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ABSTRACT

We show how to construct optimal policy projections in Ramses, the Riksbank's open-economy medium-sized DSGE model for forecasting and policy analysis. Bayesian estimation of the parameters of the model indicates that they are relatively invariant to alternative policy assumptions and supports that the model may be regarded as structural in a stable low inflation environment. Past policy of the Riksbank until 2007:3 (the end of the sample used) is better explained as following a simple instrument rule than as optimal policy under commitment. We show and discuss the differences between policy projections for the estimated instrument rule and for optimal policy under commitment, under alternative definitions of the output gap, different initial values of the Lagrange multipliers representing policy in a timeless perspective, and different weights in the central-bank loss function.

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1. Introduction

We study optimal monetary policy in Ramses, the main model used at Sveriges Riksbank for forecasting and policy analysis. Ramses is a small open-economy dynamic stochastic general equilibrium (DSGE) model estimated with Bayesian techniques. It is described in Adolfson, Laséen, Lindé, and Villani (ALLV) [4] and [5]. Relative to other estimated small open-economy DSGE models (for instance, Justiniano and Preston [19]), the international spillover effects are relatively large due to the inclusion of a worldwide stochastic technology shock. This means that the open-economy aspects are of particular importance in our setting.

By optimal monetary policy we mean policy that minimizes an intertemporal loss function under commitment. The intertemporal loss function is a discounted sum of expected future period losses. We choose a quadratic period loss function that corresponds to flexible inflation targeting and is the weighted sum of three terms: the squared inflation gap between 4-quarter CPIX inflation and the inflation target (CPIX is a core CPI concept calculated by Statics Sweden that excludes the direct effects of interest rates and indirect taxes on the CPI), the squared output gap between output and potential output, and the squared quarterly change in the Riksbank's instrument rate, the repo rate. We interpret such a loss function as consistent with flexible inflation targeting and the Riksbank's monetary-policy objective.[1]

We assume that Ramses is a structural model whose parameters are invariant to changes in monetary policy. In order to shed some light on the plausibility of that assumption, we estimate the model parameters with Bayesian techniques under different assumptions about the conduct of monetary policy. First, as in ALLV [5], we estimate the model under the assumption that the Riksbank has followed a simple instrument rule during the inflation-targeting period (starting in 1993:1) up to 2007:3.[2] Second, we estimate the model under the assumption that the Riksbank has minimized an intertemporal loss function under commitment in a time-less perspective during the inflation-targeting period. In both cases, the steady-state inflation target is assumed to be 2 percent, and most of the parameters that are estimated only affects the dynamics in the model.

[1] In some simple models, quadratic approximations of the welfare of a representative household results in a similar period loss function (Woodford [38]). Such approximations of household welfare are very model-dependent and reflect the particular distortions assumed in any given model. Household welfare is in any case hardly an operational central-bank objective, although it may be of interest and relevant to examine how household welfare in particular models is affected by central-bank policy. Such an undertaking is beyond the scope of the present paper, though.

[2] The inflation-targeting period is assumed to start 1993:1 (the sample in this paper ends 2007:3). Prior to this period (1986:1–1992:4), the conduct of policy is assumed to be given by a simple instrument rule. The switch to the inflation-targeting regime in 1993:1 is assumed to be completely unanticipated but expected to be permanent once it has occurred.

The parameters of the model turn out to be invariant to the alternative formulations about the conduct of monetary policy, which supports our assumption that Ramses can for our purposes be treated as a structural model. The estimates of the instrument-rule and loss-function parameters provide benchmarks for the subsequent policy analysis. Another interesting finding in the empirical analysis is that the Riksbank policy up to 2007:3 is empirically better characterized in the model as by following a simple instrument rule than as optimizing a loss function under commitment.[3]

We provide a detailed analysis of how to do optimal policy projections in a linear-quadratic model with forward-looking variables, extending on previous analysis by Svensson [31] and Svensson and Tetlow [36]. A key issue for a flexible inflation-targeting central bank is which measure of output is should try to stabilize. Therefore, we study alternative definitions of potential output and the output gap, in order to make an assessment to what extent the formulation of the output gap in the loss function affects the optimal policy projections. More precisely, we report results from three alternative concepts of output gaps ($y_t - \bar{y}_t$), deviations of actual (log) output (y_t) from potential (log) output ($\bar{y}_t$) in the loss function. One concept of output gap is the *trend output gap* when potential output is the trend output level, which is growing stochastically due to the unit-root stochastic technology shock in the model. This trend output level will in practice closely resemble a trend output level computed with a Hodrick-Prescott (HP) filter. This measure of the output gap appears to be extensively used in practice and often reported and discussed in various inflation and monetary policy reports. A second concept is the *unconditional output gap*, where potential output is unconditional potential output, which is defined as the hypothetical output level that would exist if the economy would have had flexible prices and wages for a long time and would have been subject to a subset of the same shocks as the actual economy. Unconditional potential output therefore presumes different levels of the predetermined variables, including the capital stock, from those in the actual economy. A third concept is the *conditional output gap*, where potential output is conditional potential output, which is defined as the hypothetical output level that would arise if prices and wages suddenly become flexible in the current period and are expected to remain flexible in the future. Conditional potential output therefore depends on the existing current predetermined variables, including the current capital stock.

We also provide alternative methods to compute the initial vector of Lagrange multipliers of the equations for the forward-looking variables, the multipliers that are inputs in the optimal policy

[3] Although not reported in the tables, we have also estimated the model under the assumption of optimization under discretion. However, this formulation of policy behavior is strongly rejected in favor of commitment behavior, which is comforting to say the least. Due to stability problems with the discretionary solution, we do not report the estimation results in the main text.

under commitment in a timeless perspective.

We report and discuss optimal policy projections for Sweden using data up to and including 2007:3. We show projections for the estimated instrument rule and for optimal policy with different output gaps in the loss function, different configurations of the initial Lagrange multipliers associated with commitment in a timeless perspective, and different weights in the loss function. Such policy projections provide good and relevant illustrations of the policy tradeoffs policymakers face. In a companion paper, Adolfson, Laséen, Lindé, and Svensson [3], we study how the policy assumption affects the propagation of shocks and the variability of some key macro variables.

The paper is organized as follows: Section 2 presents Ramses in more general modeling terms. Section 3 discusses the data and priors used in the estimation and presents estimation results for the various specifications of the model and policy. Section 4 discusses methods to construct optimal policy projections and presents and discusses the alternative projections for a policymaker in 2006:4 and 2007:4. Finally, section 6 presents a summary and some conclusions. Appendices A-D contain a detailed specification of Ramses and many other technical details. In particular, numerical demands because of the size of Ramses requires some combination of the Klein [22] algorithm and the AIM algorithm of Anderson and Moore [11] and [12].

2. The model

Ramses is a linear model with forward-looking variables. It can be written in the following state-space form,

$$\begin{bmatrix} X_{t+1} \\ Hx_{t+1|t} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + Bi_t + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1}. \tag{2.1}$$

Here, X_t is an n_X-vector of *predetermined* variables in period t (where the period is a quarter); x_t is an n_x-vector of *forward-looking* variables; i_t is an n_i-vector of *instruments* (the forward-looking variables and the instruments are the *nonpredetermined* variables);[4] ε_t is an n_ε-vector of i.i.d. shocks with mean zero and covariance matrix I_{n_ε}; A, B, and C, and H are matrices of the appropriate dimension; and, for any variable y_t, $y_{t+\tau|t}$ denotes $\mathrm{E}_t y_{t+\tau}$, the rational expectation of $y_{t+\tau}$ conditional on information available in period t. The variables can be measured as differences from steady-state values, in which case their unconditional means are zero. Alternatively, one of the elements of X_t can be unity, so as to allow the variables to have nonzero means. The elements of the matrices A, B, C, and H are estimated with Bayesian methods and are considered fixed and

[4] A variable is predetermined if its one-period-ahead prediction error is an exogenous stochastic process (Klein [22]). For (2.1), the one-period-ahead prediction error of the predetermined variables is the stochastic vector $C\varepsilon_{t+1}$.

known for the policy simulations. Then the conditions for certainty equivalence are satisfied. Thus, we abstract from any consideration of model uncertainty in the formulation of optimal policy.[5] Appendix A provides details on Ramses, including the elements of the vectors X_t, x_t, i_t, and ε_t.

The upper block of (2.1) provides n_X equations that determine the n_X-vector X_{t+1} in period $t+1$ for given X_t, x_t, i_t and ε_{t+1},

$$X_{t+1} = A_{11}X_t + A_{12}x_t + B_1 i_t + C\varepsilon_{t+1}, \tag{2.2}$$

where A and B are partitioned conformably with X_t and x_t as

$$A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{2.3}$$

The lower block provides n_x equations that determine the n_x-vector x_t in period t for given $x_{t+1|t}$, X_t, and i_t,

$$x_t = A_{22}^{-1}(Hx_{t+1|t} - A_{21}X_t - B_2 i_t). \tag{2.4}$$

We hence assume that the $n_x \times n_x$ submatrix A_{22} is nonsingular.[6]

Let Y_t be an n_Y-vector of *target* variables, measured as the difference from an n_Y-vector Y^* of *target levels*. This is not restrictive, as long as we keep the target levels time-invariant. If we would like to examine the consequences of different target levels, we can instead interpret Y_t as the absolute level of the target levels and replace Y_t by $Y_t - Y^*$ everywhere below. We assume that the target variables can be written as a linear function of the predetermined, forward-looking, and instrument variables,

$$Y_t = D \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} \equiv [D_X \quad D_x \quad D_i] \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}, \tag{2.5}$$

where D is an $n_Y \times (n_X + n_x + n_i)$ matrix and partitioned conformably with X_t, x_t, and i_t. Let the intertemporal loss function in period t be

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau}, \tag{2.6}$$

where $0 < \delta < 1$ is a discount factor, L_t is the period loss given by

$$L_t \equiv Y_t' W Y_t, \tag{2.7}$$

[5] Onatski and Williams [24] provide a thorough discussion of model uncertainty. Svensson and Williams [34] and [35] show how to compute optimal policies for Markov Jump-Linear-Quadratic systems, which provide a quite flexible way to model most kinds of relevant model uncertainty for monetary policy.

[6] Without loss of generality, we assume that the shocks ε_t only enter in the upper block of (2.1), since any shocks in the lower block of (2.1) can be redefined as additional predetermined variables and introduced in the upper block.

and W is a symmetric positive semidefinite matrix.

We consider the following target variables: the model-consistent 4-quarter CPIX inflation rate, $p_t^c - p_{t-4}^c$, where p_t^c denotes the log of CPIX; a measure of the output gap, $y_t - \bar{y}_t$, where y_t denotes output and $\bar{y}_t$ denotes potential output; and the first difference of the instrument rate, $i_t - i_{t-1}$, where i_t denotes the Riksbank's instrument rate, the repo rate. The period loss is

$$L_t = (p_t^c - p_{t-4}^c - \pi^*)^2 + \lambda_y (y_t - \bar{y}_t)^2 + \lambda_{\Delta i}(i_t - i_{t-1})^2, \tag{2.8}$$

where π^* is the 2% inflation target and λ_y and $\lambda_{\Delta i}$ are nonnegative weights on output-gap stabilization and instrument-rate smoothing, respectively. That is, the vector of target variables is $Y_t \equiv (p_t^c - p_{t-4}^c - \pi^*, y_t - \bar{y}_t, i_t - i_{t-1})'$, and the matrix W is the matrix with diagonal $(1, \lambda_y, \lambda_{\Delta i})'$. We use 4-quarter inflation as a target variable rather than quarterly inflation since the Riksbank and other inflation-targeting central banks normally specify their inflation target as a 12-month rate.[7]

We report results from three alternative concepts of output gaps $(y_t - \bar{y}_t)$ in the loss function. One concept of output gap is the *trend output gap* when potential output $(\bar{y}_t)$ is the trend output level, which is growing stochastically due to the unit-root stochastic technology shock in the model. This trend output level will in practice closely resemble a trend output level computed with a an HP filter. A second concept is the *unconditional output gap*, where potential output is unconditional potential output, which is defined as the hypothetical output level that would exist if the economy would have had flexible prices and wages for a long time and would have been subject to the same shocks as the actual economy except mark-up shocks and shocks to taxes which are held constant at their steady-state levels. Unconditional potential output therefore presumes different levels of the predetermined variables, including the capital stock, from those in the actual economy. A third concept is the *conditional output gap*, where potential output is conditional potential output, which is defined as the hypothetical output level that would arise if prices and wages suddenly become flexible in the current period and are expected to remain flexible in the future. Conditional potential output therefore depends on the existing current predetermined variables, including the

[7] Furthermore, the inflation target variable is assumed to be the model-consistent CPI inflation excluding indirect taxes. The reason is that this measure more accurately captures the true import content in the consumption basket. In the model, total consumption is a CES function of imported and domestic goods, which yields the following (log-linearized) CPI inflation (excluding taxes)

$$p_t^c - p_{t-4}^c = (1 - \omega_c)\left(p^c/p^d\right)^{-(1-\eta_c)}(p_t^d - p_{t-4}^d) + \omega_c (p^c/p^{m,c})^{-(1-\eta_c)}(p_t^{m,c} - p_{t-4}^{m,c}). \tag{2.9}$$

where ω_c is the share of expenditures in the CPI spent on imported goods, p_t^d the (log) domestic price level and $p_t^{m,c}$ the (log) price of imported goods that the consumer has to pay. (p^c/p^d) and $(p^c/p^{m,c})$ are the steady-state relative prices which adjust the expenditure shares such that the true import content of CPI inflation becomes lower (decreasing from 35% to 28%). However, qualitatively this does not affect the subsequent analysis very much.

current capital stock. Appendix C discusses alternative concepts of output gaps and potential output in some detail.[8]

3. Estimation

3.1. Data, prior distributions, and calibrated parameters

We estimate the model using a Bayesian approach by placing a prior distribution on the structural parameters. We use quarterly data for the period 1980:1-2007:3. All data are from Statistics Sweden, except the repo rate which is from the Riksbank. The nominal wage is deflated by the GDP deflator. Foreign inflation, output, and interest rate are weighted together across Sweden's 20 largest trading partners in 1991 using weights from the IMF.[9]

As in ALLV [5], we include the following $n_Z = 15$ variables among the observable variables: GDP deflator inflation (π_t^d), real wage (W_t/P_t), consumption (C_t), investment (I_t), real exchange rate ($\tilde{x}_t$), short interest rate (R_t), hours worked (H_t), GDP (Y_t), exports ($\tilde{X}_t$), imports ($\tilde{M}_t$), CPIX inflation (π_t^{cpi}), investment-deflator inflation ($\pi_t^{\text{def},i}$), foreign output (Y_t^*), foreign inflation (π_t^*), and foreign interest rate (R_t^*). As in Altig, Christiano, Eichenbaum, and Lindé [8], the unit-root technology shock induces a common stochastic trend in the real variables of the model. To make these variables stationary we use first differences and derive the state-space representation for the following vector of observed variables,

$$Z_t \equiv \begin{array}{l} (\pi_t^d, \Delta \ln(W_t/P_t), \Delta \ln C_t, \Delta \ln I_t, \hat{\tilde{x}}_t, R_t, \hat{H}_t, ... \\ \Delta \ln Y_t, \Delta \ln \tilde{X}_t, \Delta \ln \tilde{M}_t, \pi_t^{\text{cpi}}, \pi_t^{\text{def},i}, \Delta \ln Y_t^*, \pi_t^*, R_t^*)'. \end{array} \tag{3.1}$$

The growth rates are computed as quarterly log-differences, while the inflation and interest-rate series are measured as annualized quarterly rates. It should be noted that the stationary variables $\hat{\tilde{x}}_t$ and $\hat{H}_t$ are measured as deviations around the mean and the HP-filtered trend, that is, $\hat{\tilde{x}}_t \equiv (\tilde{x}_t - \tilde{x})/\tilde{x}$ and $\hat{H}_t \equiv \left(H_t - H_t^{\text{HP}}\right)/H_t^{\text{HP}}$, respectively.[10] We choose to work with per-capita hours

[8] As reported in ALLV [5], the output gap resulting from trend output seems to more closely correspond to the measure of resource utilization that the Riksbank has been responding to historically rather than the unconditional output gap. Del Negro, Schorfheide, Smets, and Wouters [15] report similar results for the US.

[9] In the data, the ratios of import and export to output are increasing from about 0.25 to 0.40 and from 0.21 to 0.50, respectively, during the sample period. In the model, import and export are assumed to grow at the same rate as output. We have removed the excess trend in import and export in the data to make the export and import shares stationary. For all other variables we use the actual series (seasonally adjusted with the X12-method, except the variables in the GDP identity which were seasonally adjusted by Statistics Sweden).

[10] The reason why we use a smooth HP-filtered trend for hours per capita, as opposed to a constant mean, is that there is a large and very persistent reduction in hours worked per capita during the recession in the beginning of the 1990s. Neglecting taking this reduction into account implies that the forecasting performance for hours per capita in the model deteriorates significantly, as documented in the forecasting exercises in ALLV [5]. Rather than imposing a discrete shift in hours in a specific time period, we therefore decided to remove a smooth HP trend from the variable. This choice is not particularly important for the parameter estimates, but has some impact on the 2-sided filtered estimates of the unobserved states of the economy.

worked, rather than total hours worked, because this is the object that appears in most general-equilibrium business cycle models. Finally, all real variables are measured in per-capita units.

In comparison with prior literature, such as Justiniano and Preston [18] and Lubik and Schorfheide [21], we have chosen to work with a large number of observable variables in order to facilitate identification of the parameters and shocks we estimate. For instance, despite the fact that the foreign variables are exogenous, we include them as observable variables as they enable identification of the asymmetric technology shock and are informative about the parameters governing the transmission of foreign impulses to the domestic economy. We estimate 13 structural shocks, of which 8 follow AR(1) processes and 5 are assumed to be i.i.d. In addition to these, there are 8 shocks provided by the exogenous (pre-estimated) fiscal and foreign VARs, whose parameters are kept fixed throughout the estimation of the model (uninformative priors are used for these stochastic processes). The shocks enter in such a way that there is no stochastic singularity in the likelihood function.[11] To compute the likelihood function, the reduced-form solution of the model is transformed into a state-space representation that maps the unobserved state variables into the observed data. We use the Kalman filter to calculate the likelihood function of the observed variables. The period 1980:1–1985:4 is used to form a prior on the unobserved state variables in 1985:4, and the period 1986:1-2007:3 is used for inference. The posterior mode and Hessian matrix evaluated at the mode is computed by standard numerical optimization routines (see Smets and Wouters [27] and the references there for details).

The parameters we choose to estimate pertain mostly to the nominal and real frictions in the model and the exogenous shock processes. Table 3.1 shows the assumptions for the prior distribution of the estimated parameters.[12] The prior distributions for the estimated parameters are in most cases identical to those assumed in ALLV [5], with the exception of the priors for the persistence coefficients of the exogenous shocks which are all centered around 0.75 instead of 0.85 in an attempt to reduce the amount of persistence stemming from the exogenous shocks in the model. For the model with a simple instrument rule, we choose identical priors for the parameters

[11] See ALLV [5] for details.

[12] We choose to calibrate those parameters that we think are weakly identified by the variables that we include in the vector of observed data. These parameters are mostly related to the steady-state values of the observed variables (that is, the great ratios: C/Y, I/Y, and G/Y). The parameters that we calibrate are set as follows: the money growth $\mu = 1.010445$; the discount factor $\beta = 0.999999$; the steady state growth rate of productivity $\mu_z = 1.005455$; the depreciation rate $\tilde{\delta} = 0.025$; the capital share in production $\alpha = 0.25$; the share of imports in consumption and investment $\omega_c = 0.35$ and $\omega_i = 0.50$, respectively; the share of wage bill financed by loans $\nu = 1$; the labour supply elasticity $\sigma_L = 1$; the wage markup $\lambda_w = 1.30$; inflation target persistence $\rho_\pi = 0.975$; the steady-state tax rates on labour income and consumption $\tau^y = 0.30$ and $\tau^c = 0.24$, respectively; government expenditures-output ratio 0.30; the subsitution elasticity between consumption goods $\eta_c = 5$; and the capital utilization parameter $\sigma_a = 10^6$ so that there is no variable capital utilization.

in the instrument rule before and after the adoption of an inflation target in 1993:1. For the model with optimal policy during the inflation-targeting regime, we use very uninformative priors for the loss-function parameters (λ_y and $\lambda_{\Delta i}$), as indicated by the high standard deviations. As mentioned in the introduction, the switch from the simple instrument rule to the inflation-targeting regime in 1993:1 is modelled as unanticipated and expected to last forever once it has occurred.

3.2. Estimation results

In table 3.1, we report the prior and estimated posterior distributions. Three posterior distributions are reported. The first, labeled "Policy rule," is under the assumption that the Riksbank has followed a simple instrument rule during the inflation-targeting period. The estimated instrument-rule parameters are reported. The second, labeled "Commitment," is under the assumption that the Riksbank has minimized a quadratic loss function under commitment during the inflation-targeting period, with the output gap in the loss function being the trend output gap. The estimated loss-function parameters are reported at the bottom of table 3.1.

There are two important facts to note in these first two posterior distributions. First, it is clear that the version of the model where policy is characterized with the simple instrument rule is a much better characterization of how the Riksbank has conducted monetary policy during the inflation-targeting period. The difference between log marginal likelihoods is almost 23 in favor of the model with the instrument rule compared to the model with optimal policy. In terms of Bayesian posterior odds, this is overwhelming evidence against the loss-function characterization of the Riksbank's past policy behavior.

Second, and perhaps even more interesting from our perspective, is that the non-policy parameters of the model are fairly invariant to the specification of how monetary policy has been conducted during the inflation-targeting period. Thus, the estimation results are not particularly sensitive to the policy specification. Hence, the common argument that likelihood-based methods in econometrics are sensitive to specification of policy does not apply here. We interpret this result as support for our assumption that the non-policy parameters can be considered structural and unaffected by the alternative policy assumptions we will consider. As can be seen from table 3.1, the largest changes in parameters pertain to the degree of habit persistence in the model, where the estimated value increase from 0.63 in the simple rule model to 0.73 in the Commitment version. Another parameter that is relatively more affected is the sensitivity of the risk-premium with respect to net foreign assets, which increases from 0.03 to about 0.14. Turning to the shock process parameters, we see that labor supply shocks are somewhat more persistent under the simple rule (ρ_{ζ_h}) , which is compensated with a somewhat lower standard deviation of the unit-root technology shocks (σ_{μ_z}). One way to assess the quantitative importance of the parameter modes is to calibrate all parameters except the coefficients in the loss function to the estimates obtained in the simple rule specification of the model, and then reestimate the coefficients in the loss function conditional on these parameters. If the loss function parameters are similar, we know that differences are not quantitatively unimportant. The results of this experiment is reported in the last column in table 3.1, and, as we hoped and anticipated, the resulting loss function parameters are very similar to the ones obtained when estimating all parameters jointly. Therefore, we conclude that the parameter changes are very modest.

As the log marginal likelihood strongly favored the specification of the model with the simple rule specification, we will in the subsequent analysis use the posterior mode estimates of the non-policy parameters resulting under the assumption of an instrument rule. In most cases, the simple rule specification of the model will also be used to generate the (partly) unobserved state variables and for consistency reasons, the associated estimated loss function parameters $\lambda_y = 1.102$ and $\lambda_{\Delta i} = 0.369$ are therefore used as benchmarks in some of the optimal policy projections below.

Table 3.1: Prior and posterior distributions

Parameter		Prior distribution			Posterior distribution					
					Policy rule		Loss function		Loss params.	
		type	mean	std.d. /df	mode	std.d. (Hess.)	mode	std.d. (Hess.)	mode	std.d. (Hess.)
Calvo wages	ξ_w	beta	0.750	0.050	0.719	0.045	0.719	0.042		
Calvo domestic prices	ξ_d	beta	0.750	0.050	0.712	0.039	0.737	0.043		
Calvo import cons. Prices	$\xi_{m,c}$	beta	0.750	0.050	0.868	0.018	0.859	0.016		
Calvo import inv. Prices	$\xi_{m,i}$	beta	0.750	0.050	0.933	0.010	0.929	0.011		
Calvo export prices	ξ_x	beta	0.750	0.050	0.898	0.019	0.889	0.025		
Indexation wages	κ_w	beta	0.500	0.150	0.445	0.124	0.422	0.115		
Indexation prices	κ_d	beta	0.500	0.150	0.180	0.051	0.173	0.050		
Markup domestic	λ_f	truncnormal	1.200	0.050	1.192	0.049	1.176	0.050		
Markup imported cons.	$\lambda_{m,c}$	truncnormal	1.200	0.050	1.020	0.028	1.021	0.029		
Markup.imported invest.	$\lambda_{m,i}$	truncnormal	1.200	0.050	1.137	0.051	1.154	0.049		
Investment adj. cost	$\tilde{S}''$	normal	7.694	1.500	7.951	1.295	7.684	1.261		
Habit formation	b	beta	0.650	0.100	0.626	0.044	0.728	0.035		
Subst. elasticity invest.	η_i	invgamma	1.500	4.0	1.239	0.031	1.238	0.030		
Subst. elasticity foreign	η_f	invgamma	1.500	4.0	1.577	0.204	1.794	0.318		
Risk premium	$\tilde{\phi}$	invgamma	0.010	2.0	0.038	0.026	0.144	0.068		
UIP modification	$\tilde{\phi}_s$	beta	0.500	0.15	0.493	0.067	0.488	0.029		
Unit root tech. shock	ρ_{μ_z}	beta	0.750	0.100	0.790	0.065	0.765	0.072		
Stationary tech. shock	ρ_{ε}	beta	0.750	0.100	0.966	0.006	0.968	0.005		
Invest. spec. tech shock	ρ_{Υ}	beta	0.750	0.100	0.750	0.077	0.719	0.067		
Asymmetric tech. shock	$\rho_{\tilde{\phi}}$	beta	0.750	0.100	0.852	0.059	0.885	0.041		
Consumption pref. shock	ρ_{ζ_c}	beta	0.750	0.100	0.919	0.034	0.881	0.038		
Labour supply shock	ρ_{ζ_h}	beta	0.750	0.100	0.382	0.082	0.282	0.064		
Risk premium shock	$\rho_{\tilde{z}*}$	beta	0.750	0.100	0.722	0.052	0.736	0.058		
Unit root tech. shock	σ_{μ_z}	invgamma	0.200	2.0	0.127	0.025	0.201	0.039		
Stationary tech. shock	σ_{ε}	invgamma	0.700	2.0	0.457	0.051	0.516	0.054		
Invest. spec. tech. shock	σ_{Υ}	invgamma	0.200	2.0	0.441	0.069	0.470	0.065		
Asymmetric tech. shock	$\epsilon_{\tilde{z}*}$	invgamma	0.400	2.0	0.199	0.030	0.203	0.031		
Consumption pref. shock	σ_{ζ_c}	invgamma	0.200	2.0	0.177	0.035	0.192	0.031		
Labour supply shock	σ_{ζ_h}	invgamma	1.000	2.0	0.470	0.051	0.511	0.053		
Risk premium shock	$\sigma_{\tilde{\phi}}$	invgamma	0.050	2.0	0.454	0.157	0.519	0.067		
Domestic markup shock	σ_{λ_d}	invgamma	1.000	2.0	0.656	0.064	0.667	0.068		
Imp. cons. markup shock	$\sigma_{\lambda_{m,c}}$	invgamma	1.000	2.0	0.838	0.081	0.841	0.084		
Imp. invest. markup shock	$\sigma_{\lambda_{m,i}}$	invgamma	1.000	2.0	1.604	0.159	1.661	0.169		
Export markup shock	σ_{λ_x}	invgamma	1.000	2.0	0.753	0.115	0.695	0.122		
Interest rate smoothing	$\rho_{R,1}$	beta	0.800	0.050	0.912	0.019	0.900	0.023		
Inflation response	$r_{\pi,1}$	truncnormal	1.700	0.100	1.676	0.100	1.687	0.100		
Diff. infl response	$r_{\Delta\pi,1}$	normal	0.300	0.100	0.210	0.052	0.208	0.053		
Real exch. rate response	$r_{x,1}$	normal	0.000	0.050	-	0.032	-	0.036		
					0.042		0.053			
Output response	$r_{y,1}$	normal	0.125	0.050	0.100	0.042	0.082	0.043		
Diff. output response	$r_{\Delta y,1}$	normal	0.063	0.050	0.125	0.043	0.133	0.042		
Monetary policy shock	$\sigma_{R,1}$	invgamma	0.150	2.0	0.465	0.108	0.647	0.198		
Inflation target shock	$\sigma_{\bar{\pi}^c,1}$	invgamma	0.050	2.0	0.372	0.061	0.360	0.059		
Interest rate smoothing 2	$\rho_{R,2}$	beta	0.800	0.050	0.882	0.019				
Inflation response 2	$r_{\pi,2}$	truncnormal	1.700	0.100	1.697	0.097				
Diff. infl response 2	$r_{\Delta\pi,2}$	normal	0.300	0.100	0.132	0.024				
Real exch. rate response 2	$r_{x,2}$	normal	0.000	0.050	-	0.029				
					0.058					
Output response 2	$r_{y,2}$	normal	0.125	0.050	0.081	0.040				
Diff. output response 2	$r_{\Delta y,2}$	normal	0.063	0.050	0.100	0.012				
Monetary policy shock 2	$\sigma_{R,2}$	invgamma	0.150	2.0	0.135	0.029				
Inflation target shock 2	$\sigma_{\bar{\pi}^c,2}$	invgamma	0.050	2.0	0.081	0.037				
Output stabilization	λ_y	truncnormal	0.5	100.0			1.091	0.526	1.102	0.224
Interest rate smoothing	$\lambda_{\Delta i}$	truncnormal	0.2	100.0			0.476	0.191	0.369	0.061
Log marg likelihood laplace					−2631.56		−2654.45			

4. Optimal policy projections

This section first explains the concept of optimal policy projections together with the information and data used and then presents the results for the optimal policy projections in 2007:4, using data up to and including 2007:3.

4.1. Information and data

First we clarify the information assumptions underlying the conditional expectation E_t used in the loss function (2.6). Let $\mathrm{E}_t[\cdot] \equiv \mathrm{E}[\cdot|\mathcal{I}_t]$ where $\mathcal{I}_t$ denotes the information set in period t. In the standard and simple case when all variables are observed, we can specify

$$\mathcal{I}_t \equiv \{\varepsilon_t, X_t, i_t; \varepsilon_{t-1}, X_{t-1}, x_{t-1},\, i_{t-1}; \varepsilon_{t-2}, X_{t-2}, x_{t-2}, i_{t-1}; ...\}.$$

This can be interpreted as information in the *beginning* of period t. We can understand this as the agents of the model (the central bank, the private sector, the fiscal authority, and the rest of the world) entering the beginning of period t with the knowledge of past realizations of shocks, variables, and instruments, $C\varepsilon_{t-1}, X_{t-1}, x_{t-1},\, i_{t-1}, C\varepsilon_{t-2}, X_{t-2}, x_{t-2}, i_{t-2}$,...[13] Then, at the beginning of period t, the shock ε_t is realized and observed, and X_t is determined by (2.2) and observed. We also assume that the agents know the model, including the matrices A, B, C, D, H, and W and the scalar δ, so either $C\varepsilon_t$ or X_t is sufficient for inferring the other from (2.2), given that previous realizations are known. Then the central bank determines, announces, and implements its instrument setting, i_t, which is hence observed by the other agents. After this, the expectations $x_{t+1|t}$ are formed, and x_t is determined by (2.4).[14] In equilibrium, both i_t and x_t will be a function of X_t and previous realizations of X_t, consistent with this specification of the information set.

However, Ramses makes more elaborate and realistic information assumptions. The variables X_t and x_t include (serially correlated) shocks and some other unobservable variables for which no data exists. Furthermore, the elements of X_t, x_t, and i_t are in many cases quarterly averages, which have not been realized until the end of quarter t. To model this, let the n_Z-vector of observable variables, Z_t, for which data exists, satisfy

$$Z_t = \bar{D} \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} + \eta_t,$$

[13] Only the linear combination $C\varepsilon_t$ of the shocks ε_t matters, not the individual shocks ε_t.
[14] More precisely, the expectations $x_{t+1|t}$ and x_t are simultaneously determined.

where $\bar{D}$ is a given matrix and η_t is an n_Z-vector of i.i.d. period-t measurement errors with the distribution $N(0, \Sigma_\eta)$. Then the information set in the beginning of period (quarter) t, just after the instrument setting for quarter t has been announced, is instead specified as

$$\mathcal{I}_t \equiv \{i_t, Z_{t-1}, Z_{t-2}, ...\}.$$

More precisely, for the monetary-policy decision at the beginning of quarter t, Bayesian methods and data on $Z_{t-\tau}$ for $\tau = 1, ..., T$ are used to estimate the parameters forming the matrices A, B, C, and H as well as the realizations of $X_{t-\tau}$, $x_{t-\tau}$, $C\varepsilon_{t-\tau}$, and $\eta_{t-\tau}$ for $\tau \geq 1$, denoted $X_{t-\tau|t}$, $x_{t-\tau|t}$, $C\varepsilon_{t-\tau|t}$, and $\eta_{t-\tau|t}$ for $\tau \geq 1$ (we assume that $i_{t-\tau}$ is observed, so $i_{t-\tau} = i_{t-\tau|t}$ for $\tau \geq 0$).[15] We also specify the estimate of X_t, denoted $X_{t|t}$, as

$$X_{t|t} = A_{11}X_{t-1|t} + A_{12}x_{t-1|t} + B_1 i_{t-1|t}, \tag{4.1}$$

where the estimated shocks $C\varepsilon_{t|t} = 0$ since Z_t is not in the information set $\mathcal{I}_t$ (Z_t is assumed to be observed at the end of period t and is in the information set $\mathcal{I}_{t+1}$).[16]

In practice, the parameters of Ramses are not necessarily reestimated each quarter, only the state of the economy, $X_{t+1-\tau|t}$, $X_{t-\tau|t}$, $x_{t-\tau|t}$, $C\varepsilon_{t-\tau|t}$, and $\eta_{t-\tau|t}$ for $\tau \geq 1$. For our purposes, we do not date the estimate of the matrices A, B, C, and H. Furthermore, with regard to projections in period t, we regard these matrices as certain and known. Then we can rely on certainty equivalence—under which conditional means of the relevant variables are sufficient for determining the optimal policy—and compute the optimal projections accordingly.[17]

4.2. The projection model and optimal projections

Let $y^t \equiv \{y_{t+\tau,t}\}_{\tau=0}^{\infty}$ denote a *projection* in period t for any variable y_t, a mean forecast conditional on information in period t. The *projection model* for the projections (X^t, x^t, i^t, Y^t) in period t is

$$\begin{bmatrix} X_{t+\tau+1,t} \\ Hx_{t+\tau+1,t} \end{bmatrix} = A \begin{bmatrix} X_{t+\tau,t} \\ x_{t+\tau,t} \end{bmatrix} + Bi_{t+\tau,t}, \tag{4.2}$$

[15] The estimated shocks $C\varepsilon_{t-\tau|t}$ for $\tau \geq 1$ satisfy $C\varepsilon_{t-\tau|t} = X_{t-\tau|t} - A_{11}X_{t-\tau-1|t} + A_{12}x_{t-\tau-1|t} + B_1 i_{t-\tau-1}$.

[16] Thus, the estimated/expected shock $C\varepsilon_{t+\tau|t}$ and $\eta_{t+\tau|t}$ for $\tau \geq 0$ are zero, whereas the estimated shocks $C\varepsilon_{t-\tau|t}$ and $\eta_{t-\tau|t}$ for $\tau \geq 1$ are given by $C\varepsilon_{t-\tau|t} = X_{t-\tau|t} - A_{11}X_{t-\tau-1|t} - A_{12}x_{t-\tau-1|t} - B_1 i_{t-\tau-1}$ and $\eta_{t-\tau|t} = Z_{t-\tau} - \bar{D}(X'_{t-\tau|t}, x'_{t-\tau|t}, i_{t-\tau})'$, and are normally nonzero.

[17] It should be noted that the setup here differs compared to what was used in, for example, ALLV [5] and [6], which examine forecasts using an instrument rule. There uncertainty about both parameters, the current state of the economy, the sequence of future shocks as well as the measurement errors were allowed for (see Adolfson, Lindé and Villani [7] for a description). However, this uncertainty is additive so certainty equivalence holds. Also our timing convention for the projections differs. In ALLV [5], [6], and [7] it is assumed that the projections are carried out at the end of period t (using the estimated instrument rule). That is, Z_t is observed and considered to be known at the time of the projection in ALLV [5], [6], and [7].

$$Y_{t+\tau,t} = D \begin{bmatrix} X_{t+\tau,t} \\ x_{t+\tau,t} \\ i_{t+\tau,t} \end{bmatrix} \tag{4.3}$$

for $\tau \geq 0$, where

$$X_{t,t} = X_{t|t}, \tag{4.4}$$

where $X_{t|t}$ is given by (4.1). Thus, we let "$,t$" and "$|t$" in subindices refer to projections and estimates in the beginning of period t, respectively. The *feasible set of projections* for given $X_{t|t}$ is the set of projections that satisfy (4.2)-(4.4). Thus, the projections in period t are conditional on the estimates of the matrices A, B, and H; the estimates of the past realizations of the unobserved predetermined and forward-looking variables, shocks, and measurement errors and the observed instrument settings, $X_{t-\tau|t}$, $x_{t-\tau|t}$, $C\varepsilon_{t-\tau|t}$, $\eta_{t-\tau|t}$, and $i_{t-\tau}$ for $\tau \geq 1$; and the estimate of the current realization of the predetermined variables $X_{t|t}$ in (4.1). These estimates are conditional on the information set $\mathcal{I}_t$ consisting of information available up to the beginning of period t.

The policy problem in period t is to determine the optimal projection in period t, denoted $(\check{X}^t, \check{x}^t, \check{i}^t, \check{Y}^t)$. The optimal projection is the projection that minimizes the intertemporal loss function,

$$\sum_{\tau=0}^{\infty} \delta^{\tau} L_{t+\tau,t}, \tag{4.5}$$

where the period loss, $L_{t+\tau,t}$, is specified as

$$L_{t+\tau,t} = Y_{t+\tau,t}{}' W Y_{t+\tau,t}. \tag{4.6}$$

The minimization is subject to the projection being in the feasible set of projections for given $X_{t|t}$.[18]

When the policy problem is formulated in terms of projections, we can allow $0 < \delta \leq 1$, since the above infinite sum will normally converge also for $\delta = 1$. The optimization is done under commitment in a timeless perspective (Woodford [38]). The optimization results in a set of first-order conditions, which combined with the model equations, (2.1), yields a system of difference equations (see Söderlind [29] and Svensson [32]). The system of difference equations can be solved with several alternative algorithms, for instance those developed by Klein [22] and Sims [26] (see Svensson [31] and [32] for details of the derivation and the application of the Klein algorithm). The system of difference equations can also be solved with the AIM algorithm of Anderson and Moore [11] and [12] (see Anderson [9] for a recent formulation). Whereas the Klein algorithm is easy to apply directly to the system of difference equations and the AIM algorithm requires some rewriting

[18] Policy projections when monetary policy is characterized by a simple instrument rule are described in appendix B.

of the difference equations, the AIM algorithm appears to be significantly faster for large systems (see Anderson [10] for a comparison between AIM and other algorithms). Appendix D discusses the relation between the Klein and AIM algorithms and shows how the system of difference equations can be rewritten to fit the AIM algorithm.

Under the assumption of optimization under commitment in a timeless perspective, one way to describe the optimal projection is by the following difference equations,

$$\begin{bmatrix} \check{x}_{t+\tau,t} \\ \check{i}_{t+\tau,t} \end{bmatrix} = F \begin{bmatrix} \check{X}_{t+\tau,t} \\ \Xi_{t+\tau-1,t} \end{bmatrix}, \tag{4.7}$$

$$\begin{bmatrix} \check{X}_{t+\tau+1,t} \\ \Xi_{t+\tau,t} \end{bmatrix} = M \begin{bmatrix} \check{X}_{t+\tau,t} \\ \Xi_{t+\tau-1,t} \end{bmatrix}, \tag{4.8}$$

for $\tau \geq 0$, where $\check{X}_{t,t} = X_{t|t}$. The Klein algorithm returns the matrices F and M. These matrices depend on A, B, H, D, W, and δ, but they are independent of C. The independence of C demonstrates the certainty equivalence of the projections. The n_X-vector $\Xi_{t+\tau,t}$ consists of the Lagrange multipliers of the lower block of (4.2), the block determining the projection of the forward-looking variables. The initial value for $\Xi_{t-1,t}$ is discussed below.

The optimal projection can also be described as

$$\begin{bmatrix} \check{X}_{t+\tau+1,t} \\ \Xi_{t+\tau,t} \\ \check{x}_{t+\tau+1,t} \\ \check{i}_{t+\tau+1,t} \end{bmatrix} = \bar{B} \begin{bmatrix} \check{X}_{t+\tau,t} \\ \Xi_{t+\tau-1,t} \\ \check{x}_{t+\tau,t} \\ \check{i}_{t+\tau,t} \end{bmatrix}, \tag{4.9}$$

for $\tau \geq 0$, where

$$\begin{aligned} \check{X}_{t,t} &= X_{t|t}, \\ \Xi_{t-1,t} &= \Xi_{t-1,t-1}, \\ \begin{bmatrix} \check{x}_{t,t} \\ \check{i}_{t,t} \end{bmatrix} &= \begin{bmatrix} \bar{B}_{x\cdot} \\ \bar{B}_{i\cdot} \end{bmatrix} \begin{bmatrix} \check{X}_{t-1,t-1} \\ \Xi_{t-2,t-1} \\ \check{x}_{t-1,t-1} \\ \check{i}_{t-1,t-1} \end{bmatrix} + \begin{bmatrix} \Phi_{x\cdot} \\ \Phi_{i\cdot} \end{bmatrix} \begin{bmatrix} X_{t|t} - \check{X}_{t,t-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

where the $\bar{B}$ and Φ are matrices returned by the AIM algorithm and partitioned conformably with X_t, Ξ_{t-1}, x_t, and i_t.[19] Here, $X_{t|t} - \check{X}_{t,t-1}$ denotes the difference between the estimate $X_{t|t}$, which is based on the information set $\mathcal{I}_t$, and the projection $\check{X}_{t,t-1}$, which is based on the information set $\mathcal{I}_{t-1}$. The new information in $\mathcal{I}_t$ relative to $\mathcal{I}_{t-1}$ is i_t and Z_{t-1}; only the latter matters for the

[19] The notation $\Phi_{x\cdot}$ refers to the larger sub matrix $[\Phi_{xX}\ \Phi_{x\Xi}\ \Phi_{xx}\ \Phi_{xi}]$; Φ_{xX} is the leftmost submatrix of this submatrix.

difference between $X_{t|t}$ and $\check{X}_{t,t-1}$. Equivalently, $\check{X}_{t,t}$, $\Xi_{t-1,t}$, $\check{x}_{t,t}$, and $\check{i}_{t,t}$ can be written as

$$\begin{bmatrix} \check{X}_{t,t} \\ \Xi_{t-1,t} \\ \check{x}_{t,t} \\ \check{i}_{t,t} \end{bmatrix} = \bar{B} \begin{bmatrix} \check{X}_{t-1,t-1} \\ \Xi_{t-2,t-1} \\ \check{x}_{t-1,t-1} \\ \check{i}_{t-1,t-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ \Phi_{xX} \\ \Phi_{iX} \end{bmatrix} \begin{bmatrix} X_{t|t} - \check{X}_{t,t-1} \end{bmatrix}.$$

The relations between the matrices matrices F, M, $\bar{B}$, and Φ are discussed in appendix D.[20]

4.3. Initial Lagrange multipliers

As discussed in Svensson [31, Appendix A],[21] the value of the initial Lagrange multiplier, $\Xi_{t-1,t}$, is zero, if there is commitment from scratch in period t, that is, if any previous commitment is disregarded. This reflects a time-consistency problem when there is reoptimization and recommitment in later periods, as is inherently the case in practical monetary policy. Instead, we assume that the optimization is under commitment in a timeless perspective. Then the commitment is considered having occurred some time in the past, and the initial value of the Lagrange multiplier satisfies

$$\Xi_{t-1,t} = \Xi_{t-1,t-1}, \tag{4.10}$$

where $\Xi_{t-1,t-1}$ denotes the Lagrange multiplier of the lower block of (4.2) for the determination of $x_{t-1,t-1}$ in the decision problem in period $t-1$. The dependence of the optimal policy projection in period t on this Lagrange multiplier from the decision problem in the previous period makes the optimal policy projection depend on previous projections and illustrates the history dependence of optimal policy under commitment in a forward-looking model shown in Backus and Driffill [13] and Currie and Levine [14] and especially examined and emphasized in Woodford [38].

Alternative methods to compute an initial $\Xi_{-1,0}$ when no explicit optimization was done in previous periods are discussed in appendix B.1. One method assumes that past policy has been optimal. Another method assumes that past policy has been systematic but not necessarily optimal.

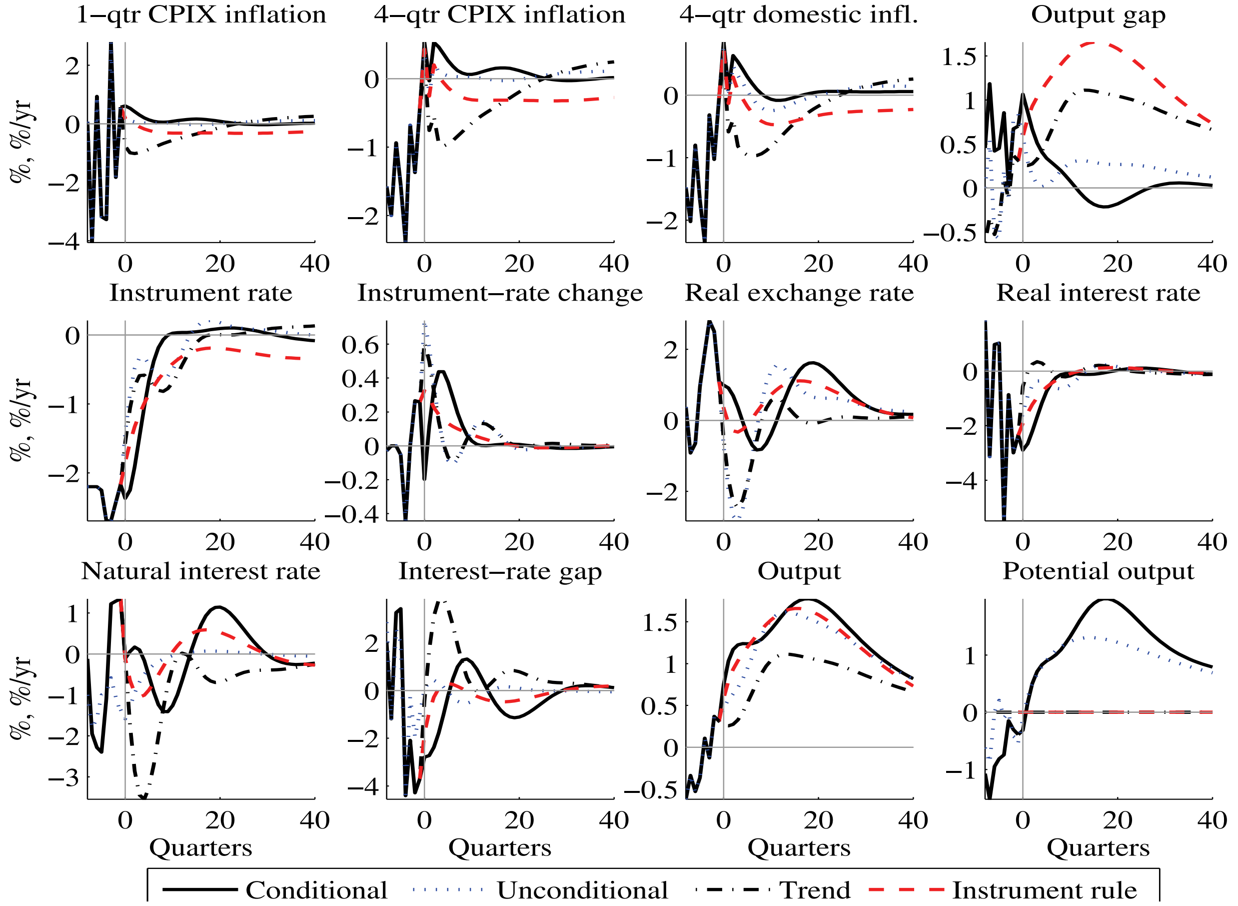
5. Results

Figure 5.1 shows four alternative policy projections for twelve different variables of interest, namely the three target variables (4-quarter CPIX inflation, the output gap, and the first-difference of

[20] As discussed in appendix D, for the AIM algorithm, all nonpredetermined variables as well as lagged one-period-ahead expectations of the nonpredetermined variables need for technical reasons to be included when the AIM form of the solution is used. In the above matrices and vectors, all those variables, including the elements corresponding to the Lagrange multipliers of the equation block (2.2) (the upper block of 4.2)), have for simplicity not been explicitly entered.

[21] The appendix is availabale at www.princeton.edu/svensson.

Figure 5.1: Projections in 2006:3 for optimal policy for different output gaps and for the instrument rule ($\lambda_y = 1.1, \lambda_{\Delta i} = 0.37$)



the instrument rate) and nine additional variables. The nine additional variables are 1-quarter CPIX inflation, 4-quarter domestic inflation, the instrument rate, the real exchange rate, the real interest rate (the instrument rate less 1-quarter-ahead CPIX inflation expectations), the neutral (real) interest rate associated with the specific definition of potential output, the interest-rate gap between the real interest rate and the neutral interest rate, output, and conditional potential output.[22] The plots show deviations from trend, where by trend we mean the (stochastic) steady state. All variables are measured in percent or percent per year. The thin vertical line marks 2006:3, which is quarter 0 in the figure. The curves to the left of the thin vertical line shows actual or (for the neutral interest rate and potential output) estimated history of data up to and including 2006:2, which is quarter −1 in the figure. As in all cases below, the estimated history and the state

[22] Then neutral interest rate is defined in appendix C.5 as the real interest rate in a flexprice equilibrium and is hence the interest rate associated with potential output. The interest-rate gap between the real interest rate and the neutral interest rate is arguably the best single indicator of monetary-policy stance.

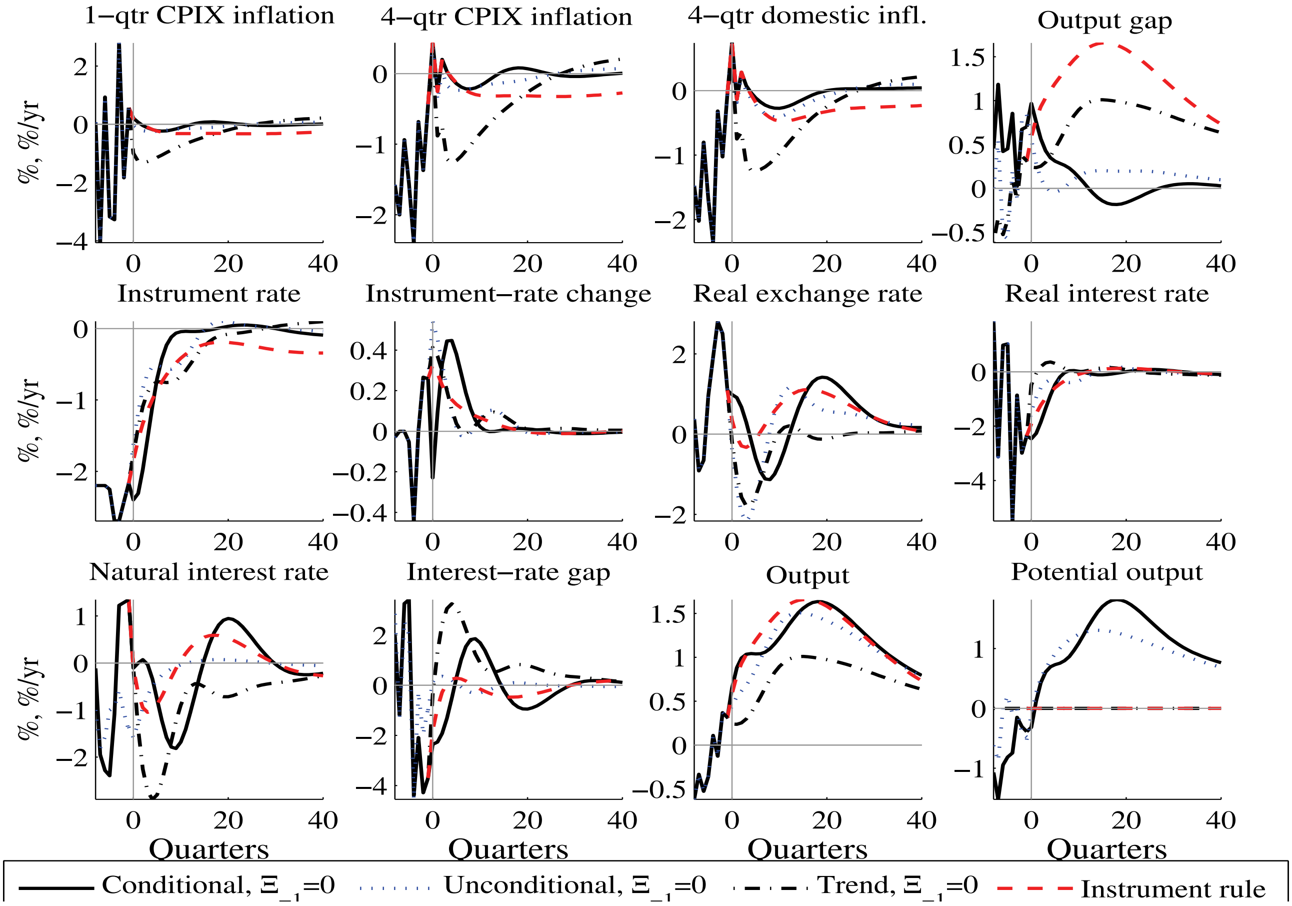
in 2006:2 is estimated with the simple instrument rule. Since the fit of the specification with the simple instrument rule is best, we consider its estimate of the state in 2006:2 better.

The projections in figure 5.1, the curves to the right of the thin vertical line, should be thought of as being considered by a policymaker in 2006:3, quarter 0 in the figure. The information available in 2006:3, quarter 0, then includes data up to and including 2006:2, quarter -1. The projections start from an estimated initial state (predetermined variables, $X_{0|0}$) in 2006:3. This initial state is projected from the given state of the economy in 2006:2 ($X_{-1|0}, x_{-1|0}, i_{-1|0}$) under the assumption that policy follows the instrument rule in 2006:2 (and is expected to follow the policy rule forever). Hence, the estimated predetermined variables ($X_{0|0}$) are given and are the same for the three policy alternatives considered in 2006:3, whereas the realization in 2006:3 of the forward-looking variables (x_0) and the instrument rate (i_0) will depend on the policy alternatives.

The dashed curves show the policy projection when policy follows the instrument rule. The monetary-policy shock in the instrument rule is set to zero, and the inflation target is assumed to be constant and equal to 2% per year. We see that the instrument rule results in an inflation projection for 4-quarter CPIX inflation that remains at around 0.4 percent per year below the inflation target during the 40 quarters plotted (the plots of inflation show deviations from the steady state which equals the inflation target). The estimated simple instrument rule is obviously not successful in keeping inflation close to the inflation target or even making it return to the inflation target in the 40 quarters shown. For the instrument-rule projection, the panel for the output gap shows the trend output gap, which is then identical to output, since the latter is measured as deviation from trend. We see that the instrument-rule projection results in a relatively large positive output gap that remains large for a long time.

The dash-dotted curves show the optimal policy projection when the trend output gap enters the loss function. The weights in the loss function, as for all optimal projections in this figure, are $\lambda_y = 1.1$, and $\lambda_{\Delta i} = 0.37$, the estimated weights in table 3.1 when the deep parameters are kept at their posterior modes obtained under the instrument rule. Furthermore, all the optimal policy projections in this figure assume commitment in a timeless perspective when the initial vector of Lagrange multipliers for the equations for the forward-looking variables (Ξ_{-1}) have been calculated under the assumption that policy during the inflation-targeting period has been optimal (see appendix B.1.1). We see that this optimal policy projection reduces the size of the output gap and initially leads to lower CPIX inflation than the instrument-rule projection. This requires initially tighter monetary policy than the instrument-rule projection, as demonstrated by an initially

Figure 5.2: Projections in 2006:3 for optimal policy for different output gaps (initial Lagrange multipliers zero) and for the instrument rule ($\lambda_y = 1.1, \lambda_{\Delta i} = 0.37$)



higher nominal and real interest rate and a large positive initial interest-rate gap.

The solid curves to the right of the thin vertical line show the optimal policy projection when the output gap in the loss function is the conditional output gap. Potential output plotted in the figure is then conditional potential output, and the output gap plotted is the conditional output gap. We see that this optimal projection successfully stabilizes 4-quarter CPIX inflation around the inflation target. It also successfully stabilizes the output gap from an initial high level, and we see that the conditional output gap is much smaller than the trend output gap. This requires initially more expansionary monetary policy, as shown by an initially lower nominal and real interest rate and an initially lower interest-rate gap.

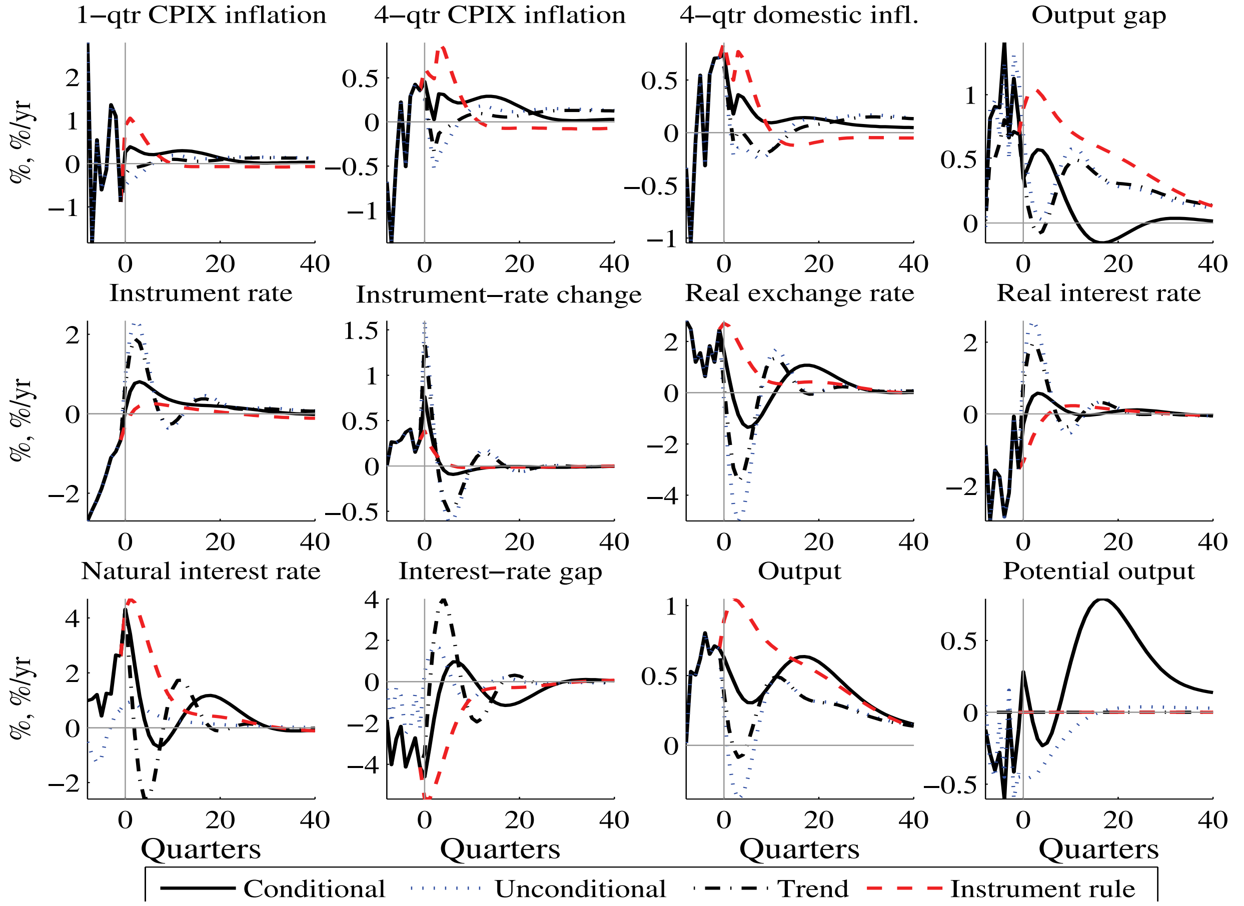
The reason that both inflation and the conditional output gap can be successfully stabilized and that the conditional output gap is much smaller than the large positive trend output gap is that conditional potential output becomes high and substantially exceeds trend output (the bottom-right panel of figure 5.1). The main reason that conditional potential output exceeds trend output is that the stationary technology shock (ε_t) is estimated to be quite high in 2006:2 (about 2%) and also quite persistent ($\rho_\varepsilon = 0.966$, see table 3.1, the column "Policy rule"). Potential output increases with the stationary technology shock, whereas trend output by definition is independent of the stationary technology shock and only depends on the permanent technology shock (μ_{zt}).

Thus, in this case it matters quite a bit for policy and the outcome for the economy whether the loss function has the conditional or trend output gap as an argument. When the conditional output gap is the target variable, optimal policy takes into account that conditional potential output is high because productivity is temporarily high and therefore allows more expansionary policy and higher actual output than when the trend output gap is the target variable or when the instrument rule is followed.

That the instrument rule is specified in terms of the trend output gap (rather than, for instance, the conditional output gap) is also one of the main reasons why the instrument rule does not bring inflation quickly back to target in this particular situation. When the stationary technology shock is high and since it is very persistent, by using the instrument rule with the trend output gap as an argument, the central bank creates an inefficient trade-off between output and inflation stabilization, which results in inflation below the target and a positive trend output gap for a long time from 2006:3. Had the rule instead been specified with a stronger inflation response or using a response to the conditional output gap, this trade-off would be less pronounced.

The dotted curves in figure 5.1 show the optimal projection when the output gap in the loss function is the unconditional output gap (computed using equation (C.7) with flexible prices from 1993:1 and onwards to form the state vector under flexible prices). Comparing with the optimal projection for the conditional output gap in the loss function, we see that 4-quarter CPIX inflation and initially the output gap is stabilized better with the unconditional output gap in the loss function. We see that the projections of conditional and unconditional potential output are in this case initially similar (the bottom-right panel of the figure) but that conditional potential output exceeds unconditional potential output after some ten quarters. The conditional and unconditional potential output levels are computed from different predetermined variables (those in the actual economy, and those in the hypothetical economy with flexible prices and wages in the past and present, respectively), whereas the exogenous shocks are the same in the two cases (with the exception of the markup shocks and the tax rates which are held constant when prices and wages

Figure 5.3: Projections in 2007:4 for optimal policy for different output gaps and for the instrument rule ($\lambda_y = 1.1, \lambda_{\Delta i} = 0.37$)

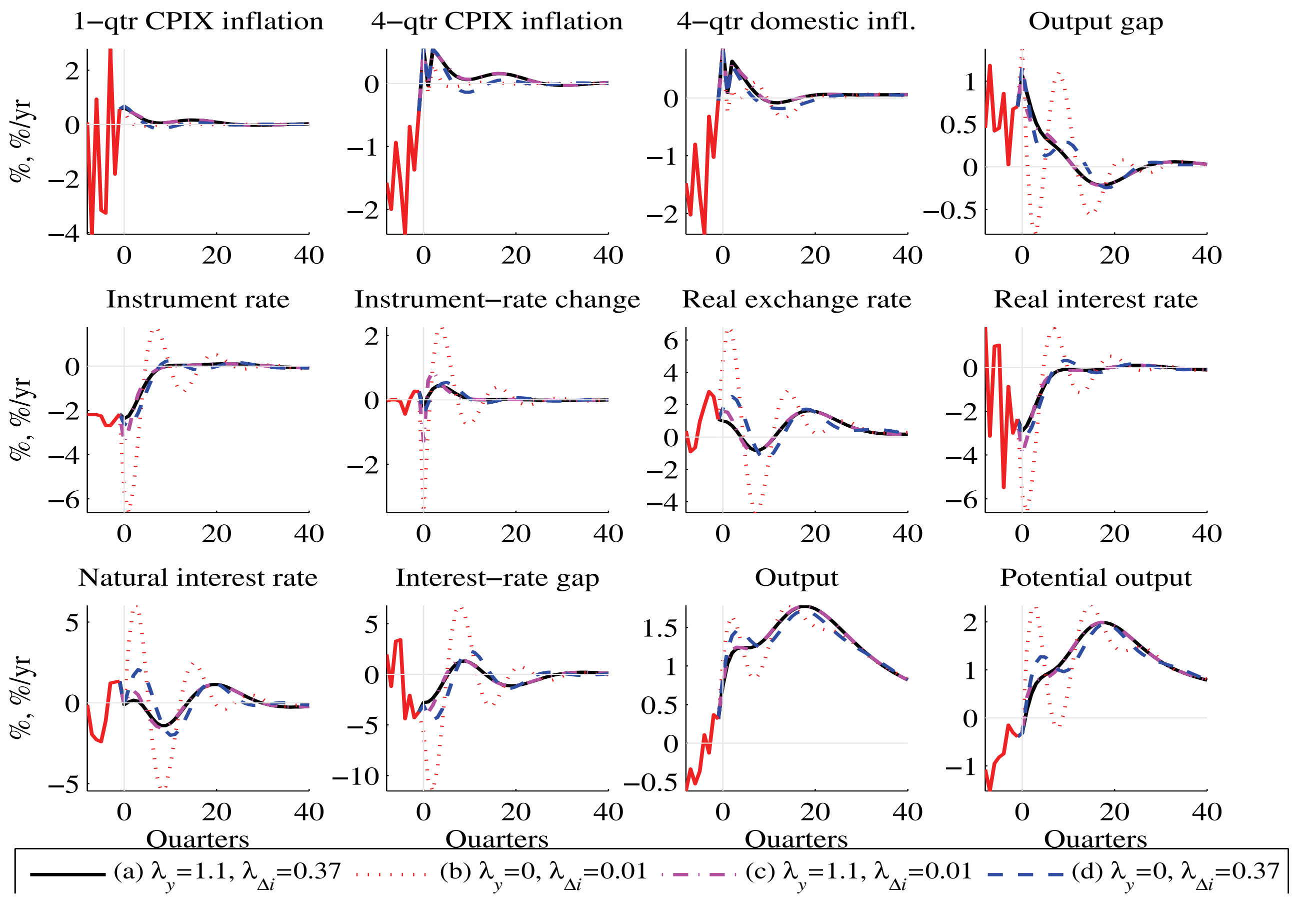


are flexible). Thus, unconditional potential output is independent of policy, whereas conditional potential output depends on policy through the endogenous predetermined variables. Higher initial output levels for the optimal policy with conditional output seem to lead to higher conditional potential output later on.

Figure 5.2 shows the same projections as figure 5.2 except that the initial Lagrange multipliers are set to zero. In this case this does not change the projections much.

Figure 5.3 shows the same projections for a policy maker in 2007:4, using data up to 2007:3, with estimated nonzero Lagrange multipliers as in figure 5.1. In this case, the stationary technology shock is estimated to be very close to zero, in which case there is less difference between the different potential output concepts and trend output, and consequently less difference between the alternative optimal policy projections. However, the optimal policy projections are more effective in stabilizing

Figure 5.4: Optimal projections in 2006:3 for different loss functions (conditional output gap)



both 4-quarter CPIX inflation around the target and the output gap than the instrument-rule projection. The latter is characterizes by much smaller movements in the instrument rate.

Figure 5.4 shows optimal policy projections for four cases with different loss-function parameters, all with the conditional output gap and nonzero initial Lagrange multipliers calculated as in figure 5.1. Projections for different loss-function parameters give policymakers information about the tradeoffs between stabilization of inflation, stabilization of the output gap, and interest-rate smoothing. We refer to the cases of loss-function parameters examined as (a) historical ($\lambda_y = 1.1$, $\lambda_{\Delta i} = 0.37$) (solid curves), (b) strict inflation targeting ($\lambda_y = 0$, $\lambda_{\Delta i} = 0.01$) (dotted curves), (c) no instrument-rate smoothing ($\lambda_y = 1.1$, $\lambda_{\Delta i} = 0.01$) (dashed-dotted curves), and (d) no output-gap stabilization ($\lambda_y = 0$, $\lambda_{\Delta i} = 0.37$).[23] First, we see that strict inflation targeting (b) is successful in stabilizing CPIX inflation but implies high variability of other variables. Second, the historical loss

[23] Setting $\lambda_{\Delta i} = 0$ rather than 0.01 leads to convergence problems in some cases.

function, with a higher weight on output-gap stabilization and instrument-rate smoothing, tolerates some deviation of CPIX inflation from target but stabilizes the output gap effectively. Third, we see that the case of no instrument-rate smoothing (c) implies more instrument-rate variability than the historical loss function but the projections of the other variables are not very different. Fourth, no output-gap stabilization (d) results in more effective stabilization of CPIX inflation, at the cost of more variability in the output gap, but otherwise not very different projections than with the historical loss function.

6. Conclusions

In this paper, we have shown how to construct operational optimal policy projections in the Riksbank's model Ramses, a linear-quadratic open-economy DSGE model. By optimal policy projections we mean projections of the target variables and the instrument rate that minimize a loss function under commitment in a timeless perspective. We have illustrated the use of different output-gap concepts in the loss function and clarified the difference between output gaps relative to conditional potential output, trend output, and unconditional potential output, where conditional refers to the existing endogenous predetermined variables, such as the capital stock. When productivity is temporarily high, conditional potential output exceeds trend output. Then optimal policy projections in this case differ substantially depending on whether conditional or trend output gaps enter the loss function.

Optimal policy under commitment in a timeless perspective also raises the question of what initial set of Lagrange multipliers to use to represent the commitment in the first period of optimization. We have illustrated the difference in projections when these multipliers are either zero (corresponding to the case of no previous commitment in the first period of optimization) or computed under the assumption that past policy was optimal. In the cases we examine, the optimal policy projections are not that much affected by whether the initial Lagrange multipliers are nonzero or zero. We show in the appendix an alternative method to compute these multipliers that does not presume that past policy was optimal. We discuss many other technical issues in the appendix, including details of Ramses and how the algorithms of Klein [22] and Anderson and Moore [11] and [12] can be combined for increased speed.

We have illustrated optimal policy projections only for a policymaker making policy decisions in two particular quarters, 2006:3 and 2007:4, and hence only in two particular initial situations. The results from these two quarters should be interpreted as illustrations and not general results.

However, with the tools we have demonstrated, we believe optimal policy projections in Ramses and similar DSGE models can now be applied in real-time policy processes and provide policy-makers with useful advice for their decisions—together with the usual input of detailed analysis and estimation of the initial state of the economy, policy simulations with historical policy reaction functions, other forecasting models, judgment, concerns about model uncertainty, and so forth.

Appendix

A. Ramses in some detail

This appendix presents the loglinear approximation of the model. For a more detailed description of the complete model and the derivation of the loglinear approximation, see ALLV [4].

There are four Phillips curves pertaining to Calvo-style price-setting firms in four sectors $j \in \{d, \mathrm{mc}, \mathrm{mi}, x\}$: domestic (production) (d), consumer-goods import (mc), investment-goods import (mi), and export (x),

$$\begin{aligned} \xi_j(\hat{\pi}_t^j - \widehat{\bar{\pi}}_t^c) \;=\;& \frac{\xi_j \beta}{1+\kappa_j \beta}(\hat{\pi}_{t+1|t}^j - \rho_{\bar{\pi}^c}\widehat{\bar{\pi}}_t^c) + \frac{\xi_j \kappa_j}{1+\kappa_j \beta}(\hat{\pi}_{t-1}^j - \widehat{\bar{\pi}}_t^c) \\ & - \frac{\xi_j \kappa_j \beta\,(1-\rho_{\bar{\pi}^c})}{1+\kappa_j \beta}\widehat{\bar{\pi}}_t^c + \frac{(1-\xi_j)(1-\beta\xi_j)}{1+\kappa_j \beta}(\widehat{\mathrm{mc}}_t^j + \hat{\lambda}_t^j), \end{aligned} \tag{A.1}$$

where a hat denotes the deviation of a loglinearized variable from a steady-state level ($\hat{v}_t \equiv dv_t/v$ for any variable v_t, where v is the steady-state level); $\hat{\pi}_t^j$, $j \in \{d, \mathrm{mc}, \mathrm{mi}, x\}$, denotes the deviation in quarter t of the log of gross inflation in sector j from its steady state ($\hat{\pi}_t^j \equiv d\pi_t^j/\pi^j$, where $\pi_t^j \equiv P_t^j/P_{t-1}^j$ is gross inflation, P_t^j is the price level in sector j, and π^j is the steady-state inflation in sector j); $\widehat{\bar{\pi}}_t^c$ is a (perceived) exogenous CPIX inflation target whose stochastic process is given by

$$\widehat{\bar{\pi}}_t^c = \rho_{\bar{\pi}^c}\widehat{\bar{\pi}}_{t-1}^c + \varepsilon_{\bar{\pi}^c t}$$

where $\varepsilon_{\bar{\pi}^c t}$ is i.i.d. and $N(0, \sigma_{\bar{\pi}^c}^2)$; β is a discount factor; κ_j, $\rho_{\bar{\pi}^c}$, and ξ_j are parameters (ξ_j is the Calvo price-stickiness parameter); $\widehat{\mathrm{mc}}_t^j$ denotes the real marginal cost of firms in sector j; and $\hat{\lambda}_t^j$ denotes a time-varying markup in sector j, assumed to be i.i.d. and $N\left(0, \sigma_{\lambda^j}^2\right)$. Firms in sector j that do not optimize their price in a given quarter are assumed to index them to the previous quarter's inflation in the sector and the inflation target and to set the rate of change of the firm's individual price equal to

$$\kappa_j \hat{\pi}_{t-1}^j + (1-\kappa_j)\widehat{\bar{\pi}}_t^c, \tag{A.2}$$

so the parameter κ_j is an indexation parameter. The firms' marginal costs are defined as

$$\begin{aligned} \widehat{\mathrm{mc}}_t^d &\equiv \alpha(\hat{\mu}_{zt} + \hat{H}_t - \hat{k}_t) + \widehat{\bar{w}}_t + \hat{R}_t^f - \hat{\varepsilon}_t, \\ \widehat{\mathrm{mc}}_t^{\mathrm{mc}} &\equiv \hat{P}_t^* + \hat{S}_t - \hat{P}_t^{\mathrm{mc}} \;\equiv\; -\widehat{\mathrm{mc}}_t^x - \hat{\gamma}_t^{x*} - \hat{\gamma}_t^{\mathrm{mc}d}, \\ \widehat{\mathrm{mc}}_t^{\mathrm{mi}} &\equiv \hat{P}_t^* + \hat{S}_t - \hat{P}_t^{\mathrm{mi}} \;\equiv\; -\widehat{\mathrm{mc}}_t^x - \hat{\gamma}_t^{x*} - \hat{\gamma}_t^{\mathrm{mi}d}, \\ \widehat{\mathrm{mc}}_t^x &\equiv \hat{P}_t^d - \hat{S}_t - \hat{P}_t^x \;\equiv\; \widehat{\mathrm{mc}}_{t-1}^x + \hat{\pi}_t - \hat{\pi}_t^x - \Delta\hat{S}_t, \end{aligned}$$

respectively; where α denotes the elasticity of output with respect to capital services; $\hat{\mu}_{zt}$ denotes the stochastic growth rate in quarter t of a unit-root technology shock ($\mu_{zt} \equiv z_t/z_{t-1}$ where z_t is the technology shock); $\hat{H}_t$ denotes hours worked; $\hat{k}_t$ denotes the capital-services flow, which may differ from the capital stock, $\hat{\bar{k}}_t$, since we allow for varying capital utilization; $\hat{\overline{w}}_t$ denotes the real wage; $\hat{R}^f_t$ denotes the effective nominal interest rate paid by the firms, reflecting the assumption that a fraction ν of the firms' wage bill has to be financed in advance, and is given by

$$\hat{R}^f_t = \frac{\nu R}{v(R-1)+1}\hat{R}_{t-1} + \frac{\nu(R-1)}{v(R-1)+1}\hat{\nu}_t$$

where $\hat{\nu}_t$ is a shock to the fraction (throughout we set $\nu = 1$); $\hat{\varepsilon}_t \equiv (\varepsilon_t - 1)/1$ is a stationary technology shock; $\hat{P}^*_t$ denotes the foreign price level; $\hat{\pi}^*_t$ denotes foreign inflation ($\hat{\pi}^*_t = d\pi^*_t/\pi^*$, where $\pi^*_t \equiv P^*_t/P^*_{t-1}$ and π^* is the foreign steady-state inflation); and $\hat{S}_t$ is the nominal exchange rate (domestic currency per unit of foreign currency) so $\Delta\hat{S}_t$ is the rate of currency depreciation ($\Delta v_t \equiv v_t - v_{t-1}$ for any variable v_t). Furthermore, $\hat{\gamma}^{x*}_t$, $\hat{\gamma}^{\mathrm{mc}d}_t$, $\hat{\gamma}^{\mathrm{mi}d}_t$, and $\hat{\gamma}^f_t$ are relative prices defined as

$$\begin{aligned}
\hat{\gamma}^{x*}_t &\equiv \hat{P}^x_t - \hat{P}^*_t \equiv \hat{\gamma}^{x*}_{t-1} + \hat{\pi}^x_t - \hat{\pi}^*_t,, \\
\hat{\gamma}^{\mathrm{mc}d}_t &\equiv \hat{P}^{\mathrm{mc}}_t - \hat{P}^d_t \equiv \hat{\gamma}^{\mathrm{mc}d}_{t-1} + \hat{\pi}^{\mathrm{mc}}_t - \hat{\pi}^d_t, \\
\hat{\gamma}^{\mathrm{mi}d}_t &\equiv \hat{P}^{\mathrm{mi}}_t - \hat{P}^d_t \equiv \hat{\gamma}^{\mathrm{mi}d}_{t-1} + \hat{\pi}^{\mathrm{mi}}_t - \hat{\pi}^d_t, \\
\hat{\gamma}^f_t &\equiv \hat{P}^d_t - \hat{S}_t - \hat{P}^*_t \equiv \widehat{\mathrm{mc}}^x_t + \hat{\gamma}^{x*}_t.
\end{aligned}$$

The stochastic processes for $\hat{\mu}_{zt}$, $\hat{\varepsilon}_t$, and $\hat{\pi}^*_t$ are specified below.

Since all real variables grow with the non-stationary technology shock z_t, we have to divide all quantities with the trend level of technology to make them stationary. We denote the resulting stationary variables by lower-case letters, that is, $c_t = C_t/z_t$, $\tilde{\imath}_t = I_t/z_t$ (we denote investment by $\tilde{\imath}_t$ to avoid confusion with the monetary-policy instrument i_t), $k_{t+1} = K_t/z_t$, $\bar{k}_{t+1} = \bar{K}_{t+1}/z_t$, $\overline{w}_t = W_t/(P_t z_t)$.

Households are Calvo-style wage setters. Households that are not allowed to reoptimize their nominal wage in the current quarter are assumed to index their nominal wage to the previous-quarter's rate of CPIX inflation, the inflation target, and productivity growth and set the rate of increase of their individual nominal wage as

$$\kappa_w \hat{\pi}^c_{t-1} + (1-\kappa_w)\hat{\overline{\pi}}^c_t + \hat{\mu}_{zt},$$

where κ_w is a wage-indexation parameter, $\hat{\pi}^c_t$ denotes CPIX inflation and satisfies

$$\hat{\pi}^c_t = (1-\omega_c)(\gamma^{cd})^{-(1-\eta_c)}\hat{\pi}^d_t + \omega_c(\gamma^{\mathrm{cmc}})^{-(1-\eta_c)}\hat{\pi}^{\mathrm{mc}}_t,$$

where ω_c is the (standardized) weight of imported goods in the CPIX, and γ^{cd} and γ^{cmc} are the steady-state relative price between the CPIX and the domestic good and the imported consumer good, respectively. The resulting real-wage equation can be written

$$\begin{aligned}
0 \;=\; & b_w \xi_w \widehat{\bar{w}}_{t-1} + [\sigma_L \lambda_w - b_w \left(1 + \beta \xi_w^2\right)] \widehat{\bar{w}}_t + b_w \beta \xi_w \widehat{\bar{w}}_{t+1|t} \\
& - b_w \xi_w \left(\hat{\pi}_t^d - \widehat{\bar{\pi}}_t^c\right) + b_w \beta \xi_w \left(\hat{\pi}_{t+1}^d - \rho_{\bar{\pi}^c} \widehat{\bar{\pi}}_t^c\right) \\
& + b_w \xi_w \kappa_w \left(\hat{\pi}_{t-1}^c - \widehat{\bar{\pi}}_t^c\right) - b_w \beta \xi_w \kappa_w \left(\hat{\pi}_t^c - \rho_{\bar{\pi}^c} \widehat{\bar{\pi}}_t^c\right) \\
& + (1 - \lambda_w) \hat{\psi}_{zt} - (1 - \lambda_w) \sigma_L \hat{H}_t \\
& - (1 - \lambda_w) \frac{\tau^y}{1 - \tau^y} \hat{\tau}_t^y - (1 - \lambda_w) \frac{\tau^w}{1 + \tau^w} \hat{\tau}_t^w - (1 - \lambda_w) \hat{\zeta}_t^h,
\end{aligned}$$

where

$$b_w \equiv \frac{\lambda_w \sigma_L - (1 - \lambda_w)}{(1 - \beta \xi_w)(1 - \xi_w)},$$

$\hat{\psi}_{zt}$ denotes the marginal utility of income, $\hat{\tau}_t^y$ denotes a labor income tax, $\hat{\tau}_t^w$ denotes a payroll tax assumed to be paid by the households, and $\hat{\zeta}_t^h$ denotes a labor supply shock to be specified below. The additional parameters are the Calvo wage-stickiness parameter ξ_w, the steady-state pay-roll tax τ^w, the steady-state labor income tax τ^y, the labor supply elasticity σ_L, and the wage markup λ_w.

The households' consumption preferences are subject to internal habit formation, which yields the following Euler equation for consumption expenditures,

$$\begin{aligned}
0 \;=\; & - b \beta \mu_z \hat{c}_{t+1|t} + \left(\mu_z^2 + b^2 \beta\right) \hat{c}_t - b \mu_z \hat{c}_{t-1} + b \mu_z \left(\hat{\mu}_{zt} - \beta \hat{\mu}_{z,t+1|t}\right) \\
& + \left(\mu_z - b\beta\right)\left(\mu_z - b\right) \hat{\psi}_{zt} + \frac{\tau^c}{1 + \tau^c} \left(\mu_z - b\beta\right)\left(\mu_z - b\right) \hat{\tau}_t^c \\
& + \left(\mu_z - b\beta\right)\left(\mu_z - b\right) \hat{\gamma}_t^{cd} - \left(\mu_z - b\right) \left(\mu_z \hat{\zeta}_t^c - b\beta \hat{\zeta}_{t+1|t}^c\right),
\end{aligned}$$

where b denotes the habit persistence parameter, μ_z denotes the steady-state growth rate, $\hat{c}_t$ denotes consumption, $\hat{\tau}_t^c$ denotes a consumption tax,

$$\hat{\gamma}_t^{cd} \equiv \hat{P}_t^c - \hat{P}_t^d \equiv \hat{\gamma}_{t-1}^{cd} + \hat{\pi}_t^c - \hat{\pi}_t^d$$

denotes the relative price between consumption and domestically produced goods, and $\hat{\zeta}_t^c$ denotes a consumption preference shock (specified below).

The household's first-order conditions with respect to investment $\tilde{\imath}_t$, the physical capital stock $\bar{k}_{t+1}$, and the utilization rate

$$\hat{u}_t \equiv \hat{k}_t - \widehat{\bar{k}}_t,$$

are, in their loglinearized forms, given by

$$\hat{P}_{kt} + \hat{\Upsilon}_t - \hat{\gamma}_t^{id} - \mu_z^2 \tilde{S}'' \left[\left(\widehat{\hat{\imath}}_t - \widehat{\hat{\imath}}_{t-1} \right) - \beta \left(\widehat{\hat{\imath}}_{t+1|t} - \widehat{\hat{\imath}}_t \right) + \hat{\mu}_{zt} - \beta \hat{\mu}_{z,t+1|t} \right] = 0, \tag{A.3}$$

$$\hat{\psi}_{zt} + \hat{\mu}_{z,t+1|t} - \hat{\psi}_{z,t+1|t} - \frac{\beta(1-\delta)}{\mu_z} \hat{P}_{k,t+1|t} + \hat{P}_{kt} - \frac{\mu_z - \beta(1-\delta)}{\mu_z} \hat{r}^k_{t+1|t} + \frac{\tau^k}{1-\tau^k} \frac{\mu_z - \beta(1-\delta)}{\mu_z} \hat{\tau}^k_{t+1|t} = 0,$$

$$\hat{u}_t = \frac{1}{\sigma_a} \hat{r}^k_t - \frac{1}{\sigma_a} \frac{\tau^k}{(1-\tau^k)} \hat{\tau}^k_t, \tag{A.4}$$

where P_{kt} is the price of capital goods in terms of domestic goods,[24] $\hat{\Upsilon}_t = (\Upsilon_t - 1)/1$ is a stationary investment-specific technology shock specified below, $\tilde{S}''$ is a parameter of the cost-of-investment function, δ is the depreciation rate, and $\hat{r}^k_t$ denotes the loglinearized expression for the real rental rate of capital and satisfies

$$\hat{r}^k_t = \hat{\mu}_{zt} + \widehat{\bar{w}}_t + \hat{R}^f_t + \hat{H}_t - \hat{k}_t.$$

The loglinearized law of motion for capital is given by

$$\widehat{\bar{k}}_{t+1} = (1-\delta)\frac{1}{\mu_z} \widehat{\bar{k}}_t - (1-\delta)\frac{1}{\mu_z} \hat{\mu}_{zt} + \left[1 - (1-\delta)\frac{1}{\mu_z} \right] \hat{\Upsilon}_t + \left[1 - (1-\delta)\frac{1}{\mu_z} \right] \widehat{\hat{\imath}}_t. \tag{A.5}$$

The output gap satisfies,

$$\hat{y}_t = \lambda_d \hat{\epsilon}_t + \lambda_d \alpha \hat{k}_t - \lambda_d \alpha \hat{\mu}_{zt} + \lambda_d (1-\alpha) \hat{H}_t, \tag{A.6}$$

where λ_d is the steady-state markup.

By combining the first-order conditions for the holdings of domestic and foreign bonds, we obtain an uncovered interest parity (UIP) condition. The UIP condition is modified to account for a negative correlation between the risk premium and expected exchange rate changes following the empirical evidence in, for example, Duarte and Stockman [16] (for details about this modification, see ALLV [5]),

$$\left(1 - \tilde{\phi}_s\right) \mathrm{E}_t \Delta \hat{S}_{t+1} - \tilde{\phi}_s \Delta \hat{S}_t - \left(\hat{R}_t - \hat{R}^*_t \right) - \tilde{\phi}_a \hat{a}_t + \widehat{\tilde{\phi}}_t = 0, \tag{A.7}$$

where $\hat{R}_t$ denotes the domestic nominal interest rate, $\hat{R}^*_t$ denotes the foreign nominal interest rate, $\hat{a}_t$ denotes the net foreign asset position, $\widehat{\tilde{\phi}}_t$ denotes a shock to the risk premium, and $\tilde{\phi}_s$ and $\tilde{\phi}_a$ are parameters. To ensure a well-defined steady state in the model, we assume that there is a premium on foreign bond holdings depending on the net foreign asset position, which explains the presence of the net foreign asset position. The real exchange rate, $\tilde{x}_t$ (we denote the real exchange rate by $\tilde{x}_t$ to avoid confusion with the vector of forward-looking variables x_t),[25] satisfies

$$\widehat{\tilde{x}}_t \equiv \hat{S}_t + \hat{P}^*_t - \hat{P}^c_t \equiv \widehat{\tilde{x}}_{t-1} + \hat{\pi}^*_t - \hat{\pi}^c_t.$$

[24] The price of capital goods is denoted $P_{k't}$ in ALLV [5].

[25] In ALLV [5], $\tilde{x}$ denotes technology-scaled export, but we do not need any notation for that variable here.

The loglinearized version of the first-order conditions for money balances m_{t+1} and cash holdings q_t are, respectively,

$$-\mu\hat{\psi}_{zt}+\mu\hat{\psi}_{z,t+1|t}-\mu\hat{\mu}_{z,t+1|t}+\left(\mu-\beta\tau^{k}\right)\hat{R}_t-\mu\hat{\pi}^{d}_{t+1|t}+\frac{\tau^{k}}{1-\tau^{k}}\left(\beta-\mu\right)\hat{\tau}^{k}_{t+1|t}=0,$$

$$\hat{q}_t=\frac{1}{\sigma_q}\left[\hat{\zeta}^{q}_t+\frac{\tau^{k}}{1-\tau^{k}}\hat{\tau}^{k}_t-\hat{\psi}_{zt}-\frac{R}{R-1}\hat{R}_{t-1}\right], \tag{A.8}$$

where μ is steady-state money growth and $\hat{\zeta}^{q}_t$ is a cash preference shock, to be specified below.

The following aggregate resource constraint must hold in equilibrium

$$(1-\omega_c)\left(\gamma^{cd}\right)^{\eta_c}\frac{c}{y}\left(\hat{c}_t+\eta_c\hat{\gamma}^{cd}_t\right)+(1-\omega_i)\left(\gamma^{id}\right)^{\eta_i}\frac{\tilde{\imath}}{y}\left(\hat{\tilde{\imath}}_t+\eta_i\hat{\gamma}^{id}_t\right)+\frac{g}{y}\hat{g}_t+\frac{y^{*}}{y}\left(\hat{y}^{*}_t-\eta_f\hat{\gamma}^{x*}_t+\hat{\tilde{z}}^{*}_t\right)$$

$$=\lambda_d[\hat{\varepsilon}_t+\alpha(\hat{k}_t-\hat{\mu}_{zt})+(1-\alpha)\hat{H}_t]-\left(1-\tau^{k}\right)r^{k}\frac{\bar{k}}{y}\frac{1}{\mu_z}(\hat{k}_t-\hat{\bar{k}}_t);$$

where η_c, η_i, and η_f are elasticities of substitution between domestic and imported consumer goods, domestic and imported investment goods, and domestic and foreign goods (in foreign consumption), respectively; c, $\tilde{\imath}$, y, y^{*}, g, and $\bar{k}$ are steady-state levels of consumption, investment, domestic output, foreign output, government expenditure, and the capital stock, respectively (when scaled with z_t, the technology level); $\hat{g}_t$ is government expenditure, and

$$\hat{\gamma}^{id}_t\equiv\hat{P}^{i}_t-\hat{P}^{d}_t\equiv\hat{\gamma}^{id}_{t-1}+\hat{\pi}^{i}_t-\hat{\pi}^{d}_t$$

is the relative price between investment and domestically produced goods.

We also need to relate money growth μ_t to real balances (where real balances $\bar{m}_{t+1}\equiv(M_{t+1}/P^{d}_t)/z_t$ are scaled by the technology shock z_t and M_{t+1} denote nominal balances) and domestic inflation,

$$\mu_t\equiv\frac{M_{t+1}}{M_t}\equiv\frac{\bar{m}_{t+1}z_tP^{d}_t}{\bar{m}_tz_{t-1}P^{d}_{t-1}}\equiv\frac{\bar{m}_{t+1}\mu_{zt}\pi^{d}_t}{\bar{m}_t}.$$

Loglinearizing, we have

$$\hat{\bar{m}}_{t+1}=\hat{\bar{m}}_t-\hat{\mu}_{zt}+\hat{\mu}_t-\hat{\pi}^{d}_t. \tag{A.9}$$

To clear the loan market, the demand for liquidity from the firms (which are financing their wage bills) must equal the supplied deposits of the households plus the monetary injection by the central bank:

$$\nu\bar{w}H\left(\hat{\nu}_t+\hat{\bar{w}}_t+\hat{H}_t\right)=\frac{\mu\bar{m}}{\pi^{d}\mu_z}\left(\hat{\mu}_t+\hat{\bar{m}}_t-\hat{\pi}^{d}_t-\hat{\mu}_{zt}\right)-q\hat{q}_t, \tag{A.10}$$

where $\bar{w}$, $\bar{m}$ and q are steady-state levels of real wages, real balances, and cash holdings (when scaled by z_t), H is the steady-state level of hours worked, and π^{d} is the steady-state inflation of domestic goods.

The evolution of net foreign assets at the aggregate level satisfies

$$\begin{aligned}\hat{a}_t &= -y^* \widehat{\mathrm{mc}}_t^x - \eta_f y_t^* \hat{\gamma}_t^{x*} + y^* \hat{y}_t^* + y^* \widehat{\tilde{z}}_t^* + \left(c^m + \tilde{\imath}^m\right) \hat{\gamma}_t^f \\ & \quad -c^m \left[-\eta_c \left(1-\omega_c\right) \left(\gamma^{cd}\right)^{-(1-\eta_c)} \hat{\gamma}_t^{\mathrm{mcd}} + \hat{c}_t\right] \\ & \quad -\tilde{\imath}^m \left[-\eta_i \left(1-\omega_i\right) \left(\gamma^{id}\right)^{-(1-\eta_i)} \hat{\gamma}_t^{\mathrm{mid}} + \widehat{\tilde{\imath}}_t\right] + \frac{R}{\pi \mu_z} \hat{a}_{t-1};\end{aligned} \tag{A.11}$$

where c^m and i^m are steady-state levels of consumption and investment of imported goods (when scaled by z_t), R is the steady-state nominal interest-rate level, $\hat{y}_t^*$ denotes foreign output, and $\widehat{\tilde{z}}_t^*$ is a stationary shock that measures the degree of asymmetry in the technological level between the domestic economy and the foreign economy.

The exogenous shocks of the model are AR(1). Those not explained above are given by the representation

$$\hat{\varsigma}_t = \rho_\varsigma \hat{\varsigma}_{t-1} + \varepsilon_{\varsigma t},$$

where $\varepsilon_{\varsigma t}$ is i.i.d. and $N\left(0, \sigma_\varsigma^2\right)$, for $\varsigma_t \in \{\, \mu_{zt}, \varepsilon_t, \lambda_t^j, \zeta_t^c, \zeta_t^h, \zeta^q, \Upsilon_t, \tilde{\phi}_t, \bar{\pi}_t^c, \tilde{z}_t^*\}$ and $j \in \{d, \mathrm{mc}, \mathrm{mi}, x\}$. Furthermore, as discussed in ALLV [5], for estimation purposes it is convenient to rescale the markup shock $\hat{\lambda}_t^j$ $j = \{d, \mathrm{mc}, \mathrm{mi}, x\}$ in the Phillips curves so as to include the coefficient $\frac{(1-\xi_j)(1-\beta\xi_j)}{\xi_j(1+\kappa_j\beta)}$ in the markup shock. Then the new coefficient on them in the Phillips curve is unity. Similarly, we rescale the investment specific technology shock $\hat{\Upsilon}_t$, the labor supply shock ζ_t^h, and the consumption preference shock ζ_t^c, so that these shocks enter in an additive fashion as well.[26]

The government collects tax revenues resulting from taxes on capital income τ_t^k, labor income τ_t^y, consumption τ_t^c, and the pay-roll τ_t^w, and spends resources on government consumption, G_t. The resulting fiscal surplus/deficit plus the seigniorage are assumed to be transferred back to the households in a lump sum fashion. Consequently, there is no government debt. The fiscal policy variables are assumed to be exogenously given by an identified VAR model with two lags and an uninformative prior. Let $\tau_t \equiv (\hat{\tau}_t^k, \hat{\tau}_t^y, \hat{\tau}_t^c, \hat{\tau}_t^w, \hat{g}_t)'$, where $\hat{g}_t$ denotes HP-detrended government expenditures. The fiscal policy VAR(2)-model is given by

$$\Theta_0 \tau_t = \Theta_1 \tau_{t-1} + \Theta_2 \tau_{t-2} + S_\tau \varepsilon_{\tau t}, \tag{A.12}$$

where $\varepsilon_{\tau t} \sim N\left(0, I_\tau\right)$ and $\Theta_0^{-1} S_\tau \varepsilon_{\tau t} \sim N\left(0, \Sigma_\tau\right)$.

[26] Although this is not of any major importance for the baseline estimation of the model in ALLV [5], it is important when carrying out the sensitivity analysis there because the effective prior standard deviation of the shocks changes with the value of the nominal and real friction parameters. Smets and Wouters [28] adopt the same strategy. Therefore, to obtain the size of the four truly fundamental markup shocks, the estimated standard deviations reported in table 3.1 should be divided by their respective scaling parameter (for instance, $\frac{(1-\xi_d)(1-\beta\xi_d)}{\xi_d(1+\kappa_d\beta)}$ in the case of the domestic markup shock).

Foreign prices π_t^*, (HP-detrended) output $\hat{y}_t^*$, and the interest rate R_t^* are exogenously given by an identified VAR model with four lags using an uninformative prior.[27] Given our assumption of equal substitution elasticities in foreign consumption and investment, these three variables suffice to describe the foreign economy in our model setup. Let $X_t^* \equiv (\pi_t^*, \hat{y}_t^*, R_t^*)'$, where π_t^* and R_t^* are quarterly foreign inflation and interest rates, and $\hat{y}_t^*$ is foreign output. The foreign economy is modeled as a VAR(4) model,

$$\Phi_0 X_t^* = \Phi_1 X_{t-1}^* + \Phi_2 X_{t-2}^* + \Phi_3 X_{t-3}^* + \Phi_4 X_{t-4}^* + S_{x^*} \varepsilon_{x^*t}, \tag{A.13}$$

where $\varepsilon_{x^*t} \sim N(0, I_{x^*})$ and $\Phi_0^{-1} S_{x^*} \varepsilon_{x^*t} \sim N(0, \Sigma_{x^*})$. When estimating the VAR, we assume and do not reject that Φ_0 in (A.13) has the following structure

$$\Phi_0 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma_{\pi 0}^* & -\gamma_{y0}^* & 1 \end{bmatrix}.$$

The model can then be written on the form (2.1), where we specify the predetermined variables $X_t = (X_t^{\text{ex}\prime}, X_t^{\text{pd}\prime})'$, the forward-looking variables x_t, the policy instrument i_t, and the shock vector ε_t as follows (the number of the element in each vector is also listed):

$$X_t^{\text{ex}} \equiv \begin{bmatrix} \widehat{\epsilon}_t & 1 & \hat{\tau}_t^y & 21 \\ \widehat{\epsilon}_{t-1} & 2 & \hat{\tau}_t^c & 22 \\ \hat{\mu}_{zt} & 3 & \hat{\tau}_t^w & 23 \\ \hat{\mu}_{z,t-1} & 4 & \hat{g}_t & 24 \\ \hat{\nu}_t & 5 & \hat{\tau}_{t-1}^k & 25 \\ \hat{\zeta}_t^c & 6 & \hat{\tau}_{t-1}^y & 26 \\ \hat{\zeta}_t^h & 7 & \hat{\tau}_{t-1}^c & 27 \\ \hat{\zeta}_t^q & 8 & \hat{\tau}_{t-1}^w & 28 \\ \hat{\lambda}_t^d & 9 & \hat{g}_{t-1} & 29 \\ \hat{\lambda}_t^{\text{mc}} & 10 & \hat{\pi}_t^* & 30 \\ \hat{\lambda}_t^{\text{mi}} & 11 & \hat{y}_t^* & 31 \\ \widehat{\tilde{\phi}}_t & 12 & \hat{R}_t^* & 32 \\ \hat{\Upsilon}_t & 13 & \hat{\pi}_{t-1}^* & 33 \\ \widehat{\tilde{z}}_t^* & 14 & \hat{y}_{t-1}^* & 34 \\ \widehat{\tilde{z}}_{t-1}^* & 15 & \hat{R}_{t-1}^* & 35 \\ \hat{\lambda}_{xt} & 16 & \hat{\pi}_{t-2}^* & 36 \\ \hat{\varepsilon}_{Rt} & 17 & \hat{y}_{t-2}^* & 37 \\ \widehat{\bar{\pi}}_t^c & 18 & \hat{R}_{t-2}^* & 38 \\ \widehat{\bar{\pi}}_{t-1}^c & 19 & \hat{\pi}_{t-3}^* & 39 \\ \hat{\tau}_t^k & 20 & \hat{y}_{t-3}^* & 40 \\ & & \hat{R}_{t-3}^* & 41 \end{bmatrix}, \quad X_t^{\text{pd}} \equiv \begin{bmatrix} \widehat{\bar{k}}_t & 1 & 42 & \hat{k}_{t-1} & 16 & 57 \\ \widehat{\bar{m}}_t & 2 & 43 & \hat{q}_{t-1} & 17 & 58 \\ \hat{R}_{t-1} & 3 & 44 & \hat{\mu}_{t-1} & 18 & 59 \\ \hat{\pi}_{t-1}^d & 4 & 45 & \hat{a}_{t-1} & 19 & 60 \\ \hat{\pi}_{t-1}^{\text{mc}} & 5 & 46 & \hat{\gamma}_{t-1}^{\text{mc}d} & 20 & 61 \\ \hat{\pi}_{t-1}^{\text{mi}} & 6 & 47 & \hat{\gamma}_{t-1}^{\text{mi}d} & 21 & 62 \\ \hat{y}_{t-1} & 7 & 48 & \hat{\gamma}_{t-1}^{x*} & 22 & 63 \\ \hat{\pi}_{t-1}^x & 8 & 49 & \widehat{\tilde{x}}_{t-1} & 23 & 64 \\ \widehat{\bar{w}}_{t-1} & 9 & 50 & \widehat{\text{mc}}_{t-1}^x & 24 & 65 \\ \hat{c}_{t-1} & 10 & 51 & \widehat{\bar{k}}_{t-1} & 25 & 66 \\ \widehat{\tilde{i}}_{t-1} & 11 & 52 & \hat{u}_{t-1} & 26 & 67 \\ \hat{\psi}_{z,t-1} & 12 & 53 & \hat{\pi}_{t-2}^d & 27 & 68 \\ \hat{P}_{k'\,t-1} & 13 & 54 & \hat{\pi}_{t-2}^{\text{mc}} & 28 & 69 \\ \Delta\hat{S}_{t-1} & 14 & 55 & \hat{\pi}_{t-3}^d & 29 & 70 \\ \hat{H}_{t-1} & 15 & 56 & \hat{\pi}_{t-3}^{\text{mc}} & 30 & 71 \end{bmatrix}, \tag{A.14}$$

[27] The reason why we include foreign output HP-detrended and not in growth rates in the VAR is that the foreign output gap enters the log-linearized model (for instance, in the aggregate resource constraint). This also enables identification of the asymmetric technology shock.

$$x_t \equiv \begin{bmatrix} \hat{\pi}_t^d & 1 & \hat{k}_t & 13 \\ \hat{\pi}_t^{\mathrm{mc}} & 2 & \hat{q}_t & 14 \\ \hat{\pi}_t^{\mathrm{mi}} & 3 & \hat{\mu}_t & 15 \\ \hat{y}_t & 4 & \hat{a}_t & 16 \\ \hat{\pi}_t^x & 5 & \hat{\gamma}_t^{\mathrm{mc}d} & 17 \\ \widehat{\bar{w}}_t & 6 & \hat{\gamma}_t^{\mathrm{mi}d} & 18 \\ \hat{c}_t & 7 & \hat{\gamma}_t^{x*} & 19 \\ \widehat{\tilde{\imath}}_t & 8 & \widehat{\tilde{x}}_t & 20 \\ \hat{\psi}_{zt} & 9 & \widehat{\mathrm{mc}}_t^x & 21 \\ \hat{P}_{k't} & 10 & \widehat{\bar{k}}_{t+1} & 22 \\ \Delta\hat{S}_t & 11 & \widehat{\bar{m}}_{t+1} & 23 \\ \hat{H}_t & 12 & & \end{bmatrix}, \quad x_t^{\mathrm{def}} \equiv \begin{bmatrix} \hat{r}_t^k & 1 \\ \hat{\gamma}_t^{cd} & 2 \\ \hat{\gamma}_t^f & 3 \\ \hat{\gamma}_t^{id} & 4 \\ \widehat{\mathrm{mc}}_t^d & 5 \\ \widehat{\mathrm{mc}}_t^{\mathrm{mc}} & 6 \\ \widehat{\mathrm{mc}}_t^{\mathrm{mi}} & 7 \\ \hat{\pi}_t^{\mathrm{cpi}} & 8 \end{bmatrix}, \quad i_t \equiv \hat{R}_t, \tag{A.15}$$

$$\varepsilon_t \equiv \begin{bmatrix} \varepsilon_{\epsilon t} & \varepsilon_{zt} & \varepsilon_{\nu t} & \varepsilon_{\zeta^c t} & \varepsilon_{\zeta^h t} & \varepsilon_{\zeta^q t} & \varepsilon_{\lambda^d t} & \varepsilon_{\lambda^{\mathrm{mc}} t} & \varepsilon_{\lambda^{\mathrm{mi}} t} & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ \varepsilon_{\tilde{\phi} t} & \varepsilon_{\Upsilon t} & \varepsilon_{\tilde{z}^* t} & \varepsilon_{\lambda_x t} & \varepsilon_{\varepsilon_R t} & \varepsilon_{\widehat{\bar{\pi}} t} & \varepsilon_{\tau t}' & \varepsilon_{x^* t}' & & \\ 10 & 11 & 12 & 13 & 14 & 15 & 16:20 & 21:23 & & \end{bmatrix}'. \tag{A.16}$$

The variables in x_t^{def} are only functions of the variables in x_t and can therefore be expressed in terms of x_t to reduce the number of forward-looking variables, n_x. See the technical appendix [2] for the detailed specification of the matrices A, B, C, H, and $\bar{D}$.

B. Optimal projections in some detail

As shown in Svensson [32], the $(n_X + n_x + n_i)$ first-order conditions for minimizing (2.6) under commitment subject to (2.1) can be written

$$\bar{A}' \begin{bmatrix} \xi_{t+1|t} \\ \Xi_t \end{bmatrix} = \bar{W} \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} + \frac{1}{\delta} \bar{H}' \begin{bmatrix} \xi_t \\ \Xi_{t-1} \end{bmatrix}, \tag{B.1}$$

where the elements of the n_X-vector ξ_{t+1} are the Lagrange multipliers for the upper block of (2.1) (the dating of ξ_{t+1} emphasizes that this is a restriction that applies in period $t+1$), the elements of the n_x-vector Ξ_t are the Lagrange multipliers for the lower block of (2.1) (the dating of Ξ_t emphasizes that this is restriction that applies in period t), and the matrices $\bar{A}$, $\bar{W}$, and $\bar{H}$ are defined by

$$\bar{A} \equiv [A \; B], \quad \bar{W} \equiv D'WD, \quad \bar{H} \equiv \begin{bmatrix} I & 0 & 0 \\ 0 & H & 0 \end{bmatrix}. \tag{B.2}$$

The first-order conditions can be combined with the $n_X + n_x$ model equations (2.1) to get a system of $2(n_X + n_x) + n_i$ difference equations for $t \geq 0$,

$$\begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{A}' \end{bmatrix} \left[\begin{array}{c} X_{t+1} \\ x_{t+1|t} \\ i_{t+1|t} \\ \hline \xi_{t+1|t} \\ \Xi_t \end{array} \right] = \begin{bmatrix} \bar{A} & 0 \\ \bar{W} & \frac{1}{\delta}\bar{H}' \end{bmatrix} \left[\begin{array}{c} X_t \\ x_t \\ i_t \\ \hline \xi_t \\ \Xi_{t-1} \end{array} \right] + \left[\begin{array}{c} C \\ 0 \\ \hline 0 \\ 0 \end{array} \right] \varepsilon_{t+1}. \tag{B.3}$$

Here, X_t and Ξ_{t-1} are predetermined variables ($n_X + n_x$ in total), and x_t, i_t, and ξ_t are non-predetermined variables ($n_x + n_i + n_X$ in total). This system can be rewritten as

$$\begin{bmatrix} \tilde{H}_{11} y_{1,t+1} + \tilde{H}_{12} y_{2,t+1|t} \\ \tilde{H}_{21} y_{1,t+1|t} + \tilde{H}_{22} y_{2,t+1|t} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} + \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix} \varepsilon_{t+1}, \tag{B.4}$$

where $y_{1t} \equiv (X_t', \Xi_{t-1}')' \equiv \tilde{X}_t'$ is the vector of predetermined variables and $y_{2t} \equiv (x_t', i_t', \xi_t')' \equiv (\tilde{x}_t', \xi_t')'$ is the vector of non-predetermined variables. The matrices are defined as

$$\tilde{H}_{11} \equiv \begin{bmatrix} I & 0 \\ 0 & A_{22}' \end{bmatrix}, \quad \tilde{H}_{12} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{12}' \end{bmatrix}, \quad \tilde{H}_{21} \equiv \begin{bmatrix} 0 & 0 \\ 0 & A_{21}' \\ 0 & B_2' \end{bmatrix}, \quad \tilde{H}_{22} \equiv \begin{bmatrix} H & 0 & 0 \\ 0 & 0 & A_{11}' \\ 0 & 0 & B_1' \end{bmatrix},$$

$$\tilde{A}_{11} \equiv \begin{bmatrix} A_{11} & 0 \\ \bar{W}_{xX} & \frac{1}{\delta}H' \end{bmatrix}, \quad \tilde{A}_{12} \equiv \begin{bmatrix} A_{12} & B_1 & 0 \\ \bar{W}_{xx} & \bar{W}_{xi} & 0 \end{bmatrix}, \quad \tilde{A}_{21} \equiv \begin{bmatrix} A_{21} & 0 \\ \bar{W}_{XX} & 0 \\ \bar{W}_{iX} & 0 \end{bmatrix},$$

$$\tilde{A}_{22} \equiv \begin{bmatrix} A_{22} & B_2 & 0 \\ \bar{W}_{Xx} & \bar{W}_{Xi} & \frac{1}{\delta}I \\ \bar{W}_{ix} & \bar{W}_{ii} & 0 \end{bmatrix}, \quad \tilde{C} \equiv \begin{bmatrix} C \\ 0 \end{bmatrix}.$$

The solution to (B.4) can be written

$$y_{2t} \equiv \begin{bmatrix} \tilde{x}_t \\ \xi_t \end{bmatrix} = F_2 y_{1t} \equiv \begin{bmatrix} F \\ F_\xi \end{bmatrix} \tilde{X}_t, \tag{B.5}$$

$$y_{1,t+1} \equiv \tilde{X}_{t+1} = M y_{1t} + \tilde{H}_{11}^{-1} \tilde{C} \varepsilon_{t+1} \equiv M \tilde{X}_t + \tilde{H}_{11}^{-1} \tilde{C} \varepsilon_{t+1} \tag{B.6}$$

for $t \geq 0$, where $y_{10} \equiv \tilde{X}_0 \equiv (X_0', \Xi_{-1}')'$ is given. We note that our assumption that A_{22} is nonsingular implies that $\tilde{H}_{11}$ is nonsingular.

B.1. Determination of the initial Lagrange multiplier, $\Xi_{t-1,t}$, when there is no explicit previous optimization

We discuss two possible methods of computing the initial Lagrange multiplier, $\Xi_{t-1,t}$. The second method seems less restrictive.

B.1.1. Assuming optimal policy in the past

Note that (4.8) and (B.5) imply that, in real time, the Lagrange multiplier Ξ_t satisfies

$$\Xi_{t,t} = M_{\Xi X} X_{t|t} + M_{\Xi\Xi} \Xi_{t-1,t-1} = \sum_{\tau=0}^{T} (M_{\Xi\Xi})^{\tau} M_{\Xi X} X_{t-\tau|t-\tau},$$

if the commitment occurred from scratch in period $t-T$, so $\Xi_{t-T-1,t-T} = 0$. Here M is partitioned conformably with X_t and Ξ_{t-1} so

$$M \equiv \begin{bmatrix} M_{XX} & M_{X\Xi} \\ M_{\Xi X} & M_{\Xi\Xi} \end{bmatrix}.$$

Recall that that M will depend on the weight matrix W in the period loss function and hence vary with the assumed or estimated loss function.

Given this, one possible initial value for $\Xi_{t-1,t} = \Xi_{t-1,t-1}$ is

$$\Xi_{t-1,t-1} = \sum_{\tau=0}^{T} (M_{\Xi\Xi})^{\tau} M_{\Xi X} X_{t-1-\tau|t-1-\tau}. \tag{B.7}$$

This treats estimated realized $X_{t-\tau}$ ($\tau = 0, 1, ..., T$) as resulting from optimal policy under commitment. The lag T may be chosen such that $(M_{\Xi\Xi})^T M_{\Xi X}$ is sufficiently small.

B.1.2. Assuming any (systematic) policy in the past

Another possible initial value for $\Xi_{t-1,t}$ follows from the following reasoning. The first-order condition (B.1) can be written

$$\bar{A}' \begin{bmatrix} \xi_{t+1|t} \\ \Xi_t \end{bmatrix} = \frac{1}{\delta} \bar{H}' \begin{bmatrix} \xi_t \\ \Xi_{t-1} \end{bmatrix} + \bar{W} z_t, \tag{B.8}$$

where $z_t \equiv (X_t', x_t', i_t')' \equiv (X_t', \tilde{x}_t')$. It should be noted that we have added an equation determining the interest rate using an instrument rule $i_t = f_i \ (X_t' \ , \ x_t')'$ such that Ξ_t contains an additional element. The matrices $\bar{A}$, $\bar{W}$, and $\bar{H}$ are therefore defined as

$$\bar{A} \equiv \begin{bmatrix} A & B \\ f_i & -1 \end{bmatrix}, \quad \bar{W} \equiv D'WD, \quad \bar{H} \equiv \begin{bmatrix} I & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that we can interpret (B.8) as an equation for the endogenous stochastic process for ξ_t and Ξ_{t-1} for a given stochastic process for z_t. This can be rewritten as

$$-\frac{1}{\delta} \begin{bmatrix} \bar{H}_{11}' & \bar{H}_{12}' \\ \bar{H}_{21}' & \bar{H}_{22}' \end{bmatrix} \begin{bmatrix} \xi_t \\ \Xi_{t-1} \end{bmatrix} + \begin{bmatrix} \bar{A}_{11}' & \bar{A}_{12}' \\ \bar{A}_{21}' & \bar{A}_{22}' \end{bmatrix} \begin{bmatrix} \xi_{t+1|t} \\ \Xi_t \end{bmatrix} = \bar{W} z_t,$$

where $\bar{H}'$ and $\bar{A}'$ are partitioned conformably with ξ_t and Ξ_{t-1} and $\bar{H}'_{jk}$ and $\bar{A}'_{jk}$ denote the submatrices $(\bar{H}')_{jk}$ and $(\bar{A}')_{jk}$ of $\bar{H}'$ and $\bar{A}'$ (not the transpose $(\bar{H}_{jk})'$ of the submatrix $\bar{H}_{jk}$ of $\bar{H}$, etc.). To write this on AIM form we need to use the correct timing of the variables. Reordering implies

$$H_{-1}\left[\begin{array}{c}\xi_{t-1}\\ \Xi_{t-1}\end{array}\right]+H_0\left[\begin{array}{c}\xi_t\\ \Xi_t\end{array}\right]+H_1\left[\begin{array}{c}\xi_{t+1|t}\\ \Xi_{t+1|t}\end{array}\right]=\bar{W}z_t,$$

where

$$H_{-1}\equiv-\frac{1}{\delta}\left[\begin{array}{cc}0 & \bar{H}'_{12}\\ 0 & \bar{H}'_{22}\end{array}\right],\quad H_0\equiv\left[\begin{array}{cc}-\frac{1}{\delta}\bar{H}'_{11} & \bar{A}'_{12}\\ -\frac{1}{\delta}\bar{H}'_{21} & \bar{A}'_{22}\end{array}\right],\quad H_1\equiv\left[\begin{array}{cc}\bar{A}'_{11} & 0\\ \bar{A}'_{21} & 0\end{array}\right].$$

As noted in appendix D.2 and equation (D.18), the solution to this equation can be written

$$\left[\begin{array}{c}\xi_t\\ \Xi_t\end{array}\right]=\bar{B}\left[\begin{array}{c}\xi_{t-1}\\ \Xi_{t-1}\end{array}\right]+\sum_{s=0}^{\infty}\bar{F}^s\Phi\bar{W}z_{t+s|t}. \tag{B.9}$$

Here, $\bar{B}$ is returned by the AIM algorithm, and Φ and $\bar{F}$ depend on $\bar{B}$, H_0, and H_1 according to (D.16) and (D.19). Furthermore, $z_{t+s|t}$ denotes rational expectations in period t of the realization in period $t+s$ of the stochastic process for $\{z_{t+s}\}_{s=0}^{\infty}$. Thus, for given realizations of $z_{t-\tau}$ and corresponding expectations $\{z_{t-\tau+s|t-\tau}\}_{s=0}^{\infty}$ for $\tau=0,1,...,T$, and by setting $\Xi_{t-T-1}=0$, we can solve for $\{\xi'_{t-\tau},\Xi'_{t-1-\tau}\}_{\tau=0}^{T}$ and use the resulting Ξ_{t-1} as the initial value for $\Xi_{t-1,t}$.

Solving the first-order conditions for a given stochastic process can be interpreted as finding the shadow prices of the model equations (2.1) without necessarily assuming optimal policy. That is, we only assume that z_t follows *some* known stochastic process, not necessarily a stochastic process corresponding to optimal policy. Under the assumption of optimal policy, there is no additional information in the realizations of $x_{t-\tau}$ and $i_{t-\tau}$; hence only realizations of the predetermined variables are used in (B.7). When policy is not necessarily optimal, also realizations of $x_{t-\tau}$ and $i_{t-\tau}$ are used in calculating $\Xi_{t-1,t}$. (However, it should be noted that these are still functions of the predetermined variables.) This method seems less restrictive than assuming past optimal policy. Note that we still assume that policy is systematic during periods $t-T$, $t-T+1$, ..., $t-1$, and that it follows a stochastic process that is understood by the private sector and allows a rational-expectations equilibrium. Furthermore, the shift to optimal policy in period t is not anticipated in previous periods.

It follows from equations (4.7) and (4.8) that

$$z_{t+s|t}=\left[\begin{array}{c}I\\ F_{\mathrm{IR}}\end{array}\right]X_{t+s|t}=\left[\begin{array}{c}I\\ F_{\mathrm{IR}}\end{array}\right](M_{\mathrm{IR}})^s X_t,$$

where F_{IR} and M_{IR} are the reduced-form (solution) matrices obtained under the instrument rule.

Equation (B.9) can then be rewritten as

$$\left[\begin{array}{c} \xi_{t-\tau} \\ \Xi_{t-\tau} \end{array}\right] = \bar{B} \left[\begin{array}{c} \xi_{t-\tau-1} \\ \Xi_{t-\tau-1} \end{array}\right] + GX_{t-\tau}, \qquad \tau = 0, 1, ..., T,$$

where

$$G = \sum_{s=0}^{\infty} \bar{F}^s \Phi \bar{W} \left[\begin{array}{c} I \\ F_{\mathrm{IR}} \end{array}\right] (M_{\mathrm{IR}})^s .$$

B.2. Projections with an arbitrary instrument rule

With a constant (that is, time-invariant) arbitrary instrument rule, the instrument rate satisfies

$$i_t = [f_X \quad f_x] \left[\begin{array}{c} X_t \\ x_t \end{array}\right]$$

for $\tau \geq 0$, where the $n_i \times (n_X + n_x)$ matrix $[f_X \quad f_x]$ is a given (linear) instrument rule and partitioned conformably with X_t and x_t.[28] If $f_x \equiv 0$, the instrument rule is an *explicit* instrument rule; if $f_x \neq 0$, the instrument rule is an *implicit* instrument rule. In the latter case, the instrument rule is actually an equilibrium condition, in the sense that in a real-time analogue the instrument rate in period t and the forward-looking variables in period t would be simultaneously determined.

The instrument rule that is estimated for Ramses is of the form (see appendix A for the notation)

$$\begin{aligned} i_t \quad = \quad & \rho_R i_{t-1} + (1 - \rho_R) \left[\hat{\bar{\pi}}_t^c + r_\pi \left(\hat{\pi}_{t-1}^c - \hat{\bar{\pi}}_t^c\right) + r_y \hat{y}_{t-1} + r_x \hat{\tilde{x}}_{t-1}\right] \\ & + r_{\Delta\pi} \left(\hat{\pi}_t^c - \hat{\pi}_{t-1}^c\right) + r_{\Delta y} \left(\hat{y}_t - \hat{y}_{t-1}\right) + \varepsilon_{Rt}, \end{aligned} \tag{B.10}$$

where $i_t \equiv \hat{R}_t$ (the notation for the short nominal interest rate in Ramses). Since $\hat{\pi}_t^c$ and $\hat{y}_t$ are forward-looking variables, this is an implicit instrument rule.

An arbitrary more general (linear) policy rule (G, f) can be written as

$$G_x x_{t+1|t} + G_i i_{t+1|t} = f_X X_t + f_x x_t + f_i i_t, \tag{B.11}$$

where the $n_i \times (n_x + n_i)$ matrix $G \equiv [G_x \quad G_i]$ is partitioned conformably with x_t and i_t and the $n_i \times (n_X + n_x + n_i)$ matrix $f \equiv [f_X \quad f_x \quad f_i]$ is partitioned conformably with X_t, x_t, and i_t. This general policy rules includes explicit, implicit, and forecast-based instrument rules (in the latter the instrument rate depends on expectations of future forward-looking variables, $x_{t+1|t}$) as well as targeting rules (conditions on current or expected future target variables). When this general policy rule is an instrument rule, we require the $n_x \times n_i$ matrix f_i to be nonsingular, so (B.11) determines i_t for given X_t, x_t, $x_{t+1|t}$, and $i_{t+1|t}$.

[28] For projections with arbitrary time-varying policy rules, see Laséen, Lindé, and Svensson [20].

The general policy rule can be added to the model equations (2.1) to form the new system to be solved. With the notation $\tilde{x}_t \equiv (x_t', i_t')'$, the new system can be written

$$\begin{bmatrix} X_{t+1} \\ \tilde{H}\tilde{x}_{t+1|t} \end{bmatrix} = \tilde{A} \begin{bmatrix} X_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} C \\ 0_{(n_x+n_i)\times n_\varepsilon} \end{bmatrix} \varepsilon_{t+1},$$

$$Y_t = D \begin{bmatrix} X_t \\ \tilde{x}_t \end{bmatrix}, \quad Z_t = \bar{D} \begin{bmatrix} X_t \\ \tilde{x}_t \end{bmatrix},$$

where

$$\tilde{H} \equiv \begin{bmatrix} H & 0 \\ G_x & G_i \end{bmatrix}, \quad \tilde{A} \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ f_X & f_x & f_i \end{bmatrix},$$

where $\tilde{H}$ is partitioned conformably with x_t and i_t and $\tilde{A}$ is partitioned conformably with X_t, x_t, and i_t. The corresponding projection model can then be written

$$\begin{bmatrix} X_{t+\tau+1,t} \\ \tilde{H}\tilde{x}_{t+\tau+1,t} \end{bmatrix} = \tilde{A} \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix},$$

$$Y_{t+\tau,t} = D \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix}, \quad Z_{t+\tau,t} = \bar{D} \begin{bmatrix} X_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix}$$

for $\tau \geq 0$, where $X_{t,t} = X_{t|t}$.

Then, (under the usual assumption that the policy rule gives rise to the standard saddlepoint property for the system's eigenvalues) there exist matrices M and F such that the resulting *equilibrium projection* satisfies

$$X_{t+\tau+1,t} = MX_{t+\tau,t},$$

$$\tilde{x}_{t+\tau,t} \equiv \begin{bmatrix} x_{t+\tau,t} \\ i_{t+\tau,t} \end{bmatrix} = FX_{t+\tau,t} \equiv \begin{bmatrix} F_x \\ F_i \end{bmatrix} X_{t+\tau,t}$$

for $\tau \geq 0$, where the matrices M and F depend on $\tilde{A}$ and $\tilde{H}$, and thereby on A, B, H, G, and f.[29]

[29] Equivalently, the resulting equilibrium projection satisfy

$$\begin{bmatrix} \check{X}_{t+\tau+1,t} \\ \check{x}_{t+\tau+1,t} \\ \check{i}_{t+\tau+1,t} \end{bmatrix} = \bar{B} \begin{bmatrix} \check{X}_{t+\tau,t} \\ \check{x}_{t+\tau,t} \\ \check{i}_{t+\tau,t} \end{bmatrix}$$

for $\tau \geq 0$, where

$$\begin{aligned} \check{X}_{t,t} &= X_{t|t}, \\ \begin{bmatrix} \check{x}_{t,t} \\ \check{i}_{t,t} \end{bmatrix} &= \begin{bmatrix} \bar{B}_x \\ \bar{B}_i \end{bmatrix} \begin{bmatrix} \check{X}_{t-1,t-1} \\ \check{x}_{t-1,t-1} \\ \check{i}_{t-1,,t-1} \end{bmatrix} + \begin{bmatrix} \Phi_{xX} \\ \Phi_{iX} \end{bmatrix} (X_{t|t} - \check{X}_{t,t-1}). \end{aligned}$$

C. Flexprice equilibrium and alternative concepts of potential output

Under the assumption of flexible prices and wages and an additional equation that determines nominal variables (inflation, the price level, the exchange rate, or some other nominal variable), the *flexprice model* can be written

$$\begin{bmatrix} X_{t+1} \\ H^f x_{t+1|t} \end{bmatrix} = A^f \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} + \begin{bmatrix} C \\ 0_{n_x \times n_\varepsilon} \\ 0_{n_i \times n_\varepsilon} \end{bmatrix} \varepsilon_{t+1}, \tag{C.1}$$

with the same variables X_t, x_t, and i_t and the same i.i.d. shocks ε_t as in the sticky-price model but with the new $(n_x+n_i)\times n_x$ matrix H^f and $(n_X+n_x+n_i)\times(n_X+n_x+n_i)$ matrix A^f. There are hence n_i extra equations added to the lower block of the equations (the block of equations determining the forward-looking variables), as many equations as the number of policy instruments. The discussion here is restricted to the case $n_i = 1$ (as is Ramses), so only one equation needs to be added, such as

$$\hat{\pi}_t^{\text{cpi}} = 0 \;\text{ or }\; \hat{\pi}_t^d = 0. \tag{C.2}$$

In the latter case,

$$H^f \equiv \begin{bmatrix} H \\ 0_{1\times n_x} \end{bmatrix}, \quad A^f \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ 0_{1\times n_X} & e_1 & 0 \end{bmatrix},$$

where e_1 here denotes a row n_x-vector with the first element equal to unity and the other elements equal to zero (reflecting that domestic inflation, $\hat{\pi}_t^d$, by (A.15) is the first forward-looking variable).

The matrix A^f has been modified so there is no effect on the endogenous variables of the four time-varying markups $\hat{\lambda}_j$, $j \in \{d, \text{mc}, \text{mi}, x\}$, since these time-varying distortions would introduce undesirable variation in the difference between the (loglinearized) efficient and flexprice output (see more discussion in section C.6). This is achieved by setting the corresponding elements in A_{11} and A_{21} equal to zero. With the same argument one may also want to eliminate the effect on the endogenous variables of time-varying tax rates.

Let $\tilde{x}_t^f \equiv (x_t^{f\prime}, i_t^{f\prime})'$ and X_t^f denote the realizations of the nonpredetermined and predetermined variables, respectively, in a flexprice equilibrium for $t \geq t_0$, where t_0 is some period in the past with given predetermined variables X_{t_0} from which we compute the flexprice equilibrium. The flexprice equilibrium is the solution to the system of difference equations (C.1) for $t \geq t_0$. It can be written

$$\begin{aligned} \tilde{x}_t^f &\equiv \begin{bmatrix} x_t^f \\ i_t^f \end{bmatrix} = F^f X_t^f \equiv \begin{bmatrix} F_x^f \\ F_i^f \end{bmatrix} X_t^f, \\ X_{t+1}^f &= M^f X_t^f + C^f \varepsilon_{t+1}, \end{aligned}$$

for $t \geq t_0$, where F^f and M^f are matrices returned by the Klein [22] algorithm and $X^f_{t_0} = X_{t_0}$. In particular, by (A.15), element number four of x^f_t (and $\tilde{x}^f_t$) is $\hat{y}^f_t$, the output level in the flexprice equilibrium. We can therefore write flexprice output in period $t \geq t_0$ as

$$\hat{y}^f_t = F^f_{4\cdot} X^f_t,$$

where the row vector $F^f_{4\cdot}$ is the fourth row of the matrix F^f.

C.1. Unconditional potential output

Consider now a hypothetical flexprice equilibrium that has lasted forever, the unconditional flexprice equilibrium. The hypothetical realization of the predetermined variables in period t, denoted $X^f_{t;-\infty}$, then satisfies

$$X^f_{t;-\infty} = M^f X^f_{t-1;-\infty} + C^f \varepsilon_t = \sum_{s=0}^{\infty} (M^f)^s C^f \varepsilon_{t-s},$$

and hence depends on the realizations of the shocks ε_t, ε_{t-1}, ... The corresponding flexprice output is

$$\hat{y}^f_{t;-\infty} \equiv F^f_{4\cdot} X^f_{t;-\infty} = F^f_{4\cdot} \sum_{s=0}^{\infty} (M^f)^s C^f \varepsilon_{t-s}. \tag{C.3}$$

We refer to this output level as *unconditional potential output*. It hence corresponds to the output level in a hypothetical economy that has always had flexible prices and wages but is subject to the same shocks as the actual economy. It hence has a different capital stock and different realizations of the endogenous both predetermined and nonpredetermined variables compared to the actual economy. It corresponds to the natural rate of output consistent with the definition of the natural rate of interest in Neiss and Nelson [23].

C.2. Conditional potential output

Consider also the hypothetical situation in which prices and wages in the actual economy unexpectedly become flexible in the current period t and are then expected to remain flexible forever. The corresponding flexprice output in this economy is denoted $\hat{y}^f_{t;t}$ and given by

$$\hat{y}^f_{t;t} \equiv F^f_{4\cdot} X_t. \tag{C.4}$$

We refer to this output level as *conditional potential output* (conditional on prices and wages becoming flexible in the same period and therefore conditional on the existing predetermined variables, including the capital stock). It corresponds to the definition of the natural rate of output presented in Woodford [38, section 5.3.4].

We then realize that we can define *conditional-h potential output*, where $h \in \{0, 1, ...\}$ refers to prices and wages unexpectedly becoming flexible and expected to remain flexible forever in period $t-h$, that is, h periods before t. This concept of potential output is denoted $\hat{y}^f_{t;t-h|t}$ and is given by

$$\hat{y}^f_{t;t-h} \equiv F^f_{4\cdot} X^f_{t;t-h} \equiv F^f_{4\cdot}(M^f X^f_{t-1;t-h} + C^f \varepsilon_t) \equiv F^f_{4\cdot}[(M^f)^h X_{t-h} + \sum_{s=0}^{h-1} (M^f)^s C^f \varepsilon_{t-s}]. \qquad \text{(C.5)}$$

Conditional-h potential output hence depends on the state of the economy (the predetermined variables) in period $t-h$ and the shocks from $t-h+1$ to t. Conditional-0 potential output is obviously the same as conditional potential output. Unconditional potential output is the limit of conditional-h potential output when h goes to infinity,

$$\hat{y}^f_{t;-\infty} = \lim_{h\to\infty} \hat{y}^f_{t;t-h}.$$

C.3. Projections of potential output

Consider now projections in period t of these alternative concepts of potential output. The projection in period t of *unconditional potential output*, $\{\hat{y}^f_{t+\tau;-\infty,t}\}^\infty_{\tau=0}$ (where the first subindex, $t+\tau$, refers to the future period for which potential output is projected; the second subindex, $-\infty$, indicates that unconditional potential output is considered; and the third subindex, t, refers to the period in which the projection is made and for which information is available), is related to the projection of the unconditional flexprice predetermined variables, $\{X^f_{t+\tau;-\infty,t}\}^\infty_{\tau=0}$, and is given by

$$\hat{y}^f_{t+\tau;-\infty,t} \equiv F^f_{4\cdot} X^f_{t+\tau;-\infty,t} \equiv F^f_{4\cdot}(M^f)^\tau X^f_{t;-\infty|t} \qquad \text{(C.6)}$$

for $\tau \geq 0$, where $X^f_{t;-\infty|t}$ denotes the estimated realization of the unconditional flexprice predetermined variables in period t conditional on information available in period t.

Here, $X^f_{t;-\infty|t}$ can be estimated from

$$X^f_{t;-\infty|t} = (M^f)^{t-t_0} X_{t_0|t} + \sum_{s=0}^{t-1-t_0} (M^f)^s C^f \varepsilon_{t-s|t}, \qquad \text{(C.7)}$$

where the unconditional flexprice equilibrium is approximated by a flexprice equilibrium that starts in a particular period $t_0 < t$, $X_{t_0|t}$ denotes the estimate conditional on information available in period t of the predetermined variables in period t_0, and $\varepsilon_{t-s|t}$ denotes the estimate conditional in information available in period t of the realization of the shock in period $t-s$.

The projection in period t of *conditional potential output* (that is, conditional-0 potential output), $\{\hat{y}^f_{t+\tau;t+\tau,t}\}^{\infty}_{\tau=0}$, is given by

$$\hat{y}^f_{t+\tau;t+\tau,t} \equiv F^f_{4\cdot} X_{t+\tau,t} \tag{C.8}$$

for $\tau \geq 0$. Note that the projection of conditional potential output in period $t+\tau$ then refers to the flexprice output for a flexprice equilibrium that starts in the future period $t+\tau$, not in the current period period t. In the latter case, it would instead be the projection of conditional-τ potential output in period $t+\tau$. Therefore, the projection of the predetermined variables in period $t+\tau$ of the actual economy, $X_{t+\tau,t}$, enters in (C.8), not the projection of the predetermined variables in period $t+\tau$ of the flexprice equilibrium starting in period t, $X^f_{t+\tau;t,t}$.

The projection in period t of *conditional-h potential output*, $\{\hat{y}^f_{t+\tau;t+\tau-h,t}\}^{\infty}_{\tau=0}$, is related to the projection of the conditional-h flexprice predetermined variables, $\{X^f_{t+\tau;t+\tau-h,t}\}^{\infty}_{\tau=0}$, and is given by

$$\hat{y}^f_{t+\tau;t+\tau-h,t} \equiv F^f_{4\cdot} X^f_{t+\tau;t+\tau-h,t} \tag{C.9}$$

for $\tau \geq 0$. That is, the projection of the conditional-h predetermined variables in period $t+\tau$ enters in (C.9), the predetermined variables in the flexprice equilibrium that starts h periods earlier, in period $t+\tau-h$. Furthermore, this projection is given by

$$X^f_{t+\tau;t+\tau-h,t} \equiv (M^f)^h X_{t+\tau-h,t} + \sum_{s=\tau}^{h-1} (M^f)^s C^f \varepsilon_{t+\tau-s|t},$$

where the summation term is zero when $\tau \geq h$. Thus, the projection in period t of the realization of conditional-h potential output in period $t+\tau$, $\hat{y}^f_{t+\tau;t+\tau-h,t}$, depends on the projection of the predetermined variables in period $t+\tau-h$ of the *actual* economy (with sticky prices and wages), $X_{t+\tau-h,t}$, and when $\tau < h$ also on the estimated shocks $\varepsilon_{t-s|t}$ for $s = 0, 1, ..., h-\tau-1$. (For $\tau > h$, $\varepsilon_{t+\tau-h|t} = 0$).

C.4. Output gaps

We can then consider several concepts of output gaps: We have the *trend* output gap, the gap between actual output and trend output, $\hat{y}_t$ (recall that $\hat{y}_t$ is the deviation from trend). For each concept of potential output, we have a corresponding concept of output gap: The *unconditional* output gap,

$$\hat{y}_t - y^f_{t;-\infty} \equiv \hat{y}_t - F^f_{4\cdot} X^f_{t;-\infty};$$

the *conditional* output gap,

$$\hat{y}_t - y^f_{t;t} \equiv \hat{y}_t - F^f_{4\cdot} X_t;$$

and the *conditional-h* output gap,

$$\hat{y}_t - y^f_{t;t-h} \equiv \hat{y}_t - F^f_{4\cdot} X^f_{t;t-h}.$$

The projections of the different output gaps are then defined in analogy with the projections of the different potential outputs.

C.5. Neutral interest rates and interest-rate gaps

For each concept of potential output, we have a corresponding concept of (short) *neutral nominal interest rate*. If the flexprice equilibrium is defined for a constant CPIX inflation, the neutral *nominal* interest rate is also the neutral *real* (CPIX) interest rate. For each neutral interest rate we can define a corresponding *interest-rate gap* as the difference between the actual interest rate and the neutral interest rate. The interest-rate gaps can be seen as measures of monetary-policy stance.

C.6. Efficient and potential output

As discussed in Svensson [33], *efficient* output can be defined as the welfare-maximizing output level. Efficient output can be conditional (conditional on current predetermined variables), conditional-h (conditional on the predetermined variables h periods ago and the shocks realized after that period), or unconditional. It could be constrained-efficient output (that is, second-best output), depending on what distortions are taken as given.

For each concept of efficient output, a corresponding concept of potential output can be constructed from a welfare-optimizing point view, that is, from the point of view of the suitable output target in a quadratic loss function corresponding to flexible inflation targeting. Svensson [33] suggests choosing the (detrended) output target at a constant distance from efficient output such that its unconditional mean equals the unconditional mean of (detrended) actual output. (This potential output can be called *constant-distance* potential output.) This should correspond to a concept of flexprice output for an economy where some distortions that affect the distance between flexprice output and efficient output are held constant, such as the price and wage markups and tax rates. The resulting constant-distance output gap would then have a welfare basis.

For each concept of constant-distance potential output, one can then define a constant-distance neutral interest rate and a corresponding interest-rate gap. The potential-output concepts discussed above are defined for constant price and wage markups and tax rates.

D. The Klein and AIM algorithms

The state-space form (2.1) is convenient for deriving the first-order conditions for optimal policy as in Söderlind [29] and Svensson [32] and the resulting state-space form of the combined model and first-order conditions are suitable for the application of the Klein [22] algorithm. However, the Klein algorithm appears to be slower than the alternative so-called AIM algorithm of Anderson and Moore [11], especially for large systems (see Anderson [10]). This may make it more practical, especially for estimation and Monte-Carlo simulations, to use the AIM algorithm to solve the system of difference equations. We therefore clarify the relation between the inputs and outputs of the two algorithms and how they can be converted to each other. Anderson [10] provides a comparison between AIM and other algorithms, including Klein's.

D.1. The Klein algorithm

Consider a system of stochastic difference equations written in the state-space form

$$\begin{bmatrix} y_{1,t+1} \\ Hy_{2,t+1|t} \end{bmatrix} = \tilde{A} y_t + \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix} \varepsilon_{t+1}, \quad t = 0, 1, ..., \tag{D.1}$$

where $y_t \equiv (y_{1t}', y_{2t}')'$ is an n-vector, y_{1t} is an n_1-vector of predetermined variables, y_{2t} is an n_2-vector of non-predetermined variables, $n = n_1 + n_2$, $y_{2,t+1|t} \equiv \mathrm{E}_t y_{2,t+1}$, and y_{10} is given. All linear stochastic difference equations can be written in this form, by the appropriate addition of new variables. In particular, for the model (2.1) with an arbitrary instrument rule of the form,

$$i_t = f_i \begin{bmatrix} X_t \\ x_t \end{bmatrix}, \tag{D.2}$$

the system of difference equations can be written as (D.1) with $y_{1t} \equiv X_t$, $y_{2t} \equiv x_t$, and $\tilde{A} \equiv A + Bf$.

The upper block of (D.1) can be written

$$y_{1,t+1} = \tilde{A}_{11} y_{1t} + \tilde{A}_{12} y_{2t} + \tilde{C} \varepsilon_{t+1}, \tag{D.3}$$

where the matrix $\tilde{A}$ is partitioned conformably with y_{1t} and y_{2t} as

$$\tilde{A} \equiv \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}.$$

The system (D.3) determines $y_{1,t+1}$ in period $t+1$ for given y_t and ε_{t+1}. We note that the one-period-ahead prediction error in $y_{1,t+1}$, the innovation in $y_{1,t+1}$, is given by

$$y_{1,t+1} - y_{1,t+1|t} = \tilde{C}\varepsilon_{t+1} \tag{D.4}$$

and is exogenous. Having exogenous one-period-ahead prediction errors is a general definition of predetermined variables (Klein [22]).

The lower block of (D.1) can be written

$$y_{2t} = \tilde{A}_{22}^{-1}(Hy_{2,t+1|t} - \tilde{A}_{21}y_{1t}). \tag{D.5}$$

The system (D.5) determines y_{2t} given y_{1t} and expectations $y_{2,t+1|t}$. The matrix $\tilde{A}_{22}$ is assumed to be nonsingular.

Take expectations of (D.1) and consider the system

$$\tilde{H}y_{t+1|t} = \tilde{A}y_t$$

for $t \geq 0$ with y_{10} given, where in this case $\tilde{H}$ satisfies

$$\tilde{H} \equiv \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & H \end{bmatrix} \tag{D.6}$$

when partitioned conformably with y_{1t} and y_{2t}. In the more general case examined below, $\tilde{H}$ need not have the upper left submatrix be the identity matrix, and the upper right and lower left submatrices need not be zero matrices, as long as the upper left submatrix $\tilde{H}_{11}$ is nonsingular. Consider the generalized eigenvalues of pair of matrices $(\tilde{H}, \tilde{A})$.[30] Under the assumption that there are exactly n_2 generalized eigenvalues with modulus larger than unity, the system (D.1) has a unique solution for y_{20} and y_t $(t \geq 1)$ for given y_{10}.

The Klein [22] algorithm uses the matrices $\tilde{H}$ and $\tilde{A}$ and the scalar n_1 (the number of predetermined variables) as inputs and returns the $n_1 \times n_1$ matrix M and the $n_2 \times n_1$ matrix F such that, given that the one-period-ahead prediction errors of y_{1t} are given by (D.4), the solution to this system can be written

$$y_{2t} = Fy_{1t}, \tag{D.7}$$

$$y_{1,t+1} = My_{1t} + \tilde{C}\varepsilon_{t+1}, \tag{D.8}$$

[30] The generalized eigenvalues of the matrices $(\tilde{H}, A)$ are the complex numbers λ that satisfy $\lambda\tilde{H}x = Ax$ for some $x \neq 0$, that is, the complex numbers that satisfy $\det(\lambda\tilde{H} - A) = 0$.

for $t \geq 0$. The matrices M and F depend on the matrices $\tilde{A}$ and $\tilde{H}$ but not on the matrix $\tilde{C}$. They satisfy the identity

$$(\tilde{H}M - \tilde{A}) \begin{bmatrix} I \\ F \end{bmatrix} \equiv 0, \tag{D.9}$$

which can be used to test the solution.

The one-period-prediction errors for $y_{2,t+1}$ satisfy

$$y_{2,t+1} - y_{2,t+1|t} = F(y_{1,t+1} - y_{1,t+1|t}) = F\tilde{C}\varepsilon_{t+1}.$$

Since the matrix F is endogenous and part of the solution, the one-period-prediction errors for y_{2t} are endogenous. Having endogenous one-period-ahead prediction errors is a general definition of non-predetermined variables (Klein [22]).

We can also consider a system in the more general state-space form

$$\begin{bmatrix} \tilde{H}_{11}y_{1,t+1} + \tilde{H}_{12}y_{2,t+1|t} \\ \tilde{H}_{21}y_{1,t+1|t} + \tilde{H}_{22}y_{2,t+1|t} \end{bmatrix} = \tilde{A}y_t + \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix} \varepsilon_{t+1}. \tag{D.10}$$

Note that the upper block has $y_{1,t+1}$ and $y_{2,t+1|t}$ on the left side, where as the lower block has $y_{1,t+1|t}$ and $y_{2,t+1|t}$. Such a system results from combining the model equations and the first-order equations from optimization under commitment in a stochastic linear-quadratic system with forward-looking variables, (B.4), (see Söderlind [29] and Svensson [31] and [32]). Such a system also results from the model (2.1) with an arbitrary instrument rule of the form

$$i_t = f_i \begin{bmatrix} X_t \\ x_t \end{bmatrix} + f_{i2}x_{t+1|t}. \tag{D.11}$$

When $f_{i2} \neq 0$, this rule is sometimes called a "forecast-based" instrument rule; some problems with such instrument rules are discussed in Svensson [30]. For (2.1) with a forecast-based instrument rule (D.11), we have $y_{1t} \equiv X_t$, $y_{2t} \equiv x_t$, $\tilde{A} \equiv A + Bf$, $\tilde{H}_{11} = I$, $\tilde{H}_{12} = -B_1 f_{i2}$, $\tilde{H}_{21} = 0$, and $\tilde{H}_{22} = H - B_2 f_{i2}$ (where $B \equiv [B_1' \;\; B_2']'$ is partitioned conformably with X_t and x_t).

In the general form (D.10), the submatrices $\tilde{H}_{11}$ and $\tilde{A}_{22}$ of $\tilde{H}$ and $\tilde{A}$ are assumed to be nonsingular, although $\tilde{H}_{11}$ need not be the identity matrix. The matrices $\tilde{H}_{12}$ and $\tilde{H}_{21}$ need not be zero. The upper block of (D.10) is an equation system that determines $y_{1,t+1}$ in period $t+1$ for given y_t, $y_{2,t+1|t}$, and ε_{t+1}. Therefore the system must be possible to solve for $y_{1,t+1}$, which requires that $\tilde{H}_{11}$ is nonsingular. Hence, the one-period-ahead prediction errors for $y_{1,t+1}$ satisfy

$$\tilde{H}_{11}(y_{1,t+1} - y_{1,t+1|t}) = \tilde{C}\varepsilon_{t+1},$$

so they remain exogenous and can be written

$$y_{1,t+1} - y_{1,t+1|t} = \tilde{H}_{11}^{-1}\tilde{C}\varepsilon_{t+1}.$$

The lower block of (D.10) is an equation system determining y_{2t} in period t for given y_{1t} and $y_{2,t+1|t}$. This requires that $\tilde{A}_{22}$ is nonsingular.

The Klein algorithm applied to the general system (D.10) uses $\tilde{H}$, $\tilde{A}$, and n_1 as inputs and returns M and F. The solution for y_{20} and y_t $(t \geq 1)$ for given y_{10} is given by (D.7) and

$$y_{1,t+1} = My_{1t} + \tilde{H}_{11}^{-1}\tilde{C}\varepsilon_{t+1} \tag{D.12}$$

for $t \geq 0$. The identity (D.9) is also satisfied for the more general case.

The one-period-prediction errors for $y_{2,t+1}$ satisfy

$$y_{2,t+1} - y_{2,t+1|t} = F(y_{1,t+1} - y_{1,t+1|t}) = F\tilde{H}_{11}^{-1}\tilde{C}\varepsilon_{t+1}$$

and are endogenous.

We show below that the system (D.10) can always be written in the form (D.1) by the introduction of an additional forward-looking variable, $q_t \equiv y_{2,t+1|t}$, and some reshuffling.

D.2. The AIM algorithm

Anderson [9] and [10] specify the Anderson and Moore [11] and [12] algorithm (also called the AIM algorithm). It solves the deterministic problem

$$\sum_{i=-\tau}^{\theta} H_i x_{t+i} = \Psi z_t \qquad (t \geq 0), \tag{D.13}$$

for the endogenous n_x-vectors x_t for $t \geq 0$, where z_t for $t \geq 0$ is an exogenous n_z-vector, H_i for $i = -\tau, -\tau+1, ..., \theta$ are $n_x \times n_x$ matrices, Ψ is an $n_x \times n_z$ matrix, $\theta \geq 0$, $\tau \geq 1$, and x_t is given for $t = -\tau, -\tau+1, ..., -1$.

The solution from the AIM algorithm is simplest for the special case of $\tau = \theta = 1$.[31] All linear systems can be reduced to that case by introducing variables with longer leads or lags as separate variables. Let us consider the related system of stochastic difference equations,

$$H_{-1}x_{t-1} + H_0 x_t + H_1 x_{t+1|t} = \Psi\varepsilon_t \qquad (t \geq 0), \tag{D.14}$$

where $x_{t+1|t} \equiv \mathrm{E}_t x_{t+1}$, the n_ε-vector ε_t is an i.i.d. shock in period t with mean zero and covariance matrix I_{n_ε}, and x_{-1} is given. The system (D.14) is completely described by the $n_x \times n_x$ matrices H_{-1}, H_0, and H_1, the $n_x \times n_\varepsilon$ matrix Ψ, and the given x_{-1}.

[31] The AIM Matlab program available at www.federalreserve.gov/pubs/oss/oss4/aimindex.html requires $\tau \geq 1$ and $\theta \geq 1$.

For the special case of $\tau = \theta = 1$, the AIM algorithm uses the matrices H_{-1}, H_0, and H_1 as inputs and returns the $n_x \times n_x$ matrix $\bar{B}$, called the convergent autoregressive matrix, such that the solution to (D.14) can be written

$$x_t = \bar{B}x_{t-1} + \Phi\Psi\varepsilon_t, \tag{D.15}$$

where

$$\Phi \equiv (H_0 + H_1\bar{B})^{-1}. \tag{D.16}$$

We can see that Φ must satisfy (D.16) by substituting $x_{t+1|t} = \bar{B}x_t$ into (D.14) and solving for x_t:

$$\begin{aligned} H_{-1}x_{t-1} + H_0x_t + H_1\bar{B}x_t &= \Psi\varepsilon_t, \\ H_{-1}x_{t-1} + (H_0 + H_1\bar{B})x_t &= \Psi\varepsilon_t, \end{aligned}$$

$$x_t = -(H_0 + H_1\bar{B})^{-1}H_{-1}x_{t-1} + (H_0 + H_1\bar{B})^{-1}\Psi z_t.$$

The matrices $\bar{B}$ and Φ depend on H_{-1}, H_0, and H_1 but not on Ψ, consistent with certainty equivalence of the system. The matrix $\bar{B}$ satisfies the identify

$$H_{-1} + H_0\bar{B} + H_1\bar{B}^2 \equiv 0, \tag{D.17}$$

which can be used to test the solution.

For the case when the right side of (D.14) is Ψz_t and z_t is an arbitrary exogenous stochastic process, the solution can be written

$$x_t = \bar{B}x_{t-1} + \sum_{s=0}^{\infty} \bar{F}^s \Phi\Psi z_{t+s|t}, \tag{D.18}$$

where[32]

$$\bar{F} \equiv -\Phi H_1. \tag{D.19}$$

We can show that the solution can be written on the form (D.18) by assuming that the solution can be written

$$x_t = \bar{B}x_{t-1} + R_t, \tag{D.20}$$

$$x_{t+1|t} = \bar{B}x_t + R_{t+1|t}, \tag{D.21}$$

and substituting (D.21) into (D.14) with the right side equal to Ψz_t,

$$H_{-1}x_{t-1} + H_0x_t + H_1(\bar{B}x_t + R_{t+1|t}) = \Psi z_t.$$

[32] Note that there seems to be a typo in the expression for F in Anderson [10, p. 5].

Solving for x_t gives

$$x_t = -(H_0 + H_1\bar{B})^{-1}H_{-1}x_{t-1} + (H_0 + H_1\bar{B})^{-1}\Psi z_t - (H_0 + H_1\bar{B})^{-1}H_1 R_{t+1|t}.$$

Identification with (D.20) and using (D.16) give that $\bar{B}$ satisfies

$$\bar{B} \equiv -\Phi H_{-1}$$

(which by (D.16) is just the identity (D.17)) and that R_t satisfies the stochastic difference equation

$$R_t = \Phi\Psi z_t + \bar{F}R_{t+1|t}, \tag{D.22}$$

where $\bar{F}$ satisfies (D.19). Assuming that the eigenvalues of $\bar{F}$ have modulus less than unity and that $z_{t+s|t}$ for $s \geq 0$ is sufficiently bounded, we can solve (D.22) forward to get

$$R_t = \sum_{s=0}^{\infty} \bar{F}^s \Phi\Psi z_{t+s|t},$$

which confirms (D.18).

If z_t follows an exogenous autoregressive process,

$$z_t = A_z z_{t-1} + C_z \varepsilon_t,$$

it is arguably easiest to incorporate it into the predetermined variables y_{1t} and expand the matrices H_{-1}, H_0, and H_1 accordingly.

D.3. Relations between Klein and AIM inputs and outputs

In order to transform a system in the state-space form (D.1) to the AIM form (D.14), the system (D.1) can be written as

$$y_{1t} - \tilde{A}_{11}y_{1,t-1} - \tilde{A}_{12}y_{2,t-1} = \tilde{C}\varepsilon_t, \tag{D.23}$$

$$Hy_{2,t+1|t} - \tilde{A}_{21}y_{1t} - \tilde{A}_{22}y_{2t} = 0, \tag{D.24}$$

for $t \geq 0$, with y_{-1} and ε_0 given (hence y_{10} is also given by (D.23) for $t = 0$). Then we can identify the vector x_t and the matrices H_{-1}, H_0, H_1, and Ψ directly from (D.23) and (D.24) as

$$x_t \equiv \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}, \quad H_{-1} \equiv \begin{bmatrix} -\tilde{A}_{11} & -\tilde{A}_{12} \\ 0 & 0 \end{bmatrix}, \quad H_0 \equiv \begin{bmatrix} I & 0 \\ -\tilde{A}_{21} & -\tilde{A}_{22} \end{bmatrix}, \tag{D.25}$$

$$H_1 \equiv \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}, \quad \Psi \equiv \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix}. \tag{D.26}$$

The above identification of H_{-1}, H_0, and H_1 exploits the property (D.6) of $\tilde{H}$ that the upper left submatrix is the identity matrix and the upper right and lower left submatrices are zero.

Given (D.7) and (D.23), we realize that the convergent autoregressive matrix $\bar{B}$ will satisfy

$$\bar{B} \equiv \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ F\tilde{A}_{11} & F\tilde{A}_{12} \end{bmatrix}.$$

Furthermore, the matrix Φ will in this case satisfy

$$\Phi \equiv \begin{bmatrix} I & 0 \\ F & \Phi_{22} \end{bmatrix}. \tag{D.27}$$

Thus, we can extract F from the lower left submatrix Φ_{21} of Φ,

$$F \equiv \Phi_{21}.$$

The submatrix Φ_{22} corresponds to a hypothetical i.i.d. shock on the right side of (D.24) and is irrelevant under the assumption that any such shocks are zero since they have already been incorporated among the predetermined variables.

In order to extract the Klein output matrix M, partition $\bar{B}$ as

$$\bar{B} \equiv \begin{bmatrix} \bar{B}_{\cdot 1} & \bar{B}_{\cdot 2} \end{bmatrix},$$

where $\bar{B}_{\cdot 1}$ is the submatrix consisting of the n_1 first columns of $\bar{B}$. Since by (D.7) all x_t that are solutions to the system satisfy

$$[F \quad -I]\, x_t = 0$$

for $t \geq 0$, we have

$$\bar{B} + \bar{B}_{\cdot 2}\,[F \quad -I] \equiv \begin{bmatrix} \bar{B}_{\cdot 1} + \bar{B}_{\cdot 2}F & 0 \end{bmatrix} \equiv \begin{bmatrix} M & 0 \\ FM & 0 \end{bmatrix}.$$

Then M can be extracted from the upper left submatrix of the matrix specified on the left side of this identity.

The more general system (D.10) can be written in a form suitable for conversion to the AIM form by lagging the first block one period and introducing the new vector nonpredetermined variables $q_t \equiv y_{2,t+1|t}$. Then the equation system can be written as

$$\tilde{H}_{11}y_{1t} - \tilde{A}_{11}y_{1,t-1} - \tilde{A}_{12}y_{2,t-1} + \tilde{H}_{12}q_{t-1} = \tilde{C}\varepsilon_t, \tag{D.28}$$

$$\tilde{H}_{21}y_{1,t+1|t} + \tilde{H}_{22}y_{2,t+1|t} - \tilde{A}_{21}y_{1t} - \tilde{A}_{22}y_{2t} = 0, \tag{D.29}$$

$$y_{2,t+1|t} - q_t = 0. \tag{D.30}$$

This then allows us to identify $x_t \equiv (x'_{1t}, x'_{2t})'$ and the matrices H_{-1}, H_0, H_1, and Ψ as

$$x_t \equiv \begin{bmatrix} y_{1t} \\ y_{2t} \\ q_t \end{bmatrix}, \quad H_{-1} \equiv \begin{bmatrix} -\tilde{A}_{11} & -\tilde{A}_{12} & \tilde{H}_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_0 \equiv \begin{bmatrix} \tilde{H}_{11} & 0 & 0 \\ -\tilde{A}_{21} & -\tilde{A}_{22} & 0 \\ 0 & 0 & -I \end{bmatrix}, \tag{D.31}$$

$$H_1 \equiv \begin{bmatrix} 0 & 0 & 0 \\ \tilde{H}_{21} & \tilde{H}_{22} & 0 \\ 0 & I & 0 \end{bmatrix}, \quad \Psi \equiv \begin{bmatrix} \tilde{C} \\ 0 \\ 0 \end{bmatrix}, \tag{D.32}$$

where $x_{1t} = y_{1t}$ and $x_{2t} = (y'_{2t}, q'_t)'$.

Given (D.7) and (D.8), we have

$$y_{2,t+1|t} = F y_{1,t+1|t} = FM y_{1t}.$$

From (D.28) we then realize that the convergent autoregressive matrix $\bar{B}$ satisfies

$$\bar{B} \equiv \begin{bmatrix} \tilde{H}_{11}^{-1}\tilde{A}_{11} & \tilde{H}_{11}^{-1}\tilde{A}_{12} & -\tilde{H}_{11}^{-1}\tilde{H}_{12} \\ F\tilde{H}_{11}^{-1}\tilde{A}_{11} & F\tilde{H}_{11}^{-1}\tilde{A}_{12} & -F\tilde{H}_{11}^{-1}\tilde{H}_{12} \\ FM\tilde{H}_{11}^{-1}\tilde{A}_{11} & FM\tilde{H}_{11}^{-1}\tilde{A}_{12} & -FM\tilde{H}_{11}^{-1}\tilde{H}_{12} \end{bmatrix}, \tag{D.33}$$

and that the matrix Φ satisfies

$$\Phi \equiv \begin{bmatrix} \tilde{H}_{11}^{-1} & 0 & 0 \\ F\tilde{H}_{11}^{-1} & \Phi_{22} & \Phi_{23} \\ FM\tilde{H}_{11}^{-1} & \Phi_{32} & \Phi_{33} \end{bmatrix}. \tag{D.34}$$

The four lower right submatrices of Φ correspond to hypothetical i.i.d. shocks on the right side of (D.29) and (D.30) and are irrelevant under our assumptions that such i.i.d. shocks are zero. It follows that the Klein output matrix F can be extracted from the middle left submatrix Φ_{21} of Φ according to

$$F = \Phi_{21}\tilde{H}_{11}.$$

In order to extract the Klein output matrix M, partition $\bar{B}$ according to

$$\bar{B} \equiv \begin{bmatrix} \bar{B}_{\cdot 1} & \bar{B}_{\cdot 2} & \bar{B}_{\cdot 3} \end{bmatrix},$$

where $\bar{B}_{\cdot 1}$ is the submatrix consisting of the n_1 first columns of $\bar{B}$, $\bar{B}_{\cdot 2}$ is the submatrix consisting of the next n_2 columns, and $\bar{B}_{\cdot 3}$ is the submatrix consisting of the last n_2 columns. Since x_t that are solutions to the system will satisfy

$$\begin{bmatrix} F & -I & 0 \\ FM & 0 & -I \end{bmatrix} x_t = 0,$$

it follows that

$$\bar{B} + \begin{bmatrix} \bar{B}_{\cdot 2} & \bar{B}_{\cdot 3} \end{bmatrix} \begin{bmatrix} F & -I & 0 \\ FM & 0 & -I \end{bmatrix} \equiv \begin{bmatrix} \bar{B}_{\cdot 1} + \bar{B}_{\cdot 2}F + \bar{B}_{\cdot 3}FM & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} M & 0 & 0 \\ FM & 0 & 0 \\ FM^2 & 0 & 0 \end{bmatrix}.$$

Then M can be extracted from the upper left submatrix of the matrix specified on the left side of the identity.[33]

We can transform the general system (D.10) to the simpler system (D.1) by multiplying (D.28) by $\tilde{H}_{11}^{-1}$ and leading it one period, resulting in (D.35), substituting the expectation of the right side of (D.35) for $y_{1,t+1|t}$ in (D.29), and rewriting the system as

$$y_{1,t+1} = \hat{A}_{11}y_{1t} + \hat{A}_{12}y_{2t} + \hat{A}_{13}q_t + \hat{C}\varepsilon_{t+1}, \tag{D.35}$$

$$\tilde{H}_{22}y_{2,t+1|t} = \hat{A}_{21}y_{1t} + \hat{A}_{22}y_{2t} + \hat{A}_{23}q_t, \tag{D.36}$$

$$y_{2,t+1|t} = q_t, \tag{D.37}$$

where

$$\hat{A}_{11} \equiv \tilde{H}_{11}^{-1}\tilde{A}_{11}, \quad \hat{A}_{12} \equiv \tilde{H}_{11}^{-1}\tilde{A}_{12}, \quad \hat{A}_{12} \equiv -\tilde{H}_{11}^{-1}\tilde{H}_{12},$$

$$\hat{A}_{21} \equiv \tilde{A}_{21} - \tilde{H}_{21}\hat{A}_{11}, \quad \hat{A}_{22} \equiv \tilde{A}_{22} - \tilde{H}_{21}\hat{A}_{12}, \quad \hat{A}_{23} \equiv -\tilde{H}_{21}\hat{A}_{13}, \quad \hat{C} = \tilde{H}_{11}^{-1}\tilde{C}.$$

A system in the AIM form (D.14), where the $x_t \equiv (x_{1t}', x_{2t}')'$ and x_{1t} and x_{2t} are predetermined and nonpredetermined variables, respectively, can always be written in the state-space form (D.10) with the non-predetermined variables $y_{2t} \equiv x_{2t}$ and the appropriate definition of additional predetermined variables y_{1t}, for instance,

$$y_{1t} \equiv \begin{bmatrix} x_{1t} \\ \Psi_2\varepsilon_t \\ x_{t-1} \end{bmatrix}, \tag{D.38}$$

where Ψ is partitioned conformably with x_{1t} and x_{2t},

$$\Psi \equiv \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$$

and the matrix $\tilde{C}$ is given by

$$\tilde{C} \equiv \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ 0 \end{bmatrix}.$$

(The details of the corresponding matrices $\tilde{H}$ and $\tilde{A}$ are not reported here.)

D.4. Klein with arbitrary exogenous stochastic process

Note that, for the case when the i.i.d. shock ε_{t+1} is replaced by the arbitrary exogenous stochastic process z_t, conversion of Klein input and output to AIM output, including $\bar{B}$, Φ, and $\bar{F}$, allows

[33] Note that this way of extracting F and M seems more general than that suggested in Anderson [10]. The latter requires that M is invertible, which is generally not the case.

the convenient representation of the solution as (D.18). This approach to the solution for this case seems simpler and cleaner than the derivation of the corresponding solution in Klein [22] and Svensson [31].

In order to sort this out, consider the Klein system of stochastic difference equations,

$$\begin{bmatrix} \tilde{H}_{11}y_{1,t+1} + \tilde{H}_{12}y_{2,t+1|t} \\ \tilde{H}_{21}y_{1,t+1|t} + \tilde{H}_{22}y_{2,t+1|t} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} y_t + \begin{bmatrix} \tilde{C}_1 z_{t+1} \\ \tilde{C}_2 z_t \end{bmatrix}, \tag{D.39}$$

for $t \geq 0$, where z_t is an arbitrary stochastic n_z-vector process, $y_t = (y_{1t}', y_{2t}')'$, y_{1t} is an n_1-vector of predetermined variables, y_{2t} is an n_2-vector of nonpredetermined variables, and y_{10} and z_0 are given. It follows that the prediction errors for y_{1t} satisfy

$$y_{1t} - y_{1t|t-1} = \tilde{H}_{11}^{-1}\tilde{C}_1(z_t - z_{t|t-1}). \tag{D.40}$$

We can rewrite this to a form suitable for AIM,

$$\begin{aligned} \tilde{H}_{11}y_{1t} - \tilde{A}_{11}y_{1,t-1} - \tilde{A}_{12}y_{2,t-1} + \tilde{H}_{12}q_{t-1} &= \tilde{C}_1 z_t, \\ \tilde{H}_{21}y_{1,t+1|t} + \tilde{H}_{22}y_{2,t+1|t} - \tilde{A}_{21}y_{1t} - \tilde{A}_{22}y_{2t} &= \tilde{C}_2 z_t, \\ y_{2,t+1|t} - q_t &= 0. \end{aligned} \tag{D.41}$$

We can then identify the $(n_1 + 2n_2)$-vector $x_t = (y_{1t}', y_{2t}', q_t')'$, and the $(n_1 + 2n_2) \times (n_1 + 2n_2)$ matrices H_{-1}, H_0, and H_1 as in (D.31) and (D.32). However, the $(n_1 + 2n_2) \times n_z$ matrix Ψ is now given by

$$\Psi \equiv \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ 0 \end{bmatrix}.$$

AIM output consists of the matrices $(n_1 + 2n_2) \times (n_1 + 2n_2)$ matrices $\bar{B}$, Φ, and $\bar{F}$, where $\bar{B}$ satisfies (D.33) and Φ and $\bar{F}$ satisfy (D.16) and (D.19). We then have the solution

$$y_t = \bar{B}y_{t-1} + \sum_{s=0}^{\infty} \bar{F}^s \Phi \Psi z_{t+s|s}.$$

This solution is an alternative way to reach the solution for arbitrary exogenous disturbances derived in Klein [22] and Svensson [31].

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