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A UNIFIED MODEL OF ENTREPRENEURSHIP DYNAMICS

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ABSTRACT

We develop an incomplete-markets *q*-theoretic model to study entrepreneurship dynamics. Precautionary motive, borrowing constraints, and capital illiquidity lead to underinvestment, conservative debt use, under-consumption, and less risky portfolio allocation. The endogenous liquid wealth-illiquid capital ratio *w* measures time-varying financial constraint. The option to accumulate wealth before entry is critical for entrepreneurship. Flexible exit option is important for risk management purposes. Investment increases and the private marginal value of liquidity decreases as *w* decreases and exit becomes more likely, contrary to predictions of standard financial constraint models. We show that the idiosyncratic risk premium is quantitatively significant, especially for low *w*.

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1. Introduction

Entrepreneurs face significant nondiversifiable business risks and liquidity constraints, both of which we refer to as frictions.[1] These frictions are important determinants for the economics of entrepreneurship. They result in incomplete markets and cause business decisions (e.g., capital accumulation and entry/exit) and household decisions (e.g., consumption/saving and asset allocation) to be highly linked, invalidating the standard complete-markets profit-maximizing analysis for entrepreneurial firms.

We develop an intertemporal model of entrepreneurship to study interdependent household and business decision making from the pre-entry stage to the post-exit stage. We model entrepreneurship as a career choice followed by a capital accumulation/business growth problem in an incomplete-markets consumption/portfolio choice framework.

Becoming an entrepreneur often requires substantial start-up costs in terms of effort, time, attention, commitment, and resources. Additionally, doing so often means giving up the outside option of being a worker elsewhere and earning wages. Thus, becoming an entrepreneur is effectively exercising a real option, which incurs both the business start-up cost and the opportunity costs of giving up the alternative career/job. Unlike standard real options, the entrepreneur's option is non-tradable, illiquid, intertwined with other decisions, and subject to important incomplete-markets frictions. Additionally, we show that the flexibility of entry timing, (i.e., the "American" feature of the option) is critically important.

By backward induction, we first study the post-entry decision making. Then, using the post-entry value function as the payoff of being an entrepreneur, we characterize the optimal entry into entrepreneurship. After setting up the firm and optimally choosing the initial size, the entrepreneur makes optimal firm investment as well as consumption-saving and portfolio choice decisions. By making entrepreneurial business illiquid, capital adjustment costs constrain the rate of investment and thus prevent the entrepreneur from targeting the ideal level of capital stock. Therefore, liquid financial wealth becomes more valuable than its pure face value because it mitigates the impact of frictions/financial constraints.

The modern q theory of investment studies optimal capital accumulation and the value of capital with costly capital adjustments. However, much of the q theory was developed for

[1] For example, see Evans and Jovanovic (1989), Gentry and Hubbard (2004), and Cagetti and De Nardi (2006). In a recent survey of research on entrepreneurship in macroeconomics, Quadrini (2009) discusses entrepreneurial career choice, entrepreneurial saving/investment, and economic development/growth.

firms owned by and run in the interest of well-diversified investors, where financial frictions do not matter and the Modigliani-Miller (MM) theorem holds.[2] However, around the world, firms are often run by entrepreneurs, founders, families, and controlling shareholders, even in publicly traded firms. La Porta, López-de-Silanes, and Shleifer (1999) document ownership concentration by controlling shareholders for large publicly traded firms around the world.[3]

One important contribution of this paper is to develop the counterpart of the modern q theory of investment for private firms run by nondiversified entrepreneurs/controlling shareholders. We do so by incorporating incomplete-markets frictions into a stochastic version of Hayashi (1982), a classic investment model with adjustment costs. We show that the *interaction* between incomplete markets and capital adjustment costs makes *liquidity* a critical determinant of corporate investment and liquidation policies.

A natural measure of liquidity is the ratio w between liquid financial wealth and illiquid physical capital. Intuitively, a larger business requires more liquid wealth for the entrepreneur to achieve the same level of financial strength, ceteris paribus. The higher the liquidity w, the less constrained the entrepreneurial firm. Liquid wealth is thus more valuable than its nominal/face value and the marginal value of liquid wealth is larger than unity.

We define enterprise value, average q, and marginal q for firms owned and run by nondiversified entrepreneurs. The entrepreneur's certainty equivalent valuation of illiquid business is the "private" enterprise value. Average q is the private enterprise value per unit of physical capital. Marginal q measures the sensitivity (marginal changes) of private enterprise value with respect to marginal changes in capital stock.

Without frictions, and with the additional assumption of convex and homogeneous capital adjustment costs as in Hayashi (1982), marginal q equals average q and investment is determined by q. When productivity shocks are independently and identically distributed (iid), the investment-capital ratio and Tobin's q are constant at all times. Moreover, prof-

[2]Brainard and Tobin (1968) and Tobin (1969) define the ratio between the firms market value to the replacement cost of its capital stock, as q and propose to use this ratio to measure the firms incentive to invest in capital. This ratio has become known as Tobin's average q. Hayashi (1982) provides conditions under which average q is equal to marginal q. Abel and Eberly (1994) develop a unified q theory of investment in neoclassic settings. Lucas and Prescott (1971) and Abel (1983) are important early contributors.

[3]See Burkart, Panunzi, and Shleifer (2003) for a model of a family firm. The law and finance/investor protection literature documents compelling evidence on concentrated ownership around the world. Lan, Wang, and Yang (2012) develop a dynamic model of investment and Tobin's q for firms run by entrenched nondiversified controlling shareholders under weak investor protection.

itability is uncorrelated with investment and the firm never gets liquidated regardless of the size of realized losses. These predictions are obviously simplistic.[4] However, we intentionally choose this stylized frictionless benchmark in order to focus on the effects of idiosyncratic risk and borrowing constraints on investment, average q, and marginal q.

With incomplete-markets frictions, investment, marginal q, and average q are all stochastic and vary with liquidity w. Investment depends on both marginal q and the marginal value of liquid wealth. Moreover, the wedge between marginal q and average q is stochastic and nonmonotonic. The option to liquidate the firm is critical for the entrepreneur to manage business downside risk.[5] The entrepreneur may prefer liquidation over continuation even before exhausting the debt capacity for risk management considerations. Liquidation becomes increasingly attractive as the entrepreneur's liquidity dries up. The optionality of liquidation makes firm value convex in liquidity w and causes corporate investment to be nonmonotonic in liquidity w, as the firm gets close to liquidation.

From the perspective of dynamic asset allocation, the entrepreneur not only chooses the risk/return profile of the portfolio (the optimal mix between the risky market portfolio and the risk-free asset), as in Merton (1971), but also determines the portfolio's optimal liquidity composition (the combination of liquid assets and the illiquid entrepreneurial business). Incomplete-markets frictions make both systematic and idiosyncratic risks matter for asset allocation.

Entrepreneurial finance, as an academic field, so far offers no apparent theoretical guidance on the cost of capital for entrepreneurial firms. We deliver an operational and analytically tractable framework to calculate the cost of capital for entrepreneurial firms. Idiosyncratic business risks as well as systematic ones have important effects on firm investment, financing, and the private equity premium. Our model provides a guideline for empirical research on private equity premium. There is much debate on the size of the idiosyncratic

[4]The Arrow-Debreu theorem holds under complete markets. Thus, consumption smoothing (utility maximization) is independent of total wealth maximization. The capital asset pricing model (CAPM) holds for the firm. Our q theory of investment under complete markets extends Hayashi (1982) to account for the (systematic) risk premium.

[5]For simplicity, we focus on the liquidation option as the exit option for downside risk protection. Without changing the analysis in any fundamental way, we can extend our model to allow the entrepreneur to have an exit option when doing well. For example, selling to diversified investors or via an initial public offering (IPO) are two ways for the entrepreneur to exit when doing well. See Pástor, Taylor, and Veronesi (2009) and Chen, Miao, and Wang (2010) for models with IPO as an exit option in good times.

risk premium. For example, Moskowitz and Vissing-Jørgensen (2002) find a low value, while Mueller (2011) finds the opposite.

We also show that the option value of waiting to become an entrepreneur (entry timing) is valuable. Before becoming an entrepreneur, the key state variable is the liquid financial wealth. We solve for the optimal cutoff level for liquid wealth and initial project size for the to-be entrepreneur. Intuitively, this cutoff wealth level depends on the outside option, fixed start-up cost, risk aversion, and other important preference and technology parameters. The initial project size trades off liquidity needs and business profitability. Cross-sectional heterogeneity among entrepreneurs along preferences, business ideas/production technology, and outside options gives rise to different entrepreneurial entry, consumption-saving, portfolio choice, capital accumulation, and business exit decisions.

While almost all existing work on the dynamics of entrepreneurship uses numerical programming, our model is analytically tractable. We provide an operational and quantitative framework to value these illiquid non-tradable options, which interact with other important decisions such as consumption-saving, firm investment, and portfolio allocation.

Quantitatively, we show that there are significant welfare costs for the entrepreneur to bear nondiversifiable idiosyncratic risk. For an expected-utility entrepreneur who has no liquid wealth and whose coefficient of relative risk aversion is two, as in our baseline calculation, the certainty equivalent valuation of the entrepreneurial business is about 11% lower than the complete-markets benchmark.

Some predictions of our model have been empirically confirmed. For example, our model predicts that the entrepreneur significantly underinvests in business, consumes less, and invests less in the market portfolio than a similarly wealthy household. Indeed, Heaton and Lucas (2000) find that entrepreneurs with variable business income hold less wealth in stocks than other similarly wealthy households.

1.1. Related literature

Our paper links to several strands of literature in finance, macroeconomics, and entrepreneurship. As we have noted earlier, our paper extends the modern q theory of investment to firms run by nondiversified entrepreneurs/controlling shareholders.

The economics of entrepreneurship literature is fast growing. Hall and Woodward (2010)

analyze the effects of nondiversifiable risk for venture capital-backed entrepreneurial firms. Heaton and Lucas (2004) show that risky non-recourse debt helps the entrepreneur diversify business risk in a static framework with capital budgeting, capital structure, and portfolio choice decisions. Chen, Miao, and Wang (2010) study the effects of nondiversifiable risk on entrepreneurial finance by building on Leland (1994).[6] They show that more risk-averse entrepreneurs borrow more in order to lower their business risk exposure.[7] Herranz, Krasa, and Villamil (2009) assess the impact of legal institutions on entrepreneurial firm dynamics.

Evans and Jovanovic (1989) show the importance of wealth and liquidity constraints for entrepreneurship.[8] Cagetti and De Nardi (2006) quantify the importance of liquidity constraints on aggregate capital accumulation and wealth distribution by constructing a model with entry, exit, and investment decisions. Hurst and Lusardi (2004) challenge the importance of liquidity constraints and provide evidence that the start-up sizes of entrepreneurial firms tend to be small. We develop a unified model of entrepreneurship and show the importance of wealth effects by incorporating endogenous entry/exit in a model with nondiversifiable risk and liquidity constraints.

Most models on portfolio choice with non-tradable income assume exogenous income.[9] Our model endogenizes the non-marketable income from business via optimal entrepreneurial decisions. The endogenous business entry/exit and consumption/portfolio decisions are important margins for the entrepreneur to manage risk. The entry/exit options significantly alter the entrepreneur's decision making. Some of our results are also related to the real options analysis under incomplete markets. Miao and Wang (2007) and Hugonnier and Morellec (2007) study the impact of nondiversifiable risk on real options exercising. These papers show that the nondiversifiable risk significantly alters option exercising strategies.

Our model also relates to recent work on dynamic corporate finance.[10] Rampini and Viswanathan (2010) develop a dynamic model of collateralized financing when contractual

[6]Morellec (2004) extends the framework to analyze managerial agency issues and leverage.

[7]For tractability, Chen, Miao, and Wang (2010) adopt exponential utility, while this paper uses non-expected Epstein-Zin utility. Also, the economic issues addressed in these two papers are rather different.

[8]See Holtz-Eakin, Joulfaian, and Rosen (1994) and Blanchflower and Oswald (1998) for empirical evidence.

[9]Merton (1971), Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), Viceira (2001), and others study consumption and portfolio choices with isoelastic utility and non-tradable labor income. Farhi and Panageas (2007) incorporate an endogenous retirement decision into a portfolio choice framework.

[10]For example, see Whited (1992), Gomes (2001), Hennessy and Whited (2005, 2007), Gamba and Triantis (2008), Riddick and Whited (2009), Eisfeldt and Rampini (2009), and Bolton, Chen, and Wang (2011).

enforcements are limited. Lorenzoni and Walentin (2007) study the relation between investment and Tobin's q in a limited-enforcement framework which generates an endogenous borrowing constraint. These papers assume risk-neutral entrepreneurs so as to focus on contractual (limited enforcements) frictions, and thus do not study the effect of the entrepreneur's precautionary demand on corporate investment.

Bolton, Chen, and Wang (2011), henceforth BCW, analyze optimal investment, financing, and risk management decisions and valuation for a financially constrained *risk-neutral* firm. BCW focus on various transaction costs that a firm incurs when raising external funds. DeMarzo, Fishman, He, and Wang (2010), henceforth DFHW, integrate the *risk-neutral* dynamic agency framework of DeMarzo and Fishman (2007b) and DeMarzo and Sannikov (2006) with the neoclassic q theory of investment. DFHW derive an optimal dynamic contract and provide financial implementation.[11]

Unlike BCW and DFHW, our paper studies entrepreneurial finance. The empirical predictions of our model also fundamentally differ from theirs. For example, our model predicts that the liquidation option makes the entrepreneurial firm value convex in liquidity near the liquidation boundary, while the liquidation options in both BCW and DFHW often make firm value concave in liquidity. We separate risk aversion from the elasticity of intertemporal substitution (EIS) by using non-expected recursive utility (Epstein and Zin, 1989). Also, we quantify the entrepreneurial firm's idiosyncratic private equity risk premium. Neither BCW nor DFHW address these issues given their risk-neutrality assumptions. The entrepreneur's risk management considerations fundamentally change the economics of business investment and entry/exit decisions. While all three papers use the seminal model of Hayashi (1982) as the MM benchmark, most importantly, the financing frictions and thus the economic issues/mechanisms in these papers are fundamentally distinct.

2. The model

We first introduce the agent's preferences and then set up the optimization problem.

Preferences. The agent has a preference featuring both constant relative risk aversion and constant EIS (Epstein and Zin, 1989; Weil, 1990). We use the continuous-time formulation

[11] DeMarzo and Fishman (2007a) analyze the impact of agency on investment dynamics in discrete time.

of this non-expected utility introduced by Duffie and Epstein (1992). That is, the agent has a recursive preference defined as follows

$$J_t = \mathbb{E}_t \left[\int_t^\infty f(C_s, J_s) ds \right], \tag{1}$$

where $f(C, J)$ is known as the normalized aggregator for consumption C and the agent's utility J. Duffie and Epstein (1992) show that $f(C, J)$ for Epstein-Zin non-expected homothetic recursive utility is given by

$$f(C, J) = \frac{\zeta}{1 - \psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1-\gamma)J)^{\chi}}{((1-\gamma)J)^{\chi-1}}, \tag{2}$$

where

$$\chi = \frac{1 - \psi^{-1}}{1 - \gamma}. \tag{3}$$

The parameter $\psi > 0$ measures the EIS, and the parameter $\gamma > 0$ is the coefficient of relative risk aversion. The parameter $\zeta > 0$ is the agent's subjective discount rate.

The widely used time-additive separable constant-relative-risk-averse (CRRA) utility is a special case of the Duffie-Epstein-Zin-Weil recursive utility specification where the coefficient of relative risk aversion is equal to the inverse of the EIS ψ, i.e., $\gamma = \psi^{-1}$ implying $\chi = 1$.[12] In general, with $\gamma \neq 1/\psi$, we can separately study the effects of risk aversion and the EIS.

Career choice and initial firm size. The agent is endowed with an entrepreneurial idea and initial wealth W_0. The entrepreneurial idea is defined by a productive capital accumulation/production function to be introduced soon. To implement the entrepreneurial idea, the agent chooses a start-up time T^0, pays a one-time fixed start-up cost Φ, and also chooses the initial capital stock K_{T^0}. One example is becoming a taxi/limo driver. The agent can first start with a used car. After building up savings, the agent tolerates risk better and potentially upgrades the vehicle. With even more savings, the agent may further increase firm size by hiring drivers and running a limo service.

Before becoming an entrepreneur, the agent can take an alternative job (e.g., to be a worker) to build up financial wealth. Being an entrepreneur is a discrete career decision.[13]

[12] For this special case, we have $f(C, J) = U(C) - \zeta J$, where $U(C)$ is the expected CRRA utility with $\gamma = \psi^{-1}$ and hence, $U(C) = \zeta C^{1-\psi^{-1}}/(1-\psi^{-1})$. Note that for CRRA utility, $f(C, J)$ is additively separable. By integrating Eq. (1) forward for this CRRA special case, we obtain $J_t = \max_C \mathbb{E}_t \left[\int_t^\infty e^{-\zeta(s-t)} U(C(s)) ds \right]$.

[13] We do not allow the agent to be a part-time entrepreneur and a part-time worker at the same time. This is a standard and reasonable assumption. For example, see Vereshchagina and Hopenhayn (2009) for a dynamic career choice model featuring the same assumption.

We naturally assume that being an entrepreneur offers potentially a higher reward at a greater risk than being a worker. Hamilton (2000) finds that earnings of the self-employed are smaller, on average, and have higher variance than earnings of workers using data from the US Census Bureau Survey of Income and Program Participation. To contrast the earnings profile differences between an entrepreneur and a worker, we assume that the outside option (by being a worker) gives the agent a constant flow of income at the rate of $r\Pi$.

At the optimally chosen (stochastic) entry time T^0, the agent uses a combination of personal savings and collateralized borrowing to finance $(K_{T^0} + \Phi)$. Lenders make zero profit in competitive capital markets. If the entrepreneur reneges on debt, creditors can always liquidate the firm's capital and recover fraction $l > 0$ per unit of capital. The borrower thus has no incentive to default on debt and can borrow up to lK at the risk-free rate by using capital as the collateral.

We will show that initial wealth W_0 plays a role in how long it takes the agent to become an entrepreneur and the choice of the firm's initial size. Borrowing constraints and nondiversifiable risk are conceptually and quantitatively important. Moreover, these two frictions interact and generate economically significant feedback effects on entrepreneurship.

Entrepreneurial idea: capital investment and production technology. The entrepreneurial idea is defined by a capital accumulation/production function.

Let I denote the gross investment. As is standard in capital accumulation models, the change of capital stock K is given by the difference between gross investment and depreciation, in that

$$dK_t = (I_t - \delta K_t)\, dt, \quad t \geq 0, \tag{4}$$

where $\delta \geq 0$ is the rate of depreciation. The firm's productivity shock dA_t over the period $(t, t + dt)$ is independently and identically distributed (iid), and is given by

$$dA_t = \mu_A dt + \sigma_A dZ_t, \tag{5}$$

where Z is a standard Brownian motion, $\mu_A > 0$ is the mean of the productivity shock, and $\sigma_A > 0$ is the volatility of the productivity shock. The firm's operating revenue over time period $(t, t + dt)$ is proportional to its time-t capital stock K_t, and is given by $K_t dA_t$. The firm's operating profit dY_t over the same period is given by

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t) dt, \tag{6}$$

where the price of the investment good is set to unity and $G(I, K)$ is the adjustment cost.

The *cumulative* productivity shock A follows an arithmetic Brownian motion process, which implies that the productivity shock for the period $(t, t + dt)$, dA_t, is iid. We thus save a state variable in the optimization problem and focus on the effects of incomplete-markets frictions on investment and the value of capital. Our specification of the productivity process differs from the conventional practice, which directly postulates a stochastic process for productivity and, thus, productivity naturally appears as a state variable in conventional q-theory models.[14]

Following Hayashi (1982), we assume that the firm's adjustment cost $G(I, K)$ is homogeneous of degree one in I and K, and write $G(I, K)$ in the following homogeneous form

$$G\left(I, K\right) = g(i)K, \tag{7}$$

where $i = I/K$ is the firm's investment-capital ratio and $g(i)$ is an increasing and convex function. With homogeneity, Tobin's average q is equal to marginal q under perfect capital markets. However, as we will show, the nondiversifiable risk drives a wedge between Tobin's average q and marginal q for the entrepreneur. For simplicity, we assume that

$$g\left(i\right) = \frac{\theta i^2}{2}, \tag{8}$$

where the parameter θ measures the degree of the adjustment cost. A higher value θ implies a more costly adjustment process.

The liquidation/exit option. The entrepreneur has an option to liquidate capital at any moment. Liquidation is irreversible and gives a terminal value lK, where $l > 0$ is a constant. Let T^l denote the entrepreneur's optimally chosen stochastic liquidation time. To focus on the interesting case, we assume capital is sufficiently productive. Thus, liquidating capital when capital markets are perfect is not optimal because doing so destroys going-concern value. However, when the entrepreneur is not well-diversified, liquidation provides an important channel for the entrepreneur to manage the downside business risk exposure. As we show later, this liquidation option is critical for the entrepreneur's optimization problem to be well-defined.[15]

[14]Specifically, a common specification of the operating profit is $Y_t = \pi_t K_t - I_t - G(I_t, K_t) - OC_t$, where OC refers to operating costs including wages, and the productivity π follows a stochastic process.

[15]Incomplete-markets frictions and the convex adjustment costs limit the rate at which the entrepreneur can adjust the firm investment in response to productivity shock. Therefore, without the liquidation/exit

Our production specification features the widely used "AK" technology augmented with capital adjustment costs. Our specification is a reasonable starting point and is also analytically tractable. Next, we turn to the agent's financial investment opportunities.

Financial investment opportunities. The agent can invest in a risk-free asset which pays a constant rate of interest r and the risky market portfolio (Merton, 1971). Assume that the incremental return dR_t of the market portfolio over time period dt is iid,

$$dR_t = \mu_R dt + \sigma_R dB_t \,, \tag{9}$$

where μ_R and σ_R are constant mean and volatility parameters of the market portfolio return process, and B is a standard Brownian motion. Let

$$\eta = \frac{\mu_R - r}{\sigma_R} \tag{10}$$

denote the Sharpe ratio of the market portfolio. Let ρ denote the correlation coefficient between the shock to the entrepreneur's business and the shock to the market portfolio. With incomplete markets ($|\rho| < 1$), the entrepreneur cannot completely hedge business risk. Nondiversifiable risk will thus play a role in decision making and private valuation.

Let W and X denote the agent's financial wealth and the amount invested in the risky asset, respectively. Then, $(W - X)$ is the remaining amount invested in the risk-free asset.

Before becoming an entrepreneur ($t \leq T^0$), the wealth accumulation is given by

$$dW_t \;=\; r\,(W_t - X_t)\,dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt + r\Pi dt\,, \quad t < T^0\,. \tag{11}$$

While being an entrepreneur, the liquid financial wealth W evolves as follows:

$$dW_t \;=\; r\,(W_t - X_t)\,dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt + dY_t\,, \quad T^0 < t < T^l. \tag{12}$$

Finally, after exiting from the business, the retired entrepreneur's wealth evolves as follows:

$$dW_t \;=\; r\,(W_t - X_t)\,dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt\,, \quad t > T^l\,. \tag{13}$$

The entrepreneur can borrow against capital K at all times, and hence, wealth W can be negative. To ensure that entrepreneurial borrowing is risk-free, we require that the

option, sufficiently large negative productivity shocks may cause the entrepreneur's total net worth to be negative and make the problem undefined for certain preferences.

liquidation value of capital lK is greater than outstanding liability, in that

$$W_t \geq -lK_t, \quad T^0 \leq t \leq T^l. \tag{14}$$

Despite being able to borrow up to lK_t at the risk-free rate r, the entrepreneur may rationally choose not to exhaust the debt capacity for precautionary reasons.[16] Without capital as collateral, the agent cannot borrow: $W_t \geq 0$ for $t \leq T^0$ and $t \geq T^l$.

The optimization problem. The agent maximizes the utility defined in Eqs. (1)–(2). The timeline can be described in five steps. First, before becoming an entrepreneur ($t \leq T^0$), the agent collects income as a worker and chooses consumption and portfolio allocations. Second, the agent chooses the optimal entry time T^0 to start up the firm and the initial firm size K_{T^0} by incurring the fixed start-up cost Φ, and financing the total costs $(K_{T^0} + \Phi)$ with savings and potentially some collateralized borrowing. Third, the agent chooses consumption and portfolio choice while running the firm subject to the collateralized borrowing limit (14). Fourth, the agent optimally chooses the stochastic liquidation time T^l. Finally, after liquidating capital, the agent collects the liquidation proceeds, retires, allocates wealth between the risk-free and the risky market portfolio, and consumes.

3. Benchmark: Complete markets

With complete markets, the entrepreneur's optimization problem can be decomposed into two separate ones: total wealth maximization and utility maximization. We will show that our model has the homogeneity property. The lower case denotes the corresponding variable in the upper case scaled by K. For example, w denotes the liquid wealth-illiquid capital ratio, $w = W/K$. The following proposition summarizes main results under complete markets.

Proposition 1. *The entrepreneur's value function $J^{FB}(K, W)$ is given by*

$$J^{FB}(K, W) = \frac{(bP^{FB}(K, W))^{1-\gamma}}{1-\gamma}, \tag{15}$$

[16] In corporate finance, collateral often plays a critical role for debt capacity. For example, see Rampini and Viswanathan (2010, 2011) for models of collateral, debt capacity, capital structure, and risk management when contractual enforcement is limited.

where the total wealth $P^{FB}(K,W)$ *is given by the sum of* W *and firm value* $Q^{FB}(K)$

$$P^{FB}(K,W) = W + Q^{FB}(K) = W + q^{FB}K\,, \tag{16}$$

and

$$b = \zeta\left[1 + \frac{1-\psi}{\zeta}\left(r - \zeta + \frac{\eta^2}{2\gamma}\right)\right]^{\frac{1}{1-\psi}}\,. \tag{17}$$

Firm value $Q^{FB}(K)$ *is equal to* $q^{FB}K$*, where Tobin's q,* q^{FB}*, is given by*

$$q^{FB} = 1 + \theta i^{FB}\,, \tag{18}$$

where the first-best investment-capital ratio i^{FB} *is given by*

$$i^{FB} = (r+\delta) - \sqrt{(r+\delta)^2 - \frac{2}{\theta}\left(\mu_A - \rho\eta\sigma_A - (r+\delta)\right)}. \tag{19}$$

The optimal consumption C *is proportional to* K*, i.e.,* $C(K,W) = c^{FB}(w)K$*, where*

$$c^{FB}(w) = m^{FB}\left(w + q^{FB}\right), \tag{20}$$

and m^{FB} *is the marginal propensity to consume (MPC) and is given by*

$$m^{FB} = \zeta + (1-\psi)\left(r - \zeta + \frac{\eta^2}{2\gamma}\right)\,. \tag{21}$$

The market portfolio allocation X *is also proportional to* K*,* $X(K,W) = x^{FB}(w)K$*, where*

$$x^{FB}(w) = \left(\frac{\mu_R - r}{\gamma\sigma_R^2}\right)\left(w + q^{FB}\right) - \frac{\rho\sigma_A}{\sigma_R}\,. \tag{22}$$

The capital asset pricing model (CAPM) holds for the firm with its expected return given by

$$\xi^{FB} = r + \beta^{FB}\left(\mu_R - r\right)\,, \tag{23}$$

where the firm's beta, β^{FB}*, is constant and given by*

$$\beta^{FB} = \frac{\rho\sigma_A}{\sigma_R}\frac{1}{q^{FB}}. \tag{24}$$

Eqs. (18) and (19) give Tobin's q and the investment-capital ratio, respectively. The adjustment cost makes installed capital earn rents and, hence, Tobin's q differs from unity. Note that the average q is equal to the marginal q as in Hayashi (1982). The entrepreneur's total wealth is given by $p^{FB}(w) = w + q^{FB}$, the sum of q^{FB} and liquidity measure w.

Eq. (20) gives consumption, effectively the permanent-income rule under complete markets. The entrepreneur's MPC out of wealth m^{FB} generally depends on the risk-free rate r, the EIS ψ, the coefficient of risk aversion γ, and the Sharpe ratio $\eta = (\mu_R - r)/\sigma_R$. Eq. (22) gives $x(w)$, the portfolio allocation to the market portfolio. The first term in Eq. (22) is the well-known mean-variance allocation, and the second term is the intertemporal hedging demand.

We explicitly account for the effects of risk on investment and Tobin's q. We decompose the total volatility of the productivity shock into systematic and idiosyncratic components. The systematic volatility is equal to $\rho\sigma_A$ and the idiosyncratic component is given by

$$\epsilon = \sigma_A \sqrt{1-\rho^2}\,. \tag{25}$$

The standard CAPM holds in our benchmark. The expected return is given in Eq. (23) and β is given by Eq. (24). As in standard asset pricing theory, the idiosyncratic volatility ϵ carries no risk premium and plays no role under complete markets. However, importantly, the idiosyncratic volatility ϵ will play a significant role in our incomplete-markets setting.

4. Incomplete-markets model solution: Post-entry

Having characterized the complete-markets solution, we now turn to the incomplete-markets setting. We first consider the agent's decision problem after liquidation, and then derive the entrepreneur's interdependent decision making before exit.

The agent's decision problem after exiting entrepreneurship. After exiting from entrepreneurship, the entrepreneur is no longer exposed to the business risk and faces a classic Merton consumption/portfolio allocation problem with non-expected recursive utility. The solution is effectively the same as the complete-markets results in Proposition 1 (without physical capital). We summarize the results as a corollary to Proposition 1.

Corollary 1. *The entrepreneur's value function takes the following homothetic form,*

$$V(W) = \frac{(bW)^{1-\gamma}}{1-\gamma}, \tag{26}$$

where b is a constant given in Eq. (17). The optimal consumption C and allocation amount X in the risky market portfolio are respectively given by

$$C = m^{FB} W \,, \tag{27}$$

$$X = \left(\frac{\mu_R - r}{\gamma \sigma_R^2} \right) W \,, \tag{28}$$

where m^{FB} is the MPC out of wealth and is given in Eq. (21).

The entrepreneur's decision problem while running his business. Let $J(K, W)$ denote the entrepreneur's value function. The entrepreneur chooses consumption C, real investment I, and the allocation to the risky market portfolio X by solving the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\begin{aligned} 0 = \max_{C, I, X} \; & f(C, J) + (I - \delta K) J_K + (rW + (\mu_R - r) X + \mu_A K - I - G(I, K) - C) J_W \\ & + \left(\frac{\sigma_A^2 K^2 + 2\rho \sigma_A \sigma_R K X + \sigma_R^2 X^2}{2} \right) J_{WW} \,. \end{aligned} \tag{29}$$

The entrepreneur's first-order condition (FOC) for consumption C is given by

$$f_C(C, J) = J_W(K, W) \,. \tag{30}$$

The above condition states that the marginal utility of consumption f_C is equal to the marginal utility of wealth J_W. The FOC with respect to investment I gives

$$(1 + G_I(I, K)) \, J_W(K, W) = J_K(K, W) \,. \tag{31}$$

To increase capital stock by one unit, the entrepreneur needs to forgo $(1 + G_I(I, K))$ units of wealth. Therefore, the entrepreneur's marginal cost of investing is given by the product of $(1 + G_I(I, K))$ and the marginal utility of wealth J_W. The marginal benefit of adding a unit of capital is J_K. At optimality, the entrepreneur equates the two sides of Eq. (31).

The FOC with respect to portfolio choice X is given by

$$X = -\frac{\mu_R - r}{\sigma_R^2} \frac{J_W(K, W)}{J_{WW}(K, W)} - \frac{\rho \sigma_A}{\sigma_R} K \,. \tag{32}$$

The first term in Eq. (32) is the mean-variance demand, and the second term captures the hedging demand. Using the homogeneity property, we conjecture that the value function $J(K, W)$ is given by

$$J(K, W) = \frac{(bP(K, W))^{1-\gamma}}{1 - \gamma} \,, \tag{33}$$

where b is given in Eq. (17). Comparing Eq. (33) with the value function without the business Eq. (26), we may intuitively refer to $P(K,W)$ as the entrepreneur's certainty equivalent (CE) wealth, the minimal amount of wealth for which the agent is willing to permanently give up the business and liquid wealth W. Let $\underline{W}$ denote the entrepreneur's *endogenous* liquidation boundary and $\underline{w} = \underline{W}/K$. The following theorem summarizes the entrepreneur's decision making and scaled CE wealth $p(w) = P(K,W)/K$.

Theorem 1. *The entrepreneur operates the business if and only if $w \geq \underline{w}$. The scaled CE wealth $p(w)$ solves the following ordinary differential equation (ODE),*

$$\begin{aligned} 0 = & \frac{m^{FB} p(w)(p'(w))^{1-\psi} - \psi\zeta p(w)}{\psi - 1} - \delta p(w) + (r+\delta) w p'(w) + (\mu_A - \rho\eta\sigma_A) p'(w) \\ & + \frac{(p(w) - (w+1)p'(w))^2}{2\theta p'(w)} + \frac{\eta^2 p(w) p'(w)}{2h(w)} - \frac{\epsilon^2 h(w) p'(w)}{2p(w)}, \quad \text{if } w \geq \underline{w}, \end{aligned} \tag{34}$$

where ϵ is the idiosyncratic volatility given in Eq. (25) and $h(w)$ is given by

$$h(w) = \gamma p'(w) - \frac{p(w)p''(w)}{p'(w)}\,. \tag{35}$$

When w approaches ∞, $p(w)$ approaches the complete-markets solution given by

$$\lim_{w\to\infty} p(w) = w + q^{FB}\,. \tag{36}$$

Finally, the ODE (34) satisfies the following conditions at the endogenous boundary $\underline{w}$,

$$p(\underline{w}) = \underline{w} + l, \tag{37}$$

$$p'(\underline{w}) = 1\,. \tag{38}$$

The optimal consumption $c = C/K$, investment $i = I/K$, and market portfolio allocation-capital ratio $x = X/K$ are given by

$$c(w) = m^{FB} p(w)(p'(w))^{-\psi}, \tag{39}$$

$$i(w) = \frac{1}{\theta}\left(\frac{p(w)}{p'(w)} - w - 1\right), \tag{40}$$

$$x(w) = -\frac{\rho\sigma_A}{\sigma_R} + \frac{\mu_R - r}{\sigma_R^2}\frac{p(w)}{h(w)}\,, \tag{41}$$

where $h(w)$ is given in Eq. (35). The dynamics of the wealth-capital ratio w are given by

$$dw_t = \mu_w(w_t)dt + \sigma_R x(w_t) dB_t + \sigma_A dZ_t\,, \tag{42}$$

where the drift $\mu_w(w)$ gives the expected change of w and is given by

$$\mu_w(w) = (r + \delta - i(w))w + (\mu_R - r)x(w) + \mu_A - i(w) - g(i(w)) - c(w)\,. \tag{43}$$

However, if Eqs. (37)–(38) do not admit an interior solution satisfying $\underline{w} > -l$, the optimal liquidation boundary is then given by the maximal borrowing capacity, $\underline{w} = -l$. We note that the scenario where the constraint binds can only occur when the coefficient of relative risk aversion $\gamma < 1$.[17]

To highlight the critical role played by the adjustment costs, we first analyze the case with no adjustment costs, which serves as a natural comparison benchmark.

A special case: Incomplete markets with no adjustment costs. We show that incomplete markets alone have no effects on portfolio allocation. With liquid physical capital (no adjustment costs), the entrepreneur optimally allocates a constant fraction of wealth invested in the firm: the fraction $K/(K+W)$ is constant and is given by

$$\frac{K}{K+W} \;=\; \frac{\mu_A - \rho\eta\sigma_A - (r+\delta)}{\gamma(1-\rho^2)\sigma_A^2}\,. \tag{44}$$

Intuitively, the firm needs to be sufficiently productive to ensure that the entrepreneur takes a long position in the firm. Specifically, we need the risk-adjusted productivity, $\mu_A - \rho\eta\sigma_A$, to be larger than the user cost of capital, $r + \delta$, in that

$$\mu_A - \rho\eta\sigma_A > r + \delta\,. \tag{45}$$

The entrepreneur has a time-invariant portfolio allocation rule Eq. (46) as in Merton (1971) with no adjustment costs under incomplete markets. By dynamically adjusting the size of capital stock (either upward or downward frictionlessly), markets are effectively complete for the entrepreneur. The optimal liquidity ratio w is constant and is given by

$$w \;=\; \frac{\gamma(1-\rho^2)\sigma_A^2}{\mu_A - \rho\eta\sigma_A - (r+\delta)} - 1\,. \tag{46}$$

Both marginal q and average q equal unity. Since liquidation recovers $l < 1$ per unit of capital, the entrepreneur never liquidates capital without adjustment costs even under incomplete markets.[18]

[17] When $\gamma \geq 1$ and without the exit option, the entrepreneur's utility approaches minus infinity at $\underline{w} = -l$ with positive probability. Therefore, when $\gamma \geq 1$, the exit option is necessary for the entrepreneur's optimization problem to be well-defined. The entrepreneur rationally stays away from the constraint, $\underline{w} > -l$.

[18] We are grateful to the referee for suggesting that we explicitly analyze this special case and correctly conjecturing the intuitive solution. Details are available upon request.

In our model, the adjustment cost makes it difficult for the entrepreneur to hold an optimal portfolio mix of the market portfolio, the risk-free asset, and a position in the entrepreneurial firm. Using this special no-adjustment-cost case, we show that the incomplete-markets friction alone does not distort the entrepreneur's optimal portfolio allocation. It is the *interactive* effect between the adjustment cost and incomplete markets that generate novel dynamic properties of investment, consumption, portfolio allocation, and business exit.

5. Results: Post-entry

Parameter choices. When applicable, all parameter values are annualized. The risk-free interest rate is $r = 4.6\%$ and the aggregate equity risk premium is $(\mu_R - r) = 6\%$. The annual volatility of the market portfolio return is $\sigma_R = 20\%$ implying the Sharpe ratio for the aggregate stock market $\eta = (\mu_R - r)/\sigma_R = 30\%$. The subjective discount rate is set to equal to the risk-free rate, $\zeta = r = 4.6\%$.

On the real investment side, our model is a version of the q theory of investment (Hayashi, 1982). Using the sample of large firms in Compustat from 1981 to 2003, Eberly, Rebelo, and Vincent (2009) provide empirical evidence in support of Hayashi (1982). Using their work as a guideline, we set the expected productivity $\mu_A = 20\%$ and the volatility of productivity shocks $\sigma_A = 10\%$. Fitting the complete-markets q^{FB} and i^{FB} to the sample averages, we obtain the adjustment cost parameter $\theta = 2$ and the rate of depreciation for capital stock $\delta = 12.5\%$.[19] We choose the liquidation parameter $l = 0.9$ (Hennessy and Whited, 2007). We set the correlation between the market portfolio return and the business risk $\rho = 0$, which implies that the idiosyncratic volatility of the productivity shock $\epsilon = \sigma_A = 10\%$. We consider two widely used values for the coefficient of relative risk aversion, $\gamma = 2$ and $\gamma = 4$. We set the EIS to be $\psi = 0.5$, so that the first case corresponds to the expected utility with $\gamma = 1/\psi = 2$, and the second case maps to a non-expected utility with $\gamma = 4 > 1/\psi = 2$. Table 1 summarizes the notations and if applicable, value choices for various parameters.

[19]The averages are 1.3 for Tobin's q and 0.15 for the investment-capital ratio, respectively, for the sample used by Eberly, Rebelo, and Vincent (2009). The imputed $\theta = 2$ is in the range of estimates used in the literature. See Whited (1992), Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).

5.1. The entrepreneur's welfare

The entrepreneur's welfare is measured by the value function given in Eq. (33), which is homogeneous of degree $(1-\gamma)$ in the certainty equivalent wealth $P(K,W)$.

Private enterprise value and average q. In corporate finance, enterprise value is defined as firm value excluding liquid assets (e.g., cash and other short-term marketable securities). Similarly, we may define private enterprise value $Q(K,W)$ for an entrepreneurial firm as follows,

$$Q(K,W) = P(K,W) - W. \tag{47}$$

Private average q is given by the ratio between private enterprise value $Q(K,W)$ and capital

$$q(w) = \frac{Q(K,W)}{K} = p(w) - w\,. \tag{48}$$

Importantly, private average q defined in Eq. (48) reflects the impact of nondiversifiable risk on the subjective valuation of capital. In the limit as $w \to \infty$, $q(w)$ approaches q^{FB}.

Insert Fig. 1 here.

For Figs. 1–4, we graph for two levels of risk aversion, $\gamma = 2, 4$. Panels A and B of Fig. 1 plot $p(w)$ and $q(w)$, respectively. Note that $p(w)$ and $q(w) = p(w) - w$ convey the same information. Graphically, it is easier to read Panel B for $q(w)$ than Panel A for $p(w)$, thus, we discuss $q(w)$. The less risk-averse the entrepreneur, the higher private average $q(w)$. Intuitively, $q(w)$ increases with w. As $w \to \infty$, the entrepreneur effectively attaches no premium for the nondiversifiable risk and thus, $\lim_{w\to\infty} q(w) \to q^{FB} = 1.31$. However, quantitatively, the convergence requires a high value of w. At $w = 3$, $q(3) = 1.23$ for $\gamma = 2$, and $q(3) = 1.21$ for $\gamma = 4$, both of which are significantly lower than $q^{FB} = 1.31$.

Importantly, $q(w)$ is not globally concave. Risk aversion does not imply that $q(w)$ is concave, even though the risk-averse entrepreneur's value function $J(K,W)$ is concave in W. Panel B of Fig. 1 shows that $q(w)$ is concave in w for $w \geq \widetilde{w}$ where $\widetilde{w}$ is the inflection point at which $q''(\widetilde{w}) = p''(\widetilde{w}) = 0$. The inflection point is $\widetilde{w} = -0.658$ for $\gamma = 2$, and $\widetilde{w} = -0.495$ for $\gamma = 4$. For an entrepreneur with sufficient financial slack ($w \geq \widetilde{w}$), $q(w)$ is concave in w. For an entrepreneur with low financial slack ($w \leq \widetilde{w}$), $q(w)$ is convex in w. The exit option allows the entrepreneur to eliminate the nondiversifiable business risk exposure and thus causes $q(w)$ to be convex in w for sufficiently low w.

The marginal value of wealth $P_W(K,W)$. For public firms, the marginal impact of cash on firm value is referred to as the marginal value of cash (Bolton, Chen, and Wang, 2011). For entrepreneurial firms, $P_W(K,W)$ measures the entrepreneur's marginal (certainty equivalent) value of liquid wealth, which is the natural counterpart to the marginal value of cash for public firms.

Panel C of Fig. 1 plots $P_W(K,W) = p'(w)$. With complete markets, $P_W(K,W) = 1$. With incomplete markets, $P_W(K,W) \geq 1$ because wealth has the additional benefit of mitigating financial constraints due to nondiversifiable risk on investment and consumption. Note that $p'(w) = 1$ at the liquidation boundary $\underline{w}$, because the agent is no longer exposed to nondiversifiable risk after exiting entrepreneurship. Then, $p'(w)$ increases with w up to the endogenous inflection point $\widetilde{w}$ (at which $p''(\widetilde{w}) = 0$), decreases with w for $w \geq \widetilde{w}$, and finally approaches unity as $w \to \infty$ and reaches the complete-markets solution.

Loose arguments may have led us to conclude that less constrained entrepreneurs (i.e., higher w) value their wealth less and $P_W(K,W)$ decreases with wealth ($p''(w) < 0$). This is incorrect because of the liquidation option, as we see in Panel C.

Marginal value of capital $P_K(K,W)$, also referred to as (private) marginal q. For public firms owned by diversified investors, the marginal change of firm value with respect to an increase in capital is known as marginal q. For a firm owned and managed by a nondiversified entrepreneur, we naturally refer to the marginal increase of $P(K,W)$ with respect to an increase of capital, $P_K(K,W)$, as the private (subjective) marginal q. Using the homogeneity property, we may write the private marginal q as follows,

$$P_K(K,W) \;\; = \;\; p(w) - wp'(w)\,. \tag{49}$$

Panel D of Fig. 1 plots the private marginal q, $P_K(K,W)$. Note that the private marginal q is not monotonic in w. One seemingly natural but loose intuition is that the (private) marginal q increases with w. Presumably, less financially constrained entrepreneurs face lower costs of investment and hence have higher marginal q. However, this intuition in general does not hold. Using the formula (49) for private marginal q, we obtain

$$\frac{dP_K(K,W)}{dw} = -wp''(w)\,. \tag{50}$$

Therefore, the sign of $dP_K(K,W)/dw$ depends on both the sign of w and the concavity of $p(w)$. When $w > 0$ and $p(w)$ is concave, $P_K(K,W)$ increases with w. When the entrepreneur is in debt ($w < 0$) and additionally $p(w)$ is convex, $P_K(K,W)$ also increases with w. In the intermediate region of w, $P(K,W)$ may decrease with w (e. g., when $w < 0$ and $p''(w) < 0$). Additionally, marginal q may exceed the first-best q^{FB} in the debt region, $w < 0$.

Marginal q versus average q. Under complete markets, marginal q equals average q as in Hayashi (1982). Given iid shocks, q is constant. However, with nondiversifiable risk, marginal q differs from average q. The wedge between marginal q and average q is given by

$$P_K(K,W) - q(w) = -w\left(p'(w) - 1\right) . \tag{51}$$

The sign of this wedge is given by the sign of w (note $p'(w) \geq 1$). If $w > 0$, increasing K makes the entrepreneur more constrained by mechanically lowering $w = W/K$, and thus gives rise to a negative wedge $P_K(K,W) - q(w)$. Generally, increasing K makes the entrepreneur richer. However, for an entrepreneur in debt ($W < 0$), increasing K moves w from the left towards the origin, which relaxes financial constraints and thus implies a positive wedge $P_K(K,W) - q(w)$. Therefore, the wedge between the qs is nonmonotonic in w.

5.2. Optimal investment and exercising of the liquidation option

The FOC (31) for investment may be simplified as follows,

$$1 + \theta i(w) = \frac{P_K(K,W)}{P_W(K,W)} . \tag{52}$$

The left side is the marginal cost of investing. The right side is the ratio between the marginal q, $P_K(K,W)$, and the marginal value of cash $p'(w)$. The entrepreneur equates the two sides of Eq. (52) by optimally choosing investment. Investment thus depends on not only the marginal q but also the marginal value of cash.[20] Both the private marginal q and P_W are endogenously determined. Moreover, they are highly correlated.

[20]Bolton, Chen, and Wang (2011) derive a similar FOC for investment in a dynamic corporate finance framework when cash/credit is the marginal source of financing for a (risk-neutral) financially constrained firm. In an optimal dynamic contracting framework, DeMarzo, Fishman, He, and Wang (2010) also derive a similar investment FOC under endogenous financial constraints. However, the economic settings, economic interpretations and financial frictions are different across these papers. In Bolton, Chen, and Wang (2011), the friction is costly external financing. In DeMarzo, Fishman, He, and Wang (2010), the friction is dynamic managerial agency (e.g. hidden cash flow diversion or effort choices). In our paper, it is the nondiversifiable (i.e. non-spanned) idiosyncratic business risk and liquidity constraints under incomplete markets.

We point out that marginal q plays a different role in our model as compared to the existing work on the q theory of investment. In standard q models with convex adjustment costs, the firm optimally equates the marginal cost of investing with the marginal q, and the latter encapsulates time-variation in the firm's investment opportunities. Unlike these q models, it is the incomplete-markets friction-induced demand for liquidity that drives changes in marginal q in our model. Importantly, the incomplete-markets friction alone (without adjustment costs) does not generate any interesting dynamics for investment and marginal q as we show at the end of Section 4. In our model, the liquidity ratio w determines the marginal value of liquid wealth P_W, the marginal q, and hence firm investment via the FOC (52). Our work thus complements the existing work in the q-theory literature by focusing on the incomplete-markets frictions on dynamics of marginal q and investment.

Insert Fig. 2 here.

Fig. 2 plots $i(w)$ and $i'(w)$, the sensitivity of $i(w)$ with respect to w. Nondiversifiable business risk induces underinvestment, $i(w) < i^{FB} = 0.156$. The underinvestment result (relative to the first-best MM benchmark) is common in incomplete-markets models.

However, investment-capital ratio is not monotonic in w, which implies that investment may decrease with wealth! This seemingly counterintuitive result directly follows from the convexity of $p(w)$ in w. We may characterize $i'(w)$ as follows,

$$i'(w) \quad = \quad -\frac{p(w)p''(w)}{\theta(p'(w))^2}. \tag{53}$$

Using the above result, we see that whenever $p(w)$ is concave, investment increases with wealth. However, whenever $p(w)$ is convex, investment decreases with w. Put differently, underinvestment is less of a concern when the entrepreneur is closer to liquidating the business because liquidation also has the benefit of leading the entrepreneur to exit incomplete markets. The entrepreneur has weaker incentives to cut investment if the distance to exiting incomplete markets is shorter. This explains why investment may decrease in w when the exit option is sufficiently close to being in the money (i.e., when w is sufficiently low).

Now we turn to the entrepreneur's liquidation decision. Costly liquidation of capital provides a downside risk protection for the entrepreneur. Quantitatively, this exit option is quite valuable for low w. The exit option generates convexity near the endogenous left

boundary $\underline{w}$. Recall that debt is fully collateralized and is risk-free. Thus, liquidation only provides an exit option which becomes in the money for the entrepreneur bearing significant nondiversifiable risks (i.e., being sufficiently low in w). In Zame (1993), Heaton and Lucas (2004), and Chen, Miao, and Wang (2010), the benefits of debt rely on the riskiness of debt, which creates state-contingent insurance. A liquidation option in our model provides downside protection, as a default option in risky debt models does.

Diversification is more valuable for more risk-averse entrepreneurs, therefore, a more risk-averse entrepreneur liquidates capital earlier in order to avoid idiosyncratic risk exposure and achieve full diversification. The optimal liquidation boundaries are $\underline{w} = -0.8$ for $\gamma = 2$ and $\underline{w} = -0.65$ for $\gamma = 4$, respectively. Note that the borrowing constraint does not bind even for a less risk-averse entrepreneur (e.g., $\gamma = 2$). The entrepreneur rationally liquidates capital before exhausting the debt capacity $w \geq -l = -0.9$ to ensure that wealth does not fall too low. While borrowing more to invest is desirable in terms of generating positive value for (diversified) investors, doing so may be too risky for nondiversified entrepreneurs. Moreover, anticipating that the liquidation option will soon be exercised, the entrepreneur has less incentive to distort investment when w is close to the liquidation boundary. This option anticipation effect explains the nonmonotonicity result for $i(w)$ in w. Next, we turn to the entrepreneur's portfolio choice decisions.

5.3. Optimal portfolio allocation and consumption

The entrepreneur's market portfolio allocation $x(w)$ has both a hedging demand term given by $-\rho\sigma_A/\sigma_R$ and a mean-variance demand term given by $\eta\sigma_R^{-1}p(w)/h(w)$. The hedging term $-\rho\sigma_A/\sigma_R$ is constant because of time-invariant real and financial investment opportunities (Merton, 1971). We thus focus on the more interesting mean-variance demand term.

Unlike the standard portfolio allocation, the entrepreneur incorporates the impact of nondiversifiable risk by (1) replacing $w + q^{FB}$ with $p(w)$ in calculating "total" wealth and (2) adjusting risk aversion from γ to the effective risk aversion $h(w)$ given in Eq. (35).

Insert Fig. 3 here.

Panel A of Fig. 3 plots $h(w)$ for $\gamma = 2, 4$. For the first-best (complete-markets) case, effective risk aversion $h(w)$ equals γ. In the region where $w \geq -0.584$ for $\gamma = 2$ and

$w \geq -0.437$ for $\gamma = 4$, $h(w)$ decreases with w as self insurance becomes more effective. In the limit as $w \to \infty$, $h(w) \to \gamma$. Importantly, in the region where $w \leq -0.584$ for $\gamma = 2$ and $w \leq -0.437$ for $\gamma = 4$, $h(w)$ decreases as w decreases towards the exit option. Near the optimal liquidation boundary $\underline{w}$, the effective risk aversion $h(\underline{w})$ is lower than γ, which follows from $h(\underline{w}) = \gamma - (\underline{w} + l)p''(\underline{w}) < \gamma$. Intuitively, when the liquidation option is in the money, the entrepreneur behaves in a *less* risk-averse manner than under complete markets because of the positive effect of volatility on the option value. Panel A also shows that $h(w)$ peaks at interior values of w and is nonmonotonic in w.

Panel B of Fig. 3 plots the demand for the market portfolio $x(w)$. The first-best optimal portfolio allocation $x^{FB}(w)$ is linearly increasing in w. With incomplete markets, portfolio allocation becomes more conservative, $x(w) < x^{FB}(w)$. This prediction is consistent with empirical findings. For example, Heaton and Lucas (2000) find that entrepreneurs with high and variable business income hold less wealth in stocks than other similarly wealthy households. The intuition is as follows. Incomplete-markets frictions lower the certainty equivalent wealth $p(w)$ and also make the marginal value of wealth $p'(w) > 1$, both of which lead to lower portfolio allocation. Additionally, portfolio allocation $x(w)$ becomes more aggressive as the exit option becomes deeper in the money, i.e. as w approaches the liquidation boundary $\underline{w}$. For example, $x(w)$ is decreasing in w when $w \leq -0.684$ for $\gamma = 2$ and when $w \leq -0.509$ for $\gamma = 4$.

Insert Fig. 4 here.

Consumption is also lower than the complete-markets benchmark, $c(w) < c^{FB}(w)$ because frictions imply $p(w) < p^{FB}(w)$ and $p'(w) > 1$. Panel A of Fig. 4 plots $c(w)$ for $\gamma = 2, 4$. The MPC $m^{FB} = 0.057$ for $\gamma = 2$, which is higher than $m^{FB} = 0.052$ for $\gamma = 4$. The less risk-averse entrepreneur consumes more (when $\psi < 1$). Panel B of Fig. 4 plots the MPC out of wealth, $C_W(K, W) = c'(w)$. Note that the MPC $C_W = c'(w)$ is not monotonic in w; it first increases with w and then decreases with w. With sufficiently large slack w ($w \geq -0.556$ for $\gamma = 2$ and $w \geq -0.397$ for $\gamma = 4$), the MPC $c'(w)$ decreases with w, which implies a standard concave consumption function induced by incomplete markets frictions. However, with low slack w ($w \leq -0.556$ for $\gamma = 2$ and $w \leq -0.397$ for $\gamma = 4$), the flexible exit/liquidation is sufficiently in the money and causes the entrepreneur's consumption rule to be convex in that region. This finding shows that the standard concave consumption function (Carroll

and Kimball, 1996) is not robust to a more general incomplete-markets environment where the entrepreneur has the exit option and is sufficiently constrained.

6. Entrepreneurial entry: Career choice and firm size

We have studied the agent's decision making and valuation after becoming an entrepreneur. However, what causes the agent to become an entrepreneur and when? These are clearly important questions. We analyze two cases: first, a time-0 binary career decision and then a richer model allowing for the choice of entry *timing*.

6.1. When career choice is "now or never," a binary decision

First, we consider the case where the agent has a time-0 binary choice to be an entrepreneur or take the outside option. By taking the outside option, the agent collects a constant perpetuity with payment $r\Pi$, which has present value Π. The agent's optimal consumption and portfolio choice problem gives the value function $V(W_0 + \Pi)$ where $V(\,\cdot\,)$ is given in Eq. (26).

By being an entrepreneur, the agent incurs a fixed start-up cost Φ and then chooses the initial project size K_0. Wealth immediately drops from W_0 to $W_0 - (\Phi + K_0)$ at time 0. Note that the entrepreneur can borrow up to lK, the liquidation value of capital, which implies

$$W_0 \geq \Phi + (1 - l)K_0\,. \tag{54}$$

To rule out the uninteresting case where the entrepreneur makes instant profits by starting up the business and then immediately liquidating capital for profit, we require $l < 1$.

The agent chooses K_0 to maximize value function $J(K_0, W_0 - (\Phi + K_0))$, which is equivalent to maximizing CE wealth $P(K_0, W_0 - (\Phi + K_0))$ by solving

$$\max_{K_0} \quad P(K_0, W_0 - (\Phi + K_0))\,, \tag{55}$$

subject to the borrowing constraint (54). Let K_0^* denote the optimal initial capital stock. Finally, the agent compares $P(K_0^*, W_0 - (\Phi + K_0^*))$ from being an entrepreneur with $W_0 + \Pi$, and makes the career decision. The following theorem summarizes the main results.

Theorem 2. *At time* 0*, the agent chooses to be an entrepreneur if and only if the initial wealth* W_0 *is greater than the threshold wealth level* $\overline{W}_0$*, which is given by*

$$\overline{W}_0 = \frac{\Phi p'(w^*) + \Pi}{p'(w^*) - 1}\,, \tag{56}$$

and w^* *is the solution of the following equation,*

$$p'(w^*) = \frac{p(w^*)}{1 + w^*}\,. \tag{57}$$

The entrepreneurial firm's initial size K_0^* *is given by*

$$K_0^* = \frac{W_0 - \Phi}{1 + w^*}\,. \tag{58}$$

The entrepreneur's CE wealth is then given by

$$P\left(K_0^*, W_0 - \Phi - K_0^*\right) = p(w^*)K_0^* = p'(w^*)\left(W_0 - \Phi\right)\,, \tag{59}$$

where w^* *is given by Eq. (*57*). After starting up the firm, the agent chooses consumption, portfolio allocation, and firm investment/liquidation decisions as described by Theorem* 1*.*

The optimal initial wealth-capital ratio, given by $w_0 \equiv (W_0 - \Phi)/K_0^* - 1 = w^*$, is independent of the fixed start-up cost Φ and outside option value Π, as we see from Eq. (57). The agent's certainty equivalent wealth at time 0 is then given by

$$E(W_0) = \max\{W_0 + \Pi,\, p'(w^*)\left(W_0 - \Phi\right)\}\,. \tag{60}$$

Being an entrepreneur is optimal if and only if $W_0 \geq \overline{W}_0$, where $\overline{W}_0$ is given by Eq. (56).

Insert Fig. 5 here.

Fig. 5 plots the firm's initial size K_0^* and the CE wealth $E(W_0)$ as functions of initial wealth W_0 for two levels of risk aversion, $\gamma = 2, 4$. We set the outside option value $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$. First, risk aversion plays a significant role in determining entrepreneurship (see Panel A of Fig. 5). The threshold for the initial wealth $\overline{W}_0$ to become an entrepreneur increases significantly from 2.86 to 4.60 when risk aversion γ increases from two to four. Second, entrepreneurs are wealth constrained and the initial wealth W_0 has a significant effect on initial firm size K_0^*. The initial firm size $K_0^*(W_0)$ increases linearly

by 1.57 for each unit of increase in W_0, provided that $W_0 \geq \overline{W}_0 = 2.86$ when $\gamma = 2$, while $K_0^*(W_0)$ increases linearly only by 1.07 for each unit of increase in W_0, provided that $W_0 \geq \overline{W}_0 = 4.60$ when $\gamma = 4$. Finally, the marginal effect of initial wealth W_0 is also higher for less risk-averse entrepreneurs. The CE wealth increases by 1.2 with W_0 for $\gamma = 2$, and increases by 1.12 with W_0 for $\gamma = 4$ (see Panel B of Fig. 5).

6.2. When career choice is flexible: Optimal entry timing

With flexible entry timing, we show that the option to build up financial wealth is highly valuable for the agent. For simplicity, we assume that becoming an entrepreneur is irreversible. Let $F(W)$ denote the agent's value function before becoming an entrepreneur. Using an argument similar to our earlier analysis, we conjecture that $F(W)$ is given by

$$F(W) = \frac{(bE(W))^{1-\gamma}}{1-\gamma}, \tag{61}$$

where b is the constant given by Eq. (17) and $E(W)$ is the agent's CE wealth.

We will show that the entrepreneurship decision is characterized by an endogenous cutoff threshold $\widehat{W}$. When $W_t \geq \widehat{W}$, the agent immediately enters entrepreneurship. Otherwise, the agent takes the outside option, builds up financial wealth, and becomes an entrepreneur if and when wealth reaches $\widehat{W}$. We summarize the main results below.

Theorem 3. *Provided that $W \leq \widehat{W}$, the agent's CE wealth $E(W)$ solves*

$$0 = \frac{m^{FB}E(W)(E'(W))^{1-\psi} - \psi\zeta E(W)}{\psi - 1} + r(W+\Pi)E'(W) + \frac{\eta^2}{2}\frac{E(W)E'(W)^2}{\gamma E'(W)^2 - E(W)E''(W)}, \tag{62}$$

with the following boundary conditions:

$$E(\widehat{W}) = p'(w^*)(\widehat{W} - \Phi), \tag{63}$$

$$E'(\widehat{W}) = p'(w^*), \tag{64}$$

$$E(-\Pi) = 0, \tag{65}$$

and w^* is given in Theorem 2. The agent's consumption and portfolio rules are given by

$$C(W) = m^{FB}E(W)E'(W)^{-\psi}, \tag{66}$$

$$X(W) = \frac{\mu_R - r}{\sigma_R^2}\frac{E(W)E'(W)}{\gamma E'(W)^2 - E(W)E''(W)}. \tag{67}$$

The value-matching condition (63) states that $E(W)$ is continuous at the endogenously determined cutoff level $\widehat{W}$. The smooth-pasting condition (64) gives the agent's optimal indifference condition between being an entrepreneur or not with wealth $\widehat{W}$. Finally, being indebted with amount Π implies that the agent will never get out of the debt region and cannot pay back the fixed start-up cost Φ. Thus, the CE wealth is zero as given by Eq. (65).

Insert Fig. 6 here.

Fig. 6 graphs $E(W) - (W + \Pi)$, the difference between the certainty equivalent wealth by being an entrepreneur and that by taking the outside option. The two convex curves correspond to the case where the agent has the timing flexibility (the "American" option), while the straight lines correspond to the case where entry is a "now-or-never" binary choice (the "European" option). First, the flexibility to optimally time entry is quite valuable. The timing flexibility significantly increases the cutoff wealth threshold of becoming an entrepreneur from $\overline{W_0} = 2.86$ to $\widehat{W} = 4.3$ for $\gamma = 2$, and from $\overline{W_0} = 4.6$ to $\widehat{W} = 5.7$ for $\gamma = 4$. Second, the less risk-averse agent is more entrepreneurial; with the timing option, the optimal cutoff wealth of becoming an entrepreneur is $\widehat{W} = 5.7$ for $\gamma = 4$, which is significantly higher than $\widehat{W} = 4.3$ for $\gamma = 2$.

Explicit solution for the expected utility case. Consider the widely used expected isoelastic utility, which is a special case of Epstein-Zin recursive utility with $\gamma = 1/\psi$. It turns out that our entrepreneurship entry part of the optimization problem is analogous to the optimization problem in Farhi and Panageas (2007), where an agent optimally chooses when to stop receiving a constant income stream in exchange for a terminal (retirement) value function.[21] Following Farhi and Panageas (2007) and their online appendix, we use the convex duality approach to derive closed-form solutions for the optimal entry threshold and consumption/portfolio rules. We next report these results.

The optimal entry threshold $\widehat{W}$ has the following explicit solution,

$$\widehat{W} = \frac{(a_2 - 1)p'(w^*)^{\gamma^{-1}-1}}{(1 + a_2/(\gamma^{-1} - 1))(p'(w^*)^{\gamma^{-1}-1} - 1)}(\Pi + \Phi) + \Phi\,, \tag{68}$$

[21] We are grateful to the referee for pointing out the technical similarity between the retirement timing problem in Farhi and Panageas (2007) and the optimal entry decision for the special case where the agent has an expected isoelastic utility.

where w^* is given in Theorem 2. The parameter a_2 is given by

$$a_2 = \frac{1 - 2(\zeta - r)/\eta^2 - \sqrt{(1 - 2(\zeta - r)/\eta^2)^2 + 8\zeta/\eta^2}}{2} < 0\,. \tag{69}$$

The pre-entry consumption and portfolio rules also have explicit solutions, given by

$$C(W) = (\lambda^*(W)/\zeta)^{-\gamma^{-1}}\,, \tag{70}$$

$$X(W) = \frac{\eta}{\sigma_R}\left(a_2(a_2-1)D_2\lambda^*(W)^{a_2-1} + \frac{b^{\gamma^{-1}-1}}{\gamma}\lambda^*(W)^{-\gamma^{-1}}\right)\,, \tag{71}$$

where $\lambda^*(W)$ solves the following implicit equation,

$$a_2 D_2 \lambda^*(W)^{a_2-1} - b^{\gamma^{-1}-1}\lambda^*(W)^{-\gamma^{-1}} + \Pi + W = 0\,, \tag{72}$$

and D_2 is a constant given by

$$D_2 = b^{(1-\gamma)(1-a_2)}(\gamma(1-a_2)-1)^{\gamma(1-a_2)-1}\left[\frac{1 - p'(w^*)^{\gamma^{-1}-1}}{(\gamma-1)(1-a_2)}\right]^{\gamma(1-a_2)}(\Pi+\Phi)^{1-\gamma(1-a_2)}\,. \tag{73}$$

7. Idiosyncratic risk premium

A fundamental issue in entrepreneurial finance is to determine the cost of capital for private firms owned by nondiversified entrepreneurs. Intuitively, the entrepreneur demands both the systematic risk premium and an additional idiosyncratic risk premium for nondiversifiable risk. Compared to an otherwise identical public firm held by diversified investors, the cost of capital should be higher for the entrepreneurial firm. Using our model, we provide a procedure to calculate the cost of capital for the entrepreneurial firm.

Let $\xi(w_0)$ denote the constant yield (internal rate of return) for the entrepreneurial firm until liquidation. We have made explicit the functional dependence of ξ on the initial wealth-capital ratio $w_0 = W_0/K_0$. By definition, $\xi(w_0)$ solves the following valuation equation:

$$Q(K_0, W_0) = \mathbb{E}\left[\int_0^\tau e^{-\xi(w_0)t}dY_t + e^{-\xi(w_0)\tau}lK_\tau\right]\,, \tag{74}$$

where τ is the stochastic liquidation time. The right side of Eq. (74) is the present discounted value (PDV) of the firm's operating cash flow plus the PDV of the liquidation value using the same discount rate $\xi(w_0)$. The left side is the "private" enterprise value $Q(K_0, W_0)$ that we have obtained earlier using the entrepreneur's optimality.

Recall that the firm's discount rate under complete markets, ξ^{FB}, is given in Eq. (23). We measure the idiosyncratic risk premium as the wedge between $\xi(w_0)$ and ξ^{FB}

$$\alpha(w_0) \;=\; \xi(w_0) - \xi^{FB} = \xi(w_0) - r - \beta^{FB}(\mu_R - r)\,. \tag{75}$$

There is much debate in the empirical literature about the significance of this *private* equity risk premium. For example, Moskowitz and Vissing-Jørgensen (2002) show that the risk-adjusted returns to investing in a U.S. non-publicly traded equity are not higher than the returns to private equity. Our model provides an analytical formula to calculate this private equity idiosyncratic risk premium.

Insert Fig. 7 here.

Fig. 7 plots the idiosyncratic risk premium for two levels of risk aversion, $\gamma = 2, 4$. For sufficiently high levels of wealth-capital ratio w_0, the idiosyncratic risk premium $\alpha(w_0)$ eventually disappears. Intuitively, this premium $\alpha(w_0)$ is higher for more risk-averse agents. Quantitatively, for entrepreneurs with positive wealth, we do not find a significant idiosyncratic risk premium. For both $\gamma = 2$ and $\gamma = 4$, the annual idiosyncratic risk premiums are less than 1%. Importantly, for entrepreneurs in debt, the idiosyncratic risk premium $\alpha(w_0)$ is much larger because the entrepreneurial business carries significantly more weight in the portfolio, and nondiversifiable risk becomes much more important. Near the endogenous liquidation boundary $\underline{w}$, the entrepreneur attaches a much lower valuation for the business, which implies a much larger idiosyncratic risk premium $\alpha(w_0)$. For example, the annual private equity idiosyncratic risk premium $\alpha(w_0)$ approaches to 0.0465 and 0.0475 for $\gamma = 2$ and $\gamma = 4$, respectively. Despite the significant difference in the coefficient of risk aversion γ, the two premiums are quite close as the risk-adjusted liquidation likelihoods for both cases are very high.

Our theory suggests that to better measure the size of private equity idiosyncratic risk premium (an important empirical debate in the entrepreneurship literature), it is critical to account for heterogeneity across entrepreneurs. Our model shows that even for the same entrepreneur, the idiosyncratic risk premium varies significantly with leverage, financial constraints and historical performance due to incomplete markets frictions.

8. Comparative analysis

There is significant heterogeneity in terms of preferences and production technology. In this section, we study the effects of various structural parameters, including the EIS ψ, idiosyncratic volatility ϵ, the adjustment cost parameter θ, and the liquidation parameter l, on the entrepreneur's decision making and business valuation. For all the figures, we use parameter values given in Table 1 except for the parameter under study. In the preceding analysis, we have shown that risk aversion has substantial effects. We now turn to the effects of other key parameters. Throughout this section, we set risk aversion $\gamma = 2$.

The EIS ψ. In asset pricing, a high EIS is often used in the long-run risk literature (Bansal and Yaron, 2004).However, there is much disagreement about the empirical estimates of the EIS. For example, Hall (1988), using aggregate consumption data, obtains a much smaller estimate. Our previous calculations are based on $\psi = 0.5$. We now consider two commonly used but significantly different values for the EIS: $\psi = 0.25, 2$. Fig. 8 shows that the effect of the EIS ψ on consumption $c(w)$ is quantitatively significant, while its effects on Tobin's $q(w)$, investment $i(w)$, portfolio choice $x(w)$, and the idiosyncratic risk premium $\alpha(w)$ are much less significant. The large effect on consumption is similar to the intuition under complete markets. For example, the MPC m^{FB} is only 0.014 when $\psi = 2$, which is substantially lower than the MPC $m^{FB} = 0.072$ when EIS is $\psi = 0.25$. Intuitively, an entrepreneur with a high EIS ($\psi = 2$) is willing to decrease consumption to build up wealth.

Insert Fig. 8 here.

Idiosyncratic volatility ϵ. In Fig. 9, we plot for two values of the idiosyncratic volatility, $\epsilon = 0.1, 0.2$. We find that the idiosyncratic volatility ϵ has significant effects on investment $i(w)$ and Tobin's $q(w)$. The entrepreneur invests significantly less in the firm (lower $i(w)$) and liquidates capital earlier when $\epsilon = 0.2$ than when $\epsilon = 0.1$. Panousi and Papanikolaou (2011) find that the firm's investment falls as its idiosyncratic risk rises, consistent with our model's prediction. Firm value $q(w)$ increases significantly when the idiosyncratic volatility ϵ decreases from 0.2 to 0.1. The marginal value of financial wealth $q'(w)$ also strongly depends on the idiosyncratic volatility, especially for low and intermediate values of w. Finally, the effect of ϵ on the idiosyncratic risk premium $\alpha(w)$ is large. For example, when doubling

the idiosyncratic volatility from 10% to 20%, the annual idiosyncratic risk premium for an entrepreneur with no liquid wealth ($w = 0$) increases from 0.5% to 2.3%!

Insert Fig. 9 here.

Adjustment cost parameter θ. In Fig. 10, we plot for two values of the adjustment cost parameter: $\theta = 2$ and $\theta = 8$. Whited (1992) estimates this parameter to be around $\theta = 2$. Hall (2004) argues that the parameter θ is small using U.S. aggregate data. Eberly, Rebelo, and Vincent (2009) use an extended Hayashi (1982) model and provide a larger empirical estimate of this parameter value (close to seven) for large Compustat firms. Clearly, the adjustment cost has a first-order effect on investment $i(w)$ and Tobin's $q(w)$ under incomplete markets, as in the first-best benchmark. Consumption $c(w)$ and portfolio allocation $x(w)$ depend little on θ. The effect of θ on the idiosyncratic risk premium $\alpha(w)$ is weak.

Insert Fig. 10 here.

Liquidation parameter l. In Fig. 11, we plot for two values of the liquidation parameter, $l = 0.6$ and $l = 0.9$. We show that liquidation value has a quantitatively significant impact on investment $i(w)$, Tobin's $q(w)$, consumption $c(w)$, and the idiosyncratic volatility $\alpha(w)$ when the entrepreneur is in debt (i.e., the left sides of each panel). A higher value of l provides a better downside protection for the entrepreneur and also allows the entrepreneur to borrow more (higher debt capacity). The entrepreneur thus operates the business longer with a higher l. Additionally, while running the business, the entrepreneur invests more, consumes more, and allocates more to the market portfolio with a higher value of l. A higher value of l also lowers the idiosyncratic risk premium $\alpha(w)$ by providing a better downside risk protection and mitigating entrepreneurial underinvestment. When the liquidation option is sufficiently out of the money (i.e., when w is sufficiently high), liquidation has almost no effect on entrepreneurial decision making and valuation, consistent with our intuition as in the first-best benchmark.

Insert Fig. 11 here.

9. Conclusion

We build a unified incomplete-markets entrepreneurship model with nondiversifiable risk and liquidity constraints to analyze interdependent business entry, capital accumulation/growth, portfolio choice, consumption, and business exit decisions. The core of our model is the entrepreneur's dynamic liquidity and risk management. The entrepreneur rationally reduces business investment, prudently uses debt, lowers consumption, and scales back portfolio investment in the stock market in order to preserve liquid wealth to buffer productivity shocks. The key variable is the ratio between liquid wealth and illiquid physical capital, which we refer to as liquidity w.

We develop the counterpart of the modern q theory of investment for firms run and owned by nondiversified entrepreneurs. We show that investment depends on not just the entrepreneur's marginal q but also the marginal value of liquid wealth. Time-series variation of investment and marginal q may arise not just from time-varying investment opportunities but also the time-varying liquidity w. Additionally, we show how capital adjustment costs and incomplete markets interactively influence the entrepreneur's portfolio allocation among the market portfolio, the risk-free asset, and the illiquid business exposure. We also provide an operational procedure to compute the private equity idiosyncratic risk premium, which helps us understand the empirical findings on the private equity premium.[22]

While being exposed to significant risk, the entrepreneur nonetheless has various options to manage risk. The liquidation option substantially enhances the entrepreneur's ability to manage downside risk. Importantly, the exit option makes investment, marginal value of liquid wealth, and the (private) marginal q all nonmonotonic in liquidity w.

The value of building up financial wealth before entering entrepreneurship is high. Wealth effects are significant for entrepreneurship. The entrepreneurs' various entry and exit options are illiquid/non-tradable, and fundamentally different from standard options in finance. Additionally, option exercising decisions intertwine with consumption-saving, portfolio allocation, and investment decisions in the presence of these incomplete-market frictions.

To study the impact of entrepreneurship on wealth distribution and economic growth, we need to construct a general equilibrium incomplete-markets model.[23] Our model treats

[22]See Moskowitz and Vissing-Jørgensen (2002), for example.

[23]See Cagetti and De Nardi (2006) for an application of equilibrium (Bewley) models to entrepreneurship.

informational or contracting frictions as exogenously given. Extending our model to allow for endogenous under-diversification and incomplete markets is an important direction for future research.[24] Our decision model may provide one natural starting point for general equilibrium analysis with and without informational frictions.

[24]Rampini (2004) introduces informational frictions into an equilibrium model of entrepreneurship and shows that entrepreneurial activity is pro-cyclical due to risk considerations.

Appendix A. Details for Theorem 1 and Proposition 1

We conjecture that the value function is given by Eq. (33). The FOCs for C and X are

$$f_C(C,J) = J_W(K,W)\,, \tag{A.1}$$

$$X = -\frac{\rho\sigma_A}{\sigma_R}K - \frac{(\mu_R - r)J_W(K,W)}{\sigma_R^2 J_{WW}(K,W)}\,. \tag{A.2}$$

Using the homogeneity property of $J(K,W)$, we obtain Eqs. (39) and (41) for $c(w)$ and $x(w)$, respectively. Using the FOCs for investment-capital ratio i, we obtain Eq. (40). Substituting these results into Eq. (29), we obtain the ODE (34).

Using Ito's formula, we obtain the following dynamics for the wealth-capital ratio w,

$$dw_t = d\left(\frac{W_t}{K_t}\right) = \frac{dW_t}{K_t} - \frac{W_t}{K_t^2}dK_t = \mu_w(w_t)dt + \sigma_R x(w_t)dB_t + \sigma_A dZ_t\,, \tag{A.3}$$

where $\mu_w(w)$ is given by Eq. (43).

Now consider the lower liquidation boundary $\underline{W}$. When $W \leq \underline{W}$, the entrepreneur liquidates the firm. Using the value-matching condition at $\underline{W}$, we have

$$J(K,\underline{W}) = V(\underline{W} + lK)\,, \tag{A.4}$$

where $V(W)$ given by Eq. (26) is the agent's value function after retirement and with no business. The entrepreneur's optimal liquidation strategy implies the following smooth-pasting condition at the endogenously determined liquidation boundary $\underline{W}$:

$$J_W(K,\underline{W}) = V_W(\underline{W} + lK)\,. \tag{A.5}$$

Using $\underline{W} = \underline{w}K$, Eqs. (A.4)–(A.5), and simplifying, we obtain the scaled value-matching and smooth pasting conditions given in Eqs. (37) and (38), respectively.

Complete-markets benchmark solution. As w approaches infinity, firm value approaches the complete-markets value and $\lim_{w\to\infty} J(K,W) = V(W + q^{FB}K)$, which implies Eq. (36). The CE wealth $P(K,W) = p(w)K$, where $p(w)$ is given by

$$p^{FB}(w) = w + q^{FB}\,. \tag{A.6}$$

Substituting the above into Eq. (34), taking the limit $w \to \infty$, and simplifying, we obtain formulae for b and m^{FB} given in Eqs. (17) and (21), respectively. Other results follow.

Appendix B. Details for Theorem 2 and Theorem 3

Theorem 2. The entrepreneur chooses initial size K_0^* to maximize utility, which implies

$$P_K(K_0^*, W_0 - \Phi - K_0^*) = P_W(K_0^*, W_0 - \Phi - K_0^*)\,. \tag{B.1}$$

Simplifying Eq. (B.1) gives Eq. (57). Apply the Euler's theorem for $P(K, W)$, we have

$$P(K_0^*, W_0 - \Phi - K_0^*) = P_K^* \times K_0^* + P_W^* \times (W_0 - \Phi - K_0^*) = p'(w^*)\,(W_0 - \Phi)\ , \tag{B.2}$$

where the second equality follows from Eq. (B.1). Therefore, the threshold level $\overline{W}$ satisfies $J(K_0^*, \overline{W} - (\Phi + K_0^*)) = V(\overline{W} + \Pi)$, which gives Eq. (56).

Theorem 3. Using the standard principle of optimality for recursive utility (Duffie and Epstein, 1992), the following HJB equation holds for the agent's value function $F(W)$,

$$0 = \max_{C,X}\ f(C, F) + (rW + (\mu_R - r)X + r\Pi - C)F'(W) + \frac{\sigma_R^2 X^2}{2} F''(W)\,. \tag{B.3}$$

The FOCs for C and X are given by

$$F'(W) = f_C(C, F)\,, \tag{B.4}$$

$$X(W) = -\frac{(\mu_R - r)F'(W)}{\sigma_R^2 F''(W)}\,. \tag{B.5}$$

Using value function (61) for $F(W)$, we obtain formulae (66) and (67) for $C(W)$ and $X(W)$, respectively. Substituting these results into Eq. (B.3), we obtain the non-linear ODE (62). The following value-matching and smooth-pasting conditions determine the threshold $\widehat{W}$,

$$F(\widehat{W}) = J(K^*, \widehat{W} - \Phi - K^*)\,, \tag{B.6}$$

$$F'(\widehat{W}) = J_W(K^*, \widehat{W} - \Phi - K^*)\,. \tag{B.7}$$

Simplifying the above, we obtain the value-matching and smooth-pasting conditions (63) and (64) for $E(W)$ at $\widehat{W}$. Finally, we have the absorbing condition, $E(-\Pi) = 0$.

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Table 1
Summary of Key Variables and Parameters.

This table summarizes the symbols for the key variables used in the model and the parameter values for the baseline calculation of Section 5. For each upper-case variable in the left column (except K, A, J, F, V, E, $\overline{W}$, $\widehat{W}$, and K^*), we use its lower case to denote the ratio of this variable to capital.

Variable	Symbol	Parameters	Symbol	Value
Capital stock	K	Riskfree rate	r	4.6%
Cumulative productivity shock	A	Expected return of market portfolio	μ_R	10.6%
Investment adjustment cost	G	Volatility of market portfolio	σ_R	20%
Cumulative operating profit	Y	Aggregate equity risk premium	$\mu_R - r$	6%
Financial wealth	W	Market Sharpe ratio	η	30%
Value function after entry	J	Subjective discount rate	ζ	4.6%
Value function before entry	F	Adjustment cost parameter	θ	2
Value function after exiting	V	Depreciation rate	δ	12.5%
Certainty equivalent wealth after entry	P	Mean productivity shock	μ_A	20%
Certainty equivalent wealth before entry	E	Volatility of productivity shock	σ_A	10%
Private enterprise value	Q	Correlation between market and firm	ρ	0
Effective risk aversion	h	Idiosyncratic volatility	ϵ	10%
Market portfolio allocation	X	Relative risk aversion	γ	$2, 4$
Consumption	C	Elasticity of intertemporal substitution	ψ	0.5
Business investment	I	Capital liquidation price	l	0.9
Liquidation boundary	$\underline{W}$	Outside option value	Π	0.5
Static entry threshold	$\overline{W}$	Fixed start-up cost	Φ	0.05
Flexible entry threshold	$\widehat{W}$			
Optimal initial capital stock	K^*			
MPC out of wealth	m			
Internal rate of return	ξ			
Idiosyncratic risk premium	α			

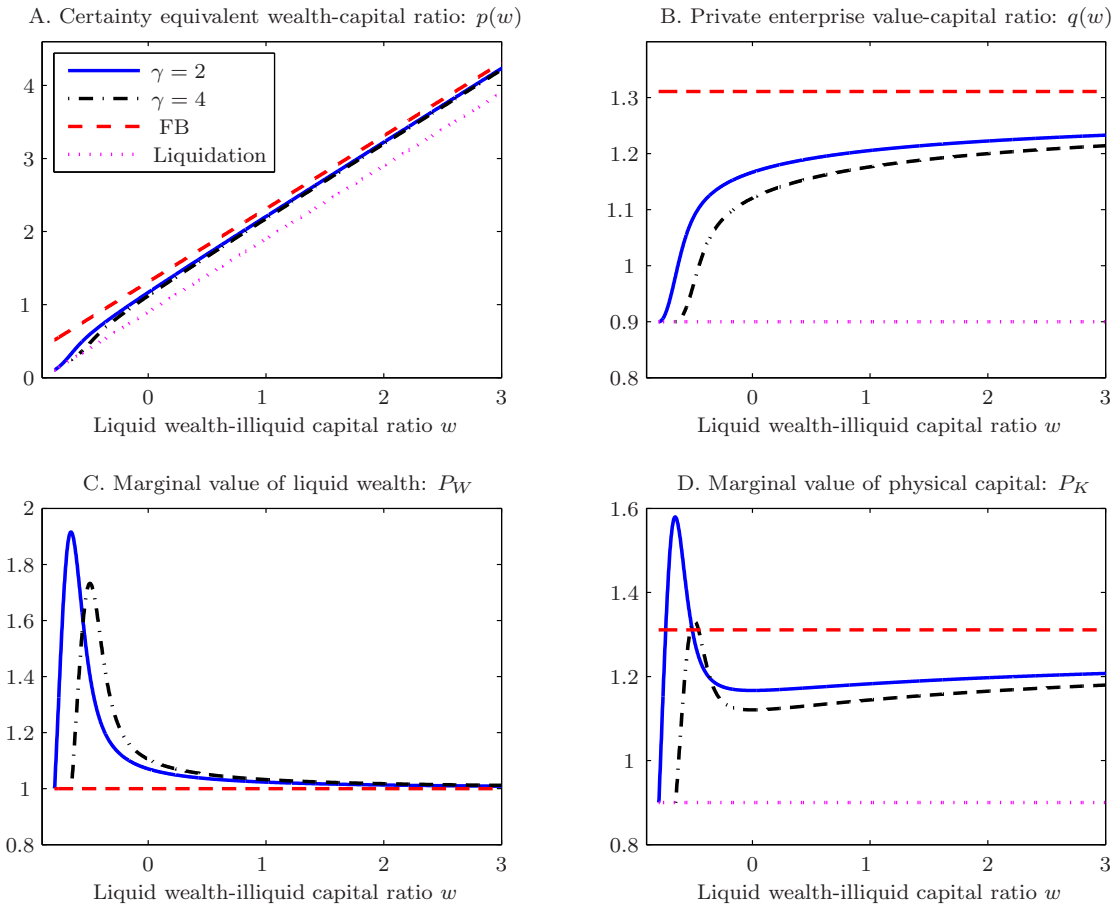

Fig. 1: The entrepreneur's scaled certainty equivalent wealth $p(w)$, private average $q(w) = p(w) - w$, private marginal value of liquid wealth $P_W(K, W) = p'(w)$, and private marginal q, $P_K(K, W) = p(w) - wp'(w)$. For the first-best (complete-markets) case, marginal q and Tobin's average q are equal and constant, $q^{FB} = 1.31$, and the marginal value of liquidity $p'(w) = 1$ for all w. With incomplete markets, the private average q is concave in w in the region $w \geq \widetilde{w}$ and convex in w when $w \leq \widetilde{w}$, where $\widetilde{w}$ is the inflection point at which $q''(\widetilde{w}) = p''(\widetilde{w}) = 0$. As the entrepreneur's risk aversion γ increases from 2 to 4, the shapes of the private average $q(w)$ and of the private marginal value of liquidity $p'(w)$ remain unchanged, but the inflection point $\widetilde{w}$ changes from -0.658 to -0.495, and the endogenous liquidation/exit boundary $\underline{w}$ changes from -0.8 to -0.65 (more risk-averse entrepreneurs use lower debt and exit risky business earlier). The private marginal q, P_K, is highly nonlinear and can exceed the first-best q^{FB} in the debt region ($w < 0$). Unless otherwise noted in the legend, all parameter values are given in Table 1.

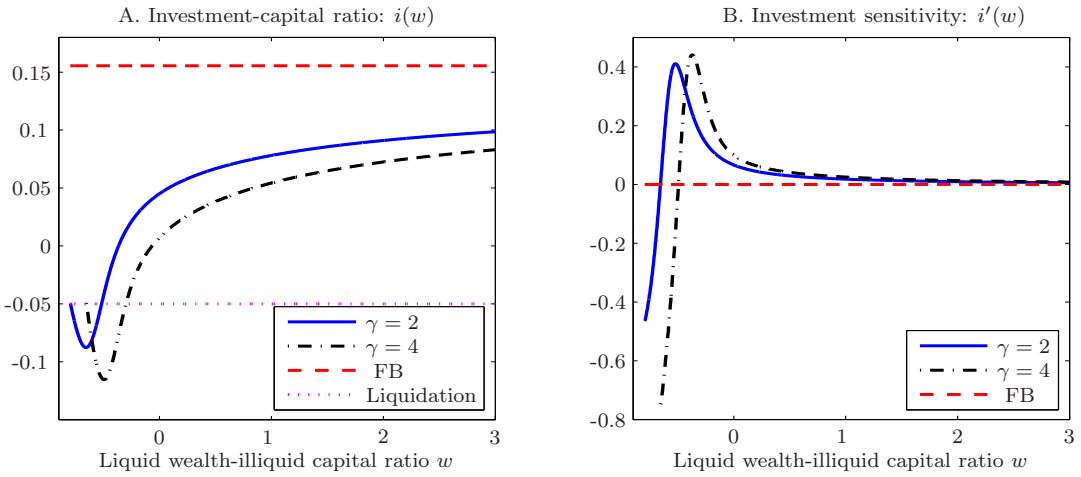

Fig. 2. Investment-capital ratio $i(w)$ and investment-liquidity sensitivity $i'(w)$. For the first-best (complete-markets) case, the investment-capital ratio is constant at all times, $i^{FB} = 0.156$. With incomplete markets, the investment-capital ratio $i(w)$ is lower than the first-best value i^{FB}. Notably, investment-capital ratio $i(w)$ is nonmonotonic in w; it is increasing in w in the region $w \geq \widetilde{w}$ and decreasing in w in the region $w \leq \widetilde{w}$, where $\widetilde{w}$ is the inflection point at which $q''(\widetilde{w}) = p''(\widetilde{w}) = 0$. As the entrepreneur's risk aversion γ increases from 2 to 4, the shapes of $i(w)$ and of investment-liquidity sensitivity $i'(w)$ remain unchanged, but the inflection point $\widetilde{w}$ changes from -0.658 to -0.495, and the endogenous liquidation/exit boundary $\underline{w}$ changes from -0.8 to -0.65 (more risk-averse entrepreneurs use lower debt and exit risky business earlier). Unless otherwise noted in the legend, all parameter values are given in Table 1.

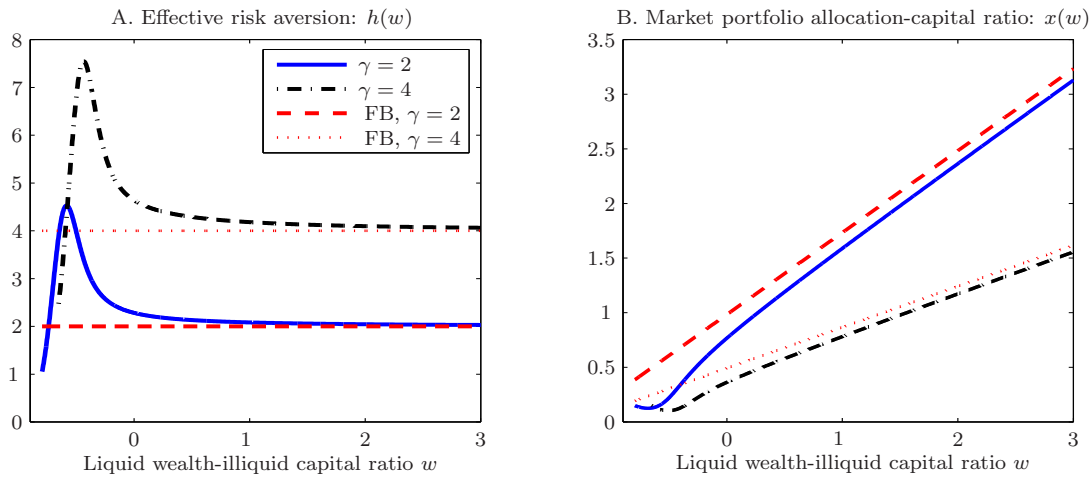

Fig. 3. The entrepreneur's "effective" risk aversion $h(w)$ and market portfolio allocation-capital ratio $x(w)$. For the first-best (complete-markets) case, effective risk aversion $h(w)$ equals the coefficient of relative risk aversion, $h(w) = \gamma$; and the optimal portfolio allocation, $x(w)$, is linearly increasing in w. With incomplete markets, effective risk aversion $h(w)$ is highly nonlinear and nonmonotonic in liquidity w. In the region where $w \geq -0.584$ for $\gamma = 2$ and $w \geq -0.437$ for $\gamma = 4$, $h(w)$ decreases with w as self insurance becomes more effective. In the region where $w \leq -0.584$ for $\gamma = 2$ and $w \leq -0.437$ for $\gamma = 4$, $h(w)$ decreases as w decreases towards the exit option. Near the exit boundary $\underline{w}$, $h(w)$ can be lower than the coefficient of relative risk aversion γ due to the flexibility/optionality of exit, $p''(\underline{w}) > 0$. The portfolio allocation to the risky market portfolio $x(w)$ is lower than the first-best level $x^{FB}(w)$. As the exit option becomes deeper in the money, i.e. as w approaches the liquidation boundary $\underline{w}$, portfolio allocation becomes more aggressive. For example, $x(w)$ is decreasing in w when $w \leq -0.684$ for $\gamma = 2$ and when $w \leq -0.509$ for $\gamma = 4$. Unless otherwise noted in the legend, all parameter values are given in Table 1.

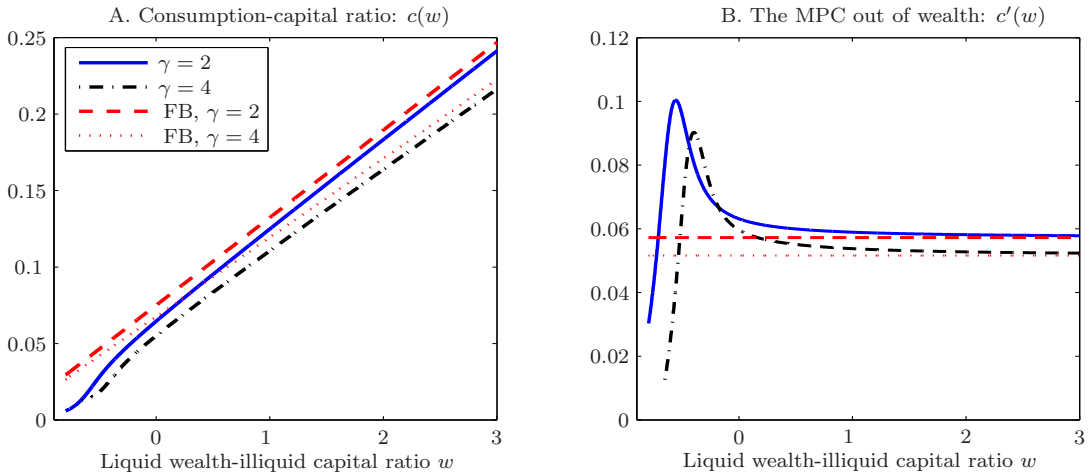

Fig. 4. Consumption-capital ratio $c(w)$ and the MPC out of wealth $c'(w)$. For the first-best (complete-markets) case, consumption-capital ratio $c^{FB}(w)$ is linearly increasing in w, which implies a constant marginal propensity to consume (MPC) out of liquid wealth. With incomplete markets, the consumption-capital ratio $c(w)$ is lower than the first-best level $c^{FB}(w)$ for a given value of γ. For $w \geq -0.556$ for $\gamma = 2$ and $w \geq -0.397$ for $\gamma = 4$, the MPC out of liquid wealth decreases in w, which implies that the consumption function is concave in this region. Surprisingly, the MPC $c'(w)$ increases for $w \leq -0.556$ for $\gamma = 2$ and $w \leq -0.397$ for $\gamma = 4$. The convexity of the consumption rule $c(w)$ in the region $w \leq -0.556$ for $\gamma = 2$ and $w \leq -0.397$ for $\gamma = 4$ is due to the flexible liquidation/exit option. Unless otherwise noted in the legend, all parameter values are given in Table 1.

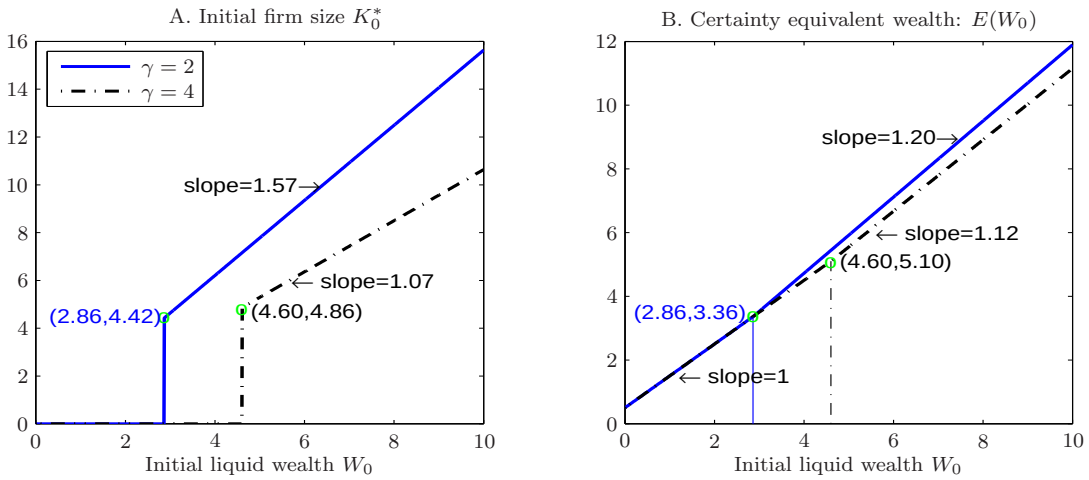

Fig. 5. Entrepreneurial entry in a time-0 (now-or-never) binary setting: Initial firm size K_0^* and certainty equivalent wealth $E(W_0)$. The threshold for the initial wealth $\overline{W}_0$ to become an entrepreneur is 2.86 and 4.60 for $\gamma = 2$ and $\gamma = 4$, respectively. Conditional on becoming an entrepreneur, the marginal effect of wealth on the initial firm size is 1.57 and 1.07 for $\gamma = 2$ and $\gamma = 4$, respectively. The marginal certainty equivalent value of liquid wealth is 1.20 and 1.12 for $\gamma = 2$ and $\gamma = 4$, respectively. Unless otherwise noted in the legend, all parameter values are given in Table 1.

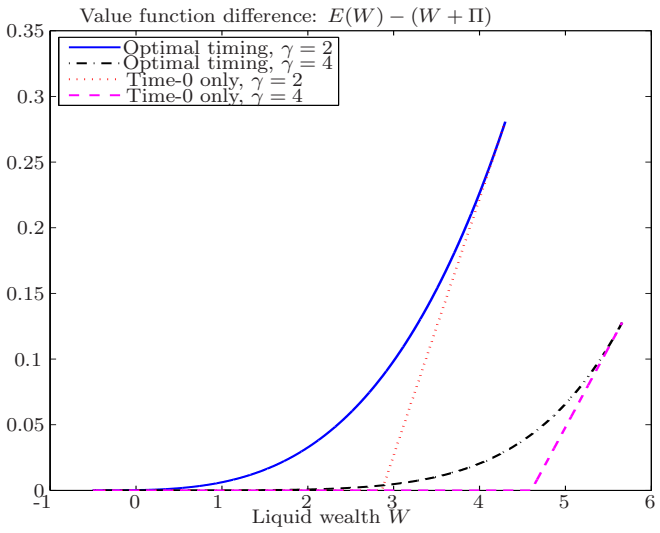

Fig. 6. The value of flexible entry: Comparing the value function differences under "optimal timing" and "time-0 binary" settings. This graph plots $E(W) - (W + \Pi)$, the difference between the certainty equivalent wealth by being an entrepreneur and the outside option value $W + \Pi$. The two convex curves correspond to the case where the agent has the timing flexibility (the American option). The two straight lines correspond to the case where entry is a now-or-never binary choice (the European option). The flexibility to time entry is valuable. The cutoff wealth threshold of becoming an entrepreneur increases from $\overline{W_0} = 2.86$ to $\widehat{W} = 4.3$ for $\gamma = 2$, and from $\overline{W_0} = 4.6$ to $\widehat{W} = 5.7$ for $\gamma = 4$ as we change the entry option from time-0 only (European) to flexible optimal timing (American). A less risk-averse agent is more entrepreneurial; with the timing option, the optimal cutoff wealth threshold of becoming an entrepreneur is $\widehat{W} = 5.7$ for $\gamma = 4$, which is significantly higher than $\widehat{W} = 4.3$ for $\gamma = 2$. Unless otherwise noted in the legend, all parameter values are given in Table 1.

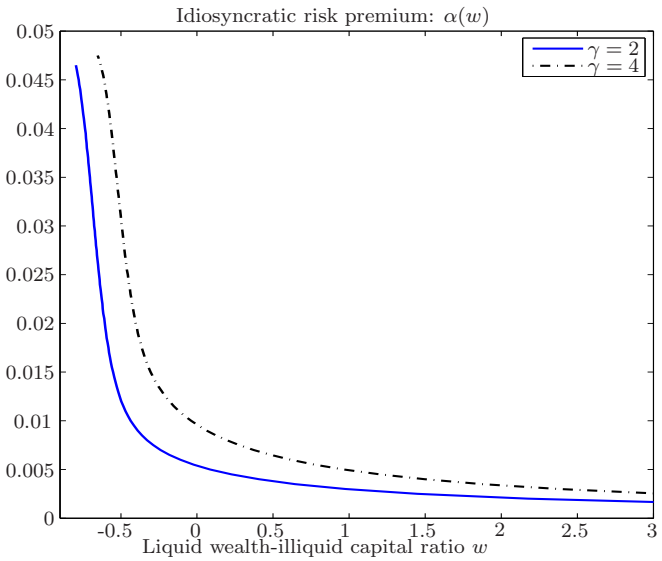

Fig. 7. Private equity idiosyncratic risk premium, $\alpha(w)$, for the coefficient of relative risk aversion $\gamma = 2, 4$. The private equity idiosyncratic risk premium $\alpha(w)$ decreases with liquidity w. As w approaches ∞, self insurance is effective and hence $\alpha(w)$ disappears. As w gets into the debt region and especially when it approaches the endogenous liquidation boundary $\underline{w}$, the idiosyncratic risk premium $\alpha(w)$ becomes much larger and also much more sensitive to the change of w. Risk aversion increases $\alpha(w)$. Unless otherwise noted in the legend, all parameter values are given in Table 1.

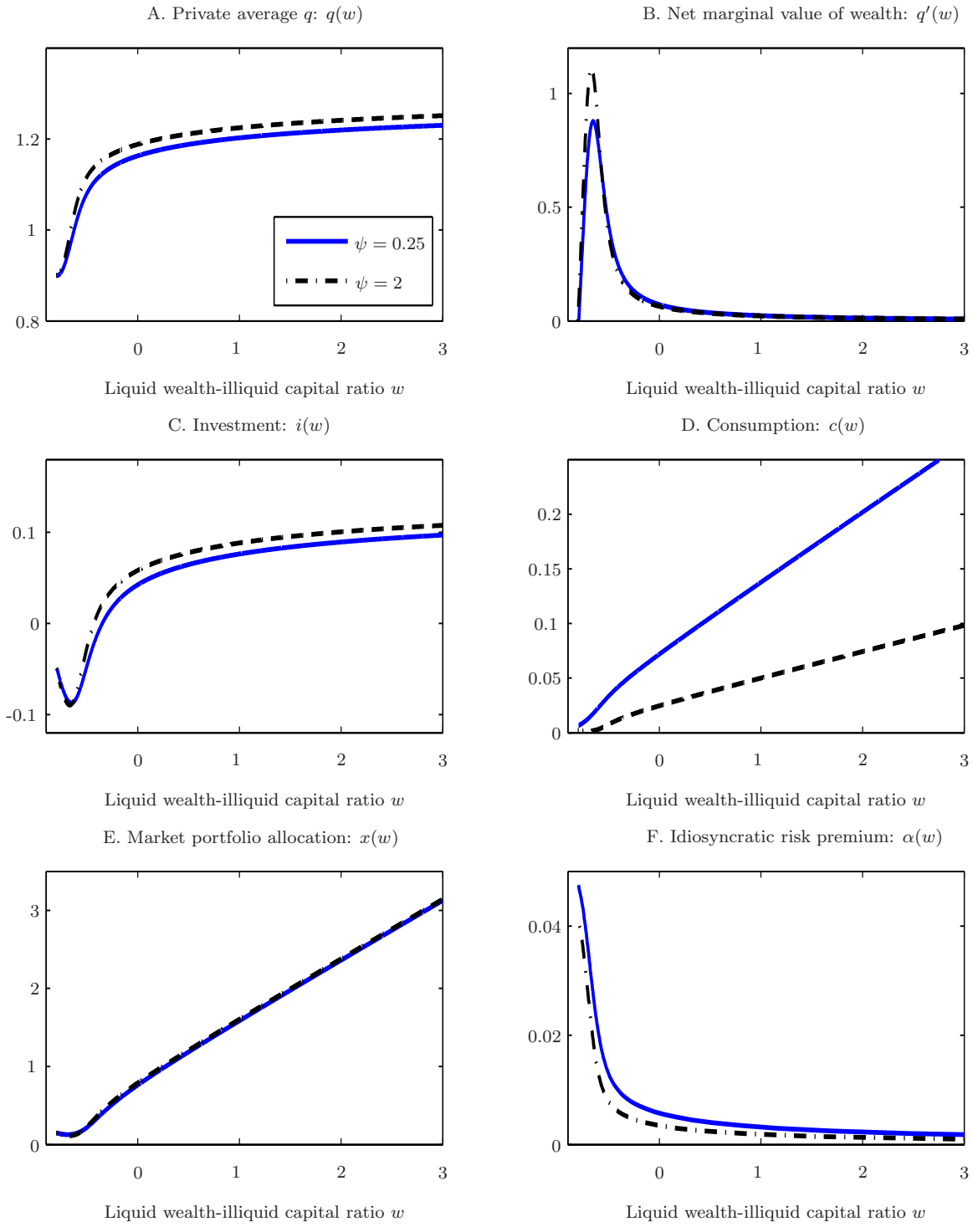

Fig. 8. The effects of EIS ψ. This figure demonstrates the comparative static effects of changing EIS ψ. The quantitative effect of EIS ψ on consumption is quite significant. Unless otherwise noted in the legend, all parameter values are given in Table 1.

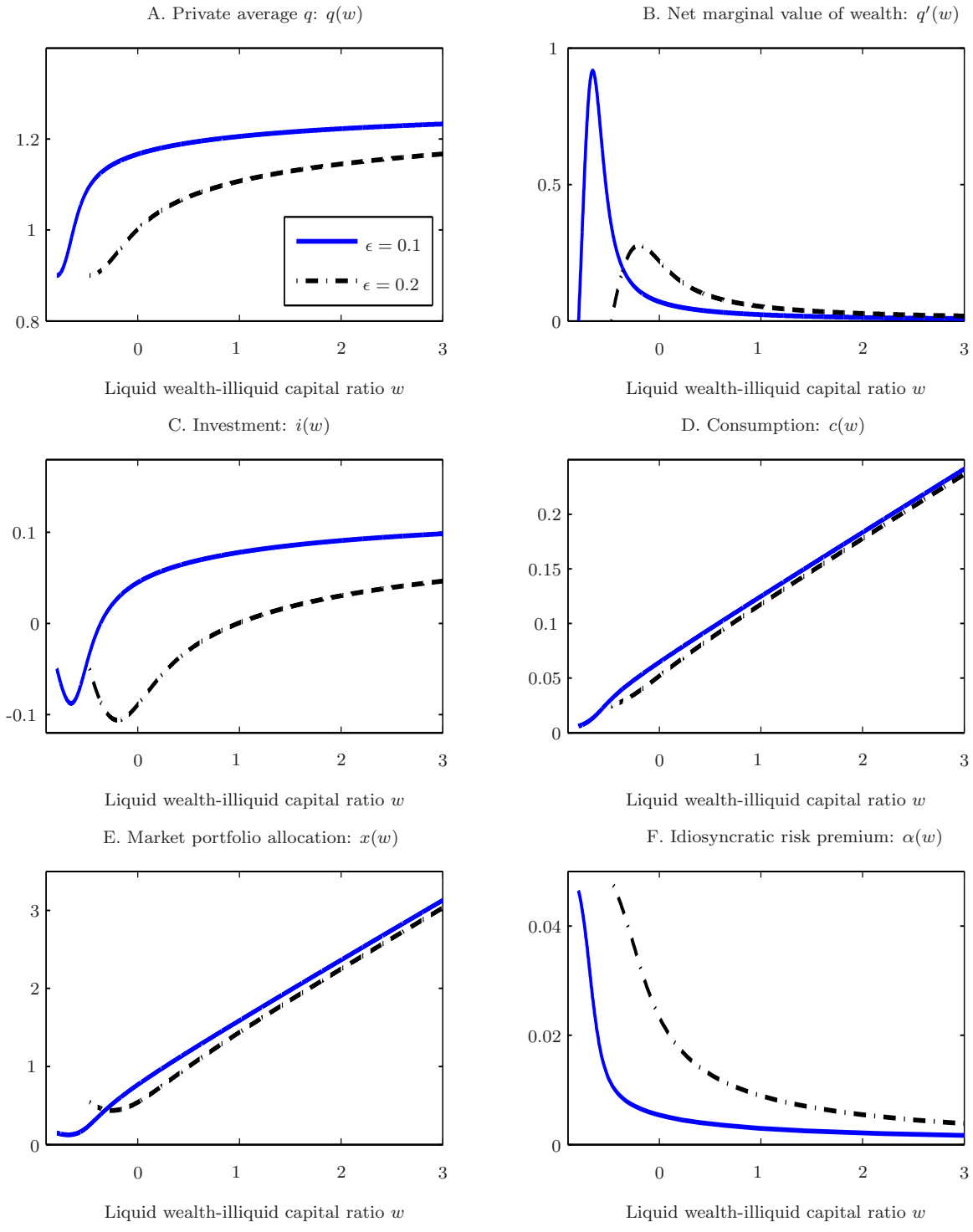

Fig. 9. The effects of idiosyncratic volatility ϵ. This figure demonstrates the comparative static effects of changing the idiosyncratic business volatility ϵ. The quantitative effects of idiosyncratic volatility ϵ on private average $q(w)$ (valuation), the net marginal value of liquidity $q'(w)$, investment $i(w)$, and the private equity idiosyncratic risk premium $\alpha(w)$ are quite significant under incomplete markets. This is in stark contrast against the first-best benchmark where idiosyncratic volatility plays no role in entrepreneurs' decision making and valuation. Unless otherwise noted in the legend, all parameter values are given in Table 1.

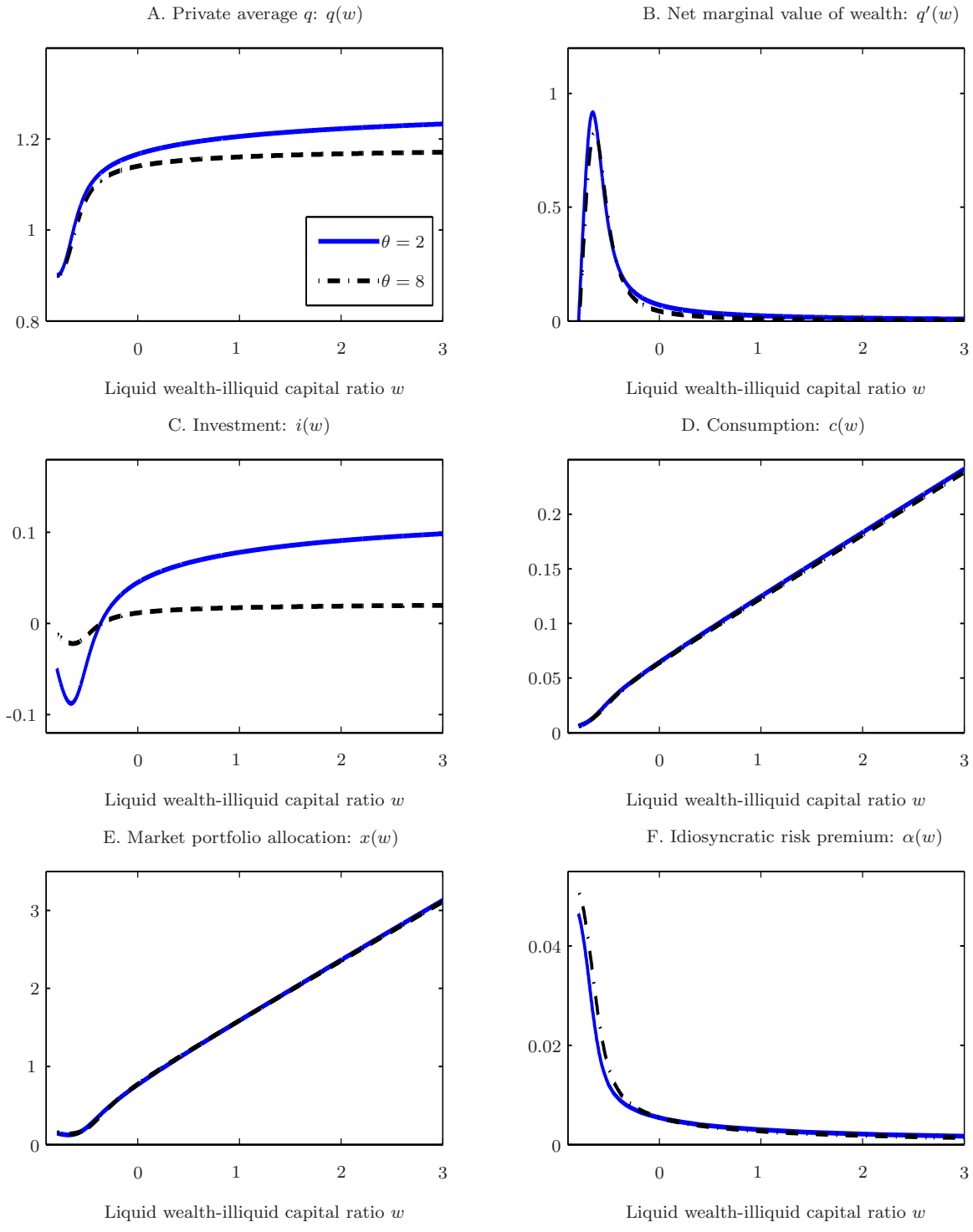

Fig. 10. The effects of the adjustment cost parameter θ. This figure demonstrates the comparative static effects of changing the adjustment cost parameter θ. The quantitative effects of θ on private average $q(w)$ (valuation) and investment $i(w)$ are significant under incomplete markets, as in the first-best benchmark. Unless otherwise noted in the legend, all parameter values are given in Table 1.

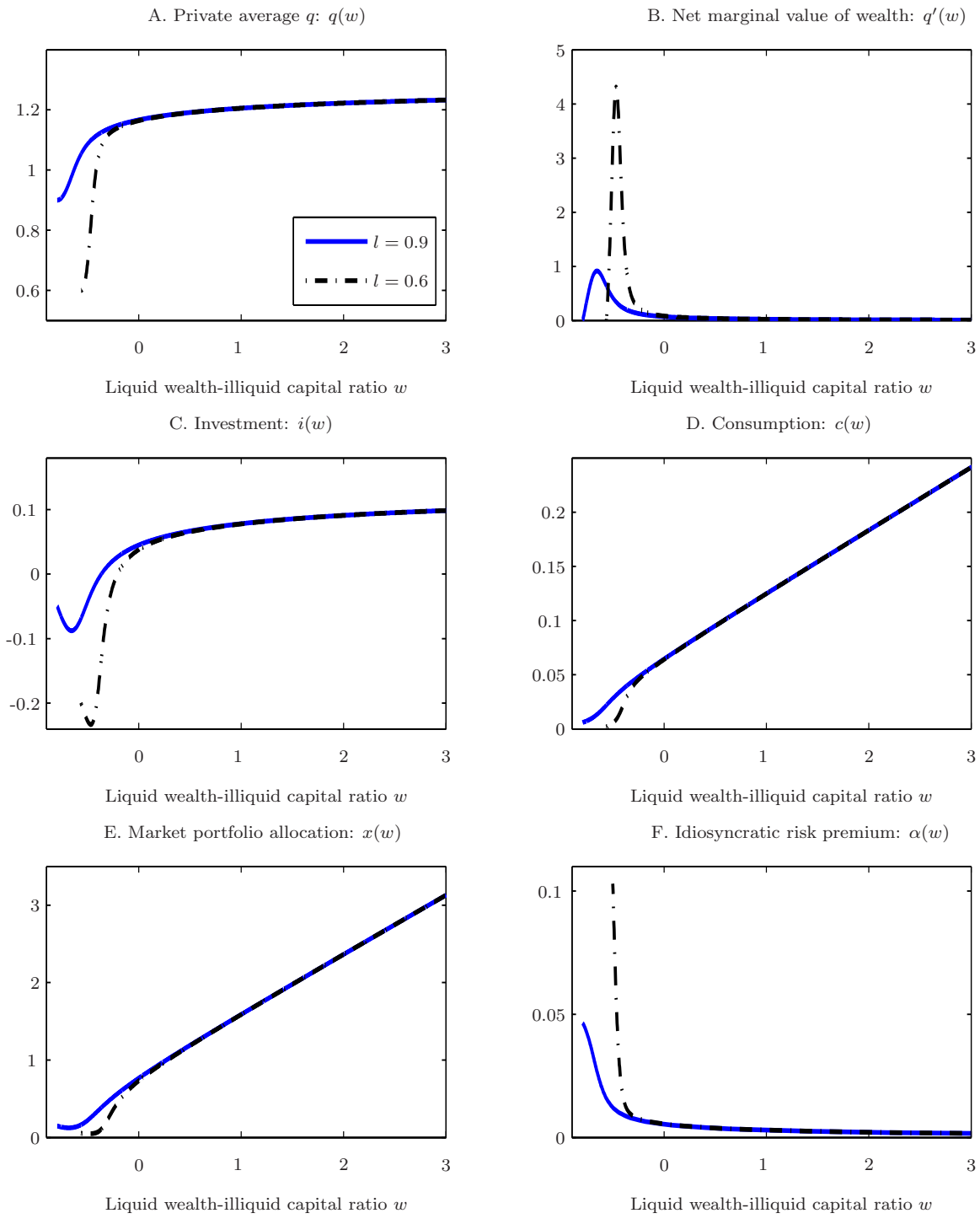

Fig. 11. The effects of the liquidation parameter l. This figure demonstrates the comparative static effects of changing the liquidation parameter l. The quantitative effects of l on all variables including private average $q(w)$ (valuation), net marginal value of liquidity $q'(w)$, investment $i(w)$, consumption $c(w)$, portfolio allocation $x(w)$, and the idiosyncratic risk premium $\alpha(w)$ are significant when the flexible liquidation/exit option are sufficiently in the money under incomplete markets, unlike in the first-best benchmark, where the exit option is completely out of the money. Unless otherwise noted in the legend, all parameter values are given in Table 1.