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MARKET VOLATILITY?

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Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility?

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**ABSTRACT**

This paper introduces a model in which the probability of a rare disaster varies over time. I show that the model can account for the high equity premium and high volatility in the aggregate stock market. At the same time, the model generates a low mean and volatility for the government bill rate, as well as economically significant excess stock return predictability. The model is set in continuous time, assumes recursive preferences and is solved in closed-form. It is shown that recursive preferences, as well as time-variation in the disaster probability, are key to the model's success.

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# 1 Introduction

The magnitude of the expected excess return on stocks relative to bonds (the equity premium) constitutes one of the major puzzles in financial economics. As Mehra and Prescott (1985) show, the standard endowment economy model predicts an equity premium that is far too small. One proposed explanation is that the return on equities is high to compensate investors for the risk of a rare disaster (Reitz (1988)). An open question has therefore been whether the risk is sufficiently high, and the rare disaster sufficiently severe, to quantitatively explain the equity premium. Recently, however, Barro (2006) shows that it is possible to explain the equity premium using such a model when the probability of a rare disaster is calibrated to international data on large economic declines.

While the models of Reitz (1988) and Barro (2006) advance our understanding of the equity premium, they fall short in other respects. Most importantly, these models predict that the volatility of stock market returns equals the volatility of dividends. Numerous studies have shown, however, that this is not the case. In fact, there is excess stock market volatility; the volatility of stock returns far exceeds that of dividends (e.g. Shiller (1981), LeRoy and Porter (1981), Keim and Stambaugh (1986), Campbell and Shiller (1988), Cochrane (1992), Hodrick (1992)). While the models of Barro and Reitz address the equity premium puzzle, they do not address this volatility puzzle.

This paper proposes two modifications to the disaster risk model of Barro (2006). First, rather than being constant, the probability of a rare disaster is stochastic and varies over time. Second, the representative agent, rather than having power utility preferences, has recursive preferences. I show that such a model can generate volatility of stock returns close to that in the data at reasonable values of the underlying parameters. Moreover, high stock market volatility does not occur because of high volatility in all assets; the volatility of the government bill rate remains low, as in the data.

Both time-varying disaster probabilities and recursive preferences are necessary to match the model to the data. Time-varying disaster probabilities are

important because they produce time-varying discount rates. If the probability of a rare disaster rises, the premium that investors require to hold equities also rises. Thus the rate at which investors discount future cash flows rises, and stock prices fall. Through this mechanism, a time-varying probability of a rare disaster generates excess stock market volatility.

Clearly time-varying disaster probabilities are important; also important, however, are recursive preferences. These preferences, introduced by Kreps and Porteus (1978) and Epstein and Zin (1989) retain the appealing scale-invariance of power utility, but allow for separation between the willingness to take on risk and the willingness to substitute over time. Power utility requires that these are driven by the same parameter. As I show, power utility implies that the response of the riskfree rate to a change in the disaster probability exceeds the response of the equity premium. This generates counterfactual predictions. Increasing the agent's willingness to substitute over time reduces the effect of the disaster probability on the riskfree rate. With recursive preferences, this can be accomplished without simultaneously reducing the agent's risk aversion.

In what follows, I solve two models. First, to highlight the importance of recursive utility, I solve a model in which the disaster probability is constant and the agent has recursive preferences. To enable comparison with the second model, I assume continuous time. Disaster risk is modeled by introducing a Poisson process for jumps into the standard diffusion model for consumption growth (as in Naik and Lee (1990)) and recursive preferences are modeled following Duffie and Epstein (1992).

The second model is the main contribution of the paper and allows for time-varying disaster probabilities and recursive utility with unit elasticity of intertemporal substitution (EIS). The assumption that the EIS is equal to 1 allows the model to be solved in closed form up to a set of ordinary differential equations. A time-varying disaster probability is modeled by allowing the intensity for jumps to follow a square-root process (Cox, Ingersoll, and Ross (1985)). The solution for the model reveals that allowing the probability of a disaster to vary not only implies a time-varying equity premium, it also

increases the level of the equity premium. This is because the risk of an increase in disaster probability leads to a fall in stock prices and therefore itself requires compensation. The dynamic nature of the model therefore leads the equity premium to be higher than what static considerations alone would predict.

Several recent papers also address the potential of rare disasters to explain the aggregate stock market. Gabaix (2007) assumes power utility for the representative agent, while also assuming the economy is driven by a linearity-generating process that combines the probability of a rare disaster with the degree to which dividends respond to a consumption disaster. This combination of assumptions allows him to derive closed-form solutions for equity prices as well as prices for other assets. In Gabaix's numerical calibration, only the degree to which dividends respond to the disaster varies over time. Therefore the economic mechanism driving stock market volatility in Gabaix's model is quite different than the one considered here. Barro (2007) and Martin (2007) solve models with a constant disaster probability and recursive utility. In contrast, the model considered here focuses on the case of time-varying disaster probabilities. In contemporaneous independent work, Gourio (2008) specifies a model in which the probability of a disaster varies between two discrete values. He solves this model numerically assuming recursive preferences. A different, though related, approach is taken by Veronesi (2004), who assumes that the drift of dividends follows a Markov switching process, with a small probability of falling into a low state. While the physical probability of a low state is constant, the representative investor's subjective probability is time-varying due to learning. Veronesi assumes exponential utility; this allows for the inclusion of learning but makes it difficult to assess the magnitude of the excess volatility generated through this mechanism.

A related literature derives asset pricing results assuming endowment processes that include jumps, with a focus on option pricing. Liu, Pan, and Wang (2005) consider an endowment process in which jumps occur with a constant intensity; their focus is on uncertainty aversion but they also consider recursive utility. My model departs from theirs in that the probability of a jump

varies over time. Drechsler and Yaron (2008) show that a model with jumps in the volatility of the consumption growth process can explain the behavior of implied volatility and its relation to excess returns. They focus on the case of EIS greater than one and derive approximate analytical and numerical solutions. Santa-Clara and Yan (2006) consider time-varying jump intensities, but restrict attention to a model with power utility and implications for options. In contrast, the model considered here focuses on recursive utility and implications for the aggregate market.

The outline of the paper is as follows. Section 2 solves a model with constant disaster probabilities and recursive utility with general elasticity of intertemporal substitution. Section 3 solves the model with time-varying disaster risk and unit elasticity of substitution. Section 4 discusses the calibration and simulation of the model, and its fit to aggregate stock market data. Section 5 concludes.

## 2 Constant disaster risk

### 2.1 Model assumptions

Assume that aggregate consumption solves the following stochastic differential equation:

$$dC_t = \mu C_{t-} dt + \sigma C_{t-} dB_t + (e^{Z_t} - 1)C_{t-} dN_t.$$

Here,  $N_t$  is a Poisson process with constant intensity  $\lambda$ .  $Z_t$  is a random variable whose time-invariant distribution  $\nu$  is independent of  $N_t$  and  $B_t$ . The notation  $C_{t-}$  denotes  $\lim_{s \uparrow t} C_s$ , while  $C_t$  denotes  $\lim_{s \downarrow t} C_s$ .

The model focuses on negative jumps, i.e. disasters, and for that reason  $Z_t$  is assumed to be negative. In what follows, I use the notation  $E_\nu$  to denote expectations of functions of  $Z_t$  taken with respect to the  $\nu$ -distribution. The  $t$  subscript on  $Z_t$  will be omitted when not essential for clarity. The diffusion term  $\mu C_{t-} dt + \sigma C_{t-} dB_t$  represents the behavior of consumption in normal times, and implies that, when no disaster takes place, log consumption growth is normally distributed with mean  $\mu - \frac{1}{2}\sigma^2$  and variance  $\sigma^2$ . Rare disasters are

captured by the Poisson process  $N_t$ . Roughly speaking,  $\lambda$  is the probability of a jump over a given unit of time.<sup>1</sup>

Following Duffie and Epstein (1992), I define the utility function  $V_t$  for the representative agent using the following recursion:

$$V_t = E_t \int_t^\infty f(C_s, V_s) ds, \quad (1)$$

where

$$f(C, V) = \frac{\beta}{1 - \frac{1}{\psi}} \frac{C^{1 - \frac{1}{\psi}} - ((1 - \gamma)V)^{\frac{1}{\theta}}}{((1 - \gamma)V)^{\frac{1}{\theta} - 1}}, \quad (2)$$

and  $\theta = (1 - \gamma)/(1 - \frac{1}{\psi})$ . Note that  $V_t$  represents continuation utility, i.e. utility of the future consumption stream. Equations (1) and (2) define the continuous-time analogue of the utility function in Epstein and Zin (1989) and Weil (1990). The parameter  $\beta > 0$  is the rate of time preference,  $\psi > 0$  can be interpreted as the elasticity of intertemporal substitution and  $\gamma > 0$  can be interpreted as relative risk aversion.

## 2.2 Solution

In what follows, I solve for asset prices using the state-price density. The advantage of this method is that it generalizes to the case of time-varying disaster probabilities. A necessary first step when assuming recursive utility is to solve for the value function. Accordingly, Section 2.2.1 describes the solution for the value function. The solution for the riskfree rate, the wealth-consumption ratio and the risk premium on the consumption claim follows easily. Section 2.2.2 describes the solution for the price-dividend ratio for the dividend claim and Section 2.2.3 describes the solution for the expected rate of return on government bills, allowing for partial default.

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<sup>1</sup>More precisely, the probability of  $k$  jumps over the course of a period  $\tau$  is equal to  $e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}$ , where  $\tau$  will be measured in units of years. In the calibrations that follow, the parameter  $\lambda$  will be set to equal 0.0170, implying a 0.0167 probability of a single jump over the course of a year, a 0.00014 probability of two jumps, and so forth.

### 2.2.1 The value function

The first step in solving the model is to solve for the value function, i.e. utility as a function of wealth. Let  $W_t$  denote the wealth of the representative agent and  $J$  the value function. I conjecture that

$$J(W) = \frac{W^{1-\gamma}}{1-\gamma} j^{1-\gamma} \quad (3)$$

for a constant  $j$ . Let  $S_t$  denote the value of the claim to aggregate consumption, and conjecture that the price-dividend ratio for this claim is constant:

$$\frac{S_t}{C_t} = l. \quad (4)$$

The process for consumption and the conjecture (4) imply that  $S_t$  satisfies

$$dS_t = \mu S_{t-} dt + \sigma S_{t-} dB_t + (e^{Z_t} - 1) S_{t-} dN_t. \quad (5)$$

Furthermore, conjecture that the riskfree rate is a constant  $r$ . The Bellman equation for an investor who allocates wealth between the consumption claim and the riskfree asset is

$$\sup_{\alpha, C} \left\{ J_W(W\alpha(\mu - r + l^{-1}) + Wr - C) + \frac{1}{2} J_{WW} W^2 \alpha^2 \sigma^2 + \lambda E_\nu [J(W(1 + \alpha(e^Z - 1))) - J(W)] \right\} + f(C, J) = 0, \quad (6)$$

where  $J_W$  is the first derivative of  $J$  with respect to wealth and  $J_{WW}$  is the second derivative. This formula follows from an application of Ito's Lemma to  $J$  (see Duffie (2001, Appendix F) for Ito's Lemma with jumps).

Substituting (3) into (6) and taking the derivative with respect to  $\alpha$  implies the following first order condition:

$$W^{1-\gamma} j^{1-\gamma} (\mu - r + l^{-1}) - \gamma W^{1-\gamma} j^{1-\gamma} \alpha \sigma^2 + W^{1-\gamma} j^{1-\gamma} \lambda E_\nu [(1 + \alpha(e^Z - 1))^{-\gamma} (e^Z - 1)] = 0.$$

In equilibrium,  $\alpha$  must equal 1. Therefore,

$$\mu + l^{-1} - r = \gamma \sigma^2 - \lambda E_\nu [e^{-\gamma Z} (e^Z - 1)]. \quad (7)$$



Let  $r^C$  denote the instantaneous expected return on the consumption claim, defined as the drift in the price, plus the dividend, plus the expectation of the jump in the price, as a proportion of the current price:

$$r^C \equiv \mu + l^{-1} + \lambda E_\nu[e^Z - 1].$$

It follows from (7) that the instantaneous equity premium on the consumption claim is given by

$$r^C - r = \gamma\sigma^2 + \lambda E_\nu[-e^{-\gamma Z}(e^Z - 1) + e^Z - 1]. \quad (8)$$

Equation (7) also has an interpretation: it is the instantaneous premium on the consumption claim conditional on no disasters.

There are several noteworthy features of (8). The first term, equal to risk aversion times consumption volatility, is the equity premium under the standard diffusion model. The second term is therefore the compensation for the possibility of a rare disaster. Because  $Z < 0$ , the contribution of the disaster risk term to the equity premium is positive and increasing in the disaster intensity  $\lambda$ . Finally, the risk premium depends only on risk aversion, not on the elasticity of intertemporal substitution. The risk premium, including the disaster risk term, is therefore the same for recursive utility as for power utility, a result also found by Liu, Pan, and Wang (2005) and Barro (2007).

In equilibrium,  $W = S$ . Therefore, the conjecture (4) is equivalent to

$$\frac{W}{C} = l, \quad (9)$$

Let  $f_C$  denote the first derivative of  $f(C, V)$  with respect to  $C$ . As shown in Appendix A.1, the envelope condition

$$J_W = f_C \quad (10)$$

together with the form of  $J$ , (3), and the conjecture that the wealth-consumption ratio is constant, implies that

$$l = \beta^{-\psi} j^{\psi-1}. \quad (11)$$

Equations for  $l$  and  $j$  follow from the Bellman equation and are given in Appendix A.1.

The equation for the riskfree rate follows from substituting the equation for  $l$  into (7). Rearranging implies that

$$r = \beta + \frac{1}{\psi}\mu - \frac{1}{2}\left(\gamma + \frac{\gamma}{\psi}\right)\sigma^2 + \lambda E_\nu \left[ -(e^{-\gamma Z} - 1) + \left(1 - \frac{1}{\theta}\right)(e^{(1-\gamma)Z} - 1) \right]. \quad (12)$$

The first three terms in (12) are the same as in the standard model without disaster risk and reflect the roles of the discount rate, intertemporal smoothing and precautionary savings respectively. The last term arises from the risk of rare disasters. To understand this term, first consider the special case of power utility ( $\theta = 1$ ). Because  $e^{-\gamma Z} > 1$ , the riskfree rate is decreasing in  $\lambda$ . An increase in the probability of a rare disaster increases the representative investor's desire to save, and thus lowers the riskfree rate. The greater is risk aversion, the greater is this effect.

In the case of recursive utility, the interpretation of (12) is more complicated. To fix ideas, assume  $\gamma > 1$ . For  $1 - \frac{1}{\theta} > 0$ , or equivalently,  $\gamma > \frac{1}{\psi}$ , an increase in  $\lambda$  has a smaller impact on the riskfree rate for recursive utility than for power utility. That is, increasing the agent's willingness to substitute over time reduces the dependence of  $r$  on  $\lambda$ .

### 2.2.2 The state-price density and the dividend claim

Given the state-price density, the price of any risky asset follows from a no-arbitrage condition. Duffie and Skiadas (1994) show that the state-price density  $\pi_t$  relates to the value function via the formula

$$\pi_t = \exp \left\{ \int_0^t f_V(C_s, V_s) ds \right\} f_C(C_t, V_t). \quad (13)$$

Let  $Y_t$  denote the dividend. I model dividends as levered consumption, i.e.  $Y_t = C_t^\phi$  as in Abel (1999) and Campbell (2003). Ito's Lemma then implies

$$dY_t = \mu_Y Y_{t-} dt + \phi \sigma Y_{t-} dB_t + (e^{\phi Z_t} - 1) Y_{t-} dN_t, \quad (14)$$

where

$$\mu_Y = \phi \mu + \frac{1}{2} \phi (\phi - 1) \sigma^2.$$

Let  $F(Y_t)$  denote the price of the dividend claim (the argument that follows will verify that the price is indeed a function of  $Y_t$ ). No arbitrage implies that

$$F(Y_t) = E_t \left[ \int_t^\infty Y_s \frac{\pi_s}{\pi_t} ds \right]. \quad (15)$$

Given a jump-diffusion process  $x_t$ , let  $Dx_t$  denote the drift of that process,  $\delta x_t$  the diffusion, and  $\mathcal{J}(x_t)$  the expected jump in the process, provided that a jump occurs. As shown in Appendix A.2, the no-arbitrage condition (15) implies that  $F$  satisfies the following:

$$\pi_t(DF_t) + F_t(D\pi_t) + Y_t\pi_t + (\delta\pi_t)(\delta F_t) + \lambda\mathcal{J}(\pi_tF_t) = 0, \quad (16)$$

where  $F_t = F(Y_t)$ . Conjecture that

$$F(Y_t) = l_Y Y_t \quad (17)$$

for a constant  $l_Y$  (which, by definition, equals the price-dividend ratio on the dividend claim). Appendix A.2 shows that substituting (17) into (16) implies

$$\mu_Y + l_Y^{-1} - r = \phi\gamma\sigma^2 + \lambda E_\nu [e^{-\gamma Z} - 1] - \lambda E_\nu [e^{(\phi-\gamma)Z} - 1]. \quad (18)$$

Let  $r^e$  denote the instantaneous expected return on the dividend claim. Analogously to the expected return on the consumption claim,  $r^e$  equals the drift in the price, plus the dividend, plus the expectation of the jump in the price, as a proportion of the current price. The conjecture that the price-dividend ratio is a constant implies that the proportional drift in the price is the same as the proportional drift in the dividend, and the expectation of the proportional jump in the price is the same as the expectation of the proportional jump in the dividend process. Therefore

$$r^e = \mu_Y + l_Y^{-1} + \lambda E_\nu [e^{\phi Z} - 1]. \quad (19)$$

It follows from (18) and (19) that the instantaneous equity premium equals

$$r^e - r = \phi\gamma\sigma^2 + \lambda E_\nu [e^{-\gamma Z} (1 - e^{\phi Z}) + e^{\phi Z} - 1]. \quad (20)$$

The first term in (20) appears in the standard diffusion model and equals risk aversion multiplied by the instantaneous covariance of the consumption

process with the dividend process. The second term accounts for jump risk. It follows from this expression that the term multiplying  $\lambda$  is positive and therefore that disaster risk raises the risk premium and is increasing in  $\lambda$ .<sup>2</sup>

Equations (12) and (18) imply that the price-dividend ratio equals

$$l_Y = \left( \beta - \mu_Y + \frac{1}{\psi} \mu - \frac{1}{2} \left( \gamma + \frac{\gamma}{\psi} - 2\phi\gamma \right) \sigma^2 + \lambda E_\nu \left[ \left( 1 - \frac{1}{\theta} \right) (e^{(1-\gamma)Z} - 1) - (e^{(\phi-\gamma)Z} - 1) \right] \right)^{-1}. \quad (21)$$

The first three terms inside the outer parentheses in (21) appear in the standard diffusion model. To understand the last term, it is useful to consider several special cases. For the remainder of this section, I assume that  $\gamma$  is greater than 1.

First consider the consumption claim, corresponding to  $\phi = 1$ . In this case, the disaster risk term inside of the parentheses in (21) reduces to  $\lambda E_\nu \left[ -\frac{1}{\theta} e^{(1-\gamma)Z} \right]$ . The term in the exponent is positive under the maintained assumption that  $\gamma > 1$ , so  $\lambda E_\nu \left[ -\frac{1}{\theta} e^{(1-\gamma)Z} \right]$  takes the sign opposite to that of  $\theta$ . It follows that the price-dividend ratio decreases in the probability of a disaster if and only if  $\theta < 0$ , i.e. if and only if the EIS is greater than 1. This observation is also made by Barro (2007).

Next consider the case of unit EIS. In the limit as the EIS approaches one,  $1 - 1/\theta$  approaches 1 and the disaster risk term inside the parentheses approaches  $\lambda E_\nu \left[ e^{(1-\gamma)Z} - e^{(\phi-\gamma)Z} \right]$ . Because  $Z < 0$ , this expression is positive if and only if  $\phi > 1$ . Therefore the price-dividend ratio decreases in  $\lambda$  if and only if equity is levered, i.e. if  $\phi > 1$ .

Finally consider power utility, for which  $\theta = 1$ . The disaster risk term reduces to  $\lambda E_\nu \left[ 1 - e^{(\phi-\gamma)Z} \right]$ , which is positive if and only if  $\phi > \gamma$ . Therefore, for power utility, the price-dividend ratio decreases in the disaster probability if and only if leverage exceeds risk aversion.

This last point suggests that the power utility model might encounter difficulties in a dynamic setting in which  $\lambda$  is allowed to vary. For values of risk

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<sup>2</sup>Note that  $e^{\phi Z}$  is between 0 and 1. Therefore  $e^{\phi Z} + (1 - e^{\phi Z}) e^{-\gamma Z}$  is a weighted average of  $e^{-\gamma Z} > 1$  and 1, and is therefore greater than 1.

aversion exceeding leverage, a low price-dividend ratio would indicate that disaster risk is low and the equity premium is low as well. When risk aversion exceeds leverage, the price-dividend ratio would predict excess returns with a positive, not a negative sign, contradicting the data. Moreover, the riskfree rate would be more volatile than the equity premium in such a model.

Equation (21) suggests that recursive utility can help solve this problem. For  $\gamma > 1$ ,  $1 - \frac{1}{\theta} > 0$  if and only if  $\gamma > \frac{1}{\psi}$ . Assuming parameter values in these ranges, the disaster risk term in (21) arising from recursive utility,  $\lambda E_{\nu} \left[ \left(1 - \frac{1}{\theta}\right) (e^{(1-\gamma)Z} - 1) \right]$ , is positive. This term therefore leads the price-dividend ratio to increase less, or possibly to decrease, in the disaster probability.

The net effect of an increase in the disaster probability on the price-dividend ratio reflects the interplay between the effects on the equity premium, on expected future cash flows, and on the riskfree rate. As shown above, the equity premium is increasing in  $\lambda$  regardless of the choice of parameters (provided, of course, that  $\gamma$  and  $\phi$  are positive). Increasing  $\lambda$  decreases future expected cash flows; this effect is relatively small because the probability of a rare disaster is low, and unlike the effect on the equity premium or riskfree rate, it is not amplified by the curvature in the utility function. Finally, the riskfree rate is decreasing in  $\lambda$  for power utility. It is also decreasing for recursive utility provided that  $\theta > 0$ . For power preferences with  $\gamma > \phi$ , the riskfree rate effect dominates the other two effects, and the price-dividend ratio increases in the probability of a disaster. For recursive preferences, it is possible for this effect to reverse because the riskfree rate effect is weaker.

### 2.2.3 Risk of default

As discussed in Barro (2006), disasters often coincide with at least a partial default on government securities. This point is of empirical relevance if one tries to match the behavior of the riskfree asset to the rate of return on government securities in the data. I therefore allow for partial default on government debt, and consider the rate of return on this defaultable security.

Let  $L_t$  be the price process for government liabilities and assume that

$$\frac{dL_t}{L_t} = r^L dt + (e^{Z_{L,t}} - 1) dN_t,$$

where  $r^L$  is the “face value” of government debt (i.e. the amount investors receive if there is no default),  $Z_{L,t}$  is a random variable whose distribution will be described shortly and  $N_t$  is the same Poisson process that drives the consumption process. Assume that, in event of a disaster, there will be a default on government liabilities with probability  $q$ . I follow Barro (2006) and assume that in the event of default, the percent loss is equal to the percent fall in consumption. Therefore,

$$Z_{L,t} = \begin{cases} Z_t & \text{with probability } q \\ 0 & \text{otherwise} \end{cases}$$

Arguments similar to those used to price the dividend claim imply that

$$r^L = r + \lambda E_\nu [e^{-\gamma Z} - 1] - \lambda ((1 - q)E_\nu [e^{-\gamma Z} - 1] + qE_\nu [e^{(1-\gamma)Z} - 1]). \quad (22)$$

Let  $r^b$  denote the instantaneous expected return on government debt. Then

$$r^b = r^L + \lambda q E_\nu [e^Z - 1], \quad (23)$$

with  $r^L$  given in (22). The instantaneous equity premium relative to the government bill rate is therefore (20), plus  $r$ , minus (23):

$$r^e - r^b = \phi \gamma \sigma^2 + \lambda E_\nu [e^{\phi Z} - e^{(\phi-\gamma)Z} + (1 - q)(e^{-\gamma Z} - 1) + q(e^{(1-\gamma)Z} - e^Z)].$$

While the the presence of default for government debt affects the equity premium, it does not affect the price-dividend ratio (21), nor for that matter the total expected return on equities. It only affects how this expected return is decomposed into the equity premium and the government bill rate. Much of the discussion in Section 2.2.2 therefore is unaffected by default. That is, the problems with the power utility model, namely that the price-dividend ratio increases rather than increases in  $\lambda$ , cannot be solved by allowing default on government bills.

Of course, the possibility of default does affect the government bill rate. The premium attached to government bills is increasing in  $\lambda$ ; the greater the risk of disaster, the greater the risk of default, and the more compensation investors require to hold bills. On the other hand, the riskfree rate is decreasing in  $\lambda$  for a wide range of parameter values. These effects offset each other, so that the expected return on government debt varies less with  $\lambda$  than the riskfree rate.

### 3 Time-varying disaster risk

Now consider the case in which the disaster intensity  $\lambda$  is not constant, but rather follows the process

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t) dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}.$$

For analytical convenience, the Brownian motion  $B_{\lambda,t}$  is assumed to be independent of  $B_t$ , the Brownian motion driving the consumption and dividend processes. Given that the risk of disasters has a much greater impact on the equity premium than does diffusion risk, this assumption is unlikely to significantly affect the results.

Figure 1 plots the probability density function for  $\lambda_t$ , assuming the parameter values discussed below. The unconditional mean of the process is  $\bar{\lambda}$  and is calibrated to 0.017 per annum (see Barro (2006)). The distribution is highly skewed; there is a long right tail of high values for  $\lambda$ . The skewness arises on account of the square root term. A high realization of  $\lambda_t$  makes the process more volatile, and thus high values are more likely than they would be under a standard auto-regressive process. Thus the model implies that there are times when “rare” disasters can occur with high probability, but that these times are themselves unlikely.

Further assume that the EIS is equal to 1. This has the advantage of leading to closed-form solutions. Moreover, independent empirical evidence suggests that this value is not unreasonable. Duffie and Epstein (1992) show that, for the limiting case of  $\psi = 1$ , preferences can be expressed using the

aggregator

$$f(C, V) = \beta(1 - \gamma)V \left( \log C - \frac{1}{1 - \gamma} \log((1 - \gamma)V) \right). \quad (24)$$

The model for consumption and dividends is the same as in the previous section, except that the disaster intensity is time-varying.

### 3.1 The value function

I follow the blueprint of the solution in the constant disaster risk case, first solving for the value function and the wealth-consumption ratio. These will allow the state-price density to be expressed in terms of the model's primitives.

Conjecture that the value function takes the form

$$J(\lambda, W) = I(\lambda) \frac{W^{1-\gamma}}{1-\gamma}, \quad (25)$$

where  $I$  is a function of  $\lambda$  that will be derived in this section. Also conjecture that the price-dividend ratio for the claim to aggregate consumption is a constant  $l$ , as in (4). The price of the consumption claim therefore follows the process given by (5) with time-varying disaster intensity  $\lambda_t$ . Let  $r_t$  denote the riskfree rate. The Bellman equation for an investor who allocates wealth between the consumption claim and the riskfree asset is therefore

$$\begin{aligned} \sup_{\alpha_t, C_t} \left\{ J_W W_t \alpha_t (\mu - r_t + l^{-1}) + J_W W_t r_t - J_W C_t + J_\lambda \kappa (\bar{\lambda} - \lambda_t) + \right. \\ \left. \frac{1}{2} J_{WW} W_t^2 \alpha_t^2 \sigma^2 + \frac{1}{2} J_{\lambda\lambda} \sigma_\lambda^2 \lambda_t + \lambda_t E_\nu [J(W_t(1 + \alpha_t(e^{Z_t} - 1)), \lambda_t) - J(W_t, \lambda_t)] \right\} \\ + f(C_t, J) = 0, \quad (26) \end{aligned}$$

where  $J_\lambda$  denotes the first derivative of  $J$  with respect to  $\lambda$  and  $J_{\lambda\lambda}$  the second derivative of  $J$  with respect to  $\lambda$ .

Reasoning identical to that in Section 2.2.1 implies that, in equilibrium,

$$\mu - r_t + l^{-1} = \gamma \sigma^2 - \lambda_t E_\nu [e^{-\gamma Z} (e^Z - 1)]. \quad (27)$$

Let  $r_t^C$  denote the instantaneous expected return on the consumption claim. As in Section 2.2.1, this is defined as the drift in the price, plus the dividend,



plus the expected jump in the price, all as a proportion of the current price:

$$r_t^C \equiv \mu + l^{-1} + \lambda_t E_\nu [e^Z - 1].$$

It then follows from (27) that the instantaneous equity premium on the consumption claim is given by

$$r_t^C - r_t = \gamma\sigma^2 + \lambda_t E_\nu [-e^{-\gamma Z}(e^Z - 1) + e^Z - 1]. \quad (28)$$

Equation (27) gives the instantaneous premium conditional on no disasters. Both terms reduce to their counterparts in Section 2.2.1 for constant  $\lambda_t$ . Because the EIS equals one, the dynamic nature of the model does not effect the premium for the consumption claim.

The value of the wealth-consumption ratio follows from the equilibrium condition  $W = S$  (and therefore  $W/C = l$ ), and the envelope condition (10). Note that

$$f_C(C, V) = \beta(1 - \gamma)\frac{V}{C}. \quad (29)$$

At the optimum,  $V$  is given by (25). The envelope condition therefore implies

$$I(\lambda)W^{-\gamma} = \beta(1 - \gamma)I(\lambda)\frac{W^{1-\gamma}}{1 - \gamma} \frac{1}{l^{-1}W}.$$

Solving for  $l$  yields  $l = \beta^{-1}$ , which equals the limit of (A.7) as  $\psi$  approaches one. The equation for the riskfree rate follows from (27):

$$r_t = \mu + \beta - \gamma\sigma^2 + \lambda_t E_\nu [e^{-\gamma Z} (e^Z - 1)], \quad (30)$$

For  $\lambda_t$  constant, this equation reduces to (12) in the case of  $\psi = 1$ .

Finally, to solve for  $I(\lambda)$ , I substitute (25) and the optimal policy functions into (26). Algebraic computations in Appendix B.1 verify that (25) is a solution to (26), with  $I$  given by

$$I(\lambda) = e^{a+b\lambda}, \quad (31)$$

where

$$\begin{aligned} a &= \frac{1 - \gamma}{\beta} \left( \mu - \frac{1}{2}\gamma\sigma^2 \right) + (1 - \gamma) \log \beta + b \frac{\kappa \bar{\lambda}}{\beta}, \\ b &= \frac{\kappa + \beta}{\sigma_\lambda^2} - \sqrt{\left( \frac{\kappa + \beta}{\sigma_\lambda^2} \right)^2 - 2 \frac{E_\nu [e^{(1-\gamma)Z} - 1]}{\sigma_\lambda^2}}. \end{aligned}$$

Unlike the value function for the model with constant disaster risk, the value function given by (25) and (31) depends on  $\lambda_t$ . In the calibration below,  $b > 0$  and  $\gamma > 1$ . Therefore an increase in disaster risk reduces utility for the representative agent. As the following section shows, the price of the dividend claim falls when the disaster probability rises. The agent requires compensation for this risk (because utility is recursive, marginal utility depends on the value function), and thus time-varying disaster risk increases the equity premium.

### 3.2 The state-price density and the dividend claim

Given the value function, it is possible to compute the process for the state-price density, and therefore to price any risky asset using the no-arbitrage condition. The state-price density  $\pi_t$  is given by (13) for both the constant disaster risk model and the time-varying disaster risk model. However, the processes  $C_t$  and  $V_t$  are different.

As in Section 2.2.2, I derive the price of the dividend claim for dividends  $Y_t = C_t^\phi$ .  $Y_t$  follows the process (14), where the intensity  $\lambda$  varies over time. I conjecture that the price of this claim can be written as a function  $F$  of  $\lambda_t$  and  $Y_t$ . As shown in Appendix B.2, the no-arbitrage condition (15) implies that

$$\pi_t(DF_t) + F_t(D\pi_t) + Y_t\pi_t + (\delta\pi_t)^\top(\delta F_t) + \lambda_t\mathcal{J}(\pi_t F_t) = 0, \quad (32)$$

where  $F_t = F(\lambda_t, Y_t)$ . Because there are two (independent) sources of uncertainty, the diffusion terms  $\delta F_t$  and  $\delta\pi_t$  are  $2 \times 1$  vectors.

In Appendix B.2, I show that (32) is solved by a function of the form

$$F(\lambda, Y) = G(\lambda)Y \quad (33)$$

where

$$G(\lambda) = \int_0^\infty \exp\{a_\phi(t) + b_\phi(t)\lambda\} dt, \quad (34)$$

and where  $a_\phi$  and  $b_\phi$  satisfy the ordinary differential equations

$$a'_\phi(t) = \mu_Y - \mu - \beta + \gamma\sigma^2 - \gamma\sigma^2\phi + \kappa\bar{\lambda}b_\phi(t) \quad (35)$$

$$b'_\phi(t) = \frac{1}{2}\sigma_\lambda^2 b_\phi(t)^2 + (b\sigma_\lambda^2 - \kappa)b_\phi(t) + E_\nu[e^{(\phi-\gamma)Z} - e^{(1-\gamma)Z}] \quad (36)$$

with boundary conditions  $a_\phi(0) = b_\phi(0) = 0$ .

The price of the claim to total dividends can be understood as the integral of prices of claims to dividends at single points in time. The function  $a_\phi(t)$  represents the effect of maturity on the price of these claims for  $\lambda_t$  equal to zero. Equation (35) shows that as maturity increases,  $a_\phi$  is incremented by the value of expected dividend growth and decremented by the values of the riskfree rate and the equity premium when  $\lambda_t = 0$ . The final term in (35) represents the effect of future changes in  $\lambda_t$  on the price. It depends on the effect of  $\lambda_t$  on the price (represented by  $b_\phi(t)$ ), on the average value of  $\lambda_t$  ( $\bar{\lambda}$ ) and on  $\kappa$ , the speed at which  $\lambda_t$  reverts to this average value.

The function  $b_\phi(t)$  represents the effect of maturity interacted with  $\lambda_t$ . The term  $E_\nu[e^{(\phi-\gamma)Z} - e^{(1-\gamma)Z}]$  in (36) summarizes the effect of  $\lambda$  on the price-dividend ratio in the static model for  $\psi = 1$  (see (21)); that is, it represents a combination of the equity premium, riskfree rate, and cash flow effect. The term  $b\sigma_\lambda^2 b_\phi(t)$  is an additional component of the equity premium in the dynamic model, and will be discussed further below. The remaining two terms,  $\frac{1}{2}\sigma_\lambda^2 b_\phi(t)^2$  and  $-\kappa b_\phi(t)$  represent the effect of future changes in  $\lambda_t$  on the price. The former is a Jensen's inequality term; all else equal, a more volatile equity premium increases the price-dividend ratio. The latter represents the fact that, if  $\lambda_t$  is high in the present,  $\lambda_t$  is likely to decrease in the future on account of mean reversion.

Figure 2 shows  $a_\phi(t)$  and  $b_\phi(t)$  as functions of time for parameter values described below. The top panel shows that  $a_\phi$  asymptotes to a decreasing linear function. The fact that this is decreasing is necessary for convergence if  $\lambda_t$  varies over time.<sup>3</sup> The asymptote is linear because  $b_\phi(t)$  asymptotes to a constant. The figure also shows that  $b_\phi(t)$  is negative, and thus the price-dividend ratio is decreasing in  $\lambda_t$ . Thus the equity premium effect, together with the cash flow effect, dominates the riskfree rate effect at these parameter values. The magnitude of  $b_\phi(t)$  increases in  $t$ . This is a duration effect. The further out in time the cash flows occur, the more the price of the claim varies

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<sup>3</sup>When  $\lambda_t$  does not vary over time, then convergence requires that  $a_\phi(t) + b_\phi(t)\bar{\lambda}$  is decreasing.

with the discount rate.

As discussed in Section 2.2, the instantaneous expected return is defined as the drift in the price (as a proportion), plus the dividend yield, plus the expected jump in in the price (as a proportion). For this more general model, this implies

$$r_t^e \equiv \frac{1}{F_t} (DF_t + Y_t + \lambda_t \mathcal{J}(F_t)). \quad (37)$$

Equation (32) can be rearranged to give a convenient characterization of the risk premium. No-arbitrage implies that the riskfree rate is characterized by

$$r_t = -\frac{D\pi_t}{\pi_t} - \lambda_t \frac{\mathcal{J}(\pi_t)}{\pi_t}. \quad (38)$$

Combining (32), (37) and (38) implies a useful characterization of the equity premium:

$$r_t^e - r_t = -\left(\frac{\delta\pi_t}{\pi_t}\right)^\top \left(\frac{\delta F_t}{F_t}\right) - \lambda_t \left(\frac{\mathcal{J}(F_t\pi_t)}{F_t\pi_t} - \frac{\mathcal{J}(F_t)}{F_t} - \frac{\mathcal{J}(\pi_t)}{\pi_t}\right).$$

The first term represents the portion of the equity premium that is compensation for diffusion risk (which includes time-varying  $\lambda_t$ ), while the second represents the portion from jump risk. The second term can be understood as the jump equivalent of a covariance. The greater the covariance of the price  $F_t$  with the marginal utility process  $\pi_t$ , the more the asset provides a hedge and the lower is the risk premium.

To write this equation in terms of the model primitives, first note that Ito's Lemma applied to the state-price density implies that

$$\frac{\delta\pi_t}{\pi_t} = \begin{bmatrix} -\gamma\sigma \\ b\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix}. \quad (39)$$

The negative of the first element of the vector (39) is the price of diffusion risk, where the negative of the second element is the price of risk associated with time-varying  $\lambda_t$ . Second, note that Ito's Lemma applied to  $F_t = F(\lambda_t, Y_t)$  implies that

$$\delta F_t = \begin{bmatrix} Y_t G(\lambda_t) \phi \sigma \\ Y_t G'(\lambda_t) \sigma_\lambda \sqrt{\lambda_t} \end{bmatrix}. \quad (40)$$

Further calculations in Appendix B.2 show that

$$r_t^e - r_t = \phi\gamma\sigma^2 - \lambda_t \frac{G'}{G} b\sigma_\lambda^2 + \lambda_t E_\nu [e^{-\gamma Z}(1 - e^{\phi Z}) + e^{\phi Z} - 1]. \quad (41)$$

The first term is identical to its counterpart in the static model and equals risk aversion multiplied by the instantaneous covariance with consumption. The second term is new to the dynamic model. This is the risk premium due to time-variation in disaster risk. Because  $b_\phi$  is negative,  $G'$  is also negative. Moreover,  $b$  is positive, so this term represents a positive contribution to the equity premium. Finally, the last term represents the portion of the equity premium arising from disaster risk itself. It takes the same form as its counterpart in the static model of Section 2, except, of course, that it varies over time.

### 3.3 Risk of default

The calculation for the government bill rate is similar to the corresponding calculation in the case of constant disaster risk. The face value of the government bill is given by

$$r_t^L = r_t + \lambda_t E_\nu [e^{-\gamma Z_t} - 1] - \lambda_t ((1 - q)E_\nu [e^{-\gamma Z_t} - 1] + qE_\nu [e^{(1-\gamma)Z_t} - 1]).$$

This is also the expected return, conditional on no disasters occurring. The instantaneous expected return on government debt is

$$r_t^b = r_t^L + \lambda_t q E_\nu [e^Z - 1]. \quad (42)$$

Figure 3 shows the face value of government debt,  $r_t^L$ , the instantaneous expected return on government debt  $r_t^b$  and the riskfree rate  $r_t$  as a function of  $\lambda_t$ . Because of the required compensation for default,  $r_t^L$  lies above  $r_t$ . The expected return lies between the two because the actual cash flow that investors receive from the government bill will be below  $r_t^L$  if default occurs.

All three rates decrease in  $\lambda_t$  because, at these parameter values, a higher  $\lambda_t$  induces a greater desire to save. However,  $r_t^L$  and  $r_t^b$  are less sensitive to changes in  $\lambda$  than  $r_t$  because of an opposing effect: the greater is  $\lambda_t$ , the greater

is the risk of default, and therefore the greater the return investors demand for holding the government bill. Because of a small cash flow effect,  $r_t^b$  decreases more than  $r_t^L$ , but still less than  $r_t$ .

The instantaneous equity premium relative to the government bill rate is equal to (41) plus  $r_t$ , minus  $r_t^b$ .

$$r^e - r^b = \phi\gamma\sigma^2 - \lambda_t \frac{G'}{G} b\sigma_\lambda^2 + \lambda_t E_\nu [e^{\phi Z} - e^{(\phi-\gamma)Z} + (1-q)(e^{-\gamma Z} - 1) + q(e^{(1-\gamma)Z} - e^Z)].$$

This instantaneous equity premium is shown in Figure 4 (solid line). The difference between the dashed line and the solid line represents the hedging component of the equity premium, namely  $-b\lambda_t\sigma_\lambda^2\frac{G'}{G}$ , and shows that this term is large in magnitude. The dashed line represents the equity premium in the standard diffusion model without disaster risk and is negligible compared with the disaster risk component.

## 4 Calibration and Simulation

This section first describes the calibration of the time-varying disaster risk model and results from simulated data. I then compare these results with those obtained from a model with constant disaster risk.

### 4.1 Calibration

Table 1 describes the parameters in the main case. Most parameters are set to values considered by Barro (2006) to highlight this model's novel implications. In the model, time is measured in units of years and parameter values should be interpreted accordingly. The drift rate  $\mu$  is calibrated so that in normal periods, the expected growth rate of log consumption is 2.5% per annum.<sup>4</sup> The standard deviation of log consumption  $\sigma$  is 2% per annum. These parameters are chosen as in Barro to match postwar data in G7 countries. The average disaster intensity is  $\bar{\lambda}$ , set equal to 0.017. The decline in consumption when a

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<sup>4</sup>The value  $\mu = 2.52\%$  reflects an adjustment for Jensen's inequality.

disaster does occur,  $Z_t$ , is calibrated to the empirical distribution of declines in GDP. The probability of default given disaster,  $q$ , is set equal to 0.4, which is the probability of default given a disaster. These values are calculated by Barro based on data for 35 countries over the period 1900–2000.

Leverage,  $\phi$ , is set equal to 2.8. This is slightly lower than the value used by Bansal and Yaron (2004) in their calibrated model of the equity premium. Given the high ratio of dividend volatility to consumption volatility, this value is conservative. Barro considers values of risk aversion equal to 3 and 4 and values of the rate of time preference equal to 0.02 and 0.03. I choose risk aversion equal to 3 and rate of time preference equal to 0.02 because, given other parameter choices, these deliver the closest match to the equity premium and the riskfree rate in the present model.

The novel parameters are the EIS  $\psi$ , the mean reversion of the disaster intensity,  $\kappa$ , and the volatility parameter for the disaster intensity,  $\sigma_\lambda$ . The EIS is set equal to 1 for tractability. A number of studies have concluded that reasonable values for this parameter lie in a range close to one, or slightly lower than 1 (see Campbell (2003) for a discussion). Mean reversion  $\kappa$  is chosen to match the autocorrelation of the price-dividend ratio in annual U.S. data. Because  $\lambda_t$  is the single state variable, the autocorrelation of price-dividend ratio implied by the model will approximately equal the autocorrelation of  $\lambda_t$ . Setting  $\kappa$  equal to 0.142 generates an autocorrelation for the price-dividend ratio equal to 0.865, its value in the data. The volatility parameter  $\sigma_\lambda$  is chosen to be 0.09; this generates a reasonable level of volatility in stock returns.

## 4.2 Results for the time-varying disaster risk model

Table 2 describes moments from a simulation of the model as well as moments from annual U.S. data. Annual U.S. data come from Robert Shiller’s website. Data are from 1890 to 2004 and are described in detail in Shiller (1989, Chap. 26).

The model is discretized using an Euler approximation (e.g. Lord, Koekkoek, and van Dijk (2006)) and simulated at a monthly frequency for 50,000 years;

simulating the model at higher frequencies produces negligible differences in the results.<sup>5</sup> The monthly results are then compounded to an annual frequency to compare with the annual data set. Two types of moments are reported. The first type (referred to as “population” in the tables) are calculated based on all years in the simulation. The second type (referred to as “conditional” in the tables) are calculated after first eliminating years in which one or more disasters took place. This second type therefore conditions on no disasters occurring. Neither corresponds exactly to U.S. data, and for this reason, the data should be viewed as an approximate benchmark.

Table 2 shows that the model generates a realistic equity premium. In population, the equity premium is 5.6%, while, conditional on no disasters, the equity premium is 6.4%. In the historical data it is 6.0%. The expected return on the government bill is 2.5% in population, 2.8% conditional on no disasters, and 2.0% in the data. The model predicts equity volatility of 18.4% per annum in population and 16.4% conditional on no disasters. The observed volatility is 18.5%. The Sharpe ratio is 0.31 in population, 0.38 conditional on no disasters and 0.32 in the data (the Sharpe ratio is substantially higher over the postwar period than in the long data set).

The model is able to generate reasonable volatility for the stock market without generating excessive volatility for the government bill or for consumption and dividends. The volatility of the government bill is 3.6% in population, much of which is due to realized disasters; it is 2.7% conditional on no disasters. This compares with a volatility of 5.9% in the data. Given that interest rate volatility in the data arises largely from unexpected inflation that is not captured by the model, the data volatility should be viewed as an upper bound on reasonable model volatility.

The volatilities for consumption and dividends predicted by the model for periods of no disasters are also below their data counterparts. Conditional on no disasters, consumption volatility is 2.0%, compared with 3.6% in the data. Dividend volatility is 5.6%, compared with 11.5% in the data. Including rare

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<sup>5</sup>The discrete-time approximation requires setting  $\lambda_t$  to equal zero in the square root when it is negative. However, this occurs in less than 1% of the simulated draws.



disasters in the data simulated from the model has a large effect on dividend volatility. When the rare disasters are included, dividend volatility is 17.2%. The difference between the effect of including rare disasters on returns as compared with the effect on fundamentals is striking. Unlike dividends, returns exhibit a relatively small difference in volatility when calculated with and without rare disasters: 18.4% versus 16.4%. This is because a large amount of the volatility in returns arises from variation in the equity premium. Risk premia are equally variable regardless of whether disasters actually occur in the simulated data or not.

The model also generates excess return predictability, as shown in Table 3. I regress long-horizon excess returns (the log return on equity minus the log return on the government bill) on the price-dividend ratio in simulated data. I calculate this regression for returns measured over horizons ranging from 1 to 10 years. Table 3 reports results for the entire simulated data set (“population moments”) for periods in the simulation in which no disasters occur (“conditional moments”) and for the historical sample.

Panel A of Table 3 shows population moments from simulated data. The coefficients on the price-dividend ratio are negative: a high price-dividend ratio corresponds to low disaster risk and therefore predicts low future expected returns on stocks relative to bonds. The  $R^2$  for the long-horizon regression is 4% at a horizon of 1 year, rising to 14% at a horizon of 10 years. Panel B reports conditional moments. The conditional  $R^2$ s are larger: 16% at a horizon of 1 year, rising to 56% at a horizon of 10 years. The unconditional  $R^2$  values are much lower because, when a disaster occurs, nearly all of the unexpected return is due to the shock to cash flows.

The data moments fall in between the population and conditional moments. As demonstrated in a number of studies (e.g. Campbell and Shiller (1988), Cochrane (1992), Fama and French (1989), Keim and Stambaugh (1986)) and replicated in this sample, high price-dividend ratios predict low excess returns. While returns exhibit predictability over a wide range of sample periods, the high persistence of the price-dividend ratio leads sample statistics to be unstable (see, for example, Lettau and Wachter (2007) for calculations of long-

horizon predictability using this data set but for differing sample periods), and unusually low when calculated over recent years. For this reason, the  $R^2$  statistics in the data should be viewed as an approximate benchmark.

Another potential source of variation in returns is variation in expected future consumption growth. According to the model, a low price-dividend ratio indicates not only that the equity premium is likely to be high in the future, but also that consumption growth is likely to be low because of the increased probability of a disaster. However, a number of studies (e.g. Campbell (2003), Cochrane (1994), Hall (1988), Lettau and Ludvigson (2001)) have found that consumption growth exhibits little predictability at long horizons, a finding replicated in Panel B of Table 4. It is therefore of interest to quantify the amount of consumption growth predictability implied by the model.

Table 4 reports the results of running long-horizon regressions of consumption growth on the price-dividend ratio in data simulated from the model and in historical data. Panel A shows the population moments implied by the model. The model does imply some predictability in consumption growth, but the effect is very small. The  $R^2$  values never rise above 6%, even at long horizons. This predictability arises entirely from the realization of a rare disaster. When these rare disasters are conditioned out, there is zero predictability because consumption follows a random walk (in simulated data, the coefficient values are less than .001 and the  $R^2$  values are less than .0001). Thus the model accounts for both the predictability in long-horizon returns and the absence of predictability in consumption growth.

### 4.3 Comparison with the constant disaster risk model

It is instructive to contrast the results in the previous section with results when the probability of a rare disaster does not vary over time. Table 5 calculates moments corresponding to those in Table 2 for the constant disaster risk model. The long-horizon regressions in Tables 3 and 4 are not repeated because, when disaster risk is constant, the predictability coefficients and  $R^2$  statistics are zero at all horizons. The first calibration (results reported in Panel A) corresponds

to the parameters in Barro (2006), and replicates the results in that paper.<sup>6</sup> The second calibration (results reported in Panel B) alters these parameters slightly: rather than assuming power utility, the EIS is set to 1. This highlights the role of recursive utility when nothing else changes. The third calibration (results reported in Panel C) maintains the assumptions of the second set, but raises leverage from 1.5 to 2.8. This shows the effect of an increase in leverage when nothing else changes.

As Panel A shows, the power utility model with constant disaster probabilities is capable of replicating the equity premium in the data, and reconciling it with a low riskfree rate. This model is not, however, capable of replicating the volatility of the stock market. Stock return volatility is about equal to dividend volatility.<sup>7</sup> That is, the conditional volatility of stock returns is 3.3%. This compares with a volatility of 18.5% in the data. The population volatility of stock returns is 6.7%, still far below the data volatility. As a consequence, the Sharpe ratio predicted by the model is 1.83 (conditional on no disasters), much higher than the observed value (0.32).

Panel B demonstrates the effect of recursive utility. This simulation is identical to the above, except the EIS is set equal to 1, the value from the base case. As this simulation verifies, recursive utility makes little difference to the equity premium. The volatility of stock returns and the Sharpe ratio are also nearly the same. However, the government bill rate is substantially lower: 1.7% per annum rather than 3.9%. The reason is that the inverse of the EIS multiplies the growth rate of the economy in the formula for the riskfree rate. Because growth is positive, power utility, with an EIS of 0.25, generates a higher riskfree rate than recursive utility with an EIS of 1.

Panel C demonstrates the effect of increasing leverage from 1.5 to 2.8. Not surprisingly, raising leverage increases the volatility of log dividends. However,

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<sup>6</sup>The model for leverage in this paper differs slightly from Barro's. However, the effects of the differences are second-order.

<sup>7</sup>Stock market volatility and dividend volatility are not exactly equal because the table reports the volatility of stock returns measured in levels, while dividends are measured in logs. The volatility of log returns is identical to the volatility of log dividends for both population and conditional moments.

this does not generate nearly enough volatility in either the conditional or full population to match the volatility of returns in the data. These results contrast with the model that incorporates both recursive utility and time-varying disaster risk, which can generate a realistic amount of stock market volatility.

## 5 Conclusion

This paper has shown that a continuous-time endowment model in which there is time-varying risk of a rare disaster can explain many features of the aggregate stock market. Besides explaining the equity premium without assuming a high value of risk aversion, it can also explain the high level of stock market volatility. The volatility of the government bill rate remains low because of a tradeoff between an increased desire to save due to an increase in the disaster probability and a simultaneous increase in the risk of default. The model therefore offers a novel explanation of volatility in the aggregate stock market that is consistent with other macroeconomic data. Moreover, the model accounts for economically significant excess return predictability found in the data, as well as the lack of long-run consumption growth predictability. Finally, the model can be solved in closed form, allowing for straightforward computation and for potential extensions. While this paper has focused on the aggregate stock market, the model could be extended to price additional asset classes, such as long-term government bonds, options and exchange rates.

# Appendix

## A Constant disaster risk model

### A.1 Value function

The envelope condition (10) can be used to derive the relation between  $l$  (the wealth-consumption ratio) and  $j$  (the constant in the value function) in (11). Equation (3) implies

$$\begin{aligned} J_W(W(C)) &= (W(C))^{-\gamma} j^{1-\gamma} \\ &= (lC)^{-\gamma} j^{1-\gamma}. \end{aligned} \quad (\text{A.1})$$

Moreover,

$$f_C(C, V) = \frac{\beta C^{-\frac{1}{\psi}}}{((1-\gamma)V)^{\frac{1}{\theta}-1}}, \quad (\text{A.2})$$

where it follows from (4) that

$$V(C) = J(W(C)) = \frac{(lC)^{1-\gamma}}{1-\gamma} j^{1-\gamma}. \quad (\text{A.3})$$

Substituting (A.3) into (A.2) implies

$$\begin{aligned} f_C(C, V(C)) &= \beta C^{-\frac{1}{\psi}} C^{(1-\frac{1}{\theta})(1-\gamma)} (lj)^{(1-\frac{1}{\theta})(1-\gamma)} \\ &= \beta C^{-\gamma} (lj)^{\frac{1}{\psi}-\gamma}. \end{aligned} \quad (\text{A.4})$$

Equating (A.4) with (A.1) and solving for  $l$  implies (11).

Given (11), the expression for  $j$  follows from the fact that  $V(C(W)) = J(W)$ , and so

$$f(C(W), J(W)) = \frac{\beta}{1-\frac{1}{\psi}} W^{1-\gamma} \frac{l^{\frac{1}{\psi}-1} - j^{1-\frac{1}{\psi}}}{j^{\gamma-\frac{1}{\psi}}}. \quad (\text{A.5})$$

Substituting into the Bellman equation (6) and dividing through by  $W^{1-\gamma}$  yields:

$$\begin{aligned} \left( \mu - \frac{1}{2}\gamma\sigma^2 + \lambda(1-\gamma)^{-1} E_\nu [e^{(1-\gamma)Z} - 1] \right) j^{1-\gamma} + \\ \frac{\beta}{1-\frac{1}{\psi}} \frac{\beta^{\psi-1} j^{(1-\frac{1}{\psi})(1-\psi)} - j^{1-\frac{1}{\psi}}}{j^{\gamma-\frac{1}{\psi}}} = 0, \end{aligned}$$

and therefore

$$\mu - \frac{1}{2}\gamma\sigma^2 + \lambda(1-\gamma)^{-1}E_\nu [e^{(1-\gamma)Z} - 1] - \frac{\beta}{1-\frac{1}{\psi}} + \frac{\beta^\psi}{1-\frac{1}{\psi}}j^{1-\psi} = 0,$$

Rearranging implies

$$j = \left[ \left( -\mu + \frac{1}{2}\gamma\sigma^2 - \lambda(1-\gamma)^{-1}E_\nu [e^{(1-\gamma)Z} - 1] + \frac{\beta}{1-\frac{1}{\psi}} \right) \frac{1-\frac{1}{\psi}}{\beta^\psi} \right]^{\frac{1}{1-\psi}}, \quad (\text{A.6})$$

and therefore that the wealth-consumption ratio  $l$  equals:

$$l = \left( -\mu + \frac{1}{2}\gamma\sigma^2 - \lambda(1-\gamma)^{-1}E_\nu [e^{(1-\gamma)Z} - 1] + \frac{\beta}{1-\frac{1}{\psi}} \right)^{-1} \left( 1 - \frac{1}{\psi} \right)^{-1}. \quad (\text{A.7})$$

## A.2 Dividend claim

By Ito's Lemma,

$$\begin{aligned} d(F_t\pi_t) + Y_t\pi_t dt &= \pi_t(DF_t) dt + F_t(D\pi_t) dt + Y_t\pi_t dt + \\ &\pi_t(\delta F_t) dB_t + F_t(\delta\pi_t) dB_t + (\delta\pi_t)(\delta F_t) dt + \\ &\lambda\mathcal{J}(\pi_t F_t) dt + ((\pi_t F_t - \pi_{t-} F_{t-})dN_t - \lambda\mathcal{J}(\pi_t F_t)) dt, \end{aligned} \quad (\text{A.8})$$

Under mild regularity conditions (see Duffie, Pan, and Singleton (2000)), the no-arbitrage condition (15) implies that the instantaneous expectation of (A.8) is equal to zero. This establishes (16).

Ito's Lemma applied to (13) implies that the diffusion term for the state-price density is given by

$$\begin{aligned} \delta\pi_t &= \frac{\pi_t}{f_C} \delta f_C \\ &= -\frac{\pi_t}{f_C} \beta \gamma C^{-\gamma-1} (lj)^{\frac{1}{\psi}-\gamma} C \sigma \\ &= -\pi_t \gamma \sigma, \end{aligned} \quad (\text{A.9})$$

where the second line follows from (A.4). No-arbitrage implies that the drift of the state-price density must equal

$$D\pi_t = -r\pi_t - \lambda\mathcal{J}(\pi_t).$$

Let

$$H_t = \exp \left\{ \int_0^t f_V(C_s, V_s) ds \right\}. \quad (\text{A.10})$$

Then

$$\begin{aligned} \mathcal{J}(\pi_t) &= H_t E_\nu [f_C(Ce^{Z_t}, V(Ce^{Z_t})) - f_C(C, V(C))] \\ &= H_t E_\nu \left[ \beta (Ce^{Z_t})^{-\gamma} (lj)^{\frac{1}{\psi} - \gamma} - \beta C^{-\gamma} (lj)^{\frac{1}{\psi} - \gamma} \right] \\ &= \pi_t E_\nu [e^{-\gamma Z_t} - 1]. \end{aligned} \quad (\text{A.11})$$

By Ito's Lemma,

$$DF_t = F_Y \mu_Y Y_t + \frac{1}{2} F_{YY} Y_t \phi^2 \sigma^2 = l_Y \mu_Y Y_t \quad (\text{A.12})$$

$$\delta F_t = F_Y \phi \sigma Y_t = l_Y \phi \sigma Y_t \quad (\text{A.13})$$

Finally,

$$\begin{aligned} \mathcal{J}(\pi_t F_t) &= H_t E_\nu [f_C(Ce^{Z_t}, V(Ce^{Z_t})) F(Y_t e^{\phi Z_t}) - f_C(C, V(C)) F(Y_t)] \\ &= \pi_t E_\nu [e^{-\gamma Z_t} F(Y_t e^{\phi Z_t}) - F(Y_t)] \end{aligned} \quad (\text{A.14})$$

$$= l_Y E_\nu [e^{(\phi - \gamma)Z} - 1] \pi_t Y_t \quad (\text{A.15})$$

Substituting (A.9 – A.15) into (16) verifies the conjecture (17), for  $l_Y$  defined implicitly by (18).

## B Time-varying disaster risk model

### B.1 Value function

Substituting the optimal policies  $\alpha = 1$  and  $W = lC$  into the Bellman equation (26) implies

$$\begin{aligned} J_W W_t \mu + J_\lambda \kappa (\bar{\lambda} - \lambda_t) + \frac{1}{2} J_{WW} W_t^2 \sigma^2 + \frac{1}{2} J_{\lambda\lambda} \sigma_\lambda^2 \lambda_t + \\ \lambda_t E_\nu [J(W_t e^{Z_t}, \lambda_t) - J(W_t, \lambda_t)] + f(C_t, J) = 0. \end{aligned} \quad (\text{B.1})$$

Using (25), the last term can be rewritten as follows:

$$\begin{aligned} f(C(W), J(\lambda, W)) &= \beta I(\lambda) W^{1-\gamma} \left( \log(\beta W) - \frac{1}{1-\gamma} \log(I(\lambda) W^{1-\gamma}) \right) \\ &= \beta I(\lambda) W^{1-\gamma} \left( \log \beta - \frac{\log I(\lambda)}{1-\gamma} \right). \end{aligned} \quad (\text{B.2})$$

Substituting (25) and (B.2) into (B.1) and dividing through by  $W^{1-\gamma}$  implies

$$\begin{aligned} I(\lambda_t)\mu + I'(\lambda_t)(1-\gamma)^{-1}\kappa(\bar{\lambda} - \lambda_t) - \frac{1}{2}\gamma I(\lambda_t)\sigma^2 + \frac{1}{2}(1-\gamma)^{-1}I''(\lambda_t)\sigma_\lambda^2\lambda_t \\ + (1-\gamma)^{-1}I(\lambda_t)\lambda_t E_\nu [e^{(1-\gamma)Z} - 1] \\ + \beta I(\lambda_t) \left( \log \beta - \frac{\log I(\lambda)}{1-\gamma} \right) = 0. \end{aligned} \quad (\text{B.3})$$

Conjecture that  $I(\lambda)$  is given by (31), implying that  $I'(\lambda) = bI(\lambda)$  and  $I''(\lambda) = b^2I(\lambda)$ . Substituting into (B.3), I find

$$\begin{aligned} \mu + b(1-\gamma)^{-1}\kappa(\bar{\lambda} - \lambda_t) - \frac{1}{2}\gamma\sigma^2 + \frac{1}{2}b^2\sigma_\lambda^2\lambda_t(1-\gamma)^{-1} + \\ (1-\gamma)^{-1}\lambda_t E_\nu [e^{(1-\gamma)Z} - 1] + \beta (\log \beta - (1-\gamma)^{-1}(a + b\lambda_t)) = 0. \end{aligned}$$

Collecting terms in  $\lambda_t$  results in the following quadratic equation for  $b$ :

$$\frac{1}{2}\sigma_\lambda^2 b^2 - (\kappa + \beta)b + E_\nu [e^{(1-\gamma)Z} - 1] = 0,$$

indicating two possible solutions

$$b_+ = \frac{\kappa + \beta}{\sigma_\lambda^2} + \sqrt{\left(\frac{\kappa + \beta}{\sigma_\lambda^2}\right)^2 - 2\frac{E_\nu [e^{(1-\gamma)Z} - 1]}{\sigma_\lambda^2}}, \quad (\text{B.4})$$

and

$$b_- = \frac{\kappa + \beta}{\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa + \beta}{\sigma_\lambda^2}\right)^2 - 2\frac{E_\nu [e^{(1-\gamma)Z} - 1]}{\sigma_\lambda^2}}. \quad (\text{B.5})$$

Collecting constant terms results in the following characterization of  $a$  in terms of  $b$ :

$$a = \frac{1-\gamma}{\beta} \left( \mu - \frac{1}{2}\gamma\sigma^2 \right) + (1-\gamma)\log \beta + b\frac{\kappa\bar{\lambda}}{\beta}. \quad (\text{B.6})$$

Following Tauchen (2005), I choose the negative root, i.e.  $b = b_-$ . The resulting solution has the desirable property of approaching a well-defined limit as  $\sigma_\lambda$  approaches 0, as shown in Appendix B.3. However, given the parameter choices in Table 1, the discriminant is of the order of  $10^{-5}$ , and therefore the asset pricing implications of this choice are negligible.



## B.2 Dividend claim

Applying Ito's Lemma to the product  $F(\lambda, Y)\pi$  in the case of time-varying disaster risk yields

$$\begin{aligned} d(F_t\pi_t) + Y_t\pi_t dt &= \pi_t(DF_t) dt + F_t(D\pi_t) dt + Y_t\pi_t dt + \\ &\pi_t(\delta F_t)^\top [dB_t dB_{\lambda,t}]^\top + F_t(\delta\pi_t)^\top [dB_t dB_{\lambda,t}]^\top + (\delta\pi_t)^\top (\delta F_t) dt + \\ &\lambda\mathcal{J}(\pi_t F_t) dt + ((\pi_t F_t - \pi_{t-} F_{t-})dN_t - \lambda\mathcal{J}(\pi_t F_t)) dt, \end{aligned} \quad (\text{B.7})$$

where I have used the fact that  $B_t$  and  $B_{\lambda,t}$  are independent. Equation (32) then follows from the argument given in Appendix A.2.

The following argument derives a differential equation for  $G$  using (32). First the drift, diffusion and jump terms for the state-price density are derived. As in the case of constant disaster risk, Ito's Lemma applied to (13) implies that the diffusion term for the state-price density is given by

$$\delta\pi_t = \frac{\pi_t}{f_C} \delta f_C$$

It follows from (25) that continuation utility equals

$$V(\lambda, C) = J(\lambda, W(C)) = J(\lambda, lC) = l^{1-\gamma} I(\lambda) \frac{C^{1-\gamma}}{1-\gamma}.$$

Therefore, by (29),

$$f_C(C, V(\lambda, C)) = \beta l^{1-\gamma} I(\lambda) C^{-\gamma} = \beta^\gamma I(\lambda) C^{-\gamma} \quad (\text{B.8})$$

It follows from Ito's Lemma that

$$\frac{\delta f_C}{f_C} = \begin{bmatrix} -\gamma\sigma \\ b\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix}$$

and therefore

$$\frac{\delta\pi_t}{\pi_t} = \begin{bmatrix} -\gamma\sigma \\ b\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix}. \quad (\text{B.9})$$

No-arbitrage implies that the drift of the state-price density is given by

$$D\pi_t = -r_t\pi_t - \lambda\mathcal{J}(\pi_t). \quad (\text{B.10})$$

Defining  $H_t$  as in (A.10), it from (13) that

$$\begin{aligned}
\mathcal{J}(\pi_t) &= H_t E_\nu [f_C(C_t e^{Z_t}, V(\lambda_t, C_t e^{Z_t})) - f_C(C_t, V(\lambda_t, C_t))] \\
&= H_t \beta^\gamma I(\lambda_t) E_\nu [C_t^{-\gamma} e^{-\gamma Z_t} - C_t^{-\gamma}] \\
&= \pi_t E_\nu [e^{-\gamma Z_t} - 1].
\end{aligned} \tag{B.11}$$

The calculation is completed by substituting (30) in for  $r_t$ .

Next the drift, diffusion and jump terms for  $F$  are calculated. It follows from (33) and Ito's Lemma that

$$\delta F_t = \begin{bmatrix} Y_t G(\lambda_t) \phi \sigma \\ Y_t G'(\lambda_t) \sigma_\lambda \sqrt{\lambda_t} \end{bmatrix}, \tag{B.12}$$

$$DF_t = G(\lambda_t) Y_t \mu_Y + G'(\lambda_t) Y_t \kappa (\bar{\lambda} - \lambda_t) + \frac{1}{2} G''(\lambda_t) Y_t \sigma_\lambda^2 \lambda_t, \tag{B.13}$$

$$\mathcal{J}(F_t) = F_t E_\nu [e^{\phi Z} - 1]. \tag{B.14}$$

Finally,

$$\begin{aligned}
\mathcal{J}(\pi_t F_t) &= H_t G(\lambda_t) E_\nu [f_C(C e^Z, V(\lambda_t, C e^Z)) Y_t e^{\phi Z} - f_C(C_t, V(\lambda_t, C_t)) Y_t] \\
&= H_t G(\lambda_t) Y_t E_\nu [\beta^\gamma I(\lambda_t) C_t^{-\gamma} e^{(\phi-\gamma)Z_t} - \beta^\gamma I(\lambda_t) C_t^{-\gamma}] \\
&= \pi_t F_t E_\nu [e^{(\phi-\gamma)Z} - 1].
\end{aligned} \tag{B.15}$$

Substituting (B.9 – B.15) into (32) implies

$$\begin{aligned}
&G \mu_Y + G' \kappa (\bar{\lambda} - \lambda_t) + \frac{1}{2} G'' \sigma_\lambda^2 \lambda_t - G (\mu + \beta - \gamma \sigma^2 + \lambda_t E_\nu [e^{-\gamma Z} (e^Z - 1)]) \\
&\quad - G \lambda_t E_\nu [e^{-\gamma Z} - 1] + 1 - G \gamma \sigma^2 \phi + G' b \sigma_\lambda^2 \lambda_t \\
&\quad + G \lambda_t E_\nu [e^{(\phi-\gamma)Z} - 1] = 0.
\end{aligned} \tag{B.16}$$

To derive the ordinary differential equations (35) and (36), note that the conjecture (34) implies

$$G'(\lambda_t) = \int_0^\infty \exp \{a_\phi(s) + b_\phi(s) \lambda_t\} b_\phi(s) ds \tag{B.17}$$

$$G''(\lambda_t) = \int_0^\infty \exp \{a_\phi(s) + b_\phi(s) \lambda_t\} b_\phi(s)^2 ds. \tag{B.18}$$

The solution must satisfy the boundary conditions  $a_\phi(0) = b_\phi(0) = 0$  and

$$\lim_{t \rightarrow \infty} \exp \{a_\phi(t) + b_\phi(t) \lambda_t\} = 0 \quad \forall \lambda_t > 0.$$

It follows from the boundary conditions and integration by parts that<sup>8</sup>

$$- \int_0^\infty \exp \{a_\phi(s) + b_\phi(s)\lambda_t\} (a'_\phi(s) + b'_\phi(s)\lambda_t) ds = 1. \quad (\text{B.19})$$

It follows from (34), (B.17), (B.18) and (B.19) that each term in (B.16) takes the form of an integral of  $\exp \{a_\phi(s) + b_\phi(s)\lambda_t\}$  multiplied by an expression that is linear in  $\lambda_t$ . Equation (34) can therefore be solved by setting the terms within the integral equal to zero, or equivalently, solving

$$\begin{aligned} \mu_Y + b_\phi(s)\kappa(\bar{\lambda} - \lambda_t) + \frac{1}{2}b_\phi^2(s)\sigma_\lambda^2\lambda_t - \mu - \beta + \gamma\sigma^2 - \lambda_t E_\nu [e^{-\gamma Z}(e^Z - 1)] \\ - \lambda_t E_\nu [e^{-\gamma Z} - 1] - a'_\phi(s) - b'_\phi(s)\lambda_t - \gamma\sigma^2\phi + b_\phi(s)b\sigma_\lambda^2\lambda_t \\ + \lambda_t E_\nu [e^{(\phi-\gamma)Z} - 1] = 0. \end{aligned} \quad (\text{B.20})$$

Collection constant terms implies (35); collecting linear terms implies (36). Thus the price of the dividend claim is characterized by (33), where  $G$  is given by (34), and  $a_\phi$  and  $b_\phi$  are characterized by (35) and (36) respectively, with  $a_\phi(0) = b_\phi(0) = 0$ .

### B.3 Limiting behavior

This section evaluates the solution in the limit as  $\sigma_\lambda$  approaches zero. When  $\lambda$  is equal to its long-run mean, the limit is shown to equal the solution to the constant disaster risk model. Choosing the solution corresponding to the negative root for  $b$ , (B.5), it follows that

$$b = \frac{\kappa + \beta}{\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa + \beta}{\sigma_\lambda^2}\right)^2 - 2\frac{E_\nu [e^{(1-\gamma)Z} - 1]}{\sigma_\lambda^2}}.$$

To evaluate  $b$  in the limit, consider a function  $g(x)$  of the form

$$g(x) = c_1 - \sqrt{c_1^2 + c_2x},$$

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<sup>8</sup>The details of this calculation are as follows. Let  $h_1(s) = \exp \{a_\phi(s) + b_\phi(s)\lambda_t\}$  and  $h_2(s) = 1$ . Integration by parts implies

$$\int_0^\infty h_1(s)h_2'(s) ds = \lim_{s \rightarrow \infty} (h_1(s)h_2(s)) - h_1(0)h_2(0) - \int_0^\infty h_1'(s)h_2(s) ds,$$

where the the left hand side of the above is identically zero.

where  $c_1$  and  $c_2$  are nonzero constants. Because  $g(0) = 0$ ,

$$\lim_{x \rightarrow 0} \frac{g(x)}{x} = g'(0) = -\frac{1}{2}c_1^{-1}c_2.$$

Now let  $c_1 = \kappa + \beta$  and  $c_2 = -2E_\nu [e^{(1-\gamma)Z} - 1]$ . Then  $b$  can be written as

$$b = \frac{1}{\sigma_\lambda^2} \left( c_1 - \sqrt{c_1^2 + c_2 \sigma_\lambda^2} \right).$$

It follows that

$$\lim_{\sigma_\lambda \rightarrow 0} b = -\frac{1}{2}c_1^{-1}c_2 = \frac{E_\nu [e^{(1-\gamma)Z} - 1]}{\kappa + \beta}. \quad (\text{B.21})$$

Combining (B.21) with (B.6), it follows that

$$\lim_{\sigma_\lambda \rightarrow 0} (a + b\bar{\lambda}) = \frac{1-\gamma}{\beta} \left( \mu - \frac{1}{2}\gamma\sigma^2 \right) + (1-\gamma) \log \beta + \bar{\lambda} \frac{E_\nu [e^{(1-\gamma)Z} - 1]}{\beta}.$$

Therefore

$$\lim_{\sigma_\lambda \rightarrow 0} J(\bar{\lambda}, W) = \exp \left\{ \frac{1-\gamma}{\beta} \left( \mu - \frac{1}{2}\gamma\sigma^2 \right) + (1-\gamma) \log \beta + \bar{\lambda} \frac{E_\nu [e^{(1-\gamma)Z} - 1]}{\beta} \right\} \frac{W^{1-\gamma}}{1-\gamma}. \quad (\text{B.22})$$

I now evaluate the solution for the constant disaster risk model in the limit as  $\psi$  approaches 1. It suffices to show that  $j^{1-\gamma}$  approaches the exponential term in (B.22). Let

$$A = -\mu + \frac{1}{2}\gamma\sigma^2 - \lambda(1-\gamma)^{-1}E_\nu [e^{(1-\gamma)Z} - 1]. \quad (\text{B.23})$$

Then, by (A.6),

$$\begin{aligned} j &= \left[ \left( A + \frac{\beta}{1 - \frac{1}{\psi}} \right) \frac{1 - \frac{1}{\psi}}{\beta^\psi} \right]^{\frac{1}{1-\psi}} \\ &= \left[ A \left( 1 - \frac{1}{\psi} \right) \beta^{-\psi} + \beta^{1-\psi} \right]^{\frac{1}{1-\psi}} \\ &= \left[ A\beta^{-1} \left( 1 - \frac{1}{\psi} \right) + 1 \right]^{\frac{1}{1-\psi}} \beta \\ &= \exp \left\{ -\frac{1}{\psi} \log \left( \left[ A\beta^{-1} \left( 1 - \frac{1}{\psi} \right) + 1 \right]^{\frac{1}{1-\frac{1}{\psi}}} \right) + \log \beta \right\}. \end{aligned}$$

From the definition of the exponential, it follows that

$$\lim_{\psi \rightarrow 1} \left[ A\beta^{-1} \left( 1 - \frac{1}{\psi} \right) + 1 \right]^{\frac{1}{1-\frac{1}{\psi}}} = \exp \{ A\beta^{-1} \}.$$

Since both this limit and the limit of  $1/\psi$  are well-defined as  $\psi$  approaches 1, I conclude that

$$\lim_{\psi \rightarrow 1} j = \exp \{ -\beta^{-1} A + \log \beta \}.$$

and thus

$$\lim_{\psi \rightarrow 1} j^{1-\gamma} = \exp \{ -(1-\gamma)\beta^{-1} A + (1-\gamma) \log \beta \}.$$

Substituting in (B.23) for  $A$  results in the exponential term in (B.22).

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Table 1: Parameters for the time-varying disaster risk model

Panel A: Cash flow parameters	
Average growth in consumption (normal times) $\mu$	0.0252
Volatility of consumption growth (normal times) $\sigma$	0.020
Leverage $\phi$	2.8
Average probability of a rare disaster $\bar{\lambda}$	0.017
Mean reversion $\kappa$	0.142
Volatility parameter $\sigma_\lambda$	0.09
Probability of default given disaster $q$	0.40
Panel B: Preference parameters	
Rate of time preference $\beta$	0.02
Relative risk aversion $\gamma$	3.0
Elasticity of intertemporal substitution $\psi$	1.0

Notes: The table shows parameter values for the time-varying disaster risk model. The process for the disaster intensity is given by

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t) dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}.$$

The consumption (endowment) process is given by

$$dC_t = \mu C_t dt + \sigma C_t dB_t + (e^{Z_t} - 1)C_{t-} dN_t,$$

where  $N_t$  is a Poisson process with intensity  $\lambda_t$ , and  $Z_t$  is calibrated to the distribution of large declines in GDP in the data. The dividend  $Y_t$  equals  $C_t^\phi$ . The representative agent has recursive utility defined by  $V_t = E_t \int_t^\infty f(C_s, V_s) ds$ , with normalized aggregator

$$f(C, V) = \beta(1 - \gamma)V \left[ \log C - \frac{1}{1 - \gamma} \log((1 - \gamma)V) \right].$$

Parameter values are in annual terms.

Table 2: Population moments from simulated data and sample moments from the historical time series

	Model		U.S. Data
	Population	Conditional	
$E[R^b]$	2.49	2.74	2.01
$\sigma(R^b)$	3.59	2.13	5.91
$E[R^e - R^b]$	5.56	6.38	5.97
$\sigma(R^e)$	18.41	16.37	18.48
Sharpe Ratio	0.31	0.38	0.32
$\sigma(\Delta c)$	6.15	1.99	3.56
$\sigma(\Delta y)$	17.23	5.56	11.51

Notes: The time-varying disaster risk model is simulated at a monthly frequency and simulated data are aggregated to an annual frequency. Moments are calculated using the annual data and (except for the Sharpe ratio) expressed in percentage terms. The second column reports population moments from simulated data. The third column reports moments from simulated data that are calculated over years in which a disaster did not occur. The last column reports sample moments from annual U.S. data from 1890 to 2004.  $R^b$  denotes the gross return on the government bond,  $R^e$  the gross equity return,  $\Delta c$  growth in log consumption and  $\Delta y$  growth in log dividends.

Table 3: Long-horizon regressions: Excess returns

	Horizon in years					
	1	2	4	6	8	10
Panel A: Model – Population moments						
$\beta_1$	-0.15	-0.28	-0.48	-0.59	-0.75	-0.83
$R^2$	0.04	0.07	0.10	0.11	0.14	0.14
Panel B: Model – Conditional moments						
$\beta_1$	-0.23	-0.43	-0.75	-0.96	-1.15	-1.27
$R^2$	0.16	0.28	0.44	0.51	0.56	0.56
Panel B: U.S. Data						
$\beta_1$	-0.09	-0.14	-0.36	-0.48	-0.73	-0.98
$t$ -stat	-2.04	-1.84	-2.91	-2.69	-3.22	-3.75
$R^2$	0.04	0.04	0.12	0.12	0.17	0.23

Notes: Excess returns are regressed on the lagged price-dividend ratio in data simulated from the model and in annual data from 1890 to 2004. Specifically, the table reports coefficients  $\beta_1$ ,  $R^2$  statistics and, for the sample, Newey-West  $t$ -statistics for regressions

$$\sum_{i=1}^h \log(R_{t+i}^e) - \log(R_{t+i}^b) = \beta_0 + \beta_1(p_t - y_t) + \epsilon_t,$$

where  $R_{t+i}^e$  and  $R_{t+i}^b$  are, respectively, the return on the aggregate market and the return on the government bill between  $t+i-1$  and  $t+i$  and  $p_t - y_t$  is the log price-dividend ratio on the aggregated market. The time-varying disaster risk model is simulated at a monthly frequency and simulated data are aggregated to an annual frequency. Panel A reports population moments from simulated data. Panel B reports moments from simulated data that are calculated over years in which a disaster does not take place (for a horizon of 2, for example, all 2-year periods in which a disaster takes place are eliminated). Panel C reports sample moments.

Table 4: Long-horizon regressions: Consumption growth

	Horizon in years					
	1	2	4	6	8	10
Panel A: Model – Population moments						
$\beta_1$	0.03	0.06	0.10	0.15	0.16	0.19
$R^2$	0.02	0.03	0.05	0.06	0.05	0.06
Panel B: U.S. Data						
$\beta_1$	0.003	-0.001	-0.007	0.002	0.006	0.040
$t$ -stat	0.45	-0.06	-0.38	0.08	0.16	0.82
$R^2$	0.0011	0.0000	0.0013	0.0001	0.0004	0.0129

Notes: Growth in aggregate consumption is regressed on the lagged price-dividend ratio in data simulated from the model and in annual data from 1890 to 2004. Specifically, the table reports coefficients  $\beta_1$ ,  $R^2$  statistics and, for the sample, Newey-West  $t$ -statistics for regressions

$$\sum_{i=1}^h \Delta c_{t+i} = \beta_0 + \beta_1(p_t - y_t) + \epsilon_t,$$

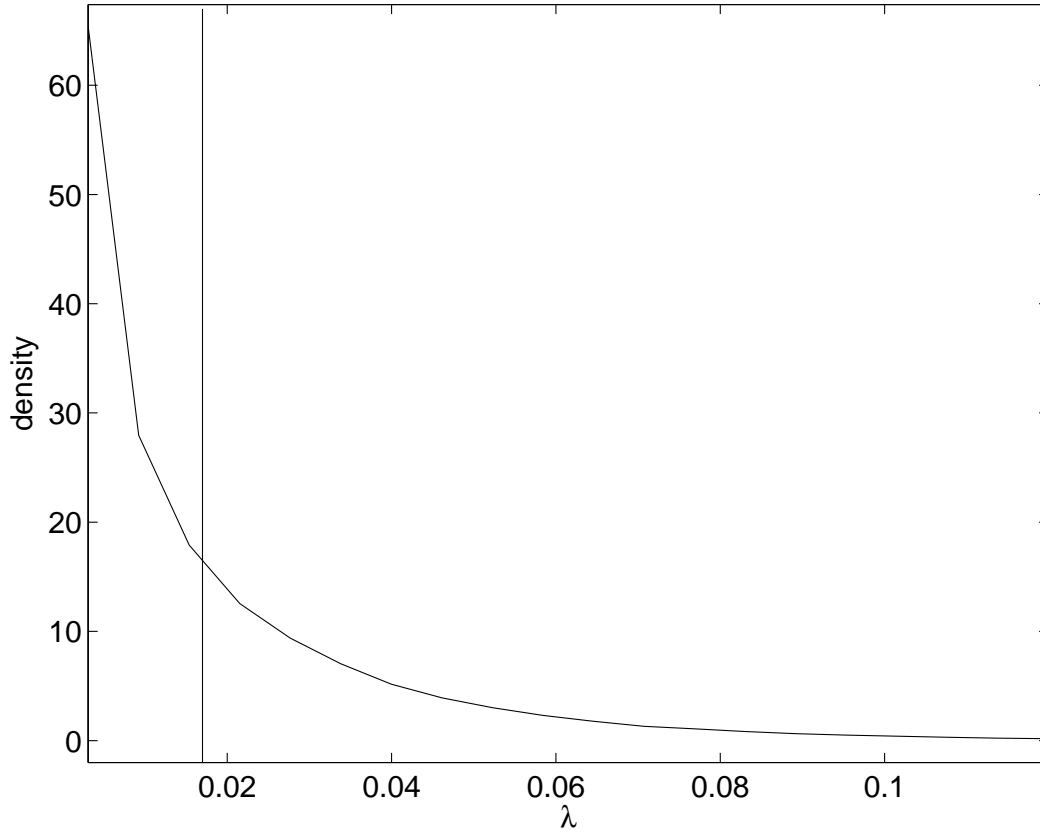
where  $\Delta c_{t+i}$  is log growth in aggregate consumption between periods  $t + i - 1$  and  $t + i$  and  $p_t - y_t$  is the log price-dividend ratio on the aggregated market. The time-varying disaster risk model is simulated at a monthly frequency and simulated data are aggregated to an annual frequency. Panel A reports population moments from simulated data. Panel B reports sample moments. The conditional moments, namely the slope coefficient and the  $R^2$  calculated over periods in the simulation without disasters, are equal to zero.

Table 5: Moments implied by the constant disaster risk model

	Population	Conditional
Panel A: $\phi = 1.5, \beta = .03, \gamma = 4, \psi = 1/4$		
$E[R^b]$	3.66	3.85
$\sigma(R^b)$	2.52	0
$E[R^e - R^b]$	5.52	6.04
$\sigma(R^e)$	6.75	3.29
Sharpe Ratio	0.95	1.83
$\sigma(\Delta y)$	8.67	3.00
Panel B: $\phi = 1.5, \beta = .03, \gamma = 4, \psi = 1$		
$E[R^b]$	1.46	1.66
$\sigma(R^b)$	2.64	0
$E[R^e - R^b]$	5.40	5.91
$\sigma(R^e)$	6.69	3.23
Sharpe Ratio	0.95	1.83
$\sigma(\Delta y)$	8.80	3.00
Panel C: $\phi = 2.8, \beta = .03, \gamma = 4, \psi = 1$		
$E[R^b]$	1.44	1.66
$\sigma(R^b)$	2.74	0
$E[R^e - R^b]$	7.89	8.79
$\sigma(R^e)$	10.62	6.19
Sharpe Ratio	0.83	1.42
$\sigma(\Delta y)$	16.59	5.60

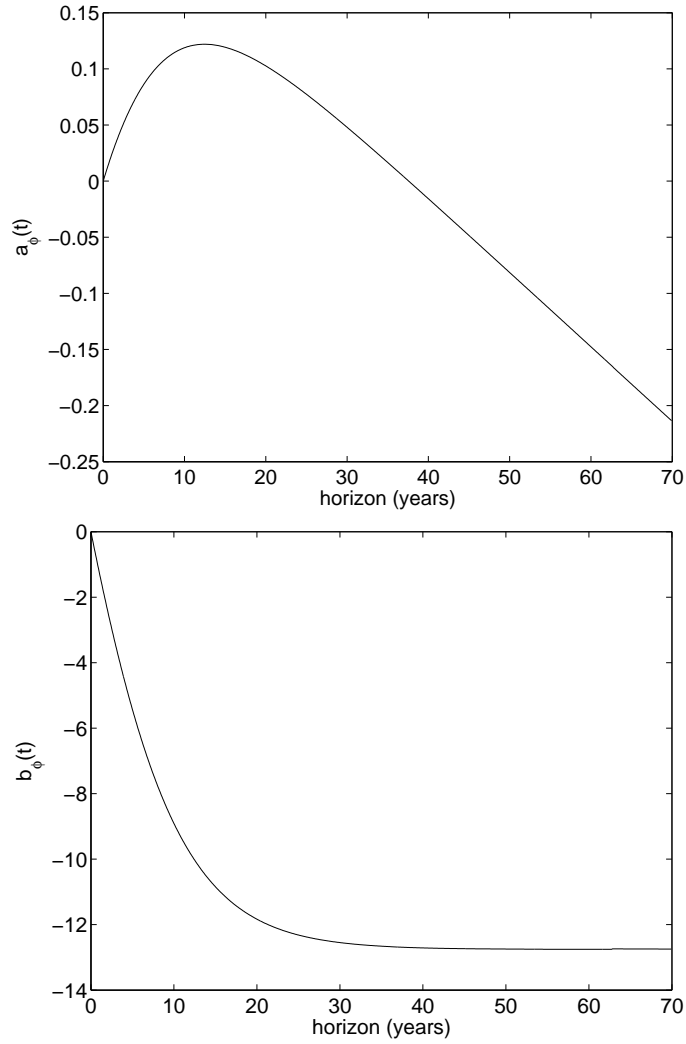
Notes: The constant disaster risk model is simulated at a monthly frequency and simulated data are aggregated to an annual frequency. Moments are calculated based on annual data and (except for the Sharpe ratio) expressed in percentage terms. All panels assume that consumption growth is i.i.d., namely that  $\sigma_\lambda = 0$  and  $\lambda_t = \bar{\lambda}$ . Other parameters (unless otherwise noted) are as in Table 1. The volatility of log consumption growth equals 5.67% per year in population and 2% per year conditional on no disasters.

Figure 1: Distribution of  $\lambda_t$



Notes: This figure shows the probability density function for  $\lambda_t$ , the time-varying intensity (per year) of a disaster. Parameter values are given in Table 1. The solid line is located at the unconditional mean  $\bar{\lambda} = 0.017$ .

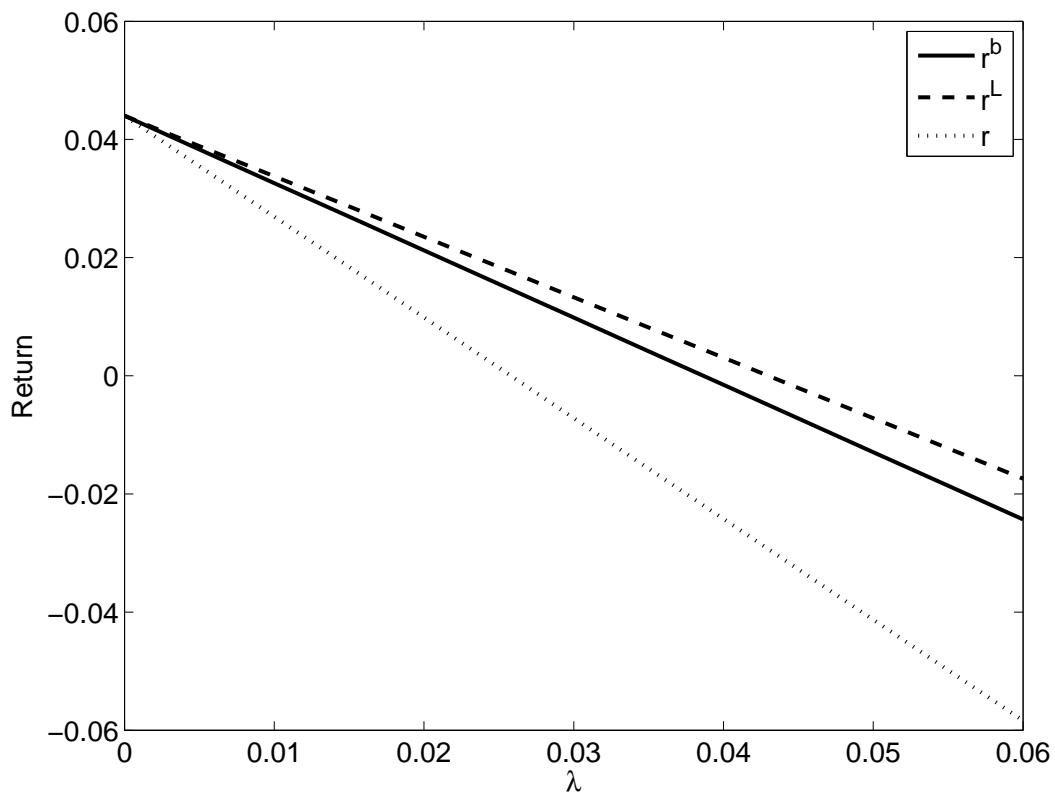
Figure 2: Components of the price-dividend ratio in the time-varying disaster risk model



Notes: The upper panel shows  $a_\phi$  as a function of horizon  $t$ ; the lower panel shows  $b_\phi$  as a function of  $t$ . These are components of the solution for the price-dividend ratio in the time-varying disaster risk model. Parameter values are given in Table 1.

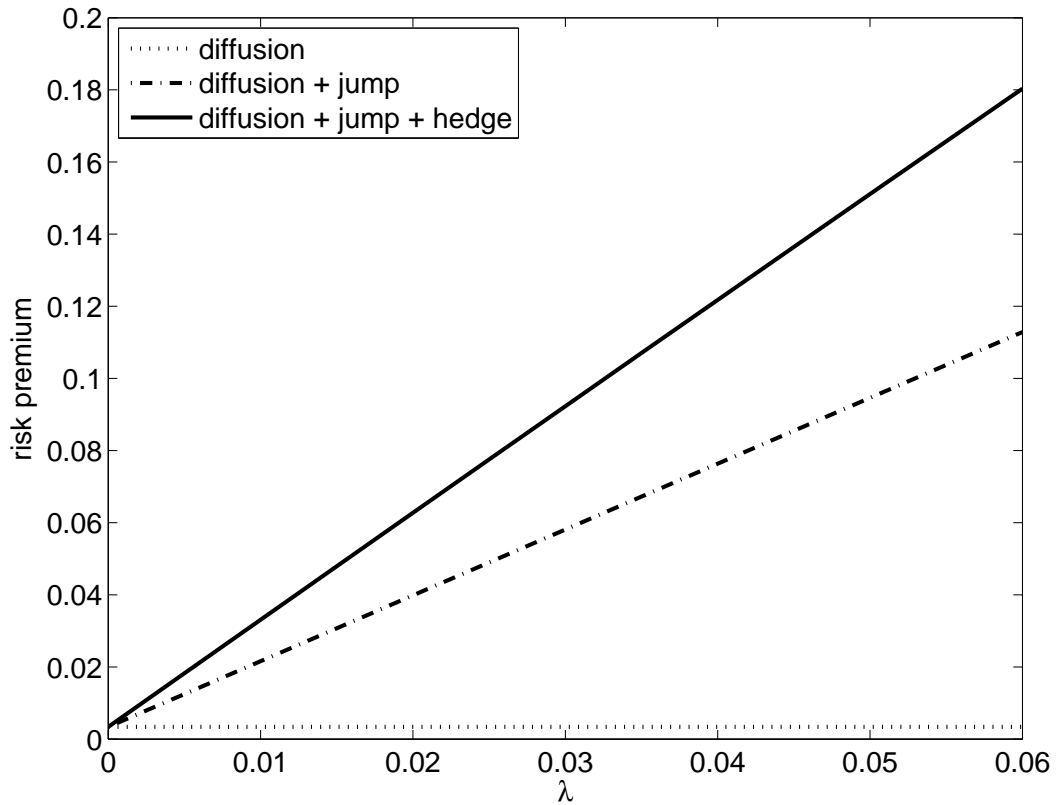


Figure 3: Government bill return in the time-varying disaster risk model



Notes: This figure shows  $r^b$ , the instantaneous expected return on a government bill;  $r^L$ , the instantaneous expected return on the bill conditional on no default and  $r$ , the rate of return on a default-free security as functions of the disaster intensity  $\lambda$ . Returns are annual.

Figure 4: Decomposition of the equity premium in the time-varying disaster risk model



Notes: This figure shows the instantaneous equity premium relative to the government bond. The solid line shows the full equity premium, the dashed line shows the equity premium without the hedging term and the dotted line shows the equity premium assuming there is no disaster risk. Returns are annual.