DYNAMICS OF ENTREPRENEURSHIP UNDER INCOMPLETE MARKETS

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## ABSTRACT

An entrepreneur faces substantial non-diversifiable business risk and liquidity constraints, both of which we refer to as frictions. We show that these frictions have significant economic effects on business start-up, capital accumulation/asset sales, portfolio allocation, consumption/saving, and business exit decisions. Compared with the complete-markets benchmark, these frictions make entrepreneurs invest substantially less in the business, consume less, and allocate less to the market portfolio. The endogenous exit option provides flexibility for the entrepreneur to manage downside risk. The entrepreneur's optimal entry decision critically depends on the outside option, the start-up cost, risk aversion, and wealth. We show that the flexibility to build up financial wealth before entering into entrepreneurship is quite valuable. Finally, we provide an operational framework to calculate the private equity idiosyncratic risk premium for an entrepreneurial firm and show that this premium depends on entrepreneurial wealth, non-diversifiable risk exposure, and risk aversion.

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# 1 Introduction

An entrepreneur owns a business and bears significant risks/rewards from the business. Casual observations and empirical studies have shown that active businesses account for a large fraction of entrepreneurs' total wealth and that entrepreneurial firms tend to have highly concentrated ownership.[1] Moreover, entrepreneurs often face liquidity constraints.[2] Lack of diversification and liquidity constraints cause business decisions (capital accumulation and entry/exit) and household decisions (consumption/saving and asset allocation) to be highly interdependent. In a recent survey of research, Quadrini (2009) discusses the importance of borrowing constraints and non-diversification on entrepreneurial career choice, entrepreneurial saving/investment, and economic development/growth.

We study the effects of liquidity constraints and non-diversification on entrepreneurial entry, capital accumulation/asset sale, consumption, portfolio allocation, and exit decisions in a dynamic framework. We then use the entrepreneur's optimal decision rules to deliver an operational and analytically tractable framework for the cost of capital as well as the private valuation of an entrepreneurial firm.

Entrepreneurial Finance, as an academic field, offers no theoretical guidance on how to calculate the cost of capital for entrepreneurial firms. Non-diversifiable risk and other frictions that entrepreneurs face invalidate the textbook risk-return and capital budgeting analysis. Consider a project valuation exercise. Suppose that the standard capital asset pricing model (CAPM) works if this project is evaluated by a firm owned by diversified investors. The standard valuation technique is based on complete diversification and thus leaves no room for an idiosyncratic risk premium. However, when the project is evaluated by an entrepreneurial firm whose owner is heavily exposed to the project's subsequent performance and cannot fully diversify, we expect that the entrepreneur not only demands the systematic risk premium but also an additional "private" equity idiosyncratic risk premium. What are the determinants of this idiosyncratic risk premium? We provide an operational and economic framework to answer this fundamental question that arises in Entrepreneurial Finance. Our calibration exercise suggests that this idiosyncratic risk premium is likely to be significant for non-diversified entrepreneurs.

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[1] For example, see Moskowitz and Vissing-Jorgensen (2002) and Gentry and Hubbard (2004).

[2] For example, see Evans and Jovanovic (1989) and Cagetti and De Nardi (2006).

Our agent has a preference for intertemporal consumption smoothing. This preference separates the elasticity of intertemporal substitution (EIS) from risk aversion, i.e. an Epstein-Zin (1989) non-expected utility. The agent optimally chooses the timing to become an entrepreneur. Doing so forgoes the outside option (e.g. earning constant wages over time) and requires a fixed start-up cost. The entrepreneur also chooses the initial firm size. The entrepreneurial project/idea is defined by a capital accumulation/production technology.

After setting up the firm, the entrepreneur accumulates capital and incurs adjustment costs as in the standard $q$ theory of investment.[3] In addition, the entrepreneur invests wealth between a risk-free asset and the risky market portfolio as in a standard consumption-portfolio choice framework (Merton (1971)). By dynamically trading the market portfolio, the entrepreneur can hedge the systematic component of the business risk, but he cannot hedge the idiosyncratic risk component.[4] Additionally, the entrepreneur can borrow up to a fixed fraction of the illiquid capital. The debt is secured by the capital. Finally, the entrepreneur can choose the optimal timing to liquidate the capital, which provides some downside risk protection for the entrepreneur. Note that the entrepreneur may also engage in asset sales (divestment) by incurring a convex adjustment cost before complete liquidation.

Our model integrates insights from several well developed literatures in economics and finance: (1) the optimal incomplete-markets consumption/savings problem;[5] (2) the $q$ theory of investment; (3) the optimal consumption and portfolio choice problem; and (4) dynamic endogenous entry and exit models. Importantly, we show that the newly derived insights critically depend on the *interaction* between the consumption smoothing motive from the consumption side and the wealth creation motive from the production side. Ignoring either the consumption or the production side of the model gives misleading predictions. Therefore, it is essential that we model the interdependence between the consumption and production

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[3]Brainard and Tobin (1968) and Tobin (1969) define the ratio between the firms market value to the replacement cost of its capital stock, as $Q$ and propose to use this ratio to measure the firms incentive to invest in capital. This ratio has become known as Tobins average $Q$. Hayashi (1982) provides conditions under which average $Q$ is equal to marginal $q$. Abel and Eberly (1994) develop a unified $q$ theory of investment in neoclassic settings. Lucas and Prescott (1971) and Abel (1983) are important early contributors.

[4]We assume that the market portfolio is not perfectly correlated with business risk, and hence markets are incomplete. This is a plausible assumption for entrepreneurs. Otherwise, the complete-market solution applies to the model. See the complete-markets solution for the benchmark model in Section 3.

[5]In macroeconomics, the literature on permanent-income and buffer-stock savings focuses on the consumption-saving margin. For example, see Hall (1978), Zeldes (1989a, 1989b), Deaton (1991), Carroll (1997), Gourinchas and Parker (2002), Storesletten, Telmer, and Yaron (2004), and Guvenen (2007), among others.

sides. Despite the richness of our modeling ingredients, we manage to solve the model in an analytically tractable way by utilizing the model's homogeneity property. The homogeneity property refers to the one that if we simultaneously double wealth and firm size, the model solution remains unchanged other than the size adjustment. We show that the entrepreneur's optimality only requires us to keep track of the ratio between wealth and capital; this wealth-capital ratio, denoted as $w$, plays a critical role in our paper.

We find that the entrepreneur significantly underinvests in business, scales back consumption, and allocates less wealth to the market portfolio in order to mitigate frictions (non-diversifiable risk and liquidation constraints). These predictions are consistent with empirical findings. For example, Heaton and Lucas (2000) find that entrepreneurs with high and variable business income hold less wealth in stocks than other similarly wealthy households. The private business valuation is also significantly lower than the complete-markets valuation. Additionally, we show that these frictions make wealth more valuable for both production and consumption purposes.

The exit option provides significant flexibility for the entrepreneur to manage downside risk.[6] The ability to liquidate capital makes the entrepreneur's certainty equivalent wealth convex in $w$ for low values of $w$, generating optionality. Similarly, the ability to time the entry decision makes the option value of waiting to become an entrepreneur quite valuable. While these entry and exit options are subject to frictions, such as non-diversifiable risk and liquidity constraints, and are not marked to the market, they are nonetheless quite valuable as risk management tools for the entrepreneur facing significant frictions under incomplete markets.

While these frictions induce underinvestment in business and the market portfolio, we cannot necessarily conclude that the degrees of underinvestment are monotonic in the wealth-capital ratio $w$. The entry and exit options make the certainty equivalent wealth convex for low values of $w$ and hence both investment and market portfolio allocation may decrease with $w$ for sufficiently low values of $w$.

We find that there are significant welfare costs for the entrepreneur to bear non-diversifiable

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[6]For simplicity, we focus on the liquidation option as the exit option for downside risk protection. Without changing the analysis in any fundamental way, we can extend our model to allow the entrepreneur to have an exit option when doing well. For example, selling to diversified investors or going to an initial public offering (IPO) are two ways for the entrepreneur to exit when doing well. See Pastor, Taylor, and Veronesi (2009) and Chen, Miao, and Wang (2010) for models with IPO as an exit option in good times.

idiosyncratic risk. For an expected-utility entrepreneur whose coefficient of relative risk aversion is equal to two and has no liquid wealth, as in our baseline example, the subjective valuation of the entrepreneurial business is about 11% lower than the complete-markets benchmark.

Our paper links to several strands of literature in finance, macroeconomics, and entrepreneurship. The economics of entrepreneurship literature is fast growing. Hall and Woodward (2010) analyze the effects of non-diversifiable risk for venture-capital-backed entrepreneurial firms. Heaton and Lucas (2004) show that risky non-recourse debt helps the entrepreneur diversify business risk in a static framework with capital budgeting, capital structure, and portfolio choice decisions. Chen, Miao, and Wang (2010) study the effects of non-diversifiable risk on entrepreneurial finance by building on the workhorse contingent-claim capital structure model.[7] They show that more risk averse entrepreneurs borrow more in order to lower their business risk exposure.[8] Herranz, Krasa, and Villamil (2009) assess the impact of legal institutions on entrepreneurial firm dynamics. They also find that more risk averse entrepreneurs default more.

Evans and Jovanovic (1989) show the importance of wealth and liquidity constraints for entrepreneurship.[9] Cagetti and De Nardi (2006) quantify the importance of liquidity constraints on aggregate capital accumulation and wealth distribution by constructing a model with entry, exit, and investment decisions. Hurst and Lusardi (2004) challenge the importance of liquidity constraints and provide evidence that the start-up sizes of entrepreneurial firms tend to be small. We provide a theory of entrepreneurship by accounting for endogenous entry/exit in a model with borrowing constraints and non-diversifiable risk.

Almost all models in the $q$ theory literature are designed for firms held by diversified investors. We extend the $q$ theory of investment to account for non-diversifiable risk and liquidity constraints. In addition, the entrepreneurial firm in our model has flexible entry and exit options. We show that non-diverisifable risk and liquidity constraints have first-order effects on capital accumulation and firm valuation. We develop the counterparts of marginal $q$ and average $q$ for *private* firms owned and operated by non-diversified entrepreneurs.

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[7]Leland (1994) is an important paper which initiated this line of research. Morellec (2004) extends the framework to analyze managerial agency issues and leverage.

[8]For tractability, Chen, Miao, and Wang (2010) adopt exponential utility, while this paper uses non-expected Epstein-Zin utility. Also, the questions addressed in these two papers are rather different.

[9]See Holtz-Eakin, Joulfaian, and Rosen (1994) and Blanchflower and Oswald (1998) for empirical evidence.

Most models on portfolio choice with non-tradable income assume exogenous income.[10] Our model endogenizes the non-marketable income from business via optimal entrepreneurial decisions. The endogenous business entry/exit and consumption/portfolio decisions are important margins for the entrepreneur to manage risk. The entry/exit options significantly alter the entrepreneur's decision making. Some of our results are also related to the real options analysis under incomplete markets. Miao and Wang (2007) and Hugonnier and Morellec (2007) study the impact of non-diversifiable risk on real options exercising. These papers show that the non-diversifiable risk significantly alters option exercising strategies.

Finally, our model also relates to recent work on dynamic corporate finance.[11] Using the same neoclassical Hayashi (1982) framework as we do, Bolton, Chen, and Wang (2011) analyze optimal investment, financing, and risk management decisions and valuation for financially constrained firms. Unlike their paper, which applies to public firms, our model focuses on entrepreneurial firms. DeMarzo, Fishman, He, and Wang (2010) integrate a dynamic moral hazard framework of DeMarzo and Fishman (2007b) and DeMarzo and Sannikov (2006) with the neoclassical $q$ theory of investment (Hayashi (1982)). They derive an optimal dynamic contract and provide financial implementation.[12]

# 2  The model

We first introduce the agent's preferences and then set up the optimization problem.

**Preferences.**  The agent has a preference featuring both constant relative risk aversion and constant EIS (Epstein and Zin (1989) and Weil (1990)). We use the continuous-time formulation of this non-expected utility introduced by Duffie and Epstein (1992a). That is, the agent has a recursive preference defined as follows

$$J_t = \mathbb{E}_t \left[ \int_t^\infty f(C_s, J_s) ds \right], \tag{1}$$

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[10]See Merton (1971) and Mayers (1974) for early contributions. Among others, Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), and Viceira (2001) study the optimal consumption and portfolio rules for an investor with isoelastic utility and non-tradable labor income risk.

[11]This is a fast growing field. For studies on investment with financial constraints, for example, see Whited (1992), Gomes (2001), Hennessy and Whited (2005, 2007), Gamba and Triantis (2008), Riddick and Whited (2009), and Bolton, Chen, and Wang (2011).

[12]DeMarzo and Fishman (2007a) analyze the impact of agency on investment dynamics in discrete time.

where $f(C, J)$ is known as the normalized aggregator for consumption $C$ and the agent's utility $J$. Duffie and Epstein (1992a, 1992b) show that $f(C, J)$ for Epstein-Zin non-expected homothetic recursive utility is given by

$$f(C, J) = \frac{\zeta}{1 - \psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1-\gamma)J)^\chi}{((1-\gamma)J)^{\chi-1}}, \qquad (2)$$

where

$$\chi = \frac{1 - \psi^{-1}}{1 - \gamma}. \qquad (3)$$

The parameter $\psi > 0$ measures the EIS, and the parameter $\gamma > 0$ is the coefficient of relative risk aversion. The parameter $\zeta > 0$ is the agent's subjective discount rate.

The widely used time-additive separable constant-relative-risk-averse (CRRA) utility is a special case of the Duffie-Epstein-Zin-Weil recursive utility specification where the coefficient of relative risk aversion is equal to the inverse of the EIS $\psi$, i.e. $\gamma = \psi^{-1}$ implying $\chi = 1$.[13] In general, with $\gamma \neq 1/\psi$, we can separately study the effects of risk aversion and the EIS.

**Career choice and initial firm size.** The agent is endowed with an entrepreneurial idea and initial wealth $W_0$. The entrepreneurial idea is defined by a productive capital accumulation/production function to be introduced soon. To implement the entrepreneurial idea, the agent chooses a start-up time $T^0$, pays a one-time fixed start-up cost $\Phi$, and also chooses the initial capital stock $K_{T^0}$. One example is being a taxi/limo driver. The agent can first start with a used car. After building up savings, the agent tolerates risk better and potentially upgrades the vehicle. With even more savings, the agent may further increase firm size by hiring drivers and running a limo service.

Before becoming an entrepreneur, the agent can take an alternative job (e.g. to be a worker) to build up financial wealth. Being an entrepreneur is a discrete career decision.[14] We naturally assume that being an entrepreneur offers potentially a higher reward at a greater risk than being a worker. Hamilton (2000) finds that earnings of the self-employed

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[13]For this special case, we have $f(C, J) = U(C) - \zeta J$, where $U(C)$ is the expected CRRA utility with $\gamma = \psi^{-1}$ and hence $U(C) = \zeta C^{1-\psi^{-1}}/(1-\psi^{-1})$. Note that for CRRA utility, $f(C, J)$ is additively separable. By integrating (1) forward for this CRRA special case, we obtain $J_t = \max_C \mathbb{E}_t \left[ \int_t^\infty e^{-\zeta(s-t)} U(C(s)) ds \right]$.

[14]We do not allow the agent to be a part-time entrepreneur and a part-time worker at the same time. This is a standard and reasonable assumption. For example, see Vereshchagina and Hopenhayn (2009) for a dynamic career choice model featuring the same assumption.

are smaller on average and have higher variance than earnings of workers using data from Survey of Income and Program Participation. To contrast the earnings profile differences between an entrepreneur and a worker, we assume that the outside option (by being a worker) gives the agent a constant flow of income at the rate of $r\Pi$.

At the optimally chosen (stochastic) entry time $T^0$, the agent uses a combination of personal savings and collateralized borrowing to finance $(K_{T^0} + \Phi)$. Lenders make zero profit in competitive capital markets. If the entrepreneur reneges on debt, creditors can always liquidate the firm's capital and recover fraction $l > 0$ per unit of capital. The borrower thus has no incentive to default on debt and can borrow up to $lK$ at the risk-free rate by using capital as the collateral.

We will show that initial wealth $W_0$ plays a role in how long it takes the agent to become an entrepreneur and the choice of the firm's initial size. Borrowing constraints and non-diversifiable risk are conceptually and quantitatively important. Moreover, these two frictions interact and generate economically significant feedback effects on entrepreneurship.

**Entrepreneurial idea: capital investment and production technology.** The entrepreneurial idea is defined by a capital accumulation/production function. Let $I$ denote the gross investment. As is standard in capital accumulation models, the change of capital stock $K$ is given by the difference between gross investment and depreciation, in that

$$dK_t = (I_t - \delta K_t)\, dt, \quad t \geq 0, \tag{4}$$

where $\delta \geq 0$ is the rate of depreciation. The firm's productivity shock $dA_t$ is independently and identically distributed (*i.i.d.*), and is given by

$$dA_t = \mu_A dt + \sigma_A dZ_t, \tag{5}$$

where $Z$ is a standard Brownian motion, $\mu_A > 0$ is the mean of the productivity shock, and $\sigma_A > 0$ is the volatility of the productivity shock. The firm's operating revenue over time period $(t, t + dt)$ is proportional to its time-$t$ capital stock $K_t$, and is given by $K_t dA_t$. The firm's operating profit $dY_t$ over the same period is given by

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t) dt, \tag{6}$$

where the price of the investment good is set to unity and $G(I, K)$ is the adjustment cost.

Following Hayashi (1982), we assume that the firm's adjustment cost $G(I, K)$ is homogeneous of degree one in $I$ and $K$, and write $G(I, K)$ in the following homogeneous form

$$G(I, K) = g(i)K, \tag{7}$$

where $i = I/K$ is the firm's investment-capital ratio and $g(i)$ is an increasing and convex function. With homogeneity, Tobin's average $q$ is equal to marginal $q$ under perfect capital markets. However, as we will show, the non-diversifiable risk drives a wedge between Tobin's average $q$ and marginal $q$ for the entrepreneur. For simplicity, we assume that

$$g(i) = \frac{\theta i^2}{2}, \tag{8}$$

where the parameter $\theta$ measures the degree of the adjustment cost. A higher value $\theta$ implies a more costly adjustment process.

The entrepreneur has an option to liquidate capital at any moment. Liquidation is irreversible and gives a terminal value $lK$, where $l > 0$ is a constant. Let $T^l$ denote the entrepreneur's optimally chosen stochastic liquidation time. To focus on the interesting case, we assume capital is sufficiently productive. Thus, liquidating capital when capital markets are perfect is not optimal because doing so destroys going-concern value. However, when the entrepreneur is not well diversified, liquidation provides an important channel for the entrepreneur to manage the downside business risk exposure.

Our production specification features the widely used "$AK$" technology[15] augmented with the adjustment cost technology. Our specification is a reasonable starting point and is also analytically tractable. Next, we turn to the agent's financial investment opportunities.

**Financial investment opportunities.** The agent can invest in a risk-free asset which pays a constant rate of interest $r$ and the risky market portfolio (Merton (1971)). Assume that the incremental return $dR_t$ of the market portfolio over time period $dt$ is i.i.d., i.e.

$$dR_t = \mu_R dt + \sigma_R dB_t, \tag{9}$$

where $\mu_R$ and $\sigma_R$ are constant mean and volatility parameters of the market portfolio return process, and $B$ is a standard Brownian motion. Let

$$\eta = \frac{\mu_R - r}{\sigma_R} \tag{10}$$

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[15]Cox, Ingersoll, and Ross (1985) feature an equilibrium production economy with the "$AK$" technology. See Jones and Manuelli (2005) for a recent survey on endogenous growth models.

denote the Sharpe ratio of the market portfolio. Let $\rho$ denote the correlation coefficient between the shock to the entrepreneur's business and the shock to the market portfolio. With incomplete markets ($|\rho| < 1$), the entrepreneur cannot completely hedge business risk. Non-diversifiable risk will thus play a role in decision making and private valuation.

Let $W$ and $X$ denote the agent's financial wealth and the amount invested in the risky asset, respectively. Then, $(W - X)$ is the remaining amount invested in the risk-free asset. Before becoming an entrepreneur ($t \leq T^0$), the wealth accumulation is given by

$$dW_t \;=\; r\left(W_t - X_t\right)dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt + r\Pi dt\,, \quad t < T^0\,. \tag{11}$$

While being an entrepreneur, the liquid financial wealth $W$ evolves as follows:

$$dW_t \;=\; r\left(W_t - X_t\right)dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt + dY_t\,, \quad T^0 < t < T^l\,. \tag{12}$$

Finally, after exiting from the business, the retired entrepreneur's wealth evolves as follows:

$$dW_t \;=\; r\left(W_t - X_t\right)dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt\,, \quad t > T^l\,. \tag{13}$$

The entrepreneur can borrow against capital $K$ at all times, and hence wealth $W$ can be negative. To ensure that entrepreneurial borrowing is risk-free, we require that the liquidation value of capital $lK$ is greater than outstanding liability, in that

$$W_t \geq -lK_t\,, \quad T^0 \leq t \leq T^l\,. \tag{14}$$

Despite being able to borrow up to $lK_t$ at the risk-free rate $r$, the entrepreneur may rationally choose not to exhaust the debt capacity for precautionary reasons. Without capital as collateral, the agent cannot borrow: $W_t \geq 0$ for $t \leq T^0$ and $t \geq T^l$.

**The optimization problem.** The agent maximizes the utility defined in (1)-(2). The timeline can be described in five steps. First, before becoming an entrepreneur ($t \leq T^0$), the agent collects income as a worker and chooses consumption and portfolio allocations. Second, the agent chooses the optimal entry time $T^0$ to start up the firm and the initial firm size $K_{T^0}$ by incurring the fixed start-up cost $\Phi$, and financing the total costs $(K_{T^0} + \Phi)$ with savings and/or potentially some collateralized borrowing. Third, the agent chooses consumption and portfolio choice while running the firm subject to the collateralized borrowing limit (14). Fourth, the agent optimally chooses the stochastic liquidation time $T^l$. Finally, after liquidating capital, the agent collects the liquidation proceeds, retires, allocates wealth between the risk-free and the risky market portfolio, and consumes.

# 3 Benchmark: Complete markets

With complete markets, the entrepreneur's optimization problem can be decomposed into two separate ones: wealth maximization and utility maximization. We will show that our model has the homogeneity property. The lower case denotes the corresponding variable in the upper case scaled by $K$. For example, $w$ denotes the wealth-capital ratio $W/K$. The following proposition summarizes our main results under complete markets.

**Proposition 1** *The entrepreneur's value function $J^{FB}(K,W)$ is given by*

$$J^{FB}(K,W) = \frac{(bP^{FB}(K,W))^{1-\gamma}}{1-\gamma}, \tag{15}$$

*where the total wealth $P^{FB}(K,W)$ is given by the sum of $W$ and firm value $Q^{FB}(K)$*

$$P^{FB}(K,W) = W + Q^{FB}(K) = W + q^{FB}K, \tag{16}$$

*and*

$$b = \zeta \left[ 1 + \frac{1-\psi}{\zeta} \left( r - \zeta + \frac{\eta^2}{2\gamma} \right) \right]^{\frac{1}{1-\psi}}. \tag{17}$$

*Firm value $Q^{FB}(K)$ is equal to $q^{FB}K$, where Tobin's q, $q^{FB}$, is given by*

$$q^{FB} = 1 + \theta i^{FB}, \tag{18}$$

*where the first-best investment-capital ratio $i^{FB}$ is given by*

$$i^{FB} = (r+\delta) - \sqrt{(r+\delta)^2 - \frac{2}{\theta} (\mu_A - \rho\eta\sigma_A - (r+\delta))}. \tag{19}$$

*The optimal consumption $C$ is proportional to $K$, i.e. $C(K,W) = c^{FB}(w)K$, where*

$$c^{FB}(w) = m^{FB} \left( w + q^{FB} \right), \tag{20}$$

*and $m^{FB}$ is the marginal propensity to consume (MPC) and is given by*

$$m^{FB} = \zeta + (1-\psi) \left( r - \zeta + \frac{\eta^2}{2\gamma} \right). \tag{21}$$

*The market portfolio allocation $X$ is also proportional to $K$, $X(K,W) = x^{FB}(w)K$, where*

$$x^{FB}(w) = \left( \frac{\mu_R - r}{\gamma\sigma_R^2} \right) \left( w + q^{FB} \right) - \frac{\rho\sigma_A}{\sigma_R}. \tag{22}$$

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*The capital asset pricing model (CAPM) holds for the firm with its expected return given by*

$$\xi^{FB} = r + \beta^{FB} \left( \mu_R - r \right) , \qquad (23)$$

*where the firm's beta, $\beta^{FB}$, is constant and given by*

$$\beta^{FB} = \frac{\rho \sigma_A}{\sigma_R} \frac{1}{q^{FB}} . \qquad (24)$$

Equations (18) and (19) give Tobin's $q$ and the investment-capital ratio, respectively. The adjustment cost makes installed capital earn rents and, hence, Tobin's $q$ differs from unity. Note that the average $q$ is equal to the marginal $q$ as in Hayashi (1982). The entrepreneur's total wealth is given by $p^{FB}(w) = w + q^{FB}$, the sum of Tobin's $q$ and $w$. Equation (20) gives consumption, effectively the permanent-income rule under complete markets. The entrepreneur's MPC out of wealth $m^{FB}$ generally depends on the risk-free rate $r$, the EIS $\psi$, the coefficient of risk aversion $\gamma$, and the Sharpe ratio $\eta = (\mu_R - r)/\sigma_R$. Equation (22) gives $x(w)$, the portfolio allocation to the market portfolio. The first term in (22) is the well-known mean-variance allocation, and the second term is the intertemporal hedging demand.

We explicitly account for the effects of risk on investment and Tobin's $q$. We decompose the total volatility of the productivity shock into systematic and idiosyncratic components. The systematic volatility is equal to $\rho \sigma_A$ and the idiosyncratic component is given by

$$\epsilon = \sqrt{1 - \rho^2} \, \sigma_A . \qquad (25)$$

The standard CAPM holds in our benchmark. The expected return is given in (23) and $\beta$ is given by (24). As in the standard finance theory, the idiosyncratic volatility $\epsilon$ carries no risk premium and plays no role under complete markets. However, importantly, the idiosyncratic volatility $\epsilon$ will play a significant role in our incomplete-markets setting.

# 4 Incomplete-markets model solution after entry

Having characterized the complete-markets solution, we now turn to the incomplete-markets setting. We first consider the agent's decision problem after liquidation, and then derive the entrepreneur's interdependent decision making before exit.

**The agent's decision problem after exiting entrepreneurship.** After exiting from entrepreneurship, the entrepreneur is no longer exposed to the business risk and faces a classic Merton consumption/portfolio allocation problem with non-expected recursive utility. The solution is effectively the same as the complete-markets results in Proposition 1 (without physical capital). We summarize the results as a corollary to Proposition 1.

**Corollary 1** *The entrepreneur's value function takes the following homothetic form*

$$V(W) = \frac{(bW)^{1-\gamma}}{1-\gamma}, \tag{26}$$

*where $b$ is a constant given in (17). The optimal consumption $C$ and allocation amount $X$ in the risky market portfolio are respectively given by*

$$C = m^{FB}W, \tag{27}$$

$$X = \left(\frac{\mu_R - r}{\gamma\sigma_R^2}\right)W, \tag{28}$$

*where $m^{FB}$ is the MPC out of wealth and is given in (21).*

We will use the value function given in (26) when analyzing the agent's pre-exit decisions.

**The entrepreneur's decision problem while running his business.** Let $J(K,W)$ denote the entrepreneur's value function. The entrepreneur chooses consumption $C$, real investment $I$, and the allocation to the risky market portfolio $X$ by solving the following Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \max_{C, I, X} f(C, J) + (I - \delta K)J_K + (rW + (\mu_R - r)X + \mu_A K - I - G(I,K) - C)J_W$$
$$+ \left(\frac{\sigma_A^2 K^2 + 2\rho\sigma_A\sigma_R KX + \sigma_R^2 X^2}{2}\right)J_{WW}. \tag{29}$$

The entrepreneur's first-order condition (FOC) for consumption $C$ is given by

$$f_C(C, J) = J_W(K, W). \tag{30}$$

The above condition states that the marginal utility of consumption $f_C$ is equal to the marginal utility of wealth $J_W$. The FOC with respect to investment $I$ gives

$$(1 + G_I(I, K)) J_W(K, W) = J_K(K, W). \tag{31}$$

To increase capital stock by one unit, the entrepreneur needs to forgo $(1 + G_I(I, K))$ units of consumption. The marginal utility of consumption is equal to the marginal utility of wealth $J_W$. Therefore, the entrepreneur's marginal cost of investing is given by the product of $(1 + G_I(I, K))$ and the marginal utility of wealth $J_W$. The marginal benefit of increasing capital stock by a unit is $J_K$. At optimality, the entrepreneur equates the two sides of (31).

The FOC with respect to portfolio choice $X$ is given by

$$X = -\frac{\mu_R - r}{\sigma_R^2} \frac{J_W}{J_{WW}} - \frac{\rho \sigma_A}{\sigma_R} K \,. \tag{32}$$

The first term in (32) is the mean-variance demand, and the second term captures the hedging demand. Because the entrepreneur's effective risk aversion depends on $w$, the mean-variance demand is much more interesting under incomplete markets than it is under complete markets. Using the homogeneity property, we conjecture that the value function $J(K, W)$ is given by

$$J(K, W) = \frac{(bP(K, W))^{1-\gamma}}{1 - \gamma} \,, \tag{33}$$

where $b$ is given in (17). Comparing (33) with the value function without the business (26), we may intuitively refer to $P(K, W)$ as the entrepreneur's certainty equivalent wealth, the minimal amount of wealth for which the agent is willing to permanently give up the business and liquid wealth $W$. Let $\underline{W}$ denote the entrepreneur's *endogenous* liquidation boundary and $\underline{w} = \underline{W}/K$. The following theorem summarizes the entrepreneur's decision making and certainty equivalent wealth $p(w) = P(K, W)/K$.

**Theorem 1** *The entrepreneur operates business if and only if $w \geq \underline{w}$. The scaled certainty equivalent wealth $p(w)$ solves the following ordinary differential equation (ODE)*

$$0 = \frac{m^{FB} p(w)(p'(w))^{1-\psi} - \psi \zeta p(w)}{\psi - 1} - \delta p(w) + (r + \delta) w p'(w) + (\mu_A - \rho \eta \sigma_A) p'(w)$$
$$+ \frac{(p(w) - (w + 1)p'(w))^2}{2\theta p'(w)} + \frac{\eta^2 p(w) p'(w)}{2h(w)} - \frac{\epsilon^2 h(w) p'(w)}{2p(w)}, \quad if \, w \geq \underline{w}, \tag{34}$$

*where $\epsilon$ is the idiosyncratic volatility given in (25) and $h(w)$ is given by*

$$h(w) = \gamma p'(w) - \frac{p(w) p''(w)}{p'(w)} \,. \tag{35}$$

*When $w$ approaches $\infty$, $p(w)$ approaches complete-markets solution given by*

$$\lim_{w \to \infty} p(w) = w + q^{FB} \,. \tag{36}$$

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*Finally, the ODE (34) satisfies the following conditions at the endogenous boundary $\underline{w}$*

$$p(\underline{w}) \;=\; \underline{w} + l, \tag{37}$$

$$p'(\underline{w}) \;=\; 1. \tag{38}$$

*The optimal consumption $c = C/K$, investment $i = I/K$, and market portfolio allocation-capital ratio $x = X/K$ are given by*

$$c(w) \;=\; m^{FB} p(w) (p'(w))^{-\psi}, \tag{39}$$

$$i(w) \;=\; \frac{1}{\theta} \left( \frac{p(w)}{p'(w)} - w - 1 \right), \tag{40}$$

$$x(w) \;=\; -\frac{\rho \sigma_A}{\sigma_R} + \frac{\mu_R - r}{\sigma_R^2} \frac{p(w)}{h(w)}, \tag{41}$$

*where $h(w)$ is given in (35). The dynamics of the wealth-capital ratio $w$ are given by*

$$dw_t = \mu_w(w_t) dt + \sigma_R x(w_t) dB_t + \sigma_A dZ_t, \tag{42}$$

*where the drift $\mu_w(w)$ gives the expected change of $w$ and is given by*

$$\mu_w(w) = (r + \delta - i(w))w + (\mu_R - r)x(w) + \mu_A - i(w) - g(i(w)) - c(w). \tag{43}$$

However, if the conditions (37)-(38) do not admit an interior solution satisfying $\underline{w} > -l$, the optimal liquidation boundary is then given by the maximal borrowing capacity, i.e. $\underline{w} = -l$.

# 5   Incomplete-markets model results after entry

We now explore the implications of the incomplete-markets model in Section 4. We choose parameter values as follows and, whenever applicable, all parameters are annualized. The risk-free interest rate is $r = 4.6\%$ and the aggregate equity risk premium is $(\mu_R - r) = 6\%$. The annual volatility of the market portfolio return is $\sigma_R = 20\%$ implying the Sharpe ratio for the aggregate stock market $\eta = (\mu_R - r)/\sigma_R = 30\%$. The subjective discount rate is set to equal to the risk-free rate, $\zeta = r = 4.6\%$.

On the real investment side, our model is a version of the $q$ theory of investment (Hayashi (1982)). Using the sample of large firms in Compustat from 1981 to 2003, Eberly, Rebelo, and Vincent (2009) provide empirical evidence in support of Hayashi (1982). Using their work

as a guideline, we set the expected productivity $\mu_A = 20\%$ and the volatility of productivity shocks $\sigma_A = 10\%$. Fitting the complete-markets $q^{FB}$ and $i^{FB}$ to the sample averages, we obtain the adjustment cost parameter $\theta = 2$ and the rate of depreciation for capital stock $\delta = 12.5\%$.[16] We choose the liquidation parameter $l = 0.9$ (Hennessy and Whited (2007)). We set the correlation between the market portfolio return and the business risk $\rho = 0$, which implies that the idiosyncratic volatility of the productivity shock $\epsilon = \sigma_A = 10\%$. We consider two widely used values for the coefficient of relative risk aversion, $\gamma = 2$ and $\gamma = 4$. We set the EIS to be $\psi = 0.5$, so that the first case corresponds to the expected utility with $\gamma = 1/\psi = 2$, and the second case maps to a non-expected utility with $\gamma = 4 > 1/\psi = 2$. Table 1 summarizes the notations and if applicable, value choices for various parameters.

## 5.1 Decomposing the entrepreneur's welfare

We measure the entrepreneur's welfare via the value function given in (33). The value function $J(K, W)$ is homogeneous of degree $(1 - \gamma)$ in the certainty equivalent wealth $P(K, W)$. Therefore, we may equivalently quantify the agent's welfare via $P(K, W)$.

**Certainty equivalent wealth and private enterprise value.** In corporate finance, enterprise value is defined as firm value excluding liquid assets (e.g. cash and short-term marketable assets). For the entrepreneurial firm, we may similarly define the entrepreneur's private enterprise value $Q(K, W)$ as follows

$$Q(K, W) = P(K, W) - W. \qquad (44)$$

Dividing the private enterprise value $Q(K, W)$ by illiquid physical capital $K$, we have

$$q(w) = \frac{Q(K, W)}{K} = p(w) - w. \qquad (45)$$

For a firm owned and managed by an entrepreneur, (45) captures the impact of non-diversifiable risk on the subjective valuation of capital. Importantly, the "private" average $q$ defined in (45) depends on the entrepreneur's preferences.

For Figures 1-5, we graph for two levels of risk aversion, $\gamma = 2, 4$. The top left and top right panels of Figure 1 plot $p(w)$ and $q(w)$, respectively. Note that $q(w) = p(w) - w$. Thus,

---

[16]The averages are 1.3 for $q$ and 15% for the investment-capital ratio, respectively, for the sample used by Eberly, Rebelo, and Vincent (2009). The imputed $\theta = 2$ is in the range of estimates used in the literature. See Whited (1992), Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).
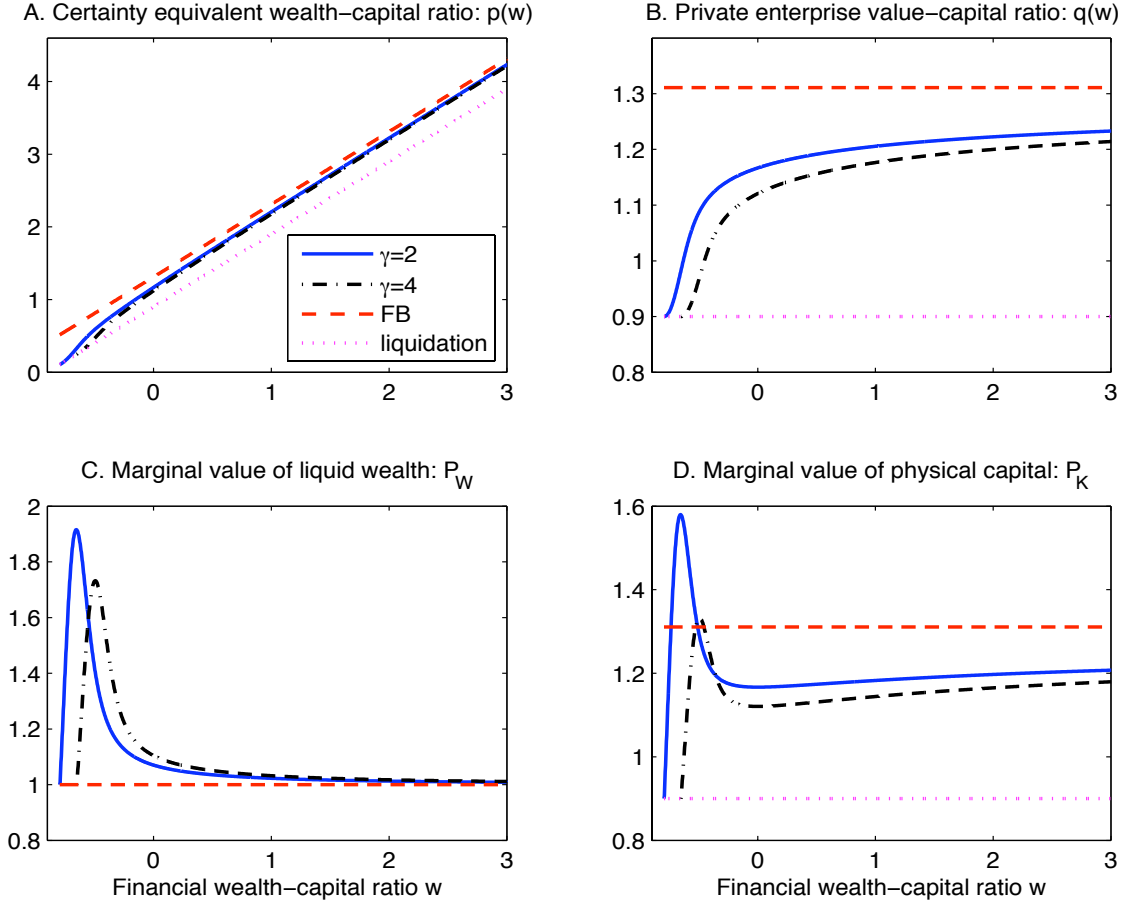
Figure 1: **Certainty equivalent wealth $p(w)$, private enterprise value ratio $q(w)$, marginal value of wealth $P_W(K,W)$, and marginal value of capital $P_K(K,W)$.**

$p(w)$ and $q(w)$ effectively convey the same information. Graphically, it is easier to read Panel B of Figure 1 for $q(w)$ than Panel A for $p(w)$, thus, we discuss $q(w)$. Recall that the first-best Tobin's $q$, $q^{FB}$, is independent of entrepreneurial preferences (complete-markets Arrow-Debreu separation results). For our baseline calculation, we have this complete-markets Tobin's $q$, $q^{FB} = 1.31$. For finitely valued $w$, $q(w)$ increases with $w$. In the limit as $w \to \infty$, the entrepreneur effectively attaches no premium for the non-diversifiable risk and $q(w)$ approaches the the complete-markets $q^{FB}$, $\lim_{w\to\infty} q(w) \to q^{FB} = 1.31$. However, quantitatively, the convergence requires a relatively high value of $w$. When $w = 3$, we have $q(3) = 1.23$ for $\gamma = 2$, and $q(3) = 1.21$ for $\gamma = 4$, both of which are significantly lower than the complete-markets benchmark value $q^{FB} = 1.31$. The less risk-averse the entrepreneur,

the higher Tobin's $q$, $q(w)$.

More interestingly, $q(w)$ is not globally concave. The second derivatives and hence concavity properties for $p(w)$ and $q(w)$ are the same because $q(w) = p(w) - w$. Risk aversion does not necessarily imply that $q(w)$ is concave. Nonetheless, the risk-averse entrepreneur's value function $J(K,W)$ is concave in $P(K,W)$ and is also concave in $W$. Figure 1 shows that $q(w)$ is concave in $w$ for sufficiently high $w$, i.e. $w \geq \widetilde{w}$ where $\widetilde{w}$ is the inflection point at which $p''(\widetilde{w}) = q''(\widetilde{w}) = 0$. For sufficiently low $w$, i.e. $w \leq \widetilde{w}$, $q(w)$ is convex in $w$.

The entrepreneur has an option to eliminate the non-diversifiable business risk exposure by liquidating the firm. The option to exit from the business causes $q(w)$ to be convex in $w$ for sufficiently low $w$. Costly liquidation of capital provides a downside risk protection for the entrepreneur. Quantitatively, this exit option is quite valuable for low $w$. Recall that debt is fully collateralized and is risk-free. Thus in our model, liquidation simply provides an exit option which becomes in the money when the entrepreneur bears significant non-diversifiable risk (i.e. with sufficiently low in $w$). In Zame (1993), Heaton and Lucas (2004), and Chen, Miao, and Wang (2010), the benefits of debt rely on the riskiness of debt, which creates state-contingent insurance. However, both arguments rely on "put" options (a liquidation option in our model and a default option in risky debt models) providing downside protection for the entrepreneur under incomplete markets. We now decompose the certainty equivalent wealth $P(K,W)$.

**Decomposing the certainty equivalent wealth $P(K,W)$.**   Using the homogeneity property, we have

$$
\begin{aligned}
P_W(K,W) &= p'(w), & (46) \\
P_K(K,W) &= p(w) - wp'(w). & (47)
\end{aligned}
$$

For public firms owned by diversified investors, the marginal increase of firm value associated with a unit increase of capital is often referred to as marginal $q$. For a firm owned and managed by a non-diversified entrepreneur, the marginal increase of $P(K,W)$ associated with a unit increase of capital, $P_K(K,W)$, is the natural counterpart to the marginal $q$ for public firms. We refer to $P_K(K,W)$ as the private (i.e. subjective) marginal $q$ for the entrepreneurial firm. For public firms, the marginal increase of firm value associated with a unit increase of cash is referred to as the marginal value of cash (Bolton, Chen, and Wang

(2011)). For entrepreneurial firms, the entrepreneur's marginal value of wealth $P_W(K, W)$ is the natural counterpart to the marginal value of cash for public firms.

**The marginal value of wealth $P_W(K, W)$.** The lower left panel (Panel C) of Figure 1 plots $P_W(K, W) = p'(w)$. In perfect capital markets, $P_W^{FB}(K, W) = 1$. With incomplete markets, $P_W(K, W)$ is greater than unity because wealth has the additional benefit of mitigating the negative impact of financial frictions on investment and consumption. Panel C shows that $p'(w)$ is equal to unity at the liquidation boundary $\underline{w}$, $p'(\underline{w}) = 1$, because the agent is no longer exposed to non-diversifiable risk after exiting entrepreneurship. Then, $p'(w)$ increases with $w$ up to the endogenous inflection point $\widetilde{w}$ (at which $p''(\widetilde{w}) = 0$), decreases with $w$ for $w \geq \widetilde{w}$, and finally approaches unity as $w \to \infty$ and non-diversifiable risk no longer matters.

Loose arguments may have led us to conclude that less constrained entrepreneurs (i.e. those with higher wealth) value their wealth less and $P_W(K, W)$ will globally decrease with wealth ($p''(w) < 0$). This is incorrect as we see from Panel C. The convexity of $p(w)$ (i.e. $p''(w) > 0$ for $w \leq \widetilde{w}$) arises because the liquidation option provides a valuable exit option for non-diversified entrepreneurs under incomplete markets.

**The marginal value of capital $P_K(K, W)$.** The lower right panel (Panel D) of Figure 1 plots the private marginal $q$, $P_K(K, W)$. Perhaps surprisingly, the private marginal $q$ is not monotonic in $w$. One seemingly natural but loose intuition is that the (private) marginal $q$ increases with financial slack measured by $w$. Presumably, less financially constrained entrepreneurs face lower costs of investment and hence have higher marginal $q$. However, this intuition in general does not hold. Using the analytical formula (47) for private marginal $q$, we obtain

$$\frac{dP_K(K, W)}{dw} = -wp''(w). \tag{48}$$

Therefore, the sign of $dP_K(K, W)/dw$ depends on both the sign of $w$ and the concavity of $p(w)$. When $w > 0$ and $p(w)$ is concave, $P_K(K, W)$ increases with $w$ (see the right end of Panel D). When the entrepreneur is in debt ($w < 0$) and additionally $p(w)$ is convex, $P_K(K, W)$ also increases with $w$ (see the left end of Panel D). In the intermediate region of $w$, $P(K, W)$ may decrease with $w$ (for example, when $w < 0$ and $p''(w) < 0$).

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The private marginal $q$, $P_K(K, W)$, is linked to the average $q$, $q(w)$, as follows,

$$P_K(K, W) = q(w) - w\left(p'(w) - 1\right). \tag{49}$$

The wedge between private marginal $q$ and private average $q$, $P_K(K, W) - q(w)$, can be either negative or positive depending on the sign of $w$ because $p'(w) \geq 1$. For entrepreneurs with positive wealth ($w > 0$), increasing $K$ mechanically lowers $w = W/K$, which further lowers the marginal product of capital and gives rise to a negative wedge $P_K(K, W) - q(w)$. However, for an entrepreneur in debt ($W < 0$), an increase $K$ leads to an increase in $w = W/K$ (by moving towards zero from the left of the origin) and hence implies a positive wedge $P_K(K, W) - q(w)$.

## 5.2  Optimal capital accumulation

Using $P(K, W)$, we rewrite the FOC (31) for investment as follows,

$$(1 + \theta i(w)) P_W(K, W) = P_K(K, W). \tag{50}$$

To install a unit of capital, the entrepreneur needs to incur cost $1 + \theta i(w)$ at the margin. The incremental cost $\theta i(w)$ is the marginal adjustment cost (e.g. over-time marginal labor costs and marginal installation costs) beyond the unit capital purchase cost. Moreover, the marginal cost of using a unit of wealth for the entrepreneur is $P_W(K, W) = p'(w)$. Therefore, the marginal cost of installing a unit of capital is given by $(1 + \theta i(w)) P_W(K, W)$, the left side of (50). The right side is the marginal value of capital $P_K$. The entrepreneur equates the two sides of (50) by optimally choosing investment. The FOC (50) states that the entrepreneur's optimal investment decision depends on the ratio between the private marginal $q$, $P_K(K, W)$, and the private marginal value of wealth $P_W(K, W)$. Both the private marginal $q$ and $P_W$ are endogenously determined. Moreover, they are highly correlated.

The left and right panels in Figure 2 plot the investment-capital ratio $i(w)$ and $i'(w)$, the sensitivity of $i(w)$ with respect to $w$. Non-diversifiable business risk induces underinvestment, $i(w) < i^{FB} = 0.15$. The underinvestment result (relative to the first-best MM benchmark) is common in incomplete-markets models.

More interestingly and less intuitively, investment-capital ratio is not monotonic in $w$. That is, investment may decrease with wealth! This seemingly counter-intuitive result di-
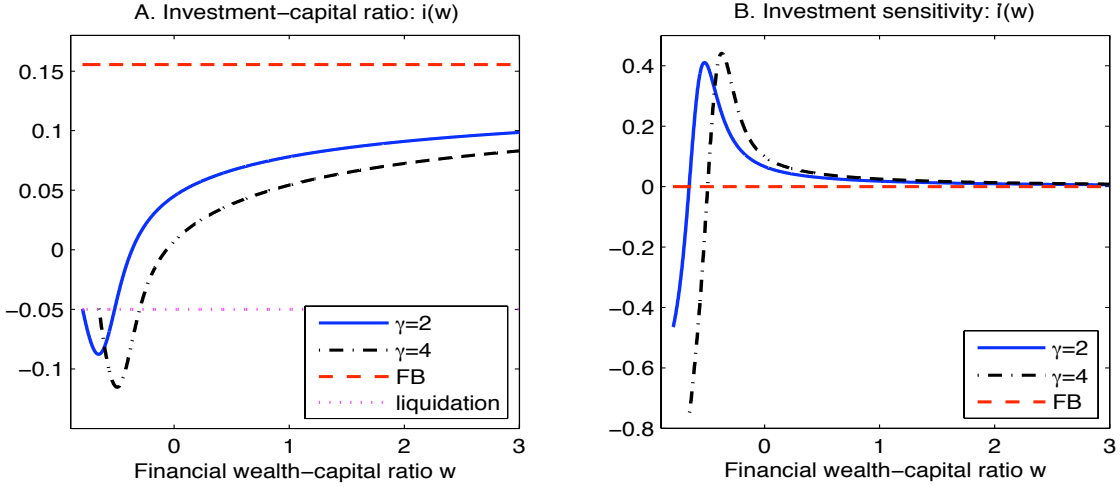
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Figure 2: **Investment-capital ratio $i(w)$ and investment-wealth sensitivity $i'(w)$.**

rectly follows from the convexity of $p(w)$ in $w$. We may characterize $i'(w)$ as follows,

$$i'(w) = -\frac{p(w)p''(w)}{\theta(p'(w))^2}. \tag{51}$$

Using the above result, we see that whenever $p(w)$ is concave, investment increases with wealth. However, whenever $p(w)$ is convex, investment decreases with $w$. Put differently, underinvestment is less of a concern when the entrepreneur is closer to liquidating the business because liquidation has also has the benefit of exiting incomplete markets. The entrepreneur has weaker incentives to cut investment if the distance to exiting incomplete markets is shorter. This explains why investment may decrease in $w$ when the exit option is sufficiently close to being in the money (i.e. when $w$ is sufficiently low).

The entrepreneur always has the option to exit the business by liquidating capital. The downside risk is thus capped by his exit option. After exiting, the entrepreneur is then only exposed to systematic shocks. This exit option induces a convexity effect of volatility on $p(w)$ for sufficiently low $w$. Note that the entrepreneur's value function $J(K, W)$ is concave in $P(K, W)$ as well as concave in $W$. However, volatility increases $p(w)$ in the region $w \le \widetilde{w}$, where the inflection point $\widetilde{w}$ is defined by $p''(\widetilde{w}) = 0$.

## 5.3  Optimal liquidation decision

Now we turn to the entrepreneur's liquidation decision. Because diversification benefits are more important for more risk-averse entrepreneurs, a more risk-averse entrepreneur liquidates capital earlier in order to avoid idiosyncratic risk exposure and achieve full diversification. For example, indeed, the optimal liquidation boundaries are $\underline{w} = -0.8$ and $\underline{w} = -0.65$ for $\gamma = 2$ and $\gamma = 4$, respectively. The (American) option to convert an illiquid risky business into liquid financial assets is more valuable for more risk-averse entrepreneurs. Note that the borrowing constraint does not bind even for a less risk-averse entrepreneur (e.g. $\gamma = 2$). The entrepreneur rationally liquidates capital before exhausting the debt capacity $w \geq -l = -0.9$ to ensure that wealth does not fall too low. While borrowing more to invest is desirable in terms of generating positive value for (diversified) investors, doing so may be too risky for non-diversified entrepreneurs. Moreover, anticipating that the liquidation option will soon be exercised, the entrepreneur has less incentive to distort investment when $w$ is close to the liquidation boundary. This option anticipation effect explains the non-monotonicity result for $i(w)$ in $w$. Next, we turn to the entrepreneur's portfolio choice decisions.

## 5.4  Optimal portfolio allocation

The entrepreneur's market portfolio allocation $x(w)$ has both a hedging demand term given by $-\rho \sigma_A / \sigma_R$ and a mean-variance demand term given by $\eta \sigma_R^{-1} p(w) / h(w)$. The constant hedging term $-\rho \sigma_A / \sigma_R$ is standard because of constant real and financial investment opportunities (Merton (1973)). We will focus on the more interesting mean-variance demand term.

Unlike the standard portfolio allocation, the entrepreneur incorporates the impact of non-diversifiable risk by $(i)$ replacing $w + q^{FB}$ with $p(w)$ in calculating "total" wealth and $(ii)$ adjusting risk aversion from $\gamma$ to the effective risk aversion $h(w)$ given in (35).

Figure 3 plots $h(w)$ for $\gamma = 2, 4$. In the limit as $w \to \infty$, idiosyncratic business risk has no role, markets are effectively complete for the entrepreneur, and $h(w)$ approaches the complete-markets value, $h(w) \to \gamma$. However, when the entrepreneur is close to liquidating the firm (i.e. $w \to \underline{w}$), the effective risk aversion $h(\underline{w})$ is lower than $\gamma$. This can be seen from $h(\underline{w}) = \gamma - (\underline{w} + l)p''(\underline{w}) < \gamma$. The intuition is as follows. When the liquidation option is in the money, the entrepreneur behaves in a *less* risk averse manner than without the
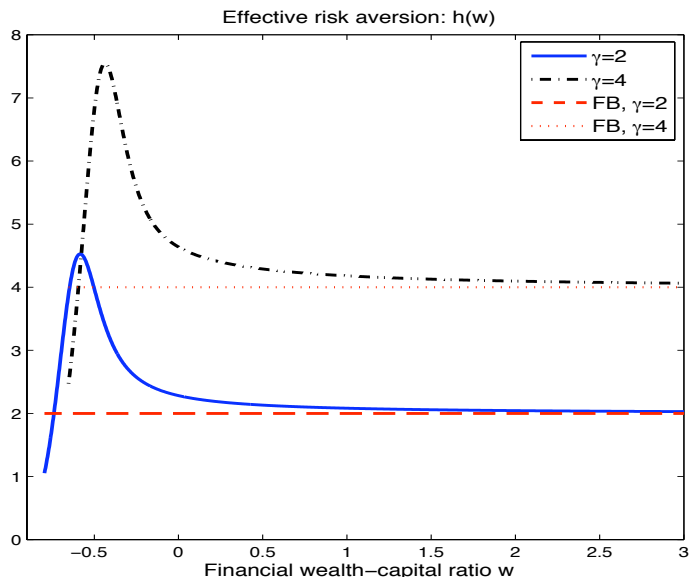
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Figure 3: **The entrepreneur's "effective" risk aversion** $h(w)$.

liquidation option under complete markets, because of the positive effect of volatility on the option value. This argument implies that at the moment of liquidation, the effective risk aversion $h(\underline{w}) < \gamma$. Figure 3 also shows that $h(w)$ is not monotonic in $w$. However, for most values of $w$ (other than near the liquidation boundary), the effective risk aversion $h(w)$ is higher than $\gamma$ due to non-diversiable risk, consistent with our intuition.

The left panel of Figure 4 plots the demand for the market portfolio $x(w)$ under incomplete markets and shows $x(w) < x^{FB}(w)$ for both $\gamma = 2$ and $\gamma = 4$. The straight lines are for the complete-markets benchmark. The demand for the market portfolio for the non-diversified entrepreneur is lower than that under the complete-markets benchmark. This is consistent with empirical findings. For example, Heaton and Lucas (2000) find that entrepreneurs with high and variable business income hold less wealth in stocks than other similarly wealthy households. Note that $x(w)$ is not monotonic in $w$ due to the optionality of liquidation. The right panel of Figure 4 shows that $x'(w)$ can be either positive or negative. On the left end, $x'(w) < 0$ implying that increasing $w$, while making the entrepreneur less financially constrained, lowers the demand for risky assets due to the reduction of optionality (further away from the liquidation boundary). Next, we turn to the entrepreneur's consumption decisions.
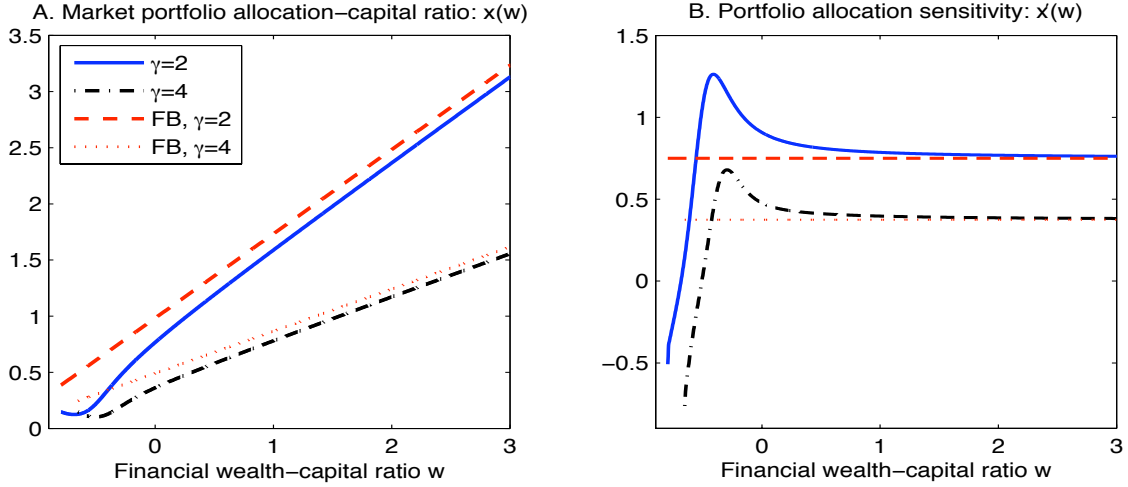
Figure 4: **Market portfolio allocation-capital ratio $x(w)$ and its sensitivity $x'(w)$.**

## 5.5 Optimal consumption and the MPCs

Recall that the consumption-capital ratio is given by $c(w) = m^{FB} p(w)(p'(w))^{-\psi}$. Consumption is thus lower under incomplete markets than under complete markets, $c(w) < c^{FB}(w)$, for two reasons: $p(w) < p^{FB}(w)$ and $p'(w) > 1$. The upper left panel of Figure 5 plots $c(w)$ for $\gamma = 2, 4$, and confirm that $c(w) < c^{FB}(w)$ for all $w$. For both $\gamma = 2$ and $\gamma = 4$, the EIS $\psi$ is set at 0.5. The MPC $m^{FB} = 0.057$ for $\gamma = 2$, which is higher than $m^{FB} = 0.052$ for $\gamma = 4$. The less risk-averse entrepreneur consumes more (if $\psi < 1$). The upper right panel of Figure 5 plots the MPC out of wealth, $C_W(K, W) = c'(w)$. Note that the MPC $C_W = c'(w)$ is not monotonic in $w$. The MPC $C_W(K, W)$ first increases with $w$ and then decreases with $w$.

Let $m(w)$ denote the ratio between $C(K, W)$ and $P(K, W)$. We have

$$m(w) = \frac{C(K, W)}{P(K, W)} = \frac{c(w)}{p(w)} = m^{FB}(p'(w))^{-\psi}. \tag{52}$$

Note that $m(w)$ is always lower than the MPC under complete markets, $m(w) \leq m^{FB}$ because $p'(w) \geq 1$. See the lower left panel (Panel C) of Figure 5. The entrepreneur has an additional motive to save, and consumption is more costly than under complete markets. Note that the sensitivity $m'(w)$ is not monotonic in $w$ (see Panel D of Figure 5). Indeed, $m'(w)$ has the same sign as that of $-p''(w)$. We know that $p(w)$ is not globally concave, and hence $m(w)$ is not monotonic in $w$.
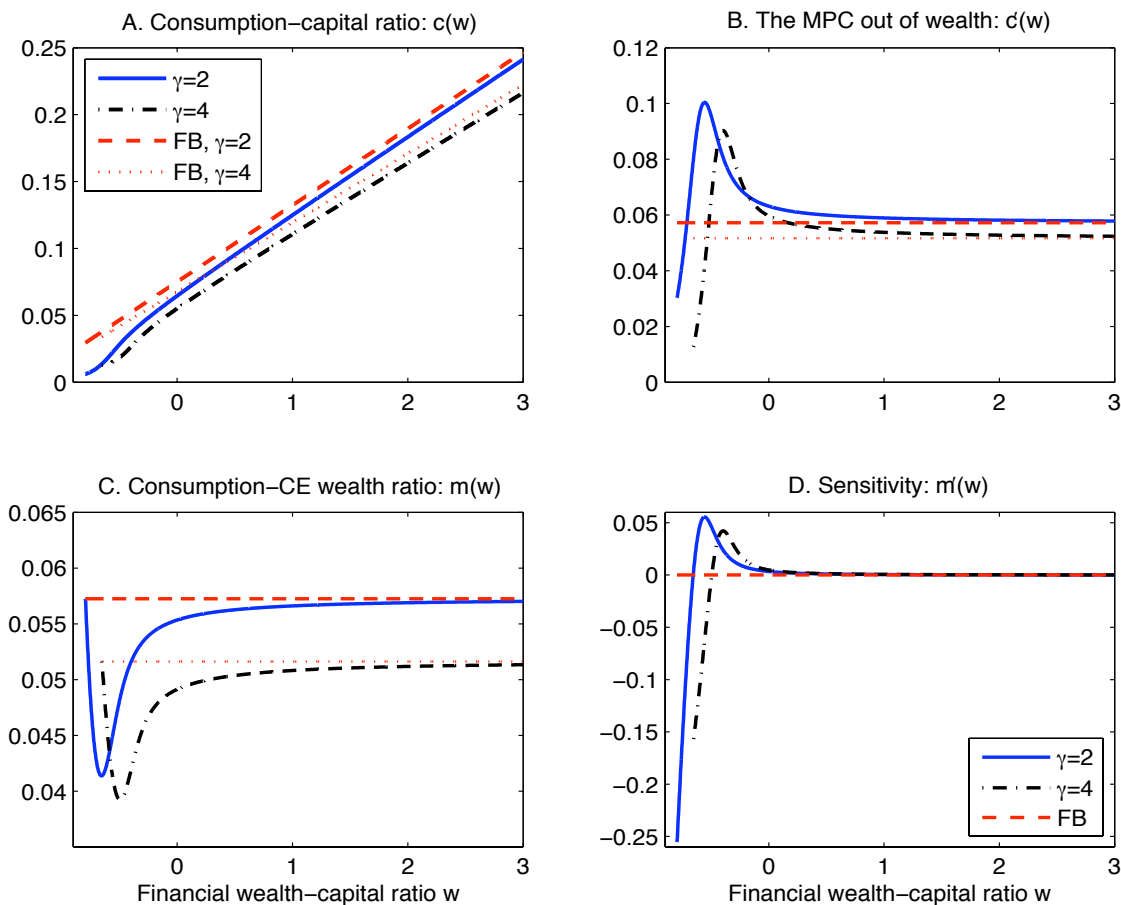
Figure 5: **Consumption-capital ratio** $c(w)$, **the MPC out of wealth** $c'(w)$, **consumption-certainty equivalent wealth ratio** $m(w) = c(w)/p(w)$, **and its sensitivity** $m'(w)$.

# 6 Entrepreneurial entry: Career choice and firm size

We have studied the agent's decision making and valuation after becoming an entrepreneur. However, what causes the agent to become an entrepreneur and when? The entrepreneurship choice is clearly an important decision. We analyze two cases: first, a time-0 binary career decision and then a richer model allowing for the choice of entry *timing*. For both cases, we will also study the determinants of the initial firm size.

## 6.1 When career choice is "now or never:" a binary decision

First consider the case where the agent has a static time-0 binary choice to be an entrepreneur or take the outside option. By taking the outside option, the agent collects a constant perpetuity with payment $r\Pi$, which has present value $\Pi$. The agent's optimal consumption and portfolio choice problem gives the value function $V(W_0 + \Pi)$ where $V(\cdot)$ is given in (26).

By being an entrepreneur, the agent incurs a fixed start-up cost $\Phi$ and then chooses the initial project size $K_0$. Wealth immediately drops from $W_0$ to $W_0 - (\Phi + K_0)$ at time 0. Note that the entrepreneur can borrow up to $lK$, the liquidation value of capital, which implies

$$W_0 \geq \Phi + (1 - l)K_0. \tag{53}$$

To rule out the uninteresting case where the entrepreneur makes instant profits by starting up the business and then immediately liquidating capital for profit, we require $l < 1$.

The value function is given by $J(K_0, W_0 - (\Phi + K_0))$, and the certainty equivalent wealth is $P(K_0, W_0 - (\Phi + K_0)) = p\,(w_0 - 1 - \Phi/K_0)\,K_0$. The agent chooses $K_0$ to maximize $J(K_0, W_0 - (\Phi + K_0))$, which is equivalent to maximizing $P(K_0, W_0 - (\Phi + K_0))$ by solving

$$\max_{K_0} \quad P(K_0, W_0 - (\Phi + K_0)), \tag{54}$$

subject to the borrowing constraint (53). Let $K_0^*$ denote the optimal initial capital stock. Finally, the agent compares $P(K_0^*, W_0 - (\Phi + K_0^*))$ from being an entrepreneur with $W_0 + \Pi$, and makes the career decision. The following theorem summarizes the main results.

**Theorem 2** *At time 0, an agent chooses to be an entrepreneur if and only if the initial wealth $W_0$ is greater than the threshold wealth level $\overline{W}_0$, which is given by*

$$\overline{W}_0 = \frac{\Phi p'(w^*) + \Pi}{p'(w^*) - 1}, \tag{55}$$

*and $w^*$ is the solution of the following equation*

$$p'(w^*) = \frac{p(w^*)}{1 + w^*}. \tag{56}$$

*The entrepreneurial firm's initial size $K_0^*$ is given by*
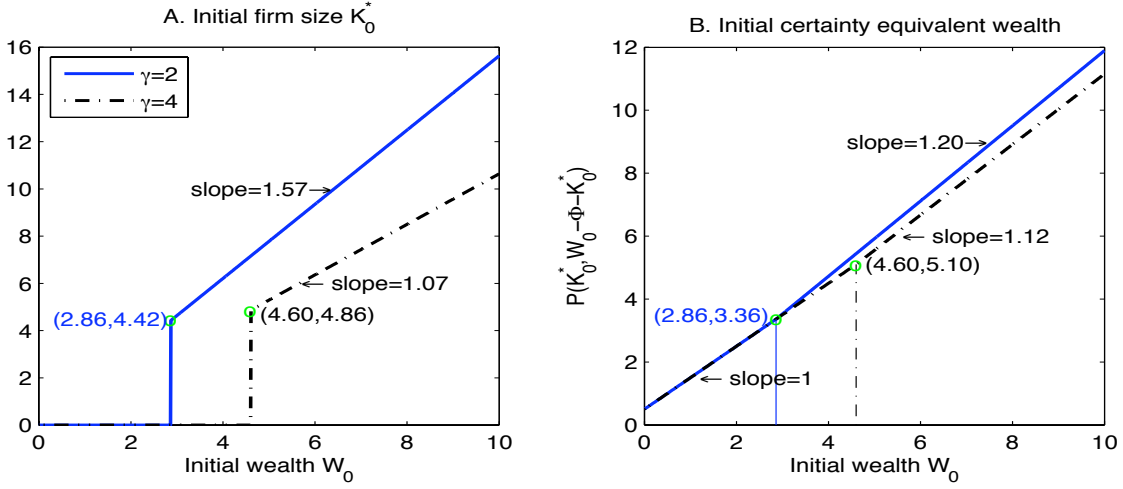
$$K_0^* = \frac{W_0 - \Phi}{1 + w^*}. \tag{57}$$

Figure 6: **Entrepreneurial entry in a time-0 (now or never) binary setting: Initial firm size $K_0^*$ and initial certainty equivalent wealth $P(K_0^*, W_0 - \Phi - K_0^*)$.** The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.

*The entrepreneur's certainty equivalent wealth is then given by*

$$P\left(K_0^*, W_0 - \Phi - K_0^*\right) = p(w^*)K_0^* = p'(w^*)\left(W_0 - \Phi\right), \tag{58}$$

*where $w^*$ is given by (56). After starting up the firm, the agent chooses consumption, portfolio allocation, and firm investment/liquidation decisions as described by Theorem 1.*

Figure 6 plots the firm's initial size $K_0^*$ and the initial certainty equivalent wealth $P(K_0^*, W_0 - \Phi - K_0^*)$ as functions of initial wealth $W_0$ for two levels of risk aversion, $\gamma = 2, 4$. First, risk aversion plays a significant role in determining entrepreneurship. The threshold for the initial wealth $\overline{W}_0$ to become an entrepreneur increases significantly from 2.86 to 4.60 when risk aversion $\gamma$ increases from 2 to 4. Second, entrepreneurs are wealth constrained and the initial wealth $W_0$ has a significant impact on firm size $K_0^*$. Note that $K_0^*$ is a linear function of $W_0$ due to the homogeneity property. Additionally, a unit increase in initial wealth $W_0$ leads to an increase in $K_0^*$ by more than unity. Moreover, this effect is greater for less risk-averse entrepreneurs. For example, the slope of the linear function for $K_0^*(W_0)$ is 1.57 for $\gamma = 2$, which is significantly higher than 1.07, the slope of $K_0^*(W_0)$ for $\gamma = 4$. Finally, the marginal effect of initial wealth $W_0$ is also higher for less risk-averse entrepreneurs if we measure in the certainty equivalent wealth $P(K_0^*, W_0 - \Phi - K_0^*)$. Note that $P(K_0^*, W_0 - \Phi - K_0^*)$

is also linear in $W_0$. The slope of the initial certainty equivalent wealth in $W_0$ is 1.2 for $\gamma = 2$ and 1.12 for $\gamma = 4$ as seen in the right panel of Figure 6. The effects of initial wealth $W_0$ on the entry decision of entrepreneurship are not the reflection of liquidity constraints *per se*, but rather the interaction between the liquidity constraints and non-diversifiable risk.

## 6.2 When career choice is flexible: Optimal entry timing

We now allow the agent to choose the optimal entry time rather than restricting the decision to be binary at time 0. We will show that this additional flexibility allows the agent to build up financial strength before becoming an entrepreneur, which is highly valuable. For simplicity, we assume becoming an entrepreneur is irreversible.

Let $F(W)$ denote the agent's value function before becoming an entrepreneur. Using an argument similar to our earlier analysis, we conjecture that $F(W)$ is given by

$$F(W) = \frac{(bE(W))^{1-\gamma}}{1-\gamma}, \tag{59}$$

where $b$ is the constant given by (17) and $E(W)$ is the agent's certainty equivalent wealth.

We will show that the entrepreneurship decision is characterized by an endogenous cutoff threshold $\widehat{W}$. When $W_t \geq \widehat{W}$, the agent immediately enters entrepreneurship. Otherwise, the agent takes the outside option, builds up financial wealth, and becomes an entrepreneur when wealth reaches $\widehat{W}$. We now summarize the results for career choice with flexible timing.

**Theorem 3** *Provided that $W \leq \widehat{W}$, the agent's certainty equivalent wealth $E(W)$ solves*

$$0 = \frac{m^{FB}E(W)(E'(W))^{1-\psi} - \psi\zeta E(W)}{\psi - 1} + r(W + \Pi)E'(W) + \frac{\eta^2}{2}\frac{E(W)E'(W)^2}{\gamma E'(W)^2 - E(W)E''(W)}, \tag{60}$$

*with the following boundary conditions*

$$E(\widehat{W}) = p'(w^*)(\widehat{W} - \Phi), \tag{61}$$

$$E'(\widehat{W}) = p'(w^*), \tag{62}$$

$$E(-\Pi) = 0, \tag{63}$$

*and $w^*$ is given in Theorem 2. The agent's consumption and portfolio rules are given by*

$$C(W) = m^{FB}E(W)E'(W)^{-\psi}, \tag{64}$$

$$X(W) = \frac{\mu_R - r}{\sigma_R^2}\frac{E(W)E'(W)}{\gamma E'(W)^2 - E(W)E''(W)}. \tag{65}$$
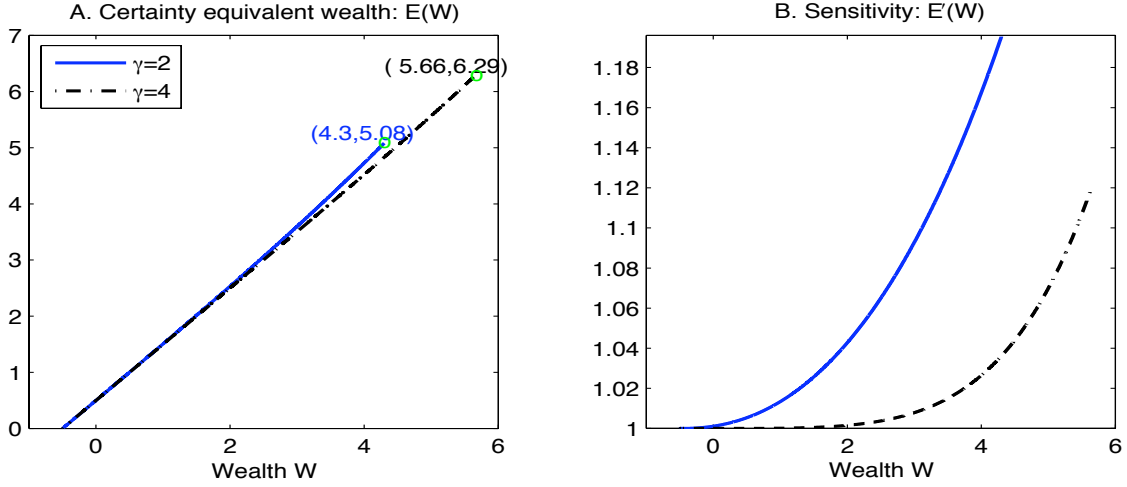
Figure 7: **Certainty equivalent wealth $E(W)$ and its sensitivity $E'(W)$ before entrepreneurial entry.** The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.

The value-matching condition (61) states that the agent's certainty equivalent wealth $E(W)$ is continuous at the endogenously determined cutoff level $\widehat{W}$. The smooth-pasting condition (62) gives the agent's optimal indifference condition between being an entrepreneur or not with wealth $\widehat{W}$. Finally, being indebted with amount $\Pi$ implies that the agent will never get out of the debt region and cannot pay back the fixed start-up cost. If this is the case, the certainty equivalent wealth is then zero as given by (63).

Figure 7 plots the agent's certainty equivalent wealth $E(W)$ before becoming an entrepreneur for two levels of risk aversion, $\gamma = 2, 4$. First, the less risk-averse agent is more entrepreneurial. For example, the threshold wealth $\widehat{W}$ is 5.66 for $\gamma = 4$, which is significantly higher than 4.3 for $\gamma = 2$. Unlike the standard real options problem, ours features incomplete markets. This means that the less risk-averse agent values the investment option more and hence exercises it earlier. Second, the less risk-averse agent values the future investment opportunity more by demanding a lower idiosyncratic risk premium as we will show in the next section. Finally, for all levels of $W$, $E'(W)$ is greater for less risk-averse agents.

Figure 8 quantifies the value of "entry timing flexibility" by comparing the time-0 binary entry with the flexible entry. For both $\gamma = 2$ and $\gamma = 4$, the convex curves in Figure 8 correspond to the the one with full timing flexibility (the "American" option), while the straight lines give the "now-or-never" time-0 binary version. First, it is immediate to see that
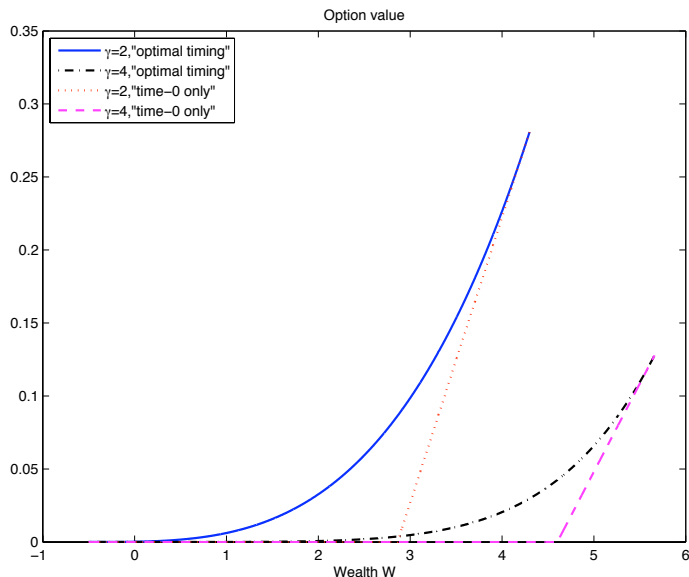
Figure 8: **The optimal entry decisions: Comparing the "optimal timing" and "time-0 binary decision" settings.** The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.

timing flexibility is valuable. Second, the value of timing flexibility is highest when the agent's wealth is in the intermediate range where building more financial strength substantially lowers the risk premium for the business project and hence enhances welfare. When the option value is sufficiently close to being in the money or deep out of the money, the wedge between the two versions of entrepreneurship entry is small. Finally, by allowing for flexible entry, the entrepreneur's cutoff level of wealth significantly increases. For example, for $\gamma = 2$, the cutoff wealth increases from 2.86 in a time-0 binary setting (see Figure 6) to 4.3. To summarize, flexibility has significant value in entrepreneurship.

Figure 9 plots the optimal consumption $C(W)$ and the MPC out of wealth $C'(W)$ for four cases: worker with $\gamma = 2$ or $\gamma = 4$; and entrepreneur-to-be with $\gamma = 2$ or $\gamma = 4$. For workers receiving constant wage income at the rate of $r\Pi$, markets are effectively complete and the MPC out of wealth is constant: $m^{FB} = 0.0573$ for $\gamma = 2$ and $m^{FB} = 0.0516$ for $\gamma = 4$, respectively. Recall that the more risk-averse agent consumes less out of wealth when the EIS $\psi < 1$. More interestingly, the entrepreneur's MPC out of wealth $C'(W)$ is lower than $m^{FB}$. Intuitively, frictions such as non-diversifiable risk and borrowing constraints make the agent consume less on the margin in order to build up wealth to become an entrepreneur
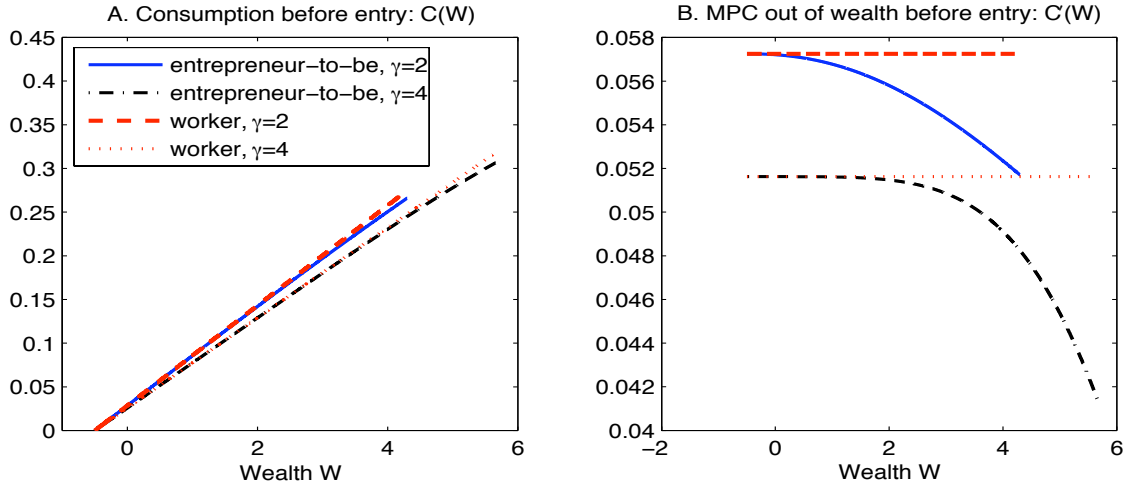
Figure 9: **Optimal consumption and the MPC before entry.** The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.
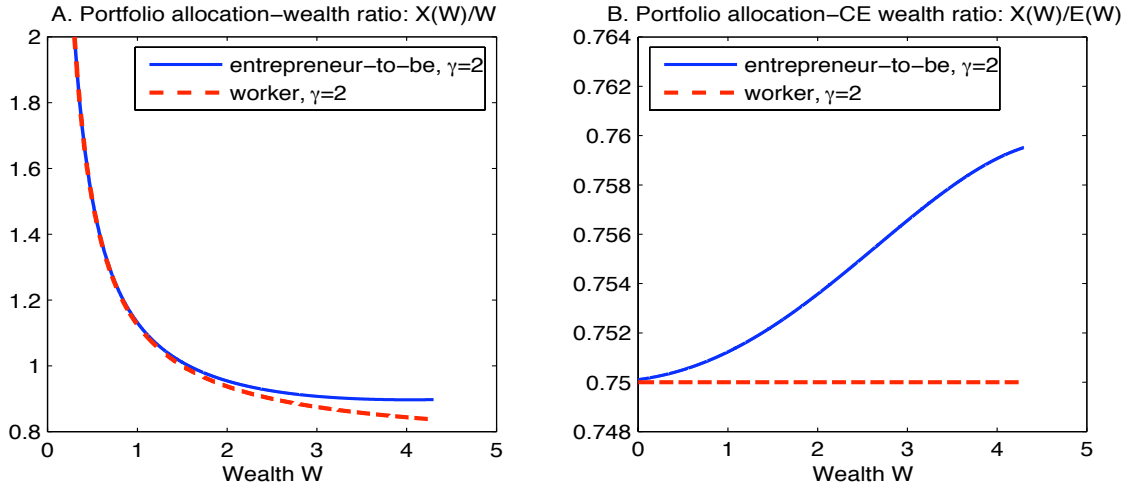


Figure 10: **Optimal market portfolio allocation before entry.** The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.

sooner. This underconsumption effect is greater the closer the agent's wealth $W$ is to the endogenous entry threshold $\widehat{W}$, i.e. when the agent's entry option is deeper in the money.

Figure 10 plots the optimal allocation to the market portfolio for a worker and for an entrepreneur-to-be with $\gamma = 2$. The left panel plots $X(W)/W$, the fraction of liquid financial wealth $W$ allocated to the risky market portfolio. The entrepreneur-to-be invests more in

the risky market portfolio, i.e. $X(W)/W$ is higher for entrepreneurs-to-be than for workers. The right panel plots the portfolio allocation as a fraction of the agent's certainty equivalent wealth $E(W)$, $X(W)/E(W)$. For workers, the certainty equivalent wealth is equal to the sum of $W$ and $\Pi$, i.e. $E(W) = W + \Pi$. For entrepreneurs-to-be, $E(W)$ is convex and the option value makes $E(W) > W + \Pi$ for $W < \widehat{W}$. The right panel shows that the entrepreneur-to-be allocates more to the market portfolio even after controlling for a higher level of $E(W)$ for workers than for entrepreneurs-to-be. However, quantitatively, the effects of the entry option on portfolio allocation are not significant. We have similar results for the comparison between the worker and the entrepreneur-to-be with $\gamma = 4$.

We next study the risk premium implications for entrepreneurial firms.

# 7 Idiosyncratic risk premium

A fundamental issue in entrepreneurial finance is to determine the cost of capital for private firms owned by non-diversified entrepreneurs. Intuitively, the entrepreneur demands both the systematic risk premium and an additional idiosyncratic risk premium for non-diversifiable risk. Compared to an otherwise identical public firm held by diversified investors, the cost of capital should be higher for the entrepreneurial firm. Using our model, we provide a procedure to calculate the cost of capital for the entrepreneurial firm.

Let $\xi(w_0)$ denote the constant yield (internal rate of return) for the entrepreneurial firm until liquidation. We have made explicit the functional dependence of $\xi$ on the initial wealth-capital ratio $w_0 = W_0/K_0$. By definition, $\xi(w_0)$ solves the following valuation equation

$$Q(K_0, W_0) = \mathbb{E}\left[\int_0^\tau e^{-\xi(w_0)t}dY_t + e^{-\xi(w_0)\tau}lK_\tau\right], \tag{66}$$

where $\tau$ is the stochastic liquidation time. The right side of (66) is the present discounted value (PDV) of the firm's operating cash flow plus the PDV of the liquidation value using the same discount rate $\xi(w_0)$. The left side is the "private" enterprise value $Q(K_0, W_0)$ that we have obtained earlier using the entrepreneur's optimality.

Recall that the firm's discount rate under complete markets, $\xi^{FB}$, is given in (23). We measure the idiosyncratic risk premium as the wedge between $\xi(w_0)$ and $\xi^{FB}$

$$\alpha(w_0) \;\; = \;\; \xi(w_0) - \xi^{FB} = \xi(w_0) - r - \beta^{FB}(\mu_R - r)\,. \tag{67}$$
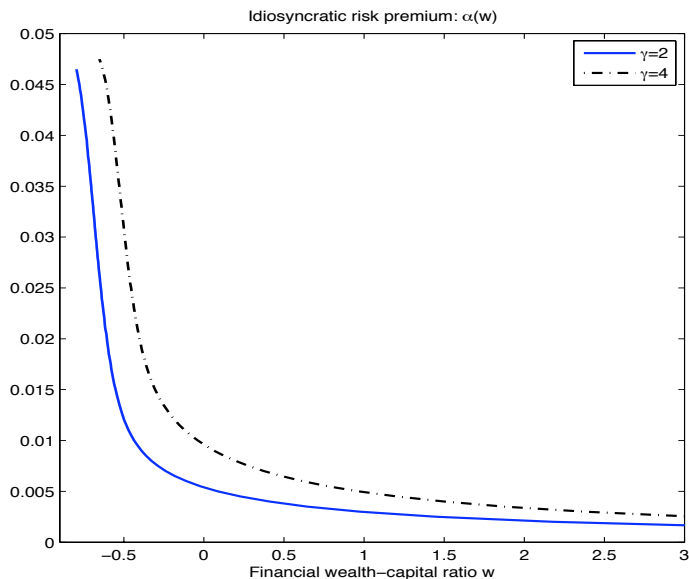
Figure 11: **The idiosyncratic risk premium, $\alpha(w)$, for two levels of risk aversion $\gamma = 2, 4$.**

There is much debate in the empirical literature about the significance of this *private* equity risk premium. For example, Moskowitz and Vissing-Jorgensen (2002) document that the risk-adjusted returns to investing in a U.S. non-publicly traded equity are not higher than the returns to private equity. Our model provides an analytical formula to calculate this private equity idiosyncratic risk premium.

Figure 11 plots the idiosyncratic risk premium for two levels of risk aversion, $\gamma = 2, 4$. For sufficiently high levels of wealth-capital ratio $w_0$, the idiosyncratic risk premium $\alpha(w_0)$ eventually disappears. Intuitively, this premium $\alpha(w_0)$ is higher for more risk-averse agents. Quantitatively, for entrepreneurs with positive wealth, we do not find a significant idiosyncratic risk premium. For both $\gamma = 2$ and $\gamma = 4$, the annual idiosyncratic risk premia are less than 1%. However, for entrepreneurs in debt, this premium $\alpha(w_0)$ is significant because the business carries significantly more weight in the entrepreneur's portfolio, and non-diversifiable risk becomes much more important as Figure 11 shows.

# 8 Comparative analysis

We now analyze the effects of structural parameters including the EIS $\psi$, idiosyncratic volatility $\epsilon$, the adjustment cost parameter $\theta$, and the liquidation parameter $l$ on the entrepreneur's decision making and business valuation. For all the figures, we fix $\gamma = 2$ and use the other parameter values given in the baseline model (Section 5). In the preceding analysis, we have shown that risk aversion has substantial effects. Next, we analyze the impact of EIS $\psi$.

**The EIS $\psi$.** In asset pricing, a high EIS is often used in the long-run risk literature (Bansal and Yaron (2004)). However, there is much disagreement about the empirical estimates of the EIS. Recall that our previous calculations are based on $\psi = 0.5$ We now consider two commonly used but significantly different values for the EIS, $\psi = 0.25$ and $\psi = 2$. Figure 12 shows that the effect of the EIS $\psi$ on consumption is quantitatively significant, while its effects on $q(w)$, $i(w)$, and $x(w)$ are much less significant. The significant effects on consumption are similar to the intuition under complete markets. For example, the MPC $m^{FB}$ is only 0.014 when $\psi = 2$, which is substantially lower than the MPC $m^{FB} = 0.072$ when EIS is $\psi = 0.25$. Intuitively, an entrepreneur with a high EIS ($\psi = 2$) is willing to decrease consumption to build up wealth.

<div align="center">Insert Figure 12 here.</div>

**Idiosyncratic volatility $\epsilon$.** In Figure 13, we plot for two cases, $\epsilon = 0.1$ and $\epsilon = 0.2$. We find that the idiosyncratic volatility $\epsilon$ has significant effects on investment $i(w)$. The entrepreneur invests significantly less in the firm (lower $i(w)$) and liquidates capital earlier when $\epsilon = 0.2$ than when $\epsilon = 0.1$. In addition, the entrepreneur invests less wealth in the market portfolio (lower $x(w)$), and consumes less (i.e. lower $c(w)$) due to more "background" risk (higher $\epsilon$). Firm value $q(w)$ depends strongly on idiosyncratic volatility $\epsilon$. Finally, the effect on the idiosyncratic risk premium $\alpha(w)$ is significant. For example, when doubling the idiosyncratic volatility from 10% to 20%, the annual idiosyncratic risk premium for an entrepreneur with no liquid wealth ($w = 0$) increases from 0.5% to 2.3%.

<div align="center">Insert Figure 13 here.</div>

**Adjustment cost parameter $\theta$.** We next demonstrate the effects of the adjustment cost parameter $\theta$. Whited (1992) estimates this parameter to be around $\theta = 2$.[17] Eberly, Rebelo and Vincent (2009) use an extended Hayashi (1982) model and provide a larger empirical estimate of this parameter value (close to seven) for large Compustat firms. Based on these studies, we consider the values $\theta = 2$ and $\theta = 8$ in Figure 14. Our main results are as follows. From Figure 14, we see that with a low $\theta$, investment $i(w)$ is highly dependent on $w$ reflecting the significant effects of non-diversifiable risk on $i(w)$, while with a high $\theta$, investment responds much less to $w$. Investment smoothing is stronger the more convex the adjustment costs. Consumption $c(w)$ and portfolio allocation $x(w)$ effectively do not depend on $\theta$. The effect of $\theta$ on the idiosyncratic risk premium also appears to be weak.

<div align="center">Insert Figure 14 here.</div>

**Liquidation parameter $l$.** In Figure 15, we plot for two values of the liquidation parameter, $l = 0.6$ and $l = 0.9$. We show that liquidation value has a quantitatively significant impact, particularly when the entrepreneur is in debt (the left sides of each panel). Increasing $l$ increases the downside protection for the entrepreneur and also allows the entrepreneur to borrow more (higher debt capacity). The entrepreneur naturally operates the business longer. In addition, while running the business, the entrepreneur invests more, consumes more, and allocates more to the market portfolio with an increase in $l$. A higher value of $l$ also lowers the idiosyncratic risk premium $\alpha$. However, when the liquidation option is sufficiently out of the money (high $w$), liquidation has almost no effect on entrepreneurial decision making and valuation.

<div align="center">Insert Figure 15 here.</div>

# 9  Conclusion

Non-diversifiable risk and liquidity constraints are important frictions in the real world. This paper provides an incomplete-markets framework with these two frictions to analyze the entrepreneur's interdependent business entry, capital accumulation/growth, portfolio

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[17]Hall (2004) argues that the parameter $\theta$ is small using U.S. aggregate data.

choice, consumption, and business exit decisions. Even when business is uncorrelated with the stock market, the entrepreneur rationally chooses to reduce business investment, lowers consumption, and scales back portfolio investment in the stock market. Non-diversifiable risk and liquidity constraints also significantly influence the private business valuation. We provide an operational procedure to compute the private equity idiosyncratic risk premium and the cost of capital, which can be used to address the empirical findings of Moskowitz and Vissing-Jorgensen (2002).

While being exposed to significant risk, the entrepreneur nonetheless has various options to manage risk. For example, the liquidation option substantially enhances the entrepreneur's ability to manage downside risk. We show that the option value of building up financial strength before entering entrepreneurship is high. We show that wealth effects are significant for entrepreneurial entry in a dynamic setting with flexible entry timing. While entrepreneurs have important entry and exit options, these options are fundamentally different from the standard (real or financial) options analyzed in finance because the entrepreneurs' entry and exit options are illiquid and not tradable. Additionally, the option exercising decisions interact with the agent's consumption, saving, portfolio allocation and capital accumulation decisions in a fundamental way when frictions (incomplete markets and liquidity constraints) are important. We provide a unified, analytically tractable dynamic framework to analyze the firm's various decision margins in a life-cycle model of entrepreneurship.

Finally, our model is a single agent's intertemporal decision problem. To study the impact of entrepreneurship on wealth distribution and economic growth, we need to construct a general equilibrium incomplete-markets model.[18] Our model may provide one natural starting point for the general equilibrium analysis.

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[18]See Aiyagari (1994) and Huggett (1993) for foundational incomplete-markets equilibrium (Bewley) models and Cagetti and De Nardi (2006) for an application with entrepreneurship.

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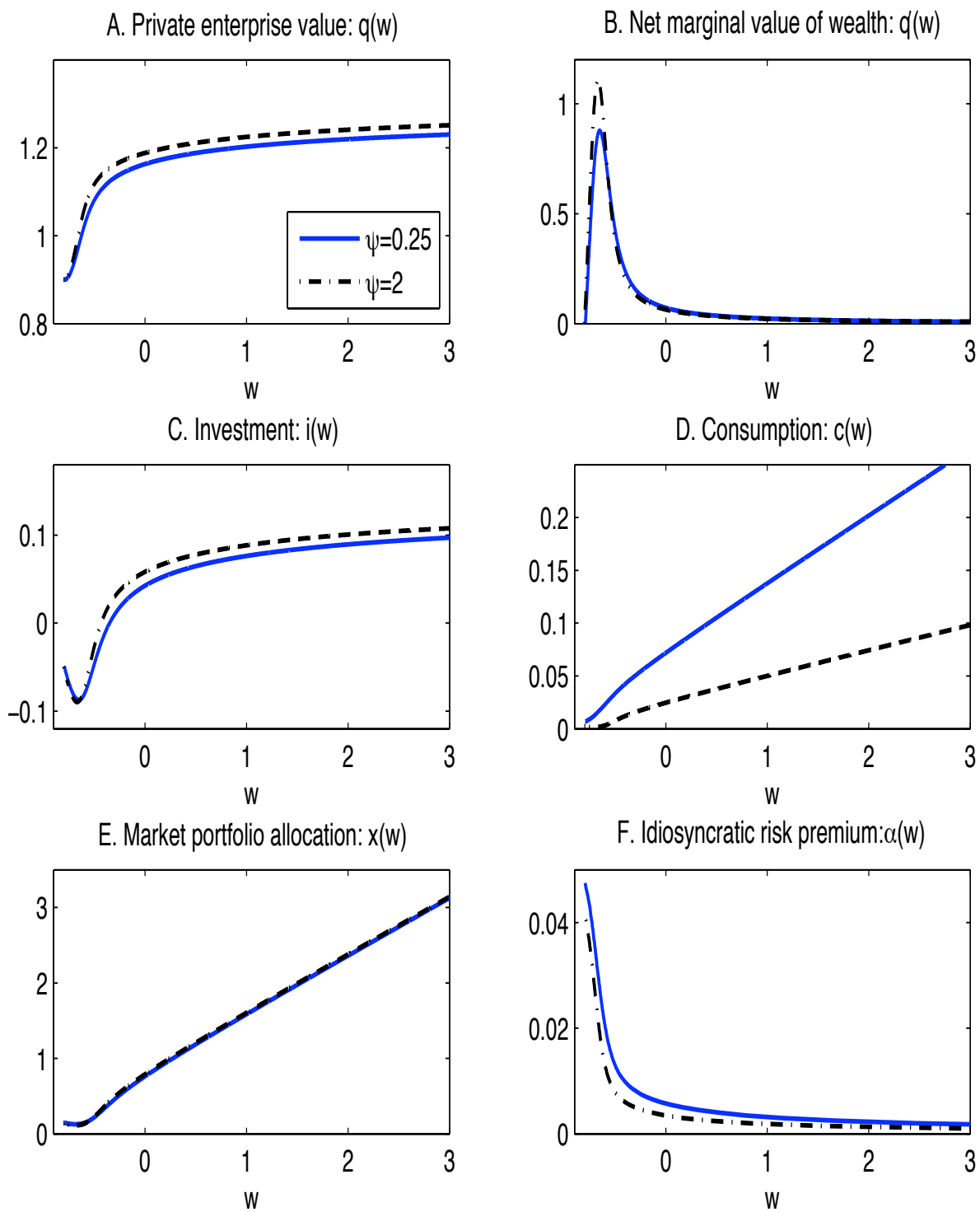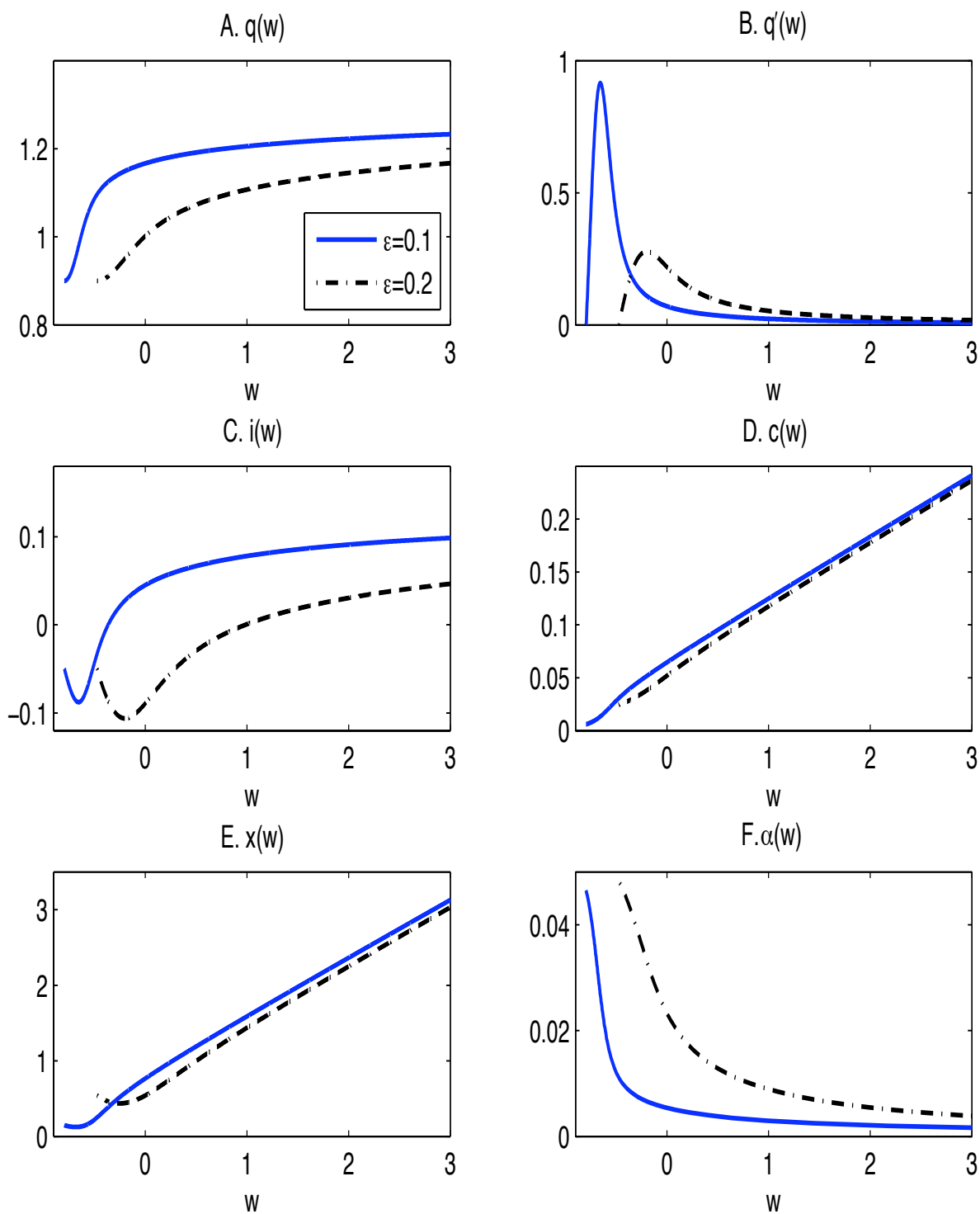
Figure 12: **The effects of the EIS** $\psi$.

Figure 13: **The effects of idiosyncratic volatility $\epsilon$. The correlation coefficient: $\rho = 0$. Other parameter values are the same as in the baseline model (Section 5).**
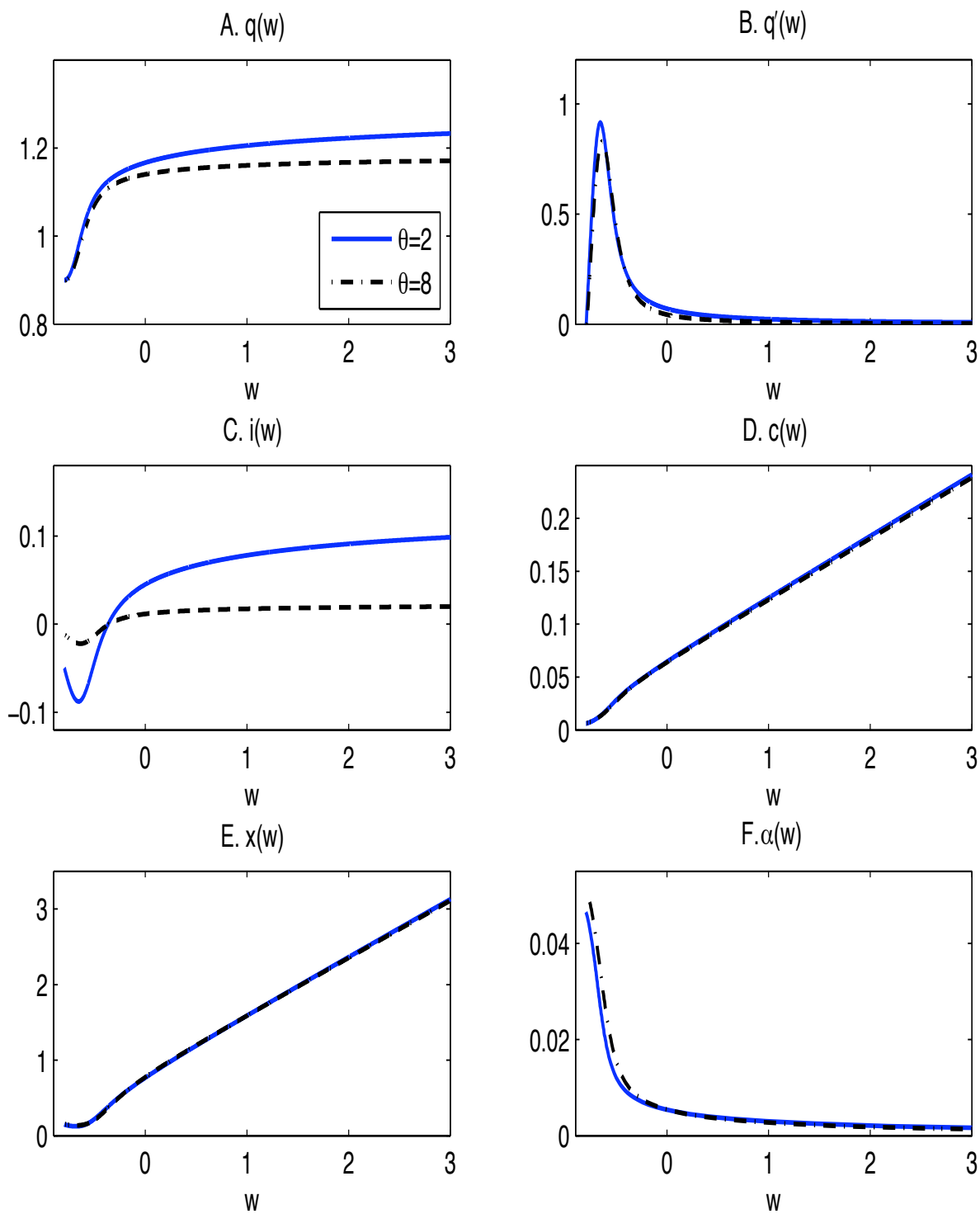
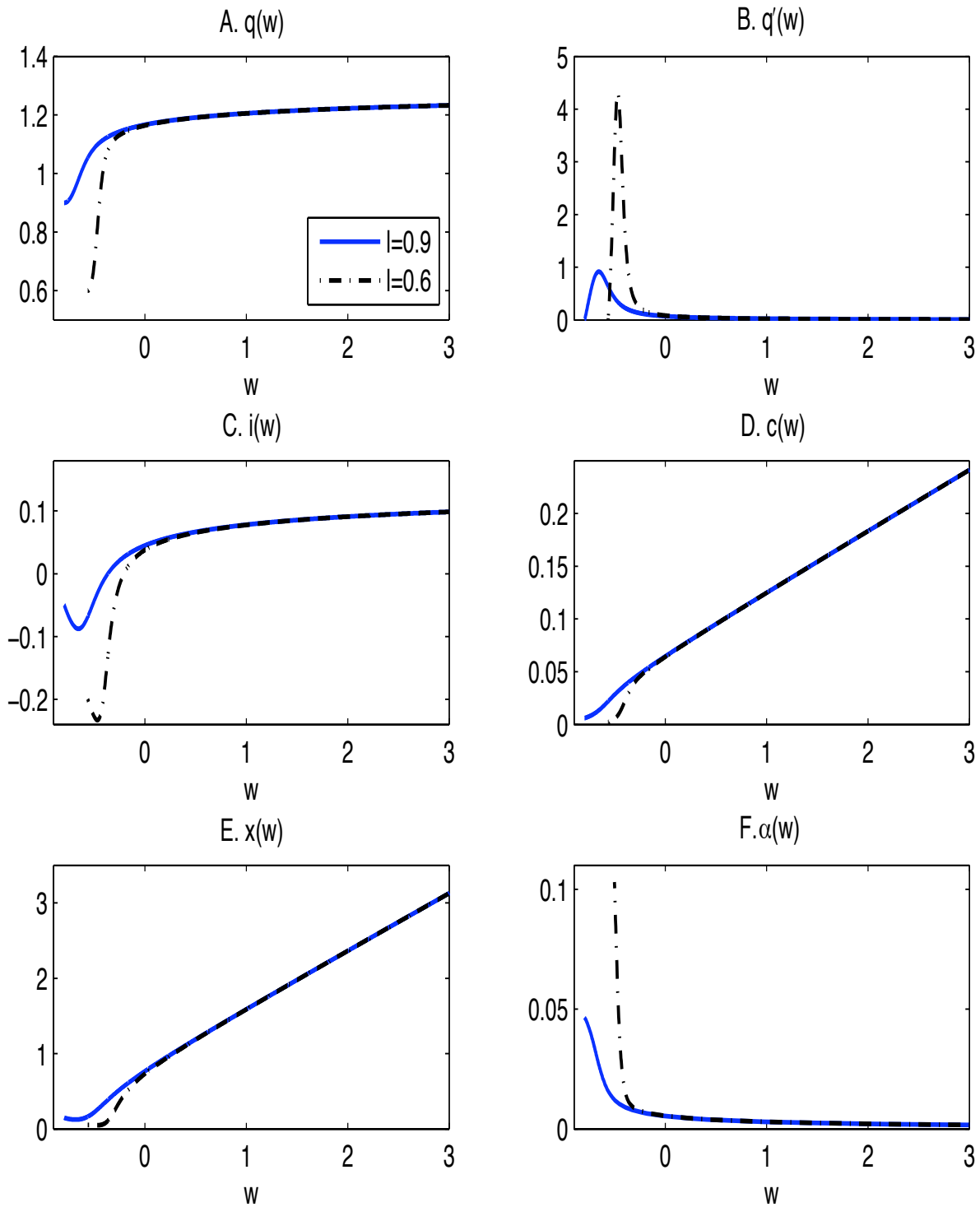Figure 14: **The effects of the adjustment cost parameter $\theta$.**

Figure 15: **The effects of the liquidation parameter** $l$.

Table 1: Summary of Key Variables and Parameters

This table summarizes the symbols for the key variables used in the model and the parameter values for the baseline calculation of Section 5. For each upper-case variable in the left column (except $K$, $A$, $J$, $F$, $V$, $E$, $\overline{W}$, $\widehat{W}$ and $K^*$), we use its lower case to denote the ratio of this variable to capital.

| Variable | Symbol | Parameters | Symbol | Value |
|---|---|---|---|---|
| Capital stock | $K$ | Riskfree rate | $r$ | 4.6% |
| Cumulative Productivity Shock | $A$ | Expected return of market portfolio | $\mu_R$ | 10.6% |
| Investment Adjustment Cost | $G$ | Volatility of market portfolio | $\sigma_R$ | 20% |
| Cumulative Operating Profit | $Y$ | Aggregate equity risk premium | $\mu_R - r$ | 6% |
| Financial wealth | $W$ | Market Sharpe ratio | $\eta$ | 30% |
| Value function after entry | $J$ | Subjective discount rate | $\zeta$ | 4.6% |
| Value function before entry | $F$ | Adjustment cost parameter | $\theta$ | 2 |
| Value function after exiting | $V$ | Depreciation rate | $\delta$ | 12.5% |
| Certainty equivalent wealth after entry | $P$ | Mean productivity shock | $\mu_A$ | 20% |
| Certainty equivalent wealth before entry | $E$ | Volatility of productivity shock | $\sigma_A$ | 10% |
| Private enterprise value | $Q$ | Correlation between market and firm | $\rho$ | 0 |
| Effective risk aversion | $h$ | Idiosyncratic volatility | $\epsilon$ | 10% |
| Market portfolio allocation | $X$ | Relative Risk Aversion | $\gamma$ | 2, 4 |
| Consumption | $C$ | Elaticity of Intertemporal Substitution | $\psi$ | 0.5 |
| Business investment | $I$ | Capital liquidation price | $l$ | 0.9 |
| Liquidation Boundary | $\underline{W}$ | Outside option value | $\Pi$ | 0.5 |
| Static entry threshold | $\overline{W}$ | Fixed start-up cost | $\Phi$ | 0.05 |
| Flexible entry threshold | $\widehat{W}$ | | | |
| Optimal initial capital stock | $K^*$ | | | |
| MPC out of wealth | $m$ | | | |
| Internal rate of return | $\xi$ | | | |
| Idiosyncratic risk premium | $\alpha$ | | | |

# Appendices

## A  Details for Theorem 1 and Proposition 1

We conjecture that the value function is given by (33). We then have

$$J_K(K, W) = b^{1-\gamma}(p(w)K)^{-\gamma}(p(w) - wp'(w)), \tag{A.1}$$

$$J_W(K, W) = b^{1-\gamma}(p(w)K)^{-\gamma}p'(w), \tag{A.2}$$

$$J_{WW}(K, W) = b^{1-\gamma}\left(\frac{(p(w)K)^{-\gamma}p''(w)}{K} - \gamma(p(w)K)^{-\gamma-1}(p'(w))^2\right). \tag{A.3}$$

The first-order conditions (FOCs) for $C$ and $X$ are

$$f_C(C, J) = J_W(K, W), \tag{A.4}$$

$$X = -\frac{\rho\sigma_A}{\sigma_R}K + \frac{(r - \mu_R)J_W(K, W)}{\sigma_R^2 J_{WW}(K, W)}. \tag{A.5}$$

Using the homogeneity property of $J(K, W)$, we obtain the follwing for $c(w)$ and $x(w)$:

$$c(w) = b^{1-\psi}\zeta^{\psi}p(w)(p'(w))^{-\psi}, \tag{A.6}$$

$$x(w) = -\frac{\rho\sigma_A}{\sigma_R} + \frac{\mu_R - r}{\sigma_R^2 h(w)}p(w). \tag{A.7}$$

Substituting $c(w)$ into (2), we have

$$f(C, J) = \frac{\zeta}{1 - \psi^{-1}}\left(\frac{(bp(w)K)^{1-\gamma}(bp'(w))^{1-\psi}}{\zeta^{1-\psi}} - (bp(w)K)^{1-\gamma}\right). \tag{A.8}$$

Substituting (A.1), (A.2), (A.3), (A.7) and (A.8) into (29) and simplifying, we obtain

$$0 = \max_i \left(\frac{\zeta^{\psi}(bp'(w))^{1-\psi}}{\psi - 1} - \frac{\psi\zeta}{\psi - 1}\right)p(w) + (i - \delta)(p(w) - wp'(w))$$

$$+ (rw + \mu_A - \rho\eta\sigma_A - i - g(i))\,p'(w) + \frac{\eta^2 p(w)p'(w)}{2h(w)} - \frac{\epsilon^2 h(w)p'(w)}{2p(w)}, \tag{A.9}$$

where $h(w)$ is given in (35). Using the FOCs for investment-capital ratio $i$, we obtain (40). Substituting it into (A.9), we obtain the ODE (34).

Using Ito's formula, we obtain the following dynamics for the entrepreneur's wealth-captial ratio $w$,

$$dw_t = d\left(\frac{W_t}{K_t}\right) = \frac{dW_t}{K_t} - \frac{W_t}{K_t^2}dK_t = \mu_w(w_t)dt + \sigma_R x(w_t)dB_t + \sigma_A dZ_t, \tag{A.10}$$

where $\mu_w(w)$ is given by (43).

Now consider the lower liquidation boundary $\underline{W}$. When $W \le \underline{W}$, the entrepreneur liquidates the firm. Using the value-match condition at $\underline{W}$, we have

$$J(K, \underline{W}) = V(\underline{W} + lK),$$ (A.11)

where $V(W)$ given by (26) is the agent's value function after retirement and with no business. The entrepreneur's optimal liquidation strategy implies the following smooth-pasting condition at the endogenously determined liquidation boundary $\underline{W}$

$$J_W(K, \underline{W}) = V_W(\underline{W} + lK).$$ (A.12)

Using $\underline{W} = \underline{w}K$, (A.11), and (A.12), and simplifying, we obtain the scaled value-matching and smooth pasting conditions given in (37) and (38), respectively.

**Complete-markets benchmark solution.** When $w$ approaches infinity, markets are effectively complete. Non-diversifiable risk no longer matters for investment and consumption. Therefore, firm value approaches the complete-markets value and $\lim_{w\to\infty} J(K, W) = V(W + q^{FB}K)$, which implies (36). The certainty equivalent wealth $P(K, W)$ is equal to the sum of the financial wealth $W$ and complete-markets firm value $q^{FB}K$, in that

$$P^{FB}(K, W) = W + q^{FB}K.$$ (A.13)

Equivalently, we have $p^{FB}(w) = w + q^{FB}$. Substituting this linear relation into (34), taking the limit $w \to \infty$, we obtain the following equation:

$$0 = \left( \frac{\zeta^\psi b^{1-\psi} - \psi\zeta}{\psi - 1} + \frac{\eta^2}{2\gamma} \right)(w + q^{FB}) + (i^{FB} - \delta)q^{FB} + rw + \mu_A - \rho\eta\sigma_A - i^{FB} - g(i^{FB}).$$ (A.14)

In order for the above to hold, we require that both the linear term coefficient and the constant term are equal to zero. This gives rise to

$$b = \zeta \left[ 1 + \frac{1-\psi}{\zeta} \left( r - \zeta + \frac{\eta^2}{2\gamma} \right) \right]^{\frac{1}{1-\psi}},$$ (A.15)

$$m^{FB} = b^{1-\psi}\zeta^\psi = \zeta + (1-\psi)\left( r - \zeta + \frac{\eta^2}{2\gamma} \right).$$ (A.16)

We may now write (A.14) as follows

$$rq^{FB} = \mu_A - \rho\eta\sigma_A - i^{FB} - g(i^{FB}) + (i^{FB} - \delta)q^{FB}. \tag{A.17}$$

Using the FOC for investment, i.e. $i^{FB} = \left(q^{FB} - 1\right)/\theta$, we obtain (19). Next, we calculate the rate of return for the firm. We have

$$\begin{aligned} dR_t^{FB} = \frac{dY_t + dQ_t^{FB}}{Q_t^{FB}} &= \frac{\mu_A dt + \sigma_A dZ_t - i^{FB}K_t dt - g(i^{FB})K_t dt}{Q_t^{FB}} + \frac{q^{FB}dK_t}{Q_t^{FB}}, \\ &= \left(r + \frac{\rho\eta\sigma_A}{q^{FB}}\right)dt + \frac{\sigma_A}{q^{FB}}dZ_t. \end{aligned} \tag{A.18}$$

Therefore, the expected return $\mu^{FB}$ is given by

$$\mu_r^{FB} = r + \beta^{FB}\left(\mu_R - r\right), \tag{A.19}$$

where $\beta^{FB}$ is given by

$$\beta^{FB} = \frac{\rho\sigma_A}{\sigma_R}\frac{1}{q^{FB}}. \tag{A.20}$$

# B    Details for Theorem 2 and Theorem 3

**Theorem 2.**  The entrepreneur chooses initial firm size $K_0^*$ to maximize utility, which gives rise to the following FOC

$$P_K(K_0^*, W_0 - \Phi - K_0^*) = P_W(K_0^*, W_0 - \Phi - K_0^*). \tag{B.1}$$

The above condition (B.1) states that the marginal value of capital $P_K(K_0^*, W_0 - \Phi - K_0^*)$ is equal to the marginal value of wealth $P_W(K_0^*, W_0 - \Phi - K_0^*)$ at the optimally chosen $K_0^*$. Simplifying (B.1) gives (56), which characterizes the optimal initial wealth-capital ratio $w_0 \equiv (W_0 - \Phi)/K_0^* - 1 = w^*$. Note that (56) implies that $w^*$ is independent of the fixed start-up cost $\Phi$ and outside option value $\Pi$.

Second, using the Euler's theorem, we write $P(K_0^*, W_0 - \Phi - K_0^*)$ as follows

$$P(K_0^*, W_0 - \Phi - K_0^*) = P_K^* \times K_0^* + P_W^* \times (W_0 - \Phi - K_0^*) = p'(w^*)(W_0 - \Phi), \tag{B.2}$$

where the second equality follows from (B.1). The entrepreneur's certainty equivalent wealth $P(K_0^*, W_0 - \Phi - K_0^*)$ is given by $p'(w^*)$ multiplied by $(W_0 - \Phi)$, the initial wealth after paying the fixed start-up cost $\Phi$. If $J(K_0^*, W_0 - (\Phi + K_0^*)) > V(W_0 + \Pi)$, the agent chooses to become an entrepreneur immediately. Otherwise, the agent will take the outside option. Therefore, the threshold level $\overline{W}$ satisfies $J(K_0^*, \overline{W} - (\Phi + K_0^*)) = V(\overline{W} + \Pi)$, which gives (55).

**Theorem 3.** Let $F(W)$ and $E(W)$ denote the agent's value function and certainty equivalent wealth before entering entrepreneurship. Using the standard principle of optimality for recursive utility (Duffie and Epstein (1992b)), the following HJB equation holds

$$0 = \max_{C,X} \ f(C,F) + (rW + (\mu_R - r)X + r\Pi - C)F'(W) + \frac{\sigma_R^2 X^2}{2}F''(W). \tag{B.3}$$

The FOCs for $C$ and $X$ are given by

$$F'(W) = f_C(C,F), \tag{B.4}$$

$$X(W) = \frac{(r - \mu_R)F'(W)}{\sigma_R^2 F''(W)}. \tag{B.5}$$

Using the conjectured value function (59) and simplifying, we obtain

$$C(W) = b^{1-\psi}\zeta^\psi E(W)(E'(W))^{-\psi}, \tag{B.6}$$

$$X(W) = \frac{(\mu_R - r)E(W)}{\sigma_R^2} \frac{E'(W)}{\gamma E(W)'^2 - E(W)E''(W)}. \tag{B.7}$$

Substituting the above into $f(C,F)$, we obtain

$$f(C,F) = \frac{\zeta}{1 - \psi^{-1}} \left[ \frac{(bE(W))^{1-\gamma}(bE'(W))^{1-\psi}}{\zeta^{1-\psi}} - (bE(W))^{1-\gamma} \right]. \tag{B.8}$$

Substituting these results into (B.3), we obtain the following non-linear ODE

$$0 = \frac{m^{FB}E(W)(E'(W))^{1-\psi} - \psi\zeta E(W)}{\psi - 1} + r(W + \Pi)E'(W) + \frac{\eta^2}{2}\frac{E(W)E'(W)^2}{\gamma E'(W)^2 - E(W)E''(W)}. \tag{B.9}$$

The following value match and smooth-pasting conditions determine the threshold $\widehat{W}$

$$F(\widehat{W}) = J(K^*, \widehat{W} - \Phi - K^*), \tag{B.10}$$

$$F'(\widehat{W}) = J_W(K^*, \widehat{W} - \Phi - K^*). \tag{B.11}$$

Using (58) and (59), we obtain the following conditions for $E(W)$ at $\widehat{W}$

$$E(\widehat{W}) = p'(w^*)\left(\widehat{W} - \Phi\right), \tag{B.12}$$

$$E'(\widehat{W}) = p'(w^*), \tag{B.13}$$

where $w^*$ is defined by (56). Finally, we have the absorbing condition, $E(-\Pi) = 0$.