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TIME TO BUILD, OPTION VALUE, AND INVESTMENT DECISIONS

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ABSTRACT

Many investment projects have the following characteristics: (i) spending decisions and cash outlays occur sequentially over time, (ii) there is a maximum rate at which outlays and construction can proceed -- it takes "time to build," and (iii) the project yields no cash return until it is actually completed. Furthermore, the pattern of investment outlays is usually flexible, and can be adjusted as new information arrives. For such projects traditional discounted cash flow criteria, which treat the spending pattern as fixed, are inadequate as a guide for project evaluation. This paper develops an explicit model of investment projects with these characteristics, and uses option pricing methods to derive optimal decision rules for investment outlays over the entire construction program. Numerical solutions are used to demonstrate how time to build, opportunity cost, and uncertainty interact in affecting the investment decision. We show that with moderate levels of uncertainty over the future value of the completed project, a simple NPV rule could lead to gross over-investment. Also, we show how the contingent nature of the investment program magnifies the depressive effect of increased uncertainty on investment spending.

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1. Introduction.

Many investment projects have the following characteristics: (i) spending decisions and associated cash outlays occur sequentially over time, (ii) there is a maximum rate at which outlays and construction can proceed -- it takes "time to build" -- and (iii) the project yields no cash return until it is completed. The firm's investment problem is then to choose a contingent plan for making these sequential -- and irreversible -- expenditures over time. The arrival of new information might lead the firm to depart from the spending scenario originally planned; the firm might accelerate or decelerate the rate of investment, or simply stop the program in midstream.

Examples of industries for which these characteristics are especially important include aircraft and mining. The production of a new line of aircraft requires engineering, prototype production, testing, and final tooling stages that together can take eight to ten years to complete. The construction of a new underground mine, or the development of a large petrochemical plant are projects that usually require at least five or six years, with clear constraints on the pattern of expenditures. In other industries the lead times may be somewhat shorter, but are still important.

Traditional discounted cash flow criteria, which treat the spending pattern as fixed, are inadequate for evaluating such projects. Likewise, neoclassical investment theory,[1] which treats individual units of capital as homogeneous, interchangeable, and individually productive, fails to provide a correct theoretical description of investment behavior under uncertainty. Introducing convex adjustment costs is of little help in this regard - most real projects are composed of heterogenous units of capital that must be installed in sequence, and are unproductive until the project is complete.[2] Indeed, the importance of a sequential investment and time to build have been demonstrated by Kydland and

Prescott (1982) in the context of a general equilibrium model. They have suggested that such a model yields a much better description of cyclical fluctuations than does the standard adjustment cost framework.

Our paper should be viewed in the context of several recent strands of research, all of which have helped to provide a better microeconomic foundation for investment behavior. First, Roberts and Weitzman (1981) examine projects with sequential outlays using a model that stresses the role of information gathering.[3] In their model, each stage of investment yields information that reduces the uncertainty over the value of the completed project. Since the project can be stopped in mid-stream, it might pay to go ahead with the early stages of the project even though *ex ante* the net present value of the entire project is negative. Hence the use of a net present value rule for such projects, particularly one based on a single risk-adjusted discount rate, might reject investments that should be undertaken.

Second, in related papers, Bernanke (1983) and Cukierman (1980) examine investment decisions for which information about project value arrives independently of the cash outlays, and consider the incentives to postpone expenditures until more information arrives. In their models the project involves a single expenditure, and there is no time to build. However, the investment expenditure is irreversible, a firm can choose only a subset of the available projects (so that investing in one set of projects excludes all others), and (unlike in Roberts and Weitzman) the firm obtains information before beginning the project. They show that uncertainty over project returns creates an incentive (an "option value") to postpone the investment and wait for more information to arrive, even if the firm is risk neutral.[4] This is just the opposite result from that in Roberts and Weitzman; here a naive net present value rule might accept projects that should be rejected or postponed. Both authors use their models to explain the cyclical nature of aggregate investment spending; a recession is associated with greater uncertainty over future cash flows because firms reduce their investment spending until some of that uncertainty is resolved.

The models developed in Roberts and Weitzman, Bernanke, and Cukierman are not explicitly based on valuation in financial markets. Thus, a manager following their investment criteria may not be maximizing the firm's value to stockholders. For example, although the assumption of risk-neutrality allows Bernanke and Cukierman to underscore the effects of irreversibility, as distinct from risk aversion, extending their models to a more general setting is not straightforward; the correct risk premium cannot be determined independently of the optimal decision rule.

The third strand of work, and that most closely associated with this paper, is best represented by McDonald and Siegel (1983).[5] They also stress the option value of postponing an irreversible investment, but not as a means of accumulating information. Instead, the payoff from completing the project has some value today which is consistent with capital market equilibrium. This value fluctuates stochastically over time (independently of any investment expenditures), so that its future value is always unknown. Access to the investment opportunity (perhaps purchased or obtained as the result of R&D) is analogous to holding a call option on a dividend-paying common stock, where "exercising" the option is equivalent to making the investment expenditure. As with such financial options, increased risk increases the incentive to delay the investment expenditure, and for any positive amount of risk, the expenditure is made only when the project's value exceeds costs by a positive amount. These results are similar to those in Bernanke and Cukierman, but for a different reason.

This paper is also concerned with the option value of being able to delay irreversible investment expenditures, but here we focus on a series of expenditures that must be made sequentially, that cannot exceed some maximum rate, and that become productive only after the entire sequence is completed. For example, a project requiring a total outlay of $5 million might take five years to build, with a maximum rate of investment of $1 million per year. Such a project can be viewed as a compound option; each unit of investment buys an option on the next unit. Evaluating the project requires a decision rule that determines whether an additional dollar should be spent given any arbitrary cumulative amount that has already been spent. That decision rule will depend on the underlying value that the project would have today if completed, the remaining expenditure required for completion, as well as parameters describing risk and the opportunity cost of delaying completion.

This paper is in the spirit of recent work on capital budgeting with option-equivalent cash flows.[6] Our approach assumes that the value of a completed project is spanned by a set of traded assets and that the distribution of future values is given. Option pricing techniques are used to derive the relationship between the value of the investment program (what a firm would pay for the right to undertake the program) and the value of the project once completed.

We have several objectives. First, we show how a decision rule, applicable to each stage in the development of a project, can be derived and applied to project evaluation. Second, we show how the value of an investment program and the decision to invest depend on the maximum rate at which expenditures can productively be made (i.e. on the "time to build"). Finally, we will see how time to build interacts with uncertainty to affect investment spending, and in particular, how the depressive effect of increased uncertainty on investment spending is magnified.

The next section describes the nature of the investment program, and our assumptions regarding the distribution of future values of a completed project. It also outlines our approach to deriving the optimal investment rule. Section 3 presents numerical results for a simple example that shows how risk, opportunity cost, and time to build interact to affect the investment decision. Section 4 uses the model to examine the economic value of construction time flexibility. The concluding section discusses some implications of our results for aggregate investment behavior.

2. The Model.

Consider a program to build a widget factory. The program involves a sequence of investment outlays, corresponding to the specific steps involved in construction. The payoff to completing the program is the market value of a completed widget factory. This market value might be calculated as the present value of the stream of uncertain future cash flows from operating the factory. Note that we are not assuming that identical factories are traded in a market and, therefore, have an observable price. We are only assuming that we could calculate the value that would prevail if it was traded by applying appropriate capital budgeting methods to the cash flow stream from the completed factory. This market value will, of course, fluctuate stochastically over time, reflecting new information about future cash flows.

We take the market value of the completed factory, which we denote by V, as exogenous, and assume it evolves according to:

$$dV = (\mu-\delta)Vdt + \sigma Vdz \tag{1}$$

where dz is the increment of a Weiner process. The last term in (1) characterizes the unexpected component of changes in V. The central feature is that future values of V are always uncertain, and the degree of uncertainty depends only on how far into the future one looks. Unlike the stylized R&D projects of Roberts and Weitzman where learning takes place at each stage of investment, uncertainty about future values of V is independent of the proportion of the project already completed.[7] Nor is such uncertainty resolved by waiting, as in the models of Bernanke and Cukierman.

The parameter μ is the expected rate of return from owning a completed factory. It is the equilibrium rate established by the capital market, and includes an appropriate risk premium. Eqn. (1) says that the expected rate of capital gain on the factory is less than μ, so that δ represents the opportunity cost of delaying completion of the project. Here, δ is the expected flow of net earnings accruing from operation of the factory. For simplicity, we assume that δ is constant.[8]

If δ is zero, the asset value V is expected to grow at the fair market return μ. There is then no cost of delaying completion of the project, but there is a savings from delaying the investment expenditure. Hence for $\delta = 0$, investment never occurs. It is because most real projects do have an associated opportunity cost of delay in the form of a foregone earnings flow that investment does occur.

An important assumption in our model is that the factory cannot be built overnight. There is a maximum rate at which construction and investment can proceed -- it takes time to build. Because completion of the project requires some minimum amount of time, the payoff from completion is unknown during the construction period. However, we assume that the total required investment is known.

We also assume that the _minimum_ rate of construction and investment is zero, and that construction can be halted and later resumed without cost. In reality, we would expect fixed costs associated with maintaining the partially completed factory (e.g., to prevent "rusting"), and with maintaining the capital and labor resources needed to resume construction. For simplicity we ignore such fixed costs here, although it is straightforward to extend our model to include them. Finally, we assume that investment is completely _irreversible_; capital in place has no alternative use, and therefore zero salvage value.

To see how the constraint of time to build affects investment decisions, we must determine the _market value of the entire investment program_. This market value is what a value-maximizing firm would pay for the right to undertake the program. It will correspond to an optimal program of investment outlays, which will, of course, be contingent on the evolution of V.

We can characterize the investment problem as one of optimal control. There are two state variables, the total amount of investment remaining for completion, K, and the current market value of a completed factory, V. The control variable is the rate of investment, I. The problem is to choose the control rule (corresponding to an investment program contingent on the state variables), $I^*(V,K)$, to maximize the value of the investment program. This is subject to the constraint $0 \leq I^*(V,K) \leq k$, where k is the maximum rate of investment.

Because there are no adjustment costs or costs associated with changing the level of investment, the problem has a "bang-bang" solution: the instantaneous level of investment will be either 0 or k. In turn, the optimal decision rule reduces to a cutoff value for a completed project, $V^*(K)$, such that $I^* = k$ for $V > V^*$, and $I^* = 0$ otherwise. As we will see, the optimal decision rule $V^*(K)$ is determined simultaneously with the current market value of the investment program.

The equilibrium market value of the investment program and the optimal current value of the control variable, I^*, will depend on the values of the two state variables, V and K. In this case I^* is either k or 0, depending on whether the current value of V is above or below the cutoff level, $V^*(K)$. We will find it convenient to denote the value of the investment program when $V > V^*$ (upper region) by F(V,K), and when $V < V^*$ (lower region) by f(V,K).

Formally, the investment program is a contingent claim. However, it is not a simple contingent claim: at every instant the manager can choose whether or not to invest and continue construction. Hence the project is a compound option, where each expenditure buys an option to make the next expenditure. Although this complicates the solution procedure, the same techniques used to value options in securities markets can be applied to value the investment program.

Using a continuous time framework, Merton (1977) derives the valuation equations for general contingent claims. His approach relies on continous trading of specified assets to replicate the payoff to the contingent claim. Nevertheless, this approach is also valid when the assets that must be included in the replicating portfolio are not traded in financial markets. What is necessary is a capital market sufficiently complete that the new project does not change the opportunity set avilable to investors. If this is the case, managers need only calculate the value of the underlying asset, V, that is consistent with the equilibrium valuation model implied by the market. For example, if the CAPM holds and the manager can estimate the underlying asset's beta from prices of traded securities, then he can correctly calculate V, as well as the value of any contingent claim on V (e.g, this investment program).[9, 10]

The option pricing approach yields a valuation equation relating the value of the contingent claim (the investment program) to the value of the underlying asset (the completed factory).[11] Since the value of the contingent claim depends on whether V is above or below V^*, for notational convenience we write a separate valuation equation for each region, i.e., for F(V,K) and f(V,K). It is straightforward to show that F and f must satisfy the following partial differential equations:

$$(1/2)\sigma^2 V^2 F_{VV} + (r-\delta)VF_V - rF - kF_K - k = 0 \tag{2a}$$

$$(1/2)\sigma^2 V^2 f_{VV} + (r-\delta)Vf_V - rf = 0 \tag{2b}$$

subject to the boundary conditions:

$$F(V,0) = V \tag{3a}$$

$$\lim_{V\to\infty} F_V(V,K) = e^{-\delta K/k} \tag{3b}$$

$$f(0,K) = 0 \tag{3c}$$

$$f(V^*,K) = F(V^*,K) \tag{3d}$$

$$f_V(V^*,K) = F_V(V^*,K). \tag{3e}$$

The first boundary condition just says that when the project is completed, the value of the investment program is the market value of a completed project.

As the value of the completed project becomes very large relative to the total investment K, the option "premium" becomes negligible, and the value of the program approaches the value of the completed project. However, the value of the investment program will increase less rapidly than the value of a completed project. As V becomes large, construction outlays will be made at the maximum rate, k, but there is still a foregone opportunity cost. Hence for very large V, the increase in the value of the investment program for a 1 dollar increase in V is given by

$$1 - \int_0^{K/k} \delta e^{(\mu-\delta)t} e^{-\mu t} dt = e^{-\delta K/k}.$$

This condition is shown as (3b) above.

Condition (3c) says that the minimum value of the investment program is zero, and is reached when V is zero. Finally, conditions (3d) and (3e) require that the value of the investment program be continuous and differentiable at the critical value V^*.[12]

Eqn. (2b) has the analytic solution:

$$f(V) = aV^{\alpha}, \tag{4}$$

where $\alpha \equiv \dfrac{-(r-\delta-\sigma^2/2) + [(r-\delta-\sigma^2/2)^2 + 2r\sigma^2]^{1/2}}{\sigma^2}$.

The coefficient a must be determined jointly with the solution for F in the upper region, via the shared boundary conditions (3d) and (3e). This would be straightforward if eqn. (2a) also had an analytical solution; as it does not, a numerical approach is required. First, we eliminate a using eqn. (4) and the boundary conditions (3d) and (3e):

$$F(V^*,K) = (V^*/\alpha)\ F_V(V^*,K) \tag{5}$$

Then eqns. (2a), (2b), and the conditions (3a) - (3c) and (5) are solved numerically using a finite difference method. The procedure transforms the continuous variables V and K into discrete variables, and the partial differential equations into finite difference equations. These equations can be solved algebraically, and the solution proceeds as a backward dynamic program which incorporates the optimal investment decisions at each point. Hence the critical cutoff level, $V^*(K)$ (the optimal boundary between the two regions), is solved for simultaneously with the value of the investment program. Details of this procedure are in the Appendix.[13]

In the next section we apply the solution procedure to a simple and stylized example. This serves to illustrate how the procedure works, and how time to build and uncertainty interact to affect investment decisions.

3. A Simple Example.

Consider a project that requires a total investment (K) of \$6 million, which can be spent productively at a rate no faster than \$1 million per year (k). We assume the riskless rate of interest (r) is 2% per year. The value of the underlying asset, V, evolves according to eqn. (1); we will vary δ and σ, but as a "base case", we take $\delta = .06$ and $\sigma = .20$ (annual rates).[14] As discussed in the Appendix, the solution procedure requires a discretization of the variables V and K; for this example, we assume investment outlays are made quarterly, i.e., K is measured in discrete units of \$0.25 million.

The base case solution is shown in Table 1. Each entry is the value of the contingent claim for different levels of V and K. Entries with an asterisk denote the critical cutoff level $V^*(K)$. For example, a project with \$5 million of investment outlays to go has a critical cutoff level $V^*(K)$ of \$9.49 million: if V is currently \$9.49 million or more it pays to invest this quarter, otherwise it does not (although one would resume investing should V later rise above \$9.49 million). At this critical level the value of the contingent claim is \$2.4 million; this is the equilibrium market value of the right to the investment program.[15]

Observe that Table 1 can be used to make optimal investment decisions as construction of this project proceeds (i.e. as K falls from \$6 million to zero). It can also be used to evaluate any project requiring a total outlay less than \$6 million, but with the same values of r, δ, σ, and in particular the same maximum rate of investment k.

We will be interested in the sensitivity of the investment decision to the parameters σ, δ, and k. That decision is summarized by the critical cutoff value $V^*(K)$. Table 2 shows, for the initial investment decision (i.e., when K = 6), how the cutoff value changes in response to changes in σ and δ. (The middle entry in Table 2 corresponds to the base case shown in Table 1.)

Observe that V^* increases when σ is increased. Like most financial options, the value of the investment program, f, is a convex function of the value of the underlying asset V, and so increases as the standard deviation of V increases. But this increase in f with increasing σ comes about because the right to the investment program can be held and not "exercised" (σ is the standard deviation of V <u>per unit of time</u>). Therefore, if δ and k are held fixed, increasing σ increases the incentive to delay investing, and raises the cutoff value V^*.[16]

The dependence of V^* on δ is less clear. A higher value of δ implies a higher opportunity cost of delaying the project, so one might expect that with σ fixed, V^* would fall as δ increased. This would indeed be the case if the project could be built instantly.[17] But the fact that it takes time to build the project creates a countervailing effect. The payoff from the project, V, is only obtained at completion and must be discounted at the risk-adjusted market rate μ. However its expected rate of growth is only $(\mu - \delta)$. Time to build therefore reduces the present value of V at completion, and as δ increases, it reduces it by a larger amount. This in turn increases the <u>current</u> critical cutoff value V^*. As Table 2 shows, for σ = .10 and .20, V^* rises when δ goes from .06 to .12.

It is useful to calculate the critical cutoff value <u>net</u> of the present value of the expected flow of opportunity cost (δV), assuming that investment expenditures are made at the maximum rate. This value, V^{**}, is simply:

$$V^{**} = V^* - \int_0^{K/k} \delta V^* e^{(\mu-\delta)t} e^{-\mu t} dt = V^* e^{-\delta K/k} \qquad (6)$$

where the second term on the right is the present value of the expected flow of opportunity cost (e.g. foregone rent) during the construction period.

Values for V^{**} are shown for each case in Table 2. Increasing δ increases the opportunity cost of delaying the project (leading to a lower critical cutoff value), and also increases the opportunity cost <u>necessarily</u> incurred because of time to build (leading to a higher cutoff value). V^{**} corrects for the latter, and, as shown in the Table, for any value of σ, it declines as δ increases.

Table 2 also shows the importance of the contingent nature of the investment program. A "naïve" discounted cash flow criterion would ignore flexibility during the construction period, and assume a fixed scenario for the investment outlays. The present value of the construction outlays under the assumed scenario would then be compared to the present value of the payoff at completion (i.e. the current value of a completed project, V, less the present value of the flow of opportunity cost incurred during construction). Assuming investment occurs at the maximum rate, under that naïve criterion one would invest at time t if $\hat{V}(t) > K^*$, where:

$$\hat{V}(t) = V(t) - \int_0^{K/k} \delta V(t) e^{-\delta\tau} d\tau = V(t) e^{-\delta K/k} \tag{7}$$

$$K^* = \int_0^{K/k} k e^{-r\tau} d\tau = (1 - e^{-rK/k}) k/r \tag{8}$$

For our example, the present value of investment outlays made at the maximum rate is $K^* = 5.65$. Even if we ignore the effect of time to build, the critical cutoff value (which would then be V^{**}) should be significantly higher than K^* for any reasonable value of σ and δ, and <u>much</u> higher if σ is large and/or δ is small. Time to build increases the threshold still further; V^* is significantly larger than V^{**}, particularly for large values of δ. Thus for our base case of $\sigma = .20$ and $\delta = .06$, V^* is 11.02, about <u>double</u> the present value of the investment outlays K^*.

We can obtain further insight into the ways in which uncertainty and time to build interact in affecting the investment decision by calculating V^* for different values of k, the maximum rate of investment. Figure 1 shows V^* as a function of k for $\delta = .03$ and .12, and K = 6.[18] Observe that if δ is small (.03), changes in k have very little effect on V^*, because the opportunity cost of time to build is small (i.e. V has an expected rate of growth close to μ, the equilibrium market rate). Hence the ability to speed up construction has little effect on the value of the investment program, or on the investment decision. However, if δ is large (.12), V^* is fairly sensitive to k. Small values of k correspond to long minimum construction times. Hence the present value of the opportunity cost during the construction period is large, reducing the value of the investment program, and increasing the <u>current</u> critical value V^*. (If V^* is adjusted for the flow of opportunity cost during the construction period, the resulting cutoff value (V^{**}) will not be very sensitive to k.) Thus time to build is more important for investment decisions where most of the return on the underlying asset is in the form of a payout stream rather than price appreciation.

Figure 2 shows V^* as a function of δ, for k = 0.5 (a 12 year minimum construction period) and 2.0 (a 3 year minimum construction period). In both cases, V^* falls as δ is increased from .01 to .04. (Remember that as $\delta \to 0$, $V^* \to \infty$, and $f(V,K) \to V$; if there is no opportunity cost to delay, one would never exercise the right to the investment program, and the value of the right would be equal to the value of the underlying asset.) However, as δ increases, the effect on V^* depends on the maximum rate of investment. If that rate is high, V^* remains low over a wide range of δ (but is still 30 - 50 percent greater than the present value of the investment stream). But if k is small, V^* can depend critically on δ. Thus for projects where the minimum time to build is long, the estimate of the rate of opportunity cost δ is a particularly critical input to the investment decision.

We have used numerical examples to illustrate how investment decisions are affected by the sequential nature of construction outlays. The difference between the results of our calculations and those based on a "naïve" application of DCF rules will depend on the parameters of the problem. As Table 2 shows, these differences can be large, even for reasonable values of σ and δ.

4. The Value of Construction Time Flexibility.

Many projects can be built with alternative construction technologies. An important way in which these technologies can differ is in terms of flexibility over the rate of construction. Generally, technologies offering greater flexibility are more costly, so that increased cost must be balanced against the value of increased construction time flexibility. Our model provides a straightforward way to determine the value of that increased flexibility.

In our model, construction time flexibility can be measured by the maximum rate of construction, k. Higher k corresponds to greater flexibility, i.e. a shorter minimum construction time K/k. The value of the investment program f(V,K) increases as k increases, and the change in f corresponding to a change in k measures the value of the extra flexibility. This value of extra flexibility will depend on V and K, as well as other parameters of the problem, such as δ and σ.

We can determine the incremental value of construction time flexibility by examining the way in which the value of the investment program f changes as k changes. We calculate the value of the investment program f for different values of k, holding all other variables, including V, constant. The incremental value of flexibility is then given by the slope of f(k).[19]

Figure 3 shows the results of such a calculation for the base case parameters from the preceding section. All values of the investment program, f, are calculated at a fixed reference value for V. We show f(k) for two different (arbitrary) reference values, $V_1 = 10$ and $V_2 = 15$. As Figure 3 shows, the value of the investment program increases as k increases. Note also that the incremental value of flexibility falls as k increases. For V = 10, the value of the investment program with <u>maximum</u> flexibility (corresponding to $k = \infty$) is 4.0, and for V = 15 it is 9.0, and these values are shown as horizontal lines in the Figure.[20]

Consider two different construction technologies with the same total construction cost K = 6, but with different maximum rates of investment (k = .5 for the first, and k = 1.0 for the second). At V = 10, the incremental value of the more flexible technology (k = 1) is $\Delta f/\Delta k = 0.977/0.5 = 1.954$. This incremental value will be higher if the value of the completed project is higher; at V = 15, the incremental value is 5.520.

In general, greater flexibility might be accompanied by a different total investment K. Because f(V,K;k) is not linear homogeneous in K (see footnote 19), we cannot isolate the value of the greater flexibility in such cases simply by comparing f(V,K;k)/K for each technology. However, we can still rank the technologies by comparing f(V,K;k).

5. Concluding Remarks.

We have shown how optimal investment rules can be determined for projects with sequential investment outlays and maximum construction rates. An important feature of such projects is that the pattern of expenditures can be adjusted as new information arrives. For such projects, we have shown that traditional discounted cash flow criteria based on a fixed pattern of expenditures can lead to grossly incorrect investment decisions. As our calculations for different values of σ, δ, and k have illustrated (Table 2, and Figures 1, 2, and 3), the effects of time to build are greatest when uncertainty is greatest, when the opportunity cost of delay is greatest, and when the maximum rate of construction is lowest.

There are some important caveats. Our optimal investment rule critically depends on the current value of a completed project, V, as well as the parameters σ and δ. We have assumed that these numbers are known, but in fact it may be difficult or impossible to estimate them accurately. Moreover, in some cases both V and the opportunity cost δ are endogenous to the problem. This would be the case, for example, if the opportunity cost corresponding to a foregone cash flow is affected by potential entry by competitors. Then the values of δ and V, as well as the optimal decision rule, must be determined simultaneously (e.g. as a Nash equilibrium for the resulting noncooperative game).

Our primary focus in this paper has been investment decisions from the point of view of a single firm. However, our results also have implications for the behavior of aggregate investment spending, and in particular the role of risk in the economy. As in the models of Bernanke, Cukierman, and McDonald and Siegel (1983), we find that investment decisions can be extremely sensitive to the level of risk (which we measure by the parameter σ). Indeed, this sensitivity is greater than that suggested by traditional investment models. In our model, this greater sensitivity is due to the flexibility that the firm has in making sequential investment outlays; in the models of Bernanke and Cukierman, it is due to the reduction of uncertainty that results from learning. For different reasons, therefore, our results reinforce the view that aggregate investment spending is likely to be highly sensitive to changes in perceived risk.[21]

TABLE 1

OPTIMAL INVESTMENT RULE FOR STYLIZED EXAMPLE

($r = .02$, $\sigma = .20$, $\delta = .06$, $K = 6$, $k = 1$/year)

Value of Investment Program: $f(V,K)$

V\K:	6	5	4	3	2	1	0
42.52	23.70	26.47	29.37	32.42	35.62	38.98	42.52
36.60	19.62	22.12	24.75	27.50	30.39	33.42	36.60
31.50	16.10	18.38	20.76	23.26	25.88	28.62	31.50
27.11	13.07	15.16	17.34	19.62	22.00	24.50	27.11
23.34	10.46	12.38	14.39	16.48	18.67	20.95	23.34
20.09	8.22	10.00	11.85	13.78	15.79	17.89	17.29
17.29	6.23	7.94	9.67	11.46	13.32	15.26	17.29
14.88	4.63	6.18	7.78	9.46	11.19	13.00	14.88
12.81	3.20	4.65	6.17	7.73	9.36	11.05	12.81
11.02	2.02*	3.34	4.77	6.25	7.79	9.37	11.02
9.49	1.22	2.23*	3.57	4.98	6.43	7.93	9.49
8.17	0.74	1.34	2.54	3.88	5.26	6.69	8.17
7.03	0.44	0.81	1.65*	2.93	4.26	5.62	7.03
6.05	0.27	0.49	1.00	2.12	3.39	4.70	6.05
5.21	0.18	0.29	0.60	1.42*	2.65	3.91	5.21
4.48	0.10	0.18	0.36	0.86	2.00	3.23	4.48
3.86	0.06	0.11	0.22	0.52	1.45	2.64	3.86
3.32	0.04	0.06	0.13	0.31	0.98*	2.13	3.32
2.86	0.02	0.04	0.08	0.19	0.59	1.70	2.86
2.46	0.01	0.02	0.05	0.11	0.36	1.32	2.46
2.12	0.01	0.01	0.03	0.07	0.21	1.00	2.12
1.82	0.00	0.01	0.02	0.04	0.13	0.73*	1.82
1.57	0.00	0.01	0.01	0.02	0.08	0.44	1.57
1.35	0.00	0.00	0.00	0.02	0.05	0.27	1.35
1.16	0.00	0.00	0.00	0.01	0.03	0.16	1.16
1.00	0.00	0.00	0.00	0.01	0.02	0.10	1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00*

TABLE 2

DEPENDENCE OF CUTOFF VALUE ON σ AND δ

(r = .02, K = 6, k = 1/year)

Standard Deviation σ	Annual rate of opportunity cost δ: .02	.06	.12
.10	V^* = 11.02 V^{**} = 9.77	V^* = 9.03 V^{**} = 6.30	V^* = 12.43 V^{**} = 6.05
.20	V^* = 20.09 V^{**} = 17.82	V^* = 11.02 V^{**} = 7.69	V^* = 12.81 V^{**} = 7.03
.40	V^* = 121.51 V^{**} = 107.77	V^* = 24.53 V^{**} = 17.11	V^* = 20.09 V^{**} = 9.78

Note: Present value of investment outflow at maximum rate = K^* = 5.65.

FOOTNOTES

1. See Hall and Jorgenson (1967).

2. For an overview of the adjustment cost literature, see Nickell (1978)

3. Their model is most applicable to R&D projects in which learning is important, i.e., where early stages of investment are intended mainly to yield information about costs and feasibility. For an application to synthetic fuels, see Weitzman, Newey, and Rabin (1981).

4. This notion of an "option value" is quite different from the one that we will develop in this paper. In Bernanke, as in earlier papers such as Arrow and Fisher (1974) and Henry (1974), the option refers to a choice of projects (or irrevocable disposition of a natural resource in Arrow and Fisher) that is foregone once the expenditure has been made.

5. Related papers include McDonald and Siegel (1982), and Paddock, Siegel, and Smith (1983).

6. For related work, see Myers and Majd (1983) and Brennan and Schwartz (1985). For an overview, see Mason and Merton (1984).

7. We could introduce learning in our model by making σ a function of the stage of completion. Letting K denote the total amount of investment remaining for completion, we would make $\sigma = \sigma(K)$, $\sigma'(K) > 0$, and $\sigma(0) = 0$. We ignore learning in this paper in order to focus on the implications of time to build.

8. Our approach can easily be extended to cases in which δ varies over time or is a function of V. Myers and Majd (1983) discuss projects for which δ represents the percentage rate of cash flow payout, and varies with time.

9. As Merton (1977) shows, it is not necessary that the contingent claim itself be traded. For example, a dynamic portfolio of stock and bonds can be created to replicate the payoff from a call option on the stock. Note that our assumption that the project does not change the opportunity set available to investors will not always hold. We will discuss this later.

10. Of course, the value of any options implicit in a completed factory, such as the option to shut down the factory, is also included in V.

11. An alternative method of arriving at the valuation equation is via stochastic dynamic programming, but that requires knowledge of the expected rate of return on the underlying asset, μ. An advantage of the option pricing approach is that given V, knowledge of μ is not required. (There are certainly cases where μ is needed to calculate V, but there are many cases for which V can be observed directly from prices of nearly equivalent assets). In addition, the option pricing approach is easier to implement.

12. See Merton (1973), footnote 6, regarding (3e). Intuitively, if a small change in the value of the contingent claim in response to a small change in the value of the underlying asset is greater in one direction than another, moving the free boundary in that direction would result in a net increase in the value of the contingent claim.

13. See Hawkins (1982) for a similar model with analytic solutions in both regions. For an overview of numerical methods for solving option problems, see Geske and Shastri (1985). For a useful discussion of finite difference methods, see Brennan and Schwartz (1978).

14. Payout rates on projects (a major component of δ) can vary enormously from one project to another, so that this value of 6 percent should be viewed as reasonable, but not necessarily representative. The standard deviation of the rate of return on the stock market as a whole has been about 20 percent on average. Although this represents a diversified portfolio of assets, it also includes the effects of leverage on equity returns, and, therefore, might be a reasonable number for an average asset.

15. To conserve space, Table 1 only shows values of the investment program for values of K in multiples of $1 million, and for values of V up to $42.5 million.

16. The only reason to exercise the option to invest at <u>any</u> value of V is that the expected rate of growth of V is less than the risk-adjusted market rate, i.e. $\delta > 0$. If δ were zero one would never invest, just as one would never exercise an infinitely-lived call option on a stock that paid no dividends. Because $\delta > 0$ there is some finite V^* at which one should invest. But as σ gets larger, there is a greater potential gain from the stochastic evolution of V, and, therefore, a greater incentive to hold the option rather than exercise it. This is an important point made by McDonald and Siegel (1983).

17. This is the case in the model of McDonald and Siegel (1983), in which one has the option to delay investing, but once the investment is made, the project is built instantly (i.e., there is no time to build).

18. Our calculations are subject to numerical error because of the finite difference approximation. Absent such errors, the points plotted in Figures 1, 2, and 3 would lie on smooth curves.

19. Another measure of the incremental value of flexibility is the change in $f(V,K;k)/K$ (the value of the investment program per dollar of total required investment) corresponding to changes in the minimum construction times K/k. Note, however, that $f(V,K;k)$ is not linearly homogeneous in K, so that the resulting measure will still depend on K.

20. The case of $k = \infty$ means there is no time to build. This corresponds to a perpetual call option on a stock paying a constant proportional dividend, with exercise price K. The analytical solution is $f(V) = aV^{\alpha}$ for $V \leq V^*$, and $f(V) = V - K$ for $V > V^*$. Here α is given in eqn. (4), $a = (V^* - K)/(V^*)^{\alpha}$, and $V^* = \alpha K/(\alpha - 1)$ is the critical value above which the option is exercised (i.e. the factory is built). In our example,

$V^* = 8.6$. Since $V = 10$ and 15 exceed this critical value, $f(V,K;\infty) = V - K$. See Merton (1973) for a derivation. Note that this is also the model used in McDonald and Siegel (1983).

21. Fischer and Merton (1984) have documented the close empirical connection between aggregate investment and the level of the stock market. Our results (like those of the authors mentioned above) suggest that aggregate investment spending might also be sensitive to the volatility of the stock market. We have attempted to test for this by estimating regression equations like those in Fischer and Merton, but including measures of stock market volatility as additional independent variables. These additional variables turned out not to be statistically significant. One reason for this may be a lack of persistence in volatility changes, as suggested by the recent work of Poterba and Summers (1985).

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APPENDIX FOR

Time to Build, Option Value, and Investment Decisions*

by

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May 1985

This Appendix shows how equations (2a) and (2b), subject to conditions (3a), (3b), (3c) and (5), can be solved numerically. We use the explicit form of the finite difference method.[1] Before implementing this scheme, we make the transformation:

$$F(V, K) \equiv e^{-rK/k} G(X, K), \qquad X \equiv \ln V \,. \tag{a-1}$$

The partial differential equation (PDE) (for $V > V^*$) and boundary conditions become:

$$\tfrac{1}{2}\sigma^2 G_{XX} + (r - \delta - \sigma^2/2) G_X - kG_K - ke^{rK/k} = 0 \,; \tag{a-2}$$

$$G(X, 0) = e^X \,; \tag{a-3}$$

$$\lim_{X\to\infty} \left[e^{-X} e^{-rK/k} G_X(X, K) \right] = e^{-\delta K/k} \,; \tag{a-4}$$

$$G(X^*, K) = G_X(X^*, K)/\alpha \,. \tag{a-5}$$

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[1] For example, see Brennan and Schwartz (1978).

Note that the coefficients of the PDE are no longer functions of V.

The finite difference method transforms the continuous variables V and K into discrete variables, and replaces the partial derivatives in the PDE by finite differences. The explicit form corresponds to a specific choice of finite differences for this substitution. Let $G(V, K) \equiv G(i\Delta X, j\Delta K) \equiv G_{i,j}$ where $-b \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. For the explicit form, substitute:

$$G_{XX} \simeq [G_{i+1,j} - 2G_{i,j} + G_{i-1,j}]/\Delta X^2 ; \qquad \text{(a-6)}$$

$$G_X \simeq [G_{i+1,j} - G_{i-1,j}]/2\Delta X ; \qquad \text{(a-7)}$$

$$G_K \simeq [G_{i,j+1} - G_{i,j}]/\Delta K . \qquad \text{(a-8)}$$

The PDE becomes the difference equation:

$$G_{i,j} = p^+G_{i+1,j-1} + p^oG_{i,j-1} + p^-G_{i-1,j-1} - \eta_{j-1} ; \qquad \text{(a-9)}$$

where

$$p^+ \equiv \Delta K[\sigma^2/\Delta X + r - \delta - \sigma^2/2]/2k\Delta X ;$$

$$p^o \equiv 1 - \sigma^2\Delta K/k\Delta x^2 ;$$

$$p^- \equiv \Delta K[\sigma^2/\Delta X - r + \delta + \sigma^2/2]/2k\Delta X ;$$

$$\eta_j \equiv \Delta K e^{rj\Delta K/k} .$$

Note that $p^+ + p^o + p^- = 1$.[2] The terminal boundary condition becomes:

$$G_{i,j=0} = e^{i\Delta X} ; \qquad \text{(a-10)}$$

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[2]See Brennan and Schwartz (1978) for a discussion of an interpretation of such difference equations based on a jump process with probabilities p^+, p^o, and p^-.

The upper boundary condition becomes:

$$\lim_{X\to\infty} \left[e^{-X}e^{-rK/k}G_X(X, K)\right] = e^{-\delta K/k} ,$$

or,
$$G_X(m\Delta X, K) = e^{m\Delta X}e^{(r-\delta)j\Delta K/k} .$$

Using the finite difference approximation (a-7) for G_X, we get:

$$[G_{m+1,j} - G_{m-1,j}]/2\Delta X = e^{m\Delta X}e^{(r-\delta)j\Delta K/k},$$

or,
$$G_{m+1,j} = (2\Delta X e^{m\Delta X}e^{(r-\delta)j\Delta K/k}) + G_{m-1,j} .$$

Substitute for $G_{m+1,j}$ in equation a-9 (setting i = m):

$$G_{m,j+1} = p^{+}G_{m+1,j} + p^{o}G_{m,j} + p^{-}G_{m-1,j} - \eta_j ,$$

or, $$G_{m,j+1} = p^{+}(2\Delta X e^{m\Delta X}e^{(r-\delta)j\Delta K/k}) + p^{o}G_{m,j} + (p^{+} + p^{-})G_{m-1,j} - \eta_j . \quad \text{(a-11)}$$

Finally, the free boundary condition becomes:

$$G_{i=i*,j} = G_{i=i*+1,j}/(\alpha\Delta X + 1) . \quad \text{(a-12)}$$

The solution proceeds as a backward dynamic program. This is illustrated in Figure A.1. First, the values of G at the terminal boundary (j = 0) are filled in using equation a-10. Stepping back to j = 1, equation a-11 is used to calculate $G_{i=m,j=1}$, and equation a-9 is used to calculate the values of G for i = m - 1, m - 2, etc. Each time a value for G is calculated, we use equation a-12 to check to see if the free boundary has been reached. However, due to discretisation error, equation a-12 is unlikely to hold exactly at any point, so we check only to see if it holds to within a specified bound using:

$$G_{i*,j} - G_{i*+1,j}/(\alpha\Delta X + 1) \leqslant \varepsilon , \quad \text{(a-13)}$$

where ε is chosen arbitrarily to be $\Delta X/2$. Once our check identifies the free boundary, we use equation 4 (in the text) to infer the value of the coefficient a, and fill in the values of G below the boundary using the analytic solution.

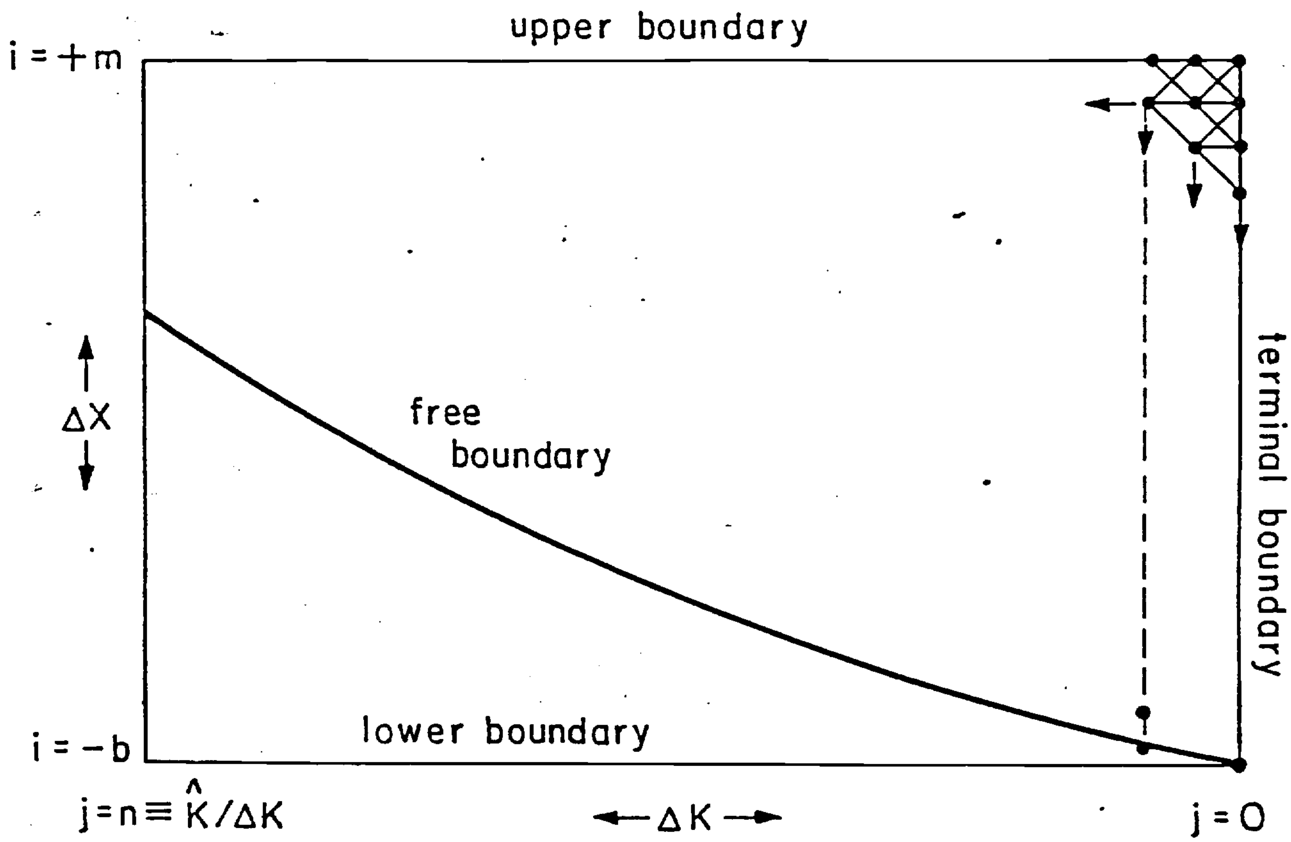

Figure A.1

Procedure:

1. Fill in terminal boundary ($j=0$) using equation a-10.

2. For $j = 1$ to $j = n$:

   a. For $i = m$, use equation a-11 to calculate $G_{m,j}$;

   b. for $i < m$, moving down the column, use equation a-9 to calculate $G_{i,j}$;

3. At the free boundary, calculate the value of the coefficient, a, and use equation 4 to fill in the values of $G_{i,j}$ in the lower region.

FIGURE 1: V* AS FUNCTION OF k

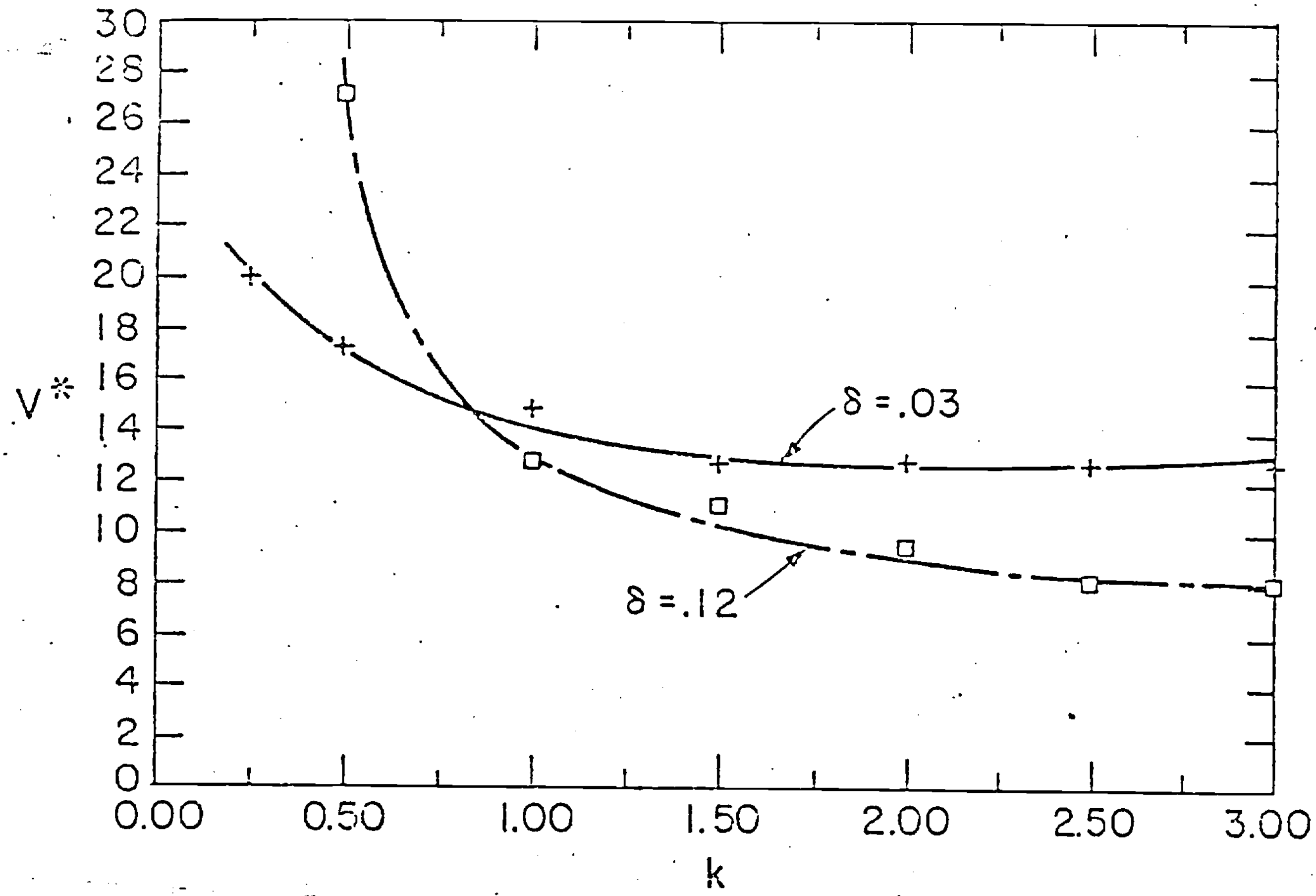

FIGURE 2: V* AS FUNCTION OF DELTA

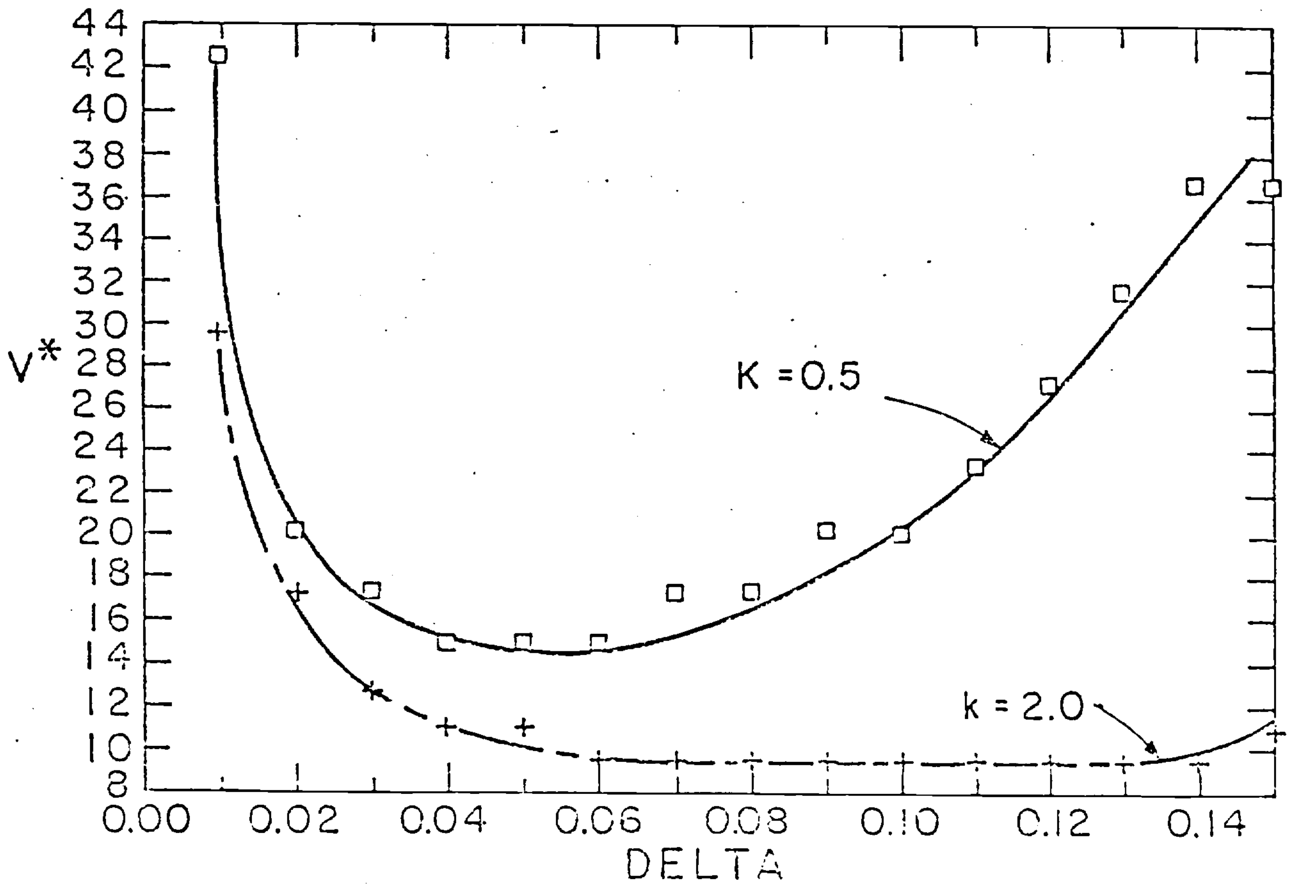

FIGURE 3: VALUE OF FLEXIBILITY