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ECONOMIC GROWTH WITH BUBBLES

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ABSTRACT

We develop a stylized model of economic growth with bubbles. In this model, financial frictions lead to equilibrium dispersion in the rates of return to investment. During bubbly episodes, unproductive investors demand bubbles while productive investors supply them. Because of this, bubbly episodes channel resources towards productive investment raising the growth rates of capital and output. The model also illustrates that the existence of bubbly episodes requires some investment to be dynamically inefficient: otherwise, there would be no demand for bubbles. This dynamic inefficiency, however, might be generated by an expansionary episode itself.

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1 Introduction

Modern economies often experience episodes of large movements in asset prices that cannot be explained by changes in economic conditions or fundamentals. It is commonplace to refer to these episodes as bubbles popping up and bursting. Typically, these bubbles are unpredictable and generate substantial macroeconomic effects. Consumption, investment and productivity growth all tend to surge when a bubble pops up, and then collapse or stagnate when the bubble bursts. Here, we address the following questions: What is the origin of these bubbly episodes? Why are they unpredictable? How do bubbles affect consumption, investment and productivity growth? In a nutshell, the goal of this paper is to develop a stylized view or model of economic growth with bubbles.

The theory presented here features two idealized asset classes: productive assets or "capital" and pyramid schemes or "bubbles". Both assets are used as a store of value or savings vehicle, but they have different characteristics. Capital is costly to produce but it is then useful in production. Bubbles play no role in production, but initiating them is costless.[1] We consider environments with rational, informed and risk neutral investors that hold only those assets that offer the highest expected return. The theoretical challenge is to identify situations in which these investors optimally choose to hold bubbles in their portfolios and then characterize the macroeconomic consequences of their choice.

Our research builds on the seminal papers of Samuelson (1958) and Tirole (1985) who viewed bubbles as a remedy to the problem of dynamic inefficiency. Their argument is based on the dual role of capital as a productive asset and a store of value. To satisfy the need for a store of value, economies sometimes accumulate so much capital that the investment required to sustain it exceeds the income that it produces. This investment is inefficient and lowers the resources available for consumption. In this situation, bubbles can be both attractive to investors and feasible from a macroeconomic perspective. For instance, a pyramid scheme that absorbs all inefficient investments in each period is feasible and its return exceeds that of the investments it replaces. This explains the origins and the effects of bubbles. Since bubbles do not have intrinsic value, their size depends on the market's expectation of their future size. In a world of rational investors, this opens the door for

[1] It is difficult to find these idealized asset classes in financial markets, of course, as existing assets bundle or package together capital and bubbles. Yet we think that much can be learned by working with these basic assets. To provide a simple analogy, we have surely learned much by studying theoretical economies with a full set of Arrow-Debreu securities, even though only a few bundles or packages of these basic securities are traded in the real world.

self-fulfilling expectations to play a role in bubble dynamics and accounts for their unpredictability.

The Samuelson-Tirole model provides an elegant and powerful framework to think about bubbles. However, the picture that emerges from this theory is hard to reconcile with historical evidence. In this model, bubbly episodes are consumption booms financed by a reduction in inefficient investments. During these episodes both the capital stock and output contract. In the real world, bubbly episodes tend to be associated with consumption booms indeed. But they also tend to be associated with expansions in both the capital stock and output. A successful model of bubbles must come to grips with these correlations.

This paper shows how to build such a model by extending the theory of rational bubbles to the case of imperfect financial markets. In the Samuelson-Tirole model, frictionless financial markets eliminate rate-of-return differentials among investments making them either all efficient or all inefficient. Introducing financial frictions is crucial because these create rate-of-return differentials and allow efficient and inefficient investments to coexist. Our key observation is quite simple: bubbles not only reduce inefficient investments, but they also increase efficient ones. In our model, bubbly episodes are booms in consumption *and* efficient investments financed by a reduction in inefficient investments. If the increase in efficient investments is sizable enough, bubbly episodes expand the capital stock and output. This turns out to be the case under a wide range of parameter values.

To understand these effects of bubbly episodes, it is useful to analyze the set of transfers that bubbles implement. Remember that a bubble is nothing but a pyramid scheme by which the buyer surrenders resources today expecting that future buyers will surrender resources to him/her. The economy enters each period with an initial distribution of bubble owners. Some of these owners bought their bubbles in earlier periods, while others just created them. When the market for bubbles opens, on the demand side we find investors who cannot obtain a return to investment above that of bubbles; while on the supply side we find consumers and investors who can obtain a return to investment above that of bubbles. When the market for bubbles closes, resources have been transferred from inefficient investors to consumers and efficient investors, leading to an increase in consumption and efficient investments at the expense of inefficient investments.

A key aspect of the theory is how the distribution of bubble owners is determined. As in the Samuelson-Tirole model, our economy is populated by overlapping generations that live for two periods. The young invest and the old consume. The economy enters each period with two types of bubble owners: the old who acquired bubbles during their youth, and the young who are lucky enough to create them. Since the old only consume, bubble creation by efficient young investors plays a crucial role in our model: it allows them to finance additional investment by selling bubbles.

Introducing financial frictions also solves a nagging problem of the theory of rational bubbles, which was first pointed out by Abel et al. (1989). In the Samuelson-Tirole model, bubbles can only exist if the investment required to sustain the capital stock exceeds the income that it produces. Abel et al. (1989) examined a group of developed economies and found that, in all of them, investment falls short of capital income. This finding has often been considered evidence that rational bubbles cannot exist in real economies. Introducing financial frictions into the theory shows that this conclusion is unwarranted. The observation that capital income exceeds investment only implies that, on average, investments are dynamically efficient. But this does not exclude the possibility that the economy contains pockets of dynamically inefficient investments that could support a bubble. Nor does it exclude the possibility that an expansionary bubble, by lowering the return to investment, creates itself the pockets of dynamically inefficient investments that support it. In such situations, the test of Abel et al. would wrongly conclude that bubbles are not possible.

Besides building on the seminal contributions of Samuelson (1958) and Tirole (1985), this paper is closely related to previous work on bubbles and economic growth. Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) extend the Samuelson-Tirole model to economies with endogenous growth due to externalities in capital accumulation. In their models, bubbles slow down the growth rate of the economy. Olivier (2000) uses a similar model to show how, if tied to R&D firms, bubbles might foster technological progress and growth.

The model developed in this paper stresses the relationship between bubbles and frictions in financial markets. Azariadis and Smith (1993) were, to the best of our knowledge, the first to show that contracting frictions could relax the conditions for the existence of rational bubbles. More recently, Caballero and Krishnamurthy (2006), Kraay and Ventura (2007), and Farhi and Tirole (2009) contain models in which the existence and economic effects of rational bubbles are closely linked to financial frictions. Whereas we derive our results in a standard growth model, these papers study economies with linear production functions or storage technology.

The paper is organized as follows: Section 2 presents the Samuelson-Tirole model, provides conditions for the existence of equilibrium bubbles and discusses their macroeconomic effects. Section 3 introduces financial frictions and contains the main results of the paper. Section 4 extends these results to an economy with long-run growth. Finally, Section 5 concludes.

2 The Samuelson-Tirole model

Samuelson (1958) and Tirole (1985) showed that bubbles are possible in economies that are dynamically inefficient, i.e. that accumulate too much capital. Bubbles crowd out capital and raise the return to investment. We re-formulate this theory in terms of bubbly episodes and provide a quick refresher of its macroeconomic implications.

2.1 Basic setup

Consider a country inhabited by overlapping generations of young and old, all with size one. Time starts at $t = 0$ and then goes on forever. All generations maximize the expected consumption when old: $U_t = E_t c_{t+1}$; where U_t and c_{t+1} are the welfare and the old-age consumption of generation t.

The output the country is given by a Cobb-Douglas production function of labor and capital: $F(l_t, k_t) = l_t^{1-\alpha} \cdot k_t^{\alpha}$ with $\alpha \in (0, 1)$, and l_t and k_t are the country's labor force and capital stock, respectively. All generations have one unit of labor which they supply inelastically when they are young, i.e. $l_t = 1$. The stock of capital in period $t + 1$ equals the investment made by generation t during its youth.[2] This means that:

$$k_{t+1} = s_t \cdot k_t^{\alpha}, \tag{1}$$

where s_t is the investment rate, i.e. the fraction of output that is devoted to capital formation. Markets are competitive and factors of production are paid the value of their marginal product:

$$w_t = (1 - \alpha) \cdot k_t^{\alpha} \quad \text{and} \quad r_t = \alpha \cdot k_t^{\alpha-1}, \tag{2}$$

where w_t and r_t are the wage and the rental rate, respectively.

To solve the model, we need to find the investment rate. The old do not save and the young save all their income. What do the young do with their savings? At this point, it is customary to assume that they use them to build capital. This means that the investment rate equals the savings of the young. Since the latter equal labor income, which is a constant fraction $1 - \alpha$ of output, the investment rate is constant as in the classic Solow (1956) model:

$$s_t = 1 - \alpha. \tag{3}$$

[2] That is, we assume that (i) producing one unit of capital requires one unit of consumption, and that (ii) capital fully depreciates in production. We also assume that the first generation found some positive amount of capital to work with, i.e. $k_0 > 0$.

Therefore, the law of motion of the capital stock is:

$$k_{t+1} = (1 - \alpha) \cdot k_t^{\alpha}. \tag{4}$$

Equation (4) constitutes a very stylized version of a standard workhorse of modern macroeconomics. A lot of progress has been made by adding more sophisticated formulations of preferences and technology, various types of shocks, a few market imperfections, and a role for money. We shall not do any of this here though.

2.2 Equilibria with bubbles

Instead, we follow the path-breaking work of Samuelson (1958) and Tirole (1985), and assume the young have the additional option of purchasing bubbles or pyramid schemes. These are intrinsically useless assets, and the only reason to purchase them is to resell them later. Let b_t be the stock of old bubbles in period t, i.e. already existing before period t or created by earlier generations; and let b_t^N be the stock of new bubbles, i.e. added in period t or created by generation t. We assume that there is free disposal of bubbles. This implies that $b_t \geq 0$ and $b_t^N \geq 0$. We also assume that bubbles are created randomly and without cost. This implies that new bubbles constitute a pure profit or rent for those that create them.[3]

Rationality imposes two restrictions on the type of bubbles that can exist. First, bubbles must grow fast enough or otherwise the young will not be willing to purchase them. Second, the aggregate bubble cannot grow too fast or otherwise the young will not be able to purchase them. Therefore, if $b_t > 0$, then

$$E_t \left\{ \frac{b_{t+1}}{b_t + b_t^N} \right\} = \alpha \cdot k_{t+1}^{\alpha - 1}, \tag{5}$$

$$b_t \leq (1 - \alpha) \cdot k_t^{\alpha}. \tag{6}$$

Equation (5) says that, for bubbles to be attractive, they must deliver the same return as capital.[4] The return to the bubble consists of its growth over the holding period. The purchase price of the bubble is $b_t + b_t^N$, and the selling price is b_{t+1}. The return to capital equals the rental rate since each

[3] Note that new bubbles cannot be linked to the ownership of objects that can be traded before the date the bubbles appear. Otherwise the bubble would have already apppeared in the first date in which the object can be traded. The reason is simple: a rational individual would be willing to pay a positive price for an object if there is some probability that this object commands a positive price in the future. See Diba and Grossman (1987).

[4] We can rule out the possibility that bubbles deliver a higher return than capital. Asssume not and let the return to the bubble to be strictly higher than the return to capital. Then, nobody would invest and the return to capital would be infinity. But this means that the bubble must grow at a rate infinity and this is not possible.

unit of capital costs one unit of consumption and it fully depreciates in one period. Equation (6) says that, for bubbles to be feasible, they cannot outgrow the economy's savings. The savings of the young consist of labor income and the value of new bubbles created by them, i.e. $(1-\alpha)\cdot k_t^{\alpha}+b_t^N$. Since the old do not save, the young must be purchasing the whole aggregate bubble, i.e. $b_t+b_t^N$.

The investment rate equals the income left after purchasing the bubbles as a share of output:

$$s_t = 1-\alpha-\frac{b_t}{k_t^{\alpha}}, \tag{7}$$

and this implies the following law of motion for the capital stock:

$$k_{t+1}=(1-\alpha)\cdot k_t^{\alpha}-b_t. \tag{8}$$

Equation (8) shows the key feature of the Samuelson-Tirole model: bubbles crowd out investment and slow down capital accumulation.

For a given initial capital stock and bubble, $k_0>0$ and $b_0\geq 0$, a competitive equilibrium is a sequence $\left\{k_t,b_t,b_t^N\right\}_{t=0}^{\infty}$ satisfying Equations (5), (6) and (8). The assumption that the young only build capital is equivalent to adding the additional equilibrium restriction that $b_t=b_t^N=0$ for all t. This restriction cannot be justified on 'a priori' grounds, but we note that there always exists one equilibrium in which it is satisfied.[5]

There are a couple of important differences between the model described here and the original ones of Samuelson (1958) and Tirole (1985). Unlike us, Samuelson analyzed an economy with a linear production function or storage technology. Tirole analyzed instead a standard growth model like the one we study. Unlike us, however, he made weak assumptions on preferences and technology. In particular, he only assumed the existence of utility and production functions, $U_t=u\left(c_t,c_{t+1}\right)$ and $F\left(l_t,k_t\right)$ with standard properties. Unlike us, both Samuelson and Tirole restricted the analysis to the subset of equilibria that are deterministic and do not involve bubble creation or destruction. That is, they imposed the additional restrictions that $E_t b_{t+1}=b_{t+1}$ and $b_t^N=0$ for all t.[6,7] Despite these differences, we label the model described above as the Samuelson-Tirole model to give due credit to their seminal contributions.

[5] This equilibrium might not exist in the presence of rents. See Tirole (1985) and Caballero et al. (2010).

[6] Under these restrictions, any bubble must have existed from the very beginning of time and it can never burst, i.e. its value can never be zero.

[7] To the best of our knowledge, Weil (1987) was the first to consider stochastic bubbles in general equilibrium.

2.3 Bubbly episodes

An important payoff of analyzing stochastic equilibria with bubble creation and destruction is that this allows us to rigorously capture the notion of a bubbly episode. Generically, the economy fluctuates between periods in which $b_t = b_t^N = 0$ and periods in which $b_t > 0$ and/or $b_t^N > 0$. We say that the economy is in the *fundamental state* if $b_t = b_t^N = 0$. We say instead that the economy is experiencing a *bubbly episode* if $b_t > 0$ and/or $b_t^N > 0$. A bubbly episode starts when the economy leaves the fundamental state and ends the first period in which the economy returns to the fundamental state.

The following proposition provides the conditions for the existence of bubbly episodes in the Samuelson-Tirole model:

Proposition 1 *Bubbly episodes are possible if and only if $\alpha < 0.5$.*

The proof of this proposition exploits a useful trick that makes the model recursive. Let x_t be the aggregate bubble as a share of the labor income, i.e. $x_t \equiv \dfrac{b_t}{(1-\alpha)\cdot k_t^{\alpha}}$ and $x_t^N \equiv \dfrac{b_t^N}{(1-\alpha)\cdot k_t^{\alpha}}$. Then, we can rewrite Equations (5) and (6) as saying that if $x_t > 0$, then

$$E_t x_{t+1} = \frac{\alpha}{1-\alpha} \cdot \frac{x_t + x_t^N}{1-x_t}, \tag{9}$$

$$x_t \leq 1. \tag{10}$$

Equations (9) and (10) describe bubble dynamics. There are two sources of randomness in these dynamics: shocks to bubble creation, i.e. x_t^N; and shocks to the value of the existing bubble, i.e. x_t. Any admissible stochastic process for x_t^N and x_t satisfying Equations (9) and (10) is an equilibrium of the model. By admissible, we mean that the stochastic process must ensure that $x_t \geq 0$ and $x_t^N \geq 0$ for all t. Conversely, any equilibrium of the model can be expressed as an admissible stochastic process for x_t^N and x_t.

To prove Proposition 1 we ask if, among all stochastic processes for x_t^N and x_t that satisfy Equation (9), there is at least one that also satisfies Equation (10). It is useful to examine first the case in which there is no bubble creation after a bubbly episodes starts. Figure 1 plots $E_t x_{t+1}$ against x_t, using Equation (9) with $x_t^N = x_s^N$ and $x_t^N = 0$ for all $t > s$, where s is the period in which the episode starts. The left panel shows the case in which $\alpha \geq 0.5$ and the slope of $E_t x_{t+1}$ at the origin is greater than or equal to one. Any initial bubble would be demanded only if it were expected to continuously grow as a share of labor income, i.e. $E_t x_{t+1} > x_t$ in all periods. But this means that in some scenarios the bubble outgrows the savings of the young in finite time, i.e. it violates Equation (10). Therefore, bubbly episodes cannot happen if $\alpha \geq 0.5$. The right panel of Figure 1 shows instead the case in which $\alpha < 0.5$. Any initial bubble $x_{s+1} > \dfrac{1 - 2 \cdot \alpha}{1 - \alpha}$ can be ruled out with the same argument. But any initial bubble $x_{s+1} \leq \dfrac{1 - 2 \cdot \alpha}{1 - \alpha}$ can be part of an equilibrium as it is possible to find a process for x_t that satisfies Equations (9) and (10) simultaneously.

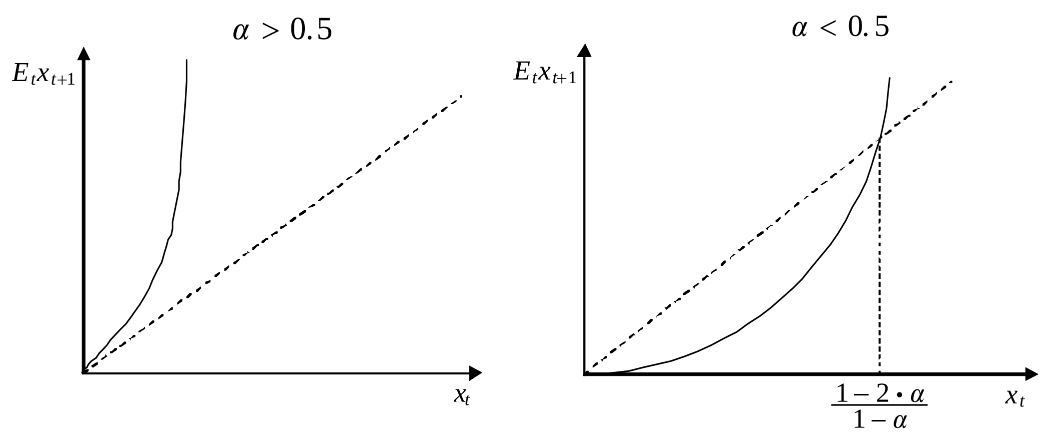

Figure 1

Allowing for bubble creation does not relax the conditions for existence of bubbly episodes. To see this, note that bubble creation shifts upwards the schedule $E_t x_{t+1}$ in Figure 1. The intuition is clear: new bubbles compete with old bubbles for the income of next period's young, reducing their return and making them less attractive. This completes the proof of Proposition 1.

This proof is instructive and helps us understand the connection between bubbles and dynamic inefficiency. To determine whether a bubbly episodes can exist, we have asked: Is it possible to construct a pyramid scheme that is attractive without exploding as a share of labor income? We have seen that the answer is affirmative if and only if there exist stochastic processes for x_t^N and x_t such that:

$$E_t x_{t+1} < x_t.$$

These processes exist if and only if $\alpha < 0.5$.

To determine whether the economy is dynamically inefficient, we ask: Is the economy accumulating too much capital? The answer is affirmative if and only if the investment required to sustain the capital stock exceeds the income that this capital produces. Investment equals $(1-x_t)\cdot(1-\alpha)\cdot k_t^\alpha$, while capital income is given by $\alpha \cdot k_t^\alpha$. Therefore, the economy is dynamically inefficient if and only if

$$(1-x_t)\cdot(1-\alpha)\cdot k_t^\alpha > \alpha \cdot k_t^\alpha. \tag{11}$$

Straightforward algebra shows that this condition is equivalent to asking whether there exist stochastic processes for for x_t^N and x_t such that:

$$E_t x_{t+1} < x_t + x_t^N.$$

These processes exist if and only if $\alpha < 0.5$. Therefore, in the Samuelson-Tirole model the conditions for the existence of bubbly episodes and dynamic inefficiency coincide.

2.4 The macroeconomic effects of bubbles

To determine the macroeconomic consequences of bubbly episodes, we rewrite the law of motion of the capital stock using the definition of x_t:

$$k_{t+1} = (1-x_t)\cdot(1-\alpha)\cdot k_t^\alpha. \tag{12}$$

Equation (12) describes the dynamics of the capital stock for any admissible stochastic process for the bubble, i.e. x_t^N and x_t; satisfying Equations (9) and (10). This constitutes a full solution to the model.

Interestingly, bubbly episodes can be literally interpreted as shocks to the law of motion of the capital stock of the Solow model. To better understand the nature of these shocks, consider the following example:

Example 1 ((n,p) episodes) *Consider the subset of bubbly episodes that are characterized by (i) a constant probability of ending, i.e.* $\Pr_t(b_{t+1}=0\,|b_t>0) = p$ *and (ii) an initial bubble* x_s^N *and then a constant rate of new-bubble creation, i.e.* $x_t^N = n\cdot x_t$.

The left panel of Figure 2 shows a (n,p) episode. The solid line represents Equation (9), i.e. the value of x_{t+1} that leaves the young indifferent between buying the bubble or investing in capital. A feature of (n,p) episodes is that the bubble declines as a share of labor income throughout the episode, i.e. $x_{t+1}\le x_t$ for all t and $x_t \to 0$. Only if the initial bubble is maximal, i.e. $x_{s+1}\to x_1^*$, this rate of decline becomes zero. This pattern of behavior is not generic, however, as the following example shows:

Example 2 ((x^N, p) episodes) *Consider the subset of bubbly episodes that are characterized by (i) a constant probability of ending, i.e.* $\Pr_t(b_{t+1} = 0 \,|b_t > 0) = p$ *and (ii) an initial bubble* x_s^N *and then a constant amount of bubble creation* $x_t^N = x^N$*.*

The right panel of Figure 2 shows an (x^N, p) episode. Any initial bubble converges to x_2^*. If $x_{s+1} < x_2^*$, the bubble grows throughout the episode. If $x_{s+1} > x_2^*$, the bubble declines throughout the episode. Once again, if the initial bubble is maximal, i.e. $x_{s+1} \to x_2^{**}$, this rate of decline becomes zero and the bubble never converges to x_2^*.

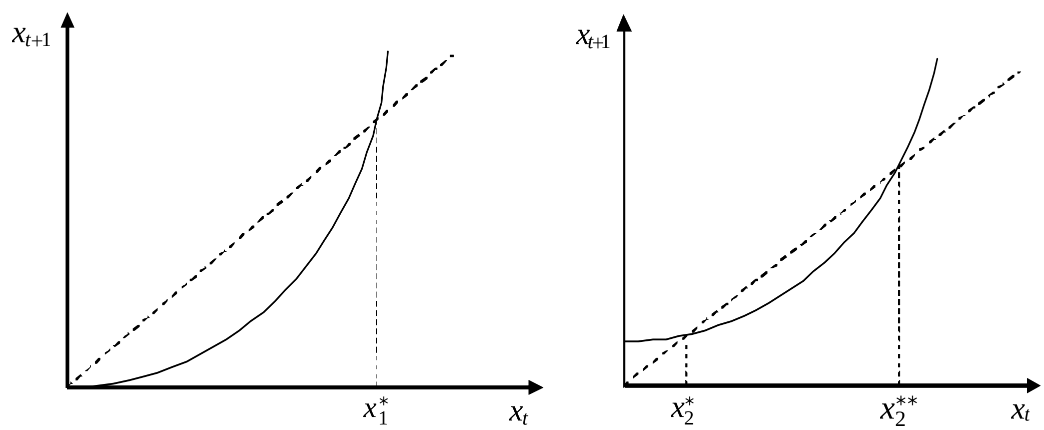

Figure 2

The only randomness in these examples refers to the periods in which they start and end. Throughout the bubbly episode, the bubble moves deterministically until the episodes ends. This need not be, of course. Assume, for instance, that bubble creation randomly switches between being a fraction of the existing bubble, i.e. $x_t^N = n \cdot x_t$; and being a constant amount, i.e. $x_t^N = x^N$. That is, the laws of motion of the bubble in the right and left panels of Figure 2 operate at different (and random) times during a given bubbly episode. Then, x_t will converge to the interval $(0, x_2^*)$, and then randomly fluctuate within it until the episode ends.

Figure 3 shows the macroeconomic effects of one of these bubbly episodes. Assume initially that the economy is in the fundamental state so that the appropriate law of motion is the one labeled k_{t+1}^F. Since the initial capital stock is below the Solow steady state, i.e. $k_t < k^F \equiv (1-\alpha)^{\frac{1}{1-\alpha}}$, the economy is growing at a positive rate. When a bubbly episode starts, the investment rate falls and the law of motion shifts below the fundamental one. In the figure, k_{t+1}^B represents the law of motion when the bubbly episode begins. The picture has been drawn so that the capital stock is above the steady state associated to k_{t+1}^B, i.e. $k_t > k^B$. As a result, growth turns negative. Throughout the episode, k_{t+1}^B may shift up or down as the bubble grows or shrinks, although it always remains below the original law of motion k_{t+1}^F. Eventually, the episode ends and the economy returns to k_{t+1}^F.

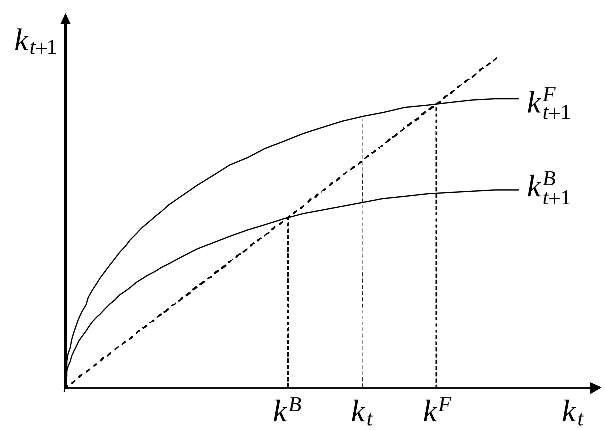

Figure 3

At first sight, one could think of bubbly episodes as akin to negative shocks to the investment rate. But this would not be quite right. Bubbles also affect consumption directly as passing the bubble across generations increases the share of output that the old receive and consume:

$$c_t = [\alpha + (1 - \alpha) \cdot x_t] \cdot k_t^{\alpha}. \tag{13}$$

The relationship between bubbles and consumption has therefore two different aspects to it. Past bubbles reduce the capital stock and, ceteris paribus, this lowers consumption. But present bubbles raise the share of output in the hands of the old and, ceteris paribus, this raises consumption.

2.5 Discussion

Bubbles affect allocations through two channels: (i) by implementing a set of intergenerational transfers, and (ii) by creating wealth shocks. The first channel is a central feature of a pyramid scheme, by which buyers surrender resources today expecting future buyers to surrender resources to them. These intergenerational transfers are feasible because the economy is dynamically inefficient. The second channel are the wealth shocks associated with bubble creation and destruction. When bubbles appear, those lucky individuals who create them receive a windfall or transfer from the future. This is another central feature of a pyramid scheme whereby the initiator claims that, by making him/her a payment now, the other party earns the right to receive a payment from a third person later. By successfully creating and selling a bubble, young individuals have assigned themselves and sold the "rights" to the income of a generation living in the very far future or, to be more exact, living at infinity. This appropriation of rights is a pure windfall or positive wealth shock for the generation that creates them. Naturally, the opposite happens when bubbles burst since this constitutes a negative wealth shock for those that are holding them and see their value collapse.

Once extended to allow for random bubbles and random bubble creation and destruction, the Samuelson-Tirole model provides an elegant and powerful framework to think about bubbly episodes. Unfortunately, the macroeconomic implications of the Samuelson-Tirole model are at odds with the facts along two key dimensions:

1. The model predicts that bubbles can only appear in dynamically inefficient economies, i.e. $\alpha \leq 0.5$. However, Abel et al. (1989) examined a group of developed economies and found that, in all of them, aggregate investment, i.e. $(1 - x_t) \cdot (1 - \alpha) \cdot k_t^{\alpha}$, falls short of aggregate capital income, i.e. $\alpha \cdot k_t^{\alpha}$.

2. The model predicts that bubbles lead to simultaneous drops in the stock of capital and output. Historical evidence suggests however that bubbly episodes are associated with increases in the capital stock and output.

We next show that these discrepancies between the theory and the facts rest on one important assumption: financial markets are frictionless.

3 Introducing financial frictions

We extend the model by introducing a motive for intragenerational trade and a financial friction that impedes this trade. We show that this relaxes the conditions for the existence of bubbly episodes. Moreover, these episodes can lower the return to investment and lead to expansions in the capital stock.

3.1 Setup with financial frictions

Assume that a fraction $\varepsilon \in [0, 1]$ of the young of each generation can produce one unit of capital with one unit of the consumption good. We refer to them as "productive" investors. The remaining young are "unproductive" investors, as they only have access to an inferior technology that produces $\delta < 1$ units of capital with one unit of the consumption good. This heterogeneity creates gains from borrowing and lending. If markets worked well, unproductive investors would lend their resources to productive ones and these would invest on everyone's behalf. This would bring us back to the Samuelson-Tirole model. We shall however assume that this is not possible because of some unspecified market imperfection. The goal here is to analyze how this financial friction affects equilibrium outcomes.

Now the evolution of the capital stock depends not only on the level of investment but also on its composition. Let A_t be the average efficiency of investment. Then, Equation (1) must be replaced by

$$k_{t+1} = s_t \cdot A_t \cdot k_t^{\alpha}. \tag{14}$$

For instance, in the benchmark case in which the young use all their savings to build capital we have that:

$$A_t = \varepsilon + (1 - \varepsilon) \cdot \delta \equiv A. \tag{15}$$

Since all individuals invest the same amount, the average efficiency of investment is determined by the population weights of both types of investors. The investment rate is still determined by Equation (3) and the dynamics of the capital stock are given by

$$k_{t+1} = (1 - \alpha) \cdot A \cdot k_t^{\alpha}. \tag{16}$$

Since $A < 1$, financial frictions lower the level of the capital stock but they do not affect the nature of its dynamics. This result does not go through once we allow for bubbles.

3.2 Equilibria with bubbles

The introduction of financial frictions forces us to make an assumption about the distribution of rents from bubble creation. In the Samuelson-Tirole model, all investment is carried out by productive investors and the distribution of rents is inconsequential. With financial frictions, this is no longer the case since the distribution of wealth – and hence of these rents – affects the average efficiency of investment. We use b_t^{NP} and b_t^{NU} to denote the stock of new bubbles created by productive and unproductive investors, respectively. Naturally, $b_t^{NP} + b_t^{NU} = b_t^N$.

Recall that rationality imposes two restrictions on the type of bubbles that can exist. First, bubbles must grow fast enough or otherwise the young will not be willing to purchase them. Second, the aggregate bubble cannot grow too fast or otherwise the young will not be able to purchase them. While the second of these restrictions still implies Equation (6), the first of them now implies that if $b_t > 0$, then

$$E_t\left\{\frac{b_{t+1}}{b_t + b_t^{NP} + b_t^{NU}}\right\}\begin{cases} = \delta \cdot \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } \dfrac{b_t + b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha}} < 1 \\ \in \left[\delta \cdot \alpha \cdot k_{t+1}^{\alpha-1}, \alpha \cdot k_{t+1}^{\alpha-1}\right] & \text{if } \dfrac{b_t + b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha}} = 1 \\ = \alpha \cdot k_{t+1}^{\alpha-1} & \text{if } \dfrac{b_t + b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha}} > 1 \end{cases}. \tag{17}$$

Equation (17) is nothing but a generalization of Equation (5) that recognizes that the marginal buyer of the bubble changes as the bubble grows. If the bubble is small, the marginal buyer is an unproductive investor and the expected return to the bubble must equal the return to unproductive investments. If the bubble is large, the marginal buyer is an productive investor and the expected return to the bubble must be the return to productive investments.

Bubbles affect both the level of investment and its composition. As in the Samuelson-Tirole model, the bubble reduces the investment rate and Equation (7) still holds. Unlike the Samuelson-Tirole model, the bubble now affects the average efficiency of investment as follows:

$$A_t = \begin{cases} \dfrac{(1-\alpha)\cdot A \cdot k_t^{\alpha} + (1-\delta)\cdot b_t^{NP} - \delta \cdot b_t}{(1-\alpha)\cdot k_t^{\alpha} - b_t} & \text{if } \dfrac{b_t + b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha}} < 1 \\ 1 & \text{if } \dfrac{b_t + b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha}} \geq 1 \end{cases}. \tag{18}$$

To understand Equation (18), note first that in the fundamental state $b_t = b_t^{NP} = b_t^{NU} = 0$ and the average efficiency of investment equals the population average A. Bubbles raise the efficiency of investment through two channels. First, existing bubbles displace a disproportionately high share of unproductive investments. This is why A_t is increasing in b_t. Second, bubble creation by productive investors raises their income and expands their investment. This is why A_t is also increasing in b_t^{NP}. When all unproductive investments have been eliminated, the average efficiency of investment reaches one.

We can thus re-write the dynamics of the capital stock as follows:

$$k_{t+1} = \begin{cases} (1-\alpha)\cdot A\cdot k_t^{\alpha} + (1-\delta)\cdot b_t^{NP} - \delta\cdot b_t & \text{if } \dfrac{b_t + b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha}} < 1 \\ (1-\alpha)\cdot k_t^{\alpha} - b_t & \text{if } \dfrac{b_t + b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha}} \geq 1 \end{cases}. \tag{19}$$

Bubbles now have conflicting effects on capital accumulation and output. On the one hand, existing bubbles reduce the investment rate. On the other hand, new bubbles raise the efficiency of investment. If the first effect dominates, i.e. $b_t^{NP} < \dfrac{\delta}{1-\delta}\cdot b_t$, bubbles are contractionary and crowd out capital. If instead the second effect dominates, i.e. $b_t^{NP} > \dfrac{\delta}{1-\delta}\cdot b_t$, bubbles are expansionary and crowd in capital.

We are ready to define a competitive equilibrium for the modified model. For a given initial capital stock and bubble, $k_0 > 0$ and $b_0 \geq 0$, a competitive equilibrium is a sequence $\left\{k_t, b_t, b_t^{NP}, b_t^{NU}\right\}_{t=0}^{\infty}$ satisfying Equations (6), (17) and (19). As we show next, there are many such equilibria.

3.3 Bubbly episodes with financial frictions

The following proposition provides the conditions for the existence of bubbly episodes in the model with financial frictions:

Proposition 2 *Bubbly episodes are possible if and only if:*

$$\alpha < \begin{cases} \dfrac{A}{A+\delta} & \text{if } A > 1-\varepsilon \\ \max\left\{\dfrac{A}{A+\delta}, \dfrac{1}{1+4\cdot(1-\varepsilon)\cdot\delta}\right\} & \text{if } A \leq 1-\varepsilon \end{cases}.$$

Proposition 2 generalizes Proposition 1 to the case of financial frictions. Once again, we use the trick of making the model recursive through a change of variables. Define now $x_t^{NP} \equiv \dfrac{b_t^{NP}}{(1-\alpha)\cdot k_t^{\alpha}}$ and $x_t^{NU} \equiv \dfrac{b_t^{NU}}{(1-\alpha)\cdot k_t^{\alpha}}$. Then, we can rewrite Equations (6) and (17) as saying that if $x_t > 0$, then

$$E_t x_{t+1} \begin{cases} = \dfrac{\alpha}{1-\alpha} \cdot \dfrac{\delta \cdot \left(x_t + x_t^{NP} + x_t^{NU}\right)}{A + (1-\delta) \cdot x_t^{NP} - \delta \cdot x_t} & \text{if } \dfrac{x_t + x_t^{NP}}{1-\varepsilon} < 1 \\ \in \left[\dfrac{\alpha}{1-\alpha} \cdot \dfrac{\delta \cdot \left(x_t + x_t^{NP} + x_t^{NU}\right)}{A + (1-\delta) \cdot x_t^{NP} - \delta \cdot x_t}, \dfrac{\alpha}{1-\alpha} \cdot \dfrac{x_t + x_t^{NU} + x_t^{NP}}{1 - x_t} \right] & \text{if } \dfrac{x_t + x_t^{NP}}{1-\varepsilon} = 1 \\ = \dfrac{\alpha}{1-\alpha} \cdot \dfrac{x_t + x_t^{NP} + x_t^{NU}}{1 - x_t} & \text{if } \dfrac{x_t + x_t^{NP}}{1-\varepsilon} > 1 \end{cases}, \tag{20}$$

$$x_t \leq 1. \tag{21}$$

Equations (20) and (21) describe bubble dynamics in the modified model. Any admissible stochastic process for x_t^{NP}, x_t^{NU} and x_t satisfying Equations (20) and (21) is an equilibrium of the model. Conversely, any equilibrium of the model can be expressed as an admissible stochastic process for x_t^{NP}, x_t^{NU} and x_t.

To prove Proposition 2 we ask again if, among all stochastic processes for x_t^{NP}, x_t^{NU} and x_t that satisfy Equation (20), there is at least one that also satisfies Equation (21). Consider first the case in which there is no bubble creation after a bubbly episode starts. Figure 4 plots $E_t x_{t+1}$ against x_t, using Equation (20) with $x_t^{NP} = x_s^{NP}$, $x_t^{NU} = x_s^{NU}$ and $x_t^{NP} = x_t^{NU} = 0$ for all $t > s$, where s is once again the period in which the episode starts. The left panel shows the case in which $\alpha \geq \dfrac{A}{A+\delta}$ and the slope of $E_t x_{t+1}$ at the origin is greater than or equal to one. This means that any initial bubble would be demanded only if it were expected to continuously grow as a share of labor income, i.e. if it violates Equation (21), and this can be ruled out. The right panel of Figure 4 shows the case in which $\alpha < \dfrac{A}{A+\delta}$. Now the slope of $E_t x_{t+1}$ at the origin is less than one and, as a result, $E_t x_{t+1}$ must cross the 45 degree line once and only once. Let x^* be the value of x_t at that point. Any initial bubble $x_{s+1} > x^*$ can be ruled out. But any initial bubble $x_s^N \leq x^*$ can be part of an equilibrium since it is possible to find a stochastic process for x_t that satisfies Equations (20) and (21).

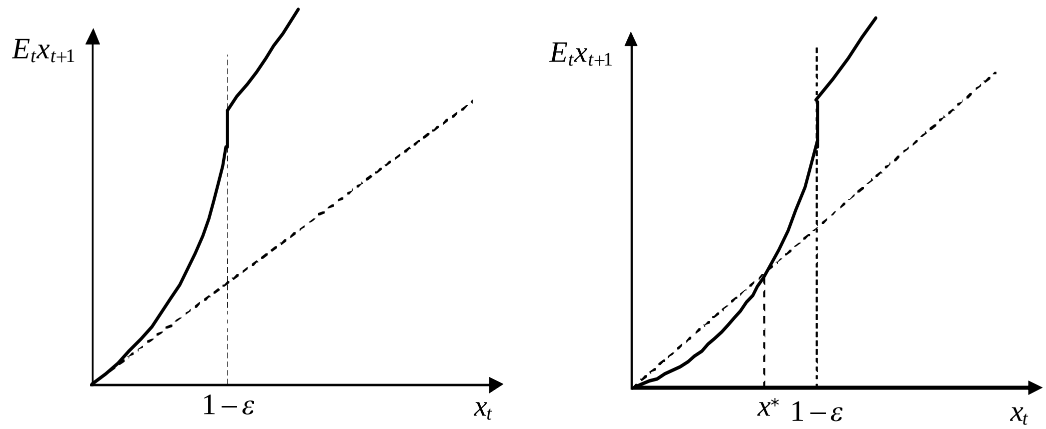

Figure 4

Is it possible that bubble creation relaxes the conditions for the existence of bubbly episodes? Consider first bubble creation by unproductive investors, i.e. x_t^{NU}. As in the Samuelson-Tirole model, this type of bubble creation shifts the schedule $E_t x_{t+1}$ upwards. The intuition is the same as before: new bubbles compete with old bubbles for the income of next period's young, reducing their return and making them less attractive. Therefore, allowing for bubble creation by unproductive investors does not relax the conditions for the existence of bubbly episodes.

Consider next bubble creation by productive investors, i.e. x_t^{NP}. This type of bubble creation shifts the schedule $E_t x_{t+1}$ upwards if $x_t \in (0, A] \cup (1-\varepsilon, 1]$, but it shifts it downwards if $x_t \in (A, 1-\varepsilon]$. To understand this result, it is important to recognize the double role played by bubble creation by productive investors. On the one hand, new bubbles compete with old ones for the income of next period's young. This effect reduces the demand for old bubbles and shifts the schedule $E_t x_{t+1}$ upwards. On the other hand, productive investors sell new bubbles to unproductive investors and use the proceeds to invest, raising average investment efficiency and the income of next period's young. This effect increases the demand for old bubbles and shifts the schedule $E_t x_{t+1}$ downwards. This second effect operates whenever $x_t \leq 1-\varepsilon$, and it dominates the first effect only if $x_t \geq A$. Hence, if $A > 1-\varepsilon$, bubble creation by productive investors cannot relax the condition for the existence of bubbly episodes.

If $A \leq 1-\varepsilon$, bubble creation does relax the condition for the existence of bubbles. Namely, this condition becomes $\alpha < \max\left\{\frac{A}{A+\delta}, \frac{1}{1+4\cdot(1-\varepsilon)\cdot\delta}\right\}$. Figure 5 provides some intuition for this result by plotting $E_t x_{t+1}$ against x_t, using Equation (20) and assuming that $x_t^{NU} = 0$ while

$x_t^{NP} = x_s^{NP}$ if $t = s$ and

$$x_t^{NP} = \begin{cases} 0 & \text{if } x_t \in (0, A] \cup (1-\varepsilon, 1] \\ 1-\varepsilon - x_t & \text{if } x_t \in (A, 1-\varepsilon] \end{cases},$$

for all $t > s$. The left panel shows the case in which bubble creation by productive investors does not affect the conditions for the existence of bubbly episodes. The right panel shows instead the case in which bubble creation by productive investors weakens the conditions for the existence of bubbly episodes. This completes the proof of Proposition 2.

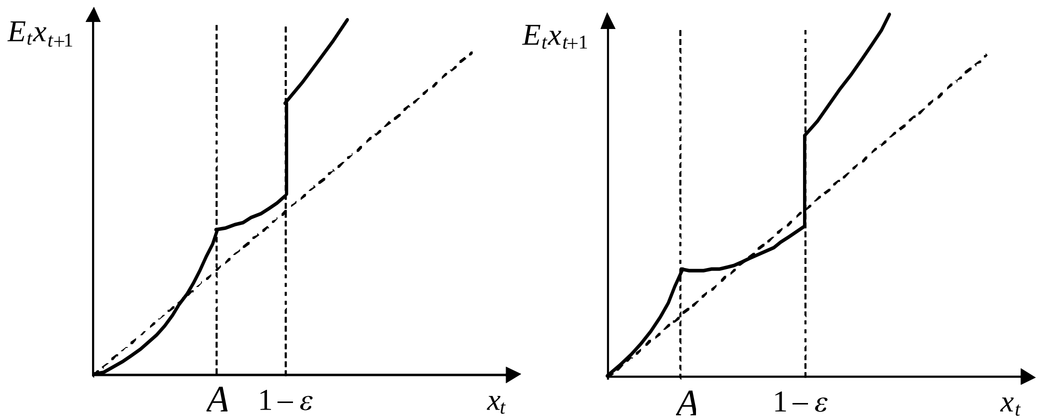

Figure 5

With financial frictions, the connection between bubbles and dynamic inefficiency becomes more subtle. To determine whether a bubbly episodes can exist, we asked again: Is it possible to construct a pyramid scheme that is attractive without exploding as a share of labor income? We have shown that the answer is affirmative if and only if there exist stochastic processes for x_t^{NP}, x_t^{NU} and x_t such that:

$$E_t x_{t+1} < x_t.$$

These processes exist if and only if α satisfies the restriction in Proposition 2.

To determine whether the economy is dynamically inefficient, we ask again: Is the economy accumulating too much capital? This question is now tricky because there are two types of investments. The condition that aggregate investment be higher than aggregate capital income, i.e.

$$(1 - x_t) \cdot (1 - \alpha) \cdot k_t^{\alpha} > \alpha \cdot k_t^{\alpha},$$

asks whether the average investment exceeds the income it produces. Even if this were not the case, the economy might still be dynamically inefficient since it might contain pockets of investments that exceed the income they produce. We need to check for this additional possibility. Investments by unproductive investors equal $\left(1-\varepsilon-x_t-x_t^{NP}\right)\cdot(1-\alpha)\cdot k_t^{\alpha}$, while their capital income is given by $\frac{\delta\cdot\left(1-\varepsilon-x_t-x_t^{NP}\right)}{A+(1-\delta)\cdot x_t^{NP}-\delta\cdot x_t}\cdot\alpha\cdot k_t^{\alpha}$. Therefore, these investors constitute a pocket of dynamic inefficiency if and only if:

$$\left(1-\varepsilon-x_t-x_t^{NP}\right)\cdot(1-\alpha)\cdot k_t^{\alpha}>\frac{\delta\cdot\left(1-\varepsilon-x_t-x_t^{NP}\right)}{A+(1-\delta)\cdot x_t^{NP}-\delta\cdot x_t}\cdot\alpha\cdot k_t^{\alpha}. \tag{22}$$

Straightforward algebra shows that this condition holds if and only if there exist stochastic processes for for x_t^{NP} and x_t such that:

$$E_t x_{t+1}<x_t+x_t^{NP}.$$

When $\frac{A}{A+\delta}<\frac{1}{1+4\cdot(1-\varepsilon)\cdot\delta}$ this restriction is weaker than the condition for the existence of bubbly episodes in Proposition 2. The intuition for this result is that sometimes bubbly episodes can only exist if there is enough bubble creation. This requires the economy to be not only dynamically inefficient, but to be sufficiently so to support bubble creation.

This discussion sheds some light on the analysis of Abel et al. (1989). The finding that aggregate investment falls short of aggregate capital income still implies that $\alpha>0.5$. Under this parameter restriction, financial frictions are crucial for bubbly episodes to exist and their removal would eliminate these episodes at once. But it does not follow that, under this parameter restriction, bubbly episodes cannot exist. This is for two reasons: (i) if $0.5<\alpha<\frac{A}{A+\delta}$, in the fundamental state there are pockets of dynamic inefficiency that would support a bubble if it were to pop up; and (ii) if $\frac{A}{A+\delta}\leq\alpha<\frac{1}{1+4\cdot(1-\varepsilon)\cdot\delta}$, there are no pockets of dynamic inefficiency in the fundamental state but an expansionary bubble that lowers the return to investment would create such pockets itself. This second case brings a simple but powerful point home: what is required for bubbly episodes to exist is that the economy be dynamically inefficient *during these episodes and not in the fundamental state*. Bubbles that crowd in capital can convert a dynamically efficient economy into a dynamically inefficient one.

3.4 The macroeconomic effects of bubbles revisited

We have shown that financial frictions weaken the conditions for bubbly episodes to exist. We show next that they also modify the macroeconomic effects of bubbly episodes. To do this, we rewrite the law of motion of the capital stock using the definition of x_t and x_t^{NP}:

$$k_{t+1} = \begin{cases} \left[A + (1-\delta) \cdot x_t^{NP} - \delta \cdot x_t\right] \cdot (1-\alpha) \cdot k_t^{\alpha} & \text{if } \dfrac{x_t + x_t^{NP}}{1-\varepsilon} < 1 \\ (1-\alpha) \cdot k_t^{\alpha} - x_t & \text{if } \dfrac{x_t + x_t^{NP}}{1-\varepsilon} \geq 1 \end{cases} . \tag{23}$$

Equation (23) describes the dynamics of the capital stock for any admissible stochastic process for the bubble, i.e. x_t^{NP}, x_t^{NU} and x_t; satisfying Equations (20) and (21). Once again, note that bubbly episodes are akin to shocks to the law of motion of capital. As in the Samuelson-Tirole model, bubbles can grow, shrink or randomly fluctuate throughout these episodes.

The macroeconomic effects of bubbly episodes depend on whether x_t^{NP} is smaller or greater than $x_t \cdot \frac{\delta}{1-\delta}$. If smaller, bubbly episodes are contractionary and they lower the capital stock and output. If greater, bubbly episodes are expansionary and they raise the capital stock and output.[8] Contractionary episodes are similar in all regards to those analyzed in the Samuelson-Tirole model. As for expansionary episodes, their macroeconomic effects are illustrated in Figure 6. Assume initially that the economy is in the fundamental state so that the appropriate law of motion is the one labeled k_{t+1}^F. This law of motion for the fundamental state is that of the standard Solow model. Assume that, initially, the capital stock is equal to the Solow steady state so that $k_t = k^F \equiv [A \cdot (1-\alpha)]^{\frac{1}{1-\alpha}}$. When an expansionary bubble pops up, it reduces unproductive investments and uses part of these resources to increase productive investments. As can be seen from Equation (23), the law of motion during the bubbly episode lies above k_{t+1}^F: in the figure, k_{t+1}^B represents the initial law of motion when the episode begins. As a result, growth turns positive. Throughout the episode, k_{t+1}^B may shift as the bubble grows or shrinks. The capital stock and output, however, unambiguously increase relative to the fundamental state. Eventually, the bubble bursts and the economy returns to the original law of motion k^F.

[8] Even within a single episode, the effects on the capital stock and output might vary through time or across states of nature.

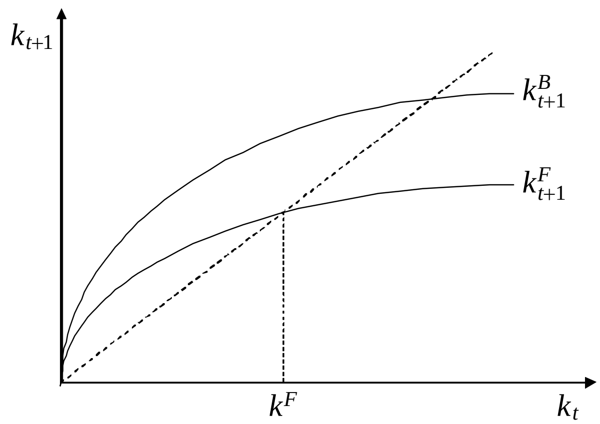

Figure 6

Expansionary and contractionary episodes differ also in their implications for consumption. In this model, consumption is still expressed by Equation (13). Therefore, regardless of their type, all present bubbles raise the income in the hands of the old and thus consumption. However, the effect of past bubbles depends on their type. If they were contractionary, the current capital stock and therefore consumption are lower. If instead they were expansionary, the current capital stock and therefore consumption are higher.

3.5 Discussion

The Samuelson-Tirole model has been criticized because the conditions for the existence of bubbly episodes and their macroeconomic effects seem both unrealistic. We have shown that these criticisms do not apply to the model with financial frictions. In particular, (i) bubbly episodes are possible even if aggregate investment falls short of capital income and; (ii) bubbly episodes can be expansionary. The critiques to the Samuelson-Tirole model therefore stem from the assumption that financial markets are frictionless and rates of return to investment are equalized across investors.

We can summarize our findings on the connection between bubbles and financial frictions with the help of Figure 7. The line labeled α_C provides, for each δ, the largest α that is consistent with the existence of contractionary episodes.[9] The line labeled α_E provides, for each δ, the largest α

[9] These episodes are possible if and only if the economy is dynamically inefficient in the fundamental state. Therefore,

$$\alpha_C = \frac{A}{A+\delta}.$$

that is consistent with the existence of expansionary episodes.[10] These lines partition the (α, δ) space into four regions.[11]

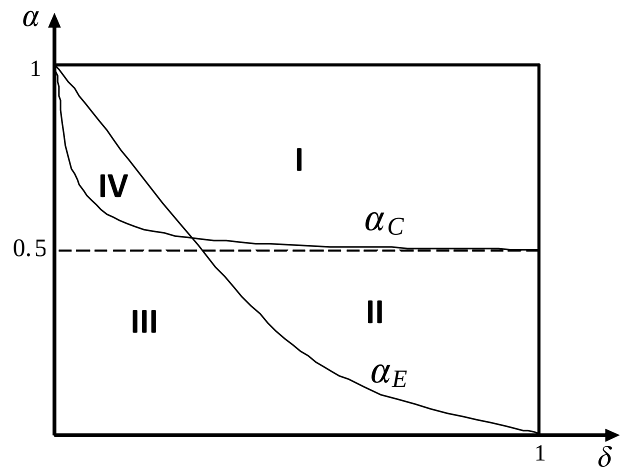

Figure 7

Bubbly episodes are possible in Regions II-IV, but not in Region I. In Regions II and III, $\alpha < \alpha_C$ and contractionary episodes are possible. In Region III and IV, $\alpha < \alpha_E$ and expansionary episodes are possible. We can think of δ as a measure of the costs of financial frictions. The higher is δ, the smaller are the gains from borrowing and lending and the smaller are the costs of financial frictions. In the limiting case of $\delta \to 1$, the Samuelson-Tirole model applies: only contractionary episodes are possible and this requires $\alpha < 0.5$. As δ decreases, the conditions for existence are relaxed for both types of bubbly episodes. In the limiting case of $\delta \to 0$, financial frictions are very severe and all types of bubbly episodes are possible regardless of α.

[10] Fomally, the existence of a expansionary bubbly episode requires the existence of a triplet $\left\{x_t, x_t^{NP}, x_t^{NU}\right\}$ satisfying $E_t x_{t+1} < x_t$ and $x_t \cdot \dfrac{\delta}{1-\delta} < x_t^{NP}$, where $E_t x_{t+1}$ is as in Equation (20). This means that:

$$\alpha_E = \begin{cases} \dfrac{1}{1 + 4 \cdot (1-\varepsilon) \cdot \delta} & \text{when } \delta \leq \dfrac{0.5-\varepsilon}{1-\varepsilon} \\ \dfrac{A \cdot (1-\delta)}{\delta + A \cdot (1-\delta)} & \text{when } \delta > \dfrac{0.5-\varepsilon}{1-\varepsilon} \end{cases}$$

[11] Figure 7 has been drawn under the assumption that $\varepsilon < 0.5$. This guarantees that Region IV exists.

4 Bubbles and long-run growth

The Samuelson-Tirole model predates the development of endogenous growth models. To maximize comparability, the model with financial frictions used the same production structure. Not surprisingly, it has little to say about the relationship between bubbles and long-run growth. We now generalize the production structure and allow for the possibility of constant or increasing returns to capital. We show that the conditions for existence of bubbly episodes do not change. However, even transitory episodes have permanent effects and can even lead the economy into or out of negative-growth traps.

4.1 Setup with long-run growth

We assume that the production of the final good consists of assembling a continuum of intermediate inputs, indexed by $m \in [0, m_t]$. This variable, which can be interpreted as the level of technology in period t, will be obtained endogenously as part of the equilibrium. The production function of the final good is given by the following symmetric CES function:

$$y_t = \eta \cdot \left(\int_0^{m_t} q_{tm}^{\frac{1}{\mu}} \cdot dm \right)^{\mu}, \tag{24}$$

where q_{tm} denotes units of the variety m of intermediate inputs and $\mu > 1$. The constant η is a normalization parameter that will be chosen later. Throughout, we assume that final good producers are competitive, and we normalize the price of the final good to one.

Production of intermediate inputs requires labor and capital. In particular, each type of intermediate input $m \in [0, m_t]$ is produced according to the following production function,

$$q_{tm} = (l_{tm,v})^{1-\alpha} \cdot (k_{tm,v})^{a}, \tag{25}$$

where $l_{tm,v}$ and $k_{tm,v}$ respectively denote the use of labor and capital to cover the variable costs of producing variety m. Besides this use of factors, the production of any given variety requires the payment of a fixed cost f_{tm} given by

$$f_{tm} = \begin{cases} 1 = (l_{tm,f})^{1-\alpha} \cdot (k_{tm,f})^{a} & \text{if } q_{tm} > 0 \\ 0 & \text{if } q_{tm} = 0 \end{cases}, \tag{26}$$

where $l_{tm,f}$ and $k_{tm,f}$ respectively denote the use of labor and capital to cover the fixed costs of producing variety m. To simplify the model, we assume that input varieties become obsolete in one generation and, as a result, all generations must incur these fixed costs. It is natural therefore to assume that the production of intermediate inputs takes place under monopolistic competition and free entry.

This production structure is a special case of that considered by Ventura (2005).[12] He shows that, under the assumptions made, it is possible to rewrite Equation (24) as

$$y_t = k_t^{\alpha \cdot \mu}, \tag{27}$$

where we have chosen units such that $\eta = (\mu)^{-\mu} \cdot (1-\mu)^{\mu-1}$. Maintaining the assumption of competitive factor markets, factor prices can now be expressed as follows

$$w_t = (1-\alpha) \cdot k_t^{\alpha \cdot \mu} \;\; \text{and} \;\; r_t = \alpha \cdot k_t^{\alpha \cdot \mu - 1}. \tag{28}$$

Equation (27) shows that there are two opposing effects of increasing the stock of physical capital. On the one hand, such increases make capital abundant and they have the standard effect of decreasing its marginal product. The strength of this diminishing-returns effect is measured by α. On the other hand, increases in the stock of capital expand the varieties of inputs produced in equilibrium, which has an indirect and positive effect on the marginal product of capital. The strength of this market-size effect is measured by μ. If diminishing returns are strong and market-size effects are weak ($\alpha \cdot \mu < 1$) increases in physical capital reduce the marginal product of capital. If instead diminishing returns are weak and market-size effects are strong ($\alpha \cdot \mu \geq 1$) increases in physical capital raise the marginal product of capital.

4.2 Equilibria with bubbles

How does this generalization of the production structure affect the dynamics of bubbles and capital? For a bubble to be attractive, its expected return must be at least equal to the return to investment.

[12] The Appendix contains a formal derivation of the equations that follow.

Formally, this requirement must now be written as follows,

$$E_t\left\{\frac{b_{t+1}}{b_t+b_t^{NP}+b_t^{NU}}\right\}\begin{cases}=\delta\cdot\alpha\cdot k_t^{\alpha\cdot\mu-1} & \text{if } \dfrac{b_t+b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}<1\\ \in\left[\delta\cdot\alpha\cdot k_t^{\alpha\cdot\mu-1},\alpha\cdot k_t^{\alpha\cdot\mu-1}\right] & \text{if } \dfrac{b_t+b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}=1\\ =\alpha\cdot k_t^{\alpha\cdot\mu-1} & \text{if } \dfrac{b_t+b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}>1\end{cases}, \tag{29}$$

which is a generalization of Equation (17). The dynamics of the capital stock, in turn, are now given by,

$$k_{t+1}=\begin{cases}(1-\alpha)\cdot A\cdot k_t^{\alpha\cdot\mu}+(1-\delta)\cdot b_t^{NP}-\delta\cdot b_t & \text{if } \dfrac{b_t+b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}<1\\ (1-\alpha)\cdot k_t^{\alpha\cdot\mu}-b_t & \text{if } \dfrac{b_t+b_t^{NP}}{(1-\varepsilon)\cdot(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}\geq 1\end{cases}, \tag{30}$$

which is a generalization of Equation (19). For a given initial capital stock and bubble, $k_0>0$ and $b_0\geq 0$, a competitive equilibrium is a sequence $\left\{k_t,b_t,b_t^{NP},b_t^{NU}\right\}_{t=0}^{\infty}$ satisfying Equations (6), (29) and (30).

4.3 Bubbly episodes with long-run growth

This generalization of the production structure expands the set of economies that can be analyzed with the model. It does not, however, affect the conditions for the existence of bubbly episodes. That is, Proposition 2 applies for any value of μ. To see this, define $x_t\equiv\dfrac{b_t}{(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}$, $x_t^{NP}\equiv\dfrac{b_t^{NP}}{(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}$ and $x_t^{NU}\equiv\dfrac{b_t^{NU}}{(1-\alpha)\cdot k_t^{\alpha\cdot\mu}}$. Once we apply our recursive trick to Equation (29), we recover Equation (20). Hence, bubble dynamics are still described by Equations (20) and (21).

It is useful at this point to provide an additional characterization of the condition for dynamic inefficiency. Combining Equations (22) and (30) we find that the economy has pockets of dynamic inefficiency if and only if the growth rate exceeds the return to unproductive investments,

$$G_{t+1}\equiv\left(\frac{k_{t+1}}{k_t}\right)^{\alpha\cdot\mu}\geq\delta\cdot\alpha\cdot k_{t+1}^{\alpha\cdot\mu-1}\equiv R_{t+1}^{U}. \tag{31}$$

This condition is quite intuitive. Remember that, for a bubble to exist, it must grow fast enough to be attractive but no too fast to outgrow its demand, i.e. its growth must be between G_{t+1} and R_{t+1}^{U}. When the condition in Equation (31) fails there is no room for such a bubble.

Consider first the case in which $\alpha\cdot\mu<1$. Contractionary bubbles reduce G_{t+1} and increase R_{t+1}^{U}.

This is why contractionary episodes are only possible if, in the fundamental state, $G_{t+1} > R^U_{t+1}$. Expansionary bubbles, however, increase G_{t+1} and lower R^U_{t+1}. This is why expansionary episodes are sometimes possible even if, in the fundamental state, $G_{t+1} < R^U_{t+1}$.

Consider next the case in which $\alpha \cdot \mu \geq 1$. Now, market-size effects dominate diminishing returns and the relationship between the capital stock and the return to investment is reversed. Contractionary bubbles still reduce G_{t+1} but now they also reduce R^U_{t+1}. Despite this, we still find that contractionary episodes are only possible if, in the fundamental state, $G_{t+1} > R^U_{t+1}$. The reason, of course, is that the decrease in R^U_{t+1} is small relative to the decrease in G_{t+1}. Expansionary bubbles still increase G_{t+1} but now they also increase R^U_{t+1}. Despite this, we still find that expansionary episodes might be possible even if, in the fundamental state, $G_{t+1} < R^U_{t+1}$. The reason, once again, is that the increase in R^U_{t+1} is small relative to the increase in G_{t+1}.

4.4 The macroeconomic effects of bubbles

We can rewrite the law of motion of the capital stock using the definitions of x_t and x_t^{NP}:

$$k_{t+1} = \begin{cases} \left[A + (1-\delta) \cdot x_t^{NP} - \delta \cdot x_t\right] \cdot (1-\alpha) \cdot k_t^{\alpha \cdot \mu} & \text{if } \dfrac{x_t + x_t^{NP}}{1-\varepsilon} < 1 \\ (1-\alpha) \cdot k_t^{\alpha \cdot \mu} - x_t & \text{if } \dfrac{x_t + x_t^{NP}}{1-\varepsilon} \geq 1 \end{cases} . \qquad (32)$$

Equation (32) describes the dynamics of the capital stock for any admissible stochastic process for the bubble, i.e. x_t^{NP}, x_t^{NU} and x_t; satisfying Equations (20) and (21). Once again, note that bubbly episodes are akin to shocks to the law of motion of capital. As in previous models, bubbles can grow, shrink or randomly fluctuate throughout these episodes. If diminishing returns are strong and market-size effects are weak, i.e. $\alpha \cdot \mu < 1$, all the analysis of Section 3 applies. Therefore, we restrict the analysis to the new case in which diminishing returns are weak and market-size effects are strong, i.e. $\alpha \cdot \mu \geq 1$.

An interesting feature of bubbly episodes when $\alpha \cdot \mu \geq 1$ is that, even if they are transitory, they can have permanent effects on the levels and growth rates of capital and output. We illustrate this with the help of Figure 9. The left panel depicts the case of an expansionary bubble. Initially, the economy is in the fundamental state so that the appropriate law of motion is the one labeled k^F_{t+1}. Since the initial capital stock is below the steady state, i.e. $k_t < k^F \equiv [(1-\alpha) \cdot A]^{\frac{1}{1-\alpha \cdot \mu}}$, growth is negative. We think of this economy as being caught in a "negative-growth trap". When an expansionary bubble pops up, it reduces unproductive investments and uses part of these resources to increase productive investments. During the bubbly episode, the law of motion of capital lies above k^F_{t+1}: in the figure, k^B_{t+1} represents the initial law of motion when the episode begins. Throughout the episode, k^B_{t+1} may shift as the bubble grows or shrinks. Growth may be positive if, during the bubbly episode, the capital stock lies above its steady-state value as shown in the figure. Eventually, the bubble bursts but the economy might keep on growing if the capital stock at the time of bursting exceeds k^F. The bubbly episode, though temporary, leads the economy out of the negative-growth trap and it has a permanent effect on long-run growth.

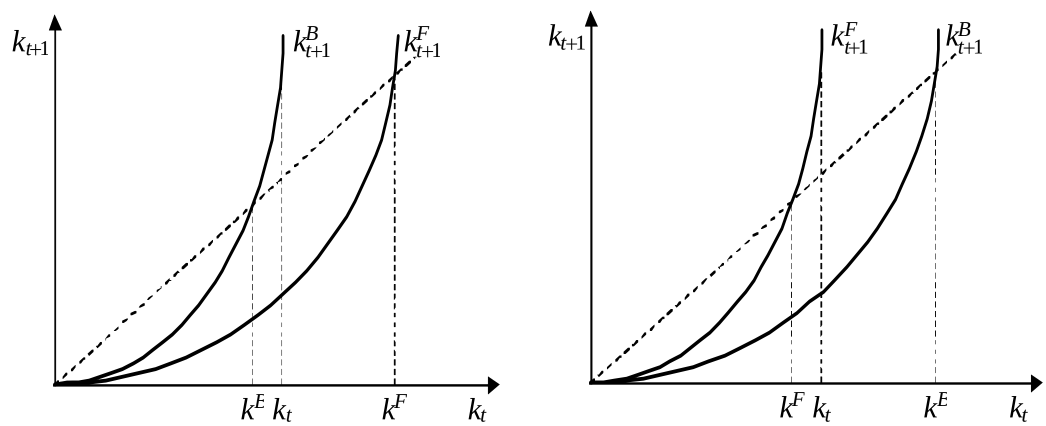

Figure 8

Naturally, it is also possible for bubbles to lead the economy into a negative growth trap thereby having permanent negative effects on long-run growth. The right panel of Figure 9 shows an example of a contractionary bubble that does this.

This section has extended the model to allow for endogenous growth. This does not affect the conditions for the existence of bubbly episodes. It does, however, affect some of their macroeconomic effects. First, the behavior of the return to investment during bubbly episodes is reversed. Second, bubbly episodes can have long-run effects on economic growth.

5 Further issues and research agenda

We have developed a stylized model of economic growth with bubbles. In this model, financial frictions lead to equilibrium dispersion in the rates of return to investment. During bubbly episodes, unproductive investors demand bubbles while productive investors supply them. Because of this, bubbly episodes channel resources towards productive investment raising the growth rates of capital and output. The model also illustrates that the existence of bubbly episodes requires some investment to be dynamically inefficient: otherwise, there would be no demand for bubbles. This dynamic inefficiency, however, might be generated by an expansionary episode itself.

Our analysis is incomplete in two important respects. The first one refers to the connection between bubbles and savings. Throughout, we have assumed that young individuals care only about old age consumption and therefore save all of their income. That is, we have assumed that their savings rate is constant and equal to one. The constancy of this savings rate has allowed us to exploit a nice analytical trick that makes the model recursive. This assumption is not only unrealistic, but also crucial for some results. For instance, a strong prediction of the model developed here is that bubbly episodes reduce investment spending. If we allowed instead for the savings rate to respond positively to income and/or negatively to the return to investment, expansionary bubbly episodes could raise savings and investment spending.

The analysis is also incomplete because we have not studied the welfare implications of bubbly episodes. Instead, we have focused exclusively on the conditions for these episodes to exist and their effects on macroeconomic aggregates. This omission does not reflect a lack of interest in our part. To the contrary, we think that a full analysis of the welfare implications of bubbly episodes is a central objective of the theory. The reason for this omission is that we have found a full treatment of the issues to be quite involved. Moreover, some of the more interesting results require us to extend the model in various directions. There is simply no space for this here.

Introducing financial frictions in the Samuelson-Tirole model shows that bubbles can also help to channel resources from inefficient to efficient investors. In our stylized model, this transfer of resources happens exclusively in the market for bubbles. But this need not be so. Consider two suggestive examples. Martin and Ventura (2010) introduce a credit market in which investors are subject to borrowing constraints. The prospect of a future bubble raises the collateral of efficient investors and allows them borrow and invest more. In this setup, bubbles help transfer resources from inefficient to efficient investors. But this happens in the credit market and not in the market for bubbles. Ventura (2004) introduces a distinction between consumption and investment goods. Bubbles reduce inefficient investments and lower the price of investment goods, allowing efficient investors to invest more. In this setup, bubbles also help transfer resources from inefficient to efficient investors. But this happens in the goods market and not in the market for bubbles. These are only two examples, and much more needs to be done. The role of bubbles in resource allocation remains an essentially unexplored topic.

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6 Appendix

Profit maximization by producers of the final good yields the following demand function for intermediate good m,

$$p_{tm} = \left(\frac{q_{tm}}{y_t}\right)^{\frac{1-\mu}{\mu}}. \tag{33}$$

Profit maximization by producers of intermediate good m can in turn be stated as

$$\max_{l_{tm,v}, l_{tm,f}, k_{tm,v}, k_{tm,f}} \left[q_{tm} \cdot p_{tm} - (l_{tm,v} + l_{tm,f}) \cdot w_t - (k_{tm,v} + k_{tm,f}) \cdot r_t\right]$$

subject to Equations (25), (26) and (33). Replacing the first and the last of these constraints into the objective function, this optimization problem yields the following first-order conditions:

$$\begin{aligned} \frac{p_{tm} \cdot q_{tm} \cdot (1-\alpha)}{\mu} &= l_{tm,v} \cdot w_t, \\ \frac{p_{tm} \cdot q_{tm} \cdot \alpha}{\mu} &= k_{tm,v} \cdot r_t, \\ \lambda_{tm} \cdot (1-\alpha) &= l_{tm,f} \cdot w_t, \\ \lambda_{tm} \cdot \alpha &= k_{tm,f} \cdot r_t, \end{aligned}$$

where λ_{tm} denotes the multiplier on the constraint imposed by Equation (26). This constraint along with the first-order conditions imply,

$$\lambda_{tm} = \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \cdot \left(\frac{r_t}{\alpha}\right)^{\alpha}, \tag{34}$$

$$p_{tm} = \mu \cdot \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \cdot \left(\frac{r_t}{\alpha}\right)^{\alpha}, \tag{35}$$

which, along with the zero profit condition of intermediate goods producers yield

$$q_{tm} = \frac{1}{\mu - 1}. \tag{36}$$

Equation (36) says that, conditional on it being produced, the amount produced of any given intermediate good is constant. We can use it to derive the number of varieties being produced in equilibrium. To do so, we replace Equations (34)-(36) in the factor demands implied by the first-order conditions in order to obtain the following expressions for equilibrium in the markets for labor and capital:

$$
\begin{aligned}
m_t \cdot \frac{\mu}{\mu-1} \cdot \left(\frac{w_t}{1-\alpha}\right)^{-\alpha} \cdot \left(\frac{r_t}{\alpha}\right)^{\alpha} &= 1, \\
m_t \cdot \frac{\mu}{\mu-1} \cdot \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \cdot \left(\frac{r_t}{\alpha}\right)^{\alpha-1} &= k_t.
\end{aligned}
$$

These conditions can be combined to derive the equilibrium number of varieties,

$$
m_t = \frac{\mu-1}{\mu} \cdot k_t^{\alpha},
$$

which, when replaced in the production function of Equation (24), delivers Equation (27).