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ABSTRACT

What is it about modern capitalist economies that allows them, in contrast to all earlier societies, to generate sustained growth in productivity and living standards? It is widely agreed that the productivity growth of the industrialized economies is mainly an ongoing intellectual achievement, a sustained flow of new ideas. Are these ideas the achievements of a few geniuses, Newton, Beethoven, and a handful of others, viewed as external to the activities of ordinary people? Are they the product of a specialized research sector, engaged in the invention of patent-protected processes over which they have monopoly rights? Both images are based on important features of reality and both have inspired interesting growth theories, but neither seems to me central. What is central, I believe, is that fact that the industrial revolution involved the emergence (or rapid expansion) of a class of educated people, thousands–now many millions–of people who spend entire careers exchanging ideas, solving work-related problems, generating new knowledge.

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In this paper I introduce and partially develop a new model of technological change, viewed as the product of a class of problem-solving producers. The model is built up from the premise that all knowledge resides in the head of some individual person, so that the knowledge of a firm, or economy, or any group of people is simply a list of the knowledge of its members. A main feature of the model will be the social or reciprocal character of intellectual activity: Each person gains from the knowledge of the people around him; his ideas in turn stimulate others.[1]

As we will see, this idea is very flexible, but to develop it we need to have a tractable mathematical idealization of an individual's knowledge, of the relation of this knowledge to his productivity, and of the process by which new knoweledge is aquired. Everything in this paper is based on the Kortum (1997) model of the *technology frontier*, a differential equation describing the evolution of the production frontier introduced in Eaton and Kortum (1999), and an adaptation of this equation due to Alvarez, Buera, and Lucas (2007). The necessary details are provided in the next section, in the context of a one-good economy with a labor-only production technology. In Section 2, a cohort structure is introduced, so that individual careers will have a beginning, middle, and end, as they do in reality. Section 3 discusses the possibility of using cross-section evidence on individual earnings, in addition to aggregate data, to calibrate the model. Section 4 considers the incorporation of schooling into the cohort model. Section 5 contains conclusions and further speculations.

[1] Jones (2005) surveys and contributes to a number of papers with a motivation similar to mine, dating back to Arrow (1962). Indeed, my title is just a permutation of Jones'. I was also influenced by Jovanovic and Rob (1989) and Chatterjee and Rossi-Hansberg (2007) who also focus, in different ways, on ideas and problem-solving.

1. The Technology Frontier

Kortum (1997) and Eaton and Kortum (1999) consider models in which different agents have different knowledge and a description of the state of knowledge of an economy requires the entire *distribution* of knowledge over individual agents. Kortum (1997) calls this distribution the *technology frontier*, a usage I will follow here. In different applications, an "agent" can be a firm, a plant, or even an entire country. In this paper, an agent will be an individual person and the knowledge of any collection of people is just the distinct, privately held knowledge of the individuals that comprise it. I will thus interpret Eaton and Kortum's theory in the style of Jovanovic (1979), as a stochastic process model of individual careers.

To begin, we consider an economy with continuum of infinitely-lived people. Each person has one unit of labor per unit of time, and labor is the only factor of production. Everyone produces the same, single, non-storable good, which I will call simply GDP. The productivity of each person—the amount he can produce per unit of time—is a random variable z, say. The distribution of these individual productivities at any date t is the technology frontier, represented by the cdf $G(\cdot,t)$ (and density $g(\cdot,t)$) of z. Then GDP per person in this economy is

$$y(t) = \int_0^\infty zg(z,t)dz.$$

The productivity of any individual evolves as follows. Suppose he has the productivity z at date t, viewed as a draw from the date-t technology frontier $G(\cdot,t)$. Over the time interval $(t,t+h)$ he gets αh independent draws from another distribution, with cdf H.[2] Let y denote the best of these draws. Then at $t+h$ his productivity will be the higher of his original productivity z and the best of his new ideas y, or $\max(z,y)$. It follows that the probability that he has the same productivity z at $t+h$ as he had at t is the probability he does not get any better ideas between t and $t+h$,

[2] A non-integer number of draws? Bear with me.

or

$$
\begin{aligned}
G(z,t+h) &= G(z,t) \times \Pr\{\text{all } \alpha h \text{ draws } \leq z\} \\
&= G(z,t)H(z)^{\alpha h}.
\end{aligned}
$$

Take logs, divide by h and let $h \to 0$ to get

$$
\frac{\partial \log\left(G(z,t)\right)}{\partial t} = \alpha \log\left(H(z,t)\right)).
$$

So if we know the source distribution $H(\cdot,t)$ of ideas and the initial frontier distribution $G(z,0)$ we know the complete evolution of the frontier.

As is clear from Eaton and Kortum's work, this set-up can be developed in a variety of ways. For my purposes here, I want to close the system and obtain an endogenous growth theory. To do this, nothing more is involved than equating H and G, which is to say assuming that the source of everyone's ideas is other people in the same economy. In this case, we have the autonomous equation

$$
\frac{\partial \log\left(G(z,t)\right)}{\partial t} = \alpha \log\left(G(z,t)\right)). \tag{1.1}
$$

Equation (1.1) tells us is that everyone's productivity is always improving, unless there is a maximum productivity level (a z value that $G(z,0) = 1$) and someone has attained it. But we already knew this much from the verbal statement of the problem.

To get beyond this, more structure is needed. I will get it by setting up a specific, parametric model of the process, also due to Eaton and Kortum. We view each person at t as a random draw x from an exponential distribution with parameter $\lambda(t) > 0$: the density function is $\lambda(t)e^{-\lambda(t)x}$. His productivity—the units of GDP he can produce in a year—is $x^{-\theta}$, $\theta > 0$. In this parameterization, the cdf G above becomes

$$
\begin{aligned}
G(z,t) &= \Pr(x^{-\theta} \leq z\} \\
&= \Pr(x \geq z^{-1/\theta}\} \\
&= e^{-\lambda(t)z^{-1/\theta}}.
\end{aligned}
$$

Then $\log(G(z,t)) = -\lambda(t)z^{-1/\theta}$ and the differential equation (1.1) becomes

$$-\frac{d\lambda(t)}{dt}z^{-1/\theta} = -\alpha\lambda(t)z^{-1/\theta}$$

or, cancelling, simply

$$\frac{d\lambda(t)}{dt} = \alpha\lambda(t). \tag{1.2}$$

We can say that productivities have a *Frechet distribution* with parameters (λ, θ). Instead of tracking the entire distribution G we can track the single parameter $\lambda(t)$, the only changing feature of the distribution. Figures 1 and 2 illustrate the implied dynamics of the technology frontier and Figure 3 provides a sample of individual realized productivity paths.

The fact that the Frechet form is "preserved" by the above matching process is a common-sense application of a well-known feature of the exponential distribution: If x and y are mutually independent random variables, exponentially distributed with parameters λ and μ, then $z = \min(x, y)$ is exponentially distributed with parameter $\lambda + \mu$. This feature carries over in an obvious way to the Frechet productivity distributions, since $\max(x^{-\theta}, y^{-\theta}) = [\min(x, y)]^{-\theta} = z^{-\theta}$. The parameter θ does not change. In the process of meeting others an exchanging ideas, both parties in the matched pair (x, y) emerge with productivity $\max(x^{-\theta}, y^{-\theta})$. The effect of this on the motion of the frontier is completely described by the motion of the single parameter $\lambda(t)$.[3]

Total GDP in the economy, so parameterized, is

[3] The differential equation (1.1) is based on the assumption that new ideas arrive at a deterministic rate. An alternate assumption, applied for example in Eaton and Kortum (1999), would be to assume stochastic, Poisson arrivals of new ideas. The practical advantage of (1.1), for my purposes here, is that it preserves the exact exponential distribution. Poisson arrivals do not. These two approaches are compared more systematically in Alvarez, Buera, and Lucas (2007).

$$
\begin{aligned}
y(t) &= \int_0^\infty x^{-\theta}\lambda(t)e^{-\lambda(t)x}dx \\
&= \lambda(t)^{\theta}\int_0^\infty z^{-\theta}e^{-z}dz \\
&= A\lambda(t)^{\theta},
\end{aligned}
$$

where A is the gamma function evaluated at $1-\theta$. Convergence of the integral requires that $\theta < 1$, which I assume to hold throughout the paper. The implied GDP growth rate is $\alpha\theta$.

One way to proceed from this point would be to view this model of idea flows and individual productivity as a theory of the growth of the effective labor input, and to introduce preferences, physical capital accumulation—the entire apparatus of neoclassical growth theory. The details of these extensions are familar, but I will continue—for simplicity only—to work with a labor-only technology. In any case, notice that with or without physical capital, the mystery of TFP growth has disappeared![4] Later on we will estimate θ from the variability of individual earnings, but the parameter α—the rate at which ideas are processed—remains free. For the U.S., for example, we could simply calibrate α to the value $(.02)/\theta$, once we have θ, and per capita GDP growth will be "explained" as much as it will ever be by a purely economic theory. Moreover, the parameterization of the model invites us to think about the *sources* of growth in an organized way, in terms of its three constituent parameters: each agent's own efforts and ability to process ideas, α, the average quality of his environment, λ, and the diversity of his environment, θ.

To this point, I have not introduced any individual allocative decisions into this economy. One could, for example, assume that a person's learning rate α varies with the way he allocates his time or with his choice of occupation, and of course such modifications would be easy to carry out. In Section 5, I illustrate by considering a

[4] See Choi (2008) for a conceptually similar approach to the study of TFP growth.

schooling choice. But it should already be clear that the social returns from actions to affect α will exceed their private return. An individual will take into account the effect of such action on his own productivity but not its external effect on the intellectual environment of others, as summarized in the parameter λ.[5]

2. A Cohort Structure

Individual contributions to the growth of knowledge vary widely, depending on differences in people's ability and in the way they allocate their time. The reason we need a theory of knowledge creation is to gain the ability to analyze the effects of such individual differences. The approach taken in the last section will be useful, then, only insofar as it can be adapted to situations in which agents differ in more fundamental respects than in their random productivity draws.

In particular, a theory of technology focused on individually held knowledge must face up to mortality and the loss of knowledge it entails. Some knowledge can be "embodied" in books, blueprints, machines, and other kinds of physical capital, and we know how to introduce capital into a growth model, but we also know that doing so do not by itself provide an engine of sustained growth. To deal with mortality we need to introduce a cohort structure with overlapping generations.[6]

To keep things simple I assume a stationary age distribution, with density $\pi(s)$ and

[5] This externality is related to the external effects in Romer (1986) and Lucas (1988), but in both these papers the effect is on the level of others' productivity, not on their learning rate. A closer precedent is the learning in Arrow (1962), though it is there linked to the accumulation of physical capital.

[6] We know that an overlapping generations model without a family structure and a bequest motive is inadequate to analyze the holdings of physical capital. These are good reasons to prefer a model of households as infinitely-lived "dynasties." But even within a perfectly functioning family, knowledge is lost with deaths and "replacement investment" in new members is needed. See Stokey (1991).

cdf $\Pi(s)$:

$$\Pi(s) = \int_0^s \pi(v)dv = \text{ fraction of people with age } \leq s.$$

The birth rate $\pi(0)$ is constant. There is no immigration, so $\pi(s)$ is non-increasing. As before, everyone's individual productivity is $x^{-\theta}$, where $0 < \theta < 1$ and x is an exponentially distributed random variable. For a person of age s at date t, we view x as a draw from an exponential distribution with parameter $\mu(t, s)$. We assume as well that the technology frontier for the entire society is Frechet distributed, with parameters $\lambda(t)$ and θ. It remains to relate the parameters $\lambda(t)$ and $\mu(t, s)$.

We first define the economy's knowledge $\lambda(t)$ in terms of the knowledge levels $x^{-\theta}$ of individuals in the economy at date t. For the cohort of age s, productivities are Frechet distributed random variables with parameters $(\mu(t, s), \theta)$. Consider an individual meeting others over $(t, t+h)$. If everyone is met with equal probability (a convenient and also easily varied assumption), then the fraction $\pi(s)\varepsilon$ of the draws an individual gets are from people of ages in $(s, s+\varepsilon)$. As discussed in the last section, I treat this as as single draw from an exponential with parameter $\alpha h\pi(s)\varepsilon\mu(t, s)$. The parameter of the distribution of the minimum of all these draws is then approximately

$$\lambda(t) = \int_0^\infty \pi(s)\mu(t, s)ds. \tag{2.1}$$

Next we define the cohort parameters $\mu(t, s)$ in terms of current and past knowledge levels of $\lambda(t)$. The initial state of the economy is specified by the cohort parameters $\mu(0, s)$, $s \geq 0$, that describe the population at $t = 0$. (These parameters determine $\lambda(0)$ as in (2.1), but in general the function $\mu(0, s)$ conveys more relevant information than can be summarized in a single number.) Those alive at t will continue to learn from the rest of the population until they die (leave the labor force), as described by:

$$\begin{aligned} \mu(t, s) &= \mu(0, s-t) + \alpha \int_0^t \lambda(v)dv \\ &= \mu(0, s-t) + \alpha \int_0^t \lambda(t-v)dv \quad \text{if } \ s > t \end{aligned} \tag{2.2}$$

Those born after date 0 will begin learning at birth (entry to the labor force) in the same way:

$$\mu(t,s) = \alpha \int_{t-s}^{t} \lambda(v)dv = \alpha \int_{0}^{s} \lambda(t-v)dv \quad \text{if} \quad s \le t. \tag{2.3}$$

Substituting from (2.2) and (2.3) into (2.1) and simplifying we have

$$\begin{aligned}
\lambda(t) &= \int_0^t \pi(s)\mu(t,s)ds + \int_t^\infty \pi(s)\mu(t,s)ds \\
&= \alpha \int_0^t \pi(s) \int_0^s \lambda(t-v)dvds + \alpha \int_t^\infty \pi(s) \int_0^t \lambda(t-v)dvds \\
&\quad + \int_t^\infty \pi(s)\mu(0,s-t)ds.
\end{aligned}$$

The first two terms can be consolidated to give

$$\lambda(t) = \alpha \int_0^\infty \pi(s) \int_0^{\min(s,t)} \lambda(t-v)dvds + Q(t), \tag{2.4}$$

where

$$Q(t) = \int_t^\infty \pi(s)\mu(0,s-t)ds = \int_0^\infty \pi(s+t)\mu(0,s)ds.$$

The integral equation (2.4) contains the complete description of equilibrium behavior. It is an immediate consequence of Picard's theorem on the existence of solutions to differential equations that for any continuous $\mu(0,s)$, $s \ge 0$, there is exactly one continuous function $\lambda(t)$, $t \ge 0$, satisfying (2.4).

In the rest of this section we characterize these solutions as fully as possible, taking up in turn the existence and character of balanced growth paths, the complete dynamics of all solution paths when the age distribution is exponential, and finally the complete dynamics with general age distributions.

Consider first candidate solutions to (2.4) of the form $\lambda(t) = Be^{\gamma t}$. I will refer to such a solution as a *balanced growth path* (BGP). Any BGP must have (B,γ) that satisfy

$$Be^{\gamma t} = \alpha \int_0^\infty \pi(s) \int_0^{\min(s,t)} Be^{\gamma(t-v)}dvds + Q(t). \tag{2.5}$$

If (2.5) holds for all t, then letting $t \to \infty$ we see that γ must be a solution to

$$1 = \alpha \int_0^\infty \pi(s) \int_0^s e^{-\gamma v} dv ds$$

or to

$$\gamma = \alpha \int_0^\infty \pi(s) \left(1 - e^{-\gamma s}\right) ds. \tag{2.6}$$

The right side of (2.6) is a strictly increasing, strictly concave function of γ. It takes the value 0 at $\gamma = 0$ and (2.6) exactly one non-zero solution as well. (See Figure 4). The derivative of the right side of (2.6) with respect to γ is

$$\alpha \int_0^\infty \pi(s) s e^{-\gamma s} ds$$

which evaluated at 0 is

$$\alpha \int_0^\infty s\pi(s) ds.$$

Hence the non-zero fixed point is positive if and only if

$$\alpha \int_0^\infty s\pi(s) ds > 1. \tag{2.7}$$

This inequality involves the product of the rate α at which ideas are processed and the mean working life $\int_0^\infty s\pi(s)ds$. Thus a BGP with positive growth can occur only if (2.7) holds and then only at the unique positive γ value that satisfies (2.6). In this theory, a productive idea needs to be in use by a living person to be acquired by someone else, so what one person learns is available to others only as long as he remains alive. If lives are too short or too dull, sustained growth at a positive rate is impossible.

We focus first on the case that is consistent with growth: $\gamma > 0$. For this case, it is convenient to use the change of variable $x(t) = e^{-\gamma t} \lambda(t)$, and to restate (2.4) as

$$x(t) = \alpha \int_0^\infty \pi(s) \int_0^{\min(s,t)} e^{-\gamma v} x(t - v) dv ds + e^{-\gamma t} Q(t). \tag{2.8}$$

Let $\mathbf{S}$ be the space of continuous, bounded functions $x : \mathbf{R}_+ \to \mathbf{R}$, with the norm

$$\|x\| = \sup_{t\geq 0} |x(t)| .$$

For given $\mu(0,\cdot)$, and hence Q, define the operator T on $\mathbf{S}_+$ by

$$(Tx)(t) = \alpha \int_0^\infty \pi(s) \int_0^{\min(s,t)} e^{-\gamma v} x(t-v) dv ds + e^{-\gamma t} Q(t),$$

so that (2.8) is equivalent to $Tx = x$.

It is evident that Tx is continuous if x is. If x is bounded in the sup norm so is Tx, since

$$\begin{aligned} |(Tx)(t)| &\leq \alpha \|x\| \int_0^\infty \pi(s) \int_0^{\min(s,t)} e^{-\gamma v} dv ds + e^{-\gamma t} Q(t) \\ &= \frac{\alpha}{\gamma} \|x\| \int_0^\infty \pi(s) \left(1 - e^{-\gamma \min(s,t)}\right) ds + e^{-\gamma t} Q(t) \\ &\leq \frac{\alpha}{\gamma} \|x\| + e^{-\gamma t} Q(t), \end{aligned}$$

which is bounded because Q is. Thus $T : \mathbf{S}_+ \to \mathbf{S}_+$. The same argument establishes that T is a contraction with modulus α/γ. For any $x, y \in \mathbf{S}_+$

$$\begin{aligned} \|Tx - Ty\| &\leq \alpha \int_0^\infty \pi(s) \int_0^\infty e^{-\gamma v} \|x - y\| dv ds \\ &\leq \frac{\alpha}{\gamma} \|x - y\| . \end{aligned}$$

We already know that (2.8) has a unique solution for any continuous, bounded $\mu(0,\cdot)$, but in view of the contraction property it also follows that for any continuous, bounded $\mu(0,\cdot)$, the unique solution x of (2.8) satisfies

$$\lim_{n\to\infty} T^n x_0 = x \tag{2.9}$$

for all $x_0 \in \mathbf{S}_+$. Thus we can calculate a fixed point x or establish its properties by using the successive approximations from any starting point x_0.

We will use this fact in a moment to establish a stability theorem for this case $\gamma > 0$. But it is instructive to begin the discussion of stability with the special case of exponential lives: $\pi(s) = \delta e^{-\delta s}$. In this case, (2.4) becomes

$$\lambda(t) = \alpha \int_0^\infty \delta e^{-\delta s} \int_0^{\min(s,t)} \lambda(t-v) dv ds + Q(t),$$

where

$$Q(t) = e^{-\delta t} Q(0) = e^{-\delta t} \lambda(0).$$

Equation (2.6) becomes

$$\gamma = \alpha \frac{\gamma}{\gamma + \delta}$$

which has the solutions $\gamma = 0$ and $\gamma = \alpha - \delta$. (Note that $\alpha - \delta > 0$ is the inequality $\alpha \int_0^\infty s\pi(s)ds > 1$ in the case of exponential π.) One can verify directly that

$$\lambda(t) = \lambda(0) e^{(\alpha-\delta)t}$$

In this case with $\gamma > 0$, then, *all* solutions are balanced growth paths, with $\lambda(t)$ growing at the constant rate γ from any initial level $\lambda(0)$. Then the normalized paths $x(t) = e^{-\gamma t}\lambda(t)$ are all constant at their initial levels $x(0)$.

In the general case, the initial conditions $\mu(s, 0)$ cannot be summarized in a single number $\lambda(0)$ or $x(0)$, but we can prove that growth rates converge to constants and levels will always depend on initial conditions. Specifically, we have

Theorem: If $\gamma > 0$ and if x is any solution to (2.8),

$$\lim_{t \to \infty} x(t) = B \quad \text{for some} \quad B > 0. \tag{2.10}$$

Proof: We first show that if any $x \in \mathbf{S}_+$ has the property (2.10), so does Tx. Since

$Q(t)$ is bounded,

$$\begin{aligned}
\lim_{t\to\infty}(Tx)(t) &= \alpha\int_0^\infty \pi(s)\lim_{t\to\infty}\int_0^{\min(s,t)} e^{-\gamma v}x(t-v)dvds \\
&= \alpha\int_0^\infty \pi(s)\int_0^s \lim_{t\to\infty} e^{-\gamma v}x(t-v)dvds \\
&= \alpha B\int_0^\infty \pi(s)\int_0^s e^{-\gamma v}dvds \\
&= \frac{\alpha}{\gamma}B\int_0^\infty \pi(s)\left(1-e^{-\gamma s}\right)ds \\
&= B.\ \square
\end{aligned}$$

Then if we let $x_0(t) = B$ for all t, (2.9) implies that solutions to (2.8) satisfy (2.10). $\square$

(Note that in general it will not be the case that $B = x(0)$, and that this theorem does not provide an algorithm for mapping initial conditions $\mu(s,0)$ into limiting values B of the corresponding solution.)

This completes the discussion of the case $\gamma > 0$. When $\gamma = 0$, a BGP $\lambda(t)$ will be constant at some level B. say. Working back, the values for $\mu(0,s)$ consistent with this are

$$\mu(0,s) = \alpha Bs,$$

in which case

$$Q(t) = \int_0^\infty \pi(s+t)\alpha Bsds.$$

Substituting this expression into (2.4) yields

$$B = \alpha\int_0^t \pi(s)\int_{t-s}^t Bdvds + \alpha\left[1-\Pi(t)\right]\int_0^t Bdv + \int_0^\infty \pi(s+t)\alpha Bsds$$

which reduces to

$$1 = \alpha\int_0^\infty \pi(s)sds.$$

Thus $\lambda(t) = B > 0$ is a solution to (2.4) if and only if $\alpha \int_0^\infty \pi(s) s ds = 1$ and if the initial conditions are $\mu(0, s) = \alpha B s$. The same method, applied to the case where $\alpha \int_0^\infty \pi(s) s ds < 1$ and thus $\gamma < 0$ solves (2.6), verifies that $\lambda(t) = 0$ is the only BGP. We have thus catalogued the possible balanced growth paths, the initial conditions that will induce them, and their stability properties.

3. Issues of Identification and Calibration

In the models of the last two sections two parameters, the idea-processing rate α and the Frechet variance parameter θ, combine to determine the rate γ of technological change. Unless we can find other sources of evidence on α and θ this hardly seems an advance over simply assuming a given γ value, as in the original Solow model. These models are a little too abstract to support a full discussion of this issue, but I want to explore it in a preliminary way in this section. The cohort model of Section 2 predicts the entire age-earnings (or experience-earnings) profiles for everyone in the economy and the way these profiles evolve over time. There is a vast amount of evidence on individual earnings for the U.S. economy. Why not use some of it to test the model and to estimate its parameters?

To do this, I will use earnings from the 1990 U.S. census, viewing these data as though they were observations on a balanced growth path of the theoretical, labor-only economy. These earnings data are broken down by sex, race and schooling levels attained as well as by age, so some decisions need to be made as to how model and data are to be matched. In this section, I treat white, male heads of households who have graduated from high school and not gone on to college as constituting the entire population and use their yearly earnings as the measure of productivity. This will be adequate to illustrate the main identification and calibration issues. In the next section, I consider the effects of schooling differences.

From the data, I calculated mean log earnings and the variance of the log of earnings for each age group (among the set of male high school gradusates). Figure 5 displays the cell log means, based on a sample extracted from the 1990 U.S. Census.[7] It also shows means for two other schooling levels (out of 17 provided), but I will not make any use of these in this section. The census provides earnings for workers up to age 90, but these fall off rapidly after age 60. I deleted these older workers from the figure since partial retirement is important for yearly earnings of old workers and the theory does not accommodate this.

Along the theoretical balanced growth path, the earnings of a person of age s are random variables $x^{-\theta}$, where x has an exponential distribution with the parameter

$$\begin{aligned} \mu(s,t) &= \alpha \int_{t-s}^{t} \lambda(v)dv \\ &= \alpha \int_{t-s}^{t} Be^{\gamma v}dv \\ &= \alpha \frac{B}{\gamma} e^{\gamma t}\left(1-e^{-\gamma s}\right), \end{aligned}$$

where γ is the positive solution to (2.6). Thus the theoretical counterparts to the within-cell means are

$$\begin{aligned} w(s,t) &= \int_0^\infty \log(x^{-\theta})\mu(s,t)e^{-\mu(s,t)x}dx \\ &= \theta \log(\mu(s,t)) - \theta \int_0^\infty \log(z)e^{-z}dz \end{aligned}$$

The integral on the right on the second line is the negative of Euler's constant: .577. Combining,

$$w(s,t) = K + \theta\gamma t + \theta \log(1-e^{-\gamma s}). \tag{3.1}$$

where K is a constant that depends on GDP units. The corresponding variance depends only on the parameter θ :

$$\mathrm{Var}(\log(x^{-\theta})) = \theta^2 \frac{\pi^2}{6}. \tag{3.2}$$

[7]The data can be found at http://usa.ipums.org/usa. This is the source used in Heckman *et al.* (2006).

Figure 6 is a plot of the standard deviations of log earnings against age for the same three schooling levels used in Figure 5. The theory is based on a constant θ but one can see a clear upward trend in the earnings variability of both high school and college graduates. I will comment on this in a moment. For now, I will press on and use the high school graduates to form an estimate of 0.6 for the log standard deviation, implying—from (3.2)—the estimate $\theta = (\sqrt{6}/\pi)(0.6) \simeq 0.5$. (If the college graduates were used instead, the estimate would be about 0.58.)

The growth rate of GDP per capita in the theoretical economy is $\theta\gamma$. Using the U.S. growth rate of about 2 percent per year, this implies the estimate $(.02)/\theta = .04$ for the parameter γ. Finally, to estimate the learning rate α we need to use equation (2.6). I specialized the age distribution $\pi(s)$ to a rectangular distribution with a working life of T years to obtain

$$\gamma = \alpha \frac{1}{T} \int_0^T \left(1 - e^{-\gamma s}\right) ds. \tag{3.3}$$

Then $\gamma = .04$ and $T = 44$ together implied $\alpha = 0.075$.

So far I have used only the standard deviations plotted in Figure 6 and an estimate of the earnings growth for the economy as a whole to estimate the parameters α, θ and γ. With these parameters determined, the only free parameter in the theoretical age-earnings profile (3.1) is the intercept $K + \theta\gamma t$: The slope and curvature are completely determined. Figure 7 plots the predicted curve against the empirical profiles for three schooling levels, with intercepts of the three curves separately determined. The fits are a striking confirmation of the theory.[8]

One can see on Figure 7 that the fits deteriorate for workers over 55. The on-the-job

[8] Park (1997) developes a model in which the young learn from the old on the job. In his analysis, these effects are internalized within firms: young workers in effect pay tuition to their older colleagues. The theory has implications for age-earnings profiles, and for the equilibrium age composition of firm workforces as well.

learning models of Rosen (1976) and Heckman (1976) fit this earnings decline phase for active workers well, with theories where workers need to devote effort (in the form of foregone production) to maintain their learning rate. As retirement nears, the incentive to do this diminishes. In my model, learning is a by-product of production, not an alternative use of time, and this effect is absent. It would be a useful but challenging task to integrate the two models.

The fact that the variance of log earnings increases with age, rather than remaining constant, does not much effect the accuracy of the calibration, but it is a symptom of a difficulty with the theory that should not be ignored. The observation itself is very solid. It shows up in earlier censuses (see Heckman *et al.* (2006)). It is documented by Deaton and Paxson (1994) using household survey data from the U.S., the U.K., and Taiwan. It should be clear from my mathematical development of this model, not to mention all of the applications in trade theory stemming from Eaton and Kortum (2002), that if most if not all of the convenience of the model is lost if θ is not constant. On the other hand, as the cohort analysis in Section 2 makes clear, the theory can easily accommodate mixtures of populations with different λ or μ parameters. Perhaps the increasing trend we see in the empirical variances can be viewed as arising from mixtures, changing over time, of different populations with constant, common θ values. At present, this is obviously pure speculation.

The calibration strategy illustrated in this section treats stochastic productivity shocks, drawn by all workers from a common distribution, as the only sources of earnings variability. This is obviously not the case for the economy as a whole, as I have already acknowledged implicitly by treating the subset of male high school

In Rosen (1972), knowledge accumulated on the job is firm-specific, owned by firms. Rosen argues that some, though not all, learning can be internalized in this way.

In Prescott and Boyd (1987) external effects among workers are also fully internalized. In their model these effects generate sustained growth.

graduates as though it were the entire labor force. But even within this group, or within any group defined by observable characteristics, unobserved differences in ability will be an added source of earnings variation. Estimation of the Frechet parameter θ, interpreted as I am doing in this paper, requires taking a position on the relative importance of different sources of earnings variability, for different subsets of the workforce.

4. A Schooling Decision

To this point I have focused on modeling a technology of producing and learning, taking individual behavior as given. In this section I introduce an element of decision making by assuming that the idea-processing rate α depends on years of schooling: $\alpha = \alpha(S)$, an idea proposed long ago by Nelson and Phelps (1966) and recently applied by Benhabib and Spiegel (2002), and that years of schooling is a matter of individual choice. Examining the schooling decision will give a clearer idea of the nature of the gap between private and social returns that I view as a central implication of the view of technology I am applying here. It will also expose some serious problems in interpreting evidence on earnings, schooling, and economic growth.

I begin by setting out the notation for a symmetric equilibrium in an economy of identical (except for age) people, where everyone begins with S years of school. This is just a slight variation on the economy of Sections 2 and 3. Then I consider an individual's choice of his own schooling S taking everyone else's choice as given. The first-order condition for this problem will help to determine the equilibrium schooling level. Then we can compare this equilibrium schooling level to the efficient level.

Let the productivity at date t of a person of age s who has S years of schooling be $x^{-\theta}$, where x is drawn from an exponential distribution with parameter $\mu(t, s, S)$. Given S, the basic equations are

$$\lambda(t) = \int_S^\infty \pi(s)\mu(t,s,S)ds \tag{4.1}$$

where

$$\mu(t,s,S) = \mu(0, s-t, S) + \alpha \int_0^t \lambda(t-v)dv \quad \text{if} \quad s - S > t. \tag{4.2}$$

and

$$\mu(t,s,S) = \alpha \int_0^{s-S} \lambda(t-v)dv \quad \text{if} \quad s - S \leq t. \tag{4.3}$$

Subtituting from (4.2) and (4.3) into (4.1) we have

$$\lambda(t) = \alpha \int_S^\infty \pi(s) \int_0^{\min(s-S,t)} \lambda(t-v)dvds + Q(t) \tag{4.4}$$

where

$$Q(t) = \int_{t+S}^\infty \pi(s)\mu(0, s-t, S)ds = \int_S^\infty \pi(s+t)\mu(0,s,S)ds.$$

The mathematical analysis of this equation proceeds exactly as the analysis of (2.4): only the parameter S is new. In particular, on a balanced growth path $Be^{\gamma t}$ with constant S, the parameter γ must satisfy

$$\gamma = \alpha(S) \int_S^\infty \pi(s) \left(1 - e^{-\gamma(s-S)}\right) ds. \tag{4.5}$$

This is one condition that a balanced growth equilibrium pair (γ, S) must satisfy.

Along such a path, a worker who has graduated after $t = 0$ has a productivity draw from a distribution with the parameter

$$\mu(t,s,S) = \alpha(S)\frac{B}{\gamma}e^{\gamma t}\left(1 - e^{-\gamma(s-S)}\right).$$

The mean log earnings of his cohort at t is

$$w(s,t) = K + \theta\gamma t + \theta \log(\alpha(S)) + \theta \log(1 - e^{-\gamma(s-S)}). \tag{4.6}$$

Equations (4.4)-(4.6) are just restatements of (2.4), (2.6), and (3.1) with a given level S of schooling imposed. Note that the experience-earnings profiles (4.6) are parallel curves—again, see Figure 6. The schooling term $\alpha(S)$ affects only the intercept. This is a point in favor of the model: See Heckman *et al.* (2006).

Suppose, then, that the balanced path corresponding to a particular S is an equilibrium, in the sense that no individual worker will choose a schooling level different from S. To describe the S value that has this property we need to describe the individual opportunity sets. For discussion purposes, assume that earnings risks can be perfectly insured and that anyone can borrow and lend any amount at a market rate r. A person born at $t = 0$ will then choose a schooling level S so as to maximize the present value of expected earnings

$$V(S) = \int_S^\infty e^{-rs} E\{x^{-\theta}(s, S)\} ds.$$

Given the individual's choice S, the random variable $x(s, S)$ is a draw from an exponential distribution with parameter $\mu(s, s, S)$, conditional on his still being alive at s. The probability of the latter event is $1 - \Pi(s)$. Thus

$$V(S) = \int_S^\infty \int_0^\infty e^{-rs} (1 - \Pi(s)) \, x^{-\theta} \mu(s, s, S) e^{-\mu(s,s,S)x} dx ds.$$

Evaluating the inner integral,

$$V(S) = A \int_S^\infty e^{-rs} (1 - \Pi(s)) \, \mu(s, s, S)^\theta ds \tag{4.7}$$

where the parameter A depends on θ but not on z.

The value of $\mu(s, s, S)$ is

$$\mu(s, s, S) = \alpha(S) \int_S^s \lambda(v) dv \ ,$$

where the variables $\lambda(v)$ are determined by the S-choices of others. In particular, along the path $\lambda(t) = Be^{\gamma t}$,

$$\mu(s, s, S) = \alpha(S) \frac{B}{\gamma} e^{\gamma s} \left(1 - e^{-\gamma(s-S)}\right).$$

Inserting this value into (4.7) gives

$$V(S) = A \left(\frac{B}{\gamma}\right)^\theta \alpha(S)^\theta \int_S^\infty e^{-(r-\theta\gamma)s} (1 - \Pi(s)) \left(1 - e^{-\gamma(s-S)}\right)^\theta ds. \tag{4.8}$$

An individual agent in this economy takes γ as a given parameter and solves

$$\max_S \alpha(S)^\theta \int_S^\infty e^{-(r-\theta\gamma)s} (1 - \Pi(s)) \left(1 - e^{-\gamma(s-S)}\right)^\theta ds.$$

A planner, selecting among balanced growth equilibria and taking r as given, would

$$\max_{S,\gamma} \alpha(S)^\theta \int_S^\infty e^{-(r-\theta\gamma)s} (1 - \Pi(s)) \left(1 - e^{-\gamma(s-S)}\right)^\theta ds$$

subject to

$$\gamma = \alpha(S) \int_S^\infty \pi(s) \left(1 - e^{-\gamma(s-S)}\right) ds.$$

That is, an efficiency-seeking planner would take into account the effects of each person's schooling on the productivity of others and well as on his own productivity.[9] This difference between the private and social returns to schooling will show up in any equilibrium in which individual agents make choices that affect their learning rates.

These issues certainly merit a quantitative investigation, but there are two reasons why the model I have sketched here is not equal to the task. In the first place, a quantitative analysis requires an estimate of the function $\alpha(S)$ relating schooling to the idea-processing rate, and hence to the growth rate. It would seem natural to use the earnings differences between, say, high school and college graduates for this purpose. For example, the college and high-school intercepts used in Figure 7 differ by 10.9 - 10.4 = 0.5: a factor of 1.65. We could think an equilibrium of identical agents, all indifferent at their high school graduation between going to work and proceeding to a four year college degree. But this thought experiment carried out with the above model implies an internal rate of return on the order of twice the U.S. after-tax return on capital: an untenable conclusion. It seems certain that part of

[9] In fact, an efficiency-seeking planner will have a discounted expected utility objective rather than take an interest rate as given, and will take transition dynamics into account as well. This is a complicated but tractable optimal growth problem—deterministic since the idiosyncratic risks can be pooled—but it is clear without working through the details that its solution will not be replicated by an equilibrium in which individuals maximize (4.8) taking γ as given.

the college premium in the census data reflects difference in abilities between high school graduates that finish college and those that do not continue past high school, or differences in school quality, or capital market imperfections, or a mix of all three. See Heckman et al. (2008) for a detailed discussion of these and other difficulties in the economic interpretation of the census evidence.

A second difficulty with the balanced growth paths in the model of this section is the joint prediction of a constant rate of productivity growth, constant schooling levels, and a positive growth effect of schooling levels—$\alpha'(S) > 0$. It is not an easy task to reconcile these features with the enormous increase in schooling over the past century. Goldin and Katz (1999) refer to "the race between schooling and technology." Perhaps it is the case that the idea processing captured by the parameter α requires more and more schooling as the technology level λ increases, and we should write $\alpha(S, \lambda)$ in place of $\alpha(S)$.[10]

5. Conclusions and Speculations

A theory of endogenous growth should offer more than a pretty story about parameters that, in practice, we simply treat as unalterable givens. It should help us to think operationally about the way changes in the way time and other resources are allocation can have differential effects on productivity growth. This is the virtue of Romer's (1990) and Grossman and Helpman's (1991) models of inventive activity in patent-protected products, set in environments of imperfect competition. But my own sense is that patents and "intellectual property" more generally play a very modest role in the overall growth of production-related knowledge. I have sought a formulation that emphasizes individual contributions of large numbers of people, in which the role of market power is minimized, models that offer natural connections between theoretical variables and observeables.

[10]There is a good discussion of this problem and possible resolutions in Jones (2002).

The view of endogenous technical change that I have examined in this paper is new to me, and my first concern was to develop the formal properties of the theory and to consider, in general way, what kind of evidence could be used to get information on its key parameters. For these purposes I used an abstract economy consisting entirely of a homogeneous class of problem-solving producers of single good and spent most of my time on matters of pure technology. The discussion of schooling in Section 4 was the only allocation problem I considered explicitly. But the linearity of the basic equations (1.2) or (2.4) makes it easy to think about multi-sectoral models, in which the class studied here is but one of several, each with its own α or λ or both. In such a setting, one could think about decisions to enter an occupation or industry, to migrate, to trade, to invest abroad, as decisions that affect the flows of ideas within and across economies.

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FIGURE 1: FRECHET DENSITIES

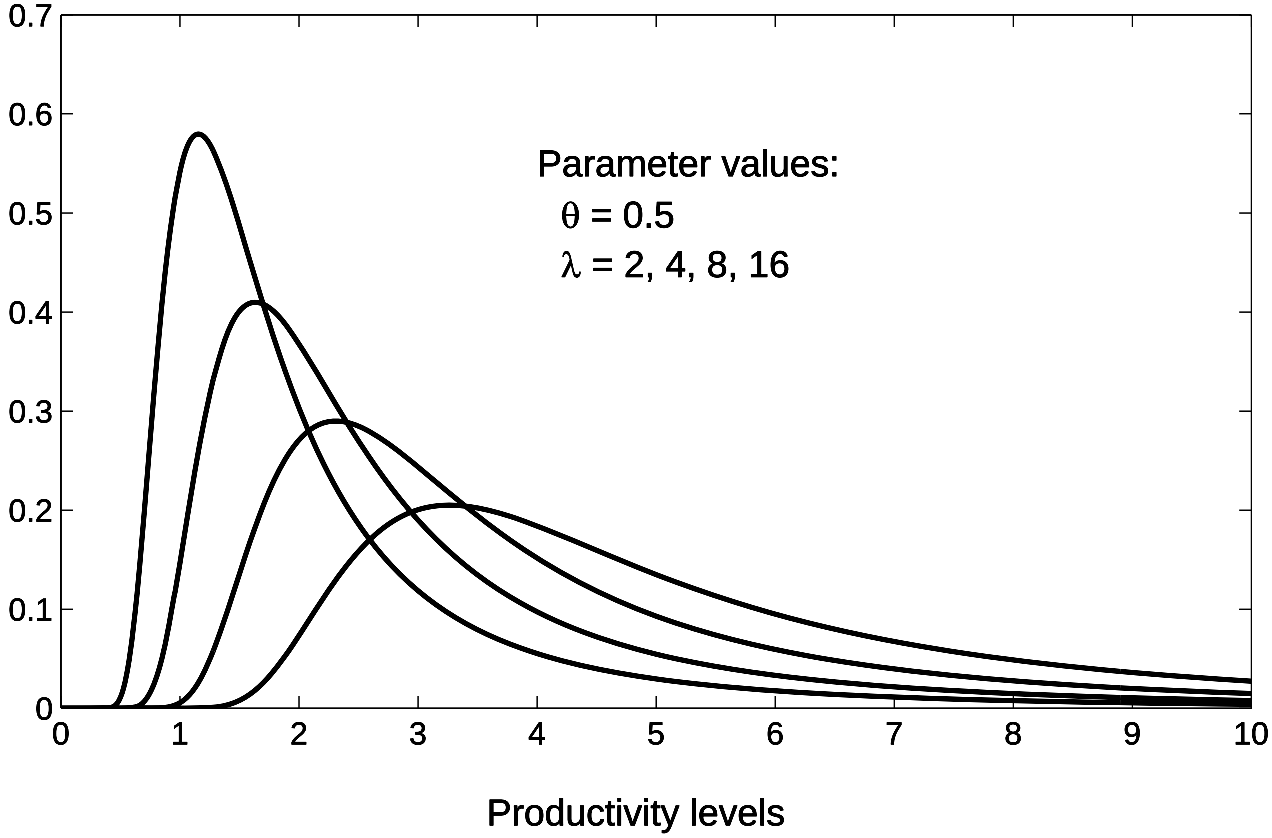

FIGURE 2: FRECHET DENSITIES, LOG SCALE

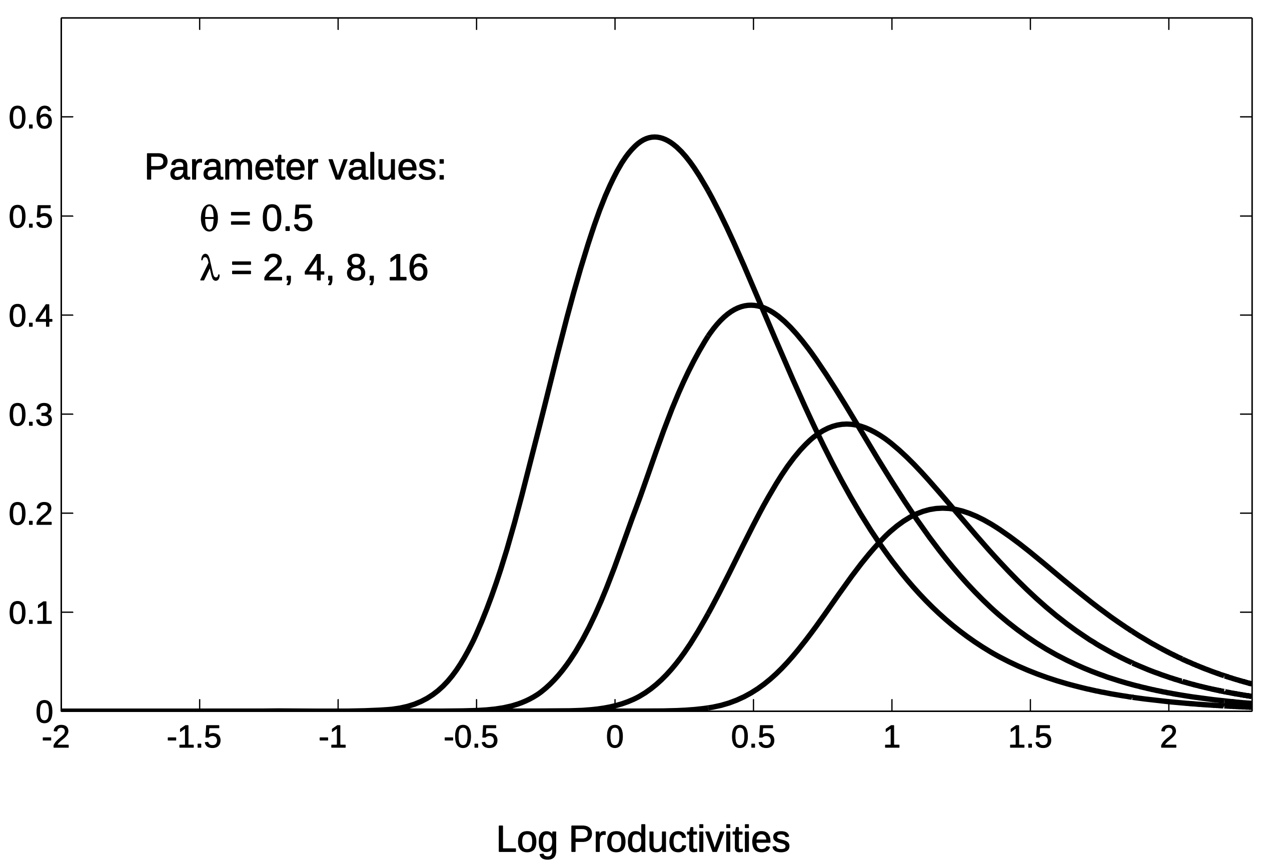

FIGURE 3: THREE STOCHASTIC CAREERS

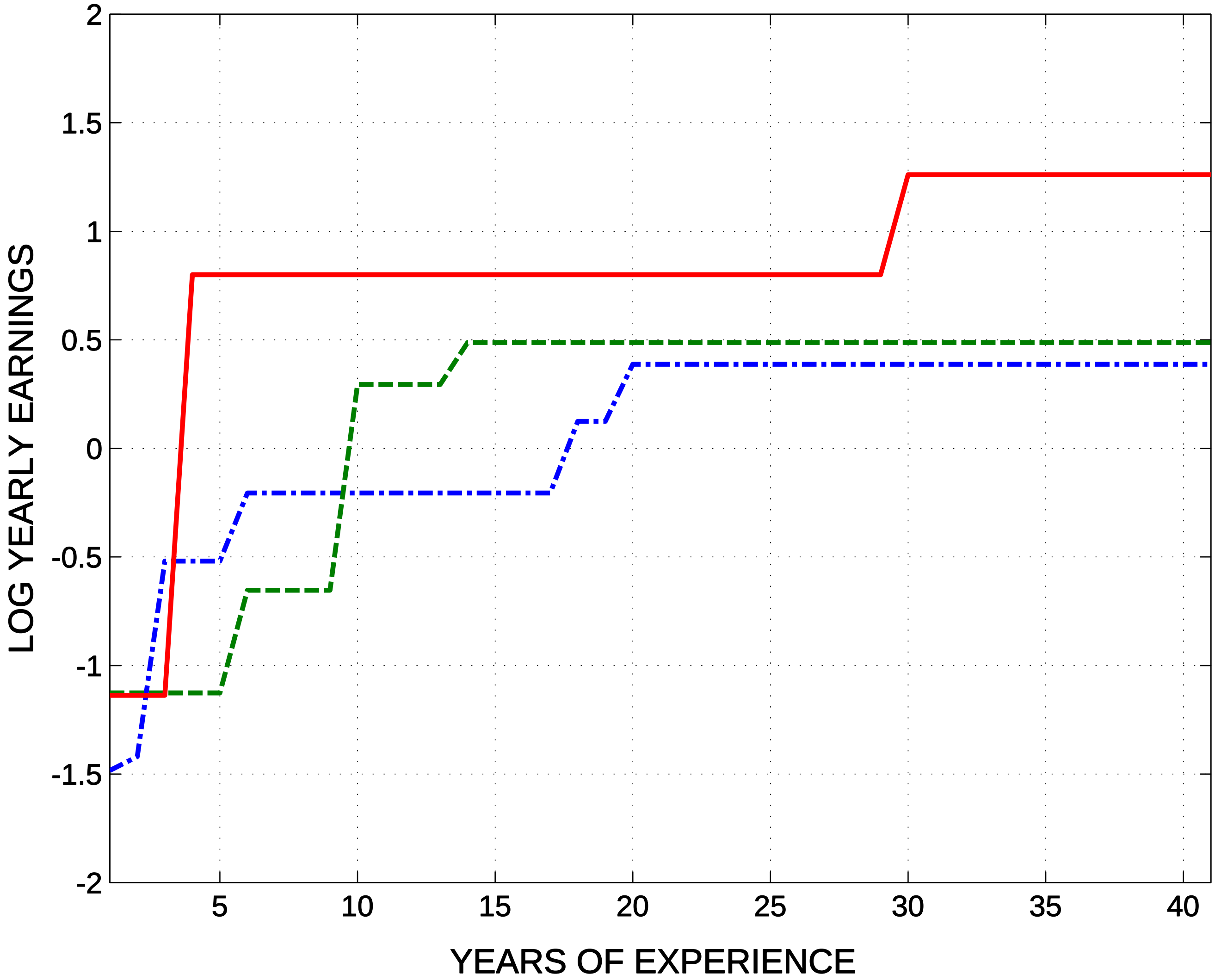

FIGURE 4: EQUILIBRIUM GAMMA POSSIBILITIES

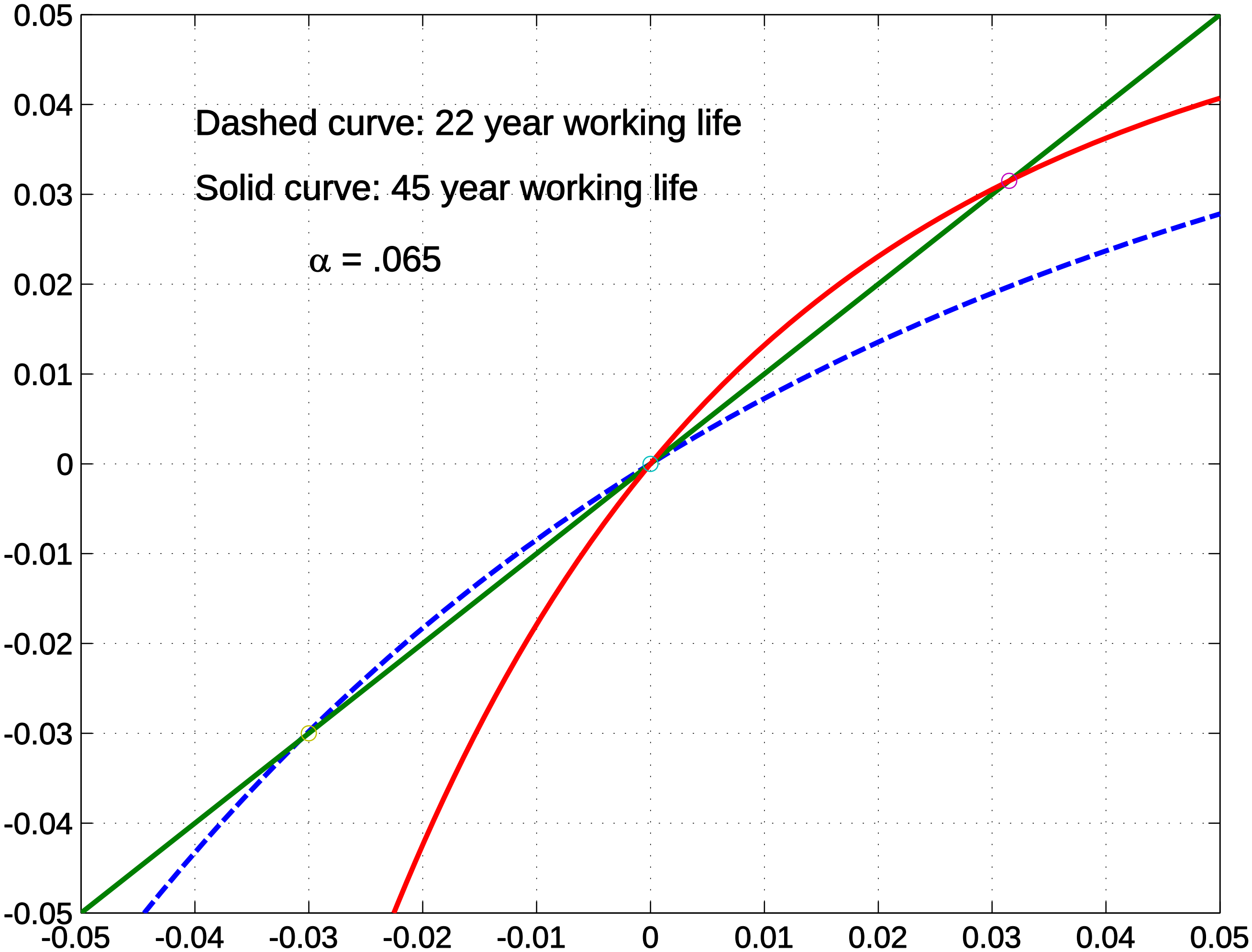

FIGURE 5: U.S. AGE-EARNINGS PROFILES, 1990 CENSUS

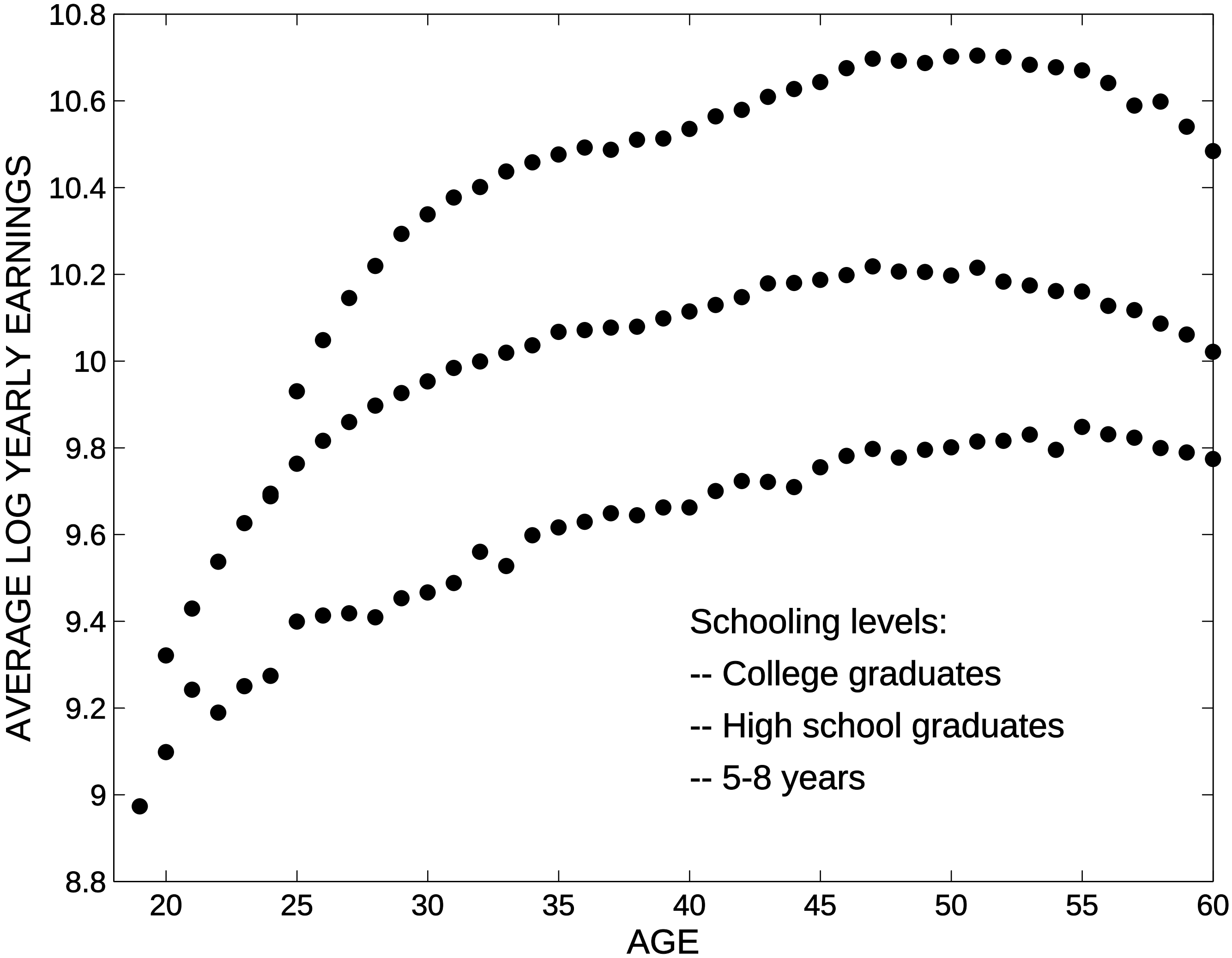

FIGURE 6: LOG EARNINGS VARIABILITY,1990 CENSUS

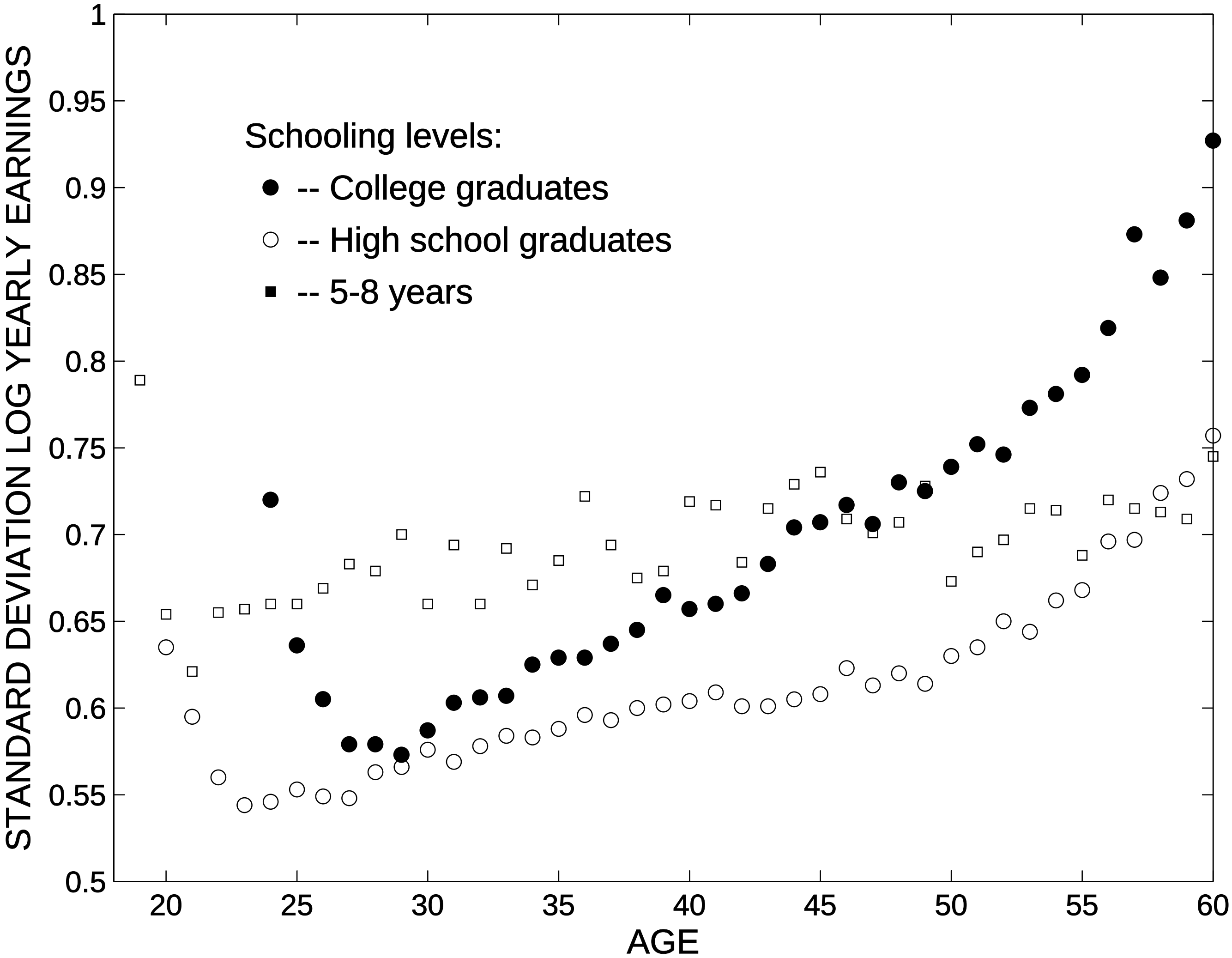

FIGURE 7: U.S. AGE-EARNINGS PROFILES, 1990 CENSUS