NBER WORKING PAPER SERIES

INTERMEDIATED QUANTITIES AND RETURNS

Rajnish Mehra
Facundo Piguillem
Edward C. Prescott

Working Paper 14351
http://www.nber.org/papers/w14351

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 2009

We thank Costas Azariadis, Sudipto Bhattacharya, George Constantinides, Cristina De Nardi, John Donaldson, Jack Favilukis, Francisco Gomes, Fumio Hayashi, Daniel Lawver, Ellen McGrattan, Jesper Rangvid, Dimitri Vayanos, the seminar participants at the Arizona State University, Bank of Korea, Charles University, Duke University, Federal Reserve Bank of Chicago, London Business School, London School of Economics, University of Mannheim, University of New South Wales, Peking University, University of Tokyo, University of Virginia, Yale University, the Economic Theory conference in Kos, and ESSFM in Gerzensee for helpful comments. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis, the Federal Reserve System, or the National Bureau of Economic Research.

Intermediated Quantities and Returns
Rajnish Mehra, Facundo Piguillem, and Edward C. Prescott
NBER Working Paper No. 14351
September 2009
JEL No. E2,E44,E6,G1,G11,G12,G23

ABSTRACT

The difference between average borrowing and lending rates in the United States is over 2 percent. In spite of this large difference, there is over 1.7 times GNP in 2007 of intermediated borrowing and lending between households. In this paper a model is developed consistent with these facts. The only difference within an age cohort is preferences for bequests. Individuals with little or no bequest motive are lenders, while individuals with strong bequest motive are borrowers and owners of productive capital. Given no aggregate uncertainty, the return on equity is the same as the household borrowing rate. The government can borrow at the household lending rate, so there is a 2 percent equity premium in our world with no aggregate uncertainty. We examine the distribution and life cycle patterns of asset holding and consumption and find there is large dispersion in asset holdings and little in consumption.

Rajnish Mehra
Department of Economics
University of California
Santa Barbara, CA 93106
and NBER
mehra@econ.ucsb.edu

Facundo Piguillem
Department of Economics
University of Minnesota
Minneapolis MN 55401
and Federal Reserve Bank of Minneapolis
fpiguill@econ.umn.edu

Edward C. Prescott
Economics Department
ASU / Main Campus
PO BOX 873806
Tempe, AZ 85287-3806
and NBER
and Federal Reserve Bank of Minneapolis
edward.prescott@asu.edu

1. Introduction

The *homogeneous household* construct is of little use in modeling borrowing and lending between households. In equilibrium, with most models using this construct the shadow price of consumption at date $t + 1$ in terms of consumption at date t is such that the amount of borrowing and lending is zero. Homogeneous household models are thus of little use in matching the *quantities* of assets held and *intermediated*.

One fact is that the average household borrowing rate is 2 percent higher than the average household lending rate. The question addressed in this study is why is there so much intermediated borrowing and lending between households even though this intermediation is so costly. To address this question we construct a model that incorporates household heterogeneity in the form of differences in the strength of preferences for bequests. Incorporating this household heterogeneity allows us to capture a key empirical fact: there is a very large amount of borrowing and lending between households.[1] This borrowing is done directly by households to finance owner-occupied housing, by proprietorships and partnerships to finance unincorporated businesses, and indirectly by shared ownership corporations to finance these businesses. We abstract from the small amount of direct borrowing and lending between households and assume that all borrowing and lending between households is intermediated through financial institutions. For the United States, in 2007 the amount intermediated was over 1.72 times the gross domestic product (GNP).[2]

[1] Age heterogeneity alone gives rise to little borrowing and lending between households, as found in Díaz-Giménez et al. (1992). Doepke and Schneider (2006) find a large amount of intermediated borrowing and lending between households, but do not address the reason for it.
[2] See Section 7 (calibration) for details.

In light of the finding that the premium for bearing non-diversifiable aggregate risk is small in worlds consistent with growth and business cycle facts, our analysis abstracts from *aggregate risk.*[3] The only uncertainty that people face is idiosyncratic risk about the duration of their lifetime after retirement. All households in an age cohort have identical preferences for consumption. They differ only with respect to their preference for making bequests. In equilibrium, those with a strong preference for bequests accumulate capital assets and borrow during their working lives, and upon retirement, use capital income for consumption and interest payment on their debt. Upon their death they bequeath all their net worth. Households with no bequest motive buy annuities during their working years and use annuity benefits to finance their consumption over their retirement years.

The intermediation technology is constant returns to scale with intermediation costs being proportional to the amount intermediated. To calibrate the constant of proportionality, we use Flow of Funds Account statistics and data from National Income and Product Accounts. The calibrated value of this parameter equals the net interest income of financial intermediaries, divided by the quantity of intermediated debt, and is approximately 2 percent.[4]

In the absence of aggregate uncertainty, the return on equity and the borrowing rate are identical, since the households who borrow are also marginal in equity markets. In our framework, government debt is intermediated at zero cost, and thus its return is

[3] Using a model with no capital accumulation, Mehra and Prescott (1985) find a small equity premium. McGrattan and Prescott (2000) find that the equity premium is small in the growth model if it is restricted to be consistent with growth and business cycle facts. Lettau and Uhlig (2000) introduce habit formation into the standard growth model and find that the equity premium is small if the model parameters are restricted to be consistent with the business cycle facts. Many others using the growth model restricted to be consistent with the macro economic growth and business cycle facts have found the same thing.

[4] See Section 7 (calibration) for details.

equal to the household lending rate. The equity premium relative to government debt equals the intermediation spread for household borrowing and lending. The divergence between borrowing and lending rates gives rise to an equity premium even in a world *without aggregate uncertainty.*

In our model, all households in a cohort have identical labor income at every point in their working life. A consequence of this is little difference in consumption cross sectionally at a point in time. However, sizable differences in net worth and large differences in capital holdings develop within a cohort over their working years. One implication is that preferences for bequests cannot be ignored when studying net worth and capital holding distributions.

The paper is organized as follows. The economy is specified in Section 2. In Section 3, we discuss the decision problem of the households. Section 4 deals with the aggregation of individual behavior, Section 5 with the relevant balance sheets, and Section 6 characterizes the balanced growth equilibrium. We calibrate the economy in Section 7. In Section 8, we present and discuss our results. Section 9 concludes the paper.

2. The Economy

In order to build a model that captures the large amount of observed borrowing and lending, as well as the large amount of resources used in this process, we introduce three key features of reality. The first feature is differences in bequest preferences, the second is an uncertain length of retirement, and the third is costly intermediation of borrowing and lending between households. This leads some households to buy *costly* annuities that make payments throughout the retirement years. Since buying an annuity is isomorphic to lending, households choosing the annuity option are the lenders in our model. Households with high bequest utility save by building equity, which is their holding of productive capital less their debt.

The model is an overlapping generations model, and we consider the balanced growth path competitive equilibrium. All households born at a given date are identical in all respects except for bequest preference parameter α. Households have identical preferences with respect to consumptions over their lifetime, so the only dimension over which they differ is α. Those with a large α (type-B) borrow and own capital; others with no preferences for bequest (type-A) lend by acquiring annuities.

What motivates bequests? While a casual consideration of bequests naturally assumes that they exist because of parents' altruistic concern for the economic well-being of their offspring, results in Menchik and David (1983), Hurd (1989), Wilhelm (1996), Laitner and Juster (1996), Altonji, Hayashi, and Kotlikoff (1997), Laitner and Ohlsson (2001), Kopczuk and Lupton (2007), and Fuster, Imrohoroglu and Imrohoroglu (2008) suggest otherwise: households with children do not, in general, exhibit behavior in greater accord with a bequest motive than do childless households. This, we think, leads us to conclude that the existing literature supports our assumption that some people have preferences for making bequests. These empirical results lead us to eschew the perspective of Barro (1974) and Becker and Barro (1988), who postulate that each generation receives utility from the consumption of the generations to follow, and simply model bequests as being motivated by a well-defined "joy of giving,"[5] as in Abel and Warshawsky (1988) and Constantinides, Donaldson, and Mehra (2007). We emphasize that our results are not sensitive to the reason why people leave bequests.

[5] See also Hurd and Mandcada (1989), De Nardi, Imrohoroglu, and Sargent (1999), De Nardi (2004), and Hansen and Imrohoroglu (2006).

Households

Any systematic consideration of bequests mandates that the analysis be undertaken in the context of an overlapping generations model. Consequently, we analyze an overlapping generations economy and determine its balanced growth behavior. Each period, a set of individuals of measure one enter the economy. Two types enter at each date: type-A, with no utility from making a bequest, and type-B, whose utility is an increasing function of the amount they bequeath.[6] The measure of type $i \in \{A, B\}$ is μ_i. The total measure of people born at each date is 1, so $\mu_A + \mu_B = 1$.

Individuals have finite expected lives. They enter the labor force at age 22, work for T years, and then retire.[7] Model age *j* is 0 when a person begins his or her working life. The first year of retirement is model age $j = T$.

All workers receive an identical wage income. Wage income grows at the economy's balanced growth rate γ. At retirement, individuals face idiosyncratic uncertainty about the length of their remaining lifetime. Their retirement lifetimes are exponentially distributed. Once individuals retire, the probability of surviving to the next period is $\sigma = (1-\delta)$, where δ is the probability of death. Expected life is $T + 1/\delta$. We emphasize that there is no aggregate uncertainty.[8]

Individuals of type α, born at time *t*, order their preferences over age-contingent consumption and bequests by[9]

---

[6] The "no utility from a bequest" assumption is a simplifying one and is not necessary for the analysis. All that is needed is the utility from bequest be sufficiently small that the type-A choose to acquire annuities.

[7] We implicitly assume that parents finance the consumption of their children under the age of 22; in other words, children's consumption is a part of their parents' consumption.

[8] The Blanchard (1985) model has individuals with exponential life. The Díaz-Giménez et al. (1992) model has individuals with both an exponential working life and an exponential retirement life.

[9] Our model has no factor giving rise to life cycle consumption patterns over the working life as in Fernández-Villaverde and Krueger (2002).

(2.1) $$\sum_{j=0}^{T}\beta^j \log c_{t+j,j} + \sum_{j=T+1}^{\infty}\beta^j \sigma^{j-T}\log c_{t+j,j} + \sum_{j=T+1}^{\infty}\alpha\delta\beta^j\sigma^{j-T-1}\log b_{t+j,j} .$$

Here $\beta < 1$ is the discount factor and α is the strength of bequest parameter. Variable $c_{t+j,j}$ is the period consumption of a *j*-year-old born at time *t*,[10] conditional on being alive at time *t* + *j*. An individual who is born at time *t* and dies at age $j-1$ consumes nothing at time *t* + *j* and bequeaths $b_{t,t+j}$ units of the period *t* + *j* consumption good and consumes nothing subsequently. Each generation supplies one unit of labor inelastically for $j = 0,1,\ldots,T-1$. Thus, aggregate labor supply is $L = T$ given that the measure of each generation is 1.

We only need to analyze the decision problems of an individual of a type α individual born at time *t* = 0. The solution to the problem for a type α born at any other time *t* can be found using the fact that along a balanced growth path

(2.2) $$c_{t,j} = (1+\gamma)^t c_{0,j} .$$

Further, to simplify the notation, we use c_j to denote the consumption of a *j*-year-old at time *j* rather than $c_{j,j}$. An analogous change of notation applies to the other *variables.*

Production Technology

The aggregate production function is

(2.3) $$Y_t = F(K_t, z_t L_t) = K_t^{\theta}(z_t L_t)^{1-\theta}$$

(2.4) $$z_{t+1} = (1+\gamma) z_t .$$

---

[10] In this paper, the first subscript represents calendar time and the second subscript represents the age at that time.

K_t is capital, L_t is labor, and z_t is the labor-augmenting technological change parameter, which grows at a rate γ. The parameter z_0 is chosen so that $Y_0 = 1$.

Output is produced competitively, so

(2.5) $$\delta_k + r_e = F_K(K_t, z_t L_t)$$

(2.6) $$e_t = z_t F_L(K_t, z_t L_t),$$

where δ_k is the depreciation rate, r_e is both the household borrowing rate and the return on equity, and e_t is the wage rate.

Income is received as either wage income E_t or gross capital income R_t. Thus,

(2.7) $$Y_t = E_t + R_t,$$

where $E_t = L_t e_t = (1-\theta)Y_t$ and $R_t = (\delta_k + r_e)K_t = \theta Y_t$. Components of output are consumption C_t, investment X_t, and intermediation services I_t; thus,

(2.8) $$Y_t = C_t + X_t + I_t.$$

Along a balanced growth path, investment $X_t = (\delta_k + \gamma)K_t$ and $K_{t+1} = (1+\gamma)K_t$.

Financial Intermediation Technology

The intermediation technology displays constant returns to scale, with the intermediation cost in units of the composite output good being proportional to the amount of borrowing and lending intermediated. The cost is ϕ times the amount of borrowing and lending between households.[11] The intermediary also intermediates between households lending to the government. There are no costs associated with this intermediation. The intermediary effectively receives interest rate r_e on its lending to

[11] Miller and Upton (1974) pioneered in having a financial sector in their dynamic general equilibrium model. They had no intermediation costs.

households and effectively pays interest rate r on its borrowing from households. Given the technology, equilibrium interest rates must satisfy

$$r_e - r = \phi .$$

The lending contract between households and intermediaries is not the standard one, but rather an annuity contract. A household can enter into an annuity contract at age 0. An annuity contract specifies an age-contingent premium payment path during working life, a benefit path contingent on being alive subsequent to retirement, and a payment upon death. The amount being lent by an individual who has chosen the annuity contract is the value of pension fund reserves for that contract at that point in time. These reserves are equal to the expected present value of future payments less the expected present value of future premium payments, if any. The present value is calculated using the lending rate r. This leads us to refer to r as the household lending rate. In equilibrium, competitive intermediaries will offer any annuity contract with the property that the expected present value of benefits is equal to the present value of the premiums using r in the present value calculations.

The alternative to entering into an annuity contract to save for retirement is to accumulate capital and to borrow to partially finance that capital. Effectively, the non-financial business sector is being consolidated with the household sector. There is also a government sector and a financial sector.

Government Policy

The government finances interest payments on its debt by issuing new debt and by taxing labor income at rate τ. The government's period t budget constraint is

$$(2.9) \qquad (1+r)D_t^G = \tau E_t + D_{t+1}^G .$$

Since $D_{t+1}^G = (1+\gamma)D_t^G$ in balanced growth,

(2.10) $$(r-\gamma)D_t^G = \tau(1-\theta)Y_t .$$

In addition, the government pursues a tax rate policy that pegs[12] r, which equals the interest rate on government debt. This being a balanced growth analysis, government debt grows at rate $\gamma > 0$, which means that the government deficits are positive and grow at rate γ as well.

Finally, the intermediary holds government debt, and there are no intermediation costs associated with holding this asset on the part of the intermediary.

Aggregate bequests at date t are

(2.11) $$B_t = B_0(1+\gamma)^t .$$

We let $\bar{b} = B_{30}$. The inheritance of a type-B born at $t = 0$ is

(2.12) $$\bar{b}^B = \bar{b}$$

and is received at date $t = 30$. The inheritance of a type-A born at $t = 0$ is

(2.13) $$\bar{b}^A = \bar{b}(1+r)/(1+r_e).$$

The reason that a type-A's inheritance is smaller than that of a type-B is that their inheritances are intermediated and intermediation is costly.

3. Optimal Individual Decisions

We consider the optimal individual decision problem, taking as given (i) the size of the inheritance the individual will receive at model age 30 (chronological age 52), (ii) wages at each date of the individual's working life, (iii) the labor income tax rate τ, and (iv) the borrowing and lending rates r_e and r. The first problem facing an individual is

[12] In this paper, we fix this at 3 percent. This is discussed further in Section 7 on calibration.

whether to choose the annuity strategy A or the no annuity strategy B. The parameters of the calibrated economy are such that a type-A will choose the annuity strategy, while a type-B will choose the no annuity strategy. The second problem is to determine the optimal lifetime consumption and savings decisions conditional on the strategy chosen. We determine, given α, the optimal consumption/saving behavior for each strategy and the resulting lifetime utility, and then determine which of the two strategies is best for that individual type.

A convention followed is that a bar over a variable denotes a constant. In the case where the constant depends upon a person's type, that is, on α, this functional dependence is indicated. This is necessary because the best strategy will differ across household types.

The Best No Annuity Strategy

This problem can be split into two sub-problems. The first problem is the one after retirement, which is stationary and is solved using recursive techniques. The state variable is net worth, which is in units of the *current period consumption good*. The value of a unit of k is $(1+r_e)k$ to a household choosing the no annuity strategy. The second problem is to determine consumptions and savings over the working life.

The problem becomes stationary and recursive at retirement age T, with net worth w being the state variable. The value function $f(w)$ is the maximal obtainable expected current and future utility flows if a retiree is alive and has net worth w. The optimality equation is

$$f(w)=\max_{c,w'}\ \{\log c+\sigma\beta f(w')+\delta\beta\alpha\log w'\}$$

(3.1)

$$\text{s.t.}\quad c+\frac{w'}{(1+r_e)}\le w\ .$$

The solution to this optimality equation has the form

(3.2) $$f(w)=\bar{f}_1(\alpha)+\bar{f}_2(\alpha)\log w,$$

where

(3.3) $$\bar{f}_2(\alpha)=\frac{1+\alpha\ \beta\delta}{1-\sigma\beta}\ .$$

The optimal consumption/saving policy for retirees is

(3.4) $$\begin{aligned}&c=w\ /\ \bar{f}_2(\alpha)\\&w'=(1+r_e)(w-c).\end{aligned}$$

The bequests, conditional on $j-1$ being the person's last year of life, is

(3.5) $$b_j=w_j\ .$$

The problem facing an individual at birth who follows the no annuity strategy (which we call strategy B because it is the one chosen by those with a sufficiently strong preference for making a bequest) is

$$U^B(\alpha)=\max_{\{c_j\}_{j=0}^{T-1},w_T}\ \{\sum_{j=0}^{T-1}\beta^j\log c_j+\beta^T[\bar{f}_1(\alpha)+\bar{f}_2(\alpha)\log w_T]\}$$

(3.6)

s.t.

$$\sum_{j=0}^{T-1}\frac{c_j}{(1+r_e)^j}+\frac{w_T}{(1+r_e)^T}\le v_0^B=\sum_{j=0}^{T-1}\frac{(1-\tau)e_0(1+\gamma)^j}{(1+r_e)^j}+\frac{\bar{b}^B}{(1+r_e)^{30}}\,.$$

Here v_0^B is the present value of wages and inheritance of an individual born at $t=0$. The solution (see Appendix 2 for more details) is

$$(3.7)\qquad \begin{aligned} c_j^B &= \bar{c}(\alpha)\beta^j(1+r_e)^j v_0^B \qquad\qquad j<T \\ w_T^B &= (1-\sum_{j=0}^{T-1}\bar{c}(\alpha)\beta^j)(1+r_e)^T\ v_0^B, \end{aligned}$$

where $$\bar{c}(\alpha) = \frac{(1-\beta)}{1-\beta^T+(1-\beta)\beta^T f_2(\alpha)} \ .$$

The preretirement age *j* net worth of an individual following this strategy satisfies

$$(3.8)\qquad \begin{aligned} w_0^B &= 0 \\ w_j^B &= (1+r_e)(w_{j-1}^B - c_{j-1}^B + (1-\tau)e_0(1+\gamma)^{j-1}) \qquad \text{for } 1 \le j < T, j \ne 30 \\ w_{30}^B &= (1+r_e)(w_{29}^B - c_{29}^B + (1-\tau)e_0(1+\gamma)^{29}) + \bar{b}^B. \end{aligned}$$

The Best Annuity Strategy

The best annuity strategy for a type α is the solution to the following:

$$U^A(\alpha) = \max_{\{b_j,c_j\}}\{\sum_{j=0}^{T}\beta^j \log c_j + \sum_{j=T+1}^{\infty}\beta^j\sigma^{j-T}\log c_j + \sum_{T+1}^{\infty}\beta^t\sigma^{j-T-1}\delta\alpha\log b_j\}$$

$$(3.9)\qquad \text{s.t.}\quad \sum_{j=0}^{T}\frac{c_j}{(1+r)^j} + \sum_{j=T+1}^{\infty}\frac{\sigma^{j-T}c_j}{(1+r)^j} + \sum_{j=T+1}^{\infty}\frac{\sigma^{j-T-1}\delta b_j}{(1+r)^j} \le v_0^A,$$

where r is the lending rate and

$$(3.10)\qquad v_0^A = \sum_{t=0}^{T-1}\frac{(1-\tau)e_0(1+\gamma)^t}{(1+r)^t} + \frac{\bar{b}^A}{(1+r)^{30}} \ .$$

The constant v_0^A is the present value of future wage income and inheritances using the lending rate r of a person born at $t=0$. The superscript *A* denotes the annuity strategy and not an individual type. In equilibrium, type-A will choose strategy A.

There are other constraints, specifically, that the worker choosing this strategy does not borrow, that is, $(1-\tau)e_j - c_j \ge 0$ for $j<T$. For the economies considered in this study, these constraints are not binding and can therefore be ignored. If, however, the

economy were such that the no-borrowing constraint were binding for some *j*, then the solution below would not be the solution to the problem formulated above.

The nature of the annuity contract is that the payment to a retiree who is alive at age $j \geq T$ is c_j. If the individual dies at age *j*, payment b_j is made to that person's estate. The solution to this program is

$$c_j^A = \bar{c}(\alpha)\beta^j(1+r)^j\, v_0^A \qquad j \geq 0 \tag{3.11}$$

$$b_j^A = \alpha\bar{c}(\alpha)(1+r)^j\beta^j v_0^A \qquad j \geq T+1 \tag{3.12}$$

The net worth of an individual choosing this strategy is the pension fund reserves associated with that individual's annuity contract. Pension fund reserves (from the point of view of the intermediary) for a given annuity contract for an individual born at $t = 0$ at age *j* in equilibrium equals the expected present value at time $t = j$ of payments that will be made less the value (at time $t = j$ as well) of premiums that will be received.

For workers, they can be determined as the present value of past premiums. Thus, pension fund reserves for individuals' annuity holders born at $t = 0$ at age *j* satisfy

$$\begin{aligned} &w_0^A = 0 \\ (3.13)\quad &w_j^A = \left(w_{j-1}^A - c_{j-1}^A + (1-\tau)e_0(1+\gamma)^{j-1}\right)(1+r) \qquad \text{for } 1 \leq j < T, j \neq 30 \\ &w_j^A = \left(w_{j-1}^A - c_{j-1}^A + (1-\tau)e_0(1+\gamma)^{j-1}\right)(1+r) + \bar{b}^A \qquad \text{for } j = 30 \end{aligned}$$

For retirees, conditional on being alive, pension fund reserves for individuals born at $t = 0$ at age *j* are equal to the expected present value of the future payments:

$$w_j^A = \sum_{t=0}^{\infty}(1-\delta)^t \frac{c_{j+t}^A}{(1+r)^t} + \sum_{t=0}^{\infty}\delta(1-\delta)^{t-1}\frac{b_{j+t}^A}{(1+r)^t} \qquad j > T \tag{3.14}$$

The Best Strategy

The best strategy is the no annuity strategy if $U^B(\alpha) > U^A(\alpha)$, while it is the annuity strategy if $U^A(\alpha) > U^B(\alpha)$. Two propositions are:

Proposition 1: If $\frac{1+r_e}{1+r} > \beta\left[\frac{1-(1-\delta)\beta}{\beta\delta}\right]$ then $\frac{\partial U^B(\alpha)}{\partial\alpha} - \frac{\partial U^A(\alpha)}{\partial\alpha} > 0$.

Proof: In Appendix 1.□

Proposition 2: For sufficiently small ϕ, $U^B(0) < U^A(0)$. For sufficiently large ϕ, $U^B(0) > U^A(0)$.

Proof outline: For small non-negative ϕ, the value of insurance associated with strategy A exceeds the value of the higher return associated with strategy B. This is why strategy A dominates for small ϕ. For large ϕ, the cost of the annuity is large and the higher return associated with the no annuity strategy dominates. This is why strategy B dominates for large ϕ.□

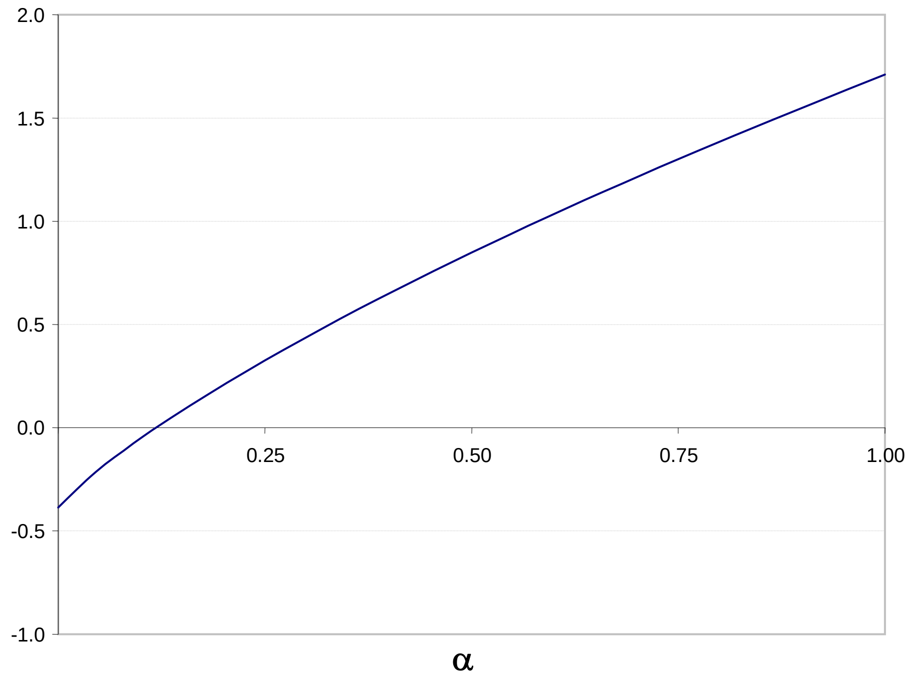
The conditions of Propositions 1 and 2 are satisfied for our calibrated economy.

Figure 1 plots the difference in utilities for the two strategies, as a function of α, for the prices, tax rate, and bequest for our calibrated economy. We see that individuals with bequest preference parameter $\alpha < 0.12$ choose to annuitize.

Figure 1

Utility Difference between the Best No Annuity and Best Annuity Strategy:

$$U^B(\alpha)-U^A(\alpha)$$



4. Aggregate Behavior of the Household Sector

Aggregate Consumption

Aggregate consumption depends upon the labor tax rate τ and inheritance factor $\bar{b}$ as well as the prices $\{e,r,r_e\}$. Equilibrium prices do not depend upon the household side, and can be determined from the policy choice of r and profit-maximizing conditions. Having formulated the optimal consumption strategies for the two types of individuals, we characterize the aggregate consumption, asset holdings, and bequest at time $t=0$ by individual type given $\bar{b}$ and τ for the equilibrium prices. Two aggregate equilibrium relations must be solved for the variables $\bar{b}$ and τ.

There are two types of households $i \in \{A, B\}$. The type-A has $\alpha_A = 0$ and will in equilibrium choose the annuity strategy A given the model economy. The type-B has $\alpha_B > 0$. The measure of type-*i* of age *j* at $t = 0$ is

(4.1)
$$\mu_j^i = \begin{cases} \mu_0^i & j \le T \\ (1-\delta)^{j-T}\,\mu_0^i & j > T \end{cases}$$

The aggregate consumption of the type-*i* households at time 0 is C^i:

(4.2)
$$C^i(\bar{b}, \tau) = \mu^i \sum_{j=0}^{T-1} c_j^i (1+\gamma)^{-j} + \mu^i \sum_{j=T}^{\infty} (1-\delta)^{j-T} c_j^i (1+\gamma)^{-j} .$$

Here we have used the fact that each subsequent generation has a consumption-age profile that is higher by a factor of $(1+\gamma)^j$ in balanced growth.

Aggregate consumption is

(4.3)
$$C(\bar{b}, \tau) = C^A(\bar{b}, \tau) + C^B(\bar{b}, \tau).$$

Aggregate Asset Holdings

The aggregate net worth at time 0 of a type $i \in \{A, B\}$ is

(4.4)
$$W(\bar{b}, \tau) = \mu_0^i \sum_{j=0}^{T} w_j^i (1+\gamma)^{-j} + \mu_0^i \sum_{j=T+1}^{\infty} (1-\delta)^{j-T} w_j^i (1+\gamma)^{-j} .$$

Net worth is prior to consumption and receipt of wage income and includes net interest income and dividend income. In the case of the intermediary, net worth includes intermediation cost liabilities. Net worth is prior to consumption and is denominated in units of the current period consumption good.

Aggregate Inheritance

At time 0 the measure of the people aged $j > T$ who *die* and leave a bequest is $\mu_0^B \delta \sigma^{j-T-1}$; thus, the total bequests given by these households is

$$B_j = \mu_0^B \delta \sigma^{j-T-1} w_j^B \quad j > T \ .$$

Hence, the *aggregate bequests* at time 0 are

(4.5) $$B_0 = \sum_{j=T+1}^{\infty} B_{0j} (1+\gamma)^{-j} \ .$$

Aggregate Private Debt

The aggregate indebtedness of a type-B satisfies

(4.6) $$D^B(\bar{b},\tau) = K - W^B(\bar{b},\tau)/(1+r_e),$$

because the price of existing capital in terms of the consumption good is $(1+r_e)$ and the household is obligated to make a payment of $(1+r_e)D^B(\bar{b},\tau)$.

5. Balance Sheets

Assets and liabilities are beginning of period numbers and are in units of the consumption good. We consider only economies for which there is intermediated borrowing and lending in equilibrium. Given there is a large amount of intermediated borrowing and lending, these economies are the ones of empirical interest.

Type-A Sector: The assets of the type-A consist of pension fund reserves. They have no liabilities. The value of these pension reserves (in terms of the consumption good) is: Pension fund reserves = $(1+r)D^B(\bar{b},\tau) + (1+r)D^G(\bar{b},\tau)$. Their balance sheet is as follows:

Balance Sheet of Type-A Households

Assets	Liabilities
Pension fund reserves	0 Net worth

Hence, their net worth satisfies

$$W^A(\bar{b},\tau) = (1+r)D^B(\bar{b},\tau) + (1+r)D^G(\bar{b},\tau).$$

Type-B Sector: Those following the no annuity strategy have debt $D^B(\bar{b},\tau)$ and hold all the economy's capital, K. Their balance sheet is as follows:

Balance Sheet of Type-B Households

Assets	Liabilities
$(1+r_e)K$	$(1+r_e)D^B(\bar{b},\tau)$ Net worth

Here we have adjusted the assets and liabilities by a factor $(1+r_e)$ to get the net worth in units of the consumption good. Their net worth is

$$W^B(\bar{b},\tau) = (1+r_e)K - (1+r_e)D^B(\bar{b},\tau).$$

Financial Intermediary Sector: The assets of the financial intermediary are the liabilities of the government and the type-B households, while its liabilities are the pension assets of type-A households and the amount payable for intermediation services. The net worth of the financial intermediaries is zero.

Balance Sheet of the Intermediaries

Assets	Liabilities
Government debt = $(1+r)D^G(\bar{b},\tau)$	Pension promises = $(1+r)[D^B(\bar{b},\tau)+D^G(\bar{b},\tau)]$
Private debt = $(1+r_e)D^B(\bar{b},\tau)$	Amounts payable for intermediation services = $D^B(\bar{b},\tau)(r_e-r)$
	Net worth = 0

Government: The assets of the government are the present value of the tax receipts on labor income, while its liabilities are the debt it has outstanding.

Balance Sheet of the Government

Assets	Liabilities
$\dfrac{\tau(1-\theta)Y}{r-\gamma}$	$D^G(\bar{b},\tau)$
	Net worth = 0

Since labor is supplied inelastically and taxed at a rate τ, the government effectively owns a fraction τ of an individual's time endowment (now and in all future periods). In our model economy, the net worth of the government is zero and government debt is an asset for debt holders in our model.

6. Equilibrium Relations

From the Production Side

We determine the value of a set of balanced growth variables at $t=0$. All variables grow at rate γ except aggregate labor supply, which is constant and equal to 40. Given that Y has been normalized to 1 at time 0, the cost share relationships determine time 0 capital stock K and wage e:

(6.1) $$(r_e + \delta_k)K = \theta Y$$

(6.2) $$eL = (1-\theta)Y$$

From the intermediary's problem, the lending rate satisfies

(6.3) $$r_e = r + \phi .$$

Three Equilibrium Conditions

Prices $\{e, r, r_e\}$ are determined from policy and technology. Therefore, only $\bar{b}$ and τ are needed to completely specify the household budget constraints. Conditional on these variables, aggregate consumption, $C(\bar{b},\tau)$, and aggregate intermediation, $I(\bar{b},\tau)$, will be determined by aggregating individual household variables. Aggregation, given the individual decisions conditional on $\bar{b}$ and τ, is specified in Appendix 2.

One aggregate equilibrium condition is the aggregate resource constraint,

(6.4) $$C(\bar{b},\tau) + X + \phi I(\bar{b},\tau) = K^{\alpha} L^{1-\alpha} ,$$

where $X = (\delta_k + \gamma)K$ is investment. Intermediation services satisfy

(6.5) $$I(\bar{b},\tau) = K - \frac{W^B(\bar{b},\tau)}{(1+r_e)} .$$

We assume that type-B households hold all the capital and the intermediary none. This is done to resolve the unimportant indeterminacy. Increasing the amount of capital held by a type-B and that type-B indebtedness by the same value amount does not affect that type-B net worth, which is what matters. This portfolio shift of the type-B is offset by a portfolio shift by some other type-B household. The aggregate indebtedness of a type-B is denoted by $D^B(\bar{b},\tau)$ and is equal to $I(\bar{b},\tau)$.

The second equilibrium condition is that the inheritance of people at a point in time equals aggregate bequests at that point in time. We consider $t=0$ and let $B(\bar{b},\tau)$ be the aggregate bequest at that time. The second equilibrium condition is

(6.6) $$\bar{b} = B(\bar{b},\tau)(1+\gamma)^{30}.$$

There is a third equilibrium condition, namely, the government's budget constraint. Equating payments to receipts, $(1+r)D_t^G = \tau E_t + D_{t+1}^G$. Given $D_{t+1}^G = (1+\gamma)D_t^G$, $E_0 = (1-\theta)Y_0$, and Y_0 has been normalized to 1.0, the time 0 government budget constraint is

(6.7) $$(r-\gamma)D^G(\bar{b},\tau) = \tau(1-\theta).$$

Equation (6.7) determines government debt.

Equilibrium

The first two equilibrium conditions are linear in $(\bar{b},\tau)$, so solving for a candidate solution is straightforward. This solution is the equilibrium only if in addition (i) the best strategy for type-B households is the no annuity strategy; (ii) the best strategy for type-A households is the annuity strategy; (iii) $D^B > 0$; and (iv) $c_{0,j}^A < (1-\tau)e_o$. The reason for the last constraint is that these equilibrium conditions hold provided that the no-borrowing constraint on annuity holders is not binding and it will not be binding if (iv) holds.

7. Calibration

The parameters that need to be “calibrated” are those related to the households $\{\alpha^A, \alpha^B, \beta, \mu^A, \mu^B, T, \delta\}$; the intermediation technology parameter $\{\phi\}$; the goods technology parameters $\{\theta, \delta_k, \gamma\}$; and the policy parameter $\{r\}$. The other policy parameters $\{\tau, D^G\}$ are endogenous. Many of these parameters are well documented in the literature; others are not.

We proceed by listing them with selected values and a brief motivation.

Parameters Associated with Individuals

$\beta = 0.99$ (Annuity holders *c* grow at almost 2 percent over their lifetimes)

$\delta = 0.05$ (Implies a postretirement life expectancy of 20 years)

$\alpha^A = 0$ (Assumption: Type-A individuals have low bequest intensity)

$\alpha^B = 1$ (Assumption: Type-B individuals have high bequest intensity)

$T = 40$ (Retire at chronological age 63) $\mu^A = 1 - \mu^B$

$\mu^B = 0.154$ (Specified so that the amount intermediated matched U.S. data)

$\mu^A = 1 - \mu^B$

Intermediation parameters

$\phi = .02$ (Consistent with the average difference in borrowing and lending rates)

Policy parameters

$r = 0.03$ (Assumption about government fiscal policy)

The motivation for this policy is that this has been the approximate return on lending by households.

Goods production parameters

$\theta = 0.3$ (Capital cost share)

$\gamma = 0.02$ (Average growth rate of U.S. per capita output)

$\delta_k = 0.05$ (Consistent with capital output ratio = 3, given $r_e = .05$)

In calibrating ϕ we proceed as follows. Our model economy has household, government, and financial intermediary sectors. All nonfinancial business borrowing is added to the household sector. We start with the net interest income of the financial intermediation sector. Fees are a small part of this sector's product and most of them are for transaction services, which is not intermediation in the sense used in this study. Using data from NIPA[13] for year 2007, the interest received amounted to 0.165 times gross national product (GNP)[14] and interest paid amounted to 0.110 times GNP. To estimate the services associated with intermediating borrowing and lending, we first subtracted intermediation services furnished without payment to households as we did not want to include implicit purchases of transaction services by the household. We also subtracted part of bad debt viewing it as interest not received by the intermediary to obtain an estimate of the cost of intermediating borrowing and lending between households of 3.4 percent of GNP in 2007. See Table 1.

Using data from the Flow of Funds,[15] we found the debt outstanding of our household sector, which includes nonfinancial businesses, equals 1.72 times GNP. The implied intermediation spread is thus 2.0 percent and in turn the calibrated $\phi = 0.02$. This

[13] Source: NIPA (U.S. Department of Commerce, 2000) Tables 7.11 and 2.4.5.
[14] Source: NIPA Table 1.7.5.
[15] Source: Flow of Funds (Board of Governors, 2000) Table D.3.

number results in the after-tax returns being close to their historical averages (see McGrattan and Prescott (2003, 2005)).

Table 1

Financial Intermediary Sector Accounts Relative to GNP Year 2000

Interest received	0.165	Table 7.11 NIPA line 28
Less interest paid	0.110	Table 7.11 NIPA line 4
Equals net interest income	0.055	
Less services furnished without payment	0.016	Table 2.4.5 NIPA line 89
Less bad debt expenses	0.005	Table 7.16 NIPA line 12*
Equals services for intermediating household borrowing and lending	0.034	
Amount intermediated between households	1.721	Table D.3 Flow of Funds (Total amount in column 1 less state, local, and federal government)

*This datum is for 2005, the latest for which this datum is currently available.

Transaction costs incurred by households associated with buying and selling assets are not part of intermediation costs. The assets in our model are capital *K*, government debt, the debt of type-B households, and pension fund reserves. With regard to *K* transactions, say the brokerage fees associated with transferring ownership of an owner occupied house, NIPA treats these costs as an investment and justifies this as putting the house to more productive use. With government debt transfer of ownership costs are zero in our model and virtually zero in fact. Pension fund reserves are not traded between households, and therefore there are almost no costs associated with transfer of ownership. The total costs of buying and selling of household debt between financial intermediaries are small and are part of intermediation costs. Households incur brokerage fees associated with transferring ownership of financial securities between households. These fees are not payment for intermediating debt between households and therefore not part of the cost of intermediated borrowing and lending between households. Brokerage fees paid by intermediaries are part of the costs of intermediating borrowing and lending between households.

8. Results

We considered four values for α^B, a parameter for which we have little information. For each value of α^B we search for the μ^B for which the intermediated borrowing and lending between households is 1.721 times GNP. The results are summarized in Table 2, which shows that the aggregate results are not sensitive to the size of the bequest preference parameter α^B. Given that the aggregate results are insensitive to α^B, subsequently we deal only with the case $\alpha^B = 1$.[16]

[16] Like Cagetti and De Nardi (2006), there is little consequence of inheritance for the net worth distribution.

Table 2

Summary of Aggregate Results

Economy	$\alpha^B = 1/3$	$\alpha^B = 1$	$\alpha^B = 3$	$\alpha^B = 6$
μ^A	0.879	0.882	0.892	0.903
μ^B	0.121	0.118	0.108	0.097
National Accounts				
C_A	0.660	0.662	0.670	0.679
C_B	0.094	0.092	0.084	0.075
X	0.210	0.210	0.210	0.210
I	0.036	0.036	0.036	0.036
Y	1.000	1.000	1.000	1.000
Depreciation	0.15	0.15	0.15	0.15
Compensation	0.70	0.70	0.70	0.70
Profits	0.15	0.15	0.15	0.15
Net Worth				
Type-A	6.79	6.81	6.88	6.96
Type-B	1.26	1.26	1.26	1.26
Government Debt/Y	4.81	4.83	4.90	4.97
Bequest/Y	0.0243	0.0249	0.0262	0.0279
Tax rate	0.0687	0.0690	0.0699	0.0711

Bequests

Total bequests in our model, as seen in Table 2, are 0.025 times GNP for $\alpha^B = 1$. This is the aggregate value of estates in the mid 1970s that exceeded $40,000.[17] Some of these estates are interposal and should not be included. Adding in vivos transfers and underreporting of gifts associated with the transfer of family businesses to the younger generation would result in aggregate bequests being close to model aggregate bequests. Modigliani's (1988) estimate of bequest flows is close to the model flow. He reports bequests of 0.02 times GDP and adds life insurance, death benefits and newly established trusts to conclude that bequests are 0.027 times GDP.

Another measure of the size of bequests is the amount an individual inherits in units of the annual wage at time of inheritance. Each individual receives at chronological age 52 an amount equal to 1.42 times their annual wage at that time. Menchick and David (1983) estimate average inheritance received by all males to be $20,000 (in 1967 dollars). Correcting by inter-spousal transfers this number is reduced to $13,220. We estimate the average gross annual wage for that year as $8840, arriving at a ratio of inheritance received to annual wage equal to 1.4.[18] These considerations suggest that inheritances are roughly in line with the predictions of our model.[19]

[17] Department of Treasury (2007), Historic Table 17, p. 203.
[18] Nominal GDP in 1967 was $833 billion. Assuming that 70 percent of GDP is labor income (consistent with our model economy) we obtain an estimate of total wage income of $583 billion in 1967. Then, since the total employment in that year was 65.9 million, the average gross annual wage income is $8840.
[19] We examined the consequence of population growth and found that they were small. Bequests fall to 0.023 times GNP as the population growth increases to the point at which the growth rate of the economy equals the interest rate.

Inheritance

Another variable of interest is the fraction of wealth that is inherited. A significant component of wealth is human capital, which is the present value of wages in our model world where labor is supplied inelastically. The other part is the present value of inheritance. As shown in Table 3, human capital is about 96.7 percent of wealth at entry into the workforce and would be higher if there were population growth. These results are for a type-A households, who discount using a 3 percent rate. The share is a little lower for type-B households who use a 5 percent discount rate. Anything that reduces the ratio of bequests to GNP reduces this number, so for the model with a 1 percent population growth rate, as in the United States, this ratio is near 98 percent.

Table 3

Inheritance as Fraction of Wealth at Entry into Workforce

	$\alpha^B = 1/3$	$\alpha^B = 1$	$\alpha^B = 3$	$\alpha^B = 6$
Type-A	0.032	0.033	0.034	0.036
Type-B	0.025	0.026	0.027	0.029

The issues as to the importance of bequest for the size of the capital stock are mute in our model, as policy determines the capital stock and not the nature of preferences for bequests. However, a statistic of interest is the one estimated by Kotlikoff and Summers (1981). This statistic is the present value of inheritances people alive have received, using a 3 percent interest rate. Their estimate of this number is 0.80 times the total household net worth. Modigliani's (1988) estimate of this number is much smaller: 0.20. Modigliani (Table 1, page 19) presents a number of other estimates, all of which range between 0.10 and 0.20. This ratio number for our model economy is 0.13, which is in line with these estimates.

In our model economy 93 percent of bequests are accidental. We came up with this number as follows. Setting $\alpha = 0$ for type-B households and requiring type-B households to follow the no annuity strategy results in this number. Treating these accidental bequests as savings for retirement along with all type-A savings implies that 99 percent of savings is for retirement purposes and 1 percent is for bequests.

Government Debt

Government debt in our model may appear large relative to U.S. federal, state and local government, which was 0.52 times GNP in 2007. But, there are huge implicit liabilities of the U.S. government. The present value of the implicit Social Security Retirement and Medicare promises are over three times GNP by most estimates. If these promises are treated as government debt, actual government debt is at the level in the model. Thus, stock of government debt in our model is reasonable.

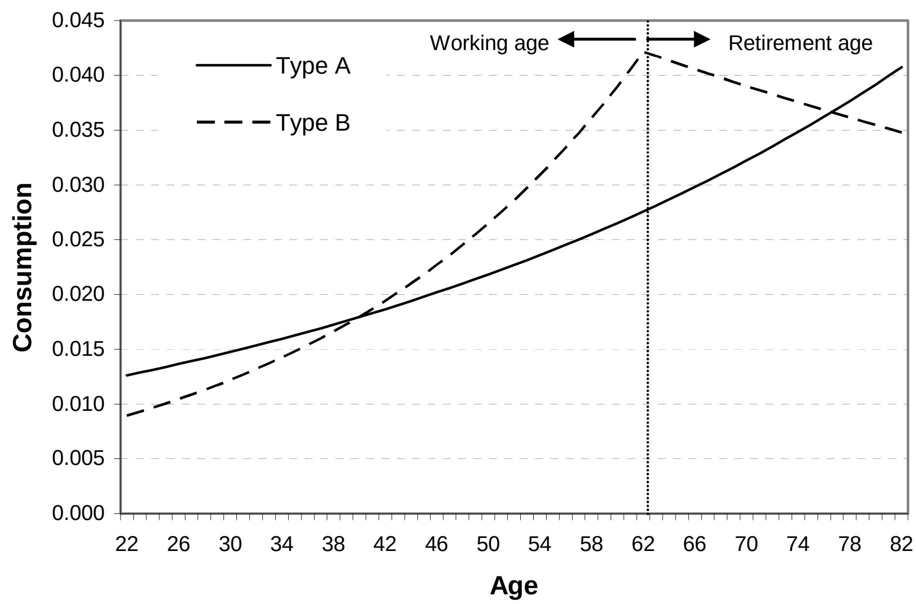
An additional point is that if no one had a bequest motive, the steady-state capital stock would be the same, namely, three times GNP, and government debt in our model would be slightly larger.

Some Micro Findings

Our abstraction has implications for micro observations. Unlike the macro findings, the model's micro findings are not a quantitative theory of the consequence of the bequest motive for the distributions of consumption, net worth, and equity holdings and consequently must be interpreted with care. They do, however, show that the bequest motive, or for that matter any factor that leads people to partially finance their capital acquisitions with debt, is quantitatively important for these statistics. With this caveat, the micro distributional relations for our model economy are as follows.

Figure 2 plots the lifetime consumption patterns of the two types of households. Type-A's consumption grows at a constant annual rate of 1.97 percent throughout their lifetime. Type-B's starts out lower and grows more rapidly during their working life, with this growth rate being 3.95 percent. Upon retirement the consumption growth rate turns negative, falling to -0.95 percent. At retirement a type-B retiree's consumption is higher than an equal age type-A retiree.

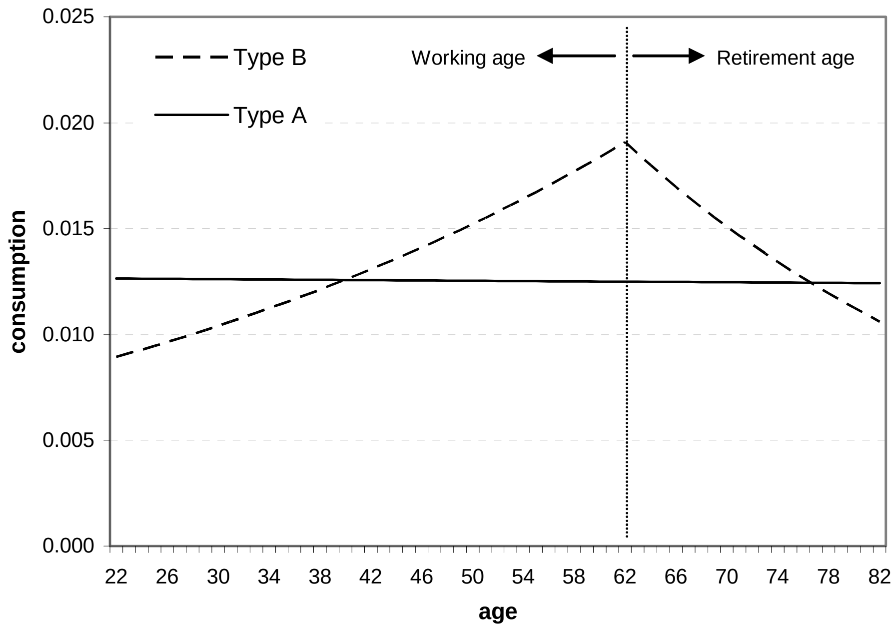
Figure 2
Lifetime Consumption Pattern



Cross-sectional consumption

Figure 3 plots cross-sectional consumption by age for the two types. All type-A that are alive have virtually the same consumption. Young type-B workers have lower consumption and older workers have higher consumption. For the type-B retirees, consumption level declines with age.

Figure 3
Cross-Sectional Consumption by Age

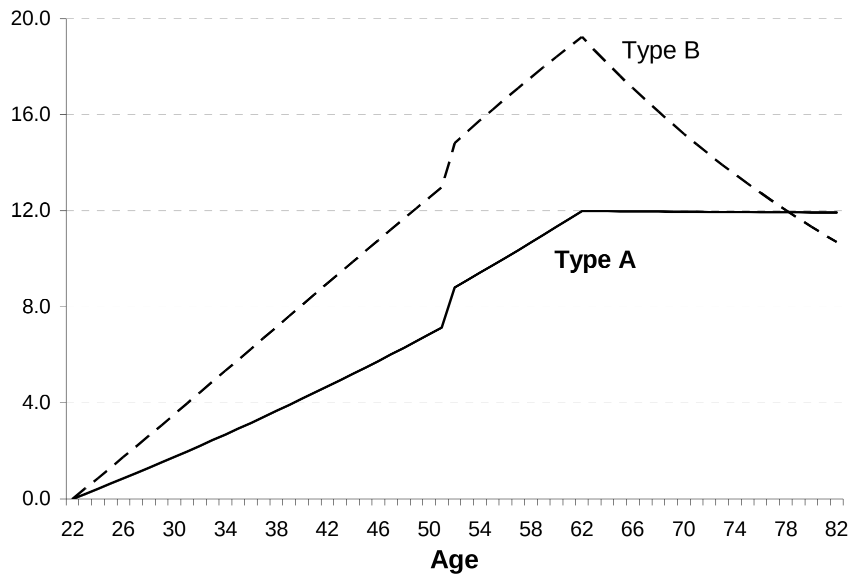


Net worth by age

In Figure 4 we plot net worth relative to current annual wage income, which has a stationary distribution. At retirement the net worth of a type-A household is 12 times the annual wage, and that of a type-B is 19 times the annual wage. The disparity in net worth (corrected for age) is modest, being a maximum of about 1.6 at retirement age. After retirement disparity falls until age 78, and then starts growing with the type-A household becoming the one with the greater net worth. The jump in net worth at chronological age 52 is due to inheritance.

Figure 4

Net Worth as a Function of Age in Units of Annual Wage Income



Lorenz curves

Figure 5 plots the Lorenz curves for consumption, net worth, and capital or equity holdings. In the case of capital, we assume all type-B households have the same ratio of debt liabilities to capital in their portfolios in order to resolve the portfolio indeterminacy at the individual level. We truncate the distribution at age 112, so the curves are not exact, but are very good approximations given the small fraction of population over this age.

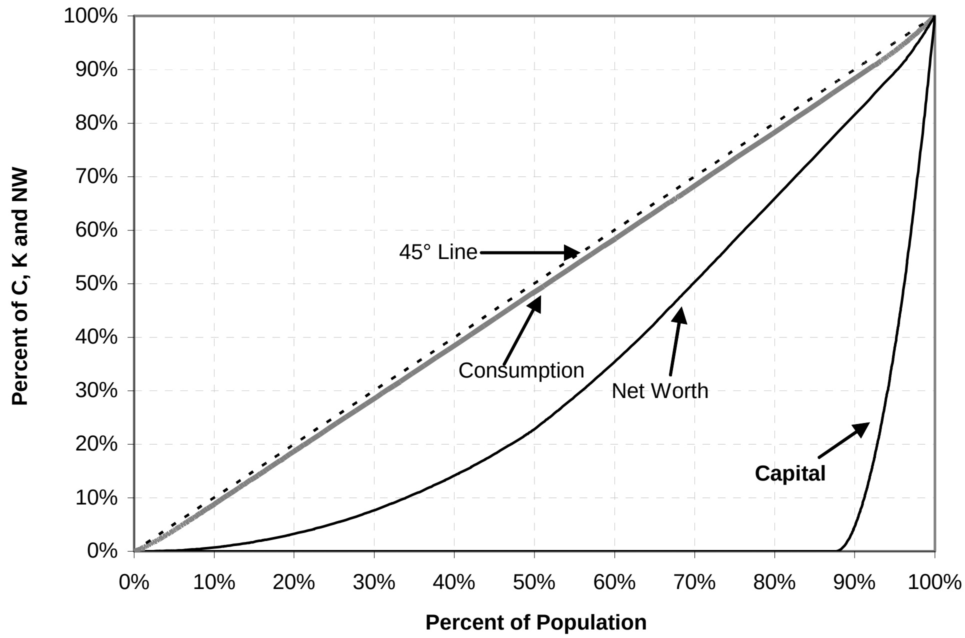
Our principal findings are that there is almost no disparity in consumption levels, modest disparities in net worth levels, and large disparity in capital holdings. Type-B households are the only ones holding the capital stock, so 12 percent of the population owns 100 percent of the capital. There is some dispersion in capital holdings within the type-B sub-population, and 3.6 percent of the population own half the capital stock. This shows that the dispersion in capital holdings is a bad proxy for dispersion in consumption.[20]

In our model economy, all individuals have the same human capital endowment. If the model were modified to have people earn proportionally different wages, to a first approximation for the equilibrium an individual's allocation is proportional to that individual's wage.[21] Introducing wage disparity would add disparity in consumption, net worth, and capital stock holdings. Introducing entrepreneurs (Cagetti and De Nardi (2006)) and idiosyncratic risk (Castãneda, Díaz-Giménez, and Ríos-Rull (2003) and Chatterjee et al. (2007)) would increase disparity as well.

[20] The Gini coefficients for the Consumption, Net Worth, and Capital Lorenz's curves are 0.03, 0.34, and 0.92, respectively.
[21] If bequests were distributed proportional to the human capital factor, the scaling result would hold exactly.

Figure 5

Lorenz curve for Consumption, Net Worth, and Capital



Cost of financial market constraints

What are the gains to a household of having access to the equity market at no intermediation cost? Table 4 reports the cost of not having this access (which was the case for most Americans prior to the development of low-cost indexed mutual funds) as being about 4.3 percent of wealth at time of entry into the workforce. This wealth is the present value of labor income and inheritance.

Table 4

Cost to a Type-A of Not Having Access to the
Annuity Market in Units of Wealth at Entry into Workforce

α^B	Change in v_0^A
1/3	0.52%
1	0.54%
3	0.57%
6	0.62%

Table 5

Cost to a Type-B of Not Being Permitted to Hold Equity Directly
in Units of Wealth at Entry into Workforce

α^B	Change in v_0^B
1/3	1.51%
1	4.25%
3	10.00%
6	16.06%

Tables 4 and 5 show the percentage increase in either e_0 or v_0^k necessary to compensate an $i \in \{A, B\}$ in wealth equivalents if forced to switch to a system other than their preferred choice. Since both consumption and bequest are linear functions of initial wealth, the percentage changes in both consumption and bequest are the same as the percentage change in initial wealth.

What are the costs to a type-A if for some reason, such as adverse selection problems or legal constraints, they do not have access to annuity markets and must use the equity option for saving? The cost is small, being approximately 0.54 percent of lifetime consumption.

9. Concluding Comments

In this paper, we develop a heterogeneous household economy where households differ in only one dimension: their preferences for bequest. In equilibrium, households with a low desire to bequeath lend and hold annuities, while those with a high desire to bequeath borrow and own capital. This is important because the total amount of borrowing by households and the government must equal the amount lent by households. Our simple framework mimics reality with respect to *both* the amount of intermediated borrowing and lending between households *and* the average spread in borrowing and lending rates resulting from intermediation costs.

We find that incorporating the divergence between household borrowing and lending rates can account for half of the historically observed equity premium, which we define as the difference between the average return on equity and the lending rate. We emphasize that lenders are receiving annuity services, which they value, and if the value of these services is included, the return on lending is the same as or higher than the borrowing rate and the return on equity.

Our analysis in this paper is admittedly stylized. However, we believe the abstraction is well suited to address the impact of the costs associated with financial intermediation on the equity premium and for enhancing our quantitative economic intuition as to the reason for the high disparity in net worth and capital holdings. We view this as a first step in what we think may prove to be a productive research program.

Possible extensions include building in differential survival rates and addressing the issues of adverse selection and moral hazard when pricing annuities. This extension might justify our requirement that people choose between the annuity and the no annuity strategies early in their careers. This research program, if successful, will require extension of the theory of household lifetime consumption behavior because the bequest motive is not the only salient factor that differentiates people. Differences in preferences with respect to consumption today versus consumption in the future and differences in preferences that give rise to differences in lifetime labor supply are likely to be important.

Another possible extension is to model non-steady-state behavior. For example, Geanakoplos, Magill, and Quinzii (2004) consider the importance of demographic waves for stock market valuation and Braun, Ikeda, and Joines (2007) for saving behavior within the overlapping generation framework.

References

Abel, A. B., and M. Warshawsky. 1988. "Specification of the Joy of Giving: Insights from Altruism," *Review of Economics and Statistics* 70(1), 145–149.

Altonji, J. G., F. Hayashi, and L. J. Kotlikoff. 1997. "Parental Altruism and Inter Vivos Transfers: Theory and Evidence," *Journal of Political Economy* 105(6), 1121–1166.

Barro, R. J. 1974. "Are Government Bonds Net Wealth?" *Journal of Political Economy* 82(6), 1095–1117.

Becker, G. S., and R. J. Barro. 1988. "A Reformulation of the Economic Theory of Fertility," *Quarterly Journal of Economics* 103(1), 1–25.

Blanchard, O. J. 1985. "Debt, Deficits, and Finite Horizons," *Journal of Political Economy* 93(2), 223–247.

Board of Governors of the Federal Reserve System. 2000. "Flow of Funds Accounts of the United States," Tables B.100, B.100.e, and B.100b.e.

Braun, G. A., D. Ikeda, and D. H. Joines, 2007. "The Saving Rate in Japan: Why It Has Fallen and Why It Will Remain Low," CIRIE-F Working Paper F-535.

Cagetti, M., and M. De Nardi, 2006. "Entrepreneurship, Frictions, and Wealth," *Journal of Political Economy* 114(5), 835–870.

Castãneda, A., J. Díaz-Giménez, and J. V. Ríos-Rull. 2003. "Accounting for U.S. Earnings and Wealth Inequality," *Journal of Political Economy* 111(4), 818–857.

Chatterjee, S., D. Corbae, M. Nakajima, and J.-V. Ríos-Rull. 2007. "A Quantitative Theory of Unsecured Consumer Credit with Risk of Default," *Econometrica* 75(6), 1525–1589.

Constantinides, G. M., J. B. Donaldson, and R. Mehra. 2007. "Junior Is Rich: Bequests as Consumption," *Economic Theory* 32(1), 125–155.

De Nardi, M. 2004. "Wealth Inequality and Intergenerational Links," *Review of Economic Studies* 71(7), 743–768.

De Nardi, M., S. Imrohoroglu, and T. J. Sargent. 1999. "Projected U.S. Demographics and Social Security," *Review of Economic Dynamics* 2(3), 575–615.

Díaz-Giménez, J., T. J. Fitzgerald, E. C. Prescott, and F. Alvarez. 1992. "Banking in Computable General Equilibrium Economies," *Journal of Economic Dynamics and Control* 16(3–4), 533–559.

Doepke, M., and M. Schneider. 2006. "Inflation and the Redistribution of Nominal Wealth," *Journal of Political Economy* 114(6), 1069–1097.

Fernández-Villaverde, J., and D. Krueger. 2002. "Consumption over the Life Cycle: Some Facts from Consumer Expenditure Survey Data," NBER Working Paper 9382.

Fuster, L., A. Imrohoroglu, and S. Imrohoroglu. 2008. "Altruism, Incomplete Markets, and Tax Reform," *Journal of Monetary Economics* 55(1), 65–90.

Geanakoplos, J., M. Magill, and M. Quinzii. 2004. "Demography and the Long-Run Predictability of the Stock Market," Cowles Foundation Discussion Paper 1380R.

Hansen, G. D., and S. Imrohoroglu. 2006. "Consumption over the Life Cycle: The Role of Annuities," NBER Working Paper W12341.

Hurd, M. D. 1989. "Mortality Risk and Bequests," *Econometrica* 57(4), 779–813.

Hurd, M. D., and B. G. Mundaca. 1989. "The Importance of Gifts and Inheritances among the Affluent," in *The Measurement of Saving, Investment and Wealth*, Lipsey, R. E., and H. S. Tice (editors), Chicago: University of Chicago Press.

Kopczuk, W., and J. Lupton. 2007. "To Leave or Not to Leave: The Distribution of Bequest Motives," *Review of Economic Studies* 74(1): 207–235.

Kotlikoff, L. J., and L. H. Summers. 1981. "The Role of Intergenerational Transfers in Aggregate Capital Accumulation," *Journal of Political Economy* 89(4), 706–732.

Laitner, J., and F. T. Juster. 1996. "New Evidence on Altruism: A Study of TIAA-CREF Retirees," *American Economic Review* 86(4), 893–908.

Laitner, J., and H. Ohlsson. 2001. "Bequest Motives: A Comparison of Sweden and the United States," *Journal of Public Economics* 79(1), 205–236.

Lettau, M., and H. Uhlig. 2000. "Can Habit Formation be Reconciled with Business Cycle Facts?" *Review of Economic Dynamics* 3(1), 79–99.

McGrattan, E. R., and E. C. Prescott. 2000. "Is the Stock Market Overvalued?" *Federal Reserve Bank of Minneapolis Quarterly Review* 24(4), 20–40.

McGrattan, E. R., and E. C. Prescott. 2003. "Average Debt and Equity Returns: Puzzling?" *American Economic Review* 93(2), 392–397.

McGrattan, E. R., and E. C. Prescott. 2005. "Taxes, Regulations, and the Value of U.S. and U.K. Corporations," *Review of Economic Studies* 72(3), 767–796.

Mehra, R., and E. C. Prescott. 1985. "The Equity Premium: A Puzzle," *Journal of Monetary Economics* 15(2), 145–161.

Menchik, P. L., and M. David. 1983. "Income Distribution, Lifetime Savings, and Bequests," *American Economic Review* 73(4), 672–690.

Miller, M. H., and C. W. Upton. 1974. *Macroeconomics: A Neoclassical Introduction*, Homewood, IL: Richard D. Irwin.

Modigliani, F. 1986. “Life Cycle, Individual Thrift, and the Wealth of Nations,” *American Economic Review* 76(3), 297–313.

Modigliani, F. 1988. “The Role of Intergenerational Transfers and Life Cycle Saving in the Accumulation of Wealth,” *Journal of Economic Perspectives* 2(2), 15–40.

U.S. Department of Commerce. Bureau of Economic Analysis. 2000. “National Income and Product Accounts,” Tables 7.11, 2.4.5.

U.S. Department of Treasury. Internal Revenue Service. 2007. *Statistics of Income Bulletin* 21(1).

Wilhelm, M. O. 1996. “Bequest Behavior and the Effect of Heirs’ Earnings: Testing the Altruistic Model of Bequests,” *American Economic Review* 86(4), 874–892.

Yaari, M. E. 1965. “Uncertain Lifetime, Life Insurance, and the Theory of the Consumer,” *Review of Economic Studies* 32(2), 137–150.

Appendix 1: Proof of Proposition 1

The prices (r, r_e, e_0) , tax rate τ , and inheritance implied by $\bar{b}_0$ are given to an individual. Note $0 < r < r_e$. Let $U_A(\alpha)$ and $U_B(\alpha)$ represent the maximum attainable utility of an agent of measure zero in this economy who follows strategy A (annuity) or B (bequest) respectively as a function of $\alpha \in \mathbb{R}_+$. Define $\Delta(\alpha) = U_B(\alpha) - U_A(\alpha)$.

Proposition 1: If $\dfrac{1+r_e}{1+r} > \beta\left[\dfrac{1-\sigma\beta}{\beta\delta}\right]^{1-\beta\sigma}$ then

$$\frac{\partial \Delta U(\alpha)}{\partial \alpha} > 0 \quad \forall \alpha \in \mathbb{R}_+ .$$

*Proof***:** The maximum utility as a function of α attainable by an agent who follows an annuity strategy (A), taking as given the parameters of the economy, can be expressed as:

$$U_A(\alpha) = \sum_{j=0}^{T-1} \beta^j \log(c_j^A) + \beta^T (\phi_A(\alpha) + \theta_A(\alpha) log(w_T^A)),$$

where

$$\theta_A(\alpha) = \frac{1+\beta\alpha\delta}{1-\beta(1-\delta)}$$

$$\phi_A(\alpha) = \frac{(\theta_A(\alpha)-1)\log[(1+r)\beta] - \log[\theta_A(\alpha)] + \beta\alpha\delta[\log(\alpha) - \log[\theta_A(\alpha)]]}{1-\beta(1-\delta)}$$

$$c_j^A = \bar{c}(\alpha)\beta^j (1+r)^j v_0^A \qquad\qquad j < T$$
$$w_T^A = \theta_A(\alpha)\bar{c}(\alpha)\beta^T (1+r)^T v_0^A$$

($\bar{c}(\alpha)$ and v_0^A are defined in Section 3).

Similarly, the maximum utility as a type α who follows an annuity strategy (B) is

$$U_B(\alpha) = \sum_{j=0}^{T-1} \beta^j \log(c_j^B) + \beta^T (\phi_B(\alpha) + \theta_B(\alpha) log(w_T^B)),$$

where

$$\theta_B(\alpha) = \frac{1+\beta\alpha\delta}{1-\beta(1-\delta)}$$

$$\phi_B(\alpha) = \frac{(\theta_B(\alpha)-1)\log(1+r_e) + (\theta_B(\alpha)-1)\log[(\theta_B(\alpha)-1)] - \theta_B(\alpha)\log[(\theta_B(\alpha)]}{1-\beta(1-\delta)}$$

$$c_j^B = \bar{c}(\alpha)\beta^j(1+r_e)^j v_0^B \qquad j < T$$
$$w_T^B = \theta_B(\alpha)\bar{c}(\alpha)\beta^T(1+r_e)^T v_0^B$$

($\bar{c}(\alpha)$ and v_0^B are defined in Section 3).

Using the properties of the logarithm function and defining $\theta(\alpha) = \theta_A(\alpha) = \theta_B(\alpha)$

(A1.1) $$\Delta(\alpha) = \sum_{j=0}^{T-1} \beta^j \log\left(\frac{(1+r_e)^j v_0^B}{(1+r)^j v_0^A}\right) + \beta^T(\phi_B(\alpha) - \phi_A(\alpha) + \theta(\alpha) log(\frac{w_T^B}{w_T^A}))$$

Since the first term is independent of α it follows that

(A1.2) $$\frac{\partial\Delta(\alpha)}{\partial\alpha} = \beta^T \frac{\partial[\phi_B(\alpha) - \phi_A(\alpha)]}{\partial\alpha} + \beta^T \theta'(\alpha) log\left(\frac{w_T^B}{w_T^A}\right),$$

where $\theta'(\alpha) = \dfrac{\beta\delta}{1-\beta\sigma} > 0$ which does not depend on α.

$$\frac{w_T^B}{w_T^A} = \frac{v_0^B(1+r_e)^T}{v_0^A(1+r)^T} = \frac{\displaystyle\sum_{j=0}^{T-1} \frac{(1-\tau)e_0(1+\gamma)^j}{(1+r_e)^{j-T}} + \frac{\bar{b}}{(1+r_e)^{30-T}}}{\displaystyle\sum_{j=0}^{T-1} \frac{(1-\tau)e_0(1+\gamma)^j}{(1+r)^{j-T}} + \frac{\bar{b}}{(1+r)^{30-T}}} > 1$$ since $r_e > r, j < T$ and $30 < T$ Th

is implies the second term in (A1.2) is positive, i.e., $\beta^T \theta'(\alpha) log\left(\frac{w_T^B}{w_T^A}\right) > 0$

To prove our assertion that $\dfrac{\partial\Delta U(\alpha)}{\partial\alpha} > 0$ is positive, we proceed in three steps:

a. We show that $\lim\limits_{\alpha\to 0} \dfrac{\partial\Delta(\alpha)}{\partial\alpha} > 0$;

b. We show that $\dfrac{\partial^2\Delta(\alpha)}{\partial\alpha^2} < 0$; and that

c. $\lim\limits_{\alpha\to\infty} \dfrac{\partial\Delta(\alpha)}{\partial\alpha} > 0$

Some straightforward algebra yields

(A1.3)

$$\frac{\partial[\phi_B(\alpha)-\phi_A(\alpha)]}{\partial\alpha}=\frac{\theta'(\alpha)}{1-\beta\sigma}\left(\log\left(\frac{1+r_e}{(1+r)\beta}\right)+\log\left(\frac{\theta(\alpha)-1}{\alpha}\right)-\beta\sigma\log\left(\frac{\theta(\alpha)}{\alpha}\right)\right)$$

From (A1.3) it is readily seen that $\lim_{\alpha\to 0}\frac{\partial[\phi_B(\alpha)-\phi_A(\alpha)]}{\partial\alpha}\to+\infty$. This follows since the last term tends to $+\infty$ and all the other terms are bounded. This coupled with the fact that $\beta^T\theta'(\alpha)log\left(\frac{w_T^B}{w_T^A}\right)>0$ proves that $\lim_{\alpha\to 0}\frac{\partial\Delta(\alpha)}{\partial\alpha}>0$.

The second derivative $\frac{\partial^2\Delta(\alpha)}{\partial\alpha^2}$ is negative by direct differentiation,

$$\frac{\partial^2\Delta(\alpha)}{\partial\alpha^2}=\frac{-\beta^{T+1}\delta(1-\delta)}{\alpha(1-\beta(1-\delta))((1+(\alpha-1)\delta)(1+\alpha\beta\delta)}<0\ ,$$

since the denominator is always positive and the numerator is negative.

Finally it can be shown that $\lim_{\alpha\to\infty}\frac{\partial\Delta(\alpha)}{\partial\alpha}>0$ under the condition stated in the theorem. Notice that (taking the limit of A1.3) when $\alpha\to\infty$) equation A1.2) is positive if and only if

$$\frac{1}{1-\beta\sigma}\log\left(\frac{(1+r_e)}{(1+r)\beta}\right)+\log\big(\theta'(\alpha)\big)+\log\left(\frac{v_0^B(1+r_e)^T}{v_0^A(1+r)^T}\right)>0\,.$$

The last term in the above expression has already been shown to be positive. Thus a sufficient condition for this inequality is

$$\frac{1}{1-\beta\sigma}\log\left(\frac{(1+r_e)}{(1+r)\beta}\right)+\log\big(\theta'(\alpha)\big)>0\ .$$

This inequality can be written as $\frac{1+r_e}{1+r}>\beta\left[\frac{1-\sigma\beta}{\beta\delta}\right]^{1-\beta\sigma}$

Since a), b), and c) are satisfied, it follows that $\frac{\partial\Delta U(\alpha)}{\partial\alpha}>0\quad\forall\alpha\in\mathbb{R}_+$. **QED**

Appendix 2: Aggregation

General formulas

There are two types $i \in \{A, B\}$. The A-type has $\alpha^A = 0$ and in equilibrium choose the annuity strategy given the model economy. The measure of type *i* of age *j* at $t = 0$ is

(A2.1) $$\mu_j^i = \begin{cases} \mu_0^i & j \le T \\ (1-\delta)^{j-T}\mu_0^i & j > T \end{cases}$$

Aggregate quantity for variable *Z* of type $i \in \{A, B\}$ agents at $t = 0$ is Z_0^k,

(A2.2) $$Z_0^i = \mu_0^i \sum_{j=0}^{T-1} z_j^i (1+\gamma)^{-j} + \mu_0^i \sum_{j=T}^{\infty} (1-\delta)^{j-T} z_j^i (1+\gamma)^{-j} ,$$

where z_j^i is the individual allocation of type-*i* at age *j* born at $t = 0$. Notice that we have used the fact that each subsequent generation has a consumption-age profile that is higher by a factor of $(1+\gamma)^j$ under balanced growth. Aggregate quantity of *Z* at time 0, Z_0 is

$$Z_0 = Z_0^A + Z_0^B$$

Agent Type-B

Aggregate assets of agent type-B and aggregate bequest

The aggregate assets for B-type agents are computed using the law of motion of Net Worth. From the individual problem,

$$w_0^B = 0$$
$$w_j^B = \left(w_{j-1}^B - c_{j-1}^B + (1-\tau)e_0(1+\gamma)^{j-1}\right)(1+r_e) \quad \text{for } j \le T \;\&\; j \ne 30$$
$$w_j^B = \left(w_{j-1}^B - c_{j-1}^B + (1-\tau)e_0(1+\gamma)^{j-1}\right)(1+r_e) + \overline{b} \quad \text{for } j = 30$$
$$w_j^B = \left(w_{j-1}^B - c_{j-1}^B\right)(1+r_e) \quad \text{for } j > T$$

From equations (3.4) and (3.7), the consumption for type B is given by

(A2.3) $$c_j^B = \begin{cases} [\beta(1+r_e)]^j \bar{c}^B(\alpha^B) v_0^B & j < T \\ \dfrac{w_j^B}{f_2(\alpha^B)} & j \geq T \end{cases}$$

where

(A2.4) $$\bar{c}^B(\alpha) = \frac{(1-\beta)}{1-\beta^T + (1-\beta)\beta^T f_2(\alpha)}$$

$$v_0^B = \sum_{j=0}^{T-1} \frac{(1-\tau)e_0(1+\gamma)^j}{(1+r_e)^j} + \frac{\bar{b}}{(1+r_e)^{30}}$$

and $$\bar{f}_2(\alpha) = \frac{1+\alpha\ \beta\delta}{1-\sigma\beta}$$

Using (A2.2) aggregate net worth is

$$W^B(\bar{b},\tau) = \mu_0^B \sum_{j=0}^{T-1} w_j^B (1+\gamma)^{-j} + \mu_0^B \sum_{j=T}^{\infty} \sigma^{j-T} w_j^B (1+\gamma)^{-j}$$

The summation over *j=0,…,T-1* is performed numerically, while for total net worth of the retirees is

(A2.5) $$\mu_0^B \sum_{j=T}^{\infty} \sigma^{j-T} w_j^B (1+\gamma)^{-j} = \frac{w_T^B \mu_0^B (1+\gamma) f_2(\alpha^B)}{(1+\gamma)^T [(1+\gamma) f_2(\alpha^B) - \sigma(1+r_e)(f_2(\alpha^B)-1)]} \ ,$$

where from the individual problem

$$w_T^B = f_2(\alpha^B)[\beta(1+r_e)]^T \bar{c}^B(\alpha^B) v_0^B$$

Since $\alpha^A = 0$ all bequests are coming from the type-B, and as shown in Section 3.1 is given by

$$b_j^B = w_j^B \qquad j \geq T+1$$

if a type-B dies prior to the end of the previous period subsequent to consuming, and zero otherwise.

Since the measure of agents dying at each age $j \geq T+1$ is $\mu_0^B \delta \sigma^{j-T-1} = \delta \mu_{j-1}^B$ the aggregate bequest is

$$B_0(\bar{b},\tau) = \sum_{j=T+1}^{\infty} \delta \frac{\mu_{j-1}^B b_j^B}{(1+\gamma)^j} = \sum_{j=T+1}^{\infty} \delta \frac{\mu_{j-1}^B w_j^B}{(1+\gamma)^j} = \frac{\delta}{\sigma} \sum_{j=T+1}^{\infty} \frac{\mu_j^B w_j^B}{(1+\gamma)^j}$$

Using (A2.5) it is straightforward to find that

$$B_0(\bar{b},\tau) = \frac{\delta w_T^B \mu_0^B}{\sigma(1+\gamma)^T}\left[\frac{(1+\gamma) f_2(\alpha^B)}{[(1+\gamma) f_2(\alpha^B) - \sigma(1+r_e)(f_2(\alpha^B)-1)]} - 1\right]$$

or

(A2.6) $$B_0(\bar{b},\tau) = \frac{\delta w_T^B \mu_0^B}{(1+\gamma)^T}\left[\frac{(f_2(\alpha^B)-1)(1+r_e)}{[(1+\gamma) f_2(\alpha^B) - \sigma(1+r_e)(f_2(\alpha^B)-1)]}\right]$$

Aggregate consumption type B

Similarly, using (A2.2) and (A2.3) the aggregate consumption of type B agents at time 0 can be expressed as

(A2.7) $$C_0^B = \Phi_1^B v_0^B,$$

where

$$\Phi_1^B = \bar{c}(\alpha^B)\left[\sum_{j=0}^{T-1}\left(\frac{\beta(1+r_e)}{1+\gamma}\right)^j + \beta^T \sum_{j=T}^{\infty}\left(\frac{(1+r_e)}{1+\gamma}\right)^j \left(\frac{f_2(\alpha^B)-1}{f_2(\alpha^B)}\right)^{j-T} \sigma^{j-T}\right]\mu_0^B$$

or

$$\Phi_1^B = (1+\gamma)\bar{c}^B(\alpha)\left[\frac{1-\left[\frac{\beta(1+r_e)}{1+\gamma}\right]^T}{(1+\gamma)-\beta(1+r_e)} + \frac{(1+\alpha^B\beta\delta)\left[\frac{\beta(1+r_e)}{1+\gamma}\right]^T}{(1+\gamma)-\beta\sigma^2+\alpha^B\beta\delta(\gamma+\delta)-\beta\sigma(1-\delta(1-\alpha^B))r_e}\right]\mu_0^B$$

Agent Type A

Aggregate assets of agent type A

The aggregate bequest is measured in units of agent type B assets, therefore the inheritance received by agent type A measured in her assets' units is $\bar{b}^A = \bar{b}(1+r)/(1+r_e)$. The aggregate assets for agents type A are computed using the law of motion of Net Worth. From the individual problem,

$$
\text{(A2.8)} \qquad
\begin{aligned}
& w_0^A = 0 \\
& w_j^A = \left(w_{j-1}^A - c_{j-1}^A + (1-\tau)e_0(1+\gamma)^{j-1}\right)(1+r) && \text{for } j \le T \;\&\; j \ne 30 \\
& w_j^A = \left(w_{j-1}^A - c_{j-1}^A + (1-\tau)e_0(1+\gamma)^{j-1}\right)(1+r) + \bar{b}^A && \text{for } j = 30 \\
& w_j^A = \sum_{t=0}^{\infty}(1-\delta)^t \frac{c_{j+t}^A}{(1+r)^t} + \sum_{t=0}^{\infty}\delta(1-\delta)^{t-1}\frac{b_{j+t}^A}{(1+r)^t} && j > T
\end{aligned}
$$

Using (A2.2) aggregate net worth is calculated as

$$W^A(\bar{b},\tau) = \mu_0^A \sum_{j=0}^{T-1} w_j^A (1+\gamma)^{-j} + \mu_0^A \sum_{j=T}^{\infty} \sigma^{j-T} w_j^A (1+\gamma)^{-j}$$

As for type B, the summation for $j=0,\ldots,T$ is performed numerically. Since in the calibration, $\alpha^A = 0$. From equation (3.11) consumption for type A agents, born at period zero when they reach age *j* (at time *j*), is

$$c_j^A = \bar{c}(\alpha^A)(1+r)^j \beta^j v_0^A \qquad j \ge 0$$

Then, agents alive at time 0 of age *j* consume

$$\text{(A2.9)} \qquad c_{0,j}^A = \bar{c}(\alpha^A) v_0^A \left[\frac{\beta(1+r)}{(1+\gamma)}\right]^j \qquad j \ge 0$$

Using (A2.8) and (A2.9) net worth for retired agents can be written as

$$w_j^A = \frac{c_{j,0}^A}{1\text{-}\beta\sigma} \qquad j > T \quad .$$

Then

$$\mu_0^A \sum_{j=T+1}^{\infty} \sigma^{j-T} \frac{w_j^A}{(1+\gamma)^{-j}} = \frac{\mu_0^A \bar{c}(\alpha^A) v_0^A}{1\text{-}\beta\sigma} \left[\frac{(1+r)\beta}{1+\gamma}\right]^{T+1} \left[\frac{1+\gamma}{1+\gamma-\beta(1+r)\sigma}\right]$$

Aggregate consumption type A

Again, using (A2.2) and (A2.9), the aggregate consumption of type A agents at time 0 can be expressed as

(A2.10) $$C_0^A = \Phi_1^A v_0^A ,$$

where

$$\Phi_1^A = \bar{c}(\alpha^A)\left[\sum_{j=0}^{T-1}\left(\frac{\beta(1+r)}{1+\gamma}\right)^j + \sum_{j=T}^{\infty}\left(\frac{\beta(1+r)}{1+\gamma}\right)^j \sigma^{j-T}\right]\mu_0^A$$

or

$$\Phi_1^A = (1+\gamma)\bar{c}(\alpha^A)\left[\frac{1-\left[\frac{\beta(1+r)}{1+\gamma}\right]^T}{(1+\gamma)-\beta(1+r)} + \frac{\left[\frac{\beta(1+r)}{1+\gamma}\right]^T}{(1+\gamma)-\beta(1+r)\sigma}\right]\mu_0^A ,$$

where

$$v_0^A = \sum_{t=0}^{T-1}\frac{(1-\tau)e_0(1+\gamma)^j}{(1+r)^j} + \frac{\bar{b}^A}{(1+r)^{30}}$$

Balance Sheets

Type B: $(1+r_e)K = (1+r_e)D^B(\bar{b},\tau) + W^B(\bar{b},\tau)$

Type A: $(1+r)A^A(\bar{b},\tau) = W^A(\bar{b},\tau)$

Intermediary: $(1+r_e-\phi)D^B(\bar{b},\tau) + (1+r)G^G(\bar{b},\tau) = (1+r)A^A(\bar{b},\tau)$

Notice that both the net worth of the intermediary and the government are 0.

Equilibrium Conditions

There are three equilibrium conditions that can potentially be used to solve the model:

1) Feasibility: $\mathrm{Y} = C_0(\bar{b},\tau) + \mathrm{X} + \phi\left[K - \frac{W^B(\bar{b},\tau)}{1+r_e}\right]$,

where $C_0(\bar{b},\tau) = C_0^A(\bar{b},\tau) + C_0^B(\bar{b},\tau)$

2) Bequest=inheritance: $\bar{b} = B_0(\bar{b},\tau)(1+\gamma)^{30}$

3) Assets Markets $\frac{W^B(\bar{b},\tau)}{1+r_e} + \frac{W^A(\bar{b},\tau)}{1+r} = G^G(\bar{b},\tau) + \mathrm{K}$

Since this is a linear system in $(\bar{b},\tau)$ one equation is redundant, and the solution is straightforward. We chose to use the first two equilibrium conditions, and then we check that the third one is satisfied as well.