

NBER WORKING PAPER SERIES

GROWTH AND EXTERNAL DEBT UNDER
RISK OF DEBT REPUDIATION

Daniel Cohen

Jeffrey Sachs

Working Paper No. 1703

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 1985

Presented at the International Seminar on Macroeconomics, Chateau de Ragny, France, June 1985. The authors would like to thank Mr. Atish Ghosh, who designed and implemented the numerical dynamic programming algorithm used in section 4 of the paper. The research reported here is part of the NBER's research program in International Studies. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

Growth and External Debt Under
Risk of Debt Repudiation

ABSTRACT

We analyze the pattern of growth of a nation which borrows abroad and which has the option of repudiating its foreign debt. We show that the equilibrium strategy of competitive lenders is to make the growth of the foreign debt contingent on the growth of the borrowing country. We give a closed-form solution to a linear version of our model. The economy, in that case, follows a two-stage pattern of growth. During the first stage, the debt grows more rapidly than the economy. During the second stage, both the debt and the economy grow at the same rate, and more slowly than in the first stage. During this second stage, the total interest falling due on the debt is never entirely repaid; only an amount proportional to the difference of the rate of interest and the rate of growth of the economy is repaid each period.

Daniel Cohen
CEPREMAP, Paris
140 rue du Chevaleret
Paris 75013
France

Jeffrey Sachs
National Bureau of
Economic Research
1050 Massachusetts Avenue
Cambridge, MA 02138
(617) 868-3900

1. Introduction

When borrowers have an option to repudiate their debts, the interactions of borrowers and lenders over time presents a strategic situation of enormous complexity. Consider the case of a sovereign borrowing country, raising loans from the world capital market. Following Eaton and Gersovitz (1981), suppose that repudiation of the debt results in financial autarky and a loss of productive efficiency of the defaulting country. An indebted country must balance these costs with the direct costs of debt repayment, in considering the option of repudiation. In turn, credit rationing will emerge from the lenders' decision to limit their exposure to a level low enough to render debt repudiation an inferior option to the borrowing country.

It turns out that the lenders' strategy is not easy to compute. The simple maxim is to lend freely to the point where the country is indifferent between debt repayment and repudiation. But what point is that? One of the incentives for debt repayment is the option to borrow again in the future. But the value of that option depends on how much lenders will lend in the future. That in turn will depend on the future lenders' assessment of the point of indifference of the country between repayment and repudiation, which in turn will depend on the capacity of the country to borrow in the still further distant future. Thus we have a problem of infinite regress, and the method of dynamic programming is needed to solve the lenders' problem.

In this paper, we examine a growth model in which sufficiently many linearities allow us to give a closed form solution to the lender's and borrower's strategies. We also describe an algorithm and a suitable numerical

technique to solve for borrower and lender behavior in more complex, non-linear settings. In the analytical version of the model, the borrowing country goes through two borrowing stages: (1) a stage with unconstrained borrowing and a rising external-debt-to-GDP ratio, with the rate of GDP growth falling progressively; and (2) a stage with a constant, low rate of GDP growth and constrained (rationed) borrowing. During the second stage (which always occurs after a finite time), the country never repays the full amount of interest falling due but only the fraction which is necessary to make the debt grow at the same rate as the GDP of the country.

Section 2 sets out the model. Section 3 analyzes the optimal lending strategy when default is an available alternative to the borrower, and presents analytical results in the linear version of the model. Section 4 briefly discusses the use of numerical techniques for more general models. Finally, section 5 examines the sensitivity of the equilibrium to alternative lending strategies. A conclusion summarizes the paper. All technical derivations are presented in four appendixes.

2. The Model

We study a discrete-time growth model of a small country which has access to an international capital market. We assume that the consumption, investment and borrowing decisions are made by a social planner maximizing an intertemporal utility function. The numeraire (both national and international) is a good which serves for both consumption and investment. The international rate of interest is assumed to be a constant, r . In this paper we do not deal with

the important distinction between traded and non-traded goods. This will enable us to use GDP as a measure of the potential foreign-exchange revenues of the country. In a more general framework, one should be careful to use the production of traded goods as the measure of the revenues which enable the country to repay its debt (see Cooper and Sachs (1984) and Dornbusch (1983)).

a. Technology and Capital Installation

The technology available to the country is described by a linear production function with only one input, capital:

$$(1) \quad Q_t = aK_t$$

Q_t is GDP at time t , and K_t installed capital. (In section 4, we briefly examine the case of more general technologies.) Installed capital depreciates at rate d , so that the net increment to installed capital is:

$$(2) \quad K_{t+1} = I_t + (1-d)K_t$$

where I_t is the flow of newly installed capital. If there were no installation costs, a social planner able to borrow at a world interest rate r would choose either to invest or disinvest at an infinite rate depending on whether r is less than or greater than $a - d$. In what follows, we shall assume the inequality

$$(3) \quad r < a - d$$

so that the planner would choose to invest at an infinite rate. In fact, we shall also assume that installation of capital is costly. Following Abel and

Hayashi, we let J_t , total investment expenditure, and I_t , the flow of new installed capital, be related by the functional form

$$(4) \quad J_t = [1 + (1/2)\phi(I_t/K_t)]I_t$$

The unit cost to install capital is therefore $(1/2)\phi(I_t/K_t)$, which increases with the rate of capital accumulation. With this formulation, the planner will always choose to invest at a finite rate.

b. The Wealth of the Country

We shall define the productive wealth W of the country as the discounted value of current and future production net of the investment expenditure, using the world interest rate as the discount factor:

$$(5) \quad W_t = \sum_{i=t}^{\infty} (1+r)^{-(i-t)} (aK_i - J_i)$$

It is easy to show (see Appendix 2, equations (A2.5) to (A2.10)) that the wealth so defined is maximized by selecting a fixed rate of capital accumulation

$$(6) \quad x^* = I_t/K_t$$

Capital stock growth is then given by:

$$(7) \quad K_{t+1}/K_t - 1 = g^* \equiv x^* - d$$

From now on, we shall assume that the value of g^* satisfies $g^* < r$ so that the infinite sum in (5) is meaningful. We show in Appendix 2 (A2.9) that the value x^* maximizing W_t and satisfying (3) is

$$(8) \quad x^* = (r+d)[1 - \sqrt{1 - (2/\phi)(a-d-r)/(r+d)}]$$

This value of x^* exists under the condition

$$(9) \quad \phi > 2(a-d-r)/(r+d)$$

(9) says that installation cost must be sufficiently large so as to make the maximum rate of capital accumulation lower than the rate of interest. We henceforth assume that (9) holds.

c. Optimizing Behavior of the Country

We shall assume that the country is governed by a social planner maximizing a time-separable utility function of the form

$$(10) \quad U = \sum_{t=0}^{\infty} \beta^t \log C_t$$

The choice of logarithmic utility helps to preserve key linearities necessary for a closed-form solution.

When the constraint on foreign borrowing is not binding, the optimal consumption path will satisfy

$$(11) \quad C_{t+1}/C_t = \beta(1+r)$$

From now on we shall assume

$$(12) \quad \beta(1+r) < 1 + g^*$$

which guarantees that the planner discounts the future sufficiently to desire to be a net borrower.

d. Foreign Borrowing

The international capital market is assumed to be competitive, and to supply loans at a fixed interest rate r (though in a rationed quantity, yet to

be determined). All loans are for one period, with D_t denoting the principal repayments due at time t , and rD_t the interest payments due at time t . The balance of payments equation is:

$$(13) \quad D_{t+1} = (1+r)D_t + C_t + I_t[1 + (\phi/2)(I_t/K_t)] - Q_t$$

In general, as described in Section 3, new borrowing will be limited by a constraint of the form $D_{t+1} \leq h(K_{t+1})$. The goal is to characterize the $h(K_{t+1})$ function, and to study optimal borrowing under that constraint.

d. The Threat of Repudiation

We shall assume that the planner has the option of repudiating the country's external debt. If it does so, it suffers two penalties. First, the country permanently loses its access to international capital markets so that it is forced into financial autarky. Second, there is a direct penalty on production following debt repudiation (due, for example, to a loss of efficiency following increased difficulties in foreign trade) so that the productivity of capital becomes $a(1-\lambda)$ for some $\lambda \in [0,1]$. GDP hence becomes

$$(14) \quad Q_t = a(1-\lambda)K_t$$

Given these constraints, a defaulting country can reach an autarkic intertemporal utility level, which we name $V^D(K_t)$. Note that this utility level is a function of the stock of capital in place when the decision to default is taken. We show in Appendix 1, equation (A1.8), that $V^D(K_t)$ is of the form:

$$(15) \quad V^D(K_t) = \log(K_t)/(1-\beta) + \text{Constant}$$

The constant term is a decreasing function of λ . We shall assume full information of both creditors and debtors of the value of λ , as well as of the level of the capital stock. Given this absence of uncertainty, default will never happen: the lenders will select a supply of credit such that the borrower will never find it profitable to repudiate its debt. The main purpose of Section 3 will be to find this loan supply and to derive its consequences for the growth of the country.

3. Equilibrium with Repudiation

We now examine the consequences of a threat of debt repudiation on the strategies available to the lenders, and on the rate of growth of the borrowing economy. At each point of time, the country has built up both external debt and productive capital. In order to decide whether to default or to repay the debt coming due, the country has to compare the autarkic utility level $V^D(K)$, which is a function of installed capital only, with the utility it can derive by servicing the debt and at least postponing the decision to default. The subtlety of the problem is that the country's decision depends in part on the future credit lines it expects from the lenders, who must relate their lending decisions to the country's present and future choices. Technically, the solution must be obtained by backward recursion. Given the constraints on lending that the country is expected to face in the future, present lenders must keep their exposure to a level which keeps the country from defaulting on its current debt.

Solving the problem in its full generality is a hard task and

simplifications are needed to find a closed form solution. An example is solved in Eaton and Gersowitz by assuming a fixed cyclical pattern to the country's output, and no capital accumulation. In our previous work (1984, 1985), a three-period horizon allowed us to solve a model with uncertainty via backward recursion. In the model here, the many linearities which we imposed on consumer tastes and on production technology enable us to give a closed-form solution to the problem in an infinite-horizon model with capital accumulation. Before we solve our specific model, we first set the problem in its general form.

a. Finitely-Lived Economy

In order to show how backward recursion is used to find the optimal decision rule of the lenders, we first assume that the horizon of the country ends at some finite time T .

Let us assume that the country has not defaulted before the last period and that it has accumulated by then a capital stock K_T and an external debt D_T (with debt D_T , the country owes $(1+r)D_t$ in the last period). During the last period, the decision to repay or to default is straightforward since the country has only to compare the costs and benefits of defaulting in one period. We define (following Eaton and Gersowitz (1981))

$$(16) \quad V_T(K_T, D_T) = \max\{V_T^R(K_T, D_T), V_T^D(K_T)\}$$

with

$$(17) \quad V_T^R(K_T, D_T) = \log[aK_T - (1+r)D_T]$$

and

$$(18) \quad V_T^D(K_T, D_T) = \log[(1-\lambda)aK_T]$$

where V_T^R denotes the utility that the country can reach by repaying its debt at

time T , and V_T^D the utility it can reach by defaulting at T . V_T is the maximum of V_T^R and V_T^D , and is thus the maximum utility achievable at time T . We assume that when repudiation and repayment yield exactly the same utility (i.e. $V^D = V^R$), the country repays the debt. Let us call $h_T(\)$ the lending limit which keeps the country from defaulting at T . h_T is defined implicitly by the condition:

$$D_T \leq h_T(K_T) \Leftrightarrow V_T^R(K_T, D_T) \geq V_T^D(K_T)$$

From (17) and (18), h_T is simply:

$$(19) \quad h_T(K_T) = \lambda a K_T / (1+r)$$

Thus, the country will repay D_T if and only if $D_T \leq h_T(K_T) = \lambda a K_T / (1+r)$.

Let us now analyze the problem at time $T-1$. We define $V_{T-1}^R(D_{T-1}, K_{T-1})$ as

$$(20) \quad V_{T-1}^R(D_{T-1}, K_{T-1}) = \max_{C_{T-1}, I_{T-1}, D_T} \{ \log C_T + \beta V_T(K_T, D_T) \}$$

subject to

$$(2) \quad K_T = I_{T-1} + (1-d)K_{T-1}$$

$$(4) \quad J_T = [1 + (1/2)\phi(I_{T-1}/K_{T-1})]I_{T-1}$$

$$(13) \quad D_T = (1+r)D_{T-1} + C_{T-1} + I_{T-1}[1 + (\phi/2)(I_{T-1}/K_{T-1})] - Q_{T-1}$$

$$\text{and } D_T \leq h_T(K_T)$$

$V_{T-1}^R(K_{T-1}, D_{T-1})$ is the utility that the country can reach by repaying the debt, D_{T-1} , when it faces a new supply of credit which is constrained by $D_T \leq h_T(K_T)$.

Let us finally define:

$$(21) \quad V_{T-1}(D_{T-1}, K_{T-1}) = \max \{ V_{T-1}^R(K_{T-1}, D_{T-1}), V_{T-1}^D(K_{T-1}) \}$$

with $V_{T-1}^D(K_{T-1})$ the utility level that the country can reach by defaulting on the debt D_{T-1} . V_{T-1} is the best outcome that the country can reach at time t given its ability to repudiate the debt. We now define $h_{T-1}(\cdot)$ as the rule which will keep the country from defaulting at time $T-1$. We want:

$$(22) \quad D_{T-1} \leq h_{T-1}(K_{T-1}) \Leftrightarrow V_{T-1}^R(K_{T-1}, D_{T-1}) \geq V_{T-1}^D(K_{T-1})$$

By backward recursion, we can clearly define a sequence of lending rules $\{h_T(\cdot), h_{T-1}(\cdot), \dots, h_0(\cdot)\}$ such that (22) holds for all $t \geq 1$, and V_t^R , V_t^D and V_t defined analogously to the expressions in (20) and (21). Equation (22), when generalized for all t , is the condition for the country to repay its debt D_t falling due at time t when it expects at time t, \dots, T , the lending rules h_t, \dots, h_T to apply.

It is interesting to consider the case when $\lambda = 1$, i.e. when all output is lost in the event of default. In this case, a default at time t results in $C_{t+i} = 0$ for all $i \geq 0$. Thus, $\lambda = 1$ is tantamount to assuming that the country has no option to repudiate. It is easy to see that the planner will always choose to service the debt as long as it is feasible to pay $(1+r)D_t$ out of net exports at time t plus new borrowing. At $T-1$, therefore, lenders limit loans D_T such that $(1+r)D_T \leq aK_T$ (clearly, for $(1+r)D_t > aK_t$, repayment in period T is not feasible, as there is no lending in $T+1$). At $T-2$, lenders require that $(1+r)D_{T-1} \leq \max(aK_{T-1} - J_{T-1} + D_T)$, such that $D_T \leq aK_T/(1+r)$. Thus, $(1+r)D_{T-1} \leq \max[aK_{T-1} - J_{T-1} + aK_T/(1+r)]$. In other words, $(1+r)D_{T-1}$ will be kept less than or equal to the maximum productive wealth at time $T-1$. By induction, it is easy to prove this result for all $t \leq T$:

$$(23) \quad (1+r)D_t \leq \max \sum_{i=t}^T (aK_i - J_i)(1+r)^{-(i-t)}$$

subject to (2), (4)

or, using earlier notation:

$$(24) \quad (1+r)D_t \leq \max W_t$$

To sum up, when $\lambda = 1$, borrowing is limited only by the maximum productive wealth of the economy.

b. A Digression on the Precommitment of Capital Installation.

In the maximization problem which we set in (20), we implicitly assumed that the investment and borrowing decisions are made simultaneously, since the lenders at time t can condition their loans on the investment decision of the country. The loan at time t , D_{t+1} , is therefore made a function of the capital stock that will be in place at time $t+1$. If the lending decision must be made before I_t is observed, there is a moral hazard problem. The country, even if it wishes to precommit itself to investing a certain amount I_t in light of the borrowing constraint $D_{t+1} \leq h_{t+1}(K_{t+1})$, may find it profitable not to do so after after the loan has been granted. In such situations, the lending strategy by the banks should be written

$$D_t \leq h_t(K_{t-1})$$

and the borrower will typically reach a lower level of utility because of this lagged relationship. In all that follows, we shall ignore this problem by assuming that the loan can be predicated on the investment decision each period. We refer the reader to our previous work (1984) for further discussion.

c. Infinite-Horizon Economies

When one lets the time horizon of the economy go to infinity, the definition of the lending rule remains the same except that the time subscript may be dropped, since $h_t(K_t)$ will be the same for all t . An equilibrium will be defined by the four functions $V^D(K_t)$, $V^R(K_t, D_t)$, $V(K_t, D_t)$ and $h(K_t)$, such that:

$$(25) \quad V^R(D_t, K_t; h(\cdot)) = \max_{C_t, I_t, D_{t+1}} [\log C_t + \beta V(D_{t+1}, K_{t+1}; h(\cdot))]$$

subject to (2), (4), (13) and $D_{t+1} \leq h(K_{t+1})$

$$V(D_t, K_t) = \max\{V^R(\cdot); V^D(\cdot)\}$$

$$D_t \leq h(K_t) \Leftrightarrow V^R(K_t, D_t; h(\cdot)) \geq V^D(K_t)$$

$h(\cdot)$ is therefore a stationary rule such that if future lenders are to apply it as a criterion for future lending, it will also be used as a criterion for current lending.

Before proceeding to the specific solution for V^D , V^R , and h in the case of $\lambda < 1$, it is useful to make one small amendment to (25). Though V^R equals $\log C_t + \beta V_{t+1}$, and $V_{t+1} = \max(V_{t+1}^R, V_{t+1}^D)$, the lending constraint $D_{t+1} \leq h(K_{t+1})$ guarantees that $V_{t+1}^R \geq V_{t+1}^D$ as long as the debt is repaid at time t . In other words, as long as the debt is repaid once, borrowing constraints will guarantee that all future debts will also be repaid (this is true only in a model without uncertainty). Therefore, V^R in (25) will also satisfy:

$$(25') \quad V^R(D_t, K_t) = \max[\log C_t + \beta V^R(D_{t+1}, K_{t+1})]$$

where V^R replaces V on the right hand side.

One case of (25) is easily solved: for $\lambda = 1$. As noted earlier, $(1+r)D_t$ will always be repaid as long as repayment is feasible out of net

exports $aK_t - J_t$ and new borrowing D_{t+1} , i.e. as long as $(1+r)D_t \leq \max(aK_t - J_t + D_{t+1})$. Of course, D_{t+1} is limited by $D_{t+1} \leq h(K_{t+1})$. Thus, the borrowing constraint is defined by a stationary h function, such that $h(K_t) = [\max(aK_t - J_t + D_{t+1})]/(1+r)$, with $D_{t+1} \leq h(K_{t+1})$. Upon substitution, for D_{t+1} we have $h(K_t) = \max[aK_t - J_t + h(K_{t+1})]/(1+r)$. The solution to this functional equation is:

$$(26) \quad h(K_t) = \max[\sum_{i=t}^{\infty} (1+r)^{-(i-t)} (aK_i - J_i)]/(1+r) = \max W_t/(1+r)$$

Thus, as in the case of finite T , the borrowing constraint when $\lambda = 1$ is simply that $D_t(1+r)$ must be less than or equal to the maximum value of national productive wealth.

For $\lambda = 1$, the problem in (24) reduces to the standard borrowing problem without repudiation risk:

$$(27) \quad \max \sum_{t=0}^{\infty} \beta^t \log C_t$$

subject to (2), (4), (13), and for all t ,

$$D_t(1+r) \leq \max \sum_{i=t}^{\infty} (aK_i - J_i)(1+r)^{-(i-t)}$$

and $D_0 = 0$, K_0 given.

It is a standard exercise to show under these circumstances that the optimal production decision and the consumption choice can be separated. Given the international rate of interest, investment is chosen to maximize the productive wealth of the country. Given the path of productive wealth, the planner then selects an optimal pattern of consumption, subject only to the constraint that $D_t \leq \max W_t$ is satisfied. We show in Appendix 3 that the equilibrium will be characterized by the two following conditions:

Proposition 1: When $\lambda = 1$ (i.e. the country can never profitably repudiate its debt)

- 1) GDP grows at the fixed rate g^* (equation (8)) which maximizes productive wealth
- 2) Consumption grows at a rate $\theta = \beta(1+r) - 1$.

The initial value of consumption is defined so that the discounted value of the infinite sequence of consumption is equal to the wealth of the country.

Under the assumption (12), consumption over GDP tends to zero in the stationary state. This says that in the long run all GDP goes to repayment of the debt. When the country has the option to repudiate its debt, even at a substantial loss, a path in which consumption asymptotically approaches zero as a fraction of GDP can never be an equilibrium, since the option to repudiate would eventually be exercised. Lenders will restrain their lending so that in the limit only a fraction of GDP goes to the repayment of the debt, and so that consumption does not tend to zero in the long run.

One additional feature of the equilibrium is the following. The productive wealth of the economy W_t is maximized by the investment path, and it grows at the same rate as GDP, i.e. at the rate g^* . Since $g^* < r$ (by our earlier assumptions on the technology parameters), and since $(1+r)D_t \leq W_t$ by the lending rule, we see that D_t must grow less rapidly than rate r in the long run. Therefore, the path of $(1+r)D_t$ satisfies the so-called "transversality condition":

$$(28) \quad \lim_{t \rightarrow \infty} D_t / (1+r)^t = 0$$

Let us define RP_t as the net repayment of debt each period, given by the difference of $(1+r)D_t$ and new borrowing D_{t+1} :

$$(29) \quad RP_t = (1+r)D_t - D_{t+1}$$

Since (29) implies $D_{t+1} = (1+r)^{t+1}D_0 - \sum_{i=0}^t (1+r)^{(t-i)} RP_i$,

we have $D_{t+1} / (1+r)^t = (1+r)D_0 - \sum_{i=0}^t (1+r)^{-i} RP_i$. Then, taking limits and applying the transversality condition, we have:

$$(30) \quad (1+r)D_0 = \sum_{i=0}^{\infty} (1+r)^{-i} RP_i$$

In effect, the transversality condition is a zero-profit condition for the sequence of loans: it guarantees that the discounted value of repayments equals the outstanding debt payment due at time t , $(1+r)D_t$.

We have found the transversality condition to be implied by the market equilibrium. If lenders expect $(1+r)D_{t+i} \leq W_{t+i}$ for all $i > 0$, they will impose $(1+r)D_{t+1} \leq W_{t+1}$ in period t . In some models, the transversality condition need not be an implication of market equilibrium (see Cohen and Sachs (1985) as an example). Later in this paper, in Section 4, we will need to impose the zero-profit condition (30) directly, rather than finding it as an implication of the model.

d. Market Equilibrium with Repudiation Risk

We now turn to the case of $\lambda < 1$, and show that the lending rule $h(K_t)$ is linear in K_t . The equilibrium path, when $\lambda < 1$ is described by the following proposition:

Proposition 2: Lending in each period is governed by a linear credit constraint:

$$(31) \quad D_t \leq h^*K_t$$

where h^* is a constant which depends both on the technology of production and on the taste parameters of the intertemporal utility function. Given this constraint, a country with no initial debt selects a path with two stages:

Stage 1: The constraint (31) on the debt is not binding. The rate of growth of the economy is high initially, and declines progressively, and the debt-to-GDP and debt-to-capital ratios increase.

Stage 2: The constraint on the debt is binding. The economy and the debt grow at a same rate n , which is lower than the growth rates in Stage 1 and below the rate of growth, g^* , which maximizes productive wealth.

According to Proposition 2, the threat of repudiation forces the economy into an equilibrium with productive inefficiency, since the rate of growth in stage 2 is below the level g^* which maximizes productive wealth. With debt growing at rate n , we have $D_{t+1} = (1+n)D_t$. Also, $D_{t+1} = (1+r)D_t - RP_t$, where RP_t is the amount of repayment at time t . Thus, $RP_t = (r-n)D_t$, or:

$$(32) \quad RP_t = \theta Q_t, \text{ with } \theta = (r-n)h^*/a$$

One sees that the repayment is always less than the amount of interest falling due, rD_t , so that the lenders always refinance some interest payments. Despite

this property, the discounted value of debt repayments equals $(1+r)D_t$, i.e., the zero profit condition in (3) holds. (We know this to be true, since with debt growing at rate $n < r$, the transversality condition in (28) is satisfied.)

Once the binding debt-to-capital ratio, h^* , has been reached, the only way the lenders can keep the country from defaulting is to continue to lend according to the same rule. If they cease to do so, and demand, for example, the repayment of all interest falling due, then the country will default. (Proof: The rule $D^* \leq h^*K$, when it is binding, leads by construction to $V^D = V^R$. Any tighter lending rule will then have $V^D > V^R$, and the country will default.)

The proof of Proposition 2 is given in Appendix 3. Here we sketch the main steps of the proof. First, we guess a function for $\tilde{h}(K)$ and solve the optimal borrowing problem under the constraint $D_t \leq \tilde{h}(K_t)$, assuming no debt repudiation. The optimal borrowing problem yields an optimal consumption plan, denoted $\{C_t^{\tilde{h}}\}$, where the superscript \tilde{h} stresses the dependence of the optimal plan on the assumed borrowing constraint. We then define the value of the optimal program (under \tilde{h}) as $V^{\tilde{h}}(K_t, D_t) = \sum_{i=0}^{\infty} \beta^{t+i} \log(C_{t+i}^{\tilde{h}})$. By construction, $V^{\tilde{h}}$ satisfies the equation (25'): $V^{\tilde{h}}(K_t, D_t) = \max \log(C_t) + \beta V^{\tilde{h}}(K_{t+1}, D_{t+1})$ such that $D_{t+1} \leq \tilde{h}(K_{t+1})$. Then, we prove for our choice \tilde{h} that $D_{t+1} \leq \tilde{h}(K_{t+1})$ if and only if $V^{\tilde{h}}(K_t, D_t) \geq V^D(K_t)$. Thus, we have found a solution to (25), with $V^R = V^{\tilde{h}}$, $h = \tilde{h}$, and $V^D = V^D$. Using $C_t^{\tilde{h}}$, we then prove the part of the proposition concerning the two-stage growth path.

Specifically, we prove that the linear credit rule in (31) satisfies the market equilibrium in (25). The properties of the borrowing equilibrium under

$D_t \leq h^*K_t$ are easy to calculate, as shown in Appendix 3. We solve the optimal control problem for foreign borrowing subject to the constraint that $D_t \leq h^*K_t$. The result is the two-stage growth process described in Proposition 2. Starting $D_0 = 0$, there is an initial period of rapid but declining growth, with D/K rising to h^* . Once the constraint is binding, D/K remains equal to h^* , and growth remains equal to x^{h^*} , where x^{h^*} is the root of equation (A3.8) in Appendix 3. This growth rate is less than g^* , the growth rate that maximizes W_t . (Remember that g^* is achieved when $\lambda = 1$, i.e. when there is no viable threat of repudiation.)

In Appendix 3, we present the non-linear equation (A3.12) that defines h^* . It is straightforward to show that h^* is: increasing in a , β , and λ ; and is decreasing in r . In words, the sustainable debt per unit of capital rises with capital productivity, the discount factor, and the penalty for default, while it falls with the world interest rate. By a simple transformation, the maximum debt-GNP ratio is a similar function of these variables, since $D/\text{GNP} = D/(aK-rD)$, which must be less than or equal to $h/(a-rh)$ when $D \leq hK$. In Table 1, we calculate the maximum debt-GNP ratios for alternative values of a , β , λ , and r .

It is also instructive to compare the equilibrium growth paths for GDP under alternative borrowing assumptions, as is done in Figure 1. The dashed line, which increases most rapidly, is for GDP in the absence of repudiation risk (i.e. for $\lambda = 1$); the solid line, which shows intermediate growth, is for GDP with repudiation risk (specifically, $\lambda = 0.4$), with D/K initially below h^* ; the dotted line, with the slowest growth, represents the case without foreign borrowing (either a closed economy, or open economy with $\lambda = 0$). There are two

Figure 1. GDP Under Alternative Borrowing Assumptions

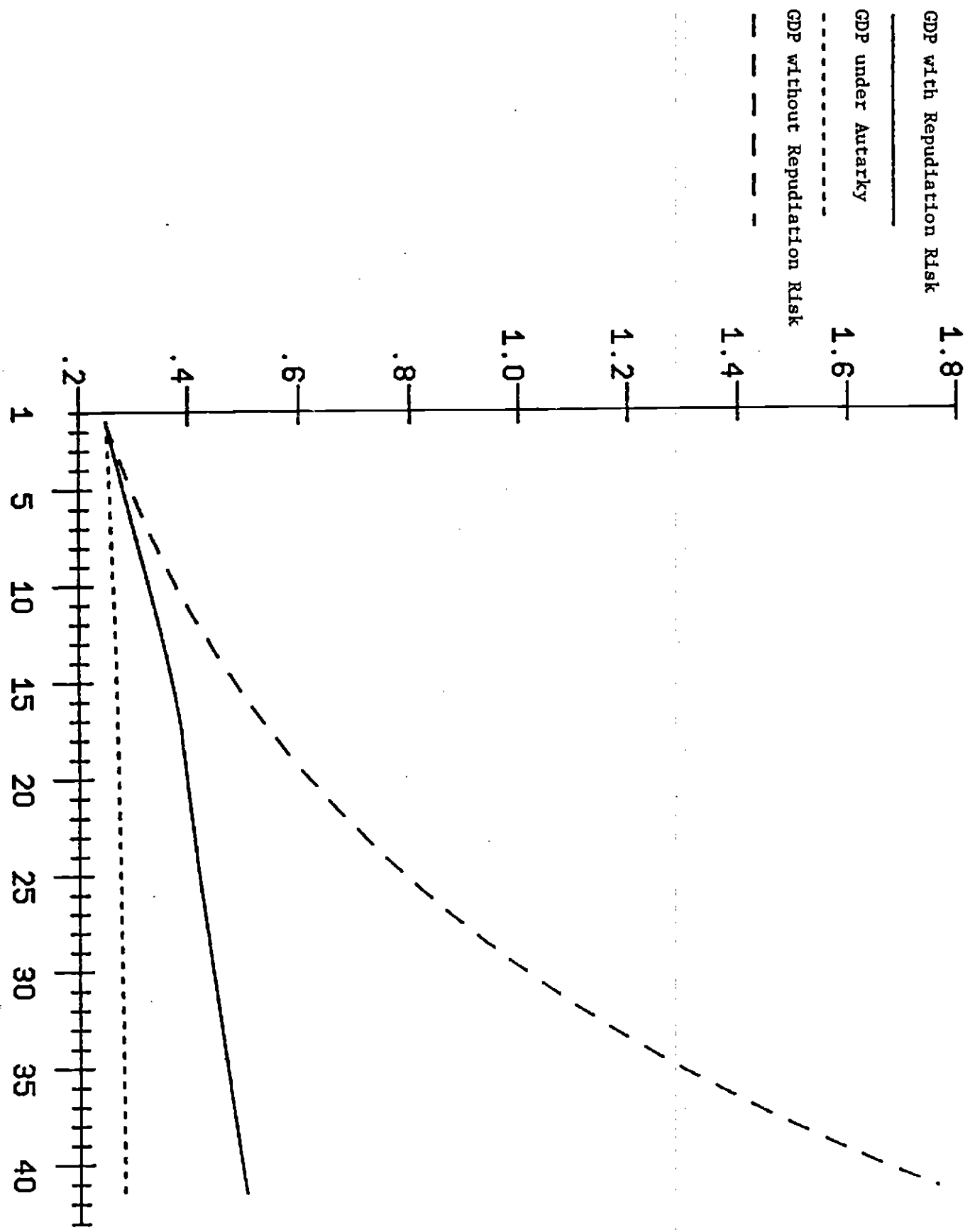


Table 1. Maximum Debt-GNP Ratios^a Various Parameter Values

I. Alternative values of capital productivity, a , and subjective discount rate, β (for $r = 0.1$, $\lambda = 0.4$).

$a \backslash \beta$	0.6	0.7	0.8	0.9
0.15	2.31	2.48	2.82	3.75
0.20	2.44	2.68	3.15	4.55
0.25	2.61	2.92	3.60	5.89
0.30	2.81	3.23	4.24	9.28

II. Alternative values of world interest rate, r , and default penalty rate, λ (for $a = 0.2$, $\beta = 0.9$).

$r \backslash \lambda$	0.1	0.2	0.3	0.4
0.10	0.95	2.02	3.22	4.55
0.15	0.67	1.42	2.28	3.26
0.20	0.51	1.10	1.78	2.58
0.25	0.41	0.90	1.46	2.14

^a The other parameter values are $d = 0.05$, $\phi = 10$

notable aspects of the figure. First, it is apparent that productive wealth is reduced by the option to repudiate. Second, we see the point of inflection (in period 17) in the intermediate growth path, which according to Proposition 2 is reached when the constraint $D_t = h(K_t)$ begins to bind.

4. Equilibrium with Repudiation in Nonlinear Models

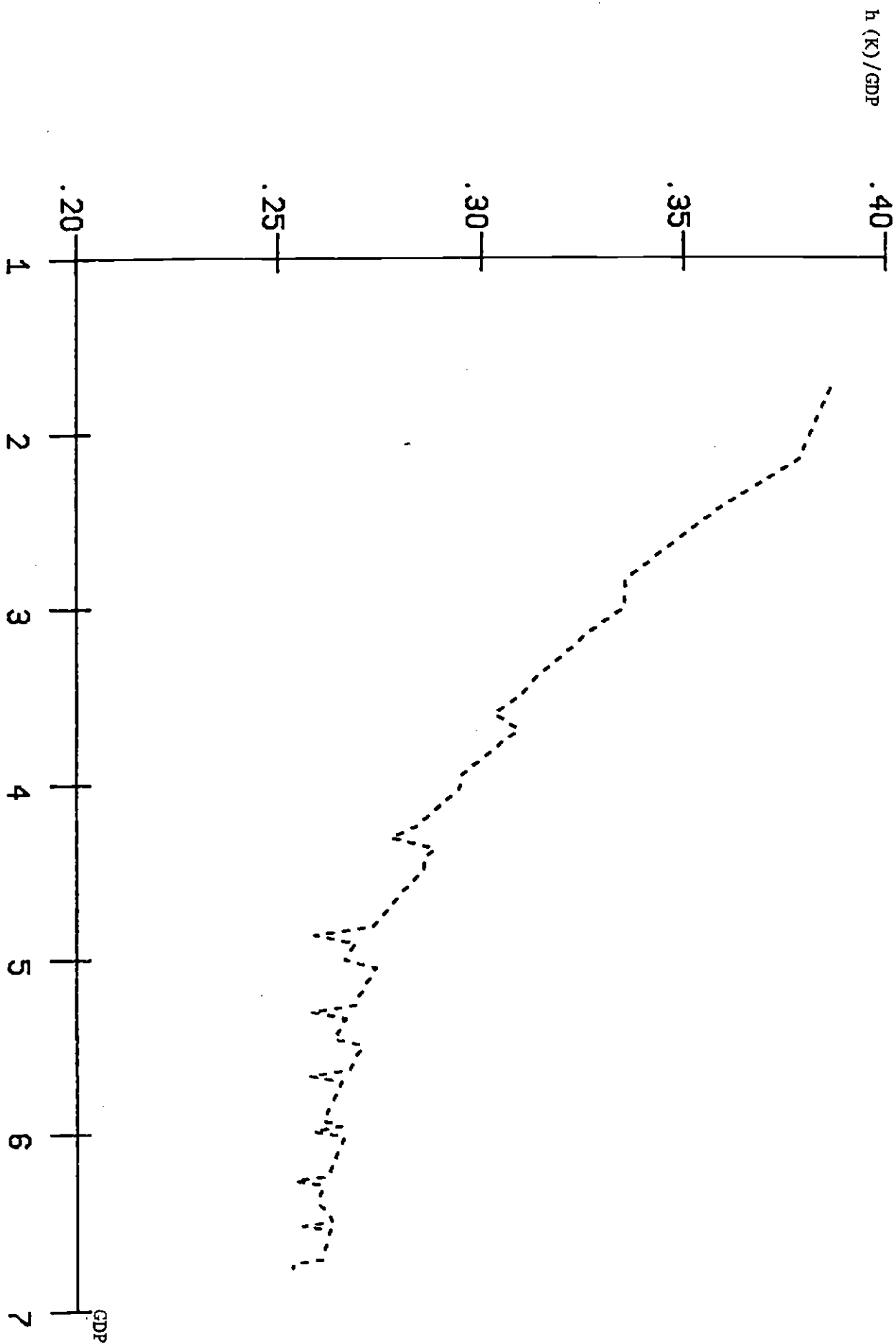
We have been able to derive a closed-form solution for the growth path in the case $Q_t = aK_t$ because the value function V^R is separable in K and D (see Appendix 3). This separability arises because two economies with the same D_t/K_t but different K_t will differ only in scale. With $D_t^1/K_t^1 = D_t^2/K_t^2$ and $K_t^2 = \lambda K_t^1$, economy 2 will simply follow a path in which all variables ($C_t^2, D_t^2, K_t^2, GDP_t^2$) are equal to λ times the values in economy 1 ($C_t^1, D_t^1, K_t^1, GDP_t^1$).

Once the linear production technology is abandoned, an analytical solution appears out of reach. Now, a rise in K is more than a scale change, since output does not (by assumption) rise in the same proportion. For a standard neoclassical production technology, $Q_t = F(K_t)$, with $F' > 0$, $F'' < 0$, a rise in K_t lowers the marginal product of capital and reduces the incentive to invest. It is not surprising, therefore, that the maximum D_t associated with K_t will not rise in equiproportion with K_t . In other words, the h function giving $D_t = h(K_t)$ will be less than unit elastic in K_t .

In order to investigate this case, we resort to numerical dynamic programming methods (designed by Mr. Atish Ghosh in conjunction with the authors). Specifically, the value function $V^D(K_t)$ is calculated numerically for a grid on K_t varying between 0 and 10.0, with steps of 0.1. Then, $V^R(K_t, D_t)$ is calculated using the backward recursion methodology outlined in Section 3. The V^R function is analyzed on a grid of 0 to 10.0 for K_t , and 0 to 2.0 for D_t , both with step sizes of 0.1. Once V_{T-t}^D and V_{T-t}^R are calculated for a period of $T-t$, as in section 3, the optimization in (20) is carried out by a numerical search procedure over I and C , using step sizes of 0.05 for these variables. The backward recursion procedure is repeated until V^D and V^R converge to steady-state functions within the specified tolerance.

An example of this numerical exercise is shown in Figures 2 and 3. We adopt a Cobb-Douglas production technology $Q_t = 3K_t^{0.3}$. The other parameter values are $a = 0.27$, $r = 0.11$, $\delta = 0.5$, $\beta = 0.85$, and $\theta = 0.3$. Figure 2 plots the maximum Debt/GDP ratio as a function of GDP. (Note that the jagged nature of the curve results only from the discrete step-sizes used in the programming algorithm. These minor blips are unimportant quantitatively. A smoother curve could be achieved by a finer grid size, but at the cost of much higher computing time.)¹ In the linear case, of course, this maximum ratio is a constant (h^*/a), independent of the level of GDP. In the case of Cobb-Douglas technology, however, the maximum ratio declines as GDP increases. This is for a straightforward, yet illuminating reason. At low levels of GDP, the marginal

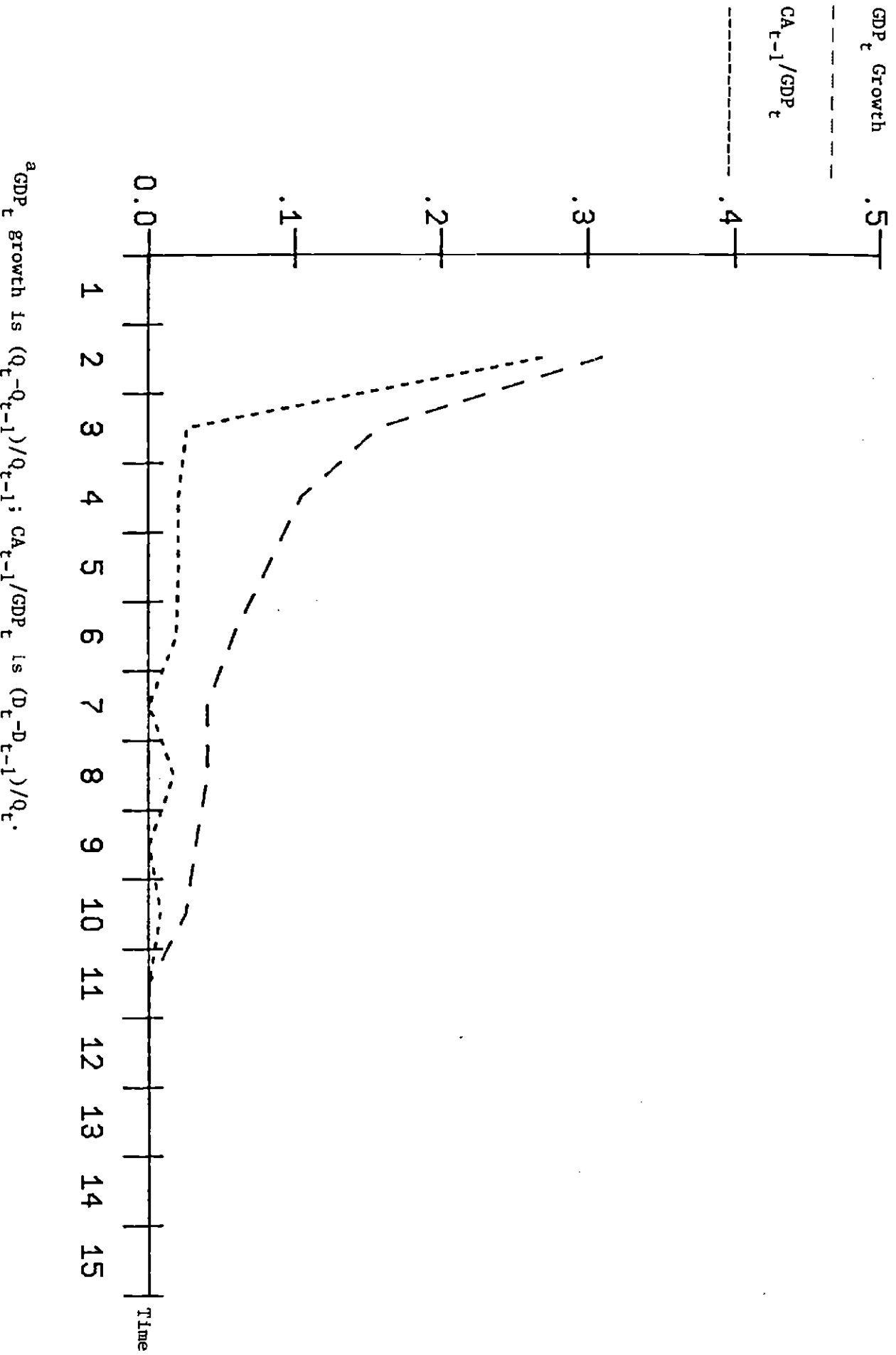
Figure 2. Maximum Debt-to-GDP Ratio



productivity of capital is very high and the investment incentives are consequently strong. The returns to borrowing abroad are large, as are the costs of debt repudiation (which would freeze out new borrowing to finance further increases in K). As the capital stock deepens, the incremental returns to investment fall, and the costs of debt repudiation as a fraction of GDP are reduced pari passu. Thus, a capital-poor country will be allowed a wide latitude in foreign borrowing, while a capital-abundant economy will be much more restricted (as a proportion of GDP). The allowable debt rises as GDP rises but less than proportionately.

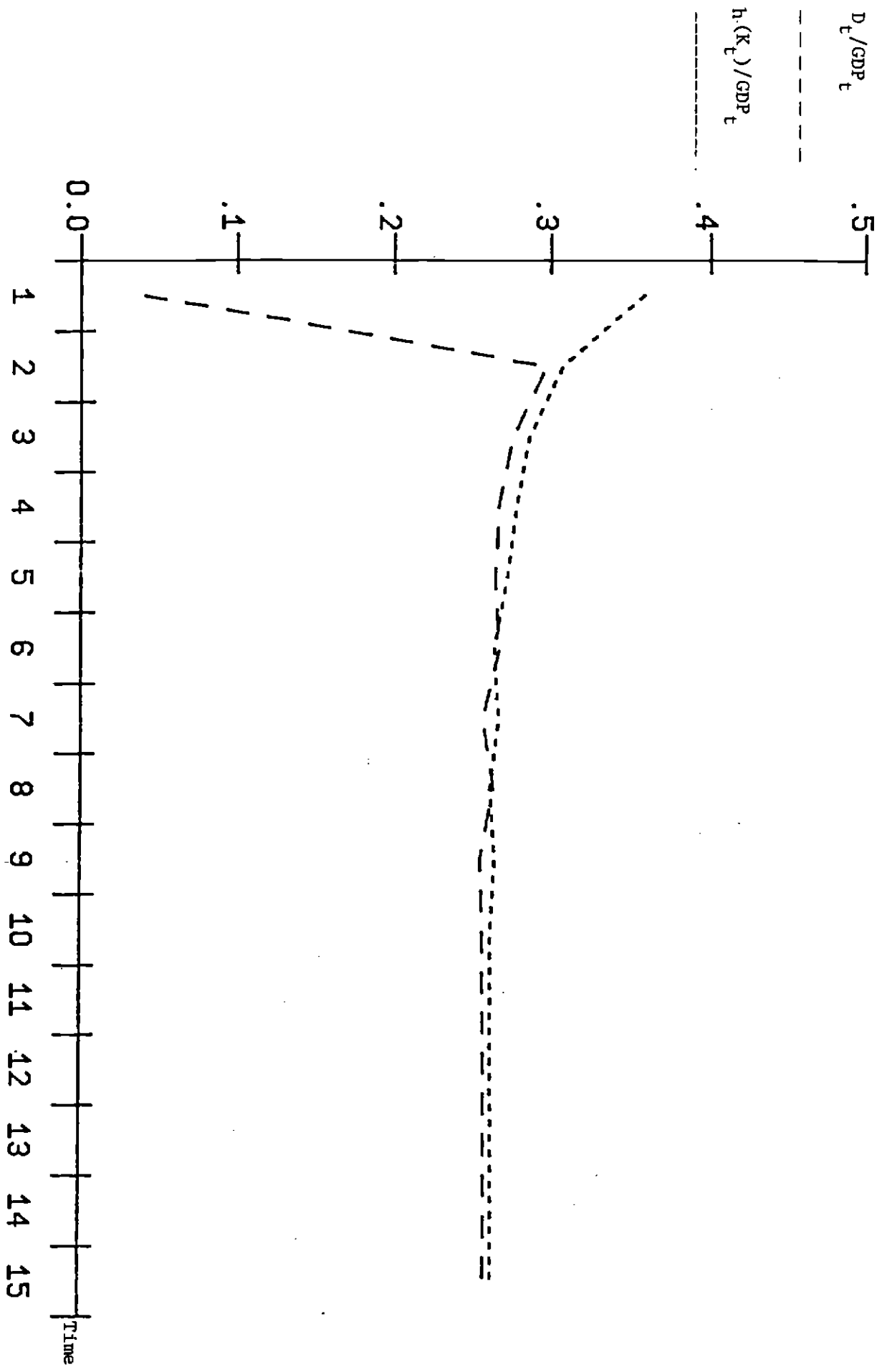
As shown in Figures 3 and 4, this feature of the $h(\)$ function has important implications for the growth path and for foreign borrowing. As in the linear model, growth and foreign borrowing start out high and then fall sharply as the debt constraint is hit (in this case as of period 2). Figure 3 shows the GDP growth rate and the current account deficit as a percentage of GDP. Figure 4 shows the actual and maximum debt/GDP ratios for periods 1 through 15. In each period, the maximum rate is calculated as $h(K_t)/GDP_t$ where K_t is the level of K_t on the equilibrium growth path. In the first period, actual $D_t/GDP_t < h(K_t)/GDP_t$. In periods 2 through 15, the debt constraint is binding as shown. Note that the Debt/GDP ratio jumps sharply between periods 1 and 2, and then declines over, as the country becomes wealthier (i.e. as the capital stock deepens). The decline reflects the already observed fact that $h(K_t)/GDP_t$ is a declining function of K_t .

Figure 3. GDP Growth and Current Account Deficits ^a



^aGDP_t growth is $(Q_t - Q_{t-1})/Q_{t-1}$; CA_{t-1}/GDP_t is $(D_t - D_{t-1})/Q_t$.

Figure 4. Actual and Maximum Debt-GDP Ratios



5. Alternative Lending Strategies

In this section we return to the linear version of the model in order to examine lending and repayment strategies which differ from the market equilibrium described above. In the first subsection, we assume that the lenders and the borrower can act cooperatively to decide upon a path of future debt, consumption and investment, subject only to the zero profit condition in (3) and to the condition that the country keep its sovereign right to repudiate the debt at any time and to suffer the penalty thereafter. (If the country can also credibly promise never to repudiate, it reaches the equilibrium of Proposition 1.) Is there any contractual scheme between the lenders and the borrowers that can dominate the market equilibrium and still keep the country from defaulting? The answer is no, as we show in the proof of Proposition 3.

In the second subsection, we analyze a repayment scheme which has been suggested by some analysts. We have seen that when $D_t = h * K_t$, in stage 2 of the market equilibrium, the country repays every period a fixed fraction θ of GDP, as in (32). What if the lenders simply demand that $RP_t = \theta Q_t$, rather than sticking to the rule $D_t = h * K_t$? It turns out that when the borrower expects to face the rule $RP_t = \theta Q_t$ rather than the rule $D_t = h * K_t$, the incentives for growth are changed, in a way that undermines this alternative lending rule.

a. Contractual Agreements Between the Lenders and the Borrower

In this subsection, we assume that the borrowing country and its lenders can design a path of future debt, investment and consumption such that: (i) the loan satisfies the zero-profit condition (30); and (ii) the country never finds it profitable to repudiate its debt. The path is set once and for

all at $t = 0$. Surprisingly, such a contractual arrangement cannot dominate the market equilibrium:

Proposition 3: The market equilibrium yields the optimal pattern of growth that a country can reach, subject to the constraint that it never prefers to default.

The proposition says, among other things, that there is no way to avoid the slowdown of growth described in Proposition 2. This slowdown is an inherent implication of the option of repudiation; there is no way that the country can credibly promise to grow at g^* forever.

To prove Proposition 3, we first define the new utility function Ω which would be attained if the country can design in advance its path of consumption and investment subject to (3) and the constraint that it never prefer to default.

$$(33) \quad \Omega[D_0, K_0] = \max_{\{C_t, I_t, D_t\}} \sum_{t=0}^{\infty} \beta^t \log C_t$$

subject to

$$(34) \quad \sum_{t=s}^{\infty} \beta^{(t-s)} \log C_t \geq V^D(K_s)$$

and subject to (2), (4), (13), (30).

transversality condition (16).

Note that (34) guarantees that the country will never choose to repudiate the debt. A priori, Ω does not coincide with the utility V achieved in the market equilibrium. V was defined as the fixed point of problem (25). Ω is defined

directly by an optimizing problem. They in fact are equal, as we show in Appendix 4. The crux of the argument is to show that inequality (34) can be written $D_t \leq h^*K_t$, as in the market equilibrium case.

b. Repayment of a Fixed Fraction of the Revenues Each Period

Let us assume that the country has already entered stage 2 of its pattern of growth. Each period, it repays a fixed fraction of its revenues to the lenders in order to equalize the growth of debt and the growth of GDP. These repayments can be written $RP_t = [(r-n)h^*/a]Q_t$, with $n = x^{h^*} - d$, where x^{h^*} is the investment rate in Stage 2.

Now, let us assume that the lenders ask to be repaid some fraction $\tilde{\theta}$ of GDP, instead of sticking to the original rule of lending $D \leq h^*K$. $\tilde{\theta}$ may or may not equal $(r-n)h^*/a$. We prove the following:

Proposition 4: When a country has reached the stage 2 of its pattern of growth, the lenders cannot get repayment of their loans by abandoning the rule $D_t \leq h^*K_t$ and by asking instead to be repaid a fixed fraction of the country's GDP each period. By doing so, they would either induce the country to default, or they would get repayment of only a portion of the outstanding debt (i.e. the zero profit condition in (3) would not hold).

The proposition stresses the importance to the creditors of stating correctly the rule governing their lending. In the market equilibrium, the lenders are ready to increase their exposure at the rate of GDP growth. By doing so, they create an incentive for the country to grow. Any slight

modification of the rule which reduces this incentive to grow yields a sub-optimal result. A repayment scheme as in Proposition 4 creates an incentive that is adverse to growth, since debt repayments are made an increasing function of the output of the economy.

To prove Proposition 4, assume that the country is asked to repay a fixed fraction $\tilde{\theta}$ of its GDP. It will select an optimal rate of growth by solving:

$$(35) \quad V_{\tilde{\theta}} = \underset{x}{\text{Max}} \log[a(1-\tilde{\theta}) - x(1 + \phi x/2)]/(1-\beta) + \beta[\log(1+x-d)]/(1-\beta)^2 + \log K_0/(1-\beta)$$

(see equation (A1.8) where $a\tilde{\theta}$ is substituted for λ)

Let $x_{\tilde{\theta}}$ be the solution to (35). We can compute the value of this repayment stream by noting that $RP_{t+i} = \tilde{\theta}GDP_{t+i} = \tilde{\theta}aK_{t+i} = \tilde{\theta}a(1+x_{\tilde{\theta}}-d)^i K_t$. Thus, the discounted value of repayments, which we denote by P_t , is given as:

$$(36) \quad P_t = \sum_{i=t}^{\infty} RP_t / (1+r)^i = [aK_t \tilde{\theta} (1+r)] / [r+d-x_{\tilde{\theta}}]$$

Now, suppose $\tilde{\theta}$ is selected so that $P_t = (1+r)D_t$, i.e. so that the debt is repaid in present value terms. Call the selected value $\hat{\theta}$. Since $D_t = h^*K_t$ by assumption, we must have $P_t = h^*(1+r)K_t$. Thus, from (36) we must have the equality $h^* = \hat{\theta} / [r + d - x_{\hat{\theta}}]$. From (35) we then see that:

$$(37) \quad V_{\hat{\theta}} = \log[a - h^*(r+d-x_{\hat{\theta}}) - x_{\hat{\theta}}(1+\phi x_{\hat{\theta}}/2)]/(1-\beta) + \beta[\log(1+x_{\hat{\theta}}-d)]/(1-\beta)^2 + \log K_t/(1-\beta)$$

Now, we prove that $V_{\hat{\theta}} < V^D$, as asserted in Proposition 4. That is, we prove

that the country would default if required to pay θ GDP, for θ high enough to yield $P_t = (1+r)D_t$. By definition of x^{h^*} and h^* (at which the country is at autarky utility in the market equilibrium), we can use (37) to write

$$(38) \quad V^D = \log[a - h^*(r+d-x^{h^*}) - x^{h^*}(1+\phi x^{h^*}/2)]/(1-\beta) \\ + \beta[\log(1+x^{h^*}-d)]/(1-\beta)^2 + \log K_t/(1-\beta)$$

Note that the RHS of (38) is the same as the RHS of (37) but with x^{h^*} replacing x_{θ}^* . But x^{h^*} maximizes the RHS of (37) for given h^* , so it must be $V^D > V_{\theta}^*$. If the creditors are to be repaid, the borrower is driven below the autarky level.

6. Conclusion

We have constructed a model with endogenous growth and endogenous credit ceilings. We have shown that the equilibrium path for the country is one with initially rapid growth followed by a permanent slowdown. The equilibrium strategy of the lenders makes the growth of debt contingent on the growth of the borrowing country. In the linear version of our model, under the equilibrium lending rule, the evolution of the debt follows two stages, a first stage in which debt grows faster than the economy, and a second stage where both the economy and the debt grow more slowly, and at the same rate. Along this path, the interest due, rD , is never repaid in full; only an amount equal to the difference between the interest due and g^*D (the growth rate times the debt) should be repaid. This permanent refinancing of part of the interest is the only way to reach the optimum pattern of growth consistent with no default by the borrowing country.

Appendix 1

Optimal Growth Under Autarky

The optimal growth problem under autarky is:

$$(A1.1) \quad V_t(K_0) = \max \sum_{t=0}^{\infty} \beta^t \log(C_t)$$

such that

$$K_{t+1} = K_t(1-d) + I_t$$

$$aK_t = C_t + I_t[1 + (\phi/2)(I_t/K_t)]; K_0 \text{ given}$$

To solve this problem, we define $c_t = C_t/K_t$, $x_t = I_t/K_t$. Note that

$$K_{t+1} = K_t(1-d+x_t), \text{ so that } K_t = K_0 \prod_{i=0}^{t-1} (1-d+x_i), \text{ and } \log K_t =$$

$$\log K_0 + \sum_{i=0}^{t-1} \log(1-d+x_i). \text{ Since } c_t = C_t/K_t, \log C_t = \log c_t + \log K_t.$$

$$\text{Hence, } \sum_0^{\infty} \beta^t \log(C_t) = \sum_0^{\infty} \beta^t \log c_t + \sum_0^{\infty} \beta^t \log K_t = \sum_0^{\infty} \beta^t \log c_t + \sum_0^{\infty} \beta^t \log K_0 +$$

$$\sum_1^{\infty} \beta^t [\sum_{i=0}^{t-1} \log(1-d+x_i)]. \text{ After a bit of manipulation, } \sum_1^{\infty} \beta^t [\sum_{i=0}^{t-1} \log(1-d+x_i)] \text{ can be}$$

$$\text{rewritten as } \sum_{t=0}^{\infty} [\beta^{t+1}/(1-\beta)] \log(1-d+x_t). \text{ Thus, the original problem (A1.1) can}$$

be rewritten as:

$$(A1.2) \quad V^A(K_0) = \max \sum_{t=0}^{\infty} \beta^t \{ \log c_t + \log K_0 + [\beta/(1-\beta)] \log(1-d+x_t) \}$$

such that

$$a = c_t + x_t [1 + (\phi/2)x_t]$$

Note that we have solved out for K_t in the objective function, so we no longer need to include $K_{t+1} = K_t(1-d+x_t)$ among the constraints. Now there are no longer any real dynamic constraints in the problem, so that we need only to maximize $\{ \log c_t + [\beta/(1-\beta)] \log(1-d+x_t) \}$ each period, subject to $a = c_t + x_t(1+\phi x_t/2)$. This maximization leads to a fixed c^A and x^A (i.e. optimal c and x under autarky) for all t , with x^A given by the largest root of

$$(A1.3) \quad (1-\beta)(1+\phi x)(1-d+x) - \beta a + \beta x(1+\phi x/2) = 0.$$

c^A is then given as:

$$(A1.4) \quad c^A = a - x^A[1 + \phi x^A/2]$$

Note that since c^A and x^A are invariant to K_0 , we have from (A1.2) that $V^A(\lambda K_0) - V^A(K_0) = \sum_{t=0}^{\infty} \beta^t \log(\lambda K_0) - \sum_{t=0}^{\infty} \beta^t \log K_0 = \log \lambda / (1-\beta)$. Thus,

$$(A1.5) \quad V^A(\lambda K_0) = V^D(K_0) + \log \lambda / (1-\beta)$$

By substitution of (A1.4) into (A1.2), we find the explicit solution for $V^A(K_0)$:

$$(A1.6) \quad V^A(K_0) = \sum_{t=0}^{\infty} \beta^t \{ \log[a - x^A(1+\phi x^A/2)] + \log K_0 + [\beta/(1-\beta)] \log(1-d+x^A) \} \\ = \log[a - x^A(1+\phi x^A/2)] / (1-\beta) + \beta \log(1-d+x^A) / (1-\beta)^2 + \log K_0 / (1-\beta)$$

The solution for the default utility $V^D(K_0)$ is the same as for $V^A(K_0)$ except that $a(1-\lambda)$ replaces a in the equation for the growth rate (A1.3). That is, x^D is the largest root of the equation

$$(A1.7) \quad (1-\beta)(1+\phi x)(1-d+x) - \beta a(1-\lambda) + \beta x(1+\phi x/2) = 0$$

Then, as in (A1.6), we may write:

$$(A1.8) \quad V^D(K_0) = \log[a(1-\lambda) - x^D(1+\phi x^D/2)] / (1-\beta) + \beta \log(1-d+x^D) / (1-\beta)^2 \\ + \log K_0 / (1-\beta)$$

It is easy to show that $x^D < x^A$, and $V^D(K_0) < V^A(K_0)$.

Appendix 2

Optimal Growth Without Repudiation

The planner solves:

$$(A2.1) \quad \max \sum_{t=0}^{\infty} \beta^t \log C_t$$

subject to

$$(A2.2) \quad \sum_{t=0}^{\infty} (1+r)^{-t} C_t = NW_0$$

where NW_0 is consumer wealth. The solution to (A1) is

$$(A2.3) \quad C_t = C_0 \beta^t (1+r)^t$$

$$(A2.4) \quad C_0 = (1-\beta)NW_0$$

Productive wealth W_0 is maximized by solving:

$$(A2.5) \quad \max W_0 = \sum_{t=0}^{\infty} (1+r)^{-t} \{aK_t - I_t[1+(1/2)\phi(I_t/K_t)]\}$$

subject to

$$(A2.6) \quad K_{t+1} = K_t(1-d) + I_t$$

If one calls q_t the Lagrange multiplier associated with (A6), one finds

$$(A2.7) \quad x_t \equiv I_t/K_t = (q_t - 1)/\phi$$

and

$$(A2.8) \quad q_{t-1}(1+r) = q_t(1-d) + a + \phi x_t^2/2$$

The optimal path is obtained by selecting the unique value of q_0 which satisfies the transversality condition. Here, q_0 is such that all q_t will stay constant, so that x_t is also a constant, x^* , which solves

$$(A2.9) \quad (1/2)x^2 - x(r+d) + (a-d-r)/\phi = 0$$

The solution, x^* , which we look for is the smaller root of (A9) (the other solution can be ruled out by the second-order conditions of the optimization problem). The solution is shown in equation (8) in the text. We let $1 + g^* \equiv 1 + x^* - d$. K_t is now defined by

$$(A2.10) \quad K_t = K_0(1+g^*)^t$$

The productive wealth of the country is now defined by

$$W_0 = [a - x^*(1+\phi x^*/2)][(1+r)/(r-g^*)]K_0$$

Net wealth (subtracting foreign debt D) is defined by NW_t :

$$(A2.11) \quad NW_t = W_t - (1+r)D_t$$

At initial time, $D_0 = 0$, so that NW_0 and W_0 coincide.

The law of motion of the debt is given by

$$(A2.12) \quad D_{t+1} = (1+r)D_t + C_t - [a - x^*(1+\phi x^*/2)]K_t$$

Since $C_t = (1-\beta)NW_t = (1-\beta)[W_t - (1+r)D_t]$, we have

$$(A2.13) \quad D_{t+1} = (1+r)D_t + (1-\beta)[a - x^*(1+\phi x^*/2)][(1+r)/(r-g^*)]K_t \\ - (1-\beta)(1+r)D_t - [a - x^*(1+\phi x^*/2)]K_t$$

Now, define $d_t = D_t/K_t$. Note that $D_{t+1}/K_t = d_{t+1}(K_{t+1}/K_t) = (1+g^*)d_{t+1}$. Then, by dividing both sides of (A13) by K_t we can write:

$$(A2.14) \quad d_{t+1} = [\beta(1+r)/(1+g^*)]d_t + [(1+g) - \beta(1+r)][a - x^*(1+\phi x^*/2)]/(r-g^*)K_t$$

We begin with $d_0 = 0$. The country becomes and remains a net debtor as long as $\beta(1+r) < (1+g^*)$. This condition, which we assume, is more likely to hold when: the world interest rate r is low; the future is heavily discounted (β small); investment is highly productive (a large); and the adjustment costs in investment, ϕ , are small.

By solving (A2.14), starting from $d_0 = 0$, we find:

$$(A15) \quad d_t = \{1 - [\beta(1+r)/(1+g^*)]^t\} [a - x^*(1+\phi x^*/2)]/(r-g^*)$$

Note that as $t \rightarrow \infty$, d_t asymptotically approaches $\bar{d} = [a - x^*(1+\phi x^*/2)]/(r-g^*)$.

When $d_t = \bar{d}$, we have $(1+r)\frac{D}{W} = 1$. Thus, consumption over GDP asymptotically approaches zero.

Appendix 3

Borrowing Equilibrium with Risk of Debt Repudiation

a. Lenders' Strategies

Consider the following optimization problem:

$$(A3.1) \quad V^h(K_0, D_0) = \max \sum_{t=0}^{\infty} \beta^t \log C_t$$

such that

$$D_{t+1} = D_t(1+r) + C_t + I_t[1 + (\phi/2)(I_t/K_t)] - aK_t$$

$$K_{t+1} = K_t(1-d) + I_t$$

$$D_{t+1} \leq hK_{t+1}$$

with $D_0 \leq hK_0$ and D_0, K_0 given

Define the following variables:

$$c_t \equiv C_t/K_t$$

$$d_t \equiv D_t/K_t$$

$$x_t \equiv I_t/K_t$$

Then, the problem in (A3.1) can be rewritten as:

$$(A3.2) \quad V^h(K_0, D_0) = \max \sum_{t=0}^{\infty} \beta^t \log c_t + \sum_{t=0}^{\infty} \beta^t \log K_t$$

such that:

$$(1-d+x_t)d_{t+1} = (1+r)d_t + c_t + x_t[1 + (\phi/2)x_t] - a$$

$$K_{t+1} = K_t(1-d+x_t)$$

$$d_{t+1} \leq h$$

with d_0, K_0 given, and $d_0 \leq h$

Using the same method as in Appendix 1, (A3.2) may be rewritten again as:

$$(A3.3) \quad V^h(K_0, D_0) = \max \sum_{t=0}^{\infty} \beta^t \log c_t + (\log K_0)/(1-\beta) \\ + \sum_{t=0}^{\infty} [\beta^{t+1}/(1-\beta)] \log(1-d+x_t)$$

such that:

$$(1-d+x_t)d_{t+1} = (1+r)d_t + c_t + x_t[1 + (\phi/2)x_t] - a$$

$$d_{t+1} \leq h$$

Note that we no longer need the equation of motion for K_t , since K_t is no longer part of the objective function.

The optimal policy for c_t, x_t in (A3.3) is clearly independent of K_0 , for a given $d_0 (= D_0/K_0)$, since the objective function is additively separable in K_0 , and the dynamic constraints do not depend on K_0 . Thus, $V^h(\lambda K_0, \lambda D_0) - V^h(K_0, D_0) = (\log \lambda K_0)/(1-\beta) - (\log K_0)/(1-\beta) = \log \lambda/(1-\beta)$.

Rearranging, we have

$$(A3.4) \quad V^h(\lambda K_0, \lambda D_0) = V^h(K_0, D_0) + \log \lambda/(1-\beta)$$

as asserted in Lemma 1.

Now, choose $\lambda = 1/K_0$, so that we find $V^h(1, D_0/K_0) = V^h(K_0, D_0) - (\log K_0)/(1-\beta)$. Since we already know from Appendix 1 that $V^D(K_0) = V^D(1) + \log(K_0)/(1-\beta)$, we see that

$$(A3.5) \quad V^h(K_0, D_0) \begin{matrix} \geq \\ < \end{matrix} V^D(K_0) \text{ if and only if } V^h(1, D_0/K_0) \begin{matrix} \geq \\ < \end{matrix} V^D(1)$$

It is easy to show that V is strictly decreasing in its second argument (more debt is strictly worse than less debt). Thus there is at most one value $\hat{h} = D_0/K_0 \leq h$ such that $V^h(1, \hat{h}) = V^D(1)$, and if such \hat{h} exists, then $V^h(1, D_0/K_0) (\geq) V^D(1)$ as $D_0/K_0 (\leq) \hat{h}$. From (A3.5), the same \hat{h} is such that $V^h(K_0, D_0) (\geq) V^D(K_0)$ as $D_0 (\leq) \hat{h}K_0$.

Now, suppose we find an h^* such that $V^{h^*}(1, h^*) = V^D(1)$. Then, $V^{h^*}(K_0, D_0) (\geq) V^D(K_0)$ as $D_0 (\leq) h^*K_0$. V^{h^*} , V^D , and h^* then constitute an equilibrium of the capital market, as defined by (24) in the text with $V^R = V^{h^*}$.

We now derive the equilibrium h^* .

Let us first derive $V^{h^*}(1, h^*)$. This is the solution to:

$$(A3.6) \quad \max \sum_{t=0}^{\infty} \beta^t \log c_t + \sum_{t=0}^{\infty} [\beta^{t+1}/(1-\beta)] \log(1-d+x_t)$$

such that

$$(1-d+x_t)d_{t+1} = (1+r)d_t + c_t + x_t[1 + \phi x_t/2] - a$$

$$d_{t+1} \leq h$$

$$d_0 = h^*$$

We set up the Lagrangian for the problem as:

$$(A3.7) \quad \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{ \log c_t + [\beta/(1-\beta)] \log(1-d+x_t) + \beta^t \lambda_t \{ (1-d+x_t)d_{t+1} - (1+r)d_t - c_t - x_t[1 + \phi x_t/2] + a \} + \beta^t \mu_{t+1} [d_{t+1} - h] \}$$

Let us first derive $V^{h^*}(1, h^*)$. This is the solution to:

$$(A3.6) \quad \max \sum_{t=0}^{\infty} \beta^t \log c_t + \sum_{t=0}^{\infty} [\beta^{t+1}/(1-\beta)] \log(1-d+x_t)$$

such that

$$(1-d+x_t)d_{t+1} = (1+r)d_t + c_t + x_t[1 + \phi x_t/2] - a$$

$$d_{t+1} \leq h$$

$$d_0 = h^*$$

We set up the Lagrangian for the problem as:

$$(A3.7) \quad \mathcal{E} = \sum_{t=0}^{\infty} \beta^t \{ \log c_t + [\beta/(1-\beta)] \log(1-d+x_t) + \beta^t \lambda_t \{ (1-d+x_t)d_{t+1} - (1+r)d_t - c_t - x_t[1 + \phi x_t/2] + a \} \beta^t \mu_{t+1} [d_{t+1} - h] \}$$

$$(a) \quad 1/c_t = \lambda_t \quad (\partial \mathcal{E} / \partial c_t = 0)$$

$$(b) \quad [\beta/(1-\beta)] / (1-d+x_t) + \lambda_t d_{t+1} - \lambda_t (1+\phi x_t) = 0 \quad (\partial \mathcal{E} / \partial x_t = 0)$$

$$(c) \quad (1-d+x_t) \lambda_{t-1} / \beta - \lambda_t (1+r) \mu_t / \beta = 0 \quad (\partial \mathcal{E} / \partial d_t = 0)$$

To these first-order conditions we add the budget constraint from (A3.6):

$$(d) \quad (1-d+x_t)d_{t+1} = (1+r)d_t + c_t + x_t[1 + \phi x_t/2] - a$$

Solutions to (a) through (d) are given by constant values of c , λ , x , and μ ,

with $d_t = h$ for all t . Using (a), (b), and (d) we find the optimal

x^h as the positive root to the following equation:

$$(A3.8) \quad [(2-\beta)\phi]x^2 + 2[(1-d)(1-\beta)\phi - (h-1)]x + 2[-\beta a + \beta h(r+d) - (1-d)(1-\beta)(h-1)]$$

Now, we rewrite the budget constraint $d_{t+1}(1-d+x_t) = (1+r)d_t + c_t + x_t[1 + (\phi/2)x_t] - a$ to give c_t as a function of x^h . Since $d_t = d_{t+1} = h$, we have:

$$(A3.9) \quad c^h = (x^h - r - d)h + a - x^h[1 + \phi x^h/2]$$

Note that c^h is written without a time subscript, now since consumption per unit

capital is a constant along the optimal path. We now plug c and x^h back into the utility function in (A3.6) to get:

$$(A3.10) \quad V^h(1,h) = \sum_{t=0}^{\infty} \beta^t c^h + \sum_{t=0}^{\infty} [\beta^{t+1}/(1-\beta)] \log(1-d+x^h)$$

or

$$(A3.11) \quad V^h(1,h) = \log[(x^h-r-d)h + a - x^h(1 + \phi x^h/2)]/(1-\beta) + [\beta/(1-\beta)^2] \log(1-d+x^h)$$

The last equation is found by substituting (A3.10) into (A3.11). Note that $d[V^h(1,h)]/dh = \partial V^h/\partial h + (\partial V^h/\partial x^h)\partial x^h/\partial h = \partial V^h/\partial h$, where the last inequality follows from the envelope theorem (i.e. from the fact that with an optimal choice of x , $\partial V^h/\partial x = 0$). Thus, $\text{sign}(dV^h/dh) = \text{sign}(x^h-r-d) < 0$, so that $V^h(1,h)$ is strictly decreasing in h .

Now, it is easy to prove that there exists a unique h^* such that $V^{h^*}(1,h^*) = V^D(1)$. First, $V^0(1,0) = V^A(1)$, since $h = 0$ implies no borrowing at all. But $V^A(1) > V^D(1)$, so that $V^h(1,h) > V^D(1)$ for $h = 0$. Now, consider $V^{\bar{d}}(1,\bar{d})$, where \bar{d} is the limit of D_t/K_t in the no-repudiation case of Appendix 1. Since c equals zero in the limit in that case, $V^{\bar{d}}(1,\bar{d}) = -\infty < V^D(1)$. Thus, since $V^0(1,0) > V^D(1) > V^{\bar{d}}(1,\bar{d})$, and since $V^h(1,h)$ is strictly decreasing in h , there must exist an h^* between 0 and \bar{d} , such that $V^{h^*}(1,h^*) = V^D(1)$. In particular h^* is the unique root of the equation:

$$(A3.12) \quad \log\{(x^h-r-d)h + a - x^h[1 + (\phi/2)x^h]\} + [\beta/(1-\beta)] \log(1-d+x^h) = \log\{a(1-\lambda) - x^D[1 + (\phi/2)x^D]\} + [\beta/(1-\beta)] \log(1-d+x^D)$$

where x^h is the positive root of (A3.8) and x^D is the positive root of (A1.7).

We can readily establish various properties of h^* . By total

differentiation of (A3.12), we have:

$$(A3.13) \quad dh^*/da = [\xi_2 - \xi_1(1-\lambda)]/[\xi_2(r+d-x^h)] > 0$$

$$dh^*/d\lambda = -a\xi_1/[\xi_2(x^h-r-d)] > 0$$

$$dh^*/d\beta = \log[(1-d+x^h)/(1-d+x^d)]/[(1-\beta)^2\xi_2(r+d-x^h)] > 0$$

$$dh^*/dr = h\xi_2/[\xi_2(x^h-r-d)] < 0$$

where

$$\xi_1 = [a(1-\lambda) - x^d(1+\phi x^d/2)]^{-1} > 0$$

$$\xi_2 = [(x^h-r-d)h + a - x^h(1+\phi x^h/2)]^{-1} > 0$$

Finally, let us show that

$$(A3.14) \quad h < 1 + \phi x^h$$

This inequality can be derived directly from equation (A3.8). Another way is as follows. Equation (A3.8) can be derived from

$$(A3.15) \quad \underset{x}{\text{Maximize}} \quad V^h(1, h)$$

(A3.15) says that x^h maximizes the utility which is reached when the constraint on the debt is binding. The first order condition (using the definition of V^h in equation (A3.11)) yields:

$$(A3.16) \quad (1+\phi x-h)(1+x-d)(1-\beta) = \beta[h(x-r-d) + a - x(1+\phi x/2)]$$

The RHS is positive (it measures consumption per unit of capital). Therefore, $(1+\phi x-h)(1+x-d)(1-\beta)$ is positive, so that (A3.14) follows. (A3.16), when developed, yields (A3.8).

b. Two-staged Growth

Now that we have characterized the lending strategy by $D_t \leq hK_t$, we can describe explicitly the optimal pattern of growth. From equations (A3.7), we can write

$$(A3.17) \quad (1+\phi x_{t-1})(1+r) = [a + \phi x_{t-1}^2/2 + (1-d)(1+\phi x_t)][1-\gamma_{t-1}] + d_t(1+r)\gamma_{t-1}$$

$$\text{with } \gamma_{t-1} = \mu_t/[\lambda_{t-1}(1+x_{t-1}-d)]$$

when the constraint $D_t \leq hK_t$ is not binding, the shadow price γ_{t-1} is zero, so that the dynamics of the system can simply be written

$$(A3.18) \quad (1+\phi x_{t-1})(1+r) = [a + \phi x_t^2/2 + (1-d)(1+\phi x_t)]$$

Let us call T the time when the constraint on the debt is first binding (i.e. the first period for which $D_t = hK_t$). For $t \leq T-2$, we have $\gamma_t = 0$, while for $t \geq T-1$, $\gamma_t > 0$. After time T , the value of x_t will be constantly held to x_h . All that we need in order to find the entire sequence $\{x_t\}$ is the value x_{T-1} and to define the sequence $\{x_t\}_{t \leq T-2}$ by backward application to equation (A3.18).

At time $T-1$, equation (A3.17) can be written

$$(A3.19) \quad (1+\phi x_{T-1})(1+r) = [a + \phi x_h^2/2 + (1+\phi x_h)(1-d)][1-\gamma_{T-1}] + h\gamma_{T-1}(1+r)$$

(Note that we have substituted $x_h = x_T$.) In order to find the equilibrium value of x_{T-1} and γ_{T-1} , we need only to consider one more equation in x_{T-1} and γ_{T-1} .

From the system (A3.7), we can derive:

$$(A3.20) \quad [c_h(1-\gamma_{T-1})]/[\beta(1+r)] = h - [(1+r)d_{T-1} + x_{T-1}(1+\phi x_{T-1}) - a]/[1+x_{T-1}-d]$$

Equation (A3.20) can be shown to be as H_{T-1} in diagram 1. It is an upward sloping line for the values of x_{T-1} which are above x^h . When $d_{T-1} = h$, equation

(A3.20) is a modified version of the current account equation when the constraint on the debt is binding: the line H_{T-1} would pass through (x^h, γ_h) , with γ_h the stationary state value of γ_t when the constraint is binding. One can also check that a lower value of d_{t-1} yields a higher value of x_{t-1} .

The system (A3.19), (A3.20) defines the equilibrium value of (x_{T-1}, γ_{T-1}) . Here, we shall prove that (A3.19) defines a downward sloping line as depicted by Γ_{T-1} in diagram 1. To see this, one need only to check that

$$(A3.21) \quad h(1+r) < (1-d)(1+\phi x^h) + a + \phi x_h^2.$$

We already know that

$$(A3.22) \quad h < 1 + \phi x^h$$

so that we only need check that

$$(1+r)(1+\phi x^h) < (1-d)(1+\phi x^h) + a + \phi x_h^2/2$$

or, equivalently, that

$$(A3.23) \quad \phi x_h^2/2 - (r+d)\phi x^h + a - r - d \geq 0$$

To see this, it is enough to write equation (A3.17) when $t = T+1$. Both values x_T and x_{T+1} are equal to the stationary state value x^h . Since $h < 1 + \phi x^h$, equation (A3.17) can be rewritten to yield the inequality in (A3.23).

We can now represent equation (A3.19) as a downward sloping line such as Γ_{T-1} in diagram 1. Its intersection with H_{T-1} defines the equilibrium value of (x_{T-1}, γ_{T-1}) . Since H_{T-1} is below H_h (defined by $d_{t-1} = h$), this shows that x_{T-1} is greater than x^h .

We can now depict, in diagram 2, the full dynamics of the system. First, note that equation (A3.18) shows that x^h is lower than x^* , the optimal rate of

capital accumulation (see diagram 3). Second, note that x_{T-1} is below x^* (from equation (A3.19)), and equation (A3.21): $(1+\phi x_{T-1})(1+r) < a + \phi x_h^2/2 + (1+\phi x^h)(1-d) < a + \phi x^{*2}/2 + (1+\phi x^*)(1-h) = (1+\phi x^*)(1+r)$, so that $x_{T-1} < x^*$. x_{T-1} takes therefore some intermediate value between x^h and x^* and the dynamics are as in the diagram.

Appendix 4

Proof of Proposition 3

Let us prove that Ω coincides with V in 5 steps.

Step 1

First, we show:

$$(A4.1) \quad \Omega(\lambda D_0, \lambda K_0) = \Omega(D_0, K_0) + \log \lambda / (1-\beta).$$

The argument is as in Appendix 3. If $\{C_t, I_t\}$ is an optimal solution to $\Omega(D_0, K_0)$, $\{\lambda C_t, \lambda I_t\}$ is an optimal solution to $\Omega(\lambda D_0, \lambda K_0)$.

Step 2

In the constraint (34), we can write $\Omega(D_s, K_s)$ in place of $\sum_{t=s}^{\infty} \beta^{(t-s)} \log C_t$. The proof uses a standard argument of dynamic programming. If the RHS did not coincide with Ω (which is the maximum one can reach at time s), then one could increase $\Omega(D_0, K_0)$ by changing $\{C_t, I_t\}_{t \geq s}$ to the solution $\{C'_t, I'_t\}_{t \geq s}$, which maximizes $\Omega(D_s, K_s)$. Clearly this change would not violate inequality (34). Thus, (33) can be rewritten as:

$$(A4.2) \quad \Omega[D_0, K_0] = \max_{\{C_t, I_t, D_t\}} \sum_{t=0}^{\infty} \beta^t \log C_t$$

subject to $\Omega[D_s, K_s] \geq V^D(K_s)$, and (2), (4), (13), (16) and (30).

Step 3

There exists an h_{Ω} such that $\Omega(D_0, K_0) \geq V^D(K_0) \Leftrightarrow D \leq h_{\Omega} K_0$. The argument is as in Appendix 3 when (A4.1) is acknowledged.

Step 4

From Steps 2 and 3, $\Omega(D_0, K_0)$ can be written $\Omega(D_0, K_0) = \text{Max} \sum_{t=0}^{\infty} \beta^t \log C_t$
subject to $D \leq h_{\Omega} K$.

Step 5

$$h_{\Omega} = h^*$$

$h_{\Omega} \geq h^*$, since Ω defines the optimum optimum. But $h^* \geq h_{\Omega}$, since h^* defines the largest D/K ratio which keeps the country from defaulting.

Footnote

1. With the grid selected, there are two thousand points in the domain of $V_t^R(K_t, D_t)$. An optimization must be performed at each of these points for each period in which V_t^R is evaluated. Even with the coarse grid that has been used, the evaluation of V^R and V^D requires substantial computer time.

List of References

- Bardhan, Pranob, "Optimum Foreign Borrowing," in Karl Shell (ed.), Essays on the Theory of Optimal Economic Growth (Cambridge: MIT Press) 1967.
- Borchard, Edwin, State Involvement and Foreign Bondholders, Vol. 1, General Principles (New Haven: Yale University Press) 1951.
- Bulow, Jeremy I. and John B. Shoven, "The Bankruptcy Decision," Bell Journal of Economics, 1978, 437-456.
- Cohen, Daniel and Jeffrey Sachs, "LDC Borrowing with Default Risk," Kredit und Kapital, Special Issue on International Banking, 1985.
- Cohen, Daniel and Jeffrey Sachs, "The Debt of Nations," processed, Harvard University, 1985.
- D'Autume, A. and Michel, P., "Epargne et investissement et monnaie dans une perspective intertemporelle," INSEE, May 1983.
- Eaton, Jonathan, and Mark Gersovitz, Poor Country Borrowing in Private Financial Markets and the Repudiation Issue (Princeton: International Finance Section) 1981a.
- Eaton, Jonathan, and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," Review of Economic Studies 48, 1981b.
- Eaton, Jonathan, and Mark Gersovitz, "Country Risk: Economic Aspects," unpublished manuscript, Yale University, 1981.
- Feis, Herbert, Europe: The World's Banker, 1870-1914 (New Haven: Yale University press, 1930.
- Foley, Duncan K. and Martin Hellwig, "A Note on the Budget Constraint in a Model of Borrowing," Journal of Economic Theory 11, 1975, 305-314.
- Grossman, Sanford J. and Hart, Oliver D. "Corporate Financial Structure and Managerial Incentives," Conference Paper No. 48, May 1980.
- Guitian, Manuel, "Policies on Access to Fund Resources - An Overview of Conditionality," International Monetary Fund manuscript, 1981.
- International Monetary Fund, International Capital Markets, Occasional Paper No. 1, IMF: Washington, D.C., September 1980.
- International Monetary Fund, External Indebtedness of Developing Countries, Occasional Paper No. 3 May 1981.

- Jensen, Michael and William Meckling, "Theory of the Firm: Managerial Behavior, Agency Costs, and Capital Structure," Journal of Financial Economics 3, 1976, 305-360.
- Madden, John T., Marcus Nadler, and Harry C. Sauvain, America's Experience as a Creditor Nation (New York: Prentice-Hall) 1937.
- Myers, Stewart C. "Determinants of Corporate Borrowing," Journal of Finance 5, 1977, 147-155.
- Sachs, Jeffrey D., "The Current Account and Macroeconomic Adjustment in the 1970s," Brookings Papers on Economic Activity, 1981:2.
- _____, "The Current Account in the Macroeconomic Adjustment Process," Scandinavian Journal of Economics 84(2), 1982a.
- _____, "Aspects of the Current Account Behavior of OECD Economics," NBER Working Paper No. 859, February 1982b.
- _____, "LDC Debt: Problems and Prospects," in P. Wachtel (ed.), Crises in the Economic and Financial Structure (Lexington Books), 1982c.
- _____, "Theoretical Issues in International Borrowing," Princeton Studies in International Finance, No. 54, 1984.
- Smith, Clifford and J. Warner, "On Financial Contracting: An Analysis of Bond Covenants," Journal of Financial Economics 7, 1979, 117-161.
- Stiglitz, J. and A. Weiss, "Credit Rationing in Markets with Imperfect Information: I," American Economic Review, 1981.
- Winkler, Max, Foreign Bonds: An Autopsy (Philadelphia: Roland Swain) 1933.
- Wynne, William H., State Insolvency and Foreign Bondholders, Vol. II, Case Histories (New Haven: Yale University Press) 1951.