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PRIZES AND INCENTIVES IN ELIMINATION TOURNAMENTS

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ABSTRACT

The role of rewards for maintaining performance incentives in multi-stage, sequential games of survival is studied. The sequential structure is a statistical design-of-experiments for selecting and ranking contestants. It promotes survival of the fittest and saves sampling costs by early elimination of weaker contenders. Analysis begins with the case where competitors' talents are common knowledge and is extended to cases where talents are unknown. It is shown that extra weight must be placed on top ranking prizes to maintain performance incentives of survivors at all stages of the game. The extra weight at the top induces competitors to aspire to higher goals independent of past achievements. In career games workers have many rungs in the hierarchical ladder to aspire to in the early stages of their careers, and this plays an important role in maintaining their enthusiasm for continuing. But the further one has climbed, the fewer the rungs left to attain. If top prizes are not large enough, those who have succeeded in attaining higher ranks rest on their laurels and slack off in their attempts to climb higher. Elevating the top prizes makes the ladder appear longer for higher ranking contestants, and in the limit makes it appear of unbounded length: no matter how far one has climbed, it looks as if there is always the same length to go. Concentrating prize money on the top ranks eliminates the no-tomorrow aspects of competition in the final stages.

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I. INTRODUCTION

Several recent papers have clarified the problem of incentives in one-shot games when competitors are paid on the basis of rank or relative performance (Lazear and Rosen [1981], Green and Stokey [1983], Nalebuff and Stiglitz [1983], Holmstrom [1982], Malcomson [1984], Carmichael [1983], O'Keefe, Viscusi and Zeckhauser [1984]). The main focus so far has been to establish the circumstances under which such schemes are efficient, for example when measurement and environmental factors exert common influences on outcomes. However, a longer tradition in statistics views the relative comparisons inherent in tournaments as a problem of experimental design for selecting and ranking contestants. These two views are joined in this work.

In what follows, I investigate the incentive properties of prizes in sequential elimination tournaments, where rewards are increasing in survival. The inherent logic of these experiments is both to determine the best contestants and promote survival of the fittest; and to maintain the "quality of play" as the game proceeds through its stages. Athletic tournaments immediately come to mind as examples, but much broader interest in this class of problems arises from its application to career games, where the tournament analogy is supported (Rosenbaum [1984]). Most organizations have a triangular structure (Beckmann [1968], Rosen [1982]) and most top level managers come up through the ranks (Murphy [1984]). A career trajectory is, in part, the outcome of competition and striving among peers to attain higher ranking and more remunerative positions over the life cycle. The structure of rewards influences the nature and quality of competition at each stage of the game.

What needs to be explained is the marked concentration of rewards in the top ranks. For example, Table I shows the percentage prize distribution in men's professional tennis tournaments. The top four receive 50 percent or more of the total prize money. Concentration is less extreme in the executive labor market, but nonetheless, those attaining top ranking positions receive more than proportionate shares of compensation. One piece of an explanation for this phenomenon is provided here, where it is shown that an elimination design requires an extra reward for the overall winner to maintain performance incentives throughout the game.

The economics of this is interesting and derives from the survival aspects of the design. A competitor's performance incentives at any stage are set by the value of continuation. This is essentially an option value. The player is guaranteed the loser's prize at that stage, but winning gives the option to continue on to all successive rungs in the ladder. As the game proceeds, there are fewer steps remaining to be attained and the option value plays out. The option value expires in the final match, where there are no further advancements to look forward to. At that point the difference in prize money between winning and losing is the sole instrument available for incentive maintenance and must incorporate the equivalent of the option value that maintained incentives at earlier stages. The extra weight of rewards at the top is fundamentally due to the no-tomorrow aspects of the game. In effect, it extends the horizon of players surviving to the final stages. Most remarkably, it makes the game appear of infinite length to a contestant in the limit, _as if_ there are always many steps to attain, independent of past achievements. In this respect the principle result bears a family resemblance to the role of a "pension" in a finitely repeated principal and agent problem (Becker and Stigler [1974], Lazear [1979]).

The next section describes the game. Section III sets forth the nature of contestants' strategies. Sections IV and V analyze the problem when the inherent talents of competitors are known, while section VI analyzes the case where talents are unknown. Conclusions are found in section VII.

II. DESIGN OF THE GAME

For analytical tractability and simplicity, the basic ideas are best revealed by analyzing a pairwise comparison structure from the statistics literature (David [1969]). The tournament begins with 2^N players and proceeds sequentially through N stages. Each stage is a set of pairwise matches, as in Figure 1. Winners survive to the next round and losers are eliminated from subsequent play. Half of all eligible contestants at the beginning of a stage survive to the next, where another pairing is drawn, and the other half are cut from further consideration. Thus in a career game those eligible for promotion to some rank have attained the rank immediately below it. Those who are passed over at any stage are out of the running for further promotions. The top prize W_1 is awarded to the winner of the final match, who has won N matches overall. The loser of the final match achieves second place overall and is awarded prize W_2 for having won N-1 matches. Losers of the semifinals are both awarded W_3, etc.

Define s as the number of stages remaining to be played. Then all players eliminated in a match where s stages remain are awarded prize W_{s+1}. There are 2^{s-1} such players, so if the total purse is R the prize distribution is constrained by

$$R = W_1 + \sum_{s=1}^{N} 2^{s-1} W_{s+1} \ .$$

Define the interrank spread $\Delta W_s = W_s - W_{s+1}$. Then ΔW_s is the marginal reward for advancing one place in the final ranking, and the total reward for achieving any final rank is the sum of the marginal increments ΔW_s up to that rank plus the lump-sum guarantee W_{N+1}. These marginal increments play a crucial role in determining incentives to advance through the stages. Substituting into the budget constraint,

$$R = \sum_{s=1}^{N} 2^{s-1} \Delta W_s + 2^N W_{N+1}. \tag{1}$$

Prizes are nondecreasing in survival: $\Delta W_s \geq 0$ for all s.

I am concerned in this work with studying how prizes affect performance; in particular with finding some rough characterization of the relative reward structure that maintains incentives as the game proceeds. This is a piece of a larger problem of the "optimal" prize structure, the study of which requires specifying how incentives affect the social value of the game. For example, most of the literature on this subject assumes that social value is reckoned on the sum of individual outcomes, but the interactions among contestants are more complex than that and not well understood. Rather than tying results to a specific and arbitrary input-output technology, a common feature of the larger problem obviously requires that players work at least as hard, if not harder, in the later stages of the game as in the early stages.

Rank-order schemes are encountered when individual output is difficult to measure on a cardinal scale, an inherent feature of managerial and many other types of talent; or when common background noise contaminates precise individual assessments of value-added. In athletic games, after which the present design is modeled, competition is inherently head-to-head and cardinality in any sense other than probability of winning has little meaning. Those games are essentially ordinal because the point scores used to calibrate performance contain many arbitrary elements, much like a classroom test. The game of tennis, for example, would be greatly affected by changing the height of the net and the size of the court. Presumably the rules of the game evolve to reward those personal dimensions of talent which have the largest social productivity. Many of these same considerations apply to selection of managerial talents: success in a lower ranking position serves as an indicator of expected success in a higher ranking one.

Given the rules, these complex issues may be finessed for studying the connection between prizes and incentives by specifying how players' actions affect the probability of winning. Let i index a player and let j index an opponent in some match. Consider a game in which there are m types of players. The ability type of the i-player is indexed by I and the ability type of the opponent j-player is indexed by J. Both I and J take on m possible values, $1,2,\ldots,m$ with $m \leq 2^N$. Let x_{si} and x_{sj} denote the intensity of effort expended by players i and j in a match when s stages remain to be played, and let γ_I and γ_J represent their abilities or natural talents for the game. The probability that a player of type I wins in a match against a player of type J (possibly the same type) is denoted by $P_s(I,J)$ and assumed to follow the law

$$(2) \qquad P_s(I,J) = \frac{\gamma_I h(x_{si})}{\gamma_I h(x_{si}) + \gamma_J h(x_{sj})}$$

with $h(x)$ strictly increasing in x and $h(0) = 0$. A player increases the probability of winning the match by exerting greater effort given the talent and effort of the opponent and own talent. To simplify a complex problem, (2) assumes that the win technology is identical at every stage (s enters only through the x's).

When both players exert the same level of effort the win probability becomes $\gamma_I/(\gamma_i + \gamma_J)$, and its inverse has the natural interpretation of a bookmaker's "morning line" or "true-to-form" actuarially fair payoffs per dollar bet on each player in this match. Notice also that (2) nicely accomodates common environmental factors that influence the nature of play. Suppose the common factor multiplies the terms in $\gamma h(x)$ for both players. Then whether the commonality is match-, stage-, or tournament-specific, it factors out of the probability calculation and has no effect on incentives.

The Poisson proportional hazards form of (2) has a racing game interpretation that has been used to great advantage in the recent literature on patent races [see especially Loury (1979) and references therein]. Let τ be the arrival time from the beginning of the match. Then $H_i = \gamma_I h(x_i)$ is the probability of "crossing the finish line" at τ given that player i has continued racing up to τ, and the unconditional duration density of finishing at τ exactly is $f_i(\tau) = H_i e^{-H_i t}$. Expected finishing time for i is $1/H_i$. The player who arrives first is declared winner of the match: (2) gives the probability of that event for player i. Expected completion time of the match itself is $(H_i + H_j)^{-1}$, so larger values of the x's are associated with higher average quality of play, though of course the realization of actual quality is a random variable.[1] However, the description of the win-technology in (2) stands irrespective of this particular interpretation: just think of it as a function of x_i and x_j and the talents of the players.

III. STRATEGIES

A player's decision of how much effort to expend in any match depends on a cost-benefit calculation. Greater effort increases the probability of surviving to a higher rank and achieving a larger reward, but involves added costs. There are two complications. First, the anticipated value of advancing to the next stage depends on how the player assesses future effort expenditure and behavior should eligibility be maintained in more advanced stages. This forward-looking interstage linkage is solved by the usual dynamic programming recursion, beginning with the final match and working backward to earlier stages one step at a time. Second, current actions depend on the behavior of the current opponent <u>and</u> on the anticipated actions of future possible opponents. The sequential character of the game allows this to be analyzed by adopting Nash noncooperative strategies as the equilibrium concept at each stage. Discounting between stages is ignored and players are assumed to be risk neutral.

Define $V_s(I,J)$ as the value to a player of type I of playing a match against an opponent of type J when s possible stages remain to be played. Assume, for now, that all players' talents are common knowledge. Let c(x) be the cost of effort in any match, assumed identical for all players irrespective of type. c(x) exhibits positive and nondecreasing marginal cost: $c'(x) \geq 0$, $c''(x) > 0$ and $c(0) = 0$. The value of the match consists of two components: One is the prize W_{s+1} earned if the match is lost and the player is eliminated, an event which occurs with probability $1 - P_s(I,J)$. The other is the value of achieving a final rank superior to s+1 if the match is won. Let $EV_{s-1}(I)$ represent the expected value of eligibility in the next stage. $EV_{s-1}(I)$ is a weighted average over J of $V_{s-1}(I\ J)$, where the weights are the probabilities that the player in question will confront an opponent of type J in the next stage. These probabilities depend on the activities of players in other matches at this stage and the rules for drawing opponents at each stage (random draw or seeding). The probability of continuation is $P_s(I,J)$, and costs c(x) are incurred for either outcome, so the fundamental equation for this problem is

$$(3) \qquad V_s(I,J) = \max_{x_{si}}[P_s(I,J)EV_{s-1}(I) + (1 - P_s(I,J))W_{s+1} - c(x_{si})].$$

The max in (3) is understood on Nash assumptions as conditioned on the given current and expected future efforts of all other players remaining alive at s and on the optimum actions taken by the player in question in subsequent matches. The sequences of solutions $\{x_i\}_s$ that are simultaneously conformable with equation (3) for all players is the equilibrium (solution) of the game.

Substituting (2) into (3) and differentiating with respect to x_{si} yields the first order condition

$$(4) \qquad \frac{\gamma_I \gamma_J h_j h_i'}{(\gamma_I h_i + \gamma_J h_j)^2}[EV_{s-1}(I) - W_{s+1}] - c_i' = 0,$$

where $h_i = h(x_{si})$, $h_i' = dh(x_{si})/dx_{si}$ etc. The second order condition is

$$(5) \qquad D = c_i'[h_i''/h_i') - 2\gamma_I h_i'/(\gamma_i h_i + \gamma_J h_j)] - c_i'' < 0.$$

where (5) is evaluated at the arguments that satisfy (4). Equation (5) certainly holds if h(x) has declining marginal product, but h" > 0 is allowed so long as h" is not too large. The marginal condition (4) indicates that effort in any match is controlled by $EV_{s-1}(I) - W_{s+1}$, the difference in value between winning and losing the match. This difference must be positive for the player to have an interest in winning and maintaining eligibility into the next stage. Otherwise, it is best to default, exert no effort, and take the loser's prize for sure.

Equation (4) defines the best response function for player i. Differentiating with respect to the current opponent's effort yields (the s subscript is suppressed but understood)

$$(6) \qquad \partial x_i/\partial x_j = \frac{c_i'(h_j'/h_j)}{-D(\gamma_I h_i + \gamma_J h_j)}(\gamma_I h_i - \gamma_J h_j).$$

Player i's best reply to the opponent's effort is increasing when the opponent is not working too hard, but is decreasing when the opponent's effort is sufficiently large. It has a turning point at $\gamma_I h(x_i) = \gamma_J h(x_j)$. Beyond that point it does not pay to keep pace with the opponent because it is too costly to do so. The turning point occurs at $x_i = x_j$ for equally talented players (i.e., $\gamma_I = \gamma_J$), from (6). It turns at some value $x_j > x_i$ when i is playing a weaker opponent ($\gamma_I > \gamma_J$) and it turns at some $x_j < x_i$ when the opponent is the stronger player ($\gamma_I < \gamma_J$). These differences are illustrated in Figure 2, given the same value of $EV_{s-1} - W_{s+1}$ in each case.[2]

IV. INCENTIVE MAINTAINING PRIZES: EQUALLY TALENTED CONTESTANTS

The complete solution to the problem is transparent when all players are equally talented (there is only one type). Then $EV_{s-1}(I)$ in (3) is simply V_{s-1} because every player knows for sure that an opponent of equal skill will be confronted at every stage of the game. We also know from Figure 2 that the best reply function has a turning point at $x_i = x_j$. It is furthermore obvious, and easily shown, that the value of eligibility at any stage is the same for all survivors. Therefore, the best reply functions for any two opponents in any match at any stage are mirror images of each other and the equilibrium is symmetric: $x_{si} = x_{sj} = x_s$ for all i and j; $P_s = 1/2$ in equilibrium, and each match is a close call in expected value. The common level of effort when s stages remain is, from (4)

$$(7) \qquad (V_{s-1} - W_{s+1})(h'(x_s)/h(x_s))/4 = c'(x_s)$$

since $h(x_{si}) = h(x_{sj})$. Define the elasticities

$$\eta(x) = xh'(x)/h(x)$$

$$(8) \qquad \varepsilon(x) = xc'(x)/c(x)$$

$$\mu(x) = \eta(x)/\varepsilon(x)$$

Then (7) may be manipulated to read as

$$(9) \qquad (V_{s-1} - W_{s+1})\mu(x_s)/4 = c(x_s)$$

in the symmetric equilibrium. Substituting (9) into (3) and using $P_s = 1/2$, we have

$$V_s = (1/2)(1 - \mu(x_s)/2)(V_{s-1} - W_{s+1}) + W_{s+1} \tag{10}$$

$$= \beta_s V_{s-1} + (1 - \beta_s)W_{s+1} ,$$

where

$$\beta_s = (1/2)(1 - \mu(x_s)/2). \tag{11}$$

The recursion in (10) holds under the assumption that (4) is a global maximum at equilibrium. For this to be true, no player can have incentives to default from x_s defined by (7), which requires from (10) that $V_s - W_{s+1} = \beta_s(V_{s-1} - W_{s+1}) > 0$. Therefore $\beta_s > 0$, or, from the definitions, $\eta(x_s)/2\varepsilon(x_s) < 1$. Otherwise $V_s - W_{s+1} < 0$, and a player is better off taking the sure loss. There can be no equilibrium in this game if any player has an incentive to default. For if the i-player defaults, then the j-player guarantees a win by exerting vanishingly small effort at vanishingly small cost. But if player j does this, then player i has incentives to put forth only a slightly larger effort, which drives the solution toward (7). However, at that point V_s is less than the value of the sure loss, both players default and so it goes.

The sense of the no-default condition $\eta(x)/\varepsilon(x) < 2$ is related to the problem of an arms race. If the elasticity of response of effort is large relative to the elasticity of its cost then players' efforts to win results in a negative sum game for which a stable equilibrium is not defined. It is not optimal to default if the opponent does, but at the local equilibrium the costs of contesting have been escalated so much that both want to default. It is in fact implicit in (10), that for a given purse and distribution of prizes, players are better off the smaller is $\mu(x)$ -- when there is less scope for actions to affect outcomes. The rules of the game must be devised to balance two conflicting forces: games which greatly constrain the effect of actions on outcomes are unproductive; whereas competition is destructive if the constraints are relaxed too much. In athletic games this problem is solved by a supreme authority, which reviews standards of play from time to time and which places limits on rules changes and the use of new equipment that would otherwise lead to problems.[3]

Assume the no-default condition holds. Then $0 < \beta_s < 1$, for all s. Using $V_0 = W_1$ as a boundary condition, the solution to (10) is

$$V_s = (\beta_1\beta_2 \ldots \beta_s)\Delta W_1 + (\beta_2\ldots\beta_s)\Delta W_2 + \ldots + \beta_s\Delta W_s + W_{s+1} \quad (12)$$

The value of maintaining eligibility at any stage is the sure prize the player has guaranteed by surviving that long plus the discounted sum of successive interrank rewards that may be achieved in future matches. The discount factor β_s defined in (11) depends on both the equilibrium probability of winning and the equilibrium elasticity parameters. Taking (12) forward one step and subtracting W_{s+1} yields an expression, which, from (7) or (9), controls incentives to perform:

$$(V_{s-1} - W_{s+1}) = (\beta_1\ldots\beta_{s-1})\Delta W_1 + (\beta_2\ldots\beta_{s-1})\Delta W_2 + \ldots + \Delta W_s. \quad (13)$$

The difference between the value of winning and losing in equilibrium is the discounted sum of the forward interrank spreads.

What reward structure maintains incentives to perform at a common value throughout all stages of the game? Here $x_s = x^*$ for all s and $\beta_s = \beta$ is a constant for all s, from (11). Then (13) becomes

$$(14) \qquad V_{s-1} - W_{s+1} = \beta^{s-1}\Delta W_1 + \beta^{s-2}\Delta W_2 + \ldots + \Delta W_s, \text{ for all } s$$

or,

$$(15) \qquad (V_{s-1} - W_{s+1}) - \beta(V_{s-2} - W_s) = \Delta W_s \quad \text{for } s = 2, 3, \ldots, N.$$

Constant performance requires that $(V_{s-1} - W_{s+1})$ is itself a constant, from condition (6). Suppose this constant value is k, where k is determined so that $x_s = x^*$ solves (6).[4] Then (15) implies

$$(16) \qquad k(1 - \beta) = \Delta W_s = \Delta W, \text{ for } s = 2, 3, \ldots, N$$

and (14) implies

$$(17) \qquad k = \Delta W_1 = \Delta W/(1 - \beta).$$

Condition (16) is independent of s, so the incentive-maintaining prize structure requires a constant interrank spread from second place down. However, from (17) it requires a larger interrank spread at the top. Prizes rise linearly in increments $\Delta W = k(1 - \beta)$ from rank N + 1 up through rank 2, but the first place prize takes a distinct jump out of sync with the general linear pattern below it. The incentive-maintaining prize distribution is convex in rank order and weighs the top prize more heavily than the rest. See figure 3.

This surprising conclusion has a nice economic interpretation. The value of playing at any stage is essentially an option value reflecting the probabilities of achieving all possible higher ranks. As the game proceeds, the option value of continuation might lose value because the end draws nearer. The option for continuation loses all value in the finals and ΔW_1 has to incorporate the equivalent of the earlier stage option. Substitute (16) and (17) into (14):

$$(18) \qquad (V_{s-1} - W_{s+1}) = \beta^{s-1}\Delta W_1 + \Delta W(\beta^{s-2} + \beta^{s-3} + \ldots + 1)$$

$$= \Delta W[(\beta^{s-1}/(1-\beta)) + \beta^{s-2} + \beta^{s-3} + \ldots + 1]$$

$$= \Delta W(1 + \beta + \beta^2 + \beta^3 + \ldots) \quad \text{for all } s,$$

where the second equality follows from (17) and the last equality from $1/(1-\beta) = 1 + \beta + \beta^2 + \ldots$. The extra increment at the top converts the value of the difference between winning and losing at each stage into a perpetuity of constant value at all stages. It effectively extends the horizon of the players and makes them behave <u>as if they are in a game which continues forever</u>. This horizon extending feature of the top prize is one of the fundamental reasons why rewards are concentrated toward the top ranks. It is clear by the nature of the proof that concentrating even more of the purse on the top creates incentives for performance to increase as the game proceeds through its stages. For example, if the winner takes all, then every term other than the one in ΔW_1 in (13) vanishes and the difference in value between winning and losing increases as the game proceeds, through the force of discounting: effort is smallest in the first stage and largest in the finals.

The result in (16) and (17) is robust to a number of modifications:

(i) <u>Risk Aversion</u>. The preference structure implicit in the problem above is strongly additive; linear in income and convex in effort. Suppose instead that preferences take the additive form $U(W) - \sum_s c(x_s)$, where $c(x)$ is as before and $U(W)$ is increasing, but not necessarily linear in W. Then the entire analysis goes through by replacing W_s with $U(W_s)$ wherever it appears. Incentive maintenance requires a constant difference in the utility of rewards $U(W_{s+1}) - U(W_{s+2})$ in all stages prior to the finals, but still requires a jump in the interrank difference in utility of winning the finals. If players are risk averse then $U''(W) < 0$ and the incentive maintenance prize structure requires strictly increasing interrank spreads, with an even larger increment between first and second place. The prize structure is everywhere convex in rank order, with greater concentration of the purse on the top prizes than when contestants are risk neutral.

The result is related to an "income effect." When $U(W)$ is concave, the relevant marginal cost of effort is (roughly) the marginal rate of substitution between W and x, or $-c'(x)/U'(W)$. At the target level of effort $c'(x^*)$ is constant, but $U'(W)$ declines as a player continues and is guaranteed a higher and higher rank. The relevant marginal cost of effort effectively increases in each successive stage. Convexity of reward is required to overcome these wealth effects and maintain a player's interest in advancing to a later stage of the game.

(ii). Symmetric win-technologies. The proof of the proposition on incentive-maintaining prizes among equally talented contestants rests only on that property that P_s is 1/2 in equilibrium. Hence the proposition is independent of the specific form of (2) and holds for any win technology resulting in a symmetric equilibrium. This would include, for example, specifications where the opponent's efforts have direct effects on own-arrival time (e.g., write the hazard for each player as $h(x_i, x_j)$ with $h_1 > 0$ and $h_2 < 0$). Furthermore, the result extends to more than pairwise comparisons: there might be n-way comparisons at each stage. In the Poisson case the probability of advancing becomes

$h(x_i)/\sum^n h(x_k)$. Then $\beta_s = (1/n)(1 - (n-1)\mu(x_s)/n)$, but the logic otherwise remains unchanged.

(iii) Stage effects. The nature of competition may vary across stages. In particular, the going may get tougher as the game proceeds. In a corporate hierarchy the pass-through rate may fall at each successive rank. Similarly the cost and elasticity of effort parameters may vary with the stage. μ_s may be smaller in the later stages because higher ranking positions are more demanding than lower ranking ones. In either case β_s decreases as the game proceeds. The argument that led from (13) to (15) is easily extended: the interrank spreads ΔW_s have to be increasing all along the line to undo the incentive dilution effects of greater discounting of the future, which otherwise reduce the option value of continuation.

Analysis becomes more complicated when there are direct interstage spillovers of effort between stages. Two effects may be distinguished: One is a force of momentum, where effort in one round increases the probability of winning the next, similar to the effect of learning. The other is a force of fatigue or depreciation, where greater effort in one match decreases the possibility of putting forth effort in the next ("burnout"). Extension of the results above at the symmetric equilibrium are straightforward.[5] The force of fatigue leads to early round "coasting," as players hold back effort, saving energy reserves for later stages, should they reach them, where the stakes are larger. Here the prize structure must be less concentrated at the top to induce contestants to put forth greater effort in the early stages. The logic is reversed in the case of momentum and learning. Then early round effort has sustained value later by either reducing future costs or increasing the productivity of future effort.

This value falls as the game proceeds and the end draws near, so the prize structure has to be more concentrated at the top to maintain nondecreasing effort.

V. HETEROGENEOUS CONTESTANTS WITH KNOWN TALENTS

The importance of the result in section IV lies in the logic of an elimination design in promoting survival of the fittest. The conditional distribution of survivors' abilities shows an increasing mean and decreasing variance at each stage because there is progressive elimination of the weaker players. If the game is sufficiently long, the contenders in the final stages are selected among those with greater ability, and differences in their talents are much smaller than among the initial field. Continuity of the best reply functions of section III in the γ's implies that the result in section IV holds in the limit in the last few stages of a large-stage game. For if the variance of talents of remaining eligibles in the final stages is small, the equilibria in those rounds must be nearly symmetric, because EV_{s-1} approximates V_{s-1}, and survival probabilities of all players remaining in these stages are close to 1/2. The extra increment at the top is required to extend the horizon and maintain performance incentives toward the end of the game.

A sequential design makes the conditional distribution of survivors exhibit a larger mean talent as the game proceeds because the value of the continuation option is larger for stronger players than for weaker ones. The weak are contending for the lower ranking prizes and the strong for top money. Analysis is complicated by these progressive strength-of-field effects. Contestants know that they are likely to encounter a stronger opponent in a later stage. This reduces the expected value of continuation. But they are likely to be matched against a weaker opponent at early stages, and it is easier to win. The analytical complexity of the problem lies in interactions among matches at any stage. A player is not only interested in what the immediate opponent is doing, but also in what opponents in other matches are doing because the outcomes of other matches determine the identities of future opponents, which in turn affect the value of the current match. Consequently the equilibrium at each stage is a simultaneous 2^s player game. This problem cannot be solved analytically, and must be simulated.[6] However, some progress can be made by examining the conditions that characterize the solution. To simplify, assume that there are two types of players, with type 1 stronger than type 2 ($\gamma_1 > \gamma_2$), and that the hazard and cost functions are of the constant elasticity type, so η, ε and μ are constants.

Begin with the finals. Here $EV_{s-1}(I) - W_{s+1}$ is ΔW_1 for all contestants independent of type and symmetry of condition (4) means that the equilibrium is symmetric irrespective of players' talents. Hence $P_1(I,J) = \gamma_I/(\gamma_I + \gamma_J)$ in equilibrium. A final match involving equally talented contestants implies greater effort than one which matches a stronger with a weaker player: see figure 4. Using the same trick that led from (7) to (9) and substituting into (3),

$$V_1(I,J) = P_1(I,J)[1 - \mu P_1(J,I)]\Delta W_1 + W_1 = \beta_1(I,J)\Delta W_1 + W_1. \tag{19}$$

Here $\beta_1(1,1) = \beta_1(2,2) = (1/2)(1 - \mu/2) = \beta$, as above. But the stronger player wins with larger probability, and $\beta_1(1,2) > \beta > \beta_1(2,1)$, because $P_1(1,2) > P_1(2,1)$.

Semifinals

Let π_1 denote the probability that the winner of the match in question will confront a strong player in the finals. This of course depends on the identities and the x's chosen by players in the other match, but by the logic of the Nash solution these actions are taken as given (at their optimized values) by the opponents in this match. By the definition of EV_{s-1} and the result above,

$$EV_{s-1}(I) = [\pi_1\beta_1(I,1) + (1-\pi_1)\beta_1(I,2)]\Delta W_1 + W_2 = \tilde{\beta}_1(I)\Delta W_1 + W_2 \tag{20}$$

where

$$\tilde{\beta}_1(I) = \pi_1\beta_1(I,1) + (1-\pi_1)\beta_1(I,2)$$

and

$$EV_1(I) - W_3 = \tilde{\beta}_1(I)\Delta W_1 + \Delta W_2. \tag{21}$$

π_1 is no less for the strong contestant so $\tilde{\beta}_1(1) > \tilde{\beta}_1(2)$ because $\beta_1(1,2) > \beta_1(2,1)$.

Using the result in (21) to examine (4) leads to two possibilities. If the two contestants in the match in question are equally talented then the equilibrium in that semifinal match is symmetric, with $P_2(1,1) = P_2(2,2) = 1/2$. But if the two contestants are not equally talented, the equilibrium is not symmetric because the stronger player has a greater value of continuation (21). The strong player exerts greater efforts to win in equilibrium and $P_2(1,2) > \gamma_1/(\gamma_1 + \gamma_2) = P_1(1,2)$: see figure 5. The usual manipulations give

$$V_2(I,J) = \beta_2(I,J)[\tilde{\beta}_1(I)\Delta W_1 + \Delta W_2] + W_3 \tag{22}$$

with

$$\beta_2(I,J) = P_2(I,J)[1 - \mu P_2(J,I)]$$

Furthermore, $\beta_2(1,1) = \beta_2(2,2) = \beta$ and

(23) $$\beta_2(1,2) > \beta_1(1,2) > \beta > \beta_1(2,1) > \beta_2(2,1).$$

(23) implies that $V_2(1,2) > V_2(2,1)$. It also implies that $\tilde{\beta}_2(1) > \tilde{\beta}_2(2)$.

The general pattern is now clear. When s stages remain to be played

(24)
$$V_s(I,J) = \beta_s(I,J)[(\tilde{\beta}_1(I)\tilde{\beta}_2(I)\ldots\tilde{\beta}_{s-1}(I))\Delta W_1 + (\tilde{\beta}_2(I)\ldots\tilde{\beta}_{s-1}(I))\Delta W_2 + \ldots + \Delta W_s] + W_{s+1}$$

$$EV_{s-1} - W_{s+1} = (\tilde{\beta}_1(I)\tilde{\beta}_2(I)\ldots\tilde{\beta}_{s-1}(I))\Delta W_1 + (\tilde{\beta}_2(I)\ldots\tilde{\beta}_{s-1}(I))\Delta W_2 + \tilde{\beta}_{s-1}(I)\Delta W_{s-1} + \Delta W_s$$

$$\tilde{\beta}_s(I) = \pi_s\beta_s(I,1) + (1 - \pi_s)\beta_s(I,2)$$

where π_s is the probability a strong opponent will be encountered at s. In addition $\tilde{\beta}_s(1) > \tilde{\beta}_s(2)$ for all s, which by (24) implies that the value of continuation is larger for stronger players at every stage of the game. Therefore the second expression in (24) implies that a strong player works harder in a strong-weak match than a weak player does, and that the weak are eliminated with probabilities in <u>excess of form</u> [$= \gamma_1/(\gamma_1+\gamma_2)$] at every stage except the last. A sequential design gives an added advantage to the stronger players. This also verifies intuition that weak players are basically competing for the lower ranking prizes.

Inequality (23) cannot be extended beyond the semifinals without additional structure. The ordering of these terms for $s > 2$ depends on the prize structure, the parameters (γ_1, γ_2, η, ε), pairing rules, and on the initial distribution of players by type. However, we do have the following analytical result for the last two stages: <u>If the prize distribution is linear at the top ($\Delta W_1 = \Delta W_2$), effort by both players in strong-weak matches is larger in the semifinals than in the finals; and in matches between similar types effort is also larger in the semis than in the finals</u>. The first part follows from the fact that $\bar{\beta}_1(1)$ necessarily exceeds $\bar{\beta}_1(2)$; while the second part follows from section IV (and in fact holds true for all stages when the prize structure is linear everywhere). The best reply for each player in any type of match is larger in the semis than in the finals when $\Delta W_1 = \Delta W_2$. Consequently, the extra incremental prize at the top remains necessary to extend the horizon and help insure that the final match is the best match.

A small simulation for a two-stage game illustrates these ideas and shows some effects of seeding. To simplify the calculations, I chose $\eta = \varepsilon = 1.0$: hazard and cost functions are linear in x. Further, $\gamma_1 = 2$ and $\gamma_2 = 1$, so the true-to-form odds in a strong-weak match are 2-to-1 in favor of type 1. The simulation assumes that the game begins with two players of each type. The total purse is fixed at 1000 and $W_3 = 0$. The parameter q refers to the ratio $\Delta W_1/\Delta W_2$, so the prize structure is linear when q = 1. The results, in Table II should be read as follows: The top numbers in each line refer to effort expended by player type I in a match against type J. The numbers in parentheses under the semifinals columns refers to the probability that I beats J, whereas the numbers in parentheses under the finals columns show the probability that the final match will be type I against type J. The "all" column under semifinals shows total effort by all four players in the two semifinal matches; the Ex_1

column under Finals shows the expected effort per player in the finals as viewed from the beginning of the game, and the Expected Total column shows expected total effort expended by all six players in all matches in the game.

The first panel shows what might happen when players are not seeded and where the draw pulls strong-to-strong and weak-to-weak in the first round. This guarantees that both a strong and a weak player advance to the finals (therefore all the finals numbers in parentheses are 1.0). Equilibria in all matches are symmetric in this case. The first line illustrates the proposition above: efforts by players of both types are larger in the semis than in the finals when the prize structure is linear. When the final spread is twice that of the semifinal spread, the weaker player exerts more effort in the finals than in the semifinals, but the strong player still exerts less in the finals. Effort is increasing in the finals for both types when the spread ratio is 3 or more. Final round effort is increasing in q because the value of winning the finals increases in q. However, semifinal effort is decreasing in q. This reflects the budget constraint (1) that with a given purse an increase in q necessarily requires decreasing ΔW_2, which decreases so much that the value of winning at $s = 2$ falls. With this parameter configuration the decline in effort with q is relatively small for the stronger players. Total effort in the semis falls with q, but the increase in final round effort with q more than offsets this. Total effort in all matches is increasing in q.

The second panel shows the effects of seeding, with first round matches assigned strong-weak. Now there are three possible matches in the finals: strong-strong, weak-weak, and strong-weak. The first line illustrates the proposition again. With linear prizes the strong player exerts 92.7 units in the (1,2) semifinal match and 74.1 units in the possible (1,2) final match: the weak exert 81.6 in the (2,1) semi match and 74.1 in the (2,1) final match. With seeding this effect is eliminated by the time q is 2 or more. The probability the strong player wins the semis increases in q and so does the probability the final match will be strong-strong, though the rate of increase is small. Again, semifinal effort is falling in q and final effort is rising in q. This reflects increasing discouragement of weak players with q in the first round, because they know they will have to work increasingly hard in the finals and still lose with the same probability of 2/3. Discouragement of the weak means the strong don't have to work as hard to gain the finals.

Comparing across panels, we see that seeding makes both types of players work less hard in the semifinals than no seeding, and no less hard and probably harder in the finals. There is less variance in semifinal effort across player types with seeding, and the final match most probably exhibits a larger quality of play and between higher quality opponents. Notice, however, the surprising result that the total effort expended in all matches for a given prize structure is larger when players are not seeded, at least with this parameter configuration. Seeding produces less variance in efforts in the first round, but a lower mean in that round, and it most likely produces a better match among more talented opponents in the finals. The final interrank spread must be greatly elevated in the seeding game to produce a level of total effort comparable to the no-seeding game. This suggests that seeds are observed when the distribution of the quality of play among players and stages, and guaranteeing the best match at the end are important for the social productivity of the game, not simply total effort expended.[7] It justifies my reluctance to specify an additive social value function for the purposes of calculating an "optimal" prize structure.

VI. HETEROGENEOUS CONTESTANTS WITH TALENTS UNKNOWN

Suppose we are interested in choosing the best out of T possible "treatments." A round-robin design matches each treatment (player) against every other and chooses the one with the largest overall win percentage. A sequential or knock-out design eliminates a treatment from further consideration after it has lost a certain number of times. The sequential design promotes the survival of the fittest and saves sampling costs by eliminating likely losers early in the game, but provides less precise information. The choice between them comes down to comparing sampling costs with the value of more precision or the loss of making errors. David [1960] suggests that knock out designs have advantages over round robins in selecting the best contestant, and Gibbons, Olkin and Sobel [1977] prove this is true on the basis of sequential statistical decision theory. There are other possibilities. For example, medical trials are crudely described by analogy to boxing: the treatment-of-choice is "king of the hill." From time to time contenders come along and occasionally knock off the existing champion. Statisticians have analyzed these kinds of problems in the method of paired comparisons (David [1969]) and there is a parallel mathematical literature from the point of view of graph theory (Moon [1970]). However, it is not possible to apply those results to selection in human population because no account is taken of the strategies and incentives of the contestants to optimize against the experimental design.

A. The Case of Symmetric Ignorance

Consider a sequential single elimination design, in which there are m types of contestants. The distribution of types is common knowledge, and there is no private information: all contestants share the same priors on who their opponents might be and, equally important, are equally ignorant about their own talent. The sequential design allows Bayesian updating of own talents and the strength of the field as the game proceeds. This information feeds back into each contestant's strategy at every stage. When contestants have no more information about themselves than their surviving opponents do, it is clear that the most interesting focus for analysis is the symmetric equilibrium. For in single elimination events all survivors share the same information set -- the same winning record, and therefore choose the same strategy.

Let $\alpha_s(I)$ denote the probability that a player is type I when s stages remain to be played, and let $\tilde{\alpha}_s(J)$ denote the player's assessment that the current opponent is type J. Then, from Bayes' rule, the player's assessment of himself when s-1 stages remain, conditional on winning at stage s, is

$$\alpha_{s-1}(I) = \Pr(\text{win at stage } s|I)\alpha_s(I)/\Pr(\text{win at stage } s) \tag{25}$$

$$= \alpha_s(I)\sum_J \tilde{\alpha}_s(J)P_s(I,J)/\sum_{IJ}\sum \alpha_s(I)\tilde{\alpha}_s(J)P_s(I,J)$$

where $P_s(I,J)$ is the win technology in (2); $\sum_J \tilde{\alpha}_s(J)P_s(I,J)$ is the conditional probability of winning given that one is type I; and the unconditional probability of winning at stage s is the denominator of (25). Assuming commonality of the initial prior distribution, information is common at all s and $\alpha_s(I) = \tilde{\alpha}_s(I)$ at the symmetric equilibrium. Furthermore, all contestants choose the same effort at each s, so $P_s(I,J) = \gamma_I/(\gamma_I + \gamma_J)$, and survival chances for each type run true-to-form at each stage irrespective of the effort levels chosen. Finally, the unconditional symmetric equilibrium probability of winning at each s is 1/2 in paired comparisons. Substituting all this into (25) we have, in the symmetric equilibrium

(26) $$\alpha_{s-1}(I) = 2\alpha_s(I)\sum_J \alpha_s(J)[\gamma_I/(\gamma_I + \gamma_J)]$$

which provides a recursion for calculating the expected survival probabilities for each type at each s, given the initial distribution of talent.

Survival probabilities are increasing for the strong players and decreasing for the weaker players as the game proceeds. There is survival of the fittest. To illustrate, suppose there are two types: $\gamma_1 > \gamma_2$, and let α_s be the expected proportion of stronger, type-1 players alive at s (so the survival proportion for the weak is $(1 - \alpha_s)$. Some manipulation of (26) yields:

(27) $$\alpha_{s-1} - \alpha_s = \alpha_s(1 - \alpha_s)\omega$$

where $\omega = (\gamma_1 - \gamma_2)/(\gamma_1 + \gamma_2)$ is the difference in form-win probabilities between types. The differential equation associated with (27) is the generating function of a logistic. The weak are eliminated at the largest rate at the "diffuse point" where $\alpha_s = 1/2$. They are eliminated at a slower pace when they are either a large or small percentage of the existing population, in the latter case because the strong knock each other off with greater frequency, and in the former case because of their large weight in the population. The rate of elimination of the weak also depends on ω. Convergence is very fast when ω is large. For example, if the initial proportion is 1/2, and ω is close to unity (its maximum possible value) over 99 percent of expected survivors are strong after three stages. More stages are required to select the fittest members of the population the smaller the initial values of α and ω. See Figure 6.

Now the value of survival depends on a player's assessment of own and opponents' talents at any stage. The problem is illustrated for the case of two types, strong (γ_1) and weak (γ_2). We have:

$$(28)\qquad V_s(\alpha_s, \tilde{\alpha}_s) = \max\{Pr(win|\alpha_s, \tilde{\alpha}_s)[V_{s-1}(\alpha_{s-1}, \tilde{\alpha}_{s-1}) - W_{s+1}] - c(x_{si})\}$$

where the win probability is conditioned on the information available at the beginning of stage s:

$$(29)\qquad Pr(win|\alpha_s, \tilde{\alpha}_s) = \alpha_s[\tilde{\alpha}_s P_s(1,1) + (1 - \tilde{\alpha}_s)P_s(1,2)] +$$

$$(1 - \alpha_s)[\tilde{\alpha}_s P_s(2,1) + (1 - \tilde{\alpha}_s)P_s(2,2)].$$

In choosing a strategy at s, the player weighs the possibilities of own and opponent's talent pairings by the information currently available. This information depends on the record of the past and is exogenous data as of stage s. The player's assessment of the future strength of an opponent $\tilde{\alpha}_{s-1}$ depends on the efforts of players in other matches, which is exogenous in the current match under Nash assumptions. However, the player's assessment of his own talent in the next match depends on today's actions and outcomes according to the Bayesian updating formula (25) and this must be taken into account in choosing effort at the current stage. The Bayesian link between stages s and s - 1 relates to the value of information in dynamic programming and introduces an interstage linkage in strategies that is not present when talents are known.

The first order condition for this problem is

$$(30)\qquad \frac{\partial \Pr(\text{win}|.)}{\partial x_{si}} [V_{s-1}(\alpha_{s-1}, \tilde{\alpha}_{s-1}) - W_{s+1}]$$

$$+\Pr(\text{win}|.)[\partial V_{s-1}(\alpha_{s-1}, \tilde{\alpha}_{s-1})/\alpha_{s-1}](\partial \alpha_{s-1}/\partial x_{si}) - c'(x_{si}) = 0$$

The derivative in the first term in (30) is calculated from (29) and $\partial\alpha_{s-1}/\partial x_{si}$ in the second term is calculated from (25), both given x_{sj}. An expression for the value term $\partial V_{s-1}/\partial\alpha_{s-1}$ is found by applying the envelope property to (28):

$$(31)\qquad \partial V_s(\alpha_s, \tilde{\alpha}_s)/\partial\alpha_s = [V_{s-1}(\alpha_{s-1}, \tilde{\alpha}_{s-1}) - W_{s-1}](\Pr(\text{win}|\alpha_s, \tilde{\alpha}_s)/\partial\alpha_s)$$

since the effect of α_s on x_s and on V_{s-1} vanish by the first order condition (30). The condition that characterizes the symmetric Nash solution is found by evaluating (30) at $\alpha_s = \tilde{\alpha}_s$ and $x_{si} = x_{sj}$ for all s. The interstage linkage is provided by the second term in (30) and is the value of information.

Writing $V_s = V_s(\alpha_s, \alpha_s)$ detailed calculations at the symmetric solution yield the following:

$$\partial V_s/\partial\alpha_s = (V_{s-1} - W_s)(\omega/2)$$

$$(32)\qquad \partial \Pr(\text{win}|.)/\partial x_{si} = (h'/h)[(1/4) - \alpha_s(1 - \alpha_s)(\omega^2/2)]$$

$$\partial\alpha_{s-1}/\partial x_{si} = -\alpha_s(1 - \alpha_s)\omega[(\omega\alpha_s - \frac{\gamma_1}{\gamma_1+\gamma_2})^2 + \frac{\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2}]$$

The first condition states that the equilibrium value of continuation is increasing in own assessment of talent, and that this incremental value is increasing in

ω, the difference in form probabilities between types. The value of information is small when contestants are not very different from each other. The marginal effect of effort on the win probability in the second condition is decreasing in the extent of heterogeneity of the population (ω) and in the degree of uncertainty with which players assess themselves at each stage (α). There is the most uncertainty when $\alpha_s = 1/2$. Uncertainty is a force that dampens incentives to perform. As the uncertainty is resolved and α_s approaches unity this dampening effect disappears. The third expression in (32) shows how x_s affects assessments of talent if one survives to the next stage.[8] The effect is unambiguously negative: given the equilibrium effort of the opponent, the winning contestant is more probably of greater talent if less effort has been expended. The magnitude of this term also depends on the extent of heterogeneity and uncertainty. It vanishes as α approaches zero or unity and is numerically largest somewhere in between. The elimination design places extra value on strength, and private incentives to experiment to discover own strength is another force tending to make players hold back efforts at earlier stages.

Plugging (32) into marginal condition (30) converting to elasticities and evaluating at the symmetric equilibrium, we have

$$(33) \qquad c(x_s) = [\tfrac{\mu}{4} - A_s](V_{s-1} - W_{s+1}) - B_s(V_{s-2} - W_s), \qquad \text{for } s \geq 2,$$

where

$$(34) \qquad \begin{aligned} A_s &= \mu\alpha_s(1 - \alpha_s)\,\frac{\omega^2}{2} \\ B_s &= \mu\,\frac{\omega^2}{4}\,\alpha_s(1 - \alpha_s)\left[(\omega\alpha_s - \frac{\gamma_1}{\gamma_1+\gamma_2})^2 + \frac{\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2}\right] \end{aligned}$$

The "law of motion" in (27) is used to calculate A_s and B_s. Condition (33) does not hold as written for the finals (s = 1). There is no private value of experimentation in the finals because additional information cannot be exploited. The boundary condition $B_1 = 0$ allows (33) to stand for all s. Substituting into the value function,

$$V_s = (\beta - A_s)(V_{s-1} - W_{s+1}) + B_s(V_{s-2} - W_s) + W_{s+1} \tag{35}$$

Subtracting W_{s+2} provides a recursion for the increments $V_s - W_{s+2}$:

$$V_s - W_{s+2} = (\beta - A_s)(V_{s-1} - W_{s+1}) + B_s(V_{s-2} - W_s) + \Delta W_{s+1} \tag{36}$$

This may be solved with another boundary condition, really a definition, that $V_0 - W_2 = \Delta W_1$.

Conditions (36) and (33) plus the boundary conditions (and the calculation of A_s and B_s from (27)) represent the complete solution of the symmetric ignorance problem for any feasible wage structure. We notice immediately, from (34), that this solution converges in the limit to that of equal known talents in section IV as α_s approaches unity. For then A_s and B_s go to zero. Hence the extra increment in the final interrank spread is required for incentive maintenance in a sufficiently long game, irrespective of the initial distribution of talents. By a similar token, the earlier result also must hold approximately when the heterogeneity parameter is small.

In fact heterogeneity must be quite large for the value of information to have much effect on the incentive maintaining prize structure of figure 3. This is illustrated by the parameters of table 2: $\gamma_1 = 2$, $\gamma_2 = 1$ and $\mu = 1$. The strong type wins two-thirds of the time and $\omega = 1/3$. Direct calculation reveals that B_s is of order 10^{-3} for any feasible value of α_s, and that A_s is of order 10^{-2}. Therefore the second difference effects in (33) and (36) are negligible and the first difference effects appear much as they did in section IV. Figure 3 is a very close approximation to the incentive maintaining prize structure under symmetric ignorance in this case. When the strong player wins three-fourths of the time, the corresponding orders of magnitude are 10^{-2} for both terms, so the approximation in Figure 3 remains very good: there are only a few minor wiggles.

Major departures occur when there are major differences between types, but this is in large measure due to the incentive dilution effects of uncertainty and in much lesser part due to the incentives to acquire private information. Thus even when the strong player wins 90 percent of the time the terms in B_s remain of order 10^{-2} and the second difference terms are negligible. But the terms in A_s show more variation with α_s, which amounts to a variable discount factor in the value of continuation formula. $(\beta - A_s)$ is smallest in those stages where uncertainty is largest and the interrank spread has to be increased in those stages to overcome larger discounting of the future. Thus consider a tournament where the known proportion of strong players is relatively small in the first round. Then early round incentive-maintaining prizes are approximately linear because there is little uncertainty. As the weak players are eliminated and α_s rises toward 1/2, uncertainty is increasing and the interrank spread has to increase to overcome this effect. If the game is long enough to pass over the diffuse point ($\alpha_s = 1/2$), uncertainty is decreasing and the interrank spreads are decreasing for incentive maintenance. They increase toward the end, due to the horizon effects, though the final round increment is shaded by the effects of experimentation. If the initial proportion of strength is in the neighborhood of 1/2, these resolution-of-uncertainty effects act to distribute the prize money more equally across the ranks and not concentrate it so heavily on the top. If the initial proportion α_N is small and the game is long the prizes redistribute from the extremes toward the middle.

One final point can be made: the expected selection recursions in (26) or (27) show that the social value of information is independent of x_s in the symmetric equilibrium: all information in selecting strong players for survival is embedded in the elimination design itself. In this respect the incentives for contestants to optimize against the design and produce private information come to naught because all players consider these possibilities in their private strategies and no one obtains an informational edge over that inherent in the design. There is a role for the prize structure to discourage these socially useless actions, and this requires less concentration of the prize money at the top. However the calculations above suggest that these effects are relatively minor unless differences in talents are enormous.

B. Private Information

The opposite extreme to the case of symmetric ignorance is when players know their own relative talent, but are informed about opponents' talents only up to a shared distribution of prior beliefs. Continuing with the case of two types, each contestant knows for sure that he is either γ_1 or γ_2 and all share the same prior that the proportion of strong players in the game at the initial round is α. The draws at each stage must be random in this case, so contestants maintain the same assessments that a strong player will be drawn at a given stage. There is no incentive to gain private own information, but each contestant updates beliefs about the strength of potential future opponents through the natural selection of the elimination design.

Analysis is conceptually straightforward, but computationally complicated because the solution does not disassemble recursively. Thus, consider survivors' strategies at some stage s. Suppose the probability that a strong player will be encountered in the current match is π_s. The ex ante value of continuation is a probability weighted average across current opponent types, and the ex ante strategy shares this feature: each player's best response function is a π_s-weighted average of the functions in Figure 2. Thus a strong player's best reply is a weighted average of the curve in the middle of the figure and the one to the right. If the probability of encountering a strong player is large, it is closer to the one in the center, and if the probability of encountering a weak player is large it is closer to the one on the right. Similarly, a weak player's reaction function is a weighted average of the curve on the left and the one in the center.

Uncertainty about the current opponent's ability is resolved ex post: either an opponent of the same type has been drawn, in which case the ex post outcome in this match is symmetric; or an opponent of the opposite type has been drawn, in which case the ex post equilibrium is not symmetric. The value at s is a weighted average of these two possibilities. The equilibrium assessment of π_s depends on the assessment of surviving strength at the beginning of the prior stage and on the equilibrium probability that a strong player won a strong-weak match at s-1. Hence the value at s depends on the actions of players in other matches in previous stages. Now in the case of known talents (section V) the equilibrium is symmetric in any match-pair in the final round. However, the final round equilibrium in a strong-weak match is not symmetric in this case, except by accident. Hence the equilibrium across all matches and all stages must be solved simultaneously.

Taking the linear prize structure as a benchmark for analyzing incentive maintenance reveals two forces: First, the value of continuation is larger for a strong player than for a weaker one, as in section V, so the best reply to an opponent of given skill (the functions in Figure 2) is larger for a strong player than for a weak one. When prizes are linear, these response functions are declining as the game proceeds through its stages, as above; and for a given assessment of field strength the weighted average ex ante strategy is also declining across successive stages. Second, the equilibrium probability weight on the presence of a strong opponent increases as the game proceeds. Hence a strong player's best reply increasingly resembles the middle curve in Figure 2 and a weak player's reply increasingly resembles the curve to the left of center in Figure 2. The first effect is a force tending to reduce the effort of all players as the game proceeds, while the second effect -- that a strong opponent is more likely to be encountered in each successive match if one survives -- is a force that tends to increase effort of the more likely strong survivors as the game proceeds. It is not possible to establish analytically which effect dominates overall. However, we have the usual limiting result that if the initial proportion of strong players is large enough, or if the game is sufficiently long to ensure that most survivors of the final stages are strong, the second effect vanishes and the top prize increment has to be large to maintain final round incentives.

VII. CONCLUSIONS

The chief result of this analysis is in identifying a unique role for top ranking prizes in maintaining performance incentives in career and other games of survival. That incentive maintenance requires extra weight on top ranking prizes rests on the plausible intuition that competitors must be induced to aspire to higher goals independent of past achievements. Competitors have many rungs in the ladder to aspire to in the early stages of the game, and this plays an important role in maintaining their enthusiasm for continuing. But after one has climbed a fair distance there are fewer rungs left to attain. If top prizes are not large enough, those that have succeeded in achieving higher ranks rest on their laurels and slack off in their attempts to climb higher. Elevating the top prizes effectively makes the ladder appear longer for higher ranking contestants, and in the limit of making it appear of unbounded length: no matter how far one has climbed, there is always the same length to go. Much attention has been paid in recent years to the question of whether or not earnings are proportional to marginal product. In problems of this type, the concept of marginal productivity has to be extended to take account of the viability of the organization in maintaining incentives and selecting the best personnel to the various rungs, not only the output produced at each step. Payments at the top have indirect effects of increasing productivity of competitors further down the ladder.

There is another interesting class of questions in this type of competition. Smith held the opinion that there is natural tendency for competitors to overestimate their survival chances ("overweaning conceit"), while Marshall held the opposite opinion. This analysis shows how biased assessments of talent affect survival chances. Analysis of the strategies in Figure 2 reveals that there is a clear disadvantage to pessimism and underestimation of own talents. Not only is there a direct effect of not trying hard enough because the typical opponent appears to be relatively stronger, but the pessimist also underrates the true value of continuation and this induces even less effort. An elimination design is a disadvantage to the timid because they are eliminated too quickly. The effects of overestimation and optimism are more subtle. For strong players and among any contestants in a field of comparable types, optimism has two countervailing forces: the optimist has a tendency to slack off due to underestimation of the relative strengths of the competition, but overestimates the own value of continuation, which induces greater effort. Optimism has no clearcut effects on altering survival probabilities for these reasons. Still, optimism has positive survival value for weak players in a strong field. A weaker player who feels closer to the average field strength than is true, works harder on both counts and is not eliminated as quickly as another weak competitor with more accurate self-assessments.

Finally, there are incentives in rank order competition for a contestant to invest in signals that mislead opponents' assessment of his strength. It is in the interest of a strong player to make rivals think his strength is greater than it truly is: the direct effect on the opponent's strategy works to induce the opponent to put forth less effort, and the indirect effect is trivial. The same is true of a weak player in a weak field. However, it is in the interests of a weak player in a strong field to give out signals that he is even weaker than true, to induce the strong opponent to slack off. To the extent that such investments are socially costly, there is a role for the prize structure to reduce them. This requires weighting the top prizes less heavily than when such effects are not present.

FOOTNOTES

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[1]That (2) is the form of a logit leads to an alternative interpretation. Think of H_i as an index of labor efficiency on an ordered, linear scale. If labor efficiency is distributed as sech^2 then (2) follows from the usual logit assumptions.

[2]The best reply function exhibits a point of discontinuity if (5) fails; then effort of the i-player jumps down when the opponent is working sufficiently hard. This can lead to random strategies in the Nash solution (Nalebuff and Stiglitz [1983]). The present analysis is confined to pure strategy solutions, which require a strict upper bound on $h''(x)$. Also one might expect a weak player to employ a riskier strategy against a stronger opponent (Bronars [1985]), but (2) is not suitably parameterized to allow for this.

[3]The rules of the game and procedures used to determine winners affect the forms of $c(x)$ and $h(x)$: see the related discussion in O'Keeffe _et al_. [1984]. In athletic games equipment producers and players have private incentives to introduce new techniques and styles of play, and complementary capital to create a winning edge. An Authority is needed to maintain the "integrity of the game" and prohibit those innovations which escalate the collective costs of competition relative to social values. In the career setting these incentives are sources of technical change. There the market disciplines "unfair" competition because such organizations have a higher supply price of new recruits.

[4]The purse must be large enough to support $V_s > 0$ for all s. It is obvious that feasible x* is bounded from above for this condition to hold. Another upper bound is implied by contestant's outside opportunities, but is ignored here.

[5]However, complete analysis is complicated because there may be asymmetric equilibria. The best response functions may intersect more than once, and the backward recursion breaks down at the asymmetric equilibria.

[6]This is a major league computational problem for many types and stages. The shape of the best reply functions in figures 2, 4, and 5 shows that the solution is not a contraction.

[7]Of course a random draw with no seeding could produce the second panel in the Table II, and does so two-thirds of the time with two players of each type. Nonetheless, the probability that the best players arrive at the final match is smaller than with seeding. For example, the .53 probability of (1,1) in the finals when $q = 3$ with seeding is decreased to .35 without seeding and the .40 probability of (1,2) in the finals with seeding is increased to .60 without seeding.

[8]Notice that the updating of own-assessment of talent conditional on losing has no value in single elimination contests because the player does not continue. It would have value in games with double, or more eliminations. However, the equilibrium would not be symmetric. Nor would it be symmetric, even with single eliminations, if contestants' observed finer information on past performances instead of only a win-loss record.

REFERENCES

Becker, Gary S. and George J. Stigler, "Law Enforcement, Malfeasance and the Compensation of Enforcers," Journal of Legal Studies 3, no. 1 (Jan., 1984), pp. 27-56.

Beckman, Martin J. Rank in Organizations. Berlin: Springer-Verlag, 1978.

Bronars, Stephen, "Underdogs and Frontrunners: Strategic Behavior in Tournaments," Texas A & M University, 1985

Carmichael, H. Lorne, "The Agent-Agents Problem: Payment by Relative Output," Journal of Labor Economics 1, no. 1 (Jan. 1983), pp. 50-65

David, H.A., "Tournaments and Paired Comparisons," Biometrika, 46, pts. 1 and 2 (June, 1959), pp. 139-49.

________, The Method of Paired Comparisons. London: Charles Griffen and Co., 1969.

Gibbons, Jean D., Ingram Olkin and Milton Sobel, Selecting and Ordering Populations: A New Statistical Methodology. New York: John Wiley and Sons, 1977.

Green, Jerry R. and Nancy L. Stokey, "A Comparison of Tournaments and Contracts," Journal of Political Economy 91, no. 3, (June, 1983), pp. 349-65.

Holmstrom, Bengt, "Moral Hazard in Teams," Bell Journal of Economics 13, no. 2 (Aut. 1982), pp. 324-40.

Lazear, Edward P., "Why Is There Mandatory Retirement?" Journal of Political Economy 89, no. 5 (Oct. 1981), pp. 841-64.

________ and Sherwin Rosen, "Rank Order Tournaments as Optimum Labor Contracts," Journal of Political Economy 89, no. 5 (Oct. 1981), pp. 841-64.

Loury, Glenn C., "Market Structure and Innovation," Quarterly Journal of Economics 94, no. 3 (Aug., 1979), pp. 395-410.

Malcomson, James M., "Work Incentievs, Hierarchy, and Internal Labor Markets," Journal of Political Economy 92, no. 3 (June, 1984), pp. 486-507.

Moon, John W., Topics on Tournaments. New York: Holt, Rinehart and Winston, 1968.

Murphy, Kevin J., "Ability, Performance and Compensation: A Theoretical and Empirical Investigation of Labor Market Contracts." Ph.D. Dissertation, University of Chicago, 1984.

Nalebuff, Barry J. and Joseph E. Stiglitz, "Prizes and Incentives: Toward a General Theory of Compensation and Competition," Bell Journal of Economics 2, no. 1 (Spring., 1984), pp. 21-43

O'Keeffe, Mary, W. Kip Viscusi and Richard J. Zeckhauser, "Economic Contests: Comparative Reward Schemes," Journal of Labor Economics 2, no. 1 (Jan., 1984), pp. 27-56.

Rosen, Sherwin, "Authority, Control and the Distribution of Earnings," The Bell Journal of Economics 13, no. 2 (Oct., 1982), pp. 311-23

Rosenbaum, James E., Career Mobility in a Corporate Hierarchy. Orlando, FL: Academic Press, 1984

TABLE 1

Men's Tennis: 1984 On-Site Prize Money Distribution Formula, Volvo Grand Prix Circuit[a]

Rank	Percent of Purse[b]			
	Grand Slam Events[c]		Other Grand Prix[d]	
	Singles (128 Draw)	Doubles (64 Draw)	Singles (64 Draw)	Doubles (32 Draw)
1	19.23	27.27	20.51	27.27
2	9.62	13.64	10.26	11.36
3-4	4.81	6.82	5.64	5.91
5-8	2.44	2.95	3.08	3.18
9-16	1.41	1.36	1.92	2.10
17-32	.77	.68	1.03	1.25
33-64	.45	.40	.43	
65-128	.22			

Notes

[a]Covers 80 international single elimination events. On-site money does not include contributions to end-of-season bonus pools. 62.5 percent of the $2.4M singles pool goes to the top 4 season ranked players and 64.2 percent of the $.6M doubles pool goes to the top 4 teams.

[b]Total tournament on-site purse split 78 percent for singles, 22 percent for doubles. Figures refer to shares of singles and doubles components of the total respectively. Each person in a tied rank receives the share indicated. Weighted shares may not sum to 100 due to rounding.

[c]French Open, Wimbledon, U.S. Open and Australian Open. Draw refers to number of players or teams. 96 draw singles events are slightly more concentrated on top ranks.

[d]On-site total purse of $25,000 or more.

Source: Official 1984 Professional Tennis Yearbook of the Men's International Professional Tennis Council. New York, 1984.

Table II: Two-Stage, Two-Types Simulation ($\gamma_1 = 2$; $\gamma_2 = 1$)[a]

q	Semifinals			Finals				Expected Totals
No Seeds	$x_2(1,1)$	$x_2(2,2)$	all	$x_1(1,1)$	$x_2(2,2)$	$X_1(2,1)$ $X_1(1,2)$	Ex_1	
1	120.3 (.5)	92.6 (.5)	425.9	---	---	74.1 (1.0)	74.1	574.1
2	118.1 (.5)	76.4 (.5)	388.9	---	---	111.1 (1.0)	111.1	611.1
3	116.6 (.5)	60.7 (.5)	366.7	---	---	133.4 (1.0)	133.4	633.3
5	115.1 (.5)	55.5 (.5)	341.3	---	---	158.8 (1.0)	158.7	658.7
8	113.9 (.5)	47.7 (.5)	322.2	---	---	177.8 (1.0)	177.8	677.8
Seeds	$x_2(1,2)$	$x_2(2,1)$	all	$x_1(1,1)$	$x_2(2,2)$	$X_1(2,1)$ $X_1(1,2)$	Ex_1	
1	92.7 (.69)	81.6 (.31)	348.6	83.3 (.48)	83.3 (.09)	74.1 (.43)	79.4	507.4
2	82.6 (.71)	66.7 (.29)	298.6	125.0 (.51)	125.0 (.08)	111.1 (.41)	119.3	537.2
3	76.1 (.73)	57.7 (.27)	267.7	150.0 (.53)	150.0 (.07)	133.4 (.40)	143.3	554.4
5	68.3 (.74)	47.4 (.26)	231.3	178.5 (.55)	178.5 (.06)	158.8 (.39)	171.0	573.3
8	62.0 (.76)	39.6 (.24)	203.2	200.0 (.57)	200.0 (.06)	177.8 (.37)	191.8	586.9

[a]Notes: $q = \Delta W_1/\Delta W_2$. Purse = 1000, $W_3 = 0$. $x_s(I,J)$ is effort expended by player of type I against opponent of type J when s stages remain. Numbers in ( ) under Semifinals is probability type I wins match against J. Numbers in ( ) in Finals is the probability that a match of type (I,J) occurs. Probability that I wins final against J is always $\gamma_I/(\gamma_I+\gamma_J)$.

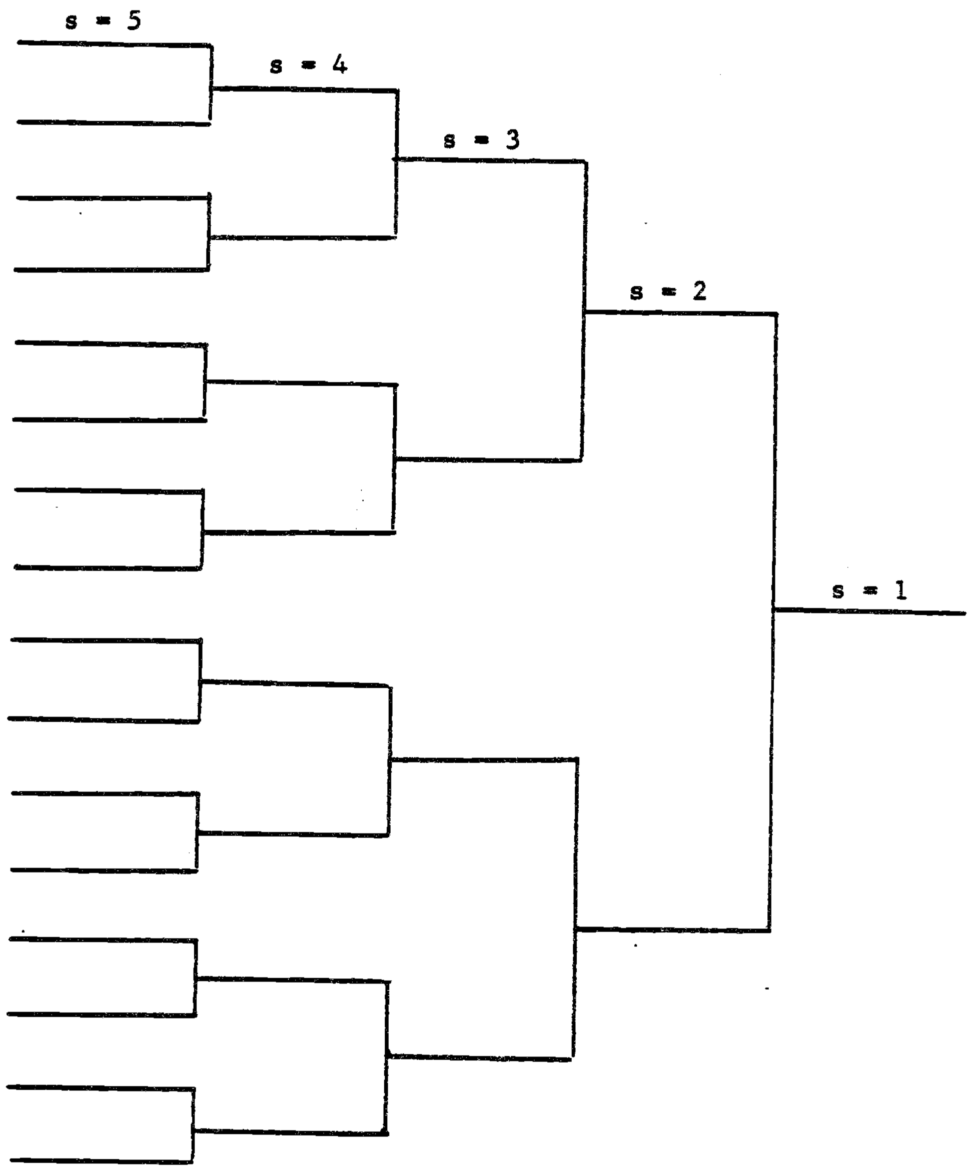

Figure 1: Design of the Game

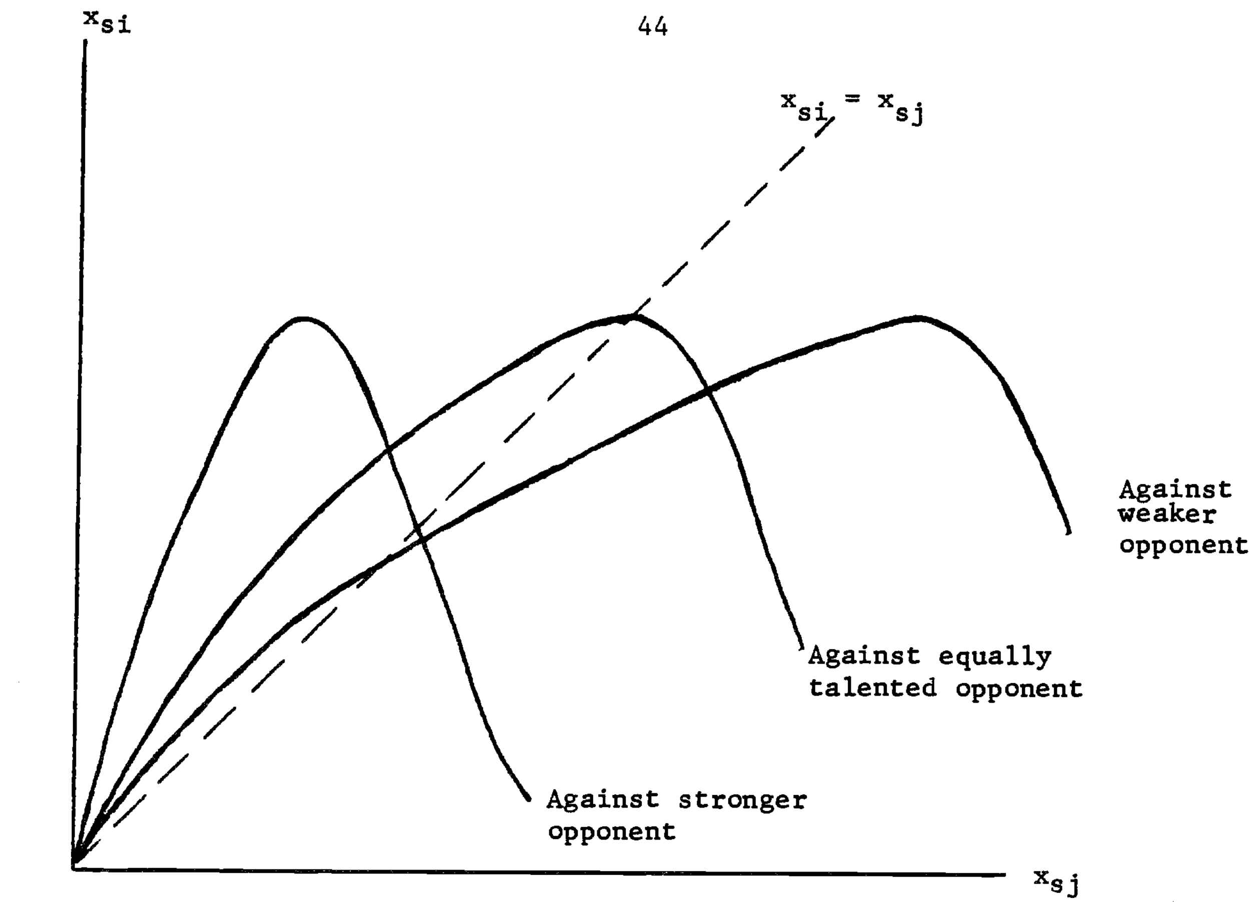

Figure 2: i-player's best response

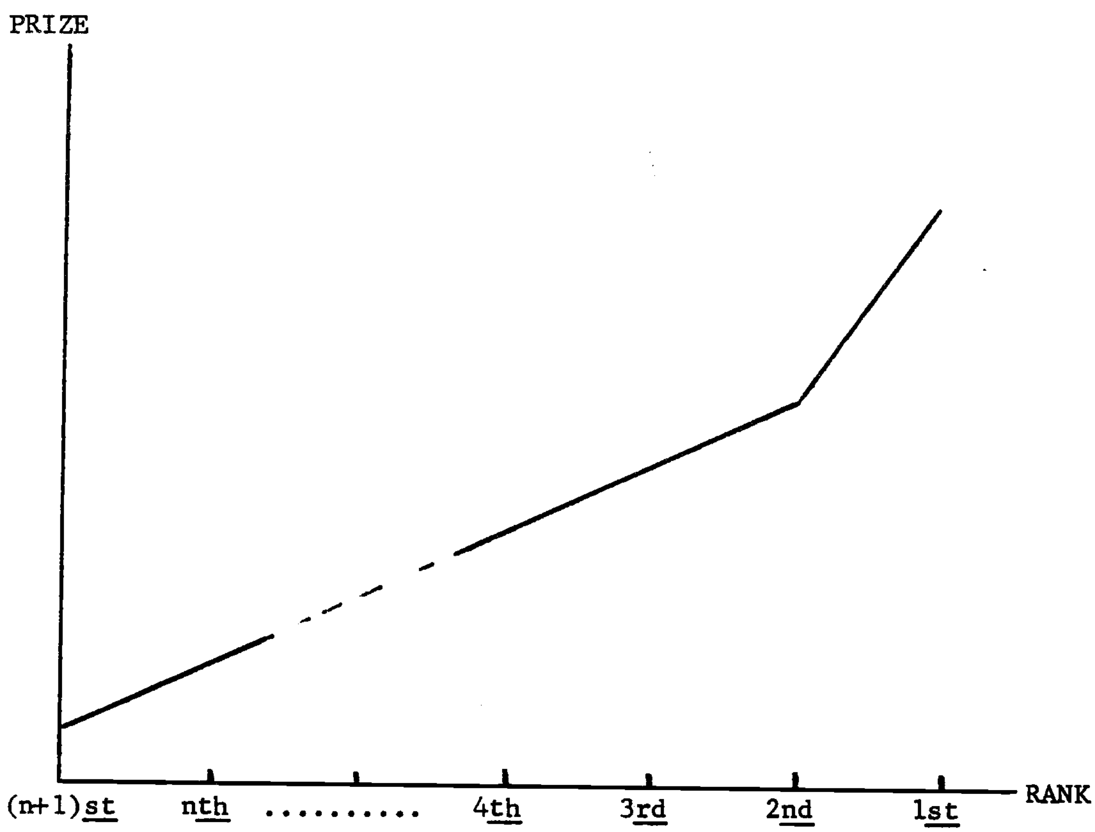

Figure 3: Incentive Maintaining Prize Structure with Equally Talented Contestants

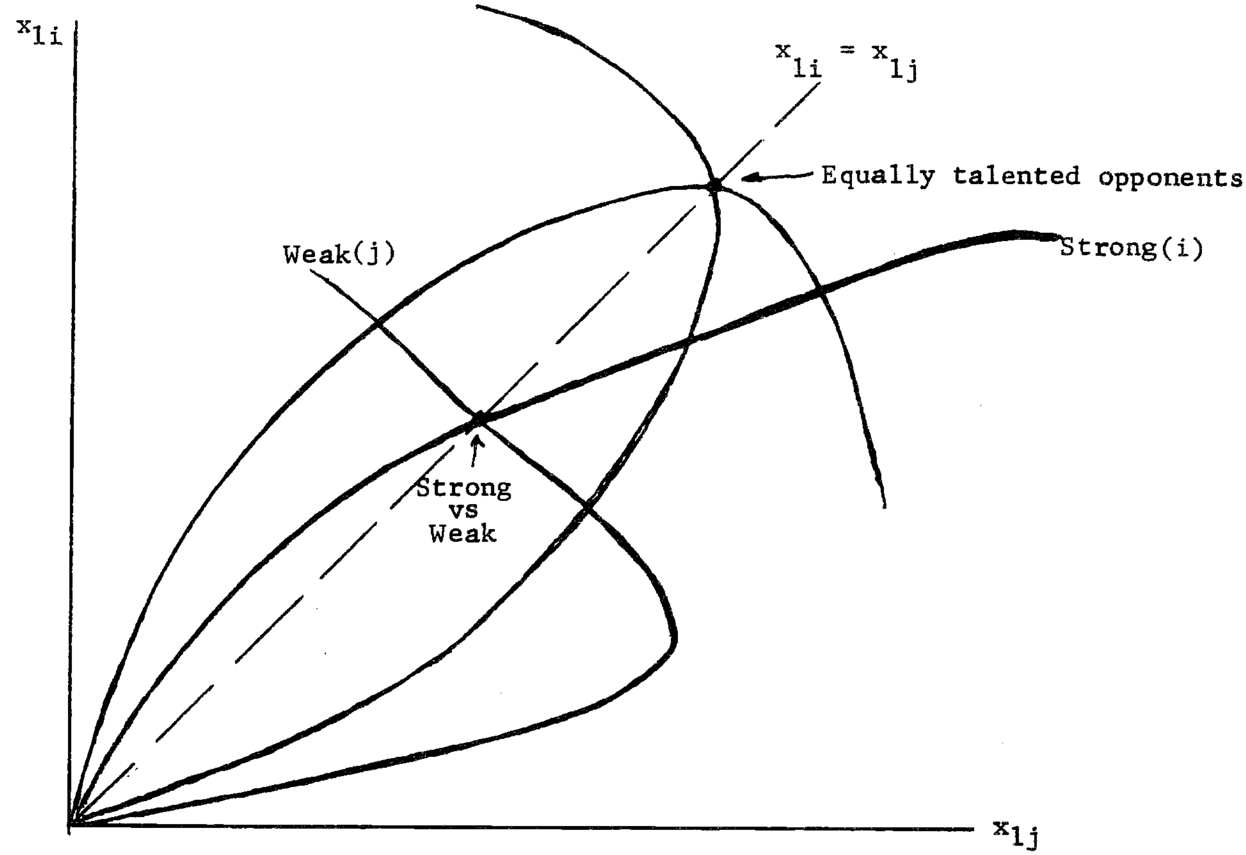

Figure 4: Equilibrium in Finals

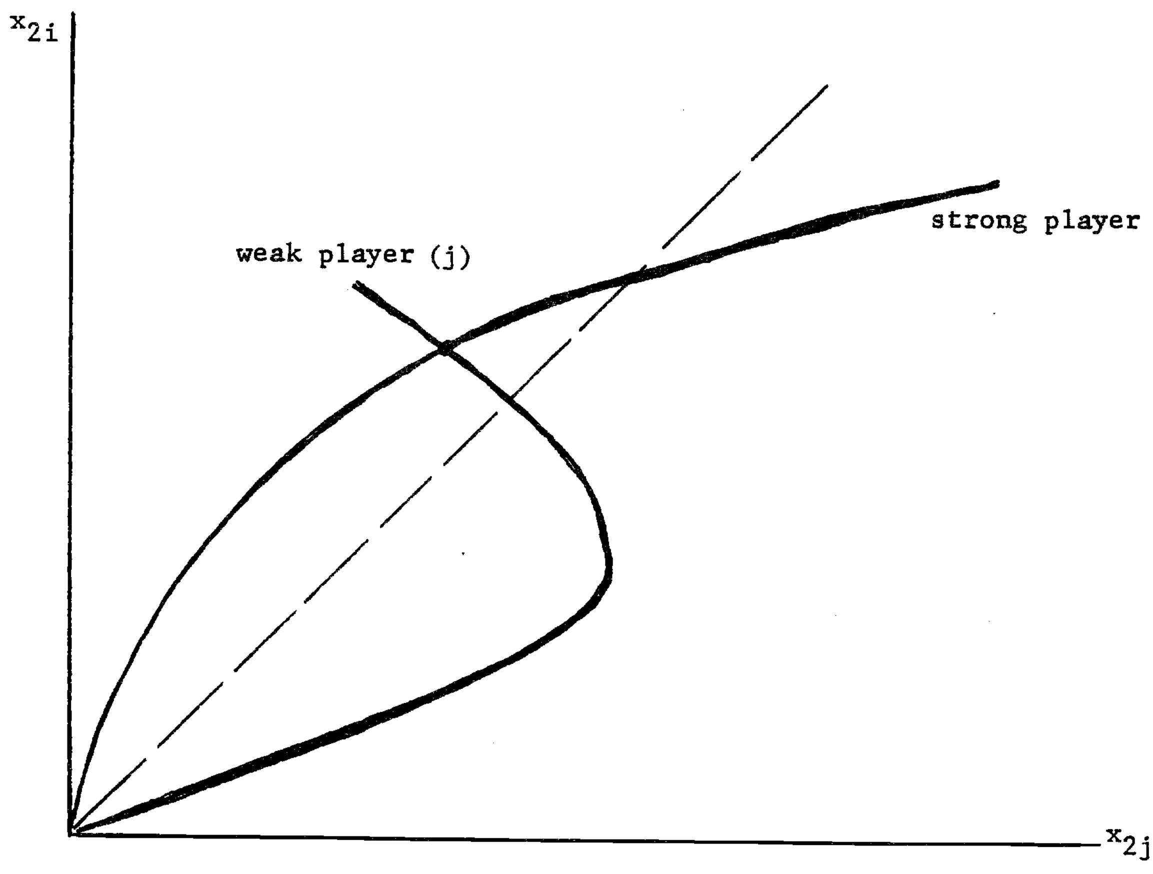

Figure 5: Equilibrium in Semifinals: Strong vs. Weak

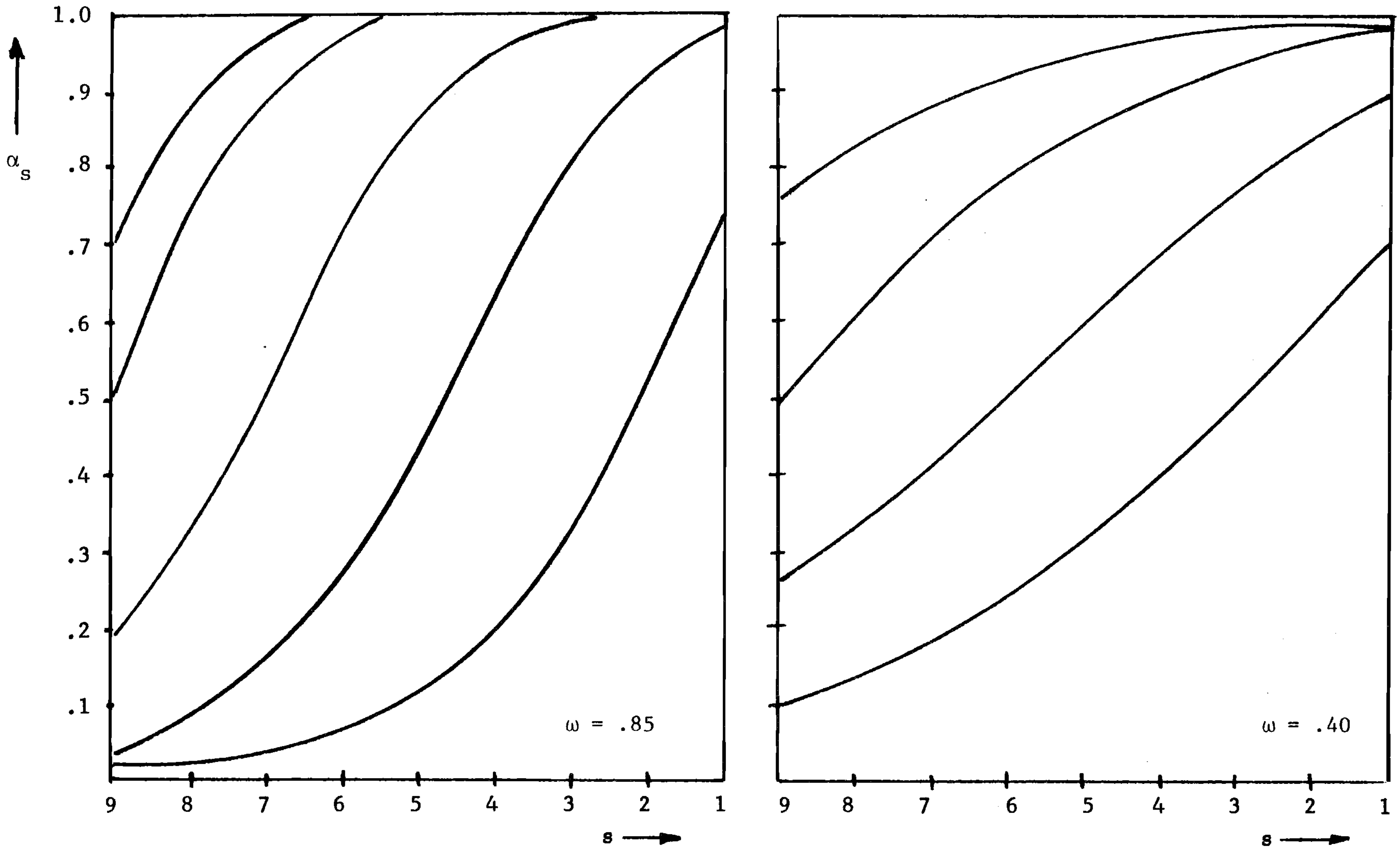

Figure 6: Expected proportion of strong players surviving: 9-stage game