j \geq 1}$ , initial physical capital stock  $K_0$ , and initial stock of scientific knowledge  $A_r$ .

Let  $V^E(\{N(j)\}_{j \geq 1}, K_0; A_r, \{\tau_t(j)\}, L_r)$  denote the welfare associated with the equilibrium allocation starting with the same initial conditions and with a sequence of innovation subsidies  $\{\tau_t(j)\}$  adjusted as necessary through time to keep equilibrium  $L_{rt} = L_r$ . By definition, we have  $V^*(\{N(j)\}_{j \geq 1}, K_0; A_r, L_r) \geq V^E(\{N(j)\}_{j \geq 1}, K_0; A_r, \{\tau_t(j)\}, L_r)$ .

To compute a bound on the change in welfare corresponding to a change in innovation policies that results in a new equilibrium level of research labor  $L'_{rt} = L'_r$ , observe that if an initial baseline equilibrium BGP allocation is conditionally efficient, then we have  $V^*(\{\bar{N}(j)\}_{j \geq 1}, \bar{K}_0; A_r, \bar{L}_r) = V^E(\{\bar{N}(j)\}_{j \geq 1}, \bar{K}_0; A_r, \{\bar{\tau}(j)\}, \bar{L}_r)$ . Thus, the change in welfare associated with a change in policies that results in a new equilibrium level of research labor  $L'_{rt} = L'_r$  is bounded by

$$V^E(\{\bar{N}(j)\}_{j \geq 1}, \bar{K}_0; A_r, \{\tau'_t(j)\}, L'_r) - V^E(\{\bar{N}(j)\}_{j \geq 1}, \bar{K}_0; A_r, \{\bar{\tau}(j)\}, \bar{L}_r) \leq V^*(\{\bar{N}(j)\}_{j \geq 1}, \bar{K}_0; A_r, L'_r) - V^*(\{\bar{N}(j)\}_{j \geq 1}, \bar{K}_0; A_r, \bar{L}_r).$$

Note that for conditionally efficient allocations, the first order impact of a policy change on welfare on the right side of this bound is the same for all changes in firms' innovative investments that induce the same alternative path for current production and research labor  $L'_t = L'_r$  regardless of their implications for the allocation of innovative investments across firms. This is because, when the initial allocation is conditionally efficient, there are no first order welfare gains from reallocating innovation expenditures across firms holding fixed the allocation of labor between production and research. Hence, one can compute the welfare gains on the right hand side of this bound corresponding to a policy induced change in the path for research labor from  $\bar{L}_r$  to  $L'_r$  to a first order approximation by specifying one particular alternative feasible allocation corresponding to that given alternative path for labor and using the functions  $G$  and  $T$  of the model to compute the



corresponding implied dynamics of aggregate productivity. Given the implied path for aggregate productivity, the associated welfare gain can be computed from equation (32).

In Appendix G we describe specifications of the models with heterogeneous innovation technologies across incumbent firms similar to those in [Lentz and Mortensen \(2014\)](#) and [Luttmer \(2011\)](#) for which the equilibrium allocation on an initial BGP with uniform innovation policies is conditionally efficient. Hence, for these models, the welfare gains from a policy-induced change in the allocation of labor to research can be bounded as above.

## 8 Conclusion

In this paper, we have derived a simple first-order approximation to the transition dynamics of aggregate productivity and GDP in response to a policy-induced change in the innovation intensity of the economy implied by a model that nests a widely-used class of models of growth through firms' investments in innovation that have been developed over the past 25 years. We see our results as a useful guide to researchers looking to use these models to address quantitative questions regarding the impact of policy induced changes in firms investments in innovation on macroeconomic dynamics and welfare.

Our first result analyzed the immediate response of aggregate productivity growth to an increase in aggregate production of the research good — a response that we termed the impact elasticity of aggregate productivity growth with respect to a change in aggregate production of the research good. We showed that if the aggregate technology relating firms' investments in innovation to aggregate productivity growth satisfies a concavity assumption and if the initial baseline allocation at which the policy change is considered maximizes current productivity growth given the innovation intensity of the economy, then this impact elasticity is bounded by the gap between the growth rate of aggregate productivity in the baseline allocation and the social depreciation of innovation expenditures, defined as the growth rate of aggregate productivity that would obtain if all firms in the economy invested nothing in innovation. Hence, Neo-Schumpeterian specifications of our model that assume no social depreciation of innovation expenditures and which are calibrated to a low initial baseline growth rate of aggregate productivity characteristic of advanced economies necessarily imply a low impact elasticity of aggregate productivity growth with respect to a change in aggregate production of the research good if the baseline allocation is conditionally efficient. In contrast, Expanding Varieties specifications of our model can imply significant social depreciation of innovation expenditures and substantially larger impact elasticities of aggregate productivity with respect to a

policy induced change in the innovation intensity of the economy. We do not intend this illustration to stand as a definitive theory of the social depreciation of innovation expenditures. We speculate that social depreciation of innovation expenditures is likely derived from the fact that productive knowledge in firms is actually embodied in individuals who are familiar both with the knowledge gained through innovation and the procedures for training new workers in that knowledge, and the work force within firms is constantly turning over and workers themselves have a life cycle.

We next examined the dynamics of aggregate productivity and GDP in response to persistent changes in the innovation intensity of the economy induced by persistent changes in innovation policies. Under the assumption made in the most tractable specifications of our model, that the technology linking firms' innovative investments to aggregate productivity growth does not depend on the distribution of incumbent firms in the economy by type, we showed that these transition dynamics are characterized entirely by the impact elasticity of aggregate productivity growth with respect to a change in aggregate production of the research good and the parameters governing the magnitude of intertemporal knowledge spillovers in the production of the research good. We characterized the socially optimal level of the innovation intensity of the intensity in terms of these statistics and the assumed patience of consumers relative to the growth rate of the economy. We showed that in specifications of our model with a low impact elasticity (due to assumptions of no social depreciation of innovation expenditures and a low initial growth rate of aggregate productivity), alternative assumptions about the extent of these intertemporal knowledge spillovers have only a small impact on the model's implications for elasticities of aggregate productivity over the medium term but a large impact on the socially optimal innovation intensity of the economy. Given our uncertainty regarding the magnitude of intertemporal knowledge spillovers in the production of research, it seems difficult to use our model to make even rough quantitative statements about the optimal innovation intensity of an economy and the welfare gains to be had from using policy to increase firms' investments in innovation.

Ideally, one might measure the impact elasticity of aggregate productivity growth with respect to policy induced changes in firms' investments in innovation directly by examining the results of policy changes (see e.g. [Bloom et al. 2013](#)). Of course, this may be difficult to do in aggregate data if the impact elasticity is small because it would take a very large policy-induced change in the innovation intensity of the economy to bring about a measurable change in aggregate productivity growth in that case (this can help rationalizing empirical studies finding a weak link between R&D and productivity growth using time series data, see e.g. [CBO 2005](#)). Likewise, ideally one might measure the degree of



intertemporal knowledge spillovers by examining the persistence of changes in aggregate productivity growth in response to persistent policy induced changes in the innovation intensity of the economy. But as we have seen in our quantitative examples, it may take many decades of data to discern the degree of intertemporal knowledge spillovers in the time series if the impact elasticity of aggregate productivity growth is small. In light of these potential obstacles to using a direct approach to measuring the impact elasticity of aggregate productivity growth and the degree of intertemporal knowledge spillovers, it may be fruitful to use models to measure the social depreciation so as to see if there is room for a substantial impact elasticity. As we saw in the more tightly specified versions of our model based on the Quality Ladders and Expanding Varieties framework, the gap between the share of jobs created in entering firms less the share of jobs displaced by entering firms as well as the share of research expenditure undertaken by entering firms are observables linked to the social depreciation of innovation expenditures. Perhaps an approach toward measuring these quantities along the lines of [Garcia-Macia et al. \(2015\)](#) may be useful here.

Under our baseline assumptions, we have abstracted from the productivity and welfare gains that might be achieved by reallocating a given level of investment in innovation across heterogeneous firms. Conceptually, the welfare gains from such a reallocation can potentially be large. Under alternative model specifications in which there are first-order gains from such a reallocation, one may wish to consider a whole range of policies aimed at reallocating innovation expenditures across firms. The research challenge here is to find reliable metrics for evaluating which firms should be doing relatively more innovation spending and by how much should these firms increase their investments in innovation. Answering these questions requires detailed knowledge of the technology linking innovative investments by different firms to aggregate productivity growth.

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## A Proofs

**Proposition 1** By the assumption that all firms increase their investment in innovation proportionally,

$$\frac{y'_{r0}(j) - \bar{y}_{r0}(j)}{\bar{y}_{r0}(j)} = \frac{N'_0(0) - \bar{N}_0(0)}{\bar{N}_0(0)} = \kappa - 1$$

for all types of firms for which the baseline level of investment is strictly positive and  $y'_{r0}(j) = \bar{y}_{r0}(j) = 0$  otherwise. Note as well that the initial distribution of firms by type is given as an initial condition,  $N'_0(j) = \bar{N}_0(j)$  for all  $j \geq 1$ . Thus, by direct calculation using (13) and (14), we have  $\mathcal{D}_{g0} = H'_0(1)(\kappa - 1)$  and  $\mathcal{D}_{Yr0} = (\kappa - 1)$  where, from the definition of  $H_0(a)$ ,

$$H'_0(1) = \sum_{j \geq 1} \frac{\partial G}{\partial y_r(j)} \bar{y}_{r0}(j) + \frac{\partial G}{\partial N(0)} \bar{N}_0(0), \quad (38)$$

where the partial derivatives are evaluated at the baseline BGP allocation (with bars). By assumption 1 regarding the concavity of  $H_t(a)$ , we have  $H'_0(1) \leq H_0(1) - H_0(0)$ . Since  $H_0(1) = \bar{g}_z$  and  $H_0(0) = G_0^0$ , this proves  $\mathcal{D}_{g0} \leq (\bar{g}_z - G_0^0) \mathcal{D}_{Yr0}$ . Since the initial distribution of firms by type is given as a state variable, we have that the initial average markup is equal to the baseline average markup ( $\mu'_0 = \bar{\mu}_0$ ) and the initial level of aggregate productivity is equal to its baseline level ( $Z'_0 = \bar{Z}_0$ ). By equation (16), we have  $\mathcal{D}_{Yr0} = \bar{L}_p \mathcal{D}_{sr0}$ . This proves the result.

**Proposition 2** The proof is similar to that for Proposition 1. By the assumption that entry increases from  $\bar{N}_t(0)$  to  $N'_t(0) = \kappa \bar{N}_t(0)$  while the investments of all incumbent firms remain constant, we have

$$\frac{N'_0(0) - \bar{N}_0(0)}{\bar{N}_0(0)} = \kappa - 1$$

and  $y'_{r0}(j) = \bar{y}_{r0}(j)$  otherwise. Since the initial distribution of firms by type is given as an initial condition,  $N'_0(j) = \bar{N}_0(j)$  for all  $j \geq 1$ . Thus, by direct calculation using (14), we have  $\mathcal{D}_{Yr0} = \frac{\bar{y}_r(0) \bar{N}_0(0)}{\bar{Y}_r} (\kappa - 1)$ . By the definition of  $\tilde{H}_t(a)$ , we have

$$\tilde{H}'_t(1) = \frac{\partial G}{\partial N_t(0)} (\bar{N}_t(0) - N_t^0(0)), \quad (39)$$

and, hence, using (13) we have

$$\mathcal{D}_{g0} = \tilde{H}'_0(1) \frac{\bar{N}_0(0)}{\bar{N}_0(0) - N_0^0(0)} (\kappa - 1).$$

Using equation (20) this gives

$$\mathcal{D}_{g0} = \tilde{H}'_0(1) \left( \frac{\bar{Y}_r}{\bar{Y}_r - Y_{r0}^0} \right) \mathcal{D}_{Yr0} \leq (\bar{g}_z - G_0^0) \left( \frac{\bar{Y}_r}{\bar{Y}_r - Y_{r0}^0} \right) \mathcal{D}_{Yr0},$$

where the inequality follows from the concavity of  $\tilde{H}_t(a)$ , which implies  $\tilde{H}'_0(1) \leq \tilde{H}_0(1) - \tilde{H}_0(0)$ . By (16), we have  $\mathcal{D}_{Yr0} = \bar{L}_p (\log s'_{r0} - \log \bar{s}_r)$ . This proves the result.

**Lemma 1** This lemma is a simple application of the envelope theorem. From assumption 2, we have that the partial derivatives of  $G$  evaluated at the baseline allocation satisfy the first order necessary conditions of the Lagrangian formed from the problem of maximizing  $G$  subject to the resource constraint for the research good. These first order conditions are

$$\frac{\partial G}{\partial y_{rt}(j)} \bar{y}_{rt}(j) = \lambda_t \bar{y}_{rt}(j) \bar{N}_t(j) \quad (40)$$

and

$$\frac{\partial G}{\partial N_t(0)} \bar{N}_t(0) = \lambda_t \bar{y}_r(0) \bar{N}_t(0), \quad (41)$$

where  $\lambda_t$  is the Lagrange multiplier on the resource constraint for the research good (7) at time  $t$ . Plugging these equations into (13) and (14) gives

$$\mathcal{D}_{gt} = \lambda_t \bar{Y}_{rt} \mathcal{D}_{Yrt}. \quad (42)$$

Finally, from (16), we have  $\mathcal{D}_{Yr0} = \bar{L}_p \mathcal{D}_{sr0}$ . Therefore,  $\Gamma_0 \equiv \frac{\mathcal{D}_{g0}}{\mathcal{D}_{sr0}}$  is independent of how the change in the output of the research good is allocated across incumbent and entering firms, which proves the result.

**Proposition 3** This result follows directly from equations (24) and (25) under the assumption that  $\mathcal{D}_{gt}/\mathcal{D}_{Yrt} = \Gamma_0/\bar{L}_p$  and  $\mu'_t = \bar{\mu}$  for all  $t \geq 0$ .

**Lemma 1** To derive expression (28), we use the fact that  $\sum_{k=0}^{\infty} \Gamma_k$  in expression (26) is equal to  $\frac{\bar{L}_p}{1-\phi}$  if that sum converges. With endogenous growth,  $\phi = 1$ , and hence  $\Gamma_k = \Gamma_0$  for all  $k$ . Equation (29) follows from taking the first difference of equation (26). Alternatively, for any transition path such that  $\lim_{t \rightarrow \infty} \mathcal{D}_{Yrt} = 0$ , then by equation (15) the result is immediate.

**Lemma 2** From Proposition (3) we have,

$$\sum_{k=0}^{T-1} \Gamma_k = \frac{\bar{L}_p}{1-\phi} \left[ 1 - \left( 1 - (1-\phi) \frac{\Gamma_0}{\bar{L}_p} \right)^T \right].$$

The result follows from the assumption that  $\Gamma_0^1 < \Gamma_0^2$  and that  $\bar{L}_p$  and  $\phi$  are common across models.

**Corollary 2** We prove this result by taking the log of GDP, which under the old measurement system is equal to the log of  $Y$ :

$$\log Y'_t - \log \bar{Y}_t = \log Z'_t - \log \bar{Z}_t + \alpha (\log K'_t - \log \bar{K}_t) + (1 - \alpha) (\log L'_{pt} - \log \bar{L}_p).$$

Using  $R_{kt} = \frac{\alpha}{\mu} \frac{Y_t}{K_t}$  and equation (12) we obtain the expression given in the statement of the Lemma. To derive the result under the new measure of GDP, we must add in expenditures on research  $P_{rt}Y_{rt}$ , which we can do by multiplying the old measure of GDP by  $(1 + s_{rt})$ .

**Lemma 3** Let the economy be on an initial BGP with the allocation marked by bars. For any alternative feasible allocation, we have that the equivalent variation is defined by

$$\zeta^{1-\eta} \sum_{t=0}^{\infty} \beta^t \bar{C}_t^{1-\eta} = \sum_{t=0}^{\infty} \beta^t C_t'^{1-\eta}.$$

Since on a BGP,  $\bar{C}_t = \exp(t\bar{g}_y)\bar{C}_0$ , we have

$$\zeta^{1-\eta} = (1 - \tilde{\beta}) \sum_{t=0}^{\infty} \tilde{\beta}^t \left( \frac{C'_t}{\bar{C}_t} \right)^{1-\eta},$$

where  $\tilde{\beta} = \beta \exp(\bar{g}_y(1 - \eta))$ . Up to a first-order approximation, around the initial BGP,

$$\log \zeta = (1 - \tilde{\beta}) \sum_{t=0}^{\infty} \tilde{\beta}^t (\log C'_t - \log \bar{C}_t). \quad (43)$$

From the resource constraint of the final good, equation (1), we have, up to a first order approximation,

$$\begin{aligned} \log C'_t - \log \bar{C}_t &= \frac{\bar{Y}_t}{\bar{C}_t} \left[ (\log Z'_t - \log \bar{Z}_t) + (1 - \alpha) (\log L'_{pt} - \log \bar{L}_p) \right] + \\ &\frac{\bar{K}_t}{\bar{C}_t} \left[ \left( \alpha \frac{\bar{Y}_t}{\bar{K}_t} + 1 - d_k \right) (\log K'_t - \log \bar{K}_t) - \exp(\bar{g}_y) (\log K'_{t+1} - \log \bar{K}_{t+1}) \right] \end{aligned}$$

so

$$\begin{aligned} \sum_{t=0}^{\infty} \tilde{\beta}^t (\log C'_t - \log \bar{C}_t) &= \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\bar{Y}_t}{\bar{C}_t} \left[ (\log Z'_t - \log \bar{Z}_t) + (1 - \alpha) (\log L'_{pt} - \log \bar{L}_p) \right] + \\ &\sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\bar{K}_t}{\bar{C}_t} \left[ \left( \alpha \frac{\bar{Y}_t}{\bar{K}_t} + 1 - d_k \right) (\log K'_t - \log \bar{K}_t) - \exp(\bar{g}_y) (\log K'_{t+1} - \log \bar{K}_{t+1}) \right] \end{aligned}$$

Since  $\log K'_0 = \log \bar{K}_0$  and, on any BGP in which the Euler equation for physical capital is undistorted

$$\exp(\bar{g}_y) = \tilde{\beta} \left( \alpha \frac{\bar{Y}_t}{\bar{K}_t} + 1 - d_k \right)$$

we have

$$\sum_{t=0}^{\infty} \tilde{\beta}^t (\log C'_t - \log \bar{C}_t) = \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\bar{Y}_t}{\bar{C}_t} \left[ (\log Z'_t - \log \bar{Z}_t) + (1 - \alpha) (\log L'_{pt} - \log \bar{L}_p) \right].$$

In combination with (43) and  $\bar{Y}_t = G\bar{D}P_t$ , we obtain equation (32). Note that  $\frac{\bar{C}_t}{\bar{Y}_t} = 1 - \frac{\bar{I}_t \bar{K}_t}{\bar{K}_t \bar{Y}_t}$ , with  $\frac{\bar{I}_t}{\bar{K}_t} = \exp(\bar{g}_y) + d_k - 1$  and if the Euler equation for physical capital is undistorted, then  $\frac{\bar{K}_t}{\bar{Y}_t} = \frac{\alpha}{(\bar{R} + d_k)}$  where  $1 + \bar{R} = \frac{\exp(\bar{g}_y)}{\beta}$ .

## B Applying our analytic procedure in standard growth model

Consider a growth model with investment-specific technical change and possible external investment effects, given by

$$Y_t = K_t^\alpha L_t^{1-\alpha} = C_t + q_t I_t$$

$$K_{t+1} = (1 - d_k) K_t + I_t$$

$$q_t = \frac{K_t^\zeta}{A_t}$$

where  $L_t = 1$  and  $\frac{A_{t+1}}{A_t} = \exp(g_A)$ . The parameter  $\zeta \geq 0$  governs the extent to which past investments increase the marginal cost of further investments (an external effect in the decentralized equilibrium). On a BGP, capital and output grow at the rate  $\bar{g}_k = \frac{1}{1+\zeta-\alpha} g_A$  and  $\bar{g}_y = \frac{\alpha}{1+\zeta-\alpha} g_A$ .

Define  $Z_t = K_t^\alpha$  (which on a BGP grows at the rate  $\bar{g}_y$ ),  $Y_{rt} = I_t/K_t$  (which is constant on a BGP), and the investment rate  $s_{rt} = \frac{q_t I_t}{Y_t}$  (also constant on a BGP). We can re-write the dynamics of  $Z_t$  as

$$\frac{Z_{t+1}}{Z_t} = (1 - d_k + Y_{rt})^\alpha$$

where

$$Y_{rt} = \frac{Y_t}{q_t K_t} s_{rt} = A_t K_t^{\alpha-1-\zeta} s_{rt}.$$

The social rate of depreciation (equal to private depreciation in this case) is given by

$$G_0^0 = \alpha \log(1 - d_k).$$

The dynamics of  $Z$  can be approximated by

$$\begin{aligned} g'_{zt} - \bar{g}_z &= \alpha \frac{\bar{Y}_r}{1 - d_k + \bar{Y}_r} (\log Y'_{rt} - \log \bar{Y}_r) \\ &= \alpha \frac{\exp(\bar{g}_z)^{\frac{1}{\alpha}} - \exp(G_0^0)^{\frac{1}{\alpha}}}{\exp(\bar{g}_z)^{\frac{1}{\alpha}}} (\log Y'_{rt} - \log \bar{Y}_r) \end{aligned}$$



and

$$\log Y'_{rt} - \log \bar{Y}_r = \frac{\alpha - 1 - \zeta}{\alpha} (\log Z'_t - \log \bar{Z}_t) + (\log s'_{rt} - \log \bar{s}_r).$$

Defining  $\phi = 1 + \frac{\alpha-1-\zeta}{\alpha}$ ,  $\rho = 1 + \frac{1}{\alpha}$ , and  $L_p = 1$ , we obtain exactly the same expression for the dynamics of aggregate productivity (Proposition 3) as in our baseline model with innovation by entrants only (in that model,  $\bar{Y}_r / (\bar{Y}_r - Y_{r0}^0) = 1$ ). Given  $\bar{g}_z$ , the impact elasticity is pinned down by  $\alpha$  and  $d_k$  which can be measured in standard ways using data on private returns and physical depreciation of physical capital. In contrast, the decay factor depends on the parameter  $\phi$ , which depends on the externality parameter  $\zeta$  which is harder to measure in practice.

## C Model examples

In this appendix we describe five model examples that we refer to in Section 6. In each case we derive the function  $G$ , which has the form in equation (35), and the condition for Assumption 2. We then derive the exact impact elasticity displayed in equations (36) and (37) and show that  $\bar{Y}_r / (\bar{Y}_r - Y_{r0}^0) \leq 1$ . In Appendix D we summarize the equilibrium characterization of these models, and discuss calibrations of these models consistent with the numerical examples in Section 6.

**Example 1: Simple quality ladders model** In the quality ladders model of [Grossman and Helpman \(1991b\)](#) and [Aghion and Howitt \(1992\)](#), incumbent firms each produce one good at a constant markup and are thus indexed by the productivity with which they can produce this good,  $\exp(z)$ . The index of firm productivity has countable support (the ladder) that we will index as  $z(j)$  for  $j = 1, 2, \dots$ . With a fixed measure of intermediate goods of size 1, there is a measure 1 of incumbents,  $\sum_{j \geq 1} N_t(j) = 1$  for all  $t$ . With constant markups across products, aggregate productivity is given by (6).

The innovation technology in this model is as follows. Incumbents do not expend resources on innovation, so  $y_{rt}(j) = 0$  for all  $t$  and  $j \geq 1$ . The measure of entering firms at each date,  $N_t(0)$ , is an endogenous variable, and the resource cost of entry is fixed in terms of the research good as a parameter at  $\bar{y}_r(0)$ . Each entering firm at  $t$  has a probability  $\sigma$  of delivering a successful innovation, in which case it is matched randomly (uniformly) to an existing intermediate good and raises the productivity with which that good can be produced from  $\exp(z(j))$  to  $\exp(z(j+1)) = \exp(z(j) + \Delta_z)$  starting at  $t+1$ . With such a successful innovation on a product produced by a firm of type  $j$ , the firm entering at  $t$  becomes a firm of type  $j+1$  at  $t+1$  and the previous incumbent firm of type  $j$  at  $t$  ceases operating. The total measure of products innovated on is  $\sigma N_t(0)$ , which is assumed to be less than one.<sup>26</sup> Thus the transition law  $T$  for the distribution of incumbent firms

<sup>26</sup>To ensure that this condition is satisfied in a discrete time model, one can simply reduce the length of a time period, scaling  $\sigma$  down appropriately.

by type is given by

$$N_{t+1}(j) = (1 - \sigma N_t(0))N_t(j) + \sigma N_t(0)N_t(j-1)$$

for all dates  $t$  and firm types  $j \geq 2$ . For firms at the bottom of the ladder ( $j = 1$ ), we have  $N_{t+1}(1) = (1 - \sigma N_t(0))N_t(1)$ . Given the definition of aggregate productivity in equation (6), we can write the function  $G$  in equation (10) simply as a function of the measure of entering firms  $N_t(0)$  and parameters:

$$G(\{y_{rt}(j)\}_{j \geq 1}, N_t(0); \{N_t(j)\}_{j \geq 1}) = \frac{1}{\rho - 1} \log \left( \sigma N_t(0) \left( \exp(\Delta_z)^{\rho-1} - 1 \right) + 1 \right). \quad (44)$$

Therefore, in terms of the form of  $G$  in equation (35), we have  $a_0 = 1$ ,  $a_1 = \sigma \left( \exp(\Delta_z)^{\rho-1} - 1 \right)$  and  $F(\cdot) = 0$ . In this model, there is no social depreciation of innovation expenditures, i.e.  $G_t^0 = 0$ .

**Example 2: Simple Quality ladders model with innovation by incumbents** This model is an extension of the simple Quality Ladders model in which both entrant and incumbent firms expend the research good in an effort to innovate, as in [Klette and Kortum \(2004\)](#). Incumbent firms are indexed now by a vector  $j = (n(j), z_1(j), z_2(j), \dots, z_{n(j)}(j))$ , for  $j \geq 1$  indicating the number of products  $n(j)$  for which this firm is the frontier producer and the vector of productivities with which this firm can produce these products. Note that since we have a continuum of measure one of products, then  $\sum_{j \geq 1} n(j)N_t(j) = 1$  for all  $t$  for all feasible allocations. Under the assumption that all firms choose a constant markup on all products that they produce and that this markup is constant over time, aggregate productivity is again given by equation (6).

Incumbent firms of type  $j$  that are the frontier produce of  $n(j)$  products have an innovation technology such that if they expend  $y_{rt}(j)$  units of the research good, they have  $\sigma d(y_{rt}(j) / n(j))n(j)$  innovations in expectation, where  $d(\cdot)$  is an increasing and concave function, with  $d(0) = 0$ . In addition, as in the simple Quality Ladders model, we have that entrants expend  $\bar{y}_r(0)$  units of the research good to have  $\sigma$  innovations in expectation, so that the total measure of product innovated on by entrants is  $\sigma N_t(0)$ . Each innovation is matched randomly (uniformly) to an existing intermediate good (produced by some other firm) raising the frontier productivity for producing that good from  $\exp(z)$  to  $\exp(z + \Delta_z)$ .<sup>27</sup>

The function  $T$  specifying the evolution of the distribution of firms across types implied by these assumptions is somewhat complex (and gives rise to rich dynamics consistent with firm-

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<sup>27</sup>This model does not have a stationary distribution of firm sizes in a balanced growth path unless  $\rho = 1$ . This is because, without this assumption, there is not a stationary distribution of expenditure across products. To ensure a stationary distribution of firm sizes, one can modify the model as follows, without changing its aggregate implications. Assume that at the end of every period  $t$ , after production and innovation occur, a measure  $\chi$  of those products that did not receive an innovation have their frontier technology  $z$  reset to a new level  $z' = \log Z_t$ . This reset frontier technology is still owned by the same incumbent firm. The transition law for  $Z_t$  is not affected by the reset probability  $\chi$ . In the equilibrium, one has to assume that at the same time as this resetting occurs, the technology freely available to other firms who may choose to produce this good is reset to  $\log Z_t - \Delta_l$ .

level data) and is discussed in [Klette and Kortum \(2004\)](#). To solve for the function  $G$ , however, we do not have to specify these dynamics in detail. In particular, because the level of aggregate productivity as specified in equation (6) depends only on the distribution of productivities across the measure one of intermediate goods being produced and because all innovations are matched randomly to existing products, then the growth rate of aggregate productivity is given as a simple function of the total measure of innovations that occur within the period as follows:

$$G = \frac{1}{\rho - 1} \log \left( \sigma \left( \sum_{j \geq 1} d(y_{rt}(j) / n(j)) n(j) N_t(j) + N_t(0) \right) \left( \exp(\Delta_z)^{\rho-1} - 1 \right) + 1 \right). \quad (45)$$

As in the basic quality ladders model, in this model there is no social depreciation of innovation expenditures, i.e.  $G_t^0 = 0$ . We assume that the baseline allocation is such that  $y_{rt}(j) / n(j) = y_r^I$  for all  $j \geq 1$ . This assumption is satisfied in an equilibrium in which all incumbents charge the same markup and face the same innovation subsidy,  $\tau_t(j) = \tau_t$  for all  $j \geq 1$ . In this case, the total measure of products innovated on by incumbents at time  $t$  is  $\sigma d(y_{rt}^I) \sum_{j \geq 1} n(j) N_t(j) = \sigma d(y_{rt}^I)$  and the  $G$  function simplifies to (35), where  $a_0 = 1$ ,  $a_1 = \sigma (\exp(\Delta_z)^{\rho-1} - 1)$  and  $F(y_r^I) = \sigma (\exp(\Delta_z)^{\rho-1} - 1) d(y_r^I)$ , a concave function. Assumption 2 is satisfied if, on the initial BGP,  $d'(\bar{y}_r^I) \bar{y}_r(0) = 1$ , which holds if entrants and incumbents face uniform innovation subsidies on the initial BGP ( $\bar{\tau}(j)$  equal for all  $j$  including entrants).

**Example 3: Simple expanding varieties model** In the three expanding varieties models we consider, and in contrast to the quality ladders model, there is no business stealing: a new product does not replace an existing product, and a lost product is not replaced by another product. We first consider now an expanding varieties model similar to that in [Luttmer \(2007\)](#), in which incumbents produce a single product and are indexed by the productivity with which they can produce it,  $\exp(z)$ . Markups are again constant across firms and over time. Hence, the level of aggregate productivity is given by equation (6).

The productivity of incumbents grows exogenously for those incumbents that survive. Specifically, each incumbent firm has exogenous probability  $\delta_f$  of exiting the market each period. If an incumbent of type  $j \geq 1$ , with productivity  $\exp(z(j))$ , in period  $t$  survives to period  $t + 1$ , then it becomes type  $j + 1$  at that date and has productivity  $\exp(z(j + 1)) = \exp(z(j) + \Delta_z)$ . Incumbents are assumed not to expend resources on innovation, so  $y_{rt}(j) = 0$  for all  $t$  and  $j \geq 1$ . In Example 4 we show how to extend this example to make the productivity growth of incumbents stochastic (which simply changes the form of the coefficient  $a_0$ ) and allow for innovation by incumbents.

We no longer assume that the measure of goods is fixed at one as in the Quality Ladders model. Instead, the measure of goods in production can expand or contract over time due to exogenous exit of incumbents and the entry of firms which produce new goods. Hence aggregate productivity can grow both due to the exogenous productivity growth in surviving incumbent

firms and due to endogenous entry of new firms producing new products.<sup>28</sup> As in the quality ladders model, there is a spillover of knowledge from incumbents to entrants. Specifically, entrants at time  $t$  start production of new goods at time  $t + 1$  with productivity drawn from a lottery with probabilities  $\eta(j; Z_t)$  over types (values of  $z(j)$ ) that satisfies  $\sum_j \exp(z(j))^{(\rho-1)} \eta(j; Z_t) = \sigma Z_t^{\rho-1}$ . Here, the transition law of the distribution of incumbent firms by type is given by

$$N_{t+1}(j) = (1 - \delta_f)N_t(j - 1) + \eta(j; Z_t)N_t(0)$$

for all dates  $t$  and firm types  $j \geq 2$ . For firms with the minimum productivity ( $j = 1$ ), we have  $N_{t+1}(1) = \eta(1; Z_t)N_t(0)$ . In this case we can write the function  $G$  as in (35), where  $a_0 = (1 - \delta_f) \exp((\rho - 1) \Delta_z)$ ,  $a_1 = \sigma$  and  $F(\cdot) = 0$ . The resource constraint for the research good, expression (7), is  $y_{rt}^I + \bar{y}_r(0) N_t(0) = Y_{rt}$ . Therefore, as long as incumbent firms shrink as a group in the sense that  $a_0 < 1$ , there is social depreciation of innovation expenditures (i.e.  $G_0^0 = \frac{1}{\rho-1} \log a_0 < 0$ ). The magnitude of  $a_0$  in this model is directly linked to the model's calibrated value for the employment share of entering firms in the BGP, which is given by

$$\sigma N_t(0) \left( \frac{\bar{Z}_t}{\bar{Z}_{t+1}} \right)^{\rho-1} = \frac{\exp(\bar{g}_z)^{\rho-1} - a_0}{\exp(\bar{g}_z)^{\rho-1}}. \quad (46)$$

**Example 4: Simple Expanding Varieties model with innovation by incumbents to improve their own products** In the simple expanding varieties model of example 3, we assumed that incumbent's productivity grows at an exogenous rate. In this example we endogenize this growth rate, following the model of [Atkeson and Burstein \(2010\)](#). Because  $z$  in this example lies in a discrete grid between  $-\infty$  and  $+\infty$ , we must slightly alter the notation relative to our baseline model. Specifically, the type of incumbent firms,  $j$ , can now take any integer value (including negative values and zero). Incumbents of type  $j$  have productivity  $\exp(j\Delta_z)$ . Entrants are denoted by  $E$  rather than by 0. Each incumbent firm has exogenous probability  $\delta_f$  of exit-

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<sup>28</sup>In this model example, in a BGP with positive entry the measure of goods is constant over time, which ensures that the distribution of firm size is time invariant and aggregate productivity grows due to the exogenous productivity growth of surviving firms. An alternative formulation of the expanding varieties model, as in [Romer \(1990\)](#), features growth in the measure of goods in the BGP. In this model, all firms have the same productivity and the cost of entry falls with past entry. Aggregate productivity is  $Z_{t+1} = M_t^{\frac{1}{\rho-1}}$  where  $M_t = \sum_{j \geq 1} N_t(j)$  denotes the measure of goods at time  $t$ , which evolves according to  $M_{t+1} = (1 - \delta_f) M_t + \sigma \tilde{N}_t(0)$ . The G function is given by  $G = \frac{1}{\rho-1} \log \left( 1 - \delta_f + \sigma \frac{\tilde{N}_t(0)}{M_t} \right)$ . The resource constraint for the research good is given by  $\bar{y}_r(0) \tilde{N}_t(0) = \tilde{Y}_{rt} = A_{rt} Z_t^{\tilde{\phi}} L_{rt}$  where  $\tilde{\phi} \leq \rho - 1$ . In the BGP,  $\tilde{N}_t(0)$  grows over time at the rate  $\frac{\rho-1}{\rho-1-\tilde{\phi}} g_{A_r}$ . With a simple transformation of variables, this alternative formulation can be mapped into our baseline framework. Specifically, define  $N_t(0) = \frac{\tilde{N}_t(0)}{M_t}$ ,  $Y_{rt} = \frac{\tilde{Y}_{rt}}{M_t}$  and  $\phi = 2 + \tilde{\phi} - \rho$ , and we obtain  $G(\cdot) = \frac{1}{\rho-1} \log \left( 1 - \delta_f + \sigma N_t(0) \right)$  and  $\bar{y}_r(0) N_t(0) = Y_{rt} = A_{rt} Z_t^{\phi-1} L_{rt}$ , as in our baseline formulation. We use a similar transformation of variables in Example 5.

ing the market each period.<sup>29</sup> To describe the incumbent's transition across types, consider an incumbent of type  $j$  at  $t$ , with productivity  $\exp(z(j))$ , that expends  $y_{rt}(j)$  units of the research good and survives to period  $t + 1$ . At that date, with probability  $d_0 + d \left( y_{rt}(j) \left( \frac{Z_t}{\exp(z(j))} \right)^{\rho-1} \right)$  it becomes a firm of type  $j + 1$  (with productivity  $\exp(z(j + 1)) = \exp(z(j) + \Delta_z)$ ) and with probability  $1 - d_0 - d \left( y_{rt}(j) \left( \frac{Z_t}{\exp(z(j))} \right)^{\rho-1} \right)$  it becomes a firm of type  $j - 1$  (with productivity  $\exp(z(j - 1)) = \exp(z(j) - \Delta_z)$ ). The function  $d(\cdot)$  is increasing and concave, and  $d(0) = 0$ . The parameter  $d_0$  controls the drift in productivity for a firm that does not invest. As in the simple expanding varieties model, there is a spillover of knowledge from incumbents to entrants, such that entrants at time  $t$  start production at time  $t + 1$  with productivity drawn from a lottery with probabilities  $\eta_t(j)$  over types (values of  $z(j)$ ) that satisfies  $\sum_j \exp(z(j))^{\rho-1} \eta_t(j) = \sigma Z_t^{\rho-1}$ . Hence, the transition law of the distribution of incumbent firms by type is given by

$$\begin{aligned} N_{t+1}(j) &= (1 - \delta_f) \left[ d_0 + d \left( y_{rt}(j-1) \left( \frac{Z_t}{\exp(z(j-1))} \right)^{\rho-1} \right) \right] N_t(j-1) \\ &+ (1 - \delta_f) \left[ 1 - \left[ d_0 + d \left( y_{rt}(j+1) \left( \frac{Z_t}{\exp(z(j+1))} \right)^{\rho-1} \right) \right] \right] N_t(j+1) + \eta_t(j) N_t^E \end{aligned}$$

for all firm types  $j$ , where  $N_t^E$  denotes the measure of entering firms. We assume that the baseline allocation is such that  $y_{rt}(j) \left( \frac{Z_t}{\exp(z(j))} \right)^{\rho-1} = y_{rt}^l$  for all  $j \geq 1$ . This assumption is satisfied in an equilibrium in which all incumbents charge the same markup and face the same innovation subsidy,  $\tau_t(j) = \tau_t$  for all  $j \geq 1$ . In this case we can write the function  $G$  in equation (10) as in (35) with  $N_t^E$  in place of  $N_t(0)$  and where  $a_0 = (1 - \delta_f) \left( d_0 \exp(\Delta_z)^{\rho-1} + (1 - d_0) \exp(-\Delta_z)^{\rho-1} \right)$ ,  $a_1 = \sigma$ , and  $F(y_r^l) = (1 - \delta_f) \left( \exp(\Delta_z)^{\rho-1} - \exp(-\Delta_z)^{\rho-1} \right) d(y_r^l)$  is a concave function. The resource constraint for the research good, expression (7), is  $y_{rt}^l + \bar{y}_r^E N_t^E = Y_{rt}$ . In this model, the extent of social depreciation depends both on the exit rate  $\delta_f$  and on the expected productivity growth of incumbents if they do not expend any resources on innovation. Assumption 2 is satisfied if, on the initial BGP,

$$(1 - \delta_f) d'(y_{rt}^l) \left( \exp(\Delta_z)^{\rho-1} - \exp(-\Delta_z)^{\rho-1} \right) \bar{y}_r^E = \sigma, \quad (47)$$

which holds if entrants and incumbents face uniform innovation subsidies on the initial BGP ( $\bar{\tau}(j)$  equal for all  $j$  and entrants). The employment share of entrants is given by  $\sigma N_t^E \left( \frac{Z_t}{Z_{t+1}} \right)^{\rho-1}$ .

**Example 5: Simple Expanding Varieties model with innovation by incumbents to create new products** We now describe a discrete time version of an expanding varieties model with innovation by incumbent firms to create new products based on [Luttmer \(2011\)](#). Here we consider the first model in [Luttmer \(2011\)](#) in which there is a single innovation technology for incumbents. In Appendix [G](#) we consider the case of heterogeneous innovation technologies. In

<sup>29</sup>It is straightforward to extend the model to let firms' investment in the research good reduce  $\delta_f$ .

order to fit this model example into our framework, we alter our notation slightly.

All products are produced with the same productivity  $\exp(z) = \exp(\theta t)$ , which grows exogenously at the rate  $\theta$  — we omit firm productivity as a firm type and index incumbent firms only by the number of products that they produce,  $n(j)$ . The total number of products is given by  $M_t = \sum_{j \geq 1} n(j)N_t(j)$ . With constant markups across products, aggregate productivity at  $t$  is given by  $Z_t = \exp(\theta t) M_t^{\frac{1}{\rho-1}}$ .

The evolution of the distribution of firms by type is as follows. Incumbent firms invest units of the research good per product that they own to increase the number of products that they produce and they invest units of the research good per product to slow down the depreciation (loss) of products that they own. As in examples 3 and 4, and in contrast to quality ladders model, there is no business stealing. The function  $T$  specifying the evolution of the distribution of firms across types depends on the stochastic process through which firms increase and lose products over time given their investment. To solve for the function  $G$ , however, we do not have to specify these dynamics in detail. We can summarize this evolution by saying that if type  $j \geq 1$  firms each invests  $y_{rt}(j)$  units of the research good, then the expected number of products that this group of firms will own next period is  $[d_0 + d(y_{rt}(j)/n(j))] n(j)N_t(j)$ , where  $d(\cdot)$  is increasing and concave,  $d(0) = 0$ , and  $d_0 \geq 0$ . We assume that the baseline allocation is such that all incumbent firms invest the same units of the research good per product,  $y_{rt}^I$ , so  $y_{rt}(j) = y_{rt}^I n(j)$ . This assumption is satisfied in an equilibrium in which all incumbents face the same innovation subsidy,  $\tau_t(j) = \tau_t$  for all  $j \geq 1$ . A measure  $\tilde{N}_t(0)$  of entrants expends  $\bar{y}_r(0)$  units of the research good each to create  $\sigma$  new products in expectation. We then have the evolution of the total number of products is given by

$$M_{t+1} = \sum_{j \geq 1} [d_0 + d(y_{rt}^I)] n(j)N_t(j) + \sigma \tilde{N}_t(0) = [d_0 + d(y_{rt}^I)] M_t + \sigma \tilde{N}_t(0).$$

The resource constraint for the research good is given by  $y_{rt}^I M_t + \bar{y}_r(0) \tilde{N}_t(0) = \tilde{Y}_{rt} = \tilde{A}_{rt} Z_t^{\tilde{\phi}} L_{rt}$  where  $\tilde{\phi} \leq \rho - 1$  and  $\tilde{A}_{rt}$  grows at the rate  $g_{\tilde{A}_r}$ .

In order to map this example to our framework (since this example features BGP growth in  $\tilde{Y}_{rt}$ ), we perform a simple transformation of variables:  $N_t(0) = \frac{\tilde{N}_t(0)}{M_t}$ ,  $Y_{rt} = \frac{\tilde{Y}_{rt}}{M_t}$ ,  $A_{rt} = \exp(\theta(\rho - 1)t) \tilde{A}_{rt}$ ,  $\phi = 2 + \tilde{\phi} - \rho$  and  $g_{A_r} = g_{\tilde{A}_r} + \theta(\rho - 1)$ . The BGP growth rate of aggregate productivity is<sup>30</sup>

$$\bar{g}_z = \frac{g_{A_r}}{1 - \phi} = \frac{g_{\tilde{A}_r} + \theta(\rho - 1)}{\rho - \tilde{\phi} - 1}.$$

The function  $G$  in equation (10) is

$$G = \theta + \frac{1}{\rho - 1} \log \left( d_0 + d(y_{rt}^I) + \sigma N_t(0) \right)$$

<sup>30</sup>Luttmer (2011) considers the case with  $\tilde{\phi} = g_{\tilde{A}_r} = 0$  and includes population growth to keep the firm size distribution unchanged over time.



and the resource constraint for the research good, expression (7), is  $y_{rt}^I + \bar{y}_r(0)N_t(0) = Y_{rt}$ . Therefore, in terms of the form of  $G$  in equation (35), we have  $a_0 = \exp(\theta)^{\rho-1} d_0$ ,  $a_1 = \exp(\theta)^{\rho-1} \sigma$  and  $F(y_r^I) = \exp(\theta)^{\rho-1} d(y_r^I)$  is a concave function. Assumption 2 is satisfied if, on the initial BGP,  $d'(\bar{y}_{rt}^I) \bar{y}_r(0) = 1$ , which holds if entrants and incumbents face uniform innovation subsidies on the initial BGP ( $\bar{\tau}(j)$  equal for all  $j$  including entrants). The employment share of entrants is given by  $\sigma N_t(0) \frac{M_t}{M_{t+1}} = \sigma N_t(0) \exp(\theta)^{\rho-1} \left(\frac{Z_t}{Z_{t+1}}\right)^{\rho-1}$ .

### Verifying Assumption 3

In order to verify Assumption 3 in our model examples, we must calculate  $\mathcal{D}_{gt}$  and  $\mathcal{D}_{Y_{rt}}$  defined in equations (13) and (14). Note that from the form of  $G$  in our examples, we have  $\sum_{j \geq 1} \frac{\partial G}{\partial N_t(j)} (N_t'(j) - \bar{N}_t(j)) = 0$  and  $\sum_{j \geq 1} \bar{y}_{rt}(j) (N_t'(j) - \bar{N}_t(j)) = 0$ , so the two expressions simplify to

$$\mathcal{D}_{gt} = \frac{\partial G}{\partial y_{rt}^I} (y_{rt}^{I'} - \bar{y}_{rt}^I) + \frac{\partial G}{\partial N_t(0)} (N_t'(0) - \bar{N}_t(0)).$$

$$\mathcal{D}_{Y_{rt}} = \frac{1}{\bar{Y}_{rt}} \left( (y_{rt}^{I'} - \bar{y}_{rt}^I) + \bar{y}_r(0) (N_t'(0) - \bar{N}_t(0)) \right)$$

In the BGP,  $\bar{N}_t(0)$  and  $\bar{y}_{rt}^I$  are constant over time, and hence so are  $\bar{Y}_{rt}$ ,  $\frac{\partial G}{\partial y_{rt}^I}$  and  $\frac{\partial G}{\partial N_t(0)}$ . However, this does not imply Assumption 3 since  $y_{rt}^{I'}$  and  $N_t'(0)$  need not be constant over time. Note that Assumption 2 is satisfied when in the BGP,  $\frac{\partial G}{\partial y_{rt}^I} = \frac{1}{\bar{y}_r(0)} \frac{\partial G}{\partial N_t(0)}$ , in which case

$$\mathcal{D}_{gt} = \frac{1}{\bar{y}_r(0)} \frac{\partial G}{\partial N_t(0)} \left( (y_{rt}^{I'} - \bar{y}_{rt}^I) + \bar{y}_r(0) (N_t'(0) - \bar{N}_t(0)) \right)$$

so that

$$\frac{\mathcal{D}_{gt}}{\mathcal{D}_{Y_{rt}}} = \frac{\bar{Y}_{rt}}{\bar{y}_r(0)} \frac{\partial G}{\partial N_t(0)},$$

which is constant.

**Exact impact elasticity** To derive the exact impact elasticity  $\Gamma_0$  we consider a form for  $G$  that is more general than the one in equation (35),

$$G = \frac{1}{\rho-1} \log(a_0 + a_1 N_t(0) + F(\{y_{rt}(j)\}_{j \geq 1}; \{N_t(j)\}_{j \geq 1})) \quad (48)$$

with  $F(\{0\}_{j \geq 1}; \{N_t(j)\}_{j \geq 1}) = 0$ . The social rate of depreciation of innovation expenditures is given by  $G_0^0 = \frac{1}{\rho-1} \log a_0$ . If assumption 2 is satisfied, we can compute the impact elasticity by varying only entry  $N_t(0)$  as we vary the research intensity of the economy  $s_{rt}$  and total output of the research good  $Y_{rt}$ . The steps in equation (37) can be applied under this more general form of  $G$  and we arrive at exactly the same expression for  $\Gamma_0$ .



We now show the equivalence of  $\Gamma_0$  in equations (36) and (37).  $N_t^0(0)$ , defined in equation (19), is given by the following expression:

$$a_0 + a_1 N_t^0(0) + F(\{\bar{y}_{rt}(j)\}_{j \geq 1}; \{\bar{N}_t(j)\}_{j \geq 1}) = \exp(G_0^0)^{\rho-1}. \quad (49)$$

$Y_{rt}^0$ , defined in (20) as  $Y_{rt}^0 = \bar{Y}_r + \bar{y}_r(0)(N_t^0(0) - \bar{N}_t)$ , can be expressed using (49) as

$$\bar{Y}_r - Y_{rt}^0 = \frac{\bar{y}_r(0)}{a_1} \left( \exp(\bar{g}_z)^{\rho-1} - \exp(G_0^0)^{\rho-1} \right). \quad (50)$$

Substituting  $a_1/\bar{y}_r(0)$  into equation (37) we obtain the alternative expression for  $\Gamma_0$  displayed in (36).

We now show that if A2 holds and  $F(\{y_{rt}(j)\}_{j \geq 1}; \{N_t(j)\}_{j \geq 1})$  is concave with respect to a proportional increase in all incumbents' innovation, then  $\bar{Y}_r / (\bar{Y}_r - Y_{rt}^0) \leq 1$ . Using equations (7), (49), and (50),

$$\begin{aligned} Y_{rt}^0 &= \bar{Y}_r + \frac{\bar{y}_r(0)}{a_1} \left( \exp(G_0^0)^{\rho-1} - \exp(\bar{g}_z)^{\rho-1} \right) \\ &= \frac{\bar{y}_r(0)}{a_1} \left[ F(\{0\}_{j \geq 1}; \{\bar{N}_t(j)\}_{j \geq 1}) - F(\{\bar{y}_{rt}(j)\}_{j \geq 1}; \{\bar{N}_t(j)\}_{j \geq 1}) \right] + \sum_{j \geq 1} \bar{y}_{rt}(j) \bar{N}_t(j) \\ &= \sum_{j \geq 1} \bar{y}_{rt}(j) \bar{N}_t(j) \left[ 1 - \frac{\bar{y}_r(0)}{a_1} \frac{F(\{\bar{y}_{rt}(j)\}_{j \geq 1}; \{\bar{N}_t(j)\}_{j \geq 1}) - F(\{0\}_{j \geq 1}; \{\bar{N}_t(j)\}_{j \geq 1})}{\sum_{j \geq 1} \bar{y}_{rt}(j) \bar{N}_t(j)} \right]. \end{aligned}$$

A2 implies that  $a_1/\bar{y}_r(0) = \frac{\partial F(\{y_{rt}(j)\}_{j \geq 1}; \{N_t(j)\}_{j \geq 1})}{\partial y_{rt}(j)}$  for  $j \geq 1$  where derivatives are evaluated on the initial BGP, which implies

$$\frac{a_1}{\bar{y}_r(0)} = \frac{\sum_{j \geq 1} \frac{\partial F(\{y_{rt}(j)\}_{j \geq 1}; \{N_t(j)\}_{j \geq 1})}{\partial y_{rt}(j)} \bar{y}_{rt}(j) \bar{N}_t(j)}{\sum_{j \geq 1} \bar{y}_{rt}(j) \bar{N}_t(j)}.$$

Concavity of  $F$  with respect to a proportional increase in all  $y_{rt}(j)$ 's implies

$$\sum_{j \geq 1} \frac{\partial F(\{y_{rt}(j)\}_{j \geq 1}; \{N_t(j)\}_{j \geq 1})}{\partial y_{rt}(j)} \bar{y}_{rt}(j) \bar{N}_t(j) \leq F(\{\bar{y}_{rt}(j)\}_{j \geq 1}; \{\bar{N}_t(j)\}_{j \geq 1}) - F(\{0\}_{j \geq 1}; \{\bar{N}_t(j)\}_{j \geq 1})$$

so  $Y_{rt}^0 \leq 0$  and  $\frac{\bar{Y}_r}{\bar{Y}_r - Y_{rt}^0} \leq 1$ .

**Appendices D, E, F and G are available online at:**

[http://www.econ.ucla.edu/arielb/innovationpolicy\\_online.pdf](http://www.econ.ucla.edu/arielb/innovationpolicy_online.pdf)