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A PIGOVIAN RULE FOR THE OPTIMUM PROVISION OF PUBLIC GOODS

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ABSTRACT

The integrated treatment of optimal taxation and public expenditure presented here is based on the dual relationship between the prices of private goods and the quantities of public goods. In this paper we derive analogues of Roy's identity and the Slutsky equation for the case of public goods. The optimal provision of public goods and the level of taxation are shown to be dual problems. The conditions for optimum public good provision can be expressed as a modification of the Samuelson conditions with extra terms representing (a) the distortionary effect of taxes on the willingness to pay for the public good, and (b) distributional effects. The former captures Pigou's notion of the indirect damage caused by the need to finance public expenditure out of distortionary taxes, and we call this the "Pigou term". In certain cases a very simple benefit-cost ratio for public projects emerges that is equivalent to measuring benefits as if they were taxed.

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1. Introduction

The analysis of the conditions for the optimum provision of a public good when its cost must be financed by distortionary taxes dates from Pigou (1947). In this second-best context Pigou argued that the "indirect damage" caused by distortionary taxes should be added to the cost of production to obtain the true opportunity cost of the public good. The indirect cost of raising revenue by distortionary taxes should clearly be measured in terms of substitution effects. Hence it should be possible to express the first-order condition for the optimum provision of a public good financed by distortionary taxes as a modified version of the Samuelson (1954, 1955) conditions for the first-best case. The sum of the marginal rates of substitution would be equal to the marginal rate of transformation plus a term representing the marginal indirect damage. But the existing formal solutions of the problem (Diamond and Mirrless 1971, Stiglitz and Dasgupta 1971, Atkinson and Stern 1974, Diamond 1975 and Starrett 1983) do not yield first-order conditions of this form. Instead the first-order conditions are expressed in terms of derivatives of total tax revenue rather than pure substitution effects and are dependent upon the optimality of the distortionary tax structure.

In this paper we show that the total social marginal cost of a public good can be expressed as the sum of its production cost and the marginal efficiency costs of the distortionary taxes required to finance the public good. The latter is a function solely of substitution effects and the interpretation of the first-order condition does not depend upon whether the distortionary taxes are set at their optimal levels. To some extent this resurrects Pigou's discussion of the issue. Where Pigou was wrong was in failing to appreciate that at a second-best optimum the marginal efficiency cost could conceivably be negative. As Atkinson and Stern (1974) have shown, such an outcome might occur if an increase in the quantity of the public good led to an increased demand for private goods that were highly taxed. This possibility is an illustration of the general theory of the second-best, and has been the focus of the critique of Pigou. In contrast, in this paper the aim is to derive a general result for the optimum provision of public goods that expresses formally the substance of Pigou's argument about the need to take into acount the "indirect damage" resulting from the use of distortionary rather than lump-sum taxes.

The key to our result is to exploit the duality that exists between the prices of private goods and the quantities of public goods, and between the quantities of private goods and the willingness to pay for public goods. Recognition of this dual relationship leads directly to the result that the conditions for the optimum provision of public goods are simply the dual of the many-person Ramsey rule for optimal commodity taxes. These conditions may then be given a straightforward interpretation as a many-person Pigovian rule for optimal public goods provision.

Although neither Pigou (1947) nor Atkinson and Stern (1974) explicitly allowed for distributional effects, the model presented here incorporates differences among agents. The use of distortionary taxes presupposes such differences if a uniform poll tax is feasible.

In Section 2 some basic duality results for public goods are presented, and the optimum conditions for the provision of public goods are derived in Section 3. An example is analysed in Section 4.

2. Duality and Public Goods

Consider an economy comprising agents whose preferences are defined over leisure, J private goods which may be traded at consumer prices $\underline{p}$, and G public goods in which trade is not possible. These preferences may be represented by the indirect utility function

$$v = v(\underline{p}, \underline{g}, y) \tag{1}$$

where $\underline{p}$ is a J x 1 vector of consumer prices

$\underline{g}$ is a G x 1 vector of the quantities of public goods

y is the agent's exogenous "full" income.

Equation (1) defines the maximum level of utility that an agent can obtain given his income, market prices, and the common vector of public goods in the economy. Corresponding to (1) is the expenditure function

$$e = e(\underline{p}, \underline{g}, v) \tag{2}$$

Following King (1983) we may define an agent's money metric utility, or "equivalent income", as the convolution of (1) and (2). Equivalent income is that level of income which, at some reference vector of consumer prices and public goods provision ($\underline{p}^R$, $\underline{g}^R$), affords the same level of utility as can be attained under the budget constraint ($\underline{p}$, $\underline{g}$, y).

$$y_E = e(\underline{p}^R, \underline{g}^R, v) \tag{3a}$$

$$= f(\underline{p}^R, \underline{g}^R, \underline{p}, \underline{g}, y) \tag{3b}$$

The function f describes individual preferences, and its arguments denote the budget constraint at which money metric utility is being evaluated.

We shall assume that v and e (and hence f) are continuous functions of the quantities of the public goods, with first and second derivatives.[1] Given this assumption we may define the marginal willingness to pay for the kth public good by the expression

$$w_k = - \frac{\partial y}{\partial g_k} \Big|_{v=\bar{v}} \tag{4}$$

If we differentiate (1) with respect to g_k holding the level of utility constant we obtain

$$\frac{\partial v}{\partial g_k} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial g_k} = 0 \tag{5}$$

Combining (4) and (5) yields the following analogue of Roy's identity for public goods.

$$w_k = \frac{\partial v}{\partial g_k} \quad \frac{\partial v}{\partial y} \tag{6}$$

Comparing this expression with Roy's identity we see that in (6) the marginal willingness to pay for the public good replaces the quantity demanded of the private good and the quantity of the public good replaces the price of the private good. The sign difference reflects the fact that v is increasing in $\underline{g}$ and decreasing in $\underline{p}$.

From (2) and (4) may be derived the analogue to Shephard's lemma

$$w_k = - \frac{\partial e}{\partial g_k} \qquad (7)$$

The compensated and uncompensated demands for private goods are given, respectively, by

$$x_i = \frac{\partial e}{\partial p_i} = x_i^c \, (\underline{p}, \underline{g}, v) \qquad i=1 \ldots J \quad (8)$$

$$x_i = - \frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial y}} = x_i \, (\underline{p}, \underline{g}, y) \qquad i=1 \ldots J \quad (9)$$

The compensated effect of a change in the quantity of the kth public good on the demand for the ith private good is denoted by s_{ik} . From (8) and (9) it is given by

$$s_{ik} = \frac{\partial x_i^c}{\partial g_k} = \frac{\partial x_i}{\partial g_k} + \frac{\partial x_i}{\partial y} \frac{\partial e}{\partial g_k} \qquad (10)$$

Combining (7) and (10) yields the Slutsky equation for public goods

$$\frac{\partial x_i}{\partial g_k} = s_{ik} + w_k \frac{\partial x_i}{\partial y} \qquad \begin{matrix} i=1 \ldots J \\ h=1 \ldots G \end{matrix} \quad (11)$$

There is a further set of equations relating changes in the quantities of public goods to changes in the willingness to pay for public goods.

$$\frac{\partial w_j}{\partial g_k} = s_{jk} + w_k \frac{\partial w_j}{\partial y} \qquad j,k = 1 \ldots G \qquad (12)$$

where s_{jk} is the change in the willingness to pay for the jth public good resulting from a compensated change in the quantity of the kth public good.

Symmetry of the substitution effects follows directly from the assumption that the expenditure function is continuous and twice differentiable. In (11) $s_{ik} = s_{ki}$, which is the change in the willingness to pay for the kth public good following a compensated change in the price of the ith private good. In (12) the own-price substitution effects, s_{kk}, are not necessarily negative. This is because the expenditure function may not be convex in the quantities of public goods (see footnote 1).

For a given choice of reference prices and quantities, the equivalent income function (3b) is defined over actual private good prices, public good quantities and income. There are two ways in which commodity demands and willingnesses to pay may be obtained from the equivalent income function. First, since it may be thought of as an expenditure function defined over reference prices then from (7) and (8) demands are given by differentiating (3a) with respect to reference prices and quantities and evaluating the derivatives at the point where reference values equal actual budget values. Secondly, we may differentiate (3a) and (3b) with respect to actual prices and quantities and substitute from Roy's identity. These yield

$$x_i = \frac{\partial f}{\partial p_i^R}\bigg|_{\substack{\underline{p}^R=\underline{p} \\ \underline{q}^R=\underline{q}}} = - \frac{\partial f}{\partial p_i} \quad \frac{\partial f}{\partial y} \tag{13}$$

$$w_k = - \frac{\partial f}{\partial q_k^R}\bigg|_{\substack{\underline{p}^R = \underline{p} \\ \underline{q}^R = \underline{q}}} = \frac{\partial f}{\partial q_k} \quad \frac{\partial f}{\partial y} \tag{14}$$

3. A Many-Person Pigovian Rule for Public Goods

Armed with the above duality results for public goods we may now analyse the optimum provision of public goods in an economy where lump-sum taxation is limited. We shall consider an economy with a continuum of agents who differ in respect of some attribute. The individual-specific attribute will be denoted by θ, and its distribution function by $F(\theta)$.[2] For example, individuals may have different wage rates. The aim of allowing for such differences is partly to motivate the problem (with identical individuals a uniform poll tax is clearly optimal and this implies the Samuelson conditions for public goods), and partly to compare the result for the optimum provision of public goods with the many-person Ramsey rule for optimal commodity taxes.

The government is assumed to choose levels of public goods provision and tax rates in order to maximise a social welfare function defined over individual levels of equivalent income.[3] For simplicity we assume also that the social welfare function is additively separable and is given by

$$SW = \int_{\theta} W\left[y_E(\theta)\right] dF(\theta) \tag{15}$$

The concavity of W describes the degree of aversion to inequality in money metric utility levels.

The government's budget constraint is that total per capita revenue from all taxes, r, must be at least as great as per capita public expenditure

$$r \geq \underline{c}.\underline{g} \qquad (16)$$

where $\underline{c}$ is a G x 1 vector of the (constant) per capita marginal costs of the public goods.

The value of r depends upon the particular tax instruments available to the government. We assume that the feasible set of taxes comprises a uniform poll tax, denoted by ℓ, and uniform specific commodity taxes, $\underline{t}$ on the J private goods. Leisure is taken as the numeraire and is assumed to be untaxed.

Producer prices are assumed to be constant, and the relationship between producer and consumer prices (denoted by $\underline{q}$ and $\underline{p}$ respectively) is given by[4]

$$\underline{p} = \underline{q} + \underline{t} \qquad (17)$$

Denoting the mean demand for good i by $\bar{x}_i$ we have that

$$r = \ell + \sum_{i=1}^{J} t_i \bar{x}_i \qquad (18)$$

The Lagrangian corresponding to the government's optimisation problem is

$$\equiv \int_{\theta} W\left[y_E(\theta)\right] dF(\theta) + \lambda \left\{\ell + \sum_{i=1}^{J} t_i \bar{x}_i - \sum_{k=1}^{G} c_k g_k\right\} \qquad (19)$$

where λ denotes the multiplier corresponding to the government's budget constraint and is the shadow price of government revenue.

The first order conditions for the optimum levels of the three sets of policy instruments (i) commodity tax rates, (ii) quantities of the public goods, and (iii) the level of the poll tax (or subsidy), are the following. They use the facts that $\partial f/\partial \ell = -\partial f/\partial y$ and $\partial f/\partial t_j = \partial f/\partial p_j$. W' denotes the derivative of W with respect to y_E .

$$\int_\theta W' \frac{\partial f}{\partial p_j}\, dF(\theta) + \lambda \left\{ \bar{x}_j + \sum_{i=1}^{J} t_i \int_\theta \frac{\partial x_i}{\partial p_j}\, dF(\theta) \right\} = 0 \quad j=1\ldots J \qquad (20)$$

$$\int_\theta W' \frac{\partial f}{\partial g_k}\, dF(\theta) + \lambda \left\{ -c_k + \sum_{i=1}^{J} t_i \int_\theta \frac{\partial x_i}{\partial g_k}\, dF(\theta) \right\} = 0 \qquad h=1\ldots G \qquad (21)$$

$$\int_\theta W' \frac{\partial f}{\partial y}\, dF(\theta) - \lambda \left\{ 1 - \sum_{i=1}^{J} t_i \int_\theta \frac{\partial x_i}{\partial y}\, dF(\theta) \right\} = 0 \qquad (22)$$

Following Diamond (1975) and Atkinson and Stiglitz (1980, p387) we define the total social marginal valuation of income of an individual with attribute θ by

$$b(\theta) = \frac{W'}{\lambda} \frac{\partial f}{\partial y} + \sum_{i=1}^{J} t_i \frac{\partial x_i}{\partial y} \qquad (23)$$

In this expression all of the derivatives depend upon θ. The value of $b(\theta)$ is the money value to the planner of an additional unit of income in the hands of a person with attribute θ. The first term measures the value to the planner of the increase in the agent's welfare, and the second term measures the money gain to the government resulting from the increased taxes paid by the agent when he spends his additional income.

The Slutsky equation for private goods is

$$\frac{\partial x_i}{\partial p_j} = s_{ij} - x_j \frac{\partial x_i}{\partial y} \tag{24}$$

Substituting this equation into (20) (noting $s_{ij} = s_{ji}$), and using equation (13) and the definition of $b(\theta)$ in (23) yields the many-person Ramsey rule for optimal commodity taxes

$$\int_\theta b(\theta)\, x_j(\theta) dF(\theta) = \bar{x}_j + \sum_{i=1}^{J} t_i \int_\theta s_{ji}(\theta)\, dF(\theta) \quad j=1\ldots J \tag{25}$$

Let the mean value of the social marginal valuation of income be denoted by

$$\bar{b} = \int_\theta b(\theta) dF(\theta) \tag{26}$$

It is possible to rewrite (25) so that it is a natural extension of the simple Ramsey rule for an economy of identical individuals by defining the normalised covariance of $b(\theta)/\bar{b}$ and $x_j(\theta)/\bar{x}_j$ as

$$\Phi_j = \text{cov}\left[\frac{b(\theta)}{\bar{b}}, \frac{x_j(\theta)}{\bar{x}_j}\right]$$

$$= \int_\theta \frac{(b(\theta)-\bar{b})}{\bar{b}} \frac{(x_j(\theta)-\bar{x}_j)}{\bar{x}_j} dF(\theta) \tag{27}$$

Hence

$$\int_{\Theta} \left[b(\theta) - \bar{b}\right] x_j(\theta) dF(\theta) = \bar{b}\ \bar{x}_j\ \Phi_j \qquad (28)$$

Substituting (28) into (25) yields the alternative form of the many-person Ramsey rule (Atkinson and Stiglitz 1980, p387)

$$-\left[\frac{\sum_i t_i \int_{\Theta} s_{ji}(\theta) dF(\theta)}{\bar{x}_j}\right] = (1-\bar{b}) - \bar{b}\ \Phi_j \qquad j=1\ldots J \qquad (29)$$

This condition states that the proportionate reduction in the compensated demand for good j resulting from the imposition of the commodity tax structure is equal to a constant that is negatively related to the covariance between the consumption of good j and the social marginal value of income.[5] The more consumption is concentrated among those with a high total social marginal valuation of income, the smaller the reduction in demand that should result from the imposition of the optimal tax structure. This result is well-known and is presented here solely for purposes of comparison with the case of public goods for which there is an exactly analagous condition.

The first-order condition for the optimal poll tax implies (from (22), (23) and (26) that

$$\bar{b} = 1 \qquad (30)$$

This equation has the simple interpretation that if the government can make uniform lump-sum transfers then the average value of the total social marginal valuation of income must equal the average cost of the marginal transfer, namely unity. Note that a linear income tax can be thought of as a combination of a poll tax (or subsidy) and uniform commodity tax rates. It is, therefore, a special case of the tax structure examined here.[6]

The first-order condition for the optimum provision of public goods, equation (21), may be combined with the analogue of Roy's identity, (14), and the Slutsky equation for public goods, (11), to give

$$\int_{\theta} b(\theta) w_k(\theta) dF(\theta) = c_k - \sum_{i=1}^{J} t_i \int_{\theta} s_{ki}(\theta) dF(\theta) \qquad k=1 \ldots G \qquad (31)$$

This equation for optimum public goods provision is exactly analagous to equation (25) for optimal commodity taxes. The quantities of the private good in the latter are replaced by the willingnesses to pay for the public good in the former, and the average quantity demanded is replaced by the cost of production. The duality between the prices and quantities of private and public goods means that (25) and (31) are dual to each other and can be interpreted as particular cases of a single general condition for optimum public intervention.

We may define the normalised covariance of the total social marginal valuation of income and the willingness to pay for the kth public good as

$$\Phi_k = \text{cov}\left[\frac{b(\theta)}{\bar{b}}, \frac{w_k(\theta)}{c_k}\right] \qquad (32)$$

Hence

$$\int_\theta \left[b(\theta) - \bar{b}\right] w_k(\theta)dF(\theta) = \bar{b}\, c_k\, \Phi_k \qquad (33)$$

Substituting (33) into (31) yields

$$\left[\frac{\sum_{i=1}^{J} t_i \int_\theta s_{ki}(\theta)dF(\theta)}{\bar{w}_k}\right] = \left[\frac{c_k}{\bar{w}_k}(1 - \bar{b}\,\Phi_k) - \bar{b}\right]$$

$$k = 1\ldots G \qquad (34)$$

where w_k is the mean willingness to pay for the kth public good. We may think of this as a many-person Pigovian rule for the optimum provision of public goods. It states that the (approximate) proportionate change in the compensated willingness to pay for the kth public good that results from the imposition of the distortionary commodity tax structure is equal to a constant that is negatively related to the covariance between the willingness to pay for the public good and the total social marginal valuation of income. The greater is the relative willingness to pay of those with a high total social marginal valuation of income the larger is the optimal proportionate reduction in the marginal willingness to pay. One would expect that this would normally be associated with a larger supply of the public good. Equation (34) is dual to the many-person Ramsey rule given by equation (29), with the exception that the ratio $c_k/\bar{w}_k$ enters into the right-hand side of (34) because, as noted above, in the government revenue constraint the average quantities of private goods are dual to the marginal production costs of public goods rather than their mean willingnesses to pay. This affects the interpretation of (34). Consider the case in which agents are identical ($\Phi_k = 0$) and there is no poll tax. The Pigou rule states that for small deviations from first-best (where $c_k = \bar{w}_k$) the proportionate change in the compensated willingness to pay is the same for all public goods. Moreover, it is exactly equal to the proportionate reduction in the compensated demands for all private goods (equal to $1-\bar{b}$) from (29)). For large changes, however, the right-hand side of (34) will depend upon the nature of the public good.

The final step in the analysis is to compare the second-best optimum described by (31) with the Samuelson first-best conditions. The latter state that in a first-best the sum of the marginal rates of substitution between the public good and the numeraire must be equal to the marginal rate of transformation between the two. This implies that

$$\int_\Theta w_k(\theta)\, dF(\theta) = c_k \qquad (35)$$

Using (33) we may rewrite (31) to give

$$\int_\Theta w_k(\theta)\, dF(\theta) = c_k + \left\{(1-\bar{b})\bar{w}_k - \bar{b}\, c_k \Phi_k\right\} - \sum_{i=1}^{J} t_i \bar{s}_{ki} \qquad (36)$$

where $\bar{s}_{ki}$ is the mean value of s_{ki} (and equals $\bar{s}_{ik}$).

Equation (36) has an appealing interpretation. The difference between the first and second best is that in the latter the effective cost of the public good is equal to its production cost plus two additional terms. The first of these terms relates to the distributional effects of the public good provision, and with the poll tax set optimally ($\bar{b}$=1) its value as a proportion of production costs is exactly equal to minus the covariance between the willingness to pay and the social marginal valuation of income. The second term we may call the Pigou term because it measures the distortion to the aggregate willingness to pay resulting from the use of distortionary taxes to finance government expenditure. It consists solely of substitution effects and is a measure of the "indirect damage" caused by taxation. For this reason it seems to capture the essence of Pigou's argument. Where Pigou's intuition let him down was in suposing that the Pigou term must necessarily be positive. By the usual second-best argument it is possible that the shadow cost of the public good should be lower than its value as given by the Samuelson condition because the provision of an extra unit may reduce distortions elsewhere. Examples have been provided by Atkinson and Stern (1974). But in many cases it is possible to prove that the Pigou term is positive and we provide examples below. It should be noted that the formulation in (36) and the definition of the Pigou term do not in any way depend upon the commodity tax structure being set optimally. The many-person Pigou rule holds for any distortionary tax structure in exactly the same way that the many-person Ramsey rule for optimal commodity taxes does not depend on whether the revenue raised is spent optimally. This follows directly from the duality of the two problems but the existing treatment of the optimum provision of public goods in the literature leans heavily on the conditions for optimality of the tax structure to interpret the first-order conditions for public goods. Such an approach is unnecessarily restrictive.

We conclude by considering some special cases that throw further light on the general second-best optimum and the nature of the Pigou term. First, it is easy to check that (36) is consistent with the Samuelson conditions. In the first-best when unrestricted lump-sum taxation is possible $b(\theta) = 1$, for all θ. This implies that $\Phi_k = 0$, $\bar{b} = 1$ and $t_i = 0$ (for all i) and with these values (36) reduces to (35).

Secondly, when agents are identical (36) yields the simple Pigou rule

$$\int_{\theta} w_k(\theta)\, dF(\theta) = \frac{c_k - \sum_{i=1}^{J} t_i \bar{s}_{ki}}{b} \qquad (37)$$

It is obvious that if a poll tax is feasible no commodity taxes would be employed and the first-best would be attainable. If, however, the poll tax was infeasible then from (22) $0 < b < 1$, and the effective cost exceeds the production cost modified by the Pigou term.

Finally, and of most interest, consider the case of equal ad valorem commodity tax rates and a poll tax (such that $\bar{b}=1$). This corresponds to a linear income tax. The duality formulation allows us to derive a very simple expression for the optimum in this case. Let the common tax rate as a percentage of the tax-inclusive consumer price (equivalent to the constant marginal income tax rate) be

$$t = \frac{t_i}{p_i} \qquad i = 1 \ldots J \tag{38}$$

The Pigou term, denoted by P, becomes

$$P = - t \sum_{i=1}^{J} p_i \bar{s}_{ik} \tag{39}$$

If the public good is a substitute for expenditure on private goods as a whole, then the Pigou term must be positive. From equation (4) and the definition of full income as total expenditure on goods and leisure, we obtain

$$w_k = - \sum_{i=1}^{J} p_i s_{ik} + s_{Lk} \tag{40}$$

where s_{Lk} is the compensated effect of an extra unit of the k^{th} public good on labour supply. The Pigou term becomes

$$P = t(\bar{w}_k - \bar{s}_{Lk}) \tag{41}$$

Substituting (41) into (36) and setting the poll tax to its optimal value yields the second-best optimum

$$\int_{\theta} w_k (\theta) \, dF(\theta) = c_k \frac{(1-\Phi_k)}{(1-t)} - \frac{t \, \bar{s}_{Lk}}{1-t} \tag{42}$$

If the degree of substitutability or complementarity of the public good and leisure is very small, then the criterion for the provision of public goods with a linear income tax is very simple. The marginal benefit-cost ratio, denoted by r, required to justify a project is given by

$$r = \frac{1-\Phi_k}{1-t} \tag{43}$$

The benefits, adjusted for distributional effects, should be equated with the production cost grossed up by the marginal tax rate. Alternatively, the benefits should be measured net of tax. In other words the Treasury should instruct those responsible for project appraisal to calculate benefits as if they were taxed at the same rate as private sector incomes.

The distributional adjustment is defined in terms of the total social marginal valuation of income as given by (23). Where social preferences take no account of the distribution of welfare, and are concerned solely with "efficiency" then only the second term in (23) varies across agents. In this case (and with uniform commodity tax rates) a sufficient condition for Φ_k to be zero is that the marginal propensity to consume leisure is the same for all agents.[7] The required marginal benefit-cost ratio is then project-independent and is given simply by $r = 1/(1-t)$. Although the value of t depends upon the full general equilibrium solution, this formulation gives a simple rule of thumb for the implementation of optimal public expenditure decisions. The cost should be grossed-up by the tax rate to allow for the marginal deadweight loss of financing the public good by distortionary taxes. For example, with a tax rate of 50 per cent, the benefits (sum of the willingnesses to pay) would have to exceed twice the cost to justify provision of the public good. An alternative interpretation is that the benefits of a project should be thought of as equivalent to other forms of income and the net of tax value taken as the relevant value.

To summarise, equation (43) describes the optimal marginal benefit-cost ratio under the following assumptions:

(i) uniform commodity taxes, or a linear income tax, are employed. It is not required that uniform tax rates be optimal, merely that they characterise the tax system used.

(ii) the public good is, on average, neither a complement not a substitute for leisure. Any pattern of complementarity or substitutability between private goods and the public good is allowed.

4. An Example

To illustrate the power of the duality formulation of the problem, we shall examine a simple example, similar to that used by Atkinson and Stern (1974), and show that explicit expressions may be derived for the optimum provision of the public good and the marginal benefit-cost ratio as functions of the underlying preference and technology parameters. Consider an economy of identical individuals whose preferences are defined over a single public good, leisure and two private goods. Preferences are assumed to be Cobb-Douglas in form and described by the indirect utility function (with the wage rate normalised to unity).

$$v(p_1, p_2, g, y) = p_1^{-\alpha_1} p_2^{-\alpha_2} g^{\beta} y \qquad 0 < \alpha_1, \alpha_2 < 1 \quad \beta > 0 \qquad (44)$$

The representative consumer's budget constraint is[8]

$$p_1 x_1 + p_2 x_2 + (H_M - L) = y \qquad (45)$$

where L is hours worked, H_M is the maximum number of hours available for work, and y is exogenous full income. Producer prices are assumed constant. We denote the reference prices of the private goods and the reference quantity of the public good by $p^R{}_1$, $p^R{}_2$ and g^R respectively. From (3) and (44) the level of equivalent income is given by

$$y_E = \left[\frac{p_1^R}{p_1}\right]^{\alpha_1} \left[\frac{p_2^R}{p_2}\right]^{\alpha_2} \left[\frac{g^R}{g}\right]^{-\beta} .y \qquad (46)$$

From (13) and (46) the demands for the private goods and the supply of labour are given by

$$x_1 = \frac{\alpha_1 y}{p_1}$$
$$x_2 = \frac{\alpha_2 y}{p_2} \qquad (47)$$
$$L = H_M - (1-\alpha_1-\alpha_2)y$$

From (14) and (46) the marginal willingness to pay for the public good (denoted by w) is

$$w = \frac{\beta y}{g} \qquad (48)$$

In the first-best case where the public good can be financed by a nondistortionary lump-sum tax then at the optimum

$$w = c \qquad (49)$$

Exogenous income is reduced by the amount of the lump-sum tax which, from the government revenue constraint, must be equal to the cost of providing the public good (cg). Hence (48) and (49) yield the first-best level of public good provision, g*, as

$$g^* = \frac{\beta}{1+\beta}\frac{y}{c} \tag{50}$$

Consider now the second-best case in which only commodity taxes may be employed.

The Ramsey rule (29) implies that

$$\frac{t_1 s_{11} + t_2 s_{12}}{x_1} = b-1 = \frac{t_1 s_{21} + t_2 s_{22}}{x_2} \tag{51}$$

where b is the social marginal value of income. The substitution effects corresponding to the demands in (47) may be inserted into (51) to yield

$$\frac{t_1}{p_1} = \frac{t_2}{p_2} = t \tag{52}$$

$$b = 1 - t(1-\alpha_1-\alpha_2) \tag{53}$$

The Cobb-Douglas preferences described by (44) imply equal commodity tax rates (as a proportion of the producer price) on the two private goods. The common tax-inclusive rate of tax is given from the government budget constraint and optimal commmodity demands as

$$t = \frac{cg}{(\alpha_1+\alpha_2)y} \tag{54}$$

From (37) the first-order condition for the provision of the public good may be written as

$$w = \frac{c - t(p_1 s_{1g} + p_2 s_{2g})}{b} \quad (55)$$

The substitution effects of the public good are given by (11)

$$\frac{\partial x_i}{\partial g} = s_{ig} + w \frac{\partial x_i}{\partial y} = 0 \qquad i=1,2 \quad (56)$$

Hence

$$s_{ig} = - \frac{\alpha_i w}{p_i} \quad (57)$$

Substituting (53) and (57) into (55) yields the following simple result.

$$w = \frac{c}{1-t} \quad (58)$$

At the second-best optimum the use of distortionary commodity taxes increases the relevant marginal cost. In effect, the marginal cost is grossed up by the rates of tax or, equivalently, the marginal benefits must be computed as if they were taxed.

To solve explicitly for the optimal quantity of the public good substitute (48) and (54) into (58). This yields the second-best optimal quantity, g**, as

$$g^{**} = \left[\frac{(\alpha_1 + \alpha_2)\beta}{\alpha_1 + a_2 + \beta} \right] \frac{y}{c} \quad (59)$$

Comparing this expression with (50)[9]

$$\frac{g^{**}}{g^{*}} = \frac{(a_1 + a_2)(1 + \beta)}{\alpha_1 + \alpha_2 + \beta} < 1 \qquad (60)$$

Less of the public good is supplied at the second-best optimum. Substituting (59) into (54) gives an explicit expression for the optimum tax rate

$$t = \frac{\beta}{\alpha_1 + \alpha_2 + \beta} \qquad (61)$$

Together with (58) this defines the marginal benefit-cost ratio to be used in evaluating public projects[10]

$$\frac{w}{c} = \frac{\alpha_1 + \alpha_2 + \beta}{\alpha_1 + \alpha_2} \qquad (62)$$

The welfare loss that results from the inability to use lump-sum taxes may be calculated by evaluating (46) at the first-and second-best optima. It is important to note that the loss arises from the impact of the distortionary tax structure on both the pattern of demand for private goods and the provision of the public good. As Hines (1984) has pointed out, conventional measures of the impact of distortionary taxes ignore their effect on the provision of public goods by examining a switch from distortionary to lump-sum taxation holding public goods provision constant. In our example the total welfare loss can be decomposed quite simply. When calculating money metric utility we take as reference prices and quantities the producer prices of private goods and the first-best quantity of the public good. With these choices of reference values money metric utility in the first-best is given (from (46)) by

$$y_{\epsilon}^{*} = y - cg^{*} = \frac{y}{1+\beta} \tag{63}$$

In the second-best optimum the prices of private goods are equal to producer prices grossed up by the tax rate t, and public good provision is g^{**}. Equivalent income is

$$y_E^{**} = (1-t)^{\alpha_1+\alpha_2}\left[\frac{g^{**}}{g^{*}}\right]^{\beta} y \tag{64}$$

Substituting from (50), (59) and (61) gives a measure of the welfare loss resulting from the use of distortionary taxes in terms of the ratio

$$\frac{y_E^{**}}{y_E^{*}} = \left[\frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + \beta}\right]^{a_1+\alpha_2+\beta} .(1+\beta)^{(1+\beta)} < 1 \tag{65}$$

In contrast, the welfare level that can be achieved by replacing commodity taxes by lump-sum taxes holding revenue and the quantity of the public good constant is measured by

$$\hat{y}_E = \left[\frac{g^{**}}{g^{*}}\right]^{\beta}\left[y - cg^{**}\right] \tag{66}$$

Substitution yields

$$\frac{y_E^{**}}{\hat{y}_E} = \frac{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}(\alpha_1+\alpha_2+\beta)^{1-\alpha_1-\alpha_2}}{\beta + (\alpha_1+\alpha_2)(1-\beta)} < 1 \tag{67}$$

The total welfare gain from removing distortionary taxes can be decomposed into that part attributable to the distortion of the pattern of demand for private goods and that part attributable to the distortion of public goods provision. This is shown by

$$\frac{y_E^{**}}{y_E^{*}} = \left[\frac{y_E^{**}}{\hat{y}_E}\right] \left[\frac{\hat{y}_E}{y_E}\right] \qquad (68)$$

5. Conclusions

The key to our results is the dual relationship between the price of private goods and the quantity of public goods. We derived analogues of Roy's identity, Shephard's lemma and the Slutsky equation for public goods. From these we showed that the determination of optimal tax rates and the optimal provision of public goods are dual problems. An integrated treatment of optimal taxation and public expenditure follows naturally. The first-order conditions for public goods provision can be expressed as a modification of the Samuelson conditions with extra terms representing (a) the distortionary effect of taxes on the willingness to pay for the public good (the "Pigou term"), and (b) the distributional effects of the government budget. Our analysis of the Pigou term (defined as the sum of substitution effects) captures Pigou's notion of the indirect damage resulting from the need to finance public expenditure by distortionary taxes. In some special cases we showed that the consequence of the indirect damage is that the benefits of a public project should be measured as if they were taxed.

FOOTNOTES

1. In the case of private goods the properties of continuity and differentiability follow from the fact that because of substitution possibilities the expenditure function is concave in prices. In the case of public goods the analagous result would be that the expenditure function was convex in the quantities of the public goods. But this is a statement about preferences for public goods and cannot be derived from the assumption of optimising behaviour on the part of individual agents as in the case of private goods. To prove that the expenditure function is concave in the prices of private goods requires the assumption of consistent individual choice over bundles of private goods (see, for example, Deaton and Muellbauer 1980 pp.39-40 and Varian 1978 p.29), but individual agents have no such choices to make when it comes to public goods. Chiappori (1984) and Hines (1984) have independently noted the duality between private and public goods, but in their formulation indirect utility is defined over the willingness to pay for public goods rather than quantities. The problem with this is that the willingness to pay is a function of preferences exactly analogous to the "virtual price" of a rationed private good.

2. For notational simplicity we take θ to be a scalar. The generalisation is straightforward.

3. Defining social welfare over levels of money metric utility rather than over indirect utilities, for example, makes it possible to distinguish the cardinality of social preferences from the form of individual preferences.

4. The assumption that producer prices are constant may be relaxed in a straightforward manner provided that any pure profits that result may be taxed at a rate of 100 per cent (Diamond and Mirrlees 1971).

5. The left-hand side of (29) is only approximately equal to the proportionate reduction in compensated demand for large taxes.

6. For a discussion of nonlinear income taxes see Atkinson and Stiglitz (1976).

7. An alternative sufficient condition is that the willingness to pay for the public good is uncorrelated with full income.

8. The preferences defined by (44) exhibit nonsatiation.

9. If labour supply is non-negative then $\alpha_1 + \alpha_2 \leq 1$.

10. The reader may easily check that if households are not identical but differ in respect of their endowments, the tax rate and marginal benefit-cost ratio are unchanged and that the quantity of the public good is given by (59) with income replaced by mean income.

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